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UMI
COMPUTER-BASED REPRESENTATIONS IN MATHEMATICS CLASSROOMS: THE EFFECTS
OF MULTIPLE LINKED AND SEMI-LINKED REPRESENTATIONS ON STUDENTS'
LEARNING OF LINEAR RELATIONSHIPS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy in the Graduate School of The Ohio State
University

By

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* * * * *

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ABSTRACT

The utilization of technology in multiple representations has become one of the significant topics in mathematics education in the last decade. Here, multiple representations are defined as external mathematical embodiments of ideas and concepts to provide the same information in more than one form. One example of this type of environment is educational software with linked multiple representations. Linked multiple representations are a group of representations in which, upon altering a given representation, every other representation is automatically updated to reflect the same change. In this study semi-linked representations were defined as those for which the corresponding updates of changes within the representations are available only upon request and are not automatic.

The focus of this study was comparing three groups of Algebra I students: one group using linked representation software, the second group using similar software but with semi-linked representations, and the control group. Briefly, the main question was what are the effects on students’ understanding of linear relationships using linked representation software compared to those using semi-linked representations. Moreover, this study investigated students’ attitudes towards and preferences for various mathematical representations.
Data collection methods included mathematics pre- and posttests, follow-up interviews with all students after the mathematics posttest, computer-based clinical interviews at the end of the treatment with 5 students from each experimental group, classroom and lab observations, and document analysis. A survey was conducted at the end of the study in order to see students’ opinions about mathematics, representations in general, and the computer environment.

Questions in all written tests used in this study were clustered into categories, and those categories were compared across the three groups—linked, semi-linked, and control. The categories were: Word Problems, Interpreting/Constructing and Reading Graphs, Solving and Constructing Equations), Reading and Constructing Tables, and Misconceptions (Height/Slope, Point/Interval, Graph as Picture). These categories were compared using a nonparametric test—Kruskal-Wallis (a test for independent samples)—to identify differences between the linked, semi-linked, and control groups. The results of this test showed that there were no significant differences in achievement between the groups in any category of problems on either the pretest or posttest. To study how the groups’ achievement changed from pretest to posttest, a nonparametric test—the Wilcoxon test for dependent samples, was performed. Some of the improvements were significant at the .05 level, such as for the experimental groups in the categories of interpreting graphs and constructing equations, the semi-linked group for the height/slope misconception category, and the linked group for the graph as picture category.
The software provided an environment that offered opportunities and constraints for students to experience assimilation and accommodation through a series of equilibrium-disequilibrium states. Linkage was helpful for some students in constructing mathematical ideas, whereas students who had already constructed those ideas and trusted them did not need the linkage. Semi-linked software forced students to be more active in their learning process, to make use of their existing knowledge and the information provided by the software, and to construct new mathematical ideas or connections among the representations with the help of new information. However, some students did not have enough basis on which to construct new mathematical ideas and that is where linkage could be very helpful. The conclusion of this study was that semi-linked representations can be as effective as linked representations and that there is a role for each in different situations, at different levels, and with different mathematical concepts.

Most of the students mentioned that they found VideoPoint helpful in learning mathematics. Students reported that easy access to all representations at once was helpful in comparing different representations or checking their answers. Personal preferences and previous experience/knowledge were the main themes emerging as reasons for preferring one representation over another, and there were specific reasons for choosing a particular representation, such as being able to find an exact answer with an equation, the visual advantages of graphs, or the organized information provided by tables.
Dedicated to my daughter Ekim Ö zgün Koca
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CHAPTER 1

STATEMENT OF THE PROBLEM

As educators, we must prepare our students and ourselves for new and exciting forms of technology that take the best of what we have to offer as teachers and apply it to our subject matter.  
(Diem, 1992, p.109)

In the age of technology, we cannot deny that computers and educational software are crucial components in the learning and teaching of mathematics. The National Council of Teachers of Mathematics (NCTM) makes strong statements on behalf of the use of technology in mathematics education. For example, they assert that "mathematics programs must take full advantage of the power of calculators and computers at all levels" (NCTM, 1980, p.1). They also state that "changes in technology and the broadening of the areas in which mathematics is applied have resulted in growth and changes in the discipline of mathematics itself" (NCTM, 1989, p.7). On the other hand, Kaput and Thompson (1994) assert that, "With few exceptions, the mathematics education community, and especially researchers, have had a passive attitude
towards technology" (p. 681) and then raise this question: "Will research in mathematics education inform the growth of technology-supported mathematics teaching and learning?" (p. 682). These contradictory statements point to a need for research in instructional aspects of technology in mathematics education. The focus of this study was specifically on computer uses in mathematics education.

Ninety-eight percent of all American schools own computers, and survey results indicate that about half of 13- and 17-year-olds have access to a computer to learn mathematics (Coley et al., 1997). The Office of Technology Assessment (OTA, 1995) also reports that mathematics is the subject that uses computers most often after English classes among non-computer subjects. Furthermore, about 54 percent of 13-year-olds and 42 percent of 17-year-olds have studied mathematics through computer instruction, and approximately 70 percent of all of them have used a computer in solving mathematics problems (Campbell et al., 1997).

Meanwhile, Mullis et al. (1998) report that in half of the countries that participated in the Third International Mathematics and Science Study (TIMSS), which took place during the 1995-1996 school year and involved 41 countries, "The students who reported using a computer most frequently were also those with the highest performance on mathematics and science literacy, but in the rest the relationship was less regular" (p. 110). Collectively, this information suggests that technology is becoming an integral part of education, especially mathematics education.
Given the increasing interest in computers within mathematics education we,
as mathematics educators, need to concentrate on the pedagogical rationale for
using computers in the mathematics classroom. Particularly, we need to concentrate
on instructional features of computers and educational software because, as Becker
succinctly puts it:

The 1990s promise to be different in at least two ways: (1) more numerous
and more powerful computers and (2) more sophisticated software will
come to be used. Major changes in math and science curricula are likely to
make computers more relevant and more naturally integral to instructional
activities in those subjects. (Becker, 1991, p.25)

Recent changes in the mathematics curriculum
also make it necessary for mathematics
educators and researchers to study the
instructional features of mathematics education
software. Although there are already a number
of instructional features of mathematics
educational software, the potential use of software in mathematics education has
only begun to be investigated. This study focused on an as yet underexplored area:
the use of multiple representations, their instructional advantages, and students’
preferences for types of representations in computer settings. The special focus was
on multiple representations because representations have always been crucial
components of mathematics, and technology opens new avenues for teaching and
learning mathematics with computer-based representations.
Representations

There are different definitions for representation in mathematics education. Most researchers differentiate between external and internal representations, where external representations are embodiments of ideas or concepts like charts, tables, graphs, etc., and internal representations are schemata or cognitive structures of a person (Janvier et al., 1993). Kaput discusses the following components which should be considered when talking about representations:

- The represented world,
- The representing world,
- What aspects of the represented world are being represented,
- What aspects of the representing world are doing the representing, and
- The correspondence between the two worlds. (Kaput, 1987, p. 23)

Types of representations also mentioned by Kaput are:

- Cognitive and perceptual representation,
- Explanatory representation involving models,
- Representation within mathematics, and
- External symbolic representation. (Kaput, 1987, p.23)

Dufour-Janvier et al. (1987) identified the roles of representations in mathematics education as follows:

- Representations are an inherent part of mathematics;
- Representations are multiple concretizations of a concept; and
- Representations are used locally to mitigate certain difficulties.
- Representations are intended to make mathematics more attractive and interesting. (pp. 110-111)
In this study, representation will refer to an external symbolic mathematical representation. Multiple representations are defined as providing the same information in more than one form of external, symbolic mathematical representation. With the advent of technology, it has become possible to link representations. Collectively, linked multiple representations are a group of representations in which every representation currently being utilized in a technological setting is automatically updated to reflect a change in a given representation (Rich, 1995/1996). Semi-linked representations are those for which the corresponding updates of changes within the representations are available only upon request and so are not automatic.

Figure 1.1. The Linkages among Linked and Semi-Linked Representations in VideoPoint
The differences between the linkages in the linked and the semi-linked representations are summarized in Figure 1.1. As one can observe, the graph, table, and movie representations are linked two-way in the linked version. This means that when the user clicks on a point in those representations, the corresponding data points in all other two representations are highlighted. Moreover, when the linked version user clicks to see the algebraic form (the equation of best fit) of the phenomena, the line of best fit is also graphed in the graph window automatically.

On the other hand, the user of the semi-linked version is not able to see any updates when s/he clicks on one representation. The only linkage that is available in the semi-linked version is between the graph and equation form. When the user estimates the coefficients in the algebraic form, s/he has an option to see the graph of the predicted equation.

**Purpose of the Study**

The use of multiple representations with or without technology is encouraged by many educators in mathematics education. Indeed, many studies in the literature, as cited in the “theoretical framework and review of literature” sections, call for further studies in order to improve the instruction in this area, and they address several recurring questions on this topic. The major questions asked in the articles are:

- How are reasoning, problem solving, and learning processes altered by the use of multiple representations compared to the use of a single representation?
• Does the simultaneous presence of manipulable, linked representations support the learning of translation skills between those modes of representations?

• How do students misinterpret representations?

• What are expected outcomes that justify a wide variety of representations?

In examining the use of multiple representations, some of the studies examined one group of students and looked for the effects of multiple representations on mathematics education in technology-rich environments (Borba, 1993; Borba & Confrey, 1993, Confrey, et al., 1991; Donnelly, 1995; Lin, 1993; O'Keefe, 1992/1993; Rizutti, 1991/1992; Yerushalmy, 1991a). There have also been studies that focused on comparing students in a traditional classroom and in a computer group (Dyer, 1994/1995; Porzio, 1994/1995; Rich, 1995/1996; Rosenheck, 1991/1992) and analyzed the effects of technology on learning. Two studies compared two technology groups: one which utilized graphing calculators and another that employed computers (Porzio, 1994/1995; Rich, 1995/1996). In these studies, students in the computer group sometimes used linked representation software and sometimes used semi-linked representation software. However, it appears there may have been no study that has compared directly the effects of linked multiple representations to semi-linked multiple representations. It was the premise of this study that semi-linked representations may be as effective as linked representations, and that there may be roles for both kinds of linking in
different instructional settings. The focus of this study was a comparison of three
groups of students: one group using linked representation software, one using
similar software but with semi-linked representations, and a control group.

Research Questions

The research questions of this study were:

1. What are the effects on students' understanding of linear relationships
   using linked representation software compared to using semi-linked
   representation software?

2. What are students' attitudes towards and preferences for mathematical
   representations—equations, tables, or graphs?

This study aimed through the first question to examine the effects of the
automatic linking feature of the computer software on student learning where linked
vs. semi-linked representation environments were provided. The second question
explored students' preferences for the use of various forms of representations.
Specifically, this dissertation studied a) students' strategies for and beliefs about the
use of representations in mathematics classrooms, and b) whether being able to
access three or four (linked) types of representations in a computer environment
would change their preferences.

Significance of the Study

Since this study focused on the effects of linked and semi-linked multiple
representational environments using mathematics education software, its results
should help mathematics educators who apply such materials for pedagogical
purposes in mathematics classrooms. For example, when teachers are aware of
what kind of learning is supported with linked and semi-linked representational environments, they can better choose or utilize an appropriate type of software to meet the needs of their students.

Moreover, the results of this study highlight new areas for mathematics education researchers to examine regarding other, related features of mathematics education software in order to empower pedagogical and theoretical development of computers in mathematics instruction.

Finally, the results of this study inform software developers in more instructive ways when they develop new mathematics education software. In particular, software developers could consider including an option that changes a semi-linked representational environment to the linked environment or vice versa in order to support different kinds of learning in mathematics classrooms.

**Limitations of the Study**

One of the limitations of this study was time constraints. Since data were collected in a ten-week period, it was difficult to generalize the effects of such technology over a longer period of time. However, triangulation of data collection and analysis provided valuable insight into how multiple representational software impacts students' mathematical learning.

Studying only one class in a quantitative-qualitative design was another limitation. However, such a focus provided deeper insight into the performance and preference of individual students. The aim of this study was to provide meaningful insight into the research questions instead of to generalize the findings.
Another limitation of this study was the software itself, since it was still in the process of being developed. A similar study with another type of software may provide different results. Moreover, the versions of semi-linked and linked software were created by the developer of VideoPoint for the present study. The semi-linked version only provided a one-way linkage between the graph and the algebraic form compared to three two-way and one-way linkages in the linked version (see Figure 1.1). A comparison of a fully linked and a fully semi-linked version that provides linkage among all representations upon request could be very interesting.
CHAPTER 2

THEORETICAL FRAMEWORK

The aim of this chapter is to provide a theoretical foundation for different aspects of this study. Moreover, this chapter will also serve as a frame of reference during the data analysis period, when the researcher examines the collected data through the lenses of theories mentioned below. The goal will be not only to see how existing theories agree or disagree with the data of this study, but also to develop a theory in this study with the help of the theories discussed in the following pages.

Firstly, a brief historical introduction to theories of representations in mathematics education will be provided. Secondly, the theories on understanding within multiple representations will be synthesized in order to present a more cohesive framework for the study. After Kaput's theory of representations in mathematics education is introduced, the place of technology in those theories will be discussed. Piaget's theory of the development of knowledge will be discussed in the context of linked and semi-linked representational environments. All these theories will be helpful in providing a basis for the first research question. Finally, a
conceputal framework for the second research question of this study will be presented through a synthesis of ideas provided by various mathematics researchers and educators.

**Theories of Representations in Mathematics Education**

Although a number of theories related to multiple representations have been proposed in the history of mathematics education, with the "Multiple Embodiment Principle" of Dienes, this issue has gained additional significance. The Multi-Embodiment Principle or Perceptual Variability Principle, suggests that learners' conceptual understanding is enhanced through a variety of physical representations of mathematical embodiments (Dienes, 1960). Reys (1972) defines *multi-embodiment* as follows: "When different, yet appropriate, concrete materials are used to develop the same mathematical idea, a *multi-embodiment* is provided" (p. 489).

Bruner provides a more detailed theory of representations in education and classifies them into three categories: enactive, iconic, and symbolic levels of representation. The enactive stage of representation is action-oriented, and representations are physically manipulated (Adler, 1963; Shulman, 1970). The second level of representation in this hierarchical process is the iconic form, in which events are summarized and manipulated by perceptions and images. At the last level of
representation, manipulation of symbols begins and certain features of symbolic structures are to be understood by the learner, thus permitting more abstract learning to occur (Bruner, 1968). Although Bruner defines these levels of representation, he notes that the boundaries of these levels are not completely clear.

In addition to Bruner’s and Dienes’ theories, a constructivist theory of learning also supports the use of multiple representations in mathematics classrooms. Constructivism suggests that students construct their knowledge by themselves while actively engaging in their experiential world (Goldin, 1990; Noddings, 1990; von Glasersfeld 1990, 1993, 1996). As von Glasersfeld (1996) observes, radical constructivism suggests that knowledge of the experiential world “is constituted by the knower’s own ways and means of perceiving” (p. 308). By communication and interaction with others, people test how similar or consistent their constructs are with others’ (Confrey, 1990; Goldin, 1990). In sum, it is not expected that everyone will understand the same concept from one representation, or that one representation will be as meaningful for one person as it is for another.

In order to demonstrate how constructivist theory supports students to construct their knowledge more effectively in multiple representational environments, the following data collected during this study’s pre-pilot investigation into students’ preferences or strategies while using representations in mathematics classrooms are presented. The students mentioned that their ways of knowing are directly related to determining the particular representation for a mathematics problem (Özgün-Koca, 1998):
I think *if I can understand* a certain representation then that is what I will use, because if I can understand how to do something a certain way that is easier for me, then that is how I will do it. (Survey, 2/27/98)

[An equation] makes more sense to me that way. (Survey, 2/27/98)

Since knowledge is the result of an active constructive endeavor, it cannot be transmitted to a *passive* learner (Confrey, 1990). Therefore, according to von Glasersfeld (1993), a teacher may establish environments in which a general direction and constraints can be arranged. We should not assume students come to the classroom as blank white paper to be written on. They possess their past experiences and knowledge which serve as a basis, both a foundation and constraint, for acquisition of future knowledge. Therefore, according to the constructivist model, it seems that presenting the information with more than one representation, or providing multi-representational environments, helps students grasp information which is meaningful to them or for which they have a base to build upon. We can see this in the following examples of students' answers to open-ended questions in the survey created and used during the pre-pilot investigation of this study related to the use of representations in mathematics education (Özgün-Koca, 1998):

[Graphs] allow me to *relate* to something I can understand better and it seems to help me better. (Survey, 2/27/98)

It all depends on *how familiar* I am with each process. (Survey, 2/27/98)
Understanding Within Multiple Representations

Part of what we mean when we say that a student 'understands' an idea like '1/3' is that: (1) he or she can recognize the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within the different representational systems, and (3) he or she can accurately translate the idea from one system to another.

(Lesh, Post, and Behr, 1987, p. 36)

Most of the theories of understanding with representations have three bases. Understanding within the representation means that students (1) identify and (2) manipulate the idea in different representational systems and (3) translate the idea from one representation to another. Schwarz and Dreyfus (1993) add another component. They state that knowing the strengths and weaknesses of various representations for a mathematical concept is another crucial component of understanding within multiple representations.

Dufour-Janvier, Bednarz, and Belanger (1987) describe expected outcomes of the use of representations in mathematics learning as follows:

- Being able to decide the appropriate representation to use,
- Grasp common properties of diverse representations,
- Grasp different aspects of them,
- Pass from one to another. (pp. 111-112)

Most theories of understanding focus on being able to transfer the mathematical ideas among different representations. Fey (1989) hypothesizes that "multiple representations are helpful and that ability to translate an idea from one notation to another is an indicator of meaningful knowledge" (p. 255). Moreover,
Harvey (1991) asserts that "one must be able to translate mathematical concepts among representations, and more importantly, reconcile the different information provided by the different representations so as to understand the common abstraction underlying all of them" (p. 3).

Another approach to this issue is Hiebert and Carpenter's (1992) theory. They define external representations as mathematical embodiments and internal representations as mental representations that a person has. They assume that

- There is a relationship between external and internal representations, and
- The internal representations can be related or connected to one another as a network or web of representations (p. 66).

Connections within the mental network can be made based on similarities and differences, and on inclusion. Hiebert and Carpenter (1992) define understanding as follows:

The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. (Hiebert and Carpenter, 1992, p. 67)

They suggest that understanding increases as the network grows and as relationships become stronger. Here growth of a network is defined as reorganizations in networks as well as additions to networks. Although the authors suggest that there is a relationship between internal and external representations, they do not reveal the nature of this relationship. They claim the relationships between external and internal representations are not direct relationships, as reflected in the following example:
We do not presume, for example, that if second graders work with bundled sticks when dealing with two-digit numbers, they represent internally all quantities more than nine as mental images of stick. We do presume, though, that students who interact with bundled sticks represent these quantities differently for themselves than students who work only with written symbols. (Hiebert and Carpenter, 1992, p. 66)

An interpretation of this theory from the perspective of multiple representations suggests that if students construct the internal connections in the mental network by being able to see the similarities and differences between types of representations, it will help them understand the representations better. Students will have a more connected mental network.

If we summarize and gather these components together, the theory of understanding within multiple representations will include the following characteristics:

- identifying the mathematical idea in a set of different representations,
- manipulating the idea within a variety of representations,
- translating the idea from one representation to another,
- constructing connections between internal representations in one’s network of representations,
- being able to decide the appropriate representation to use in a given problem,
- identifying the strengths and weaknesses, differences and similarities of various representations of a concept (Dufour-Janvier, Bednarz, and Belanger, 1987; Harvey, 1991; Hiebert and Carpenter, 1992; Lesh, Post, and Behr, 1987; Schwarz and Dreyfus, 1993).
**Kaput’s Theory of Mathematical Representations**

Another important theory of understanding relative to (multiple) representations has been developed by James Kaput (Kaput, 1989; Kaput, 1991; Kaput, 1992; Kaput, 1995). Kaput’s theory plays a significant role for this study because it elaborates the relationships between internal and external representations in great detail, which makes it possible to see how external representations affect students’ use of internal representations in their learning process. Furthermore, Kaput developed his work in a technological environment, which is crucial for this study.

Kaput (1991) first distinguishes mental structures from notational systems: “mental structures [are] means by which an individual organizes and manages the flow of experience” versus “notation systems [are] materially realizable cultural or linguistic artifacts shared by a cultural or language community” (p.55). He adds that “notation systems are used by individuals to organize the creation and elaboration of their own mental structures” (p.55).

Kaput (1995) differentiates the relationships between mental operations and physical observations. When one moves from mental operations to physical operations, “one has cognitive content that one seeks to externalize for purposes of communication or testing for viability” (p.140). On the other hand, in moving from physical observations to mental operations, “processes are based on an intent to use some existing physical material to assist one’s thinking” (p. 140). The results from Greeno and Hall’s (1997) study support this theory:
As individuals or groups work on problems, they may make drawings, write notes, or construct tables or equations. These representations help them keep track of ideas and inferences they have made and also serve to organize their continuing work. The representational work that people use often are nonstandard forms, which are constructed for the immediate purpose of developing their understanding. In addition to being a representation of something, they are for something. (p. 365)

Cobb, Yackel, and Wood (1992) discuss the shortcomings of a representational view of mind in mathematics education and outline a constructivist alternative to this view. It is argued that "learning [from a representational view of mind] is characterized as a process in which students modify mental representations that accurately mirror the mathematical features of external representations" (p. 3). Three features of the eclectic position compatible with a representational view of mind are:

1. The overall goal of instruction is to help students construct mental representations that correctly mirror mathematical relationships located outside the mind in instructional representations.
2. The goal is to develop transparent instructional representations that make it possible for students to construct correct internal representations.
3. External representations are the primary basis from which they build their mathematical knowledge. (p. 4)

However, the authors argue that

[In considering] the processes by which students modify their cognitive representations as they create external representations and use conventional symbols to express their thinking...[teachers should] consider the various ways that students actively interpret the materials as they
engage in genuine mathematical communication in the social context of the classroom. The materials would then no longer be used as a means of presenting readily apprehensible mathematical relationships but would instead be aspects of a setting in which the teacher and students explicitly negotiate their differing interpretations as they engage in mathematical activity. (Cobb, Yackel, and Wood, 1992, pp. 2-3)

**How Does Technology Help?**

*Information technology will have its greatest impact in transforming the meaning of what it means to learn and use mathematics by providing access to new forms of representation as well as providing simultaneous access to multiple, linked representations.*

(Kaput, 1986, p. 188)

First of all, being able to attend to all representations dynamically at the same time is one of the biggest advantages of computer technology. Fey (1989) indicates that computer-based representation environments are unique in that they:

1. Present dynamic computer representations that no text or chalkboard diagram can;
2. Provide individual environments for students that are flexible, but at the same time, constrained to give corrective feedback to each individual user whenever appropriate;
3. Help move students from concrete thinking to a more abstract form;
4. Feature the versatility of computer graphics; and
5. Insure machine accuracy. (pp. 255-256)

Providing immediate feedback or revealing instant implications are other important advantages of technology (Perkins & Unger, 1994). Finally, as discussed below, another

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*Dynamic representations make it easier to understand complex phenomena by making representations of them tangible and visible.*

(Schoenfeld, 1988, p. 5)
advantage of technology is the ability to link different representations to each other. A number of studies suggest that providing linked representations may help students link these representations cognitively.

In a technological context, the meaning of understanding stays the same, but the evidence of understanding changes to “the student’s ability to operate within a system of several linked representation systems” (Yerushalmy, 1991a, p. 43). Translation within the representations is replaced by the operation within the different linked representations. In order to examine this idea in more detail, an example from a technological environment is described below.

**VideoPoint**

Figure 2.1 shows a computer screen obtained from VideoPoint software which allows users to collect data from digital movies and represent them in different ways. In this case, the screen displays a movie of a candle burning out. The height of the candle was measured over time and reported in four representations—movie, table, graph, and algebraic form. As described earlier, the components of the theory of understanding within multiple representations are:

1. identifying the mathematical idea in a set of different representations,
2. manipulating the idea within a variety of representations,
3. translating the idea from one representation to another,
4. constructing connections between internal representations in one’s network of representations,
5. being able to decide the appropriate representation to use in a given problem,
6. identifying the strengths and weaknesses, differences and similarities of various representations of a concept.

Figure 2.1. Computer Screen from VideoPoint
As can be seen in Figure 2.1, this environment provides access to all four types of representation. This helps students to work easily on the first, fifth, and sixth components. The information presented by the different representations assists students in identifying effortlessly the mathematical concept in different settings. For example, while the movie presents the real-life phenomena, the mathematical model is presented in the equation form abstractly by clicking on the F button near the graph. Moreover, being able to see the same mathematical idea in different forms makes it easier for students to identify strengths and weaknesses, differences and similarities of various representations of a mathematical concept. Moreover, class discussion about the use of different representations for different mathematical problems related to a particular event may help students to decide on the appropriate representation for different situations. For example, in order to answer a question about the height of a candle at a particular time, table values could be more efficient to use, whereas height changes over time can be clearly seen with the graph.

The rest of the components of the theory emphasize the relations between the representations. As discussed above, technology provides several advantages in this regard. Many mathematics educational software programs offer linked representations. Thus, most of the translations are made by the computer. For example, VideoPoint forms tables and graphs automatically. It is crucial to draw students’ attention to particular aspects of the representations while the computer translates the ideas from one representation to another. On the other hand, this software enables students to input equations to predict a mathematical model of
the phenomena. Then it graphs the user’s prediction, and the user can decide how his or her prediction fits by looking at the data and graph of the model on the screen or by simply looking at the $r^2$ number. There is other software, such as Function Probe, that allows users to manipulate graphs dynamically, which is impossible in a paper and pencil world. Thus, manipulation of mathematical ideas is enhanced by this new form of technology.

VideoPoint has another feature worth noting in linking the representations with each other. It allows users to click on a data point in the graph and see the corresponding points in movie and table representations. Although it is not clear whether or how this feature assists students to connect their internal representations, it opens new avenues for helping students to understand the relationships between representations.

**Multiple Linked Representations**

Goldenberg et al. (1988) argues that “multiple linked representations increase redundancy and thus can reduce ambiguities that might be present in any single representation” (p.1). Therefore, multiple representations facilitate understanding. Other advantages of computer-based multiple linked representations are described by Goldenberg (1995):

> The catalyst that makes this entire analysis possible is the power of a computer or graphing utility instantly to transform algebraic expressions into visual and numerical displays for verification, exploration, and experimentation. (Vonder Embse, 1997, p.56)
The interactive nature of the computer
• Allows students to become engaged in dialogue with themselves,
• Raises conflict and surprise which leads to more thinking,
• Affirms (if not parallels) students' own internal multiple representations,
• Helps us [educators and researchers] distinguish between students' expressed models and the ones they act on,
• Provides immediacy and accuracy in tying two or more representations together,
• Helps students represent their concepts in multiple ways. (pp. 159-161)

Linked representations are also supported by Kaput (Kaput, 1985; Kaput, 1989; Kaput, 1994; Kaput, 1995; Kaput & Goldin, 1996), who advocates the use of multiple linked representations with technological aspects as follows:

To the extent that different representations of ideas and actions are important and that their connections are to be internalized, computer linked representations will encompass a growing genre of computer use in mathematics. (Kaput, 1992, p.532)

All aspects of a complex idea cannot be adequately represented within a single notation system.... Multiple, linked representations will grow in importance as an application of the new, dynamic, interactive media. (Kaput, 1992, p.530)

Kaput (1986) also notes that:

By making visually explicit the relationships between different representations and the ways that actions in one have consequences in the others, the most difficult pedagogical and curricular problem of building cognitive links between them becomes much more tractable than when representations could be tied together only by clumsy, serial illustration in static media. (p. 199)
Kaput (1989) echoes Goldenberg on the value of redundancy and goes on to say:

- Multiple representations enable us to suppress some aspects of complex ideas and emphasize others, thereby supporting different forms of learning and reasoning processes,
- In these kinds of environments the computations required to translate actions across representations are done by the computer, leaving the student free to perform the actions and to monitor their consequences across the representation,
- Appropriate experience in a multiple linked representation environment may provide webs of referential meaning missing from much of school mathematics. (pp. 179-180)

Kaput (1989 and 1994) argues further that linked representations help students develop more connected mental structures:

The manipulation of multiple representations of these concepts in a coordinated window environment appears to provide a potentially ideal vehicle for building those cognitive representations now lacking and for linking these to students' mental models. (Kaput, 1989, p. 5)

![Figure 2.2. Understanding Within the Representations.](image)
Kaput (1994) provides Figure 2.2 (p.389) to present the linkages at the level of actions. In the candle activity described above in VideoPoint, students were asked to decide the candle’s height at the beginning. The linked version of VideoPoint displays the corresponding points in the graph and table windows when students bring the movie back to the starting point. The software shows students where to look in the table and the graph to find the candle’s original height. So, changes in the movie window, i.e. displaying different frames, result in changes in the graph and table windows. The corresponding points are highlighted. This correspondence is indicated in the bottom part of Figure 2.2. Now the question is whether this link helps students to connect the movie with the graph and table mentally, as indicated in the top part of Figure 2.2. As Kaput (1994) explains:

The purpose of the physical connection is to make the relationship explicit and observable at the level of actions in order to help build the integration of knowledge structures and coordination of changes depicted at the top of the diagram (see Figure 2.2) (p. 389). By acting in one of the externally linked representations and either observing the consequences of that action in the other representational system or making an explicit prediction about the second representational system to compare with the effect produced by actions in the linked representation, one experiences the linkages in new ways and is provided with new opportunities for internal constructions. (Kaput, 1996, p. 416)

In the present study, the intention was to see the relationship between physical linkages and mental linkages presented by the arrow in the middle, which can be problematic, according to Kaput:
Computer linked manipulatives provide new opportunity to attack the problem but give us no guarantees. (Kaput, 1996, p. 417)

To what extent does the linked multiple representation software support the cognitive linking of the multiple representations? (Kaput, 1985, p. 65)

Many researchers agree on one point: that building these novel but powerful computer-based linked representations and presenting them to the students does not insure that there will be any connection among the representations in students’ minds. However, well-designed curriculum and activities are essential for enabling students to make predictions, make conjectures, look for supporting evidence, test their conjectures, and justify their results while working in these kinds of technological environments (Edwards, 1998; Perkins & Unger, 1994).

**Piaget’s Theory of Learning**

*Piaget’s stage independent theory is of most interest in the design of computer microworlds.*

(Rieber, 1993, p. 197)

First of all, Piaget’s ideas about disequilibrium, accommodation, assimilation, and equilibrium, all key ideas in knowledge development, will be described. Then the application of this theory to computer-based representations will be considered.

**Assimilation**

Piaget (1969) defines assimilation as the process whereby “reality data are treated or modified in such a way as to become incorporated into the [cognitive] structure of the subject” (p.5). Montangero & Maurice-Naville (1997) describe the term assimilation as “integration into previous [cognitive] structures” (p. 72).
Piaget (1952) posits that "assimilation is intelligence to the extent that it incorporates all the given data of experience within its framework" (p.6). In sum, assimilation is the process whereby the child fits new information into his or her pre-existing mental structures or the child gives meaning to the situation, again according to his/her pre-existing schemata.

**Accommodation**

Piaget (1969) defines accommodation as "the modification of internal schemes to fit reality" (p. 6). He mentions in his book *The Origins of Intelligence of Children* that "mental life is also accommodation to the environment; assimilation can never be pure because by incorporating new elements into its earlier schemata the intelligence constantly modifies the latter in order to adjust them to new elements" (p. 6-7).

**Adaptation**

Assimilation and accommodation constitute adaptation. Piaget (1952) states that "adaptation is equilibrium between accommodation and assimilation...adaptation is only accomplished when it results in a stable system, that is to say, when there is equilibrium between accommodation and assimilation" (p.6-7). Rieber (1993) describes adaptation as "the adaptation of an individual to survive and flourish in an ever-changing environment" (p. 197).
Organization

Piaget (1952) defines organization as “the relationships between the parts and the whole; it is sufficiently well known that every intellectual operation is always related to all the others and that its own elements are controlled by the same law” (p.7). Rieber defines organization as “stability and coherence of the world knowledge and experiences learned to date” (p. 197).

To summarize, the development of knowledge is described as a process of adaptation and organization. Adaptation occurs when the child interacts with his or her environment. The child is coping with his or her world, and this involves adjusting. Assimilation is the process whereby the child integrates new information into his or her mental structures, and it involves the interpretation of events in terms of the existing cognitive structure, whereas accommodation refers to changing the cognitive structure. Adaptation is achieved when assimilation and accommodation are brought into equilibrium. Organization is a structural concept used to describe the integration of cognitive structures.

Application of Piaget’s Theory of Learning to Computer-Based Representations

The linked versus semi-linked computer based representational systems will now be analyzed in terms of Piaget’s theory. First, the linked representational system will be examined. Students act on the representation system and observe the results of their actions, such as a change in the graph of a function. If they encounter something they cannot immediately assimilate, then an imbalance
between their structures and the environment occurs. When an imbalance occurs, students are forced to change cognitive structures to adjust to the environment through accommodation until equilibrium is achieved.

In semi-linked representational environments, when students make any changes in a representation, either the computer or the teacher can ask about students' ideas and predictions regarding the corresponding changes in other representations before displaying the relationships between the representations. This provides an environment for students to engage in a more active assimilation process in which they can use their prior knowledge and cognitive structures. Again, if their predictions are different from those in the computer feedback, disequilibrium occurs and they will go into the accommodation process as explained above. This environment creates an atmosphere for students to assimilate the information in terms of their existing knowledge every time they act.

Semi-linked representational environments put students in a more active role. Although in both situations students decide what to change in one representation in order to solve the problem, semi-linked representational environments force them to think through the results of their actions and to reflect on what they are doing rather than engaging in trial and error without thinking.

Confrey et al. (1991) state that they decided not to automate features of Function Probe, a computerized tool for studying functions, for which they believed student interaction was essential in building stronger understanding. Schwarz and Bruckheimer (1990) employed an analogue of a Piagetian idea of equilibrium while establishing the curriculum of The Triple Representation Model. This model can
While tools become ever more powerful, the question of learning with the tool remains. (Kaput, 1992, p. 547)

Although Kaput (1992) indicates that asking students about their predictions before the computer exhibits the consequences of their actions is found to be more pedagogically useful, he does not incorporate this feature in his other studies. He also mentions the translations between notations and models:

The process of relating actions in one notation A with actions or consequences in another one B often proves cognitively overwhelming. In particular, one must become engaged [with] three different activities: (i) actions in A which effect state changes in A; (ii) actions in B which effect state changes in B; and (iii) coordinations of objects, relations, and most importantly, state-changes between A and B. (p. 541)

Kaput also advocates adjusting tools according to the educational task or aim. For example, being able to enable/disable the links between notations would give teachers the opportunity to shift the balance of power from the computer to the students.
Students’ Strategies for Using External Representations

The second question of this study dealt with students’ strategies and preferences related to external (multiple) representations. Some of questions that were researched focused on factors influencing students’ choices of representation to solve any problem, and how technology affects their preferences. There are a number of study results related to these questions that are discussed in Chapter 3.

While studying the reluctance to visualize in mathematics, Eisenberg and Dreyfus (1991) proposed some reasons summarized below, why students use more dominantly algebraic representations and avoid visual ones.

- The choice of representation may depend on the problem itself and personal preferences of the student,
- Beliefs about mathematics—a visual representation is not mathematical,
- Sociological Rationale—visual representations are harder to teach, especially in a sequential curriculum,
- Cognitive Rationale—visual representations are more difficult. (p. 26-30)

Sakonidis (1994) suggests that children gain access to external representations via their internal ones: “External representations which are too distant from the child’s internal ones are either rejected or create difficulties because they do not fit what the child perceives as significant in the situation studied” (p. 42).

Dufour-Janvier, Bednarz, and Belanger (1987) share Sakonidis’ conclusion and state that “if one wants to use an external representation, he needs to take into consideration that it should be as close as possible to children’s internal
The use of different representations linked with software may help bring diversity to the classroom because students can choose to initiate their investigations with the representation that they prefer. (Borba, 1995, p. 333)

representations” (pp. 117-118). Dufour-Janvier, Bednarz, and Belanger (1987) examined whether a child really selects a representation while experiencing difficulties in solving problems. The authors reported that “the young children...who can solve the problem posed, do not select a representation...[and] retain one representation that is more familiar to them” (p. 114). When students see different representations in a problem, they think that “there are as many different problems as there are representations” (p. 114). Their work seems to suggest that students also believe that using different representations in solving problems may result in different answers. Tsamir and Tirosh (1999) found that students gave two conflicting answers to the same mathematical problem asking for the number of elements in two infinite sets presented with different representations; however, the researchers asserted that this situation could be used to encourage students to reflect on and empower their mathematical reasoning.

Studies by Donnelly (1995); Dufour-Janvier, Bednarz, and Belanger (1987); Eisenberg and Dreyfus (1991); Keller and Hirsch (1998); Poppe (1993/1994); Porzio (1994/1995); and Vinner (1989) are synthesized in Table 2.1 in order to gain a deeper understanding of students’ ways of using the multiple representations in mathematics classrooms. When the above work is combined, two main categories emerge: internal and external effects.
<table>
<thead>
<tr>
<th><strong>Internal Factors</strong></th>
<th><strong>External Factors</strong></th>
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<tr>
<td>Personal Preferences</td>
<td>Presentation of the Problem</td>
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<td>Previous Experience</td>
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<td>Previous Knowledge</td>
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<td>Beliefs about Mathematics</td>
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<td>Rote Learning</td>
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Table 2.1. Factors in Students’ Preferences for Various Representations as Reported in the Literature¹


Although a number of studies have been conducted on the issue of students’ preferences for representations, the information in this area of the mathematics education research literature is fragmented. It is hard to find a cohesive framework. One reason could be that where people’s strategies or preferences are concerned, it is hard to construct any framework that will fit every situation. Thus, possible answers to the second question in this study were in its collected data and
determined students' preferences towards representations in computer and noncomputer settings. It is obvious that there were a variety of answers; there is no one 'truth' for this issue.

Although there are a number of theories emphasizing multiple representations in the history of mathematics education, this present study concentrates on two theories by Kaput and Piaget. Kaput's discussions of understanding within representations by focusing on the interactions between internal and external representations and linkage among them is crucial to this study. At the same time, Piaget's theory of learning is used to discuss the cognitive processes in the linked and semi-linked computer environments.
Mathematics does not exist independently of its representational forms; it exists through those forms. 
(Borba & Confrey, 1993, p.31)

Representations have a significant place in mathematics education. In the following sections I will first present studies on multiple representations and learning, followed by studies involving linked multiple representations and technology, and finally, studies of representations and students’ preferences.

**Multiple Representations and Learning**

Because each representation emphasizes and suppresses various aspects of a concept, we believe that students gain a more thorough understanding of a function if it is explored using numerical, graphical, and analytical methods. 
(Piez and Voxman, 1997, p.164)

The studies related to the use of multiple representations in mathematics classrooms are divided below into two sections: non-technology oriented studies and technology oriented studies.
Non-Technology Oriented Studies

Harel (1989) studied the effects of the use of embodiments of familiar geometric representations on understanding the vector space concept rather than the embodiment of a variety of unfamiliar algebraic systems. Seventy-two sophomores in a linear algebra course were divided into two groups. Half of them were exposed to a variety of abstract ideas for two hours every week in recitation sessions while the other half was exposed to geometric representations of vector spaces for one hour and traditional abstract ideas for one hour. Both groups attended the same lecture every week. The same posttest was given to both groups. Familiar geometric representations had more effect on achievement and understanding than unfamiliar algebraic representations. Harel concluded that the multiple embodiment principle is crucial for constructing a richer concept image of abstract mathematical concepts. However, he said that special attention should be given to the mode of representation and students’ familiarity with the representations.

The purpose of Poppe’s (1993/1994) teaching experiment was to investigate students’ use of variables when instruction utilized different representations of the function concept. Algebra I students from six high schools participated. Two schools were part of a teaching experiment, and the remaining four were part of the control group. Although it is clear that instruction in the treatment groups utilized multiple
representations like mapping diagrams, graphs, and tables, the author does not provide detailed information about features of the instruction. She mentions the use of the computer room and graphing calculators, but it is not clear whether they played an important role in the teaching. The instruments used to investigate the students' use of variables were written tasks, journal entries, and interviews. The algebra test from the Chelsea Diagnostic Mathematics Tests was used as a pretest and posttest. Results showed no significant difference in achievement between the experimental and control groups. An analysis of the individual questions did indicate that the experimental groups showed a higher percent of improvement in generalizing patterns and identifying restrictions on the domain and range for varying quantities.

Brenner et al. (1995) call for instruction that fosters multiple representations of mathematical concepts and suggest a guided approach in which students are encouraged to explore different representations and to develop their own understanding of each one. In their study, a multiple representation approach was utilized for the treatment group, whereas a traditional approach was utilized for the control group. They found that the treatment group gained significantly more points from pretest to posttest than did the comparison group, and the treatment group was more likely to use appropriate representations than the comparison group. Finally, the authors concluded that the students in the multi-representational group learned not only to use different representations, but that they also gained a greater understanding of the mathematical concepts being taught.
Moseley and Brenner (1997) examined whether providing students with a multiple representations curriculum would promote a conceptual shift to thinking algebraically. This study collected data from pre- and post-instruction clinical interviews administered to a sample of middle school students experiencing their first exposure to formal pre-algebra. Fifteen students from junior high school were taught with an experimental curriculum emphasizing multiple representation skills, while 12 students received traditional instruction involving textbooks and emphasizing manipulation and translation of phrases in problems. The treatment group performed a task of discovery in which they looked for a systemic error and presented their work in the form of tables, graphs, or equations. According to results of the pre- and post-interviews, students who were provided a multiple representations curriculum were significantly more likely to show signs of algebraic reasoning. Moseley and Brenner concluded that a multiple representation curriculum can influence not only the type and frequency of representation that students use, but it can also improve students' performance at solving word problems.

Knuth (2000a) studied 284 high school students enrolled in a college-preparatory mathematics courses. Students were asked to answer written mathematical questions on connections between graphs and equations and explain their responses in order to examine students' understanding of connections between algebraic and graphical representations of functions. The researcher
classified students’ responses into two categories: algebraic and graphical solution methods. Results indicated that students used algebraic solution methods more often than the graphical methods even though questions could be solved using the graphs more easily. Students saw graphical solution methods as supports for algebraic methods instead of providing a whole solution by themselves. Knuth also indicated that “students often appear to understand connections between equations and graphs...as the results of this study suggest...their understanding of the connections is often superficial at best” (p.53).

Knuth (2000b) studied students’ understanding of the Cartesian connection in another study with 178 high school students and presented results similar to Knuth (2000a). Students solved questions on the use of Cartesian connection in determining a solution for an equation where graphs of equations were also presented to the students. He concluded that students again relied more on the equations and had difficulty connecting their internal representations because of the “constraining nature of their interactions with the external representations” (p.506).
Technology Oriented Studies

Educational technology also plays an important role in utilizing multiple representations in mathematics classrooms. The following studies have investigated the effects of technology on learning with multiple representations.

A study using a multiple representation computer environment was conducted by Confrey et al. (1991). The aim of the study was to find out what mathematics instruction in schools might be like if classrooms were provided with adequate technological resources and appropriate teacher development. Data were collected from 23 “Apple Classrooms of Tomorrow” (ACOT) students. The use of contextual problems was the main focus of this study, but I will concentrate on the results related to the use of multiple representations. The researchers’ hypothesis was that a multi-representational tool, *Function Probe*, used in a problem-solving-based curriculum would foster students’ ability to develop richer conceptualizations of linear and exponential functions. The results provided evidence that the use of multi-representational software tools, such as *Function Probe*, with appropriate curriculum and instruction can assist students in constructing viable conceptions of mathematical functions, though the coordination of multiple representations of functions takes time to develop.

O'Keefe (1992/1993) focused on dynamic representations in his study, and one of his research questions was in what ways might dynamic visual representation be used to enhance students’ understanding of the concept of function. He collected data by an introductory questionnaire, function questionnaire, exit survey, observer notes, and student worksheets from high school and college
students. The data from the exit survey, which focused on students' attitudes toward computer software, showed that students seemed very comfortable and uninhibited while working with the dynamic and interactive software, DynaGraph. This particular software provides dynamic number line mappings displaying domain and range at the same time; when students move the arrows on one of the lines, they can observe the movement of the arrow on the other line. O'Keefe indicated that dynamic representational environments might be useful at various stages of the process of learning about functions. Moreover, dynamic representation might ease the transition between students' concrete experience and more abstract notions of function.

A study by Dyer (1994/1995) used a sample of applied college algebra students from two-year colleges who were divided into computer and non-computer treatment groups. Both groups received instruction that stressed strategy-based problem solving using three representations of a function—the symbolic function rule, a graph, and a table of values. Statistically significant main treatment effects upon achievement were detected through ordered multiple regression analyses using two pretests. Qualitative research data sources such as writing assignments, observations by instructors, and follow-up interviews were coupled with elementary descriptive statistics to analyze students' ability to apply the strategies, their preferences, and their facility in translating between representations. These data indicated that the students were generally proficient with each of the approaches and that they were able to effectively translate data between representations.
Porzio (1994/1995) examined how students from (a) a calculus course with the computer software “Mathematica”, (b) a traditional calculus course, and (c) a calculus course where graphics calculators were used extensively to emphasize graphical representations differed in their abilities to use and understand connections between multiple representations when solving calculus problems. An 18-item representations test and a 4-item calculus representations test were administered. Thirty-six students from the classes, twelve per course, participated in individual interviews on mathematical concepts. Weekly observations were made by the researcher. The results indicated that students in the calculus class with Mathematica were better able to use different forms of representations and were better able to make connections between representations than the students using graphing calculators or traditional methods. Moreover, the findings showed that students were better able to use and make connections between representations when the instructional approach they experienced emphasized different representations and made them solve problems specifically designed to explore, establish, or reinforce connections between representations.

Molyneux-Hodgson et al. (1999) studied Mexican and English students while they were using spreadsheets for mathematical modeling. One part of this study investigated how working with spreadsheets enabled students to use more and different types of representations. They concluded that using spreadsheets helped both Mexican and English students to appreciate the representation types they do
not make much use of in their regular classrooms. Mexican students benefited from more graphical and numerical representations while English students made more sense of algebraic representations.

Jiang and McClintock (2000) described how students made use of a technological tool, Geometer’s Sketchpad, in solving a mathematical problem and how they integrated different ways of approaching a problem using graphical, numerical, and analytical methods. Students first used a graphical approach using the Geometer’s Sketchpad, then they utilized a spreadsheet as a numerical approach, and finally, they worked on a theoretical solution by making use of graphical and numerical representations. Researchers concluded that “we, as mathematics educators, should make the best use of multiple representations, especially those enhanced by the use of technology, encourage and help our students to apply multiple approaches to mathematical problem solving and engage them in creative thinking” (p. 19).

Noble et al. (2001) studied fifth-grade students working on mathematics of change activities in different environments. This case study focused on two students who first experienced a hands-on activity where they were trying to walk on a pattern, then approached the problem numerically, and finally worked with computer software providing a virtual environment in which computer-generated icons walk on 100-unit track with the same pattern and a table and graph presented as results. Researchers indicated that these multiple environments enabled students to identify not only one core mathematical aspect common in each environment but also a “web of family resemblances” in all three environments.
There are also some studies focused on graphing calculators and multiple representations (Porzio, 1997; Ruthven, 1990; Tolias, 1993). Porzio (1997) specifically focused on the effects of graphing calculator use on students’ understanding of mathematical representations where students heavily used calculators in their calculus course. Porzio concluded that students showed proficiency using graphical representations to solve problems but had some problems with using symbolic representation and the relationships between graphic and symbolic representations. First of all, students believed the focus of the course to be on learning about only graphical representations of Calculus concepts and not symbolic ones. Moreover, the instruction was mostly demonstration of connections between symbolic and graphical representations or solving problems during lectures and sometimes by the teaching assistant using the graphing calculator. Porzio (1997) also points out “the importance of having students solve well-chosen problems designed to help them make connections between different representations of concepts that are provided by the technology” (p. 10).

Ruthven (1990) investigated the influence of graphing calculator use on secondary school mathematics students’ translation from graphic to symbolic representations. Students were tested on two types of items: symbolization items, an algebraic description of some Cartesian graph and interpretation items calling for the extraction of information from some verbally contextualized graphs. The findings showed superior performance of treatment students over comparison students on the symbolization items, but not on the interpretation items. Ruthven explained this result as follows:
The symbolization item depends on expertise in recognizing graphic forms and relating them to appropriate symbolic forms.... Reliable access to graphic calculators is likely to encourage both students and teachers to make more use of graphing approaches in solving problems and developing new mathematical ideas, not only strengthening these specific relationships but rehearsing more general relationships between graphic and symbolic forms. (p. 447)

Tolias (1993) studied the effects of using graphing technology during the simultaneous instruction of algebraic and graphical procedures for solving equations and inequalities involving the elementary functions of precalculus. Experimental subjects received graphing calculator enhanced instruction, whereas control subjects received traditional instruction. A posttest was utilized to measure three learning outcomes of students: procedural knowledge, relational knowledge, and transfer of knowledge. Significant differences between the experimental and control groups were present on the relational knowledge measure and the transfer of knowledge measure, in favor of the experimental group. Results from the questionnaire indicated that student opinion regarding graphs and graphical procedures was positive and that the experimental treatment facilitated student understanding of the relationship between the algebraic and graphical methods for solving equations and inequalities.

The studies reviewed in this section focused on mathematical representations in technological environments. Most of the studies involved younger participants. Couple studies were at college level (Dyer, 1994/1995; Porzio, 1994/1995), but the present study focused on the freshmen students at high school
level. The reviewed studies utilized various technologies such as calculators, spreadsheets, and computers. The present study employed computer software to study the effects of computer-based representations in mathematics classrooms. Moreover, it concentrated on one aspect of computer-based representations; the linkage among them. It also combined quantitative and qualitative data collection and analysis methods.

**Linked Multiple Representations and Technology**

*Studies about learning environments in which the computer is a central tool clearly illustrate the ability of the computer to provide linked representations and thus open new avenues for creativity in mathematics.*

(Yerushalmi, 1991a, p.43)

It is claimed by many mathematics educators that a dynamically linked representational environment provides a remarkable experience for students to observe changes in other representations when they alter a given representation. Moreover, they claim it makes it possible for students to link these representations cognitively.

*Each notation system reveals more clearly than its companions some aspects of the idea while hiding some other aspects. The ability to link different representations helps reveal the different facts of complex idea explicitly and dynamically.*

(Kaput, 1992, p. 542)
There are many studies that have been conducted on this issue. The studies described below have been classified according to the nature of the study—first comparative studies and then case studies. Each section is arranged chronologically in order to illustrate the evolution of developments in the areas discussed.

Comparative Studies

Rosenheck (1991/1992) examined the effects of individualized tutorial instruction using a computer simulation program that dynamically and reversibly linked multiple representations (e.g., graphs and tables) of kinematics. Twenty eighth-grade students were individually instructed by the same tutor, who was also the experimenter, on the concepts of velocity and acceleration using either the computer simulation tool, *Moving Experience* (designed by the experimenter), or paper and pencil and calculator tools for 42 minutes over five consecutive days. Each treatment started with a video instruction of the physical phenomena and continued on to solving problems with available tools. The instruction for the paper and pencil control group (n=10) matched the instruction for the computer group (n=10) except for the availability of the software. Although it was expected that the computer students would perform better than the paper and pencil students, there were no significant differences between the groups. Rosenheck suggested that the lack of group differences could be due to the *individualized tutorial instruction*. Another reason could be that it was hard for the experimental group to understand and learn to pay attention to the links between representations in the software program in just five days.
A quasi-experimental study by Rich (1995/1996) focused on three groups of students: a) Control group with scientific calculators, b) First experimental group with graphing calculators; and c) Second experimental group with graphing calculators and dynamically linked multiple representation software, Analyzer. Each student was interviewed once after the experiment, and the traditional written posttests were completed. Data were collected over six school days. Results indicated that there were no significant differences in understanding, communication, speed, or retention between groups due to treatment.

These two studies compared two groups of students where one group was using the linked multiple representational software. However, due to the crucial differences in the treatment environments, such as the calculator group versus the computer group or the computer group versus the group receiving traditional instruction, it would be hard to determine the effects of the computerized treatment. Moreover, both studies were conducted over very short periods of time. It would be difficult for students to learn about the technology and comprehend the mathematical content in such a short amount of time. However, in the present study two groups of students using the same computer software with differences in linking property were compared over a ten-week period. Computer laboratories were embedded into normal mathematics classes. This approach allowed students sufficient time to learn about the software itself and to focus on the mathematical content.
Case Studies

Rizutti (1991/1992) examined the conceptions of function developed by high school students as they used dynamic, multi-representational computer software to coordinate tabular, graphical, and algebraic representations of mathematical functions in the solution of contextually-based problems in a precalculus classroom. This study was part of a larger research project, ACOT "Apple Classroom of Tomorrow," and an example of a case study which included an entire class. It examined students' uses of the Function Probe software and their understanding of functions through classroom observations, individual interviews, pretest and posttest results, and records of students' classroom work. Conclusions related to the use of multi-representational software were:

- Multi-representational software allows students to concentrate on thinking mathematically (making predictions, testing conjectures, looking for consistency) by providing quick, accurate feedback for their immediate consideration and use and by allowing them to look for consistency between problems and representations. (p. 141)

- Multi-representational software allows students to choose their method of solving a problem and, thus, diverse groups of students have more access to mathematics. (p. vii)

Another study using a multiple representational software environment was conducted by Yerushalmy (1991a). In this study, he collected both qualitative and quantitative data on students' concept of functions. Thirty-five eighth grade students met 20 times over a 3-month period (twice a week) in computer-facilitated...
(Function ANALYZER) lessons in which the teacher operated the software and the students viewed a single enlarged monitor screen. Different aspects of student perceptions of the concept of function were examined. Yerushalmy indicated that although the software and the method of teaching enhanced students’ understanding of the concept examined, connections between the algebraic manipulations and visual representations did not occur spontaneously. Although the author mentioned that students’ and teachers’ choice of graphs or other representations were constructed, it is not likely that students experienced every type of function they had in mind unless they had their own access to a computer. In the present study, students met regularly in a computer laboratory where each pair of students had their own computer to work with.

Yerushalmy and his colleagues (1991b, 1992, 1997) also studied the effects of providing a variety of representations using technology on students’ understanding of and performance involving algebraic concepts. Yerushalmy (1991b) focused on the influences of computerized graphical feedback on students’ ability to carry out and verify algebraic transformations. There were four ways that students related to graphical feedback, varying from ignoring it completely to recognizing and using the information provided by the graphical feedback. Yerushalmy and Gafni (1992) examined and compared the performance of in-service teachers both with and without graphic feedback provided by a linked multiple representational computerized tool. They concluded that the graphic feedback was more helpful in improving performance on the debugging tasks.
Moreover, teachers using the linked graphic representations solved problems in fewer steps with fewer errors as compared to the same task performed without linked graphic representations.

Another study was a qualitative case study of one student by Borba & Confrey (1993) using the multi-representational computer environment of Function Probe. Data were collected through clinical interviews with one student. The dynamic, linked, and multi-representational software Function Probe allows students to manipulate the graphs and see the corresponding changes in the algebraic form. There is an additional option that allows corresponding changes in graphical representation not to be automatically displayed in algebraic representations. Thus, the researchers suggest that Function Probe creates a unique environment for studying transformations of functions. The experimental model starts with visualization of graphs, then focuses on the relationship between graphs and tabular values and the relationship between graphs and algebraic representations. Content questions and some open-ended questions such as, “How would you investigate the relationship between coefficients in the equation of a function and the graph of the function?” (p.8) were asked in the interview process, which enabled students to use their reasoning skills. Borba and Confrey indicate “that visual reasoning, seeing graphical transformations as movements on or of the plane, is a powerful form of cognition; ... different representations provide a source of motivation to complete an inquiry or investigation” (p.29).
An extension of the previous study can be found in Borba (1993), who investigated the possibilities of a model for the teaching and learning of transformations of functions using Function Probe. This was a qualitative case study of two students. The data sources included clinical interviews, pre- and post-tests, audio and videotapes of each interview, and interviewer notes. The teaching experiment extended over 82 hours with each student. First, the students were asked to match one graph to another, using visualization and "direct actions" on graphs. Then, the students dealt with the linkage between transformations of graphs and tables of values. Finally, students studied the linkage between algebraic and graphical representations of the transformations. A two-step model was developed to describe students' understanding in a multi-representational environment. This model starts with the notion that understanding in such environments extends across representations and ends with students searching for justification for patterns and discrepancies that they have constructed. Although this study helps teachers to see a detailed example of students' understanding of transformations of functions in a computer setting, it is not appropriate to generalize from the results due to the one-to-one nature of the instruction and the fact they were case studies, a point Borba also makes.

Another study by Lin (1993) described teaching experiments using manipulable, dynamic, linked, multiple representation computer environments to help students understand the relationship of the movement or variation of the equation graph and its algebraic representation. In these environments, students could manipulate (by translating or scaling) graphic objects and the coordinate
system using *Geometer's Sketchpad*. Three students from different levels (Algebra I, Algebra II, and Algebra III) participated in this study. The researcher met with the students for 120-minute periods once or twice a week on weekends. Two approaches were utilized. First was the transformation approach, in which the students manipulated the points on the graph or on the coordinate system and studied the changes in equation representation. This approach was used for the study of translation. The second approach was the formula-graph approach in which students manipulated the graph until it matched another one on the screen through a vector representation. Lin concluded that the dynamic, linked, manipulable computer environments seemed to encourage and facilitate these three students' visual and operational thinking. However, the focus on just three students for a long time in a particular mathematics content makes it hard to apply the results of this study to classroom context.

Most of the computer software used in the above-mentioned studies utilizes mathematical representations in a very abstract way and focuses on the representations directly. However, the present study employed computer software—*VideoPoint*—which utilizes four different types of representations in a contextualized way. Quicktime movies provide a context to the tasks and are beneficial in helping students construct meaning for the mathematical representations: equations, tables, and graphs.
Although the studies mentioned in this section investigated the different effects of multiple representational software, they do not focus directly on the effects of the linking property of the software on students’ learning. The present study focused directly on the effects of linkages by using the same software with and without the linkage feature with two groups of students.

Lastly, some of the discussed studies utilized a paper and pencil test after the use of computerized instruction. This may have hindered the evaluation of students’ learning from the instruction. To avoid this problem, clinical interviews were main focus of the present study in which students worked on a problem with the computer. The clinical interviews, together with the support of quizzes and exams, revealed how the linking feature of the software affected students’ understanding.

**Representations and Students’ Preferences**

Although there are a number of studies on the use of representations, this part of the literature review will focus on students’ preferences and use of the representations. Some of these studies highlight the issues affecting students’ choice of representation, which was one of the interests in the present study.

Dreyfus and Eisenberg (1982) presented functional relationships either in diagrams, graphs, or tables settings and observed that students with high ability and social level tended to prefer graphs and performed better with graphs. In contrast, students with low ability and social level performed better in table settings than graph settings.
LaLomia et al. (1988) examined students’ preferences for each problem display type: table and graph. The researchers concluded that students indicated a strong preference for tables when they needed to locate a specific number, and a slight preference for tables when they were performing a trend analysis. On the other hand, students had a slight preference for graphs when they were dealing with interpolation and forecasting tasks.

Vinner (1989) studied the extent of visual considerations of college students. He administered an open-ended mathematical questionnaire asking students to prove a theorem using visual or algebraic methods after first asking them to make a suitable drawing of the theorem. The majority of the students chose to use the algebraic method. One year later, Vinner asked another group of students to decide between two given proofs as to which one was more convincing, where one proof was algebraic and the other was visual. Vinner concluded that there is an algebraic bias or visual avoidance in calculus students. He proposed that one of the reasons for this was that algebraic methods are perceived as more “mathematical” and thus more general. Therefore, he suggests using more visual approaches and representations in classrooms.

The aims of the study by Shama and Dreyfus (1994) were: (1) to address the question of whether students are avoiding visualization, even with a problem presented in a visual form, (2) to identify different visual strategies that are developed spontaneously by students for solving unfamiliar problems and (3) to classify problems according to their complexity. Their research comprised two stages, each of which included investigations with a computerized environment and
a questionnaire on linear programming problems. Computer environments made it possible to attend to visual and algebraic representations at the same time. They concluded that most students do not avoid visual strategies if the problem statement or presentation emphasizes visual forms. Moreover, students were able to develop their visual strategies, although many of them were wrong or incomplete. The authors suggest that preparing and creating suitable visual settings for students may help develop their visual reasoning ability.

Piez and Voxman (1997) showed a group of students from a graphing-calculator calculus class how to solve quadratic inequalities using case, critical-number, and graphical methods. They gave students some problems to solve using a method of the students' choice and asked them to explain why they chose the method they did. Of twenty students who participated, thirteen chose one of the two analytical methods, six chose a graphical approach, and one student chose to work from both a graphical and an analytical perspective which involved using both methods for checking his or her answer. The responses of students who chose analytic methods revealed that they found analytical methods easier and quicker than graphical methods. On the other hand, students who chose the graphical methods mentioned the importance of the visual advantages of graphics.

Keller and Hirsch (1998) investigated whether students had preferences for various representations, the extent to which preferences were contextually related, and the extent to which preferences for representations were affected by the availability of various representations. They studied two groups of students enrolled in a university calculus course. One group used graphing calculators intensively,
whereas the other group was taught traditionally. For the graphing calculator section, students enrolled voluntarily. Representation preference tests were given as pre- and posttests. Keller and Hirsch (1998) concluded that students had preferences for various representations which could differ for contextualized and purely mathematical situations. Most students preferred equations for non-contextualized situations, whereas they preferred tables on the pre-test and graphs on the posttest for contextualized problems. Students in the graphing calculator section had stronger graphical preferences for both types of problems. In technology-rich situations, students' preferences for various representations did not depend as much on whether the task was contextualized as in traditional situations. The researchers' explanation was that use of technology makes it possible for students to have equal access to the representations by removing constraints perceived by students regarding the ease with which the representation could be manipulated. They also mentioned that students' preferences moved from tables to graphs when more interpretation was necessary. Moreover, the language of the task affected students' preferences. On tasks with less formal language, students tended to select graphs or tables, whereas when a problem was presented with more formal language, they selected equations.

Results Regarding Preferences in Studies Discussed Earlier

Some of the studies reviewed under multiple representations also reported on students' preferences and attitudes towards representations. Poppe (1993/1994) reports that although students said that they found tables, graphs, and mapping
diagrams helpful, they did not utilize them to solve unfamiliar problems unless prompted to do so. When no specific method was required to solve problems, the students used algebraic algorithms or trial and error. Another interesting result from Poppe's study is that students used tables and other representations when they did not have the knowledge to solve the questions algebraically. Rizutti (1991/1992) states that students used functional thinking to solve problems by representing situations with tables of data and graphs; they build algebraic representations for functions (i.e. equations) by coordinating tabular and graphical representations. Rizutti also analyzed the amount of use of representations in a computer setting by recording the use of representations through a dribble file. Results indicated that students used tables more than graphs, and they used graphs more than calculators. Dyer (1994/1995) also studied students' preferences for representations employing three types of representations—algebraic, tabular, and graphical—in both treatment and control groups. He concluded that most students could apply all three strategies. However, students' preferences were mixed, generally in favor of symbolic, followed by numerical, and then graphical approaches to solving problems. The computer group preferred algebraic and tabular representations, whereas the non-computer group preferred algebraic representations only.

Some of these studies investigated the effects of technology on students' strategies and preferences for the use of representations. For example, in a study by Rosenheck (1991/1992), a computer group seemed to find tables easier to use and to make than did a paper and pencil group. Rich (1995/1996) mentioned that the choice of representation differed among her research participants. However,
graphical representation was used more in the experimental groups using graphing calculators and computer software than the control group using scientific calculators. Porzio (1994/1995) studied three groups of students: traditional, graphing calculator, and computer. He found that students’ initial preference in the traditional class declined for graphical representations, whereas students in the graphing calculator class showed an increasing preference for graphical representation and a significant decline in preference for symbolic representations. The group using computers showed a decline in their preference for symbolic representations but an increase in their preference for numerical representation.

Discussion

Some studies reviewed in this chapter utilized multiple, dynamic, linked representational software, whereas others used other forms of technology or traditional approaches. The results suggest that the nature of mathematics teaching and learning may change due to the availability of technology in mathematics classrooms. The aim has changed from mastering mathematics to performing and experiencing mathematics. With technology, students have time to focus on “what if” questions rather than spend their time on constructing graphs, for example. However, as mathematics educators, we cannot assume that everything that comes with technology is beneficial and effective. We need to carefully and continuously study the effects of these technologies in order to facilitate our students’ learning in more powerful ways.
The studies previously discussed indicate that the use of multiple representations with or without technology helps students construct mathematical concepts in more empowering ways. Students gain more understanding not only in mathematical concepts but also in the use of multiple representations in mathematics.

The studies utilizing the linked multiple representational software are divided into two groups: comparative studies and case studies. The former ones (Rich, 1995/1996; Rosenheck, 1991/1992) compared groups of students by using different technologies in treatments. Due to the crucial differences in the environments (e.g., computer versus non-computer or calculators versus computers), it is difficult to draw clear conclusions. In fact, results of these studies showed no significant differences between groups. On the other hand, the case studies indicated more encouraging results because of the use of linked multiple representational software (Borba, 1993; Lin, 1993; Rizutti, 1991/1992). Case studies involving only a few students help the educational community to understand student learning with technology in much deeper ways. However, researchers also need to investigate classroom situations as a whole.

Some studies reviewed in this paper investigated various effects of multiple linked representational software. However, the present study focused directly on the effects of the linking property of the software on students’ learning, by using two different groups of students in a classroom environment with the same
computer software, except for the linking property. The goal was to see how this feature of the software affected their learning and understanding of the relationships between the representations and the mathematics content itself.

A methodological conclusion from the reviewed studies involves the depth that the clinical interviews bring into the study. When the learning environment includes technology, it is essential to include technology in the assessment process also. Other data collection methods, such as paper and pencil tests, provide useful but limited data. Clinical interviews allow the researcher to gain better access to information as students interact with the technology.

On the other hand, understanding students’ ways of knowing and their preferences is crucial for teachers’ decision making processes regarding classroom instruction. As most of studies reviewed, algebraic (symbolic) representations were preferred by most of the students (Donnelly, 1995; Dyer, 1994/1995; Vinner, 1989). However, as Shama and Dreyfus (1994) indicate, more emphasis on visual settings and problems requiring the other forms of representations could help students improve their ability to use multiple representations. Another factor to help students could be technology that makes it easier for students to gain access to different types of representations. The results of some studies showed that tables and graphs were appreciated by students in computer settings (Porzio, 1994/1995; Rich, 1995/1996; Rizutti, 1991/1992; and Rosenheck, 1991/1992).

The present study focused on students’ preferences for representations in both technology and non-technology environments. This helped to investigate whether students have different preferences for representations in different
environments, perhaps because of the ease of generating certain representations. If so, it was very interesting to see how technology and classroom practices might influence their preferences differently.
CHAPTER 4

METHODOLOGY CHAPTER

The purpose of this study was to examine the effects of linked and semi-linked computer-based representations on students' learning of linear relationships and understanding of relationships between representations. The research questions were:

1. What are the effects on students' understanding of linear relationships using linked representation software compared to using semi-linked representation software?

2. What are students' attitudes towards and preferences for mathematical representations—equations, tables, or graphs?

Algebra I students were studied over a ten-week period. Students were divided into three groups—linked, semi-linked, and control—in order to work with two versions of the same software and to have a group who did not participate in any computer related activity to study if overall computer usage makes any
difference. The main data collection methods included mathematics pre- and posttests, follow-up interviews, clinical interviews, observations, interviews with the classroom teacher, and a survey. The research design is summarized in Table 4.1.

<table>
<thead>
<tr>
<th>Research Questions</th>
<th>Data Collection Methods</th>
<th>Description of the Data Collection Methods</th>
<th>Criteria or Indicators for Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. What are the effects on students' understanding of linear relationships using linked representation software compared to using semi-linked representation software?</td>
<td>Computer-Based and Follow-up Interviews</td>
<td>Five students from each experimental group were interviewed while using the computer software. Follow-up interviews after the pre- and posttests provided information about their reasoning in answering the questions.</td>
<td>Codes, patterns and themes were searched throughout the data.</td>
</tr>
<tr>
<td></td>
<td>Mathematical Pre- and Posttest</td>
<td>Students' paper and pencil performance were analyzed.</td>
<td>Descriptive statistical analysis Nonparametric tests for group differences Nonparametric test for achievement differences between pre- and posttest</td>
</tr>
<tr>
<td></td>
<td>Computer Lab Sessions</td>
<td>Students experienced the computer software. Their work was saved by the computer to be analyzed.</td>
<td>Codes, patterns and themes were searched throughout the data.</td>
</tr>
<tr>
<td></td>
<td>Teacher Interviews</td>
<td>These interviews were conducted in order to see the teacher's views about students' growth mathematically and their preferences</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Observations</td>
<td>Everyday classroom observations</td>
<td></td>
</tr>
</tbody>
</table>

continued
Table 4.1. Data Sources

<table>
<thead>
<tr>
<th>Site and Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>The site was a public school in a midwestern urban area. A freshmen Algebra I class provided the participants for this study. Consent forms from the principal, the teacher, and all participants were obtained (see Appendix A). The main reason for selecting this class was that Algebra I is the first time students learn formally about subjects like linear equations that are easy to address with multiple representations. The goal of this study was to examine the effects of technological features on students' learning when they are first encountering a topic. Linear relationships, one of the mathematical concepts included in Algebra I, was well-suited for teaching via multi-representational software because it has real-world applications and a variety of representations such as tables, graphs, or equations.</td>
</tr>
</tbody>
</table>
The class included both freshmen and sophomore students. Sophomore students were taking Algebra I for the second time. Thus they were excluded from the study. The rest of the class provided 27 potential participants for the study. One of the 27 was excluded from the study because she was absent at the time the pretest was given and she also missed other instruments. Another student who did not participate in the study was a male student who did not return the signed consent form from his parents/guardians. However, the rest of the 25 students participated in this study and were placed into three groups—linked, semi-linked, and control.

In assigning students to groups, close attention was given to students' knowledge levels. After the pretests were given, their class grades were obtained from the teacher. A weighted average grade was calculated for each student by taking into account both their pretest scores and their class grades—75% of pretest score and 25% of class grade. After sorting them according to this average score, three achievement groups were formed. Students with the highest 6 scores were placed in the high group, the students with the lowest 6 scores in the low group, and the remaining 13 students in the middle group. Using an SPSS statistical package, students from each achievement groups were randomly assigned to the three treatment groups (see Table 4.2).
<table>
<thead>
<tr>
<th></th>
<th>High</th>
<th>Middle</th>
<th>Low</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linked</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Control</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>6</td>
<td>13</td>
<td>6</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 4.2. The Distribution of Students into the Groups

**Software**

VideoPoint is a software package that allows one to collect position and time data from QuickTime movies. The data can be combined to form calculations such as distances between points. The data can then be presented using different representations such as tables, graphs, and equations. This software was originally designed as linked multiple representational software. With the help of the developers of the software, a semi-linked version of this software was developed specifically for the present study.
The differences between the linkages in the linked and the semi-linked representations are summarized in Figure 4.1. As can be observed, the graph, table, and movie representations are linked two-way in the linked version. This means that when the user clicks on a data point, movie frame, or cell in those representations, the corresponding points in the other two representations are highlighted. Moreover, when a linked version user clicks to see the algebraic form (the equation of best fit) of the phenomena, the line of best fit is also graphed in the graph window automatically. On the other hand, the user of the semi-linked version is not able to see any updates when s/he clicks on one representation. The only linkage that is available in the semi-linked version is between the graph and equation form. When the user estimates the coefficients of the algebraic form, s/he has an option to see the graph of the predicted equation. Table 4.3 illustrates some differences between the two versions. The Candle movie shows a candle burning down and being extinguished. Data on the candle’s height over time are collected.
Vertical position versus time relationship is graphed. When one clicks on a table cell, one can see the movie window and the corresponding points in the movie and on the graph.

Links are broken so when students click on any representation they do not see the corresponding point in other representations.

It is possible to access the algebraic form of the line of best fit for this phenomenon. The algebraic form (equation) can be seen at the top of the graph.

The fit button is removed. Students need to work with the modeling button in order to study the algebraic form. Students are able to see the previous and current equations they have tried and what changed in the graph. Moreover, the graph window autoscales to show the student's prediction compared to the data at the same time.

Table 4.3 Differences in Linked and Semi-Linked Versions
Although VideoPoint was designed as linked representational software, the linkage for the table representation was not two-way. Clicking on a cell in the table window produced the corresponding point in the graph and the movie windows but clicking on a point in the graph or on a movie frame did not highlight the corresponding point on the table. That is, tables were linked to the other representations but other representations were not linked to the table. Since a fully linked version was needed for the aims of this study, the software developer modified the table links to be two-way.

The changes that the software developer made, at the request of the investigator, to create the semi-linked version were:

- Breaking the links between the Movie and Table windows and the Movie and Graph windows. In the original version of the software, when one clicks on a point on the graph or in a table window, the movie shows the corresponding movie frame and vice versa. I asked the programmer to break the links in order to have discussions with the class about questions such as “Where do you think the corresponding point on the graph is when the candle burns out? Why?”

- Being able to show or not show the Fit Button, which displays the equation of the line of best fit to the data. In the original version of the software, the fit button is automatically placed in the graph window in order to show the algebraic form of the phenomenon. In the semi-linked version, the fit button does not appear, and the modeling button, which is in the graph window, gives students a chance to predict the algebraic form of the graph.
- Being able to retain the last predicted equation and corresponding graph in order to help students see the differences between their last predicted and newly predicted equation graphically and algebraically.
- Autoscaling the graph points so that the students’ predicted equation as well as the students’ last prediction show at the same time. In the original version of VideoPoint, if the predicted equation was off the scale, it was impossible to see the data collected from the movie and the new prediction together in one window. I asked the programmer to autoscale the graph window in order to show the data and the predicted equation in the same window.

**Computer Lab Sessions**

Four computer lab sessions took place during the data collection period. Since this particular school schedules its classes for 78 minutes, one group was taken out of the classroom one day per week for a 35-minute computer lab at the first part of the class session; then the second part of this class session, the other group. At this time, the control group stayed in the class for the whole period and they did extra activities or questions. The two groups used the same educational software, VideoPoint, with different linking properties, i.e., linked representations versus semi-linked representations.

The responsibility of the researcher on lab days was to conduct the computer lab sessions and assist students in the computer lab. On the other four days of the week the researcher regularly observed the class during the ten week of
data collection. Five students were chosen for clinical interviews from each group. According to Patton (1990), "Qualitative inquiry typically focuses in depth on samples...selected purposefully" (p. 169) where purposeful sampling emphasizes selecting information-rich cases for in-depth study. In choosing students to interview, some students who used the software effectively were chosen along with some who did not.

An appropriate movie was chosen for every lab session to study a topic related to those discussed in class sessions. Since linear equations was the subject of the class, movies in the computer labs were related to this subject (see Table 4.4). Since the movies that come with VideoPoint are more focused towards problems in physics the movies used in this study were obtained by using VideoPoint to collect data from other similar video analysis software depicting linear relationships by clicking on specified points in each frame of the movie. VideoPoint then generated graphs and tables representing the collected data. The movies in the Water Tank, Candle, Roller Coaster, and Car Racing activities were obtained from Measurement in Motion (1996) software. The movie in the Fish activity was obtained from Graph Action Plus (1996) software. The same movie and event were studied with both groups of students with different emphasis on linking properties. A sample lesson plan and worksheets for computer lab hours are provided in Appendix B.
<table>
<thead>
<tr>
<th>Week</th>
<th>Regular Class Subject</th>
<th>Mathematical Concept in the Lab</th>
<th>Movie</th>
<th>Lab Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Graphing Linear Equations</td>
<td>Linear Relationship with Positive Slope</td>
<td>![Image]</td>
<td>Water Tank Activity: Studying the relationship between time and the height of water in a tank as the tank is being filled.</td>
</tr>
<tr>
<td>2</td>
<td>Characteristics of Linear Relationships</td>
<td>Linear Relationship with Negative Slope</td>
<td>![Image]</td>
<td>Candle Activity: Studying the relationship between time versus height of a candle burning down.</td>
</tr>
<tr>
<td>3</td>
<td>Continuing with linear relationships and starting with linear inequalities.</td>
<td>Linear Relationship with Negative Slope (Targeting Graph as Picture Misconception)</td>
<td>![Image]</td>
<td>Roller Coaster Activity: Studying the relationship between time and the distance from the top of a roller coaster ride that travels uphill at a constant rate.</td>
</tr>
<tr>
<td>4</td>
<td>Systems of Linear Equations</td>
<td>Systems of Linear Relationships of two equations with same slope (Targeting Height/Slope Misconception)</td>
<td>![Image]</td>
<td>Car Racing Activity: Studying the relationship between time and distance between two cars traveling in the same direction at different speeds while one passes the other.</td>
</tr>
<tr>
<td>Post Interview</td>
<td>Systems of Linear Equations</td>
<td>Systems of Linear Relationships</td>
<td>![Image]</td>
<td>Fish Activity: Studying the relationship between time and distance between two fish swimming towards each other.</td>
</tr>
</tbody>
</table>

Table 4.4. Description of Computer Lab Activities
Each lab activity started by watching a movie in the software, e.g. the candle movie—a birthday candle burning out—and asking general questions about the movie such as the how does the candle’s height change as it burns? At this point, only the movie window was open on the screen. Students would open other representation windows as they went further. After the general questions, students were asked to predict a specific graph, in this case the graph of the candle’s height versus time in the first section. After sketching their graph, they opened a window to see what the computer-produced graph looked like and compared their graph and the computer’s graph. The final question in this first section asked students to identify and describe a specific point on the graph, e.g., the point at which the whole candle was burned out.

Section two followed a similar procedure as in the first section, but this time the focus was on tabular representation. First, students were asked their predictions about the table of values and to fill out a table. Then they needed to check their answers with the computer-produced table by opening the table window. This procedure was repeated after the table window was open to study the linkage between the table and graph windows and the table and the movie. Again a final question asked students to identify and describe a specific point on the table, e.g., to identify the time when half of the candle had burned down.

Section three focused on the algebraic representation in a parallel fashion. After asking their predictions about the algebraic form of the event, students in the linked group were asked to push a button to see the equation of the line of best fit. On the other hand, semi-linked students were asked to decide the form of the
equation and enter their predictions for the coefficients in the equation. When they entered their predictions, the line for their equation appeared on the graph window along with the computer-produced graph of the event, with the aim of showing students how well their predictions fitted with the graph of the event. The final question asked students to use the equation to find a specific point in the event, e.g., the height of the candle before it started burning.

A final section included a general question which allowed students to use any representation they wished to answer it. For example, the question in the candle investigation was what the candle’s height after 3 seconds. Students were asked to report which representation they used to answer the question and were encouraged to use other representations to answer the same question.

Table 4.5 summarizes and highlights the differences in the computer lab sessions between linked and semi-linked groups.

<table>
<thead>
<tr>
<th>Linked Group</th>
<th>Semi-Linked Group</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Section 1</strong></td>
<td>Predicting the graph of the phenomenon.</td>
</tr>
<tr>
<td></td>
<td>Looking at the graph on the computer, identifying a specific question.</td>
</tr>
<tr>
<td></td>
<td>[Suggesting double clicking on a point on the graph to see the movie at that time]</td>
</tr>
<tr>
<td><strong>Section 2</strong></td>
<td>Asking students’ predictions about the table of values.</td>
</tr>
</tbody>
</table>

continued
Table 4.5. Differences in the Computer Lab Sessions between Groups

<table>
<thead>
<tr>
<th>Section</th>
<th>Activity</th>
</tr>
</thead>
</table>
| **Section 3** | Asking students' predictions about the equation's slope and $y$-intercept.  
Accessing the equation immediately with Fit button.  
Interpreting the differences in the equations of the two fish? [Slope and $y$-intercept]  
Answering a specific question about the phenomenon using the equation. |
| **Section 4** | Identifying a specific point of the phenomenon using their choice of representation. [Suggesting double clicking on a point on the graph or on the table to see a specific frame of the movie] |

Table 4.5 continue

<table>
<thead>
<tr>
<th>Activity</th>
<th>Activity</th>
</tr>
</thead>
</table>
| Filling out a table on the worksheet by using the graph and checking their answers with the computer.  
[Suggesting double clicking on a point on the graph to see corresponding values in the table]  
Identifying a specific point of the phenomenon on the table. [Suggesting double click on the point on the graph that they consider to see the movie or the graph at that time] |
| Identifying a specific point of the phenomenon on the table. |

**Table 4.5. Differences in the Computer Lab Sessions between Groups**

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Data Collection Methods

Data collection methods for the first question of this study included clinical interviews with 5 students from each of the two computer groups, mathematical pre- and posttests, follow-up interviews with particular students after the mathematical pretest and with all students after the mathematical posttest, and classroom and lab observations. Classroom observations, formal and informal interviews with the teacher, and a survey were data sources for the second question.

Mathematics Pre- and Posttest

Mathematical pre- and posttests were given at the beginning and at the end of the study in order to study differences between pre- and posttest performances, if any. Since the researcher was aware that linear equations would be the main subject for this class during the study, questions in the pre- and posttest were related to linear equations as were the computer lab investigations. Moreover, group differences, semi-linked versus linked, were examined according to results gathered from pre- and posttests. Many of the questions were the same in the pre- and posttests, but the order and the numbers in the questions were changed (see Figure 4.2). Some questions were based on general achievement tests such as TIMSS, SAT, and NAEP and some were based on common misconceptions drawn from the research literature. The original and adapted versions of each question in the pre- and posttest is presented in Appendix C with their sources.
The major components in the pre- and posttest included:

- Two word problems (Verbal),
- Four graph related questions (Interpreting/Constructing and Reading Graphs),
- Three equation related questions (Solving and Constructing Equations),
- Two table related questions (Constructing and Reading Tables).

Table 4.6 presents the questions included in the pre- and posttest, referring to the Figures 4.2 and 4.3.

<table>
<thead>
<tr>
<th>Type of Question</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Word Problems</td>
<td>3, 5 in Figure 4.2.</td>
<td>Same questions with different numbers</td>
</tr>
<tr>
<td>Interpreting Graphs</td>
<td>1b, 8 in Figure 4.2.</td>
<td>Same questions with different numbers</td>
</tr>
<tr>
<td>Reading Graphs</td>
<td>1a in Figure 4.2.</td>
<td>Same question with different numbers</td>
</tr>
<tr>
<td>Solving Equations</td>
<td>7 in the Figure 4.2.</td>
<td>Same question with different numbers</td>
</tr>
<tr>
<td>Constructing Equations</td>
<td>6a, 10 in Figure 4.2.</td>
<td>Same questions with different numbers and 4b in Figure 4.3.</td>
</tr>
<tr>
<td>Reading Tables</td>
<td>2 in Figure 4.2.</td>
<td>Same question and 4a in Figure 4.3.</td>
</tr>
<tr>
<td>Constructing Tables</td>
<td>4 in Figure 4.2.</td>
<td>Same question</td>
</tr>
<tr>
<td>Misconceptions</td>
<td>Height/Slope 9 in Figure 4.2.</td>
<td>Same question and 1, 2b, 3 in the posttest column of Figure 4.3.</td>
</tr>
</tbody>
</table>
Table 4.6. Questions in the Mathematical Pre- and the Posttest

<table>
<thead>
<tr>
<th>Point/Interval</th>
<th>2 in the pretest column of Figure 4.3.</th>
<th>2a in the posttest column of Figure 4.3.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph as Picture</td>
<td>Question in Figure 4.3. titled as question in both pre- and the posttest.</td>
<td>Same question</td>
</tr>
</tbody>
</table>

Table 4.6 continue
1. In the graph shown, what is the number of bicycles built in 1975?

b) During which time period did bicycle construction increase the most?

2. Based on the table above, what is the cost of a 20 minute call from city X to Z?

3. A water tank is full with 1650 gallons of water. If we drain the water at the rate of 5 gallons per minute, estimate to the nearest half an hour how long it will take to clear the tank.

4. According to the information in the table, what is the total daily earning for Thursday?

<table>
<thead>
<tr>
<th>Time Card Name</th>
<th>Total Daily Earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>J Jasmine</td>
<td></td>
</tr>
<tr>
<td>Mon 10:00am - 3:00pm</td>
<td>27.50</td>
</tr>
<tr>
<td>Tues 9:00am - 4:00pm</td>
<td>38.50</td>
</tr>
<tr>
<td>Wed 8:00am - 4:00pm</td>
<td>44.00</td>
</tr>
<tr>
<td>Thurs 2:00pm - 8:00pm</td>
<td></td>
</tr>
<tr>
<td>Fri 3:00pm - 5:00pm</td>
<td>11.00</td>
</tr>
</tbody>
</table>

5. A certain machine produces 250 labels per minute. At this rate, how long will it take the machine to produce enough labels to fill 10 pages of labels if each page has 80 labels?

6. Bill is 600 miles away from home and travels with his car at a constant rate of 65 miles/hour towards his home.
   a) If h represents the number of hours traveled and d represents the distance from home, write the equation for distance in terms of hours.
   b) How far away from home is Bill after he traveled 2 hours?

7. If $12x = 4(x + 5)$, determine the value of $x$.

8. In the equation, $y = 2x + 7$
   a) If this equation is changed to $y = 1x + 7$, sketch the graph of the new equation and discuss how the graph of the new equation is different from the old one.

9. The figure below represents the distances traveled by car A and car B in 6 hours. Which car is going faster and by how much? Explain how you arrived at your answer.

10. Write an equation of the relationship that is expressed in the graph below.

Figure 4.2. Standardized Achievement Items
Question in the Pretest
The number of patients in Hospitals A and B over a 10-month period are provided below in the graph.

1. How many total patients were in the hospitals in January?
2. When did Hospital B have less patients than did Hospital A?
3. Which hospital had a greater increase in the number of patients from February to March?

Question in both Pre- and Posttest

1. Which graph shows the distance the person has walked with a constant rate from A to B?

Questions in the Posttest
1. If this below graph represents a wide jar being filled with water, draw a graph for narrower jar being filled with water on the same coordinate axes.

2. a) Two video rental clubs offer two different rental fee plans. For what number of video rentals is it less expensive to belong to Club A?
   b) Which plan is cheaper per a movie rental?

3. Mark is walking and Bill is riding his bicycle. At 2 minutes, who travels faster?

4. Two cable companies are offering plans for pay-per-view service as presented below in the table.
   a) After how many movies Company B's plan becomes cheaper?
   b) Write the equations for both plans offered by two companies to estimate how much one will pay if he watched 10 movies to the two companies

Figure 4.3. Misconception Items
In addition to the standardized achievement questions, items from the research literature related to common misconceptions with mathematical representations were added. The best-known misconceptions are the picture/graph interval/point, and height/slope confusions (Leinhardt, Zaslavsky, & Stein, 1990).

Janvier was one of the first mathematics educators to mention the problems that students have in interpreting graphs (Bell & Janvier, 1981 and Janvier, 1981). Mostly he argues how global meanings of graphs and interpreting graphs are left out in mathematics classrooms, while reading data and constructing and reading certain points on graphs are emphasized.

One type of confusion arises when students graph the picture of a phenomenon itself instead of the required relationship. A common example of this type of confusion is drawing the picture of the hill when one aims to draw a distance versus time graph of a bicycle going up and down a hill. Bell and Janvier (1981) and Janvier (1981) mention that the “graph as picture” misconception is problematic for many students and discuss the difficulties students have in interpreting graphs. These articles emphasize the importance of the use of situations in mathematics education and point out the importance of using everyday experiences.

Barclay (1985), Clement (1989), Dugdale (1993), Monk (1992), and Mokros and Tinker (1987) are among the other mathematics education researchers who discuss the “graph as picture” misconception. Monk (1992) reports that even beginning calculus and advanced students show a tendency towards this type of misconception. Barclay (1985),

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Dudgale (1994), and Mokros and Tinker (1987) offer the use of technology, such as computer software or Calculator-Based Laboratories, to overcome this type of misconception, and they report promising results.

Although “graph as picture” has been studied more than the other types of misconceptions, point/interval and height/slope misconceptions have also been identified by mathematics education researchers. Preece (1983b) (as cited in Leinhardt, Zaslavsky, & Stein, 1990) and Bell and Janvier (1981) studied the point/interval confusion, which occurs when students focus on a single point instead of a range of points as they interpret a graph. Leinhardt, Zaslavsky, & Stein (1990) suggest that the ambiguous meaning of the word ‘when’ plays an important role in this confusion.

The following question provides an example to illustrate the interval/point and height/slope confusions: Two video rental clubs offer two different rental fee plans. (see Figure 4.4)
When is it less expensive to belong to Club A? (Interval/point confusion) Students may answer this question as four rentals instead of 1-4 rentals.

Two examples of studies that uncovered the height/slope misconception are those done by Bell and Janvier (1981) and Clement (1989). Slope versus height confusion occurs when students pay attention to a higher point/value rather than the slope when they are asked a question that focuses on the slope itself. For instance, in the example above, the question, “Which plan is cheaper per rental when one rents 3 movies?” can reveal the height/slope confusion. For this question, students’ attention may focus on the height of the graph and they may think that Club B has a cheaper rate per movie.

Bell and Janvier (1981) conducted a teaching experiment in which students collected data and graphed their findings on a transparency; whole class discussion followed during which students compared their findings and discussed the differences.
The authors compared this group with a table group in which students performed their experiments and created tables rather than graphs but discussed their results in the same way as the graph group did. The graph group did slightly better than the table group with the questions related to height/slope misconception even though the difference was not statistically significant.

In the pretest the misconception questions were presented to students on a separate sheet from the other questions (see Figure 4.3). However, in the posttest the misconception questions and other representation related questions were mixed in one test. Misconception related questions in both the pre- and posttest are presented in Figure 4.3. Unlike the achievement items in the mathematical pre- and posttests, the questions related to misconceptions were not exactly the same in the pretest and the posttest. The numbers in the representation related questions and the order of the questions were changed in the posttest while keeping the questions basically the same so as not to change the nature and the difficulty of the test. Moreover, a few extra questions were added in the posttest, because some of the misconception related questions in the pretest proved to lead students to avoid the possible misconceptions. Thus, more ambiguous questions, drawn from the mathematics education research literature, were used in the posttest. In order to compensate for having different numbers of questions in the pretest and posttest in the data analysis, questions were categorized and one score for each category was obtained for each student by using mean statistics of the variables. A copy of the mathematics posttest is provided in Appendix D. Scoring of the pre-and posttest will be discussed in Chapter 5 in more detail.
Follow-up Interviews

Follow-up interviews were carried out after the mathematical pre- and posttests in order to observe students’ reasoning in answering the questions. After the pretest, nine students were chosen to be interviewed based on their answers on the pretest. The students selected for interviews had either provided an answer but no explanation or did not provide an answer to some of the pretest questions. Interviewed students included participants from each of the three groups—linked, semi-linked, and control. On the other hand, all participants were interviewed after the mathematical posttest. The examples of questions asked on this interviews were:

- Would you explain how you answered this question?
- Would clarify your solution for this question for me?

Document Analysis

There were different documents in this study which were a very important part of the data and data analysis, but they were not graded or scored to be included in the quantitative part of the data analysis. Lab activity sheets and computer dribble files were examples of these documents. They provided information about lab work in which it was impossible for the researcher to observe all of the students at the same time. Students’ test scores for regular class were also collected.
Computer Based Interviews

The clinical interview yields information not easily available from other sources. It gives insights into students’ experiences by permitting the teacher to understand the meanings that students find in mathematical problems and to appreciate their feelings and confidence about learning mathematics.

(Long & Ben-Hur, 1991, p. 44)

Five students were interviewed (clinically) from each computer group. Here clinical interview is defined as “an exchange between two or more people in which the interviewer seeks to elicit information from the interviewee about how the latter thinks and learns” (Long & Ben-Hur, 1991, p. 44). According to Ginsburg (1981), the aim of clinical interviews is to understand the intellectual processes underlying mathematical knowledge: “Science dictates that we use standardized, reliable, replicable procedure; the clinical method demands that we treat each child flexibly and therefore often differently from every other... The clinical method is soft; it is not a substitute for the rigorous methods of science” (p. 5). Ginsburg (1981) identifies the three aims of clinical interviews in the area of research in mathematics education as: discovery and identification of cognitive activities, and the evaluation of levels of competency.

In the present study, clinical interviews were mostly content oriented and usually began with a problem. However, before being interviewed, the aims of the interview were shared with students in order to help them understand that the interviewer was not interested in whether their answer was right or wrong but how it was obtained by the interviewee. This provided a more flexible and comfortable environment for both the interviewer and the interviewee. The oral script that was read to the interviewees before the interview is provided in Appendix E.
Hunting (1997) mentions that questions or tasks should have some novelty in order to engage students’ interest. However, the questions should also challenge students mathematically. The aim is to understand the student’s thinking process by asking questions about their ways of thinking through the problem. The main suggestions for clinical interviewing that come from Hunting (1997) and Kaplan (1994) are:

- to be open ended so students have freedom in answering questions,
- to create environments for more reflection and sharing of students’ thoughts with the interviewer such as using “why” questions, or asking students to give examples, and challenging students’ answers,
- not to generalize from students’ findings too soon and to test hypotheses with other questions or examples.

Each experimental group, linked and semi-linked, had 5 pairs of students working together—one was responsible for operating the software and the other was responsible for writing down their answers on the worksheet. Each week the students switched roles. While choosing students to interview, special attention was given to interviewing some students who were very comfortable and successful with computers and some who were not. By doing this, different reactions of all participants were examined.

In these interviews students were presented a worksheet whose format was identical to the computer lab activities but with a different movie and physical phenomenon (the fish movie; see Appendix B). They were interviewed individually and
had access to a computer. Going over a mathematics investigation with the students, observing her/his problem solving procedure, and talking about steps taken to solve a problem provided very rich data for this study. These interviews were recorded using Lotus ScreenCam, which records the audio near the computer and the user’s actions on the screen. A separate tape recorder was also used to record the audio in case something went wrong with the computer recording. The interviews were semi-structured and varied in detail from student to student. Typical questions that were used in the interviews were:

Here is the task. I am interested in your thinking, so please think out loud while you are solving this problem.

- Can you say out loud what you are doing?
- Can you tell me how you worked that out?
- Do you know a way to check whether you are right?
- Why?

Class Observations

For as long as people have been interested in the social and natural world around them, observation has served as the bedrock source of human knowledge

(Adler & Adler, 1994, p. 377)

Since it is believed that behavior is purposive and expressive of deeper values and beliefs, observations were essential for the data collection in this study (Marshall & Rossman, 1995, p. 79). The researcher observed both everyday normal class sessions four days per week and the computer lab sessions one day per week during the entire data collection process, paying special attention to the use of representations. The
researcher made field notes after the laboratory hours, and computer dribble files of students’ work provided additional data. Observing the regular class sessions helped to illuminate the differences between computer settings and non-computer settings.

Teacher Interviews

_Interviewing is one of the most common and most powerful ways we use to try to understand our fellow human beings._

(Fontana & Frey, 1994, p. 361)

Two interviews were conducted with the teacher of the class on 10/1/1999 and 10/27/1999 and included questions about his class in general, use of technology in mathematics education, and multiple representations. They were semi-structured interviews and the researcher tried to follow the interviewee’s answers, ask other questions in order to reveal more information, and create a more dynamic environment. Kvale (1996) mentions that “a good interview question should contribute thematically to knowledge production and dynamically to promoting a good interview interaction” (p. 129). All interviews were audio recorded and transcribed. A typical interview protocol is provided in Appendix F.

Survey

Since the interviews with students focused only on mathematical aspects of the study, the researcher used a survey in order to obtain information about students’ preferences and attitudes towards mathematical representations. The survey explored students’ preferences and their justifications with general open-ended questions.
The survey started by asking demographic information about students and included four-level Likert scale questions about students' attitudes towards mathematics and mathematical representations ranging from strongly agree, agree, disagree, and strongly disagree (see Appendix G). The five questions regarding students' attitudes towards mathematics were adapted from Fennema and Sherman (1976). The questions regarding students' beliefs about mathematical representations were collected from the mathematics education research literature (Dufour-Janvier, Bednarz, and Belanger, 1987) and adapted from Rosenheck (1991/1992). Table 4.7 provides detailed information about the sources of the questions. The rest of the survey consisted of open-ended questions focusing on students' justifications for their preferences regarding mathematical representations in general and in the computer lab. The survey was conducted at the end of the data collection procedure on 11/10/1999.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics makes me feel uneasy and confused.</td>
<td>Fennema and Sherman (1976)</td>
</tr>
<tr>
<td>Mathematics is a very worthwhile subject for every person.</td>
<td>Likert Scale Questions</td>
</tr>
<tr>
<td>Mathematics is enjoyable and stimulating to me.</td>
<td></td>
</tr>
<tr>
<td>Mathematics has been my worst subject.</td>
<td></td>
</tr>
<tr>
<td>Mathematics helps develop a person's mind and teaches him/her to think logically.</td>
<td></td>
</tr>
<tr>
<td>Mathematics problems can be solved in various ways by using different representations such as tables, graphs, and equations.</td>
<td>Likert Scale Question</td>
</tr>
<tr>
<td>I like using more than one representation such as graphs, tables, and equations to solve mathematics problems.</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>Given a mathematical problem, I find it easier to focus on one representation than to deal with many representations.</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>When a mathematics problem is presented with more than one representation, it means that there are as many questions as representations</td>
<td>(Dufour-Janvier, Bednarz, and Belanger, 1987)</td>
</tr>
</tbody>
</table>

continued
Table 4.7 continue

<table>
<thead>
<tr>
<th>Statement</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solving a mathematics problem with different representations such as graphs, tables, and equations results in totally different answers</td>
<td>(Dufour-Janvier, Bednarz, and Belanger, 1987)</td>
</tr>
<tr>
<td>I like using □ Equations, □ Tables, □ Graphs the most when solving math problems. Because</td>
<td>Likert Scale Question</td>
</tr>
<tr>
<td>□ Equations, □ Tables, □ Graphs make mathematical topics the easiest for me to understand. Because</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>If □ Equations, □ Tables, □ Graphs were the only option I had to solve math problems then I would have the most difficult time doing the problem. Because</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>I find it the hardest to construct □ Equations, □ Tables, □ Graphs when solving problems using paper and pencil. Because</td>
<td></td>
</tr>
<tr>
<td>I will usually start solving mathematics problems with □ Equations, □ Tables, □ Graphs. Because</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>I find □ Equations, □ Tables, □ Graphs confusing when working on a mathematics problem. Because</td>
<td></td>
</tr>
<tr>
<td>Using software, VideoPoint, in our class was □ Confusing, □ Helpful in understanding mathematical representations such as graphs, tables, and equations. Because</td>
<td></td>
</tr>
<tr>
<td>While using VideoPoint, I liked using □ Equations, □ Tables, □ Graphs the most. Because</td>
<td></td>
</tr>
<tr>
<td>VideoPoint made it the easiest to use □ Equations, □ Tables, □ Graphs when solving math problems. Because</td>
<td></td>
</tr>
<tr>
<td>Using the computer was helpful in understanding the relationships between tables, graphs, and equations. □ Yes □ No. Because</td>
<td></td>
</tr>
<tr>
<td>I liked being able to see all representations at the same time on the computer screen. □ Yes □ No. Because</td>
<td></td>
</tr>
<tr>
<td>Graphs are useful for understanding mathematical concepts. □ Yes □ No. Because</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>Tables are useful for understanding mathematical concepts. □ Yes □ No. Because</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>Equations are useful for understanding mathematical concepts. □ Yes □ No. Because</td>
<td>Rosenheck (1991/1992)</td>
</tr>
<tr>
<td>What do you like the most about using VideoPoint?</td>
<td></td>
</tr>
<tr>
<td>What do you like the least about using VideoPoint?</td>
<td></td>
</tr>
<tr>
<td>Is there anything influencing your choice of or preference for a representation, such as graph, table, or equation in solving mathematics problems? If yes, what and why?</td>
<td>Poppe (1993/1994)</td>
</tr>
</tbody>
</table>

Table 4.7. Survey Question Sources
Table 4.8 presents exact dates for administering instruments during the data collection of this study.

<table>
<thead>
<tr>
<th>Time</th>
<th>Activities at School and Data Collected</th>
</tr>
</thead>
<tbody>
<tr>
<td>May 99</td>
<td><strong>Pilot Study</strong></td>
</tr>
<tr>
<td><strong>Week 1</strong></td>
<td>Representation pretest, computer lab session, class observations and collecting class materials</td>
</tr>
<tr>
<td><strong>Week 2</strong></td>
<td>Computer lab session, class observations and collecting class materials</td>
</tr>
<tr>
<td><strong>Week 3</strong></td>
<td>Computer lab session, class observations and collecting class materials</td>
</tr>
<tr>
<td><strong>Week 4</strong></td>
<td>Computer lab session, class observations and collecting class materials, survey, teacher interview, representation posttest, clinical interviews with students</td>
</tr>
<tr>
<td>Autumn 99</td>
<td><strong>Actual Study</strong></td>
</tr>
<tr>
<td><strong>Week 1</strong></td>
<td>Consent letters given to the students (9/20)</td>
</tr>
<tr>
<td>Sept 20-24</td>
<td>Preparation for the computer lab (9/22)</td>
</tr>
<tr>
<td><strong>Week 2</strong></td>
<td>Representation pretest, mathematics pretest, written questionnaire (9/29)</td>
</tr>
<tr>
<td>Sept 27-Oct 1</td>
<td>Teacher interview (10/1)</td>
</tr>
<tr>
<td></td>
<td>Class observations and collecting class materials</td>
</tr>
<tr>
<td></td>
<td>Identifying the groups of students: linked (10); semi-linked (10); and control (5)</td>
</tr>
<tr>
<td><strong>Week 3</strong></td>
<td>Computer lab session (5 computers-2 students per computer) (10/6)</td>
</tr>
<tr>
<td>Oct 4-8</td>
<td>Class observations and collecting class materials, first quiz (10/7)</td>
</tr>
<tr>
<td></td>
<td>Follow-up interviews of selected students according to their answers in written interview questionnaire. (10/4)</td>
</tr>
<tr>
<td><strong>Week 4</strong></td>
<td>No Class—Columbus Day (10/11)</td>
</tr>
<tr>
<td>Oct 11-15</td>
<td>Class observations and collecting class materials</td>
</tr>
<tr>
<td></td>
<td>NCTM regional conference (10/12-13)</td>
</tr>
<tr>
<td><strong>Week 5</strong></td>
<td>Computer lab session (10/20)</td>
</tr>
<tr>
<td>Oct 18-22</td>
<td>Class observations and collecting class materials, second quiz (10/21)</td>
</tr>
<tr>
<td><strong>Week 6</strong></td>
<td>Computer lab session (10/27)</td>
</tr>
<tr>
<td>Oct 25-29</td>
<td>Class observations and collecting class materials, third quiz (10/29)</td>
</tr>
<tr>
<td></td>
<td>Teacher interview (member check) (10/27)</td>
</tr>
<tr>
<td><strong>Week 7</strong></td>
<td>Computer lab session (11/3)</td>
</tr>
<tr>
<td>Nov 1-5</td>
<td>Class observations and collecting class materials, fourth quiz (11/5)</td>
</tr>
</tbody>
</table>
Table 4.8 continue

<table>
<thead>
<tr>
<th>Week 8</th>
<th>Nov 8-12</th>
<th>Completed computer lab activities with students who could not finish or had missed one Survey, representation posttest, mathematics posttest (11/10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weeks 9-11</td>
<td>Nov 15-29</td>
<td>Clinical interviews with students selected for computer interviews Follow-up clinical interviews with all students</td>
</tr>
</tbody>
</table>

Table 4.8. Time Table of Data Collection

**Pilot Study**

As can be observed from the timetable, a pilot study took place in May 1999. The subjects of the pilot study were Algebra I students from the same school in which the actual study took place. All instruments were administered. More importantly, all four computer lab investigations were tried out with students so that necessary changes in the nature and the process of the investigations could be made. For example, the data from the movies in the investigations were not collected beforehand so that students could do it by themselves, but it took so much time for students to do it there was not enough time for them to study other important features of the investigations. Thus the researcher decided to provide the data and also made other small changes in the investigations according to pilot student feedback. Student feedback was also very beneficial in helping the researcher to make necessary changes in other instruments. Likewise, pilot data were analyzed thoroughly in order to examine how appropriate the data analysis methods were, and minor modifications were made. After careful
examination of the data collection and analysis processes and instruments in the pilot study, it was concluded that the experimental procedures and instruments were generally adequate for answering the research questions of this study.

**Trustworthiness**

As Guba and Lincoln (1989) state, "It is not appropriate to judge constructivists by positivist criteria or standards or vice versa" (p. 251). Although there are a lot of techniques available for increasing the credibility of a study, in this study the methods to be used include triangulation, member checks, and peer debriefing.

**Triangulation**

*Triangulation is critical in establishing data-trustworthiness, a triangulation expanded beyond the psychometric definition of multiple resources to include multiple data sources, methods, and theoretical schemes. (Lather, 1986, p. 270)*

Although Janesick (1994) and Patton (1990) state that there are many types of triangulation, such as data triangulation, investigator triangulation, theory triangulation and methodological triangulation, triangulation is interpreted only as data and methodological triangulation by many researchers. Triangulation was a major method to provide trustworthiness, and it took more than one form in this study.

First of all, having different data collection methods allowed the researcher to generate a more holistic picture of the environment and the issue (Morse, 1994). On the other hand, every data collection method had more than one sub-data collection method, which made them more informative. The data on content knowledge was provided by analysis of tests and clinical interviews. Observing the class provided the
researcher with additional insight into students and their content knowledge. While written documents guaranteed that the student had sufficient private time to think without pressure, the written documents were very structured. In contrast, clinical interviews allowed an environment where one-to-one communication could occur in a very flexible and personalized way. Thus, the three forms of data collection with different encounter levels provided rich sources of data for this study and complemented each others’ weaknesses and strengths. Face-to-face encounters with informants in a natural setting was ensured with clinical interviews, whereas observation was used for obtaining large amounts of data on nonverbal behavior. Document analysis allowed the researcher to understand more about students’ individual thinking on their own. Since participant observation and document analysis were more open to interpretation, they could cause the researcher to misinterpret some points. To correct these misinterpretations, face-to-face interviews were ultimately quite helpful. Although data collected in interviews and participant observation were often subject to observer effect, there was not this kind of concern for document analysis.

**Member Check**

[A member check is] the process of testing hypotheses, data, preliminary categories, and interpretations with members of the stakeholding groups from whom the original constructions were collected.  
(Guba and Lincoln, 1989, p. 239)

Presenting the data or interpretation of data to the participants and asking for their ideas not only ensured trustworthiness of the data but also added another layer to the data collection methods. During the data collection period, one member check was obtained with the help of the teacher. This member check was conducted with the aim
of control and checking and allowed the researcher to take immediate action according to the results. Moreover, the researcher had small member check sections at the end of the clinical interviews, using questions such as “Am I following you?” or “Do you want to add something?”

**Peer Debriefing**

Peer debriefing is described as the process of engaging with a disinterested peer in extended and extensive discussions of one’s findings or conclusions in order to test out the findings, coding, or data analysis by Guba & Lincoln (1989). A graduate student in the same program helped the researcher to check the coding and findings of this study’s data. The peer coded one full post mathematical and computer-based clinical interview. The researcher and the disinterested peer agreed on eight out of nine codings on the post mathematical interview. The one remaining coding was also agreed upon through discussion. For the computer-based clinical interview, the researcher and the disinterested peer agreed on all nine existing codings and they also agreed on five extra coding.

**Validity**

**Transferability**

*Transferability is always relative and depends on the degree to which salient conditions overlap and match.*

*(Guba & Lincoln, 1989, p. 241)*
Thick description is the major way to provide transferability. The researcher tried to establish transferability by providing rich and detailed descriptions of time, place, context, and content. If the reader can recognize similarities and differences between conditions of this study and his or her conditions, then transferability of some findings may be appropriate.

**Negative Case Analysis**

*Validation becomes the issue of choosing among competing and falsifiable interpretations, of examining and providing arguments for the relative credibility of alternative knowledge claims.*  
*(Kvale, 1995, pp. 25-26)*

Negative case analysis is defined as “a deliberate search for disconfirming evidence [and] is essential to the process of inquiry” (Erickson, 1986, p. 147). Negative case analysis was a part of this study to ensure that all angles of the study received proper attention. For instance, while choosing students to interview for clinical interviews, both successful and unsuccessful students were chosen in order to investigate different reactions by different participants and to identify cases that disagreed with the common phenomena. By paying special attention to negative cases, the strength of the study was increased.

**Validity of Instruments**

A panel of experts helped the researcher assure both the content and face validity of the instruments. Doctoral committee members of the researcher, other doctoral students in the program, and the participating teacher constituted the panel of experts. Instruments were presented to the members of the panel and according to
their suggestions changes in the instruments were made. Instruments were continuously updated according to feedback from students both during the pilot and throughout the actual study.
CHAPTER 5

RESULTS

The aim of this study was to understand the effects of the automatic linking property of mathematics educational software on students’ understanding. It examined the different effects of semi-linked and linked representations by focusing on three groups of students: one group using linked representation software, another group using similar software but with semi-linked representations, and the control group. Briefly, the research questions were:

1. What are the effects on students’ understanding of linear relationships using linked representation software compared to using semi-linked representation software?

2. What are students’ attitudes towards and preferences for mathematical representations—equations, tables, or graphs?

Subjects of this study were ninth-grade Algebra I students. The class was divided into three groups of students, two experimental groups and a control group. The two experimental groups used the same software but with different linking properties. Data collection methods for the first research question included mathematics pre- and
posttests, follow-up interviews with all students after the mathematics posttest, clinical interviews at the end of the treatment with 5 students from each experimental group, and classroom and lab observations. A survey and formal and informal interviews with the teacher were the data sources for the second research question.

Data Analysis

Qualitative design requires ongoing analysis
(Janesick, 1994, p. 212)

Ongoing analysis is one of the significant components of a research study that helps the researcher to be aware of emerging themes. Thereby, the researcher can add new components to the study or remove some parts of the study as necessary (Glesne & Peshkin, 1992; Huberman & Miles, 1994; Janesick, 1994; Strauss & Corbin, 1994). Thus, the analysis of data should not be postponed until the data gathering is finished since the researcher can come up with a new angle on the data or research questions during the data analysis and can change or extend the data gathering methods before it is too late.

Data analysis involves organizing what you have seen, heard, and read so that you can make sense of what you have learned. Working with the data, you create explanations, pose hypotheses, develop theories, and link your story to other stories. To do so, you must categorize, synthesize, search for patterns, and interpret the data you have collected. (Glesne & Peshkin, 1992, p.127)

Therefore, the researcher started preliminary data entry and data analysis during the data collection and continuously worked on the post instruments according to responses to the pre-instruments in order to get more useful information from students. For
instance, the questions related to some misconceptions on the mathematics pretest were changed in the posttest, since they did not elicit the information that the researcher was trying to investigate.

Moreover, it is necessary to organize data in such a way that it is easier to see the patterns and emerging themes. Therefore, the researcher had files for every student in the class. These files included tests, worksheets, computer dribble files of the lab hours, interview transcripts, and all other materials related to each student. Files for analysis of the data for each group were also formed in order to see themes and patterns across the linked, semi-linked, and control groups.

This chapter is organized so that the results regarding each research question are presented separately. First, the quantitative results will be discussed, followed by qualitative results relative to the first research question. Then, the same procedure will be repeated for the second research question.

**Students’ Understanding of Mathematical Concepts Using Representations**

In this section, the analysis of mathematics pre- and posttests is presented in order to see if there are any differences among the three groups according to their answers on the pre- and posttests. This will be followed with the analysis of the clinical interviews.
Students' Answers to the Questions in the Pre- and Posttest

Instead of studying each test question separately, questions in all written tests used in this study were clustered into categories and those categories were compared across the three groups—linked, semi-linked, control. The categories reflected findings in the research literature related to mathematical representations and misconceptions. The categories were:

- Word Problems (Verbal),
- Graph (Reading and Interpreting/Constructing Graphs),
- Table (Reading and Constructing Tables),
- Equation (Solving and Constructing Equations), and
- Misconceptions (Height/Slope, Point/Interval, Graph as Picture).

Before presenting results, it is important to explain how scores in each of these categories were obtained. Students’ responses to the questions in the pre- and posttests were coded as right (1), wrong (0), or in some cases partial credit (.25, .5, or .75) was given. Questions’ resources and scoring are presented in Appendix C. Each student was then given one score for each category for the pre- and posttest by taking the mean of the scores for the questions in that category. These category scores were used to study the group differences and their improvement from the pretest to posttest statistically. As mentioned in the previous chapter, it is essential to remember that in some categories extra questions were added or questions were changed from the pretest to posttest in order to obtain more information from students. Therefore, using mean statistics for each category overcame the difficulty of having different numbers of questions in different categories or in the same category on pre- and posttest exams. Each student
has a mean score for each category. When the nonparametric tests were performed to analyze group differences, these mean scores for each student were used rather than task scores or whole-test scores.

Scores on the mathematical pre- and posttests were compared using a nonparametric test—Kruskal-Wallis (a test for several independent samples)—to identify differences between the linked, semi-linked, and control groups. Nonparametric statistics were used in this study due to the limited number of participants and not being able to sample the population randomly. Nonparametric tests can be used with samples as small as N=6 (Siegel, 1956). Siegel (1956) describes Kruskal-Wallis statistics as a one-way analysis of variance by ranks that is “an extremely useful test for deciding whether $k$ independent samples are from different populations.... The test assumes that the variable under study has an underlying continuous distribution. It requires at least an ordinal measurement of that variable” (pp. 184-185).

After obtaining the results about group differences, it was important to study how the groups’ achievement changed from pretest to posttest, and whether those changes were significant. In parametric cases, there are options for before-after designs with more than one sample.

When the before-after design is modified by the addition of the control group, the resulting design is referred to as a pretest-posttest control group design. Unfortunately, researchers are not in agreement with respect to what statistical analysis is most appropriate for the latter design. Among the analytical procedures that have been recommended are the following:

a) the difference scores of the two groups can be contrasted with t-test for two independent samples;
b) the results can be evaluated by employing a factorial analysis of variance for a mixed design...;
c) an analysis of covariance. (Sheskin, 1997, pp.285-286)

However, “there is no exact nonparametric equivalence to analysis of covariance” (Black, 1999, p. 602). Neither was it expected to find nonparametric equivalences for the factorial designs. Several resources that provided nonparametric equivalences for certain parametric statistics did not provide one for factorial designs. Taking difference scores for the three groups and comparing those using the Kruskal-Wallis test seemed to be the only option; however many statisticians warn about using gain scores. Moreover, in some categories students improved and in some of them they declined. There were positive and negative difference scores and taking absolute value of those might cause the loss of some valuable information. So, it was decided to study each group’s improvement or decline in each group itself. In order to study the significance of improvement/decline from pretest to posttest within groups, the Wilcoxon matched-pairs signed-ranks test which is “a nonparametric procedure employed in a hypothesis testing situation involving a design with two dependent samples...can also be employed to evaluate a before-after design” (Sheskin, 1997, p. 291). “If the relative magnitude as well as the direction of the differences is considered, [Wilcoxon matched-pairs signed-ranks test] can be made” (Siegel, 1956, p. 75). The results of the Kruskal-Wallis and Wilcoxon tests are presented in each category below. SPSS (2000) statistical package is used to perform these nonparametric tests.
After the pretest, only selected students were interviewed whereas all students were interviewed after the posttest. In the pretest, students were asked to explain their answers on their sheets. Thus, their justifications for problem solutions were either self-explanations or follow-up interviews that required a qualitative analysis in order to understand students' reasoning. This analysis also brought deeper understanding to the researcher about how students improved their reasoning from the pretest to posttest.

**Word problems.** This category included word problems related to linear relationships. The two questions in this category were essentially the same in the pretest and posttest except the numbers were changed (see Table 5.1).

<table>
<thead>
<tr>
<th>Word Problems in the Pretest</th>
<th>Word Problems in the Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>A water tank is full with 1650 gallons of water. If we drain the water at the rate of 5 gallons per minute, estimate to the nearest half an hour how long it will take to clear the tank.</td>
<td>A water tank is full with 1845 gallons of water. If we drain the water at the rate of 15 gallons per minute, estimate how long it will take to clear the tank.</td>
</tr>
<tr>
<td>A certain machine produces 250 labels per minute. At this rate, how long will it take the machine to produce enough labels to fill 10 pages of labels if each page has 80 labels?</td>
<td>A certain machine produces 75 labels per minute. At this rate, how long will it take the machine to produce enough labels to fill 15 pages of labels if each page has 20 labels?</td>
</tr>
</tbody>
</table>

Table 5.1. Word Problems in the Pre- and Posttest

Table 5.2 presents the descriptive statistics for students' responses to the word problems both in the pretest and the posttest. Scoring each item is shown Appendix C.
Table 5.2. Descriptive Statistics for Word Problems

<table>
<thead>
<tr>
<th>Group</th>
<th>Word Problems in Pretest</th>
<th>Word Problems in Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>N  5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Mean .5000</td>
<td>.8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .3536</td>
<td>.2739</td>
</tr>
<tr>
<td>Linked</td>
<td>N  10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Mean .6000</td>
<td>.9000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .3162</td>
<td>.3162</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>N  10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Mean .4000</td>
<td>.8500</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .3162</td>
<td>.2415</td>
</tr>
</tbody>
</table>

In order to study whether there were any differences among groups in the category of word problems on either the pretest or posttest, the Kruskal-Wallis test was performed. The results of this test are presented in Table 5.3. Results indicated no significant differences existed between the three groups on either the pre- or posttest.

Table 5.3. Kruskal-Wallis Test for Word Problems

<table>
<thead>
<tr>
<th></th>
<th>Chi-Square</th>
<th>df</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Word Problems in Pretest</td>
<td>1.920</td>
<td>2</td>
<td>.383</td>
</tr>
<tr>
<td>Word Problems in Posttest</td>
<td>1.410</td>
<td>2</td>
<td>.494</td>
</tr>
</tbody>
</table>
Figure 5.1 presents the improvement in each group's scores from pretest to posttest. These improvements were derived by taking the difference between the group posttest average from the group pretest average. The semi-linked group improved the most among the three groups with a gain of .45. The control and linked groups improved by .30 each.

![Bar chart showing improvements in scores for word problems]

**Figure 5.1. Improvements in Scores for Word Problems**

In order to study whether these improvements were significant, the Wilcoxon matched-pairs signed-rank test was performed. The results of this test are presented in Table 5.4. All groups improved significantly from pretest to posttest. The semi-linked group improvement was statistically significant at the .05 level, while the other two groups improved at the .1 level (see Table 5.4).
Group | Z   | Asymp. Sig.  
----|-----|----------
Control | -1.732 | .083   
Linked   | -1.897 | .058   
Semi-linked | -2.460 | .014

Table 5.4. Wilcoxon Test for Word Problems

Since all participants answered the questions in the word problem category fairly successfully, those questions were not followed up in the interviews due to time constraints.

**Graphical representations.** Two sub-categories of graphical representation questions were formed: reading graphs and interpreting or constructing graphs. Pre- and posttest questions in this category are shown in Table 5.5.
Graph Related Questions in the Pretest

Hospital B

Hospital A

Jan Feb Mar Apr May Jun Jul Aug Sep Oct

The number of patients in Hospitals A and B over a 10-month period are provided above in the graph. How many total patients were in the hospitals in January?_________

Reading Graphs

In the graph shown, what is the number of bicycles built in 1975?

In the graph shown, what is the number of bicycles built in 1965?

Interpreting/Constructing Graphs

During which time period did bicycle construction increase the most?

During which time period did bicycle construction increase the least?

continued
Table 5.5 continue

\[ y = 2x + 7 \]
If this equation is changed to \( y = 1x + 7 \), sketch the graph of the new equation.

If \( y = 2x + 3 \) is changed to \( y = 1x + 3 \), sketch the graph of the new equation and discuss how the graph of the new equation is different from the graph of the old one.

\[ y = 2x + 7 \]
If this equation is changed to \( y = 2x - 2 \), sketch the graph of the new equation.

If \( y = 2x + 3 \) is changed to \( y = 2x + 5 \), sketch the graph of the new equation and discuss how the graph of the new equation is different from the old one.

---

**Table 5.5. Graphical Representation Questions in the Pre- and Posttest**

**Table 5.6 presents the descriptive statistics for students’ responses for problems about graphical representations both in the pretest and the posttest. In general there was an improvement in the scores from the pretest to posttest except for the control group on questions related to reading values from graphs (see Appendix C for scoring).**

<table>
<thead>
<tr>
<th>Group</th>
<th>Reading Graphs in Pretest</th>
<th>Reading Graphs in Posttest</th>
<th>Interpreting Graphs in Pretest</th>
<th>Interpreting Graphs in Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>N 5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Mean .7000</td>
<td>.6000</td>
<td>.2000</td>
<td>.8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .4472</td>
<td>.5477</td>
<td>.1826</td>
<td>.2981</td>
</tr>
<tr>
<td></td>
<td>N 10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Linked</td>
<td>Mean .6000</td>
<td>1.0000</td>
<td>.2667</td>
<td>.7000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .4595</td>
<td>.0000</td>
<td>.3063</td>
<td>.3583</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>N 10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Mean .7000</td>
<td>.8000</td>
<td>.2667</td>
<td>.8667</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .3496</td>
<td>.4216</td>
<td>.1405</td>
<td>.2331</td>
</tr>
</tbody>
</table>

**Table 5.6. Descriptive Statistics for Graphical Representation Problems**
The results of the Kruskal-Wallis test are presented in Table 5.7 in order to study whether there were any differences among groups in the pre- and posttest. Results indicated no significant differences existed between three groups for either the pre- or posttest. However, it can be noted that the linked group answered all the reading graphs questions correctly in the posttest, while the semi-linked group performed slightly better than the other two groups in the category of interpreting graphs on the posttest with a mean 0.8667 (see Table 5.6).

<table>
<thead>
<tr>
<th></th>
<th>Reading Graphs in Pretest</th>
<th>Reading Graphs in Posttest</th>
<th>Interpreting Graphs in Pretest</th>
<th>Interpreting Graphs in Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chi-Square</td>
<td>.249</td>
<td>4.000</td>
<td>.590</td>
<td>1.602</td>
</tr>
<tr>
<td>df</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Asymp. Sig.</td>
<td>.883</td>
<td>.135</td>
<td>.744</td>
<td>.449</td>
</tr>
</tbody>
</table>

Table 5.7. Kruskal-Wallis Test for Graphical Representation Problems

Figure 5.2 presents the improvement in each group’s scores from pretest to posttest. The linked group improved more in reading graphs than the other two groups, whereas the semi-linked and control groups improved more in the category of interpreting graphs than the linked group.
The results of the Wilcoxon test of whether these improvements were significant are presented in Table 5.8 and Table 5.9. Only the linked group improved significantly (p<.05) from pretest to posttest in questions related to reading values from graphs. On the other hand, the improvement of experimental groups for interpreting graphs was statistically significant at the .05 level and at the .01 level for the control group.

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-0.272</td>
<td>.785</td>
</tr>
<tr>
<td>Linked</td>
<td>-2.070</td>
<td>.038</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-0.347</td>
<td>.729</td>
</tr>
</tbody>
</table>

Table 5.8. Wilcoxon Test for Problems on Reading Graphs
<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-1.890</td>
<td>.059</td>
</tr>
<tr>
<td>Linked</td>
<td>-2.220</td>
<td>.026</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-2.719</td>
<td>.007</td>
</tr>
</tbody>
</table>

Table 5.9. Wilcoxon Test for Problems on Interpreting Graphs

Students provided the following explanations for the posttest questions on interpreting graphs, which asked students to identify the time period bicycle construction increased the least according to the provided graph (see Table 5.5):

**Student #1.** It did not increase that much. (Control group student, T2A-266)

**Student #2.** I just kind of looked to see which one has less steepness between two points. (Linked group student, T6B-357)

**Student #3.** When I looked at the graph, the line looked the most horizontal right there. (Linked group student, T6A-387)

**Student #4.** Because the line does not slope as much as other segments do. Because between 1960-1965 it has almost horizontal slope and the rest of have a more positive slope. (Semi-linked group student, T2B-123)

Another question on interpreting graphs asked students to discuss how the graph of a new equation was different from the old one when the slope and y-intercepts changed. No students from the control and semi-linked groups could answer this question correctly in the pretest, and only one student did from the linked group.
However, in all groups the majority of students answered these questions correctly in the posttest. Discussing procedural steps to explain their answers was more common among control group students:

St: This one [slope] is 2 and that one is 1. You rise 1, go over 1, and you rise 2 and go over 1.
Int: What about the y-intercepts? When they have different y-intercepts how does that affect the graph?
St: Because it is where you put the point at like 5 and 3 and they are different points but the same slope. (Control group student, T2A-279)

Visual explanations for their answers were found mostly among linked and semi-linked group students for this question:

Student #1. It'd be like turn more. It would not be steep. It'd be like less slanted. (Linked group student, T9A-085)

Student #2. Slopes were the same and they would be parallel and not intersecting. So I said they have different y-intercepts. That is the difference between them but they have the same slope. (Linked group student, T3A-432)

Student #3. The slope has stayed the same but the y-intercept changed, therefore making them parallel lines. (Semi-linked group student, written explanation on the posttest)

For all groups, examples of not being able to answer or provide a sound explanation for this question included approaching questions numerically or using the steepness concept when they were talking about the y-intercept:

Student #1. I figured it should be more slope because it crosses the y higher than this one; that means it is going up more than that. (Control group student, T2A-509)
**Student #2.** Int: When you changed $2x+3$ to $1x+3$ how is the graph going to change?
St: It would have decreased probably. Just considering you have 1 instead of 2— that is all I can think of. $1x$ is $x$ times itself so it probably decreases by half as much as $2x+3$. (Linked group student, T5B-123)

**Student #3.** It $[2x+5]$ would be slightly bigger, slightly more positive than $2x+3$ because 5 is bigger number than 3. (Semi-linked group student, T8A-154)

**Student #4.** The old one $y$-intercept is kind of lower so it was kind of straight up because it is closer to the zero, but the new one is far so it is more like slanted than the old one. (Semi-linked group student, T5B-516)

**Tabular representations.** This category included questions asking about tables in the context of linear relationships. Two sub-categories were formed: reading tables and constructing tables. Questions in the pre- and posttest are presented in Table 5.10.

<table>
<thead>
<tr>
<th>Table Related Questions in the Pretest</th>
<th>Table Related Questions in the Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Telephone Call Costs</strong></td>
<td><strong>Based on the table above, what is the cost of a 20-minute call from city X to Z?</strong></td>
</tr>
<tr>
<td>From City X to: First 3 minutes: Each additional minute:</td>
<td>Based on the table above, what is the cost of a 25-minute call from city X to Y?</td>
</tr>
<tr>
<td>City Y</td>
<td>60¢</td>
</tr>
<tr>
<td>City Z</td>
<td>20¢</td>
</tr>
</tbody>
</table>

continued
Two cable companies are offering plans for pay-per-view service as presented below in the table. After how many movies does Company B's plan become cheaper?

<table>
<thead>
<tr>
<th>Number of Movies</th>
<th>Company A ($)</th>
<th>Company B ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>29</td>
<td>22</td>
</tr>
</tbody>
</table>

Constructing Tables

According to the information in the table, what is the total daily earnings for Thursday?

According to the information in the table, what is the total daily earnings for Wednesday?
Table 5.11 presents the descriptive statistics for students’ responses to problems about tabular representations both in the pretest and the posttest. Using mean scores for each subcategory, there was a general improvement in scores from the pretest to posttest, except that the control and linked groups stayed the same on the questions related to constructing tables (see Appendix C for scoring).

<table>
<thead>
<tr>
<th>Group</th>
<th>Reading Tables in</th>
<th>Reading Tables in</th>
<th>Constructing Tables in</th>
<th>Constructing Tables in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pretest</td>
<td>Posttest</td>
<td>Pretest</td>
<td>Posttest</td>
</tr>
<tr>
<td>Control</td>
<td>N 5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Mean .8000</td>
<td>1.0000</td>
<td>.8000</td>
<td>.8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .4472</td>
<td>.0000</td>
<td>.4472</td>
<td>.4472</td>
</tr>
<tr>
<td>Linked</td>
<td>N 10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Mean .4000</td>
<td>.9500</td>
<td>.8000</td>
<td>.8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .5164</td>
<td>.1581</td>
<td>.4216</td>
<td>.4216</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>N 10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Mean .7000</td>
<td>.9000</td>
<td>.7000</td>
<td>.8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .4830</td>
<td>.2180</td>
<td>.4830</td>
<td>.4216</td>
</tr>
</tbody>
</table>

Table 5.11. Descriptive Statistics for Tabular Representation Problems

In order to study whether there were any differences among groups in the pre- and posttest, the Kruskal-Wallis test was performed. The results of this test are presented in Table 5.12. Results indicated no significant differences existed among the three groups for either the pre- or posttest.
Table 5.12. Kruskal-Wallis Test for Tabular Representations Problems

Figure 5.3 presents a graph revealing the improvement in each group's scores from pretest to posttest. The linked group improved more in reading tables than the other two groups, whereas the semi-linked group progressed more in constructing tables than the control and linked groups, who did not experience any improvement.
The Wilcoxon test was performed in order to study whether these improvements were significant. The results of this test are presented in Table 5.13 and Table 5.14. Only the linked group improved significantly at the .05 level from pretest to posttest in questions related to reading values from tables (see Table 5.13). However, none of the three groups showed statistically significant improvement in the category of constructing tables (see Table 5.14).

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-1.000</td>
<td>.317</td>
</tr>
<tr>
<td>Linked</td>
<td>-2.333</td>
<td>.020</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-1.633</td>
<td>.102</td>
</tr>
</tbody>
</table>

Table 5.13. Wilcoxon Test for Problems on Reading Tables

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Linked</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-1.000</td>
<td>.317</td>
</tr>
</tbody>
</table>

Table 5.14. Wilcoxon Test for Problems on Constructing Tables

One question was the same in both the pre- and posttest in reading/interpreting the tables category. It asked the cost for long-distance phone calls. For that question, 4 out of 5 students in the control group answered correctly in the pretest and this score
increased to 5 out of 5 in the posttest. While only 4 of 10 students from the linked group could provide a correct response for this question in the pretest, in the posttest this number increased to 9. Seven students in the semi-linked group answered correctly in the pretest, whereas 8 students did so in the posttest.

The other question for reading/interpreting the tables provided a table about two cable companies’ plans for pay-per-view service with a flat rate and fee per movie. It asks after how many movies Company B’s plan becomes cheaper. This question was not included in the pretest and was answered correctly by all participants from each group. The common explanation for their answers for all three groups was finding the first point that company B was cheaper in the table. In fact, this was the only explanation given by the control group students. However, some students in the experimental groups provided deeper explanations:

**Student #1.** Well, after three movies because at two movies Company B is more expensive than Company A, but at 3 movies Company B is one dollar less than Company A. (Control group student, T2A-390)

**Student #2.** It became cheaper after about 2 movies or 3 movies because company A’s price starts to increase and their [B’s] price has started to decrease. They start at $5 and company B starts at 10. Then slowly by ...they start to increase. They both start to increase, but company A increases more than company B. (Linked group student, T7A-438)

**Student #3.** Because their starting charge with no movie is $10 and they only rise by two...This is going by 4 and this is only rising by 2 and at 3 movies company A is $17 and at 3 movies company B is at $16. (Semi-linked group student, T8A-325)
In constructing tables, 4 out of 5 students in the control group answered correctly in the posttest, as did 8 out of 10 students from each of the experimental groups. These scores were the same as in the pretest for the control and linked groups and an improvement of 1 for the semi-linked group.

**Algebraic representations.** Two subcategories were formed: solving equations and constructing equations. Pre- and posttest questions in this category are presented in Table 5.15.

<table>
<thead>
<tr>
<th>Solving Equations</th>
<th>Algebraic Pretest Questions</th>
<th>Algebraic Posttest Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $12x = 4(x + 5)$, determine the value of $x$</td>
<td>If $2x = 18 - 3(x + 1)$, determine the value of $x$</td>
<td></td>
</tr>
<tr>
<td>Bill is 600 miles away from home and travels with his car at a constant rate of 65 miles/hour towards his home. If $h$ represents the number of hours traveled and $d$ represents the distance from home, write the equation for distance in terms of hours. How far away from home is Bill after he traveled 2 hours?</td>
<td>Bill is 350 miles away from home and travels with his car at a constant rate of 50 miles/hour towards his home. If $h$ represents the number of hours traveled and $d$ represents the distance from home, write the equation for distance in terms of hours. How far away from home is Bill after he traveled 3 hours?</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.15 continue

Write an equation of the relationship that is expressed in the graph.

Write the equations for each plan offered by two companies to estimate how much one will pay to watch 10 movies.

<table>
<thead>
<tr>
<th>Number of Movies</th>
<th>Company A ($)</th>
<th>Company B ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>29</td>
<td>22</td>
</tr>
</tbody>
</table>

Table 5.15. Equation Related Questions in the Pre- and Posttest

Table 5.16 presents the descriptive statistics for students’ responses for problems about algebraic representations both in the pretest and the posttest (see Appendix C for scoring). The mean scores show that the only improvement in the scores...
from the pretest to posttest occurred for the linked group in the category of solving equations. However, for constructing equations, the semi-linked group had a higher mean than the other two groups.

<table>
<thead>
<tr>
<th>Group</th>
<th>Solving Equations in Pretest</th>
<th>Solving Equations in Posttest</th>
<th>Constructing Equations in Pretest</th>
<th>Constructing Equations in Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N 5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Control</td>
<td>Mean .8000</td>
<td>.4000</td>
<td>.0000</td>
<td>.2167</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .4472</td>
<td>.5477</td>
<td>.0000</td>
<td>.1394</td>
</tr>
<tr>
<td></td>
<td>N 10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Linked</td>
<td>Mean .5000</td>
<td>.6000</td>
<td>.0000</td>
<td>.2833</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .5270</td>
<td>.5164</td>
<td>.0000</td>
<td>.3689</td>
</tr>
<tr>
<td></td>
<td>N 10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>Mean .7000</td>
<td>.4000</td>
<td>.0000</td>
<td>.3417</td>
</tr>
<tr>
<td></td>
<td>Std.Dev .4830</td>
<td>.5164</td>
<td>.0000</td>
<td>.4487</td>
</tr>
</tbody>
</table>

Table 5.16. Descriptive Statistics for Algebraic Representations Problems

The Kruskal-Wallis test was performed in order to study whether there were any differences among groups in the pre- and posttest (see Table 5.17). Results indicated no significant differences existed among the three groups for either the pre- or posttest.
The improvement in each group’s scores from pretest to posttest for questions related to algebraic representations are presented in Figure 5.4. The linked group improved for the questions on solving equations, while the other groups experienced a drop in their responses. The semi-linked group improved slightly more than the other two groups in constructing equations.

<table>
<thead>
<tr>
<th></th>
<th>Solving Equations in Pretest</th>
<th>Solving Equations in Posttest</th>
<th>Constructing Equations in Pretest</th>
<th>Constructing Equations in Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chi-Square</td>
<td>1.500</td>
<td>.923</td>
<td>.000</td>
<td>.107</td>
</tr>
<tr>
<td>Df</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Asymp. Sig.</td>
<td>.472</td>
<td>.630</td>
<td>1.000</td>
<td>.948</td>
</tr>
</tbody>
</table>

Table 5.17. Kruskal-Wallis Test for Algebraic Representation Problems

Figure 5.4. Improvement in Scores for Algebraic Representation Problems
In order to study whether these improvements were significant, the Wilcoxon test was performed. The results of this test are presented in Table 5.18 and Table 5.19. None of the groups improved significantly from pretest to posttest in questions related to solving equations (see Table 5.18). On the other hand, the improvement of the semi-linked and linked groups for constructing equations was statistically significant at the .05 level while the control group’s improvement was significant at the .1 level (see Table 5.19).

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-1.414</td>
<td>.157</td>
</tr>
<tr>
<td>Linked</td>
<td>-1.000</td>
<td>.317</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-1.342</td>
<td>.180</td>
</tr>
</tbody>
</table>

Table 5.18. Wilcoxon Test for Problems on Solving Equations

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-1.841</td>
<td>.066</td>
</tr>
<tr>
<td>Linked</td>
<td>-2.032</td>
<td>.042</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-2.032</td>
<td>.042</td>
</tr>
</tbody>
</table>

Table 5.19. Wilcoxon Test for Problems on Constructing Equations
Since the students' answers in the solving equations category were interpreted clearly by the researcher, those questions were not followed up in the interviews due to time constraints.

There were three types of questions regarding constructing equations in the posttest and two types in the pretest. Students were asked to construct equations from different sources: verbal statements, graph, and table. Two of those questions that took either a verbal statement or a table as a base for constructing the equation had a second part asking for a specific value requiring students to substitute into the equation. No student from any group could answer those two questions in the pretest. Thus, mean scores for all students for constructing equations in the pretest were zero. Only the experimental groups were able to improve significantly from that score on the posttest at .05 level.

There were students in all three groups who understood the concept in the questions but were not able to construct the equation algebraically. They could answer the second part of each question, where the aim was to have students substitute a value into the equation after constructing it, even if they could not construct the correct equation, by answering the question without the equation, with the wrong equation, or by ignoring the equation.

Student #1. St: [For the question asking to construct a equation from a table] See how much it went up. Every time this one went up 2, this one went up 4. I just added on until 10 min. I have got the answer 45 and 30.
Int: If I ask you to write the equations?
St: I am not good at equations so. (Control group student, T2B-088)
Student #2. St: [For the second part of the question asking to construct an equation from a verbal statement] You take for the number of hours, you put in 3, and you just work the equation out.
Int: How did you work out the equation?
St: Actually it is a different question. I just did 50*3. He is 350 miles away so 350-50*3 I think. Yeah. He is 350 miles away from home and he is going 50 mph for 3 hours. 50mph times 3 hours and subtract that from 350 miles.
Int: Why did you not write that equation for a?
St: I do not know. (Linked group student, T2B-250)

Student #3. St: [For the question asking to construct a equation from a table] I did not do the equations. I just figured out how much it would be at 10 movies by adding 4. I cannot figure out any equation for that. (Semi-linked group student, T3B-284)

The question that required students to construct the equation from a verbal statement was a distance-time question asking for an equation of Bill’s distance from home if he travels toward his home, given his rate of travel and original distance from home. Some students in all three groups ignored the fact that Bill was traveling toward his home and used the distance-time formula they learned in class: $d=50*h$. On the other hand, 3 out of 10 students in the linked group and 4 out of 10 students in the semi-linked group constructed the correct equation, whereas none of the students in the control group was able to do that.

Student #1. What we did was we took, since Bill is 350 miles away from home and travels with his car at a constant rate of 50 mph towards his home, that would mean he would be less miles away from home. We figured out that the hours he traveled was 350, which was the total miles away, minus quantity of distance from home times 50...It should be the number of hours traveled equals 350 minus the number of hours traveled times 50. (Linked group student, T5B-091)
Student #2. Ok, I just took... since he gets 50 mph, for every hour he goes 50 miles, so for the amount of the hours is 50. I took the negative because it is distance away from his home and he is traveling towards his house. So you subtract how far he went away from his house. (Semi-linked group student, T2B-472)

For the question that required students to construct the equation from a graph, procedural approaches were expected as explanations, because all of the students had experienced such in their regular classroom. Two out of five students in the control group, two out of ten students in linked group, and three out of ten students in the semi-linked group constructed the expected equation from the graph. As shown below, students in the experimental group approached the question first looking at the line and determining the slope or \( y \)-intercept and using rise over run, whereas control group student took two points and worked the formulas out to obtain the equation.

Student #1. St: I did the \( y \)-intercept that was 100. And I took two points from the graph, 2 and 80 and 60 and 4, I find the slope, and I put in the slope intercept form. (Control group student, T2A-415)

Student #2. St: Ok, what I did was I figured the slope was going to be negative because your line is going down and it seemed to be going down 20 and over 2, and it would be 20 over 2 because your rise would be negative 20 and your run would be 2
Int: what about the 100?
St: I did that because that is where it started, zero 100 for your \( y \)-intercept. (Linked group student, T6B-389)
**Student #3.** St: Because the slope is... oh, how did I do that? First the $y$-intercept is on the graph. The slope is 2 because it is like 20 between each number; it goes by 20s. Maybe it should be negative 20. Because the reason it is negative is because it is a negative slope. [Then she calculates the slope using rise over run and paying attention to the scale on the axes.] (Semi-linked student, T2B-140)

Finally, for the question that used a table as the base for an equation, one control group student, 3 linked-group students, and 3 semi-linked group students constructed the correct equation from the table. Students in the experimental group approached the question again with more visual ways instead of taking two points from it and calculating the formulas as they were taught in the class.

**Student #1.** I picked two of these points from the graph [she means table] and I found the slope and they both cost $4 per video [wrong calculation] but for the first video for club A it costs $5 and starts off for B at 10. I just put into the equation and found 10. Four times 10 plus 5 is 45. Four times 10 plus 10 is 50. (Control group student, T2A-455)

**Student #2.** I just found it is like count the slope. I guess $m$ would be the number of movies he got. Since company A, they increase by four each time and they started at 5, so it would be $4m+5$. The other one would be, since they start out and they increase by 2, it would be $2m+10$, but they start out at 10 so that would get you the total number. (Linked group student, T5B-255)

**Student #3.** I figured out that every movie you make you are going up from $5 by $4 and the same thing with company B, but you are going up $2 every movie from 10. And like that is how I figured that out. Like because of the cost it is going up by 2 and you have to add that to 10, you multiply 2 times whatever number of movies you are getting. You add to 10 because 10 is the starting number. You cannot have anything less than 10 or 5 on each. (Semi-linked group, T2B-223)
**Height/slope misconception.** Height/slope misconception questions asked about the slope of the graph instead of a higher point/value.

---

**Height/Slope Misconception Pretest Questions**

The figure above represents the distances traveled by Car A and Car B in 6 hours. Which car is going faster and by how much? Explain how you arrived at your answer.

---

**Height/Slope Misconception Posttest Questions**

The figure above represents the distances traveled by Car A and Car B in 6 hours. Which car is going faster and by how much? Explain how you arrived at your answer.

---

The number of patients in Hospitals A and B over a 10-month period are provided below in the graph. Which hospital had a greater increase in the number of patients from February to March?

---

continued
Table 5.20 continue

Mark is walking and Bill is riding his bicycle. At 2 minutes, who travels faster?

Two video rental clubs offer two different rental fee plans. Which plan is cheaper per a movie rental?

If the above graph represents a wide jar being filled with water, draw a graph for a narrower jar being filled with water on the same coordinate axes.

Table 5.20. Height/Slope Misconception Questions in the Pre- and Posttest

134
Questions were changed and added from the pretest to posttest in order to study this misconception in detail after realizing that one of the questions in the pretest was not helpful in examining this type of misconception. Both pre- and posttest questions are presented in Table 5.20.

Table 5.21 presents the descriptive statistics for students’ responses to height/slope misconception problems both in the pretest and the posttest. In general there was an improvement in the scores from the pretest to posttest for this type of questions (see Appendix C for scoring).

<table>
<thead>
<tr>
<th>Group</th>
<th>Height/Slope Scores in the Pretest</th>
<th>Height/Slope Scores in the Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>N 5</td>
<td>Mean .6000 Std.Dev .2710</td>
</tr>
<tr>
<td></td>
<td>Mean .6000 Std.Dev .2710</td>
<td>.6750</td>
</tr>
<tr>
<td>Linked</td>
<td>N 10</td>
<td>Mean .6000 Std.Dev .3900</td>
</tr>
<tr>
<td></td>
<td>Mean .6000 Std.Dev .3900</td>
<td>.7938</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>N 10</td>
<td>Mean .6375 Std.Dev .2599</td>
</tr>
<tr>
<td></td>
<td>Mean .6375 Std.Dev .2599</td>
<td>.8125</td>
</tr>
</tbody>
</table>

Table 5.21. Descriptive Statistics for Height/Slope Misconception Questions

In order to study whether there were any differences among groups in the pre- and posttest, Kruskal-Wallis test was performed. The results of this test are presented in the Table 5.22. Results indicated no significant differences existed between the three
groups for either the pre- or posttest. However, it can be noted that the semi-linked group performed slightly better than the other two groups with a mean of 0.8125, while the linked group had a mean of 0.7938 and the control group had a mean of 0.675 in the posttest (see Table 5.21).

<table>
<thead>
<tr>
<th></th>
<th>Chi-Square</th>
<th>df</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height/Slope Misconception in Pretest</td>
<td>.120</td>
<td>2</td>
<td>.942</td>
</tr>
<tr>
<td>Height/Slope Misconception in Posttest</td>
<td>2.404</td>
<td>2</td>
<td>.301</td>
</tr>
</tbody>
</table>

Table 5.22. Kruskal-Wallis Test for Height/Slope Misconception Problems

Figure 5.5. Improvement in Scores for Height/Slope Misconception Problems
Figure 5.5 presents the improvement in each group’s scores from pretest to posttest. Linked group improved the most among the three groups at .19, whereas the semi-linked group improved by .18 and control group improved only by .08.

In order to see whether these improvements were significant, the Wilcoxon test was performed. The results of this test are presented in Table 5.23. Both experimental groups, semi-linked and linked, improved significantly from the pretest to posttest at the .05 and .1 levels respectively, while the control group did not improve significantly.

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-0.406</td>
<td>.684</td>
</tr>
<tr>
<td>Linked</td>
<td>-1.873</td>
<td>.061</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-2.263</td>
<td>.024</td>
</tr>
</tbody>
</table>

Table 5.23. Wilcoxon Test for Height/Slope Misconception Problems

There were 2 height/slope misconception questions on the pretest, and 4 on the posttest. One of those questions was the same in the pretest and the posttest. However, the other question in the pretest was eliminated and 3 questions were added to the posttest in order to study this misconception. The question that appeared in both the pre- and posttest provided a graph representing the distances traveled by two cars in 6 hours and asked which car was going faster and by how much. Only two students out of five students in the control group could provide a correct answer with a clear explanation:
In two hours Car B has traveled 80 miles and Car A has traveled 100. Then it had to be going faster because it was 100. How fast it was going... I divided 80 by two. (Control group student, T2A-526)

Moreover, one of these two students was the only one in this group who improved from the pretest to posttest. The other four students provided the same level answers in the posttest as they did in the pretest. Three students in the control group showed clear signs of height/slope misconception in the posttest.

Ok, I looked at the time at 2 hours. And it was about 20 mph apart. So I put that car A was going 20 mph faster than car B. And in 6 hours it says Car B was here. Which car is going faster? I just said Car A because first two hours it was just going faster, 20 mph. (Control group student, T2A-415)

Two out of ten linked group students, on the other hand, showed signs of height/slope misconception in the posttest as opposed to four students in the pretest. Six out of ten students provided complete answers with sufficient explanations:

I chose Car A because Car A is going faster. Because they went the same amount of hours and Car A got farther than Car B. (Linked group student, T3A-191)

The number of linked group students who provided complete answers for these questions was only three in the pretest. Although two students provided the correct answer, their explanation or their answer to the second part of this question was not right.

Only one student in the semi-linked group preserved her misconception from pretest to posttest, out of five students who showed signs of this misconception in the pretest. While only one student in the pretest provided complete answers with sufficient explanation, eight out of ten students did in the posttest:
The slope of car A is more than car B. (Semi-linked group student, T2B-482)

In addition, one student provided the correct answer but she could not figure out how much faster Car A was going than Car B.

One of the other three questions was answered well by all participants, which provided a graph representing a wide jar being filled with water and asked students to draw a graph on the same coordinate axes for a narrower jar being filled with water. All students in the control and semi-linked group provided a correct answer, whereas one student from the linked group could not do it.

**Student #1.** When the jar is more narrow it is gonna get filled up faster. (Control group student, T2A 196)

**Student #2.** It was up further because it is narrow. Then the line like goes up because water fills up the small jar faster. (Linked group student, T6A-310)

**Student #3.** If the height of the jar is more, it would fill up faster because it is the narrower. If it is the same height, it would fill up faster because there is less area inside of it. It has a higher slope. (Semi-linked group student, 2B-500)

The remaining two questions were more complex questions that could lead students to answer in a way that revealed the height/slope misconception. The researcher needed to clarify the question or ask the question differently for some students in the follow-up interviews. The rates for answering those two questions correctly and the types of explanations were similar for all groups. There were students in all groups who showed signs of the height/slope misconception. There were also students who explained their answers mentioning the slope of the phenomena;
however, this was more common among students from the experimental groups than it was in the control group (one student in the control group, three students in the linked group and four students in the semi-linked group):

Student #1. [The question provided a graph of Bill and Mark’s distances over time and asked who was traveling faster at 2 minutes] Yeah. It looks like. I would say Bill because in this amount of time he is going faster. The graph is going steeper than Mark’s.
[For the video rental question] I looked at the graph and I took two points. I found the slope. That means that every time you rent a video it would go up... higher price or something higher than 4 [for Club A]. But for Club B, it only costs $4 for a video. (Control group student, T2A-430)

Student #2. St: [The question provided a graph of Bill and Mark’s distances over time and asked who was traveling faster at 2 minutes] I chose Bill because his is slanted more; that is a different rate. (Linked group student, T2B-394)

Student #3. St: [The question provided a graph of Bill and Mark’s distances over time and asked who was traveling faster at 2 minutes] I just looked at the graph and Bill’s line is steeper. (Linked group student, T2B-303)

Student #4. t: [For the video rental question] I think I went by the slope of each one to see which one is cheaper. (Linked group student, T2B-375)

Student #5. St: [The question provided a graph of Bill and Mark’s distances over time and asked who was traveling faster at 2 minutes] Bill travels faster because its slope is deeper but he [Mark] got more distance because he probably got a head start or something. (Semi-linked group student, T4B-025)

Student #6. St: [The question provided a graph of Bill and Mark’s distances over time and asked who was traveling faster at 2 minutes] I do not know. Bill should be traveling faster because he started when there was no time, and Mark had a head start, but really Bill travels faster because he has a greater slope.
[For the video rental question] It does not have a steep line...Club B is more diagonal than Club A is. (Semi-linked group student, T8A-273)

**Point/interval misconception.** Only one question was asked about a range of points instead of a single point in the graph. This question was changed from the pretest to the posttest because the wording of the question in the pretest was not helpful for examining the point/interval misconception. The pre- and posttest questions are presented in Table 5.24.

<table>
<thead>
<tr>
<th>Point/Interval Misconception Pretest</th>
<th>Point/Interval Misconception Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Question</strong></td>
<td><strong>Question</strong></td>
</tr>
<tr>
<td>Video Rental Costs</td>
<td>Video Rental Costs</td>
</tr>
<tr>
<td>Club B</td>
<td>Club A</td>
</tr>
</tbody>
</table>

The number of patients in Hospitals A and B over a 10-month period are provided in the graph. When did Hospital B have fewer patients than did Hospital A?

Two video rental clubs offer two different rental fee plans. For what number of video rentals is it less expensive to belong to Club A?

Table 5.24. Point/Interval Questions in the Pre- and Posttest
Table 5.25 presents the descriptive statistics for students’ responses for the point/interval misconception problems in both the pretest and the posttest (see Appendix C for scoring). In general there was a drop in the scores from the pretest to posttest in this type of question for all groups, which could indicate that the posttest question was just more difficult than the pretest question.

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Mean</th>
<th>Std.Dev</th>
<th>N</th>
<th>Mean</th>
<th>Std.Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>5</td>
<td>1.0000</td>
<td>.0000</td>
<td>5</td>
<td>.6500</td>
<td>.3354</td>
</tr>
<tr>
<td>Linked</td>
<td>10</td>
<td>1.0000</td>
<td>.0000</td>
<td>10</td>
<td>.9500</td>
<td>.1581</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>10</td>
<td>.9000</td>
<td>.3162</td>
<td>10</td>
<td>.8500</td>
<td>.3375</td>
</tr>
</tbody>
</table>

Table 5.25. Descriptive Statistics for Point/Interval Misconception Questions

In order to study whether there were any differences among groups in the pre- and posttest, the Kruskal-Wallis test was performed (see Table 5.26). Results indicated no significant differences existed between three groups for either the pre- or posttest. All groups did very well in the pretest. As mentioned above, wording of the question in the pretest hindered the examination of possibilities of students having this
misconception. On the other hand, it can be noted that the linked group performed somewhat better on the posttest than the other two groups with a mean of 0.95, while the semi-linked group had a mean of 0.85 and the control group had a mean of 0.65 (see Table 5.25).

<table>
<thead>
<tr>
<th></th>
<th>Chi-Square</th>
<th>df</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point/interval Misconception in Pretest</td>
<td>1.500</td>
<td>2</td>
<td>.472</td>
</tr>
<tr>
<td>Point/interval Misconception in Posttest</td>
<td>4.378</td>
<td>2</td>
<td>.112</td>
</tr>
</tbody>
</table>

Table 5.26. Kruskal-Wallis Test for Point/Interval Misconception Problems

Figure 5.6. Improvement in Scores for Point/Interval Misconception Problems
Figure 5.6 presents the improvement in each group's scores from pretest to posttest. While the semi-linked and linked groups had a drop of only .05, the control group declined by .35 in their scores.

In order to see whether these drops were significant, the Wilcoxon test was performed. The results of this test are presented in Table 5.27. None of the groups declined significantly.

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-1.633</td>
<td>.102</td>
</tr>
<tr>
<td>Linked</td>
<td>-1.000</td>
<td>.317</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-1.000</td>
<td>.317</td>
</tr>
</tbody>
</table>

Table 5.27. Wilcoxon Test for Point/Interval Misconception Problems

After changing the question from the pretest to posttest, the average scores declined for all the groups, especially for the control group. In the posttest, after follow-up interviews, 3 out of 5 control students showed signs of the point/interval misconception, whereas this number was only 1 out of 10 for the linked group and 2 out of 10 for the semi-linked group. However, it was observed that for a couple of students the interviewer asked leading questions that helped students to adjust their answers. But after these incidents, the researcher was very careful not to lead students on this question. Again there were students from the experimental groups who explained their answers in a way that showed no sign of the point/interval misconception:
Student #1. I found the number where they crossed paths and I went one above that and it would be that number or higher, how many movies you can get and be less expensive. (Linked group student, T6B-415)

Student #2. I think so. The intercept is where they are the same price. They both have 4 video rentals. Club A becomes less after that, like around 5. For five videos this one begins to become less expensive right after the intercept. (Semi-linked group student, T3A-051)

*Graph as picture misconception.* This type of confusion arises when students graph the picture of a phenomenon itself instead of the required relationship. The same question was used in both the pre- and posttest. The question is presented in Figure 5.7.

Figure 5.7. Graph as picture Misconception Question
Table 5.28 presents the descriptive statistics for students' responses for the graph as picture misconception problem in the pretest and the posttest (see Appendix C for scoring). In general there was an improvement in the scores from the pretest to posttest for this type of misconception question.

<table>
<thead>
<tr>
<th>Group</th>
<th>Graph as Picture Misconception in Pretest</th>
<th>Graph as Picture Misconception in Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>N: 5</td>
<td>N: 5</td>
</tr>
<tr>
<td></td>
<td>Mean: .4000</td>
<td>Mean: .6000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev: .5477</td>
<td>Std.Dev: .5477</td>
</tr>
<tr>
<td>Linked</td>
<td>N: 10</td>
<td>N: 10</td>
</tr>
<tr>
<td></td>
<td>Mean: .2000</td>
<td>Mean: .8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev: .4216</td>
<td>Std.Dev: .4216</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>N: 10</td>
<td>N: 10</td>
</tr>
<tr>
<td></td>
<td>Mean: .6000</td>
<td>Mean: .8000</td>
</tr>
<tr>
<td></td>
<td>Std.Dev: .5164</td>
<td>Std.Dev: .4216</td>
</tr>
</tbody>
</table>

Table 5.28. Descriptive Statistics for Graph as Picture Misconception Questions

The results of the Kruskal-Wallis test are presented in Table 5.29. Results indicated no significant differences existed between the three groups for either the pre- or posttest. However, it can be noted that the semi-linked and linked groups performed better with a mean of 0.8 than the control group with a mean of 0.6 (see Table 5.28).
Table 5.29. Kruskal-Wallis Test for Graph as Picture Misconception Problems

Figure 5.8 presents the improvement scores by groups from pretest to posttest in the graph as picture misconception question. The semi-linked and control groups improved by .20, whereas the semi-linked group improved by .60.

Figure 5.8. Improvement in Scores for Graph as Picture Misconception Problems
In order to see whether these improvements were significant, the Wilcoxon test was performed. The results of this test are presented in Table 5.30. Only the linked group improved significantly from pretest to posttest at the .05 level, while the control and semi-linked groups did not improve significantly.

<table>
<thead>
<tr>
<th>Group</th>
<th>Z</th>
<th>Asymp. Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>-1.000</td>
<td>.317</td>
</tr>
<tr>
<td>Linked</td>
<td>-2.449</td>
<td>.014</td>
</tr>
<tr>
<td>Semi-linked</td>
<td>-1.414</td>
<td>.157</td>
</tr>
</tbody>
</table>

Table 5.30. Wilcoxon Test for Graph as Picture Misconception Problems

For the graph as picture problem, 2 out of 5 students in the control group were able to answer the question correctly, one student showed signs of having the graph as picture misconception, and another student did not provide an answer in the pretest. One student chose the answer which reflected the graph as picture misconception but her self-explanation and follow-up interview explanations could not necessarily be coded as this type of misconception:

**Student #1.** Because there was a downfall or a hill in the picture and it is harder to go uphill than on a straight road. (Student Worksheet)

**Student #2.** St: I looked at this. It looks like the car goes uphill and goes downhill and straight. I think I just... This would be the time.
Int: Because it shows?
St: How time goes up and down. This could be 10 min and that could be 30 and that could be 10 min
Int: What about the distance?
St: I still say this one
Int: Why?
St: Because I don’t know. How would I know the distance?... Time would be
down here and I would be able to tell what the time was and what the distance
was. (Control group student pretest follow-up Interview, T1A-320)

In the posttest the same 2 control group students answered this question
correctly again. One who had not answered this question in the pretest and another one
who showed the signs of graph as picture misconception gave signs of this
misconception in the posttest, too. The other control group student who did not have
this misconception was able to answer the question correctly with a sound explanation:

The more he walks the distance will go up. Since he walks from point A to point
B the distance will keep going up cause he did not stop, like he just stops at
point B. (Control group student, T2A-331)

For the linked group, 2 students could answer the questions with sound
explanations, whereas 6 out of 10 students had indications of graph as picture
misconception in the pretest.

The figure looked like it was representing the same thing that answer looks like
it’s representing (Linked group student pretest follow-up Interview, T1A-294).

The other two students chose the first option as their response and provided similar but
not sound explanations for their answer, which could be the result of a graph as picture
misconception:
Student #1. Because if you straighten that out that would be diagonal...if you bring that line [he means the first part of the hill] up it would be like that [the graph that he chose.] (Linked group student pretest follow-up Interview, T1A-214)

Student #2. The way I thought about it, he is going on the hill for the distance. I just thought maybe just pull it straight that way. ...Slope... I figure that is the slope of it and that would be most likely the length. (Linked group student pretest follow-up Interview, T1A-225)

In the posttest, 8 out of 10 students in the linked group either answered correctly with firm explanations written on their sheets or they provided proper answers in the follow-up interviews:

I chose this because this is the starting point and this is the ending point right here. So I chose the line going up. Because the farther you get, the numbers are going to rise. (Linked group student, T3A-514)

One student retained his graph as picture misconception and another student, who provided one of the above explanations for choosing the first option, repeated his explanation for his answer, which could be the result of a graph as picture misconception.

Finally, 4 out of 10 students in the semi-linked group indicated signs of the graph as picture misconception in the pretest:

Graph B [middle graph] shows the distance the person walked from A to B. Graph A [first graph] displays a constant downward distance. Graph B displays a graph that resembles the actual distance. Graph C [last graph] shows an upward walk. (Student worksheet of a semi-linked student)
While 4 out of the 10 students answered correctly with reasonable explanations, 2 students provided not very clear explanations for their correct choice on this question. In the posttest, 2 students retained their misconception from pretest to posttest, whereas 8 students either answered correctly with firm explanations written on their sheets or they provided correct answers in the follow-up interviews:

St: Because the distance... The closer to your destination, the distance gets smaller.
Int: But the question was its distance from the beginning at A.
St: Oh, it should be this one. Because the distance from A, that means you are going away from it. The distance becomes farther, not closer. And the time gets longer. (Semi-linked group student, T3A-126)

Summary

The results described above are summarized in Table 5.31 which provides all significant scores and improvement scores for each category. The first two columns present the results of the Kruskal-Wallis test, which is the nonparametric equivalent to one-way ANOVA. The results of this test showed that there were no differences among groups in the pretest at either the .05 or .1 level. The next three columns display the improvement scores of the groups from pretest to posttest. Finally, the last three columns present the significance scores from the Wilcoxon nonparametric test, which is a test for dependent samples and was performed to observe whether these improvements from the pretest to posttest were significant within groups. Some of the improvements were significant at the .05 level, such as experimental groups in the categories of interpreting graphs and constructing equations, the semi-linked group for
the height/slope misconception category, and the linked group for the graph as picture category. Other improvements were significant at the .1 level, such as the linked group for the height/slope category.
<table>
<thead>
<tr>
<th>Word Problems (Verbal)</th>
<th>Group Differences</th>
<th>Improvement</th>
<th>Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Significance Scores*</td>
<td>in Scores Linked</td>
<td>Semi-Linked</td>
</tr>
<tr>
<td></td>
<td>Pretest</td>
<td>Posttest</td>
<td>Control</td>
</tr>
<tr>
<td></td>
<td>0.383</td>
<td>0.494</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>0.744</td>
<td>0.449</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>0.883</td>
<td>0.135</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>0.472</td>
<td>0.63</td>
<td>-0.4</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.948</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>0.247</td>
<td>0.529</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>0.854</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0.942</td>
<td>0.301</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>0.472</td>
<td>0.112</td>
<td>-0.35</td>
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<tr>
<td></td>
<td>0.202</td>
<td>0.656</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>0.729</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Kruskal-Wallis H (A Test for Several Independent Samples)**

**Wilcoxon (A Test for Dependent Samples)**

Table 5.31. Summary of Significance Scores
Students' Learning of Mathematical Concepts in a Computer Environment

In this section, the nature of students’ learning in the linked and semi-linked computer environments is illustrated. These analyses are based mainly on the clinical interviews conducted with students while working with computers. Five students from each experimental group were interviewed. In the interviews, students watched a movie showing two fish swimming towards each other (see Appendix B). Students tried to determine when the two fish met using the graph, table, and equation. The final question was what was the distance between the two fish at the beginning of the movie.

First, quantitative results of students’ answers to the questions are presented below. This is followed by an analysis of students’ use of the linked computer environment and then their use of the semi-linked environment.

Quantitative Results of Students’ Responses in the Clinical Interviews

Table 5.32 summarizes and compares quantitative results of students’ answers to the questions in the clinical interviews.

<table>
<thead>
<tr>
<th>Sections</th>
<th>Questions</th>
<th>Linked (n=5)</th>
<th>Semi-Linked (n=5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>Predict what graphs of distance from the left-hand side of the screen versus time look like for both fish. <strong>Correct answer:</strong> Sketch of crossing two lines: one with positive and one with negative slope <strong>Example of an incorrect answer:</strong> Two parallel lines</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 5.32 continue

Looking at the graph on the computer, identify and describe the point which represents where the two fish meet. Explain your reasoning.
Correct answer: The distance at the crossing point on the graph

Section 2 Let’s consider a table representing the data. Describe how the numerical values of these positions will behave if we put them in the table? Are they going to increase all the time or decrease? Explain your answer.
Correct answer: One increases while the other decreases Example of an incorrect answer: Both increase or decrease

Identify the point which represents where the two fish meet according to the computer produced table.
Correct answer: Identification of the cells on the table where the distance between two fish is the least.

Section 3 Do the equations indicate negative or positive slopes? Estimate the slopes.
Correct answer: One has negative and the other has positive slope.

Do they have negative or positive \( y \)-intercepts? Estimate the \( y \)-intercepts.
Correct answer: One has negative and the other has positive \( y \)-intercept.

Interpret the differences in the equations of the two fish? [SLOPE]

Interpret the differences in the equations of the two fish? [\( Y \)-INTERCEPT]

How could you find the time where the two fish meet by using the equations?
Correct answer: Setting the two equations equal and solving it \( t/x \).
Table 5.32. Students’ Responses in the Clinical Interviews

The differences in the number of correct responses to the questions between students interviewed in the experimental groups occurred in only a few categories. For instance, the total number of correct answers for students in the semi-linked group was lower than the total for linked group students in three categories. Two of those were related to predicting or interpreting the slopes and were caused mostly by one student who had difficulties with the concept of slope and could not provide a sufficient response or explanation for her answers.

On the other hand, the semi-linked group of students did better than those in the linked group in three categories: interpreting the differences in the equations, using the equations to find the point where the two fish meet, and identifying the distance between the two fish at the beginning/middle/end by solving the equation:

<table>
<thead>
<tr>
<th>Section 4</th>
<th>Identify the distance between the two fish at the beginning/middle/end. (By using TABLE)</th>
<th>5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct answer: Approximately 280 pixels</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Identify the distance between the two fish at the beginning/middle/end. (By using GRAPH)</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Correct answer: Approximately 300 pixels</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Identify the distance between the two fish at the beginning/middle/end. (By using EQUATION)</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Correct answer: Approximately 400 pixels</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Student #1. Int: How could you find out the point which represents where the two fish meet by using the equations?
St: A system of linear equations...you don’t solve them together; you solve one equation and one intercept and then the other intercept and the other equation and then you solve the problem.
Int: I do not understand.
St: I am not good with linear equations. I think you would pick one equation and solve for the intercept and you would replace the number into the other equation and solve. (Semi-linked group student, T8B-350)

Student #2. Int: How could you find out the point which represents where the two fish meet by using the equations?
St: You can make them equal to each other and side by side and solve for the $x$ and solve for the $y$. (Semi-linked group student, T4A-140)

Three of five students in the semi-linked group and two of five students in the linked group provided appropriate interpretations for the $y$-intercepts. Two students in the semi-linked group and one student in the linked group had a very clear idea when interpreting the $y$-intercepts that the $y$-intercepts were the starting points for the fish. They used that information for a later question when they were supposed to use equations to find out the distance between the two fish at the beginning:

Student #1. Int: What about the differences in the $y$-intercept?
St: That is when they start, like one started before the other...
Int: How would you use the equations then to find out the distance between the two fish?
St: I put zero for $t$ and just used the $y$-intercepts to find out where they came from. (Semi-linked group student, T4A-130)

Student #2. Int: What about the differences in the $y$-intercept?
St: They started at two separate places. $y$-intercepts would be different...
Int: How would you use the equations to find out how far they were apart at the beginning?
The y-intercepts in the equations. Two different y-intercepts tell you how far away they are. (Semi-linked group student, T4A-427)

Student #3. Int: What about the differences in the y-intercept? What does that mean?
St: It means when they cross the y-axes.
Int: In the graph. What about in the movie?
St: It would be the position when they start.
Int: How would you identify the distance between the two fish at the beginning?
St: You would find the y-intercept; oh, you would look at when the time is zero.
(Linked group student, T5B-403).

Linked Computer Environment

When a question was asked of the linked group, students either used the linkage directly to answer the question or they assimilated this new information and drew upon their previous knowledge to answer the question. For instance, when they were trying to find out where on the graph the two fish met (see Figure 5.9), some students used the linkage between the movie and the graph to find the point on the graph that represented where the two fish met, while some students used their knowledge without needing to use linkage.
Figure 5.9 Snapshot from the VideoPoint when two fish pass each other

Student #1. Pretty much... not the exact point but somewhere about 120 pixels... something like that...50 seconds. (Linked group student, T5B-302)

Student #2. I try to get it right in the middle. It is right in here; they meet right here. [He manipulates two frames of the movie where two fish pass each other.] I try to get these lined up [in the movie window]. The time is, say, about 5 or 6 seconds [He uses linkage between the movie and the graph; he uses the movie to see the point and then he goes to the graph to answer the question.] (Linked group student, T7B-047)

Student #3. St: I will say about there. That is about right.
Int: Why?
St: Because that is about the time they were, both fish were on a straight line [in the movie] and the lines cross on the graph. [In this case he clicks on the point
on the graph that he thinks could be an answer and he watches the movie. He tries a couple of points on the graph and gives his answer.] (Linked group student, T6B-526)

**Student #4.** Somewhere in the middle. Between there. [He first clicked on the graph intersection point so the movie frame changed and then he turned to the movie and played the movie by moving the button under the movie frame. He focused on two frames that showed the fish passing each other.] (Linked group student, T9A-278)

When students used the linkage, their explanation for their answer was based more on the movie. When students did not use the linkage, they explained their answer based more on the mathematical aspects of the question. For instance, two students identified the point on the table that the two fish met without using the linkage and they provided very sound explanations. When I asked if there was any way to check this, they used the linkage and their explanation became based on the movie:

**Student #1.** St: Between these two points [0.7333 and 0.8 seconds] I would think. Because the [distance] numbers are very close together.
Int: Do you have any way to check?
He used the linkage and said, “It shows on the movie where the fish are at that time and that is when they meet.” (Linked group student, T7A-038)

**Student #2.** St: Like between those two [0.7333 and 0.8 seconds], when, like, the striped fish is 112 and the gray fish is 132 [at 0.7333 seconds] ... between there and when the striped fish 129 and gray fish is 111 [at 0.8 seconds].
Int: Why?
St: Because those [distance] numbers are where they were the closest, like at .7333 seconds the striped fish was still... closer [to the origin]... than the gray fish was and by .8 seconds it was farther away than the gray fish was.
Int: Do you have any way to check that with the software?
St: If you click on it, it will show on the graph. [The striped fish] is still a little closer [to the origin] than the gray fish is and then right here [0.8 seconds], the gray fish is a little closer [to the origin] than the striped fish is. (Linked group student, T6B-240)

When students provided an inappropriate answer to a question and they saw that they were wrong through the linkage or computer feedback, disequilibrium occurred. If they could not interpret the information they had just assimilated, they needed to accommodate their preexisting knowledge in order to reach equilibrium.

The next vignette is a discussion of the graphical representation of the two fish’s distance from a certain point over time:

Student #1. The student provided an incorrect prediction for the graphs, but a correct explanation: “So as time went by, it started farther away and as... time went on, it got closer and closer to [the origin]. The striped fish, it started out really close to [the origin] right there, and as time went on, it got farther and farther away.”

She opened the computer-produced graph and she saw two graphs, one with positive and the other with negative slope as she predicted. But her predictions of distance versus time graphs of the two fish were switched.

Int: Which one is the gray fish and which is the striped fish?
St: I think that the red [line] is the gray fish and green [line] is the striped fish.
Int: Do you have any way to check with the program?
St: Probably if I click on the point on the graph it will show the thing where it is then. So the red was the striped fish. That is the opposite of what I said, isn’t it?
Int: Let’s think why?
St: Does that mean that it was getting farther away? That is what I am assuming. Because I was... mixed up. I thought that one was getting closer, that is why.
Int: When it gets closer, does its distance increase or decrease?
St: Distance from the [origin] would decrease, so that is why it was wrong. It should have been the opposite way. (Linked group student, T6B-042)
She provided an inappropriate graph for the fish's distances. They were switched. When she opened the computer-produced graphs, she thought that the gray fish's graph was the striped fish and vice versa. Without the linkage she could have been lost for the rest of the activity. However, when she used the linkage to check her answer, she saw that her answer was not right and disequilibrium occurred. She went to her existing knowledge and reevaluated the information; she came up with an interpretation that agreed with the outside information.

Sometimes students do not have enough background information to assimilate the new information into their existing knowledge structures:

**Student #1.** [I asked one student his prediction about the \(y\)-intercepts of the two fish from the equations.]
St: It would be...probably...Not really sure.
[He used the best fit button to see the equations. I asked his reaction to the \(y\)-intercepts.]
St: It is different because the striped fish has a negative \(y\)-intercept and the gray fish has a positive one, so since you are multiplying by... for the gray fish... by a negative. It would have been that it decreases probably, and then for the striped fish it increases. (Linked group student, T5B-390)

**Student #2.** [I asked her prediction about the \(y\)-intercepts of the two fish from the equations.]
St: I do not know.
[I asked her reaction to the \(y\)-intercepts after seeing the equations.]
St: The \(y\)-intercept... [a long pause.]
Int: What are the biggest differences?
St: One has negative slope and one has positive slope.
Int: When you combine this with the movie, what does that mean? Why does one have a positive slope and one have a negative slope?
St: Because one keeps getting closer to the point and the other one keeps getting farther away.

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Int: What about the differences in the $y$-intercept? What does that mean when you connect that with the movie?

St: No. [I cannot answer.](Linked group student, T6B-283)

Neither of these students could interpret the question about the $y$-intercepts well enough to provide a reasonable response. The first student came up with an explanation that was not at all clear to the researcher, and the second student could not predict or explain anything about the $y$-intercepts after seeing the equations.

One other kind of response in this environment included not using the linkage at all. When students trust their knowledge and answers, they may not need the linkage. Only one out of five students did not use the linkage at all throughout the interview. However, the other four students did utilize it at least once while working with the software.

On the other hand, some students could have benefited from using the linkage but they ignored it, as for instance, when they were trying to come up with the table values from the graph (see Figure 5.10).
When the researcher asked when the striped fish was 50 pixels away [point #1 in Figure 5.10], what was the distance of the other fish at that time [point #2 in Figure 5.10], some students had problems reading values from the graph after determining the time when the striped fish was 50 pixels away. Instead of moving vertically to the other fish in order to find out its distance at that time, some students moved horizontally and read the distance of the other fish at another time [point #3 in Figure 5.10]. However, some students clicked on the striped fish while reading the time at which it was 50
pixels away; the other fish's data point on the graph was then highlighted as in Figure 5.10 because of the linkage, so they could go to the other point and read the value directly:

**Student #1.** He clicked on a point and the other point was highlighted. It was highlighted after he moved up to the correct point but still I think the highlighting was helpful. When I asked when the striped fish was 130 pixels away, he clicked on points until he found the point he was looking for, and he answered the question for the striped fish. Surprisingly he did not pay any attention to the highlighted point for answering the other fish's distance and looked for the coordinates of a point on the graph by using the mouse. VideoPoint allows user to see the coordinates of a point on the graph when the user pointed on the point by the mouse. Finally when I asked about 1 sec, he clicked on the gray fish at 1 sec, answered the question, and immediately went to the other highlighted point and answered the question. (Observation notes for linked group student, T7A-000)

**Student #2.** We were figuring out how to fill out the table when the striped fish was 50 pixels away. She used the mouse to see the coordinates on the graph and came up with .428 seconds. She moved *horizontally* instead of moving vertically. I asked the distance of the gray fish at .428 seconds, and she moved up. The next question was to find out the time when the gray fish was 50 pixels away from the origin. She said 1.045 seconds. She again first moved horizontally and but I reminded her of the time and she moved up vertically. [She moved horizontally the first couple of times, and when I repeated the question, she corrected herself. She could have clicked on one point and the other would have been highlighted through the linkage, but she did not do it.] (Observation notes for linked group student, T6B-130)

**Student #3.** He clicked on the point and estimated the time at .5 seconds for his answer to the question when the striped fish was 50 pixels away. To estimate the gray fish's distance at that time [once he clicked, the other point highlighted] he went directly to that point and answered 210. (Observation notes for linked group student, T9A-293)
Finally, the researcher observed that students in the linked group trusted the answers provided by the software and did not question them. When their answers did not agree with the computer’s feedback, they tried to interpret the computer’s answer instead of questioning it or defending their answer.

Int: Did you see any differences between the first (semi-linked) and linked group?
Teacher: The first group seems to be a lot more independent... their thought processes and stuff. They seem like the more independent group, period. I do not know if it is partly their ability or what...The second group, like I said, seem to be a lot more dependent on either the machine making connections for them or they want to raise their hand and ask you. (Teacher mid-interview, T2A-022)

Semi-Linked Computer Environment

When a question was proposed in this environment, students mainly relied on their own existing knowledge with the help of the software. They assimilated new information in this environment and called on their existing knowledge to answer the question. Although the semi-linked environment did not provide such rich feedback as in the linked environment, a ready-made graph or table constituted powerful visual information/feedback for students to use while answering the questions. Moreover, the lack of linkage forced more mathematically-based explanations instead of the movie-based explanations provided by some of the linked students.

When students were trying to figure out when the two fish met on the graph, their explanations were based mostly on the information provided by the graph:

Student #1. St: They meet at half of a second, no .75 seconds. Because there is only about a fourth of the time left before they get to actually one second, and that is where they meet.
Int: How did you decide that this is the point that they meet?
St: Because both of the circles cross each other [on the graph].
Int: What does that mean?
St: It means that they are at the same place at the same time. (Semi-linked group student, T8B, 071)

Student #2. St: Right here [showing the intersection point of two graphs.]
Int: Why?
St: Because they are together; they are overlapping each other. Beyond this point they pass each other. Right here [showing the intersection point of two graphs.] they are together. (Semi-linked group student, T4A-253)

On the other hand, some students needed linkage in some situations in order to construct more empowering concepts. For instance when students predicted the distance versus time graphs of the two fish after watching the movie, some of them came up with incorrect graphs. When they then opened up the computer produced graphs, it was clear to them that their graphs were wrong. For example, the two graphs had positive slopes instead of one having a positive slope and one having a negative slope. When they were trying to explain the slope to me, I realized that some of them misunderstood the questions. However, if they came up with a negatively sloped graph for the fish which should have had a positively sloped graph, as did some of the students in the linked group, it could have been very hard for them to see their mistake without the linkage. When they opened the computer-produced graphs, they would see one positively and one negatively sloped graph. They could continue with the questions and construct other answers on a faulty basis.
One such example was illustrated by two students’ efforts to figure out the distance between the two fish at the beginning:

Student #1. Int: Would you identify the distance between the two fish at the beginning?
St: You can look where it starts. It starts right here. [Pointing to the striped fish’s first data point on the graph which is point #2 in Figure 5.11]...and then the other one is right there. [She moves horizontally and points to the gray fish’s last data point (point #3 in Figure 5.11), instead of moving down vertically to point at the fish’s first data point which is point #2 in Figure 5.11.]
Int: [Trying to help her see her mistake.] At the beginning is the gray fish there?
St: No, there [now indicates the correct point which is #2 in Figure 5.11.]
Int: How would you find out the distance between them?
St: You could subtract those two numbers. (Semi-linked group student T8A-101)

Figure 5.11 Graph in the Fish Activity
Student #2. Int: Would you identify the distance between the two fish at the beginning by using the graph?
St: I am not sure.
Int: What is the distance of the gray fish at the beginning? [First he goes up to the correct point which is point #1 in Figure 5.11, but then he comes down and focuses on the x-axis]
St: It is about 120 or 115; oh, no, the distance is zero.
Int: What about the striped fish?
St: It is the same. [He focuses on the x-axis and reads two fish’s distances at the x-axis, so both of them are zero which are the points #2 and #3 in Figure 5.11.]
Int: At the beginning they are at the same point? You mean that?
St: Not the same point, but...
Int: I mean at the beginning of the time.
St: The time is different, but the distance is the same.
Int: My question was at the beginning of the movie.
St: At that time the striped fish was at about .2.
Int: And the gray fish?
St: It was about 1.1.
Int: Is that the distance at the beginning of the movie?
St: I do not know. (Semi-linked group student, T4A-158)

As can be observed in both situations, if linkage was available for use, students could have clicked on the beginning of the movie and seen the corresponding data points for both fish on the graph. They may have had a chance to build more powerful concepts about how to connect a phenomenon with its graph.

On the contrary, not having the linkage empowered students to trust their answers and to construct the linkages between representations by themselves:

Student#1. Int: Identify the point which represents where the two fish meet according to the computer produced table.
St: They meet around here. [She points to .7333 seconds.] (see Figure 5.12) The numbers are not that far apart, and it seems like it would be more like .8
seconds. [.8] is closer to the [intersection point] on the graph when you use the mouse, than the other one [.7333] is. [She pointed to the intersection on the graph with the mouse to see the coordinates but the numbers shown were closer to those corresponding to .733] (Semi-linked group student, T8B-152)

Figure 5.12. Table in the Fish Activity

As can be observed, the student constructed the linkage between the table and the graph by herself. She first decided two rows on the table where the distances of two fish are closest to each other. She then focused on the intersection point on the graph and read some values to decide which point represented where the two fish met on the table.
Student#2. Int: Identify the point which represents where the two fish meet according to the computer produced table.
St: So like right there [showing .8 seconds; he checked the graph and came back to the table.]
Int: How did you figure that out?
St: Right on the graph. They crossed about right here [pointing the intersection point on the graph], and then I looked for that 130 when they crossed the distance I just looked that closer to that [on the table]. And 0.8. (Semi-linked group student, T4B-254)

The student knew the point that represents the two fish’s meeting point on the graph. So he read the distance of the intersection point from the graph and came back to the table to see the time at which the distances of the two fish was closer to the value he read from the graph. Again he built the linkage between the table and the graph.

Student#3. Int: Describe how the numerical values of these positions will behave if we put them in the table. The gray fish? Is this going to increase or decrease?
St: The distance?
Int: Yeah.
St: I think the gray fish is going to increase, because looking at the graph, it looks like it. (Semi-linked group student, T4A-253)

Student#4. Int: How do you compare these two equations?
\[
D(t) = -275.0t + 350.0 \text{ for the gray fish} \\
D(t) = 220.0t - 50.00 \text{ for the striped fish}
\]

What are the biggest differences?
St: On the first equation [equation of the gray fish] the slope has to be negative and it has to be higher [greater]. The y-intercept has to be higher on that one, but that one [the equation for the striped fish] it [slope] has to be positive and lower, and this one [y-intercept] has to be negative and lower.
Int: Being positive and negative [slopes] how does that compare with the movie?
St: The striped fish, it went up—that is positive. No. [He watches the movie again]. Yeah, the striped fish went away, so the distance increased. That was a positive slope, but for the gray fish [the distance] decreased; that was negative slope. (Semi-linked group student, T4B-360)

This student built the connection between the movie and equation and gave a meaning for the slope in real life. First he thought that slope was related to going up and down in real life but after watching the movie he constructed the correct knowledge.

Not having linkage also put students in a more active role. More predictions had to be made by the students, especially when they were trying to determine the algebraic form by predicting the coefficients. When they tried new coefficients, they saw a blue line come onto the graph window. Their aim was to match this line with the fish’s data points on the graph. This active role enabled students to assimilate and accommodate through a chain of equilibriums and disequilibriums:

Int: What is happening? [She identified the y-intercept for the gray fish and was trying to find out the slope. She was trying lower negative numbers each time.]
St: The less the negative, the more the line moves upwards.
Int: What do you want to try?
St: Negative, greater. (Semi-linked group student, T8B-300)

She knew she was right to have a negative slope and she was trying to decide its magnitude. VideoPoint helped her by graphing of her predictions and showing her current and last prediction on the same axes along with the data points. She was trying negative slope, but with lower magnitude instead of greater magnitude, and the graph of her equation did not behave as she wanted it to do. So she decided to try lower
numbers and the graphs of predictions started to move in the right direction. She was able to build the connection between the slope and how it affects the graph of an equation.

St: [He put his first prediction for the coefficients. M: 250t + 50] It would be positive [slope], since the slope [he means the line] is going up. I am trying to find it; I am starting out big first, and if it kind of goes [he means if it has more slope] or slanted [down] then I took up or down. [After seeing the graph] I need to go down; the slope is all right though.

Int: What do you want to change?
St: The y-intercept [His prediction is M: 250t + -50.]

Int: Why did you try a negative number?
St: Because that positive number was too big, so I'll just try a negative. I will change the slope to make it go down a little.

Int: How are you going to change the number? Bigger or smaller?
St: It has to be a smaller number.

Int: Why?
St: The slope is still kind of greater than the line, so I try to go low. (Semi-linked group student, T4B-310)

This student had good background knowledge to start with. His first prediction for the striped fish was pretty good and he needed to couple of adjustments to fit the data. He changed the y-intercept to a negative number because it was too big. After he decided on the y-intercept he saw that his slope was a little off and he knew he needed to try a smaller number to make the line go down.

There was also one student who took a very long time to predict the equations, since she did not use the computer feedback effectively and did not move logically to the next step. She had trouble trying negative numbers; she used decimal numbers when she
needed to use negative numbers. At last, she came up with equations, but it took much
time and many tries, which could have been impossible for her under normal lab time
limitations.

Summary
The clinical interviews were performed in order to study mathematical learning in
the computerized environment. The following bullets present crucial findings from both
linked and semi-linked environments.

The Linked Software Environment

- When a question was asked, students either used the linkage directly to
  answer the question or they assimilated this new information and drew upon
  their previous knowledge to answer the question. When they used the
  linkage, their explanation for their answer was based more on the software;
  especially the movie. However, their answers were more based on the
  mathematical aspects of the question, when they did not use the linkage.

- When students provided an inappropriate answer to a question and they saw
  that they were wrong according to the linkage or computer feedback,
  disequilibrium occurred. Then they needed to go back and interpret this new
  information with their existing knowledge. If they could not interpret the new
  information, they needed to accommodate their preexisting knowledge in
  order to reach equilibrium.
• Sometimes students did not have the enough background to interpret this new information with their existing knowledge.

• Some students did not use the linkage at all, when they trusted their knowledge and answers.

The Semi-Linked Software Environment

• When a question was proposed, students relied mainly on their own existing knowledge with the help of the software. They assimilated new information and drew upon their existing knowledge to answer the question.

• Although the semi-linked environment did not provide such rich feedback as in the linked environment, a ready-made graph or table presented powerful visual information/feedback for students to use while answering the questions.

• Lack of linkage forced more mathematically-based explanations instead of movie-based explanations and empowered students to trust their answers and convince themselves and construct the linkages between representations by themselves.

• Some students needed the linkage in some situations in order to construct more empowering mathematical concepts.
Connection Between the Class and Computer Labs

The data about the connections between the computer lab and regular class sessions will be presented in this section. The researcher tried to follow up the activities in the teacher’s regular class sessions in the computer labs, with the aim of providing students an opportunity to apply the knowledge they had learned in class. Moreover, it was hoped that students would also carry their learning from the computer labs back to the regular class sessions, which mainly consisted of paper and pencil tasks. There were a couple of incidents that showed students were indeed carrying ideas back and forth from the class to the computer lab and vice versa.

Three students mentioned that they were trying to think about computer labs while they were answering the questions in the posttest:

See, I had trouble with that the first time we did it too. Half and half I guessed; half and half I tried to think back to when we were doing the VideoPoint. I tried to think back like when we had negative slope, but I could not remember which way it would have worked, so I was kind of confused doing that one. (Linked group student, T2B-394)

I tried to remember back over the computer thing on Wednesdays. (Semi-linked group student, T3A-004)

We have been watching the movie thing in the class. I just figured that it is kind of like that. I try to compare that with what I have seen in the past in the movie. (Semi-linked group student, T8A-221)

Moreover, while they were studying in class the solution of linear systems of equations by substituting, one student mentioned the activity that we had done in the computer lab on the previous day. They had started systems of linear equations in the beginning of that week. First they solved them using graphs. After that we had the final computer
lab on systems of linear equations. The movie was of two cars going the same direction at different speeds with one passing the other. One of the questions in the computer lab activity asked how to use equations to find the point at which the jeep passed the small car. For that question one pair of students came up with the idea of putting those two equations together and solving them for time. When the teacher in the class was teaching solving systems of equations by substituting, that student raised his hand and talked about what his group had done the day before in the computer lab (Class observation notes, 11/4/99). Since the teacher had observed that lab, the teacher knew what he was talking about. When the researcher interviewed the teacher after observing the computer lab, he mentioned that there were connections made by the students between the class and the computer lab:

Int: Do you think they carry the information between the computer lab and class?

Teacher: Yes and no. That is one thing I have been thinking about a lot. I think conceptually some of them are carrying the ideas back and forth. One semi-linked group student just figured out today that what is going on in class carries over to what is going on here. Also, students have commented that they just did that with Asli's [the researcher's] group and have realized things are carried back. But how they attack problems does not necessarily carry back and forth. They have been trained from previous years...to do a problem a certain way and they have been trained to do that. More and more, though, as this is going on, kids have gotten more independent with how they want to look at problems...So I think this has been good for them, and they realize that there are other ways to do problems besides what is being taught. Although still a lot of them focus on just the way they are shown in class, rather than saying we can do this by making a table. At the same time, 5 or 6 kids now would use tables first, rather
than some drawn out method I use. Yeah, I think there is a carryover back and forth and it depends on the individual student to see exactly what is going on and where they are at. (Teacher Interview, 069)

As the teacher mentioned in his interview, there were also students applying ideas from the regular class to computer lab activities. When we were doing the computer activity on systems of linear equations, one student in the semi-linked group figured out that we would be doing something related to linear systems of equation as soon as she saw the movie of two cars mentioned above (Computer observation notes, 11/3/99). Moreover, there were clear instances in the clinical computer interviews that students used the knowledge learned in their regular mathematics class while working on problems in the computer activities:

It decreases; it gets closer to it. So I am not sure which one it was that [the teacher] taught it would be that way...It would... since it starts from there, I think it would be like that. (Linked group student computer interview, T9A-248)

There were also instances in which students provided definitions of the concept that we were working on in the computer activity questions. For instance, when the researcher was asking students how to interpret the differences in the $y$-intercepts of the equation while thinking about the real-life phenomenon, some students provided the definition of the $y$-intercept:

Int: What about the differences in the $y$-intercept? What does that mean?
St: It means when they cross the $y$ axes. (Linked group student computer interview, T5B-403)
**Analysis of the Survey**

The survey started by asking demographic information about students and included four-level Likert scale questions about students' attitudes towards mathematics and mathematical representations (see Appendix G). The five questions regarding students' attitudes towards mathematics were adapted from Fennema and Sherman (1976). The questions regarding students' beliefs about mathematical representations were collected from the mathematics education research literature (Dufour-Janvier, Bednarz, and Belanger, 1987) and adapted from Rosenheck (1991/1992). The rest of the survey consisted of open-ended questions focusing on students' justifications for their preferences regarding mathematical representations in general and in the computer lab.

**Descriptive Statistics**

There were 15 females and 10 males total in the study. Table 5.33 shows the gender distribution in the groups.

<table>
<thead>
<tr>
<th>Group</th>
<th>Female</th>
<th>Male</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Linked</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>Semi-Linked</td>
<td>7</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>10</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 5.33. Gender Distribution in the Groups
Ages of the students ranged from 13 to 15 with 21 students being 14 years old. The majority of the students had computer experience both at home (22 out of 25 students) and at school (18 out of 25 students); only two out of twenty-five students had any computer experience other than at home and school. The students in the control group had more experience at home with a mean of 3.2 years than they did at school with a mean of 2.4 years. Linked students, on the other hand, reported an equal amount of experience both at school and home (mean = 3.4 years). Students in the semi-linked group had more computer experience at school (mean = 3.5 years) with than they had at home (mean = 2.9 years). (see Table 5.34).

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Computer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experience-Home Years</td>
<td>5</td>
<td>0</td>
<td>6</td>
<td>3.2</td>
<td>2.3</td>
</tr>
<tr>
<td>Computer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experience-School Years</td>
<td>5</td>
<td>0</td>
<td>9</td>
<td>2.4</td>
<td>3.9</td>
</tr>
<tr>
<td>Linked</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Computer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experience-Home Years</td>
<td>10</td>
<td>0</td>
<td>6</td>
<td>3.4</td>
<td>2.3</td>
</tr>
<tr>
<td>Computer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experience-School Years</td>
<td>10</td>
<td>0</td>
<td>14</td>
<td>3.4</td>
<td>4.7</td>
</tr>
<tr>
<td>Semi-Linked</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Computer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experience-Home Years</td>
<td>10</td>
<td>0</td>
<td>7</td>
<td>2.9</td>
<td>2.3</td>
</tr>
<tr>
<td>Computer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experience-School Years</td>
<td>10</td>
<td>0</td>
<td>8</td>
<td>3.5</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Table 5.34. Students' Computer Experience
To analyze whether students' computer experiences were significantly different from each other according to their groups, the Kruskal-Wallis nonparametric test was used (see Table 5.35).

| Computer Experience-Home Years | .306 | 2   | .858 |
| Computer Experience-School Years | .754 | 2   | .686 |
| Computer Experience-Other Years | 1.50  | 2   | .472 |

Table 5.35. Kruskal-Wallis Test for Computer Experience

As can be observed from the above table, there were no significant differences among groups according to their computer experience, either at school or at home.

**Attitude Questions**

There were two kinds of attitude questions, one of which studied students' attitudes towards mathematics and one that looked at attitudes towards mathematical representations.

**Attitude towards mathematics.** As mentioned in the methodology chapter, there were ten 4-point Likert questions in the survey that was given at the end of the study. Responses to the questions were coded as 0 for strongly disagree, 1 for disagree, 2 for agree, and 3 for strongly agree. So it was possible to turn the categorical data into an ordinal one for purposes of statistical analysis.
Five of the Likert scale questions focused on the students' attitudes towards mathematics, and three of these were worded positively:

5. Mathematics is a very worthwhile subject for every person.
6. Mathematics is enjoyable and stimulating to me.
8. Mathematics helps develop a person's mind and teaches him/her to think logically.

As can be observed from Table 5.36, students reported positive feelings about mathematics for questions 5 and 8 with means of 2.08 and 2.28 respectively. These questions were worded as general statements about mathematics. However, when the question involved mathematics and themselves personally, as did question 6, they displayed less positive attitudes towards mathematics with a mean of 1.75. Although they believed that mathematics helps develop a person's mind and teaches him/her to think logically, apparently it is not that enjoyable and stimulating to them personally.

<table>
<thead>
<tr>
<th>Question</th>
<th>N</th>
<th>Mean (Range: 0-3)</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. Mathematics is a worthwhile subject</td>
<td>25</td>
<td>2.1</td>
<td>0.8</td>
</tr>
<tr>
<td>6. Mathematics is enjoyable to me</td>
<td>24</td>
<td>1.8</td>
<td>0.8</td>
</tr>
<tr>
<td>8. Mathematics develops minds</td>
<td>25</td>
<td>2.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 5.36. Descriptive Statistics for Positively Worded Attitude Questions
Examination of students’ answers to those questions by groups showed that control group students disagreed more than the other groups with question 6. Semi-linked group students displayed more positive attitudes towards mathematics than the control and linked group students and reported the highest means for questions 5 and 6 (see Table 5.37). However, the Kruskal-Wallis test showed that none of these differences was statistically significant.

<table>
<thead>
<tr>
<th>Group</th>
<th>5. Mathematics is a worthwhile subject</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td></td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>2.2</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>6. Mathematics is enjoyable to me</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>1.2</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>8. Mathematics develops minds</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>2.0</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>5. Mathematics is a worthwhile subject</td>
<td>10</td>
<td>1</td>
<td>3</td>
<td>1.8</td>
<td>0.9</td>
</tr>
<tr>
<td>Linked</td>
<td>6. Mathematics is enjoyable to me</td>
<td>9</td>
<td>1</td>
<td>3</td>
<td>1.8</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>8. Mathematics develops minds</td>
<td>10</td>
<td>2</td>
<td>3</td>
<td>2.4</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>5. Mathematics is a worthwhile subject</td>
<td>10</td>
<td>1</td>
<td>3</td>
<td>2.3</td>
<td>0.7</td>
</tr>
<tr>
<td>Semi-Linked</td>
<td>6. Mathematics is enjoyable to me</td>
<td>10</td>
<td>0</td>
<td>3</td>
<td>2.0</td>
<td>0.9</td>
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<td></td>
<td>8. Mathematics develops minds</td>
<td>10</td>
<td>2</td>
<td>3</td>
<td>2.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 5.37. Group Descriptive Statistics for Positively Worded Attitude Questions
There were also two Likert scale questions on students’ attitudes towards mathematics that were worded negatively about mathematics:

4. Mathematics makes me feel uneasy and confused.

7. Mathematics has been my worst subject.

Students indicated that they disagreed with the negative statements (see Table 5.38), thereby confirming their agreement with the positive statements. So, overall students displayed positive but not strong attitudes towards mathematics.

<table>
<thead>
<tr>
<th>N</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. Mathematics makes me feel uneasy and confused.</td>
<td>23</td>
<td>1.3</td>
</tr>
<tr>
<td>7. Mathematics has been my worst subject.</td>
<td>24</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Table 5.38. Descriptive Statistics for Negatively Worded Attitude Questions

Analysis by groups showed that linked group students provided more negative answers than the other groups to the negatively worded mathematics attitude questions. That is, they displayed the most positive attitudes toward mathematics in this group of questions (see Table 5.39), but again the Kruskal-Wallis test showed that the groups differences were not statistically significant.
All in all, students exhibited somewhat positive attitudes towards mathematics. There were no significant differences among the three groups—control, linked, and semi-linked—in their answers to either the positively or negatively worded statements about their attitudes towards mathematics. And students tended to agree more strongly with positive statements about mathematics in general than those involving them and mathematics.
**Attitude towards representations.** There were five questions in the survey focusing on the use of mathematical representations:

8. Mathematics problems can be solved in various ways by using different representations such as tables, graphs, and equations.

9. I like using more than one representation such as graphs, tables, and equations to solve mathematics problems.

10. Given a mathematical problem, I find it easier to focus on one representation than to deal with many representations.

11. When a mathematics problem is presented with more than one representation, it means that there are as many questions as representations.

12. Solving a mathematics problem with different representations such as graphs, tables, and equations results in totally different answers.

Most students agreed that mathematics problems can be solved in various ways by using different representations such as tables, graphs, and equations (see Table 5.40). Although they reported that they liked using more than one representation in solving mathematics problems, they also agreed that they found it easier to focus on one representation. Since the mean for question 12 was close to 1.5, i.e., between agree and disagree, the descriptive statistics for this question will be more meaningful when compared according to group. Finally, students disagreed that solving a mathematics problem with different representations such as graphs, tables, and equations results in totally different answers.
Each group separately agreed decisively on question 9 which states that mathematics problems can be solved in different ways using different representations (see Table 5.41). Although they all agreed on question 10 which asked whether they liked using more than one representation to solve problems in mathematics, the control group had a higher mean of 2.2 than those of experimental groups. On the other hand, each group also agreed that they found it easier to focus on one representation (question 11).

Table 5.40. Descriptive Statistics for Students' Attitudes Towards Representations

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>2.2</td>
<td>.4</td>
</tr>
</tbody>
</table>

9. Mathematics problems can be solved with different representations.
<table>
<thead>
<tr>
<th></th>
<th>Linked</th>
<th>Semi-Linked</th>
</tr>
</thead>
<tbody>
<tr>
<td>10. Like using different representations</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>11. Easier to focus on one representation</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>12. As many questions as representations</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>13. Different representations give different results</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>9. Mathematics problems can be solved with different representations.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Like using different representations</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>11. Easier to focus on one representation</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>12. As many questions as representations</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>13. Different representations give different results</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>9. Mathematics problems can be solved with different representations.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.41. Group Descriptive Statistics for Students' Attitudes Towards Representations
Just as in the total group, the means for question 12 were still between agree and disagree for each subgroup. In terms of frequencies, four out of five control students disagreed that there are as many as questions as representations, whereas in the linked and semi-linked groups almost half of the students agreed with the statement (see Table 5.42). It can be observed that the computer lab activities seemed to have had an effect on the answers of the students in the experimental groups because there were questions in the computer lab activities asking students to answer the same question for each representation (see Appendix B). This may have led the experimental group students to believe that there are as many questions as representations.

<table>
<thead>
<tr>
<th>Group</th>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Agree</th>
<th>Strongly Agree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Linked</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Semi-Linked</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>14</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.42. Students’ Responses to as Many Questions as Representations

Even though each group disagreed that solving a mathematics problem with different representations results in totally different answers, experimental group responses had means of 0.8 and 0.7 compared to the control group mean of 1.0. The
frequencies in Table 5.43 show that most of the students in each group disagreed with the sentence. However, there were more students in the experimental groups who strongly disagreed that different representations produce totally different answers.

<table>
<thead>
<tr>
<th>Group</th>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Agree</th>
<th>Strongly Agree</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Linked</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Semi-Linked</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>8</td>
<td>15</td>
<td>1</td>
<td>1</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 5.43. Students' Responses to Different Representations Cause Different Results

The Kruskal-Wallis test computed to see whether there were any significant differences among the three groups in mean answers to the all above mentioned attitude questions. However, there were no significant differences in the responses of control, linked, and semi-linked group students.

Overall, students in each group had similar attitudes towards the use of representations in mathematics and there were no significant differences by groups in their answers. Most students agreed that mathematics problems could be solved in various ways using different representations. Although they reported that they liked using more than one representation in solving mathematics problems, they also agreed that they found it easier to focus on one representation. They also agreed that using different representations does not lead to totally different answers.
Preferences for representations.

There were six questions exploring students’ preferences towards representations in general. Those questions referred to three choices of representation: equations, tables and graphs. Students were asked to choose one of the options that reflected their opinion the most. They were also asked to explain their answer.

One question asked what kind of representation they liked the most when solving mathematics problems. Fourteen out of 25 students chose tables, whereas ten of them indicated they liked equations the most (see Figure 5.13). Only one student reported that s/he liked graphs best when solving mathematics problems. Thirteen out of 25 students also reported that tables were the representation type easiest for them to understand. Graphs were easiest for five students and equations for six students. Equation was the type of representation that students usually chose to start with in solving mathematics problems. Tables were preferred by four students and graphs by three students.
Figure 5.13. Students’ Preferences for Initial, Easiest, and Most Liked Representation

Students’ answers to these questions are presented again in Figure 5.14 according to their groups. As shown in Figure 5.14, tables were liked the most by the linked group, whereas the control and semi-linked groups were torn between tables and equations. Tables were again easiest representation for most of the control and linked group students to understand. Semi-linked group students, on the other hand, reported that graphs and tables were easiest for them.
From these results, it can be concluded that tables and equations were a more preferred representation type than graphs. Students indicated that they liked the tables and equations more than graphs and they usually start solving mathematics problems with them.

The explanations that students provided for their preferences revealed more about their feelings. There were a couple of themes that emerged from the students’ explanations. One of the themes suggested that personal taste played an important role in preferences for a particular representation:

[Tables] work good for me.
It is easiest for me to get an answer [with equations].
I find [equations] less confusing and more easier.
[A table] is a way to organize your thoughts. (Open-Ended Survey Questions)
On the other hand, being uncomfortable with one representation was the reason some students preferred a different representation:

I do not like the pressures required [with equations] when solving by graphs. [S/he chose equations because] Tables tend to get tedious. [S/he chose equations because] Tables and graphs have bunches of numbers. So if I just jumped in the middle of class or something I would be lost but if I jumped in an equation then I could see the steps and see what is going on. (Open-Ended Survey Questions)

Previous experience/knowledge with a representation and knowing how to manage it was another common reason for students to choose one over another:

I like to solve with equations because it makes more sense to me.
I know [equations] more.
I am better with numbers than pictures except in my head. I can think of a graph better.
Graphs show pictures that I can see and understand.
It is easy for me to work out [equations]. (Open-Ended Survey Questions)

Finally, teachers have also affected students’ choice of representation. One student mentioned that he starts solving problems with equations because teachers have told him, “I have to learn to use them better.” Another student stated that s/he starts solving problems with equation, table, or a graph because “I was taught that you can use all three.”

Next to be examined were students’ responses to the questions asking which representation types they would have the most difficult time with if those were the only options, which one was the hardest to construct using paper and pencil, and which one was found to be confusing when working on a mathematics problem. The results were
compatible with students' answers to the previous questions. Eleven out of 25 students reported that they found graphs confusing when working on a mathematics problem, while for nine of them, equations were confusing (see Figure 5.15). Twelve students indicated graphs were the hardest to construct, whereas for eight of them equations were the hardest to construct. The representation students would have the most difficult time with if it were the only option was equations for ten students, graphs for nine students, and tables for six students.

Figure 5.15. Students' Identification of Confusing, Hardest, and Most Difficult Representation
Students’ answers to the questions on problematic representations are presented again in Figure 5.16 according to their groups. Graphs and equations were the representations mentioned most frequently by the students in each group as problematic. Most of the control group reported that they had the most difficult time with graphs while most of the experimental group had difficulty with equations. Graphs were the representation most frequently mentioned as the hardest to construct by most of the students in each group. Finally, most of the control and linked group students indicated graphs as the most confusing representation while semi-linked group students were divided between graphs and equations.

![Figure 5.16. Subgroup Identification of Confusing, Hardest to Construct, and Most Difficult Representation](image-url)

Figure 5.16. Subgroup Identification of Confusing, Hardest to Construct, and Most Difficult Representation
Themes similar to those identified for preferences emerged from the students’ explanations for the problematic representations. Not being able to handle a representation or not having enough previous knowledge about a representation were common justifications:

- Because I do not understand [equations].
- Equations are hard for me... I do not know what I am doing
- It is very hard to do [graphs]...It is harder to understand the problem [with graphs].
- I am very bad at using tables and organizing my data that way.
- I have a hard time doing [graphs]. (Open-Ended Survey Questions)

While talking about a representation that was not preferred, some students blamed the shortcomings of one representation as compared to others:

- It is harder to find the information [in equations] because it is not just given to you the way it is in a graph or table.
- I can not figure everything out by using equations. Most of the time I can. But tables are better sometimes. (Open-Ended Survey Questions)

Three linked group students did not choose any representation as confusing. They stated that:

- I can usually work with any of them.
- None of them. They are all easy to work with most of the time.
- I do not find them confusing. (Open-Ended Survey Questions)

One semi-linked group student stated, “None [he finds hardest to construct when solving a problem using paper and pencil], because I can do all of them.”
Attitudes towards VideoPoint

There were questions in the survey targeting students' attitudes towards the computer software, VideoPoint. First, students' thoughts about how they found VideoPoint in understanding mathematical representations was asked. They gave two viewpoints as confusing or helpful. They were expected to explain their answer.

![Bar chart showing students' attitudes towards VideoPoint by group.](image)

Figure 5.17. Students' Attitudes Towards VideoPoint by Group

As can be observed from Figure 5.17, all students in the semi-linked group found VideoPoint helpful in understanding mathematical representations, whereas only 5 students in the linked group agreed with them. One linked group student checked both confusing and helpful. She explained that "some of the equations I did not understand."
Four out of ten linked group students indicated that VideoPoint was confusing. Their reasons were:

- For some reason I was getting too many numbers mixed up.
- I could not understand it that well.
- The equations were hard.
- Too much information. (Open-Ended Survey Questions)

Some students mentioned how VideoPoint helped in learning mathematics:

- I learned how to use equations and a graph. (Linked Group Student)
- I learned more and had a visual to help me. (Semi-Linked Group Student)
- The movie, equation, and graph related to what we were doing the previous day. (Semi-Linked Group Student)
- It cleared up some questions I had. (Semi-Linked Group Student)
(Open-Ended Survey Questions)

Easy access to all representations at once was another common theme mentioned by students as a reason for finding VideoPoint helpful:

- [VideoPoint] showed the problem in a lot of different ways and it helped you to understand it. (Linked Group Student)
- [VideoPoint] showed them. (Linked Group Student)
- I could see how [representations] should have looked. (Semi-Linked Group Student)
- Because I could compare my answers with [the computer representations]. (Semi-Linked Group Student)
(Open-Ended Survey Questions)

The help that the software provided was appreciated by a couple of students:

- The computer did everything. All you had to do was put in the numbers.
- It made things easier. (Semi-Linked Group Students in Open-Ended Survey Questions)
Moreover, one linked group student was very happy to “get the right answer the first time” with the software.

Two questions in this part of the survey asked students’ preferences towards representations in the computer environment. Students reported that tables and graphs were the representations they liked the most while using VideoPoint. Six out of ten students in the linked group liked tables the most, and two of them liked graphs in VideoPoint (see Figure 5.18). On the other hand, six semi-linked group students liked graphs the most, while four of them liked tables the most.

![Figure 5.18. Students’ Preferences for Representation in a Computer Environment](image-url)
Moreover, five linked group students and six semi-linked group students indicated that VideoPoint made tables the easiest to use when solving problems (see Figure 5.18). Three linked and two semi-linked group students reported that they found graphs the easiest with VideoPoint. Finally, only one student from each experimental group thought equations were the easiest with VideoPoint.

Personal preference or knowledge required to manage a representation influenced students’ decisions about the easiest representation or the representation they liked the most in the computer environment:

[S/he liked tables the most because] the graph was hard for me to pick out numbers. (Linked Group Student)
[The graph] was easier for me. (Linked Group Student)
[The table] was the one that made the most sense to me. (Linked Group Student)
I liked the activities that correspond with the graphs. (Semi-linked Group Student)
(Open-Ended Survey Questions)

Another very common theme among students in deciding which representation was easiest or liked the best in VideoPoint was the ready-made feature:

[VideoPoint] did all the filling for you [in tables and graphs]. (Linked Group Student)
The info was all right there [in tables]. (Linked Group Student)
It gave you the answers [in tables] you needed to solve the problem. (Linked Group Student)
[Tables] basically told you the answers. (Semi-linked Group Student)
[Tables] gave me the answer easier and faster. (Semi-linked Group Student)
(Open-Ended Survey Questions)
Some students also mentioned how VideoPoint helped in constructing relationships among representations:

[VideoPoint] gave the line and all the points so I could just go to the table and get the answer. (Linked Group Student)
You could look at the graph and use the graph to find the equation. (Semi-linked Group Student)

(Open-Ended Survey Questions)

The last two questions in this part of the survey focused on students’ thoughts about how they liked being able to see or use all representations together with VideoPoint and whether VideoPoint was helpful in understanding the relationships among all representations.

Figure 5.19. Students’ Responses for Questions Related to Representations in VideoPoint
Semi-linked group students held more positive attitudes on these items than the linked group students (see Figure 5.19). All semi-linked group students agreed that using the computer was helpful in understanding the relationships among representations. However, only six linked group students agreed with this statement.

Most of the students in each experimental group (nine out of 10 in the semi-linked and eight out of ten in the linked group) indicated that they liked being able to see all representations all together (see Figure 5.19). Students reported that they liked being able to see all representations all at once since it gave them a choice to work with the one that they were more comfortable with or showed them there were various forms:

- [VideoPoint] gave me a choice of what was easiest to me. (Linked Group Student)
- It gives you the table, graph, and equation all together at the same time. (Linked Group Student)
- So it is all there to choose from. (Semi-linked Group Student)
- I had more than one way to figure out something. (Semi-linked Group Student)
- (Open-ended Survey Questions).

Several students also pointed out that VideoPoint was helpful in comparing different representations or checking their answers:

- I could compare the different representations. (Linked Group Student)
- I can check my answer over to see if it is right. (Linked Group Student)
- I could look at them and compare. (Linked Group Student)
- It was all right in front of you, you could just check to see if you were right. (Semi-linked Group Student)
- I could compare my answer. (Semi-linked Group Student)
- (Open-ended Survey Questions).
Three semi-linked group students mentioned that using the computer helped them see the relationships among representations:

You had to do equations to get the graphs and you used the tables to get values for the equation.

[VideoPoint] interrelated the three.

I can see the relationship between them. (Semi-linked Group Students in Open-ended Survey Questions)

One linked group student, on the other hand, did not find using the computer helpful in understanding the relationships among representations because he “could not see how they tied together” (Open-ended Survey Questions).

Although there were students from each group who mentioned that it was easier for them to work with representations in VideoPoint, there were also students who found it confusing:

It confuses me. (Linked Group Student)

It was still a little confusing. (Linked Group Student)

It confuses and distracts me. (Linked Group Student)

I don’t want to see everything at once. It helps me if I see it a little at a time.

(Semi-linked Group Student)

(Open-ended Survey Questions)

Finally when I asked students what they liked the most about using VideoPoint, ten out of twenty students mentioned they liked the videos. Movies in mathematics class were a novelty for them.

Three questions asked whether students found graphs, tables, and equations useful for understanding mathematical concepts. These questions were aimed at students’ reasoning for the specific types of representation (see Figure 5.20).
Most of the students in each group reported that they found all representations useful in understanding the mathematical concepts, except four out of five students in the control group and three in the semi-linked group disagreed that graphs were useful in this manner. Two control group students stated that graphs do not explain anything and one student from each group mentioned that graphs are not useful because they “cannot get them.” Again, students who did not find tables or equations useful provided the same reason for it: they are confusing or hard for them.

There were specific reasons for preferring a particular representation. Three main themes emerged in the data for equations. Being able to find an exact answer with equations was one of the themes:

Figure 5.20. Students’ Responses to Whether Representations Are Useful
[Equations] are the most exact. [Equations] are always exactly right. (Open-ended Survey Questions)

Being able to follow ready-made steps when working equations was another common theme in choosing equations:

There is not as much to think about [Equations], just solving it. [Equations] It is easy to pick out what numbers correspond with its formula. Ex: y=mx+b

Tables and graphs have bunches of numbers, so if I just jumped in the middle of class or something I would be lost, but if I jumped in on an equation then I could see the steps and see what was going on. (Open-ended Survey Questions)

However, the many steps involved in solving equations were the reasons for some students not to choose equations:

[Equations] Too many steps!!! [Equations] They involve a lot of steps. [Equations] There is memorizing needed to remember the procedure. (Open-ended Survey Questions)

Difficulties in constructing a equation was another explanation for not choosing an equation:

[Equations] I never know where to put the numbers. [Equations] I sometimes have trouble getting the information where it needs to go in an equation. (Open-ended Survey Questions)

Since tables put information in a more organized and categorized way, some of the students mention this issue when they rationalize tables as their choice of representation:

All the info is right there in an easy to read table. They are easy to understand and easy to locate the answers.

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The numbers needed to do the problem are easily seen. (Open-ended Survey Questions)

Students liked tables ready to work on, not requiring them to construct tables by themselves:

The numbers stand out more...I can see the answer without working them out
They usually already have the info.
It gives exact numbers and I don’t have to figure out any numbers. (Open-ended Survey Questions)

Visual advantages of graphs were mentioned by many students when they explained why they preferred graphs:

I can see what I am doing with a picture.
They show what it looks like.
Graphs show pictures that I can see and understand. (Open-ended Survey Questions)

However, difficulties in constructing a graph was a reason for not preferring one:

Like I said, using graphs to solve equations takes longer and requires exact accuracy... It is hard for me to read and reconstruct them.
There is a lot of lines and numbers
Sometimes it is hard to pick what numbers go where.
[Graphs] are hard to do and not precise enough. (Open-ended Survey Questions)

Another reason for not choosing graphs to work with was not being able to get the exact answer:

[Graphs] are difficult to get a direct answer from... They have to be perfect or else the answer will be wrong.
Sometimes it’s hard to see where lines intersect.
I could be off by a lot of numbers and I won’t get the right number or answer.
They are incredibly inaccurate. (Open-ended Survey Questions)
Another theme was the evidence that they could see the relationships among the representations. They rationalized some representations as helpful in getting other representations:

[Equations] are the basis for graphs and tables.
[Equations are helpful because] they help you get tables. (Open-ended Survey Questions)

Most of the students felt that they found VideoPoint helpful in learning mathematics. Students reported that easy access to all representations at once was helpful in choosing a representation to work with, comparing different representations, or checking their answers. Personal taste and previous experience/knowledge were the main themes emerging as reasons for preferring one representation over another. Specific reasons for liking or disliking a particular representation are summarized in Table 5.44.

<table>
<thead>
<tr>
<th></th>
<th>Likes</th>
<th>Dislikes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equations</td>
<td>Being able to find an exact answer</td>
<td>Difficulties in constructing an equation</td>
</tr>
<tr>
<td></td>
<td>Being able to follow ready-made steps</td>
<td>Many steps are involved in solving an equation</td>
</tr>
<tr>
<td>Tables</td>
<td>Organized information</td>
<td>Difficulties in constructing an graph</td>
</tr>
<tr>
<td></td>
<td>No requirement to construct tables usually</td>
<td>Not being able to find an exact answer</td>
</tr>
<tr>
<td>Graphs</td>
<td>Visual advantages of graphs</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.44 Specific Reasons for Liking or Disliking a Representation
Conclusions

On the basis of the analyses of both the qualitative and quantitative data, nine conclusions were drawn:

• *Experimental groups, semi-linked and linked, improved significantly from pre-test to post test in overcoming the height/slope misconception.*

Although results related to Height/Slope misconception indicated no significant differences were present among three groups in both the pre- and posttest, it was noted that the semi-linked group performed better than the other two groups with a mean of 0.8, while the linked group had a mean of 0.8 and control group had mean of 0.7 in the posttest (see Table 5.21). Moreover, the experimental groups improved significantly from pretest to posttest based on Wilcoxon test, $Z=-1.873$, $p=0.61$ for the linked group and $Z=-2.263$, $p=0.24$ for the semi-linked group.

• *The experimental groups, semi-linked and linked, performed better than the control group in the category of graph as picture misconception, and the improvement of the linked group was significant.*

The semi-linked and linked groups performed better than the control group with a mean of 0.8, while control group had mean of 0.6 (see Table 5.28). Only the linked group improved significantly from pretest to posttest (based on Wilcoxon test, $Z=-2.449$, $p=.014$), while control and semi-linked groups did not improve significantly.
• It was observed that although all students were able to provide accurate answers for the questions in the posttest, experimental group students were able to explain their answers more resourcefully than the control group students who mainly used procedural steps to explain their solutions.

• The software was helpful in providing an environment that offered resources and constraints for students to experience assimilation and accommodation through a series of equilibrium-disequilibrium states.

• Linkage was helpful for some students in constructing mathematical ideas. Students who have already constructed those ideas and trust them may not need the linkage.

• Semi-linked software forced students to be more active in their learning process, to make use of their existing knowledge and the information provided by the software and to construct new mathematical ideas or connections among the representations with the help of information provided by the software. However, some students did not have enough foundation on which to construct new mathematical ideas, and the linkage could be very helpful to them.
• Since more information was provided to students in the linked version than in the semi-linked version, linked students trusted the technology and did not question it. On the other hand, since semi-linked students needed to make use themselves of the information provided by the software, rather than reading ready-made answers, there were times they questioned the technology.

• Most of the students mentioned that they found VideoPoint helpful in learning mathematics. Students reported that easy access to all representations at once was helpful in comparing different representations or checking their answers.

• Personal preferences and previous experience/knowledge were the main themes emerging from the data as reasons for preferring a representation, and there were specific reasons for choosing a particular representation, such as being able to find an exact answer with an equation, the visual advantages of graphs, or the organized information provided by tables.
CHAPTER 6

DISCUSSION

With the new millennium, the utilization of technology and the use of multiple representations in mathematics instruction have been recommended by the NCTM (2000) in *Principles and Standards for School Mathematics*. There are special technologies with capabilities of providing multiple representations effortlessly. One particular example of this type of technology is educational software with linked multiple representations. Linked multiple representations are a group of representations in which, upon altering a given representation, every other representation is automatically updated to reflect the same change (Rich, 1995/1996). In the present study semi-linked representations were those for which the corresponding update of changes within the representations were available only upon request but were not automatic. The focus of this study was comparing three groups of students: one group using linked representation software, the second group using similar software but with semi-linked representations, and a control group. Briefly, the main research question was:

- What are the effects on students' understanding of linear relationships using linked representation software compared to those using semi-linked representations?
Participants of this study were ninth-grade Algebra I students. The two experimental groups used VideoPoint, a software package that allows one to collect position and time data from QuickTime movies of, for example, two cars driving in the same direction with different constant speeds or two fish swimming towards each other. These data can be combined to perform calculations, such as distances between points, and can be presented using different representations such as tables, graphs, and equations. This software was originally developed as linked multiple representational software. With the help of the authors of the software, a fully linked and a semi-linked version of VideoPoint was also developed.

Data collection methods included mathematics pre- and posttests, follow-up interviews with all students after the mathematics posttest, clinical interviews with 5 students from each experimental group at the end of the treatment, and classroom and lab observations. A survey was administered at the end of the study in order to see students' attitudes about mathematics, representations in general, and the computer environment. Along with the interviews, the survey was also the main data source for examining the second research question of this study:

- What are students' attitudes towards and preferences for mathematical representations—equations, tables, or graphs?

In the next section, the summary of the quantitative findings will be presented, followed by a summary of qualitative results focusing on the learning in linked and semi-linked environments. Results regarding the survey, implications for teaching, and recommendations for future research are also discussed.
Summary of Quantitative Findings

Instead of studying each question separately, questions in the mathematics pre- and posttests were clustered into categories, and those categories were compared across the three groups—linked, semi-linked, and control. The categories were: Word Problems, Interpreting/Constructing and Reading Graphs, Solving and Constructing Equations), Reading and Constructing Tables, and Misconceptions (Height/Slope, Point/Interval, Graph as Picture). These categories were compared using a nonparametric test—Kruskal-Wallis (a test for several independent samples)—to identify differences between the linked, semi-linked, and control groups. The results of this test showed that there were no significant differences in achievement between the groups in any category of problems on either the pretest or posttest.

Rosenheck (1991/1992) and Rich (1995/1996) compared groups of students when one group was using linked multiple representational software. Both of them reported that there were no significant differences among groups of students. From this perspective, this study agrees with their conclusion. But there were the technological differences in the treatment environments in Rich’s (1995/1996) study namely, as calculator groups versus a computer group. However, in the present study two groups of students using the same computer software with differences in linking property were compared. Computer laboratories were embedded into normal mathematics classes. This gave students a chance to learn about the software itself and focus on the mathematical content with the help of the software.
Perhaps that is why the groups' achievement changed from pretest to posttest in a very interesting way. A nonparametric test—the Wilcoxon Test (a test for dependent samples)—was used to identify the improvement or decline from pretest to posttest within groups in each category (see Table 6.1). Some of the improvements were significant at the .05 level, such as experimental groups in the categories of interpreting graphs and constructing equations, the semi-linked group for the height/slope misconception category, and the linked group for the graph as picture category. Other improvements were significant at the .1 level, such as the linked group for the height/slope category.

<table>
<thead>
<tr>
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<th>Improvement Significance Scores</th>
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</thead>
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<td></td>
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</tr>
<tr>
<td>Interpreting/Constructing</td>
<td>0.157</td>
</tr>
<tr>
<td>Graphs</td>
<td>0.066</td>
</tr>
<tr>
<td>Reading Graphs</td>
<td>0.785</td>
</tr>
<tr>
<td>Equation</td>
<td>0.317</td>
</tr>
<tr>
<td>Solving/Substituting Equation</td>
<td>0.102</td>
</tr>
<tr>
<td>Constructing Equation</td>
<td>0.02</td>
</tr>
<tr>
<td>Reading Tables</td>
<td>1</td>
</tr>
<tr>
<td>Constructing Tables</td>
<td>0.684</td>
</tr>
<tr>
<td>Table</td>
<td>0.317</td>
</tr>
<tr>
<td>Height/Slope Misconception</td>
<td>.1 significant</td>
</tr>
<tr>
<td>Point/Interval Misconception</td>
<td>0.102</td>
</tr>
<tr>
<td>Graph as Picture Misconception</td>
<td>0.317</td>
</tr>
</tbody>
</table>

Table 6.1. Improvement Significance Scores
It was observed that, although all students were able to provide accurate answers for the questions, the experimental group students were able to explain their answers more conceptually than the control group students who used procedural steps to explain their solutions. For instance, students in both experimental groups approached the question asking for an algebraic form of a phenomenon represented by a table with more promising ideas than the control group students, who took two points from the table and calculated the formulas as they were taught in class.

I just found it is, like count the slope. I guess \( m \) would be the number of movies he got from company A. They increase by four each time and they started at 5 so it would be \( 4m + 5 \). (Linked group student, T5B-255)

I figured out that every movie you make you are going up from $5 by $4 and same thing with company B, but you are going up $2 every movie from 10... that is how I figured that out. (Semi-linked group, T2B-223)

At this point, it is very important to recall that experimental group students had more interaction with the researcher than the control group students. Control group students were in the classroom all the time while experimental group students came to the computer lab and worked with the researcher. The language used in the computer lab hours through those interactions might have contributed to the experimental group students’ more visually and conceptually rich explanations.

**Summary of Qualitative Findings**

In order to study the mathematical learning within the computerized environment, clinical interviews were conducted. In the clinical interviews, students were presented a mathematics question, had access to a computer, and followed a procedure similar to those in lab hours. Taking into account the order of the lab activities and the
clinical interview procedure, it can be observed that the procedure in the lab activities and the clinical interviews had a progression similar to Bruner’s mode of representation mentioned in the second chapter (see Table 6.2). First, students watched a movie and were asked to think about the mathematical concepts underlying that movie. This gave them a chance to act on the mathematical concept from the outset, at an enactive level of representation in Bruner’s theory.

<table>
<thead>
<tr>
<th>Sections of the Lab Procedure</th>
<th>Questions</th>
<th>Bruner’s Levels of Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Watching the movie</td>
<td>Predicting the graph of a phenomena that they watched from the movie</td>
<td>Enactive level of Representation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Action</td>
</tr>
<tr>
<td>Section 1</td>
<td>Looking at the graph produced by the computer and answering some questions related to the graph</td>
<td>Iconic Level of Representation</td>
</tr>
<tr>
<td></td>
<td>Considering a table of representation of the data. Discussing the numerical values of the phenomena</td>
<td>Visualization</td>
</tr>
<tr>
<td>Section 2</td>
<td>Looking at the table produced by the computer and answering some questions related to the table</td>
<td>Symbolic Level of Representation</td>
</tr>
<tr>
<td></td>
<td>Trying to come up with equations</td>
<td>Symbolic Manipulation</td>
</tr>
<tr>
<td>Section 3</td>
<td>Answering some questions related to the equation of the phenomena</td>
<td></td>
</tr>
<tr>
<td>Section 4</td>
<td>Miscellaneous questions regarding representations</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2. Comparison of Bruner’s Theory with Lab Procedure.
Then students moved to the next level of representation, the iconic level of representation, where they came up with a graphical representation and used the information provided by the graph to answer questions associated with it. Moreover, they also came up with a tabular representation and worked on the related questions. Lastly, students arrived at the final stage of representations, the symbolic level, which required symbolic manipulation to find and work on the algebraic form. This theory-based progress, moving from the enactive level of representation through to the symbolic level of representation, helped students move from one representation to another gradually.

While discussing the place of constructivist theory in Chapter 2, the significant emphasis on previous knowledge and experience was discussed. According to constructivist theory, previous experiences and knowledge have an essential role in shaping learning. They form the basis, both as a foundation and as a constraint, for future knowledge and understanding. So, as predicted before, presenting information with more than one representation or providing the multi-representational environments was helpful to students in grasping information that is meaningful to them or for which they have a basis on which to build.

[VideoPoint] gave me a choice of what was easiest to me. (Linked Group Student)
So it is all there to choose from. (Semi-linked Group Student)(Open-ended Survey Questions).

In both environments, linked and semi-linked, existing knowledge was an essential part of the learning process. Although students in the linked environment could use more hints from the software than the semi-linked students could to answer
questions, existing knowledge was crucial to the process in both environments. Moreover, that existing knowledge was enhanced as a result of operating the software. More theoretical implications according to Piaget’s and Kaput’s theories will be identified in the following sections when learning in the linked and semi-linked computer environments is discussed.

**Learning in the Linked Computer Environment**

Figure 6.1 summarizes the learning process in the linked environment. When a question is asked, students either use the linkage directly to answer the question or they assimilate this new information and go to their previous knowledge to answer the question. This assimilation is described as *recognitive assimilation* in Piaget’s theory which is defined as “considering reality and selecting an appropriate scheme” (Montangero & Maurice-Naville, 1997). When students were asked a question, if they did not use the readily available linkage to answer the question, they used their existing appropriate scheme to answer it. If they used the linkage, their explanation for their answer was based more on the software. Students in the linked environment who did not choose to use the linkage, as well as students in the semi-linked environment, provided explanations for their answers based more on the mathematical aspects of the question, since they either chose or needed to use their existing scheme instead of the cues provided by the software.

When students provided an inappropriate answer to a question and they saw that they were wrong through the computer feedback, disequilibrium occurred and they needed to go back and interpret this new information through their existing knowledge; that is, they assimilated the new information. If they could not interpret this new
information, they needed to accommodate their preexisting knowledge in order to reach equilibrium, that is, in Piaget's words to modify "internal schemes to fit reality" (Piaget, 1969, p. 6).

Students who trusted their own knowledge and answers did not use the linkage at all. According to Kaput (1995), when one moves from mental operations to physical operations, "one has cognitive content that one seeks to externalize for purposes of communication or testing for viability" (p. 140). Students who trusted their mental
operations, without any need for testing for viability with the software, had the choice of not using the linkage. Other students ignored the linkage or did not use the linkage when they could have benefited from it. Now, the other direction in Kaput's theory moves from physical observations to mental operations, which were described as "processes...based on an intent to use some existing physical material to assist one's thinking" (Kaput, 1995, p. 140). Here, linkage was the "existing physical material" for students to help them further construct their incomplete scheme.

The researcher also observed that students in the linked group trusted the answers provided by the software and did not question them. When their answers did not agree with the computer's, they tried to interpret the computer's answer instead of questioning it and defending their answer.

**Learning in the Semi-Linked Computer Environment**

Results suggested that when a question was proposed in a semi-linked environment, students mainly relied on their own existing knowledge with the help of the software (see Figure 6.2). They assimilated new information in this environment and went to their existing knowledge to answer the question. Although this environment did not provide such rich feedback as in the linked environment, ready-made graphs or tables presented powerful visual information/feedback for students to use while answering the questions. Lack of linkage forced more mathematically-based explanations instead of movie-based explanations. However, some students needed linkage in some situations in order to construct more empowering concepts. As mentioned above, the linkage provided enough information to assist some students in their thinking, as in Kaput's theory, or accommodate their scheme when needed, as in
Piaget's theory. On the other hand, not having the linkage empowered some students to trust their own answers and construct the linkages between representations for themselves. Then the software was just a helper, record keeper, or representation provider for the students. So, not having linkages put students in a more active role mentally as learners.

Figure 6.2. The Nature of Learning in the Semi-Linked Computer Environment

Students need to convince themselves and construct linkages by themselves

More Prediction and Active Role

Forces Mathematical Explanations

Lack of Linkage

LOST

Comp Env. Feedback

Question

Explanation

Assimilation

Empowering

Accommodation

Disequilibrium

[Previous] Knowledge

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Finally, the researcher tried to follow up the teacher’s regular class sessions in the computer labs with the aim of giving opportunities to students to apply knowledge learned in class. Moreover, it was hoped that students would also carry their learning from the computer labs to the regular class sessions, which mainly consisted of paper and pencil tasks. There were a couple of incidents (see Chapter 5) that showed students were carrying ideas back and forth from the class to the computer lab and vice versa.

These qualitative results confirm and enhance previous research studies. For example, the present study agrees with results of Rizutti (1991/1992) which suggest that (a) multi-representational software frees students to concentrate on the mathematics itself by providing multiple representations, and (b) allows students to solve a problem by choosing the representation type that is more meaningful to them. Moreover, providing ready-made representations allows students to have easy access to visual representations and facilitates their visual thinking (Borba and Confrey, 1993; Lin, 1993).

Yerushalmy (1991a) indicates that although the software and the method of teaching enhanced students’ understanding of the concept examined, connections between the algebraic manipulations and the visual representation did not occur spontaneously. The semi-linked version of the software was fairly helpful to students in constructing that connection. Semi-linked software made students come up with the algebraic form of the phenomena rather than having it provided by the clicking of a button as in the linked-version. While students were figuring out the algebraic form, they were trying to change the coefficients to fit the graph of the data, and every time
they changed a coefficient in the equation, they saw the resulting graph and the graph from their previous try. This provided students an environment to highlight the connection between algebraic and graphical representations:

[We started to estimate the coefficients in the equations. The student tries 80 for the slope]

Int: Why did you try 80?
St: Because at 0.80 seconds they are around 80 pixels so I just...

[But it did not work so she needed to change it]

Int: How do you want to change it now? Bigger or smaller number?
St: Bigger, because the slope is greater than the other one.

[She continues and comes up with slope but she needs to work on the y-intercept now.]

Int: How do you want to change then?
St: B would become negative instead of positive.
Int: Why?
St: Because it has like a head start—at least that’s what it appears to be on the graph. [She refers to the data on the graph and tries a negative y-intercept but the magnitude was not enough. She tries a couple more y-intercepts and every time she increases the magnitude. So, she reinforces the idea that bigger negative numbers for the y-intercept would bring the line down.] (Semi-linked student interview, T8B-223).

Summary of Results from the Survey

The results from the survey revealed students’ attitudes towards mathematics and their preferences for particular representations in paper and pencil and computer environments. All students exhibited somewhat positive attitudes towards mathematics. There were no significant differences among the three groups—control, linked, and semi-linked—in their answers to either the positively or negatively worded statements about their attitudes towards mathematics.
Students in each group all had similar attitudes towards the use of representations in mathematics, and again, there were no significant differences among groups in their answers. Most students agreed that mathematics problems can be solved in various ways by using different representations. Although they reported that they liked using more than one representation in solving mathematics problems, they also agreed that they found it easier to focus on one representation. They also agreed that using different representations does not lead to totally different answers.

Students reported that they preferred tables and equations to graphs. They indicated that they usually start solving mathematics problems with tables or equations. The explanations that students provided for their preferences revealed more about their preferences. A couple of themes emerged from students’ explanations. On the one hand, being uncomfortable with one representation was the reason for some students to prefer another. Previous experience/knowledge with a representation and knowing how to manage it was another common reason for students to choose a particular representation.

The practical and theoretical articles on students’ preferences and strategies for choosing representations were synthesized in Chapter 2. Two main effects emerged: either internal or external effects influenced students’ choices when working with multiple representations. Since this study used a survey to reveal students’ attitudes towards and preferences for representations, the results of the present study and the internal factors discussed in Chapter 2 will be compared based on the self-reports provided by students in the survey. There were five components of the internal effects:
• Personal Preferences,
• Previous Experience,
• Previous Knowledge,
• Beliefs about Mathematics, and
• Rote Learning.

The results of this study confirmed most parts of this theoretical synthesis. Personal preferences, previous experience, and knowledge were the main themes that emerged from these data. Moreover, this study indicated that students’ beliefs developed over the course of their mathematical history affected their choice of representation. However, it was hard to see the effects of rote learning, if any, in preferring a representation through students’ answers to the survey questions.

Most of the students indicated that they found VideoPoint helpful in learning mathematics. Easy access to all representations at once was another common theme mentioned by students as a reason for finding VideoPoint helpful. Students reported that tables and graphs were the types of representation they liked the most while using VideoPoint. This result agrees with the results of several researchers. For example, Rizutti (1991/1992) reported that students used tables more than graphs, and they used graphs more than calculators while using the software. Graphs came to be the preferred representations due to the easy access to them with VideoPoint. Students mentioned that tables and equations were their more preferred representation type in generally worded questions. However, in a technological environment tables and graphs become more preferred than the equations. Rich (1995/1996) also found that graphical representation was used more in experimental groups who had access to the graphing
calculators and computer software than a control group using scientific calculators. Moreover, Porzio (1994/1995) found that students in a graphing calculator class showed an increasing preference for graphical representation.

Personal preference or previous knowledge required to manage a representation influenced students’ choice for the easiest representation or the representation they liked the most in the computer environment. Another common theme among students was liking a representation in VideoPoint because VideoPoint made representations easy. As Keller and Hirsch (1998) indicated, in technology-rich situations students’ preferences for various representations did not depend as much on the task but on the use of technology that made it possible for students to have equal access to the representations by removing constraints. Some students also mentioned how VideoPoint helped in constructing relationships among representations. They reported that they liked being able to see different kinds of representations all at once since it gave them a choice to work with one that they were more comfortable with or showed them there were various forms available. Several students also pointed out that VideoPoint was helpful in comparing different representations or checking their answers. Finally, when asked what they liked the most about using VideoPoint, ten out of twenty students mentioned they liked the videos. Movies were a novel representation for them to use in mathematics class.

There were specific reasons for preferring or not preferring some representations. Three main themes emerged from the data for equations. They were:

- Being able to find the exact answer with equations,
- Being able to follow ready-made steps when working equations, and
• Difficulties in constructing a equation.

Students mentioned that tables put information in a more organized and categorized way. LaLomia et al. (1988) indicated that students demonstrated a strong preference for tables when they needed to locate a specific number. This was one of the advantages of tables that presented information in an ordered way.

Many students mentioned visual advantages of graphs when they explained why they preferred graphs. Students who chose the graphical methods in Piez and Voxman’s (1997) study also mentioned the importance of the visual advantages of graphs. However, difficulties they had while constructing a graph were a reason for not preferring graphs. Another reason for not choosing graphs to work with was not being able to get the exact answer via graphs.

Another theme was the evidence that they could see the relationships among the representations. They rationalized some representations as helpful in getting other representations. For example, students believed that equations were helpful because “they help you get tables” (Open-ended survey questions).

Conclusions

On the basis of the analyses of both the qualitative and quantitative data, nine conclusions were drawn (see Chapter 5). More compact list is provided one more time in Table 6.3.
The experimental groups, semi-linked and linked, improved significantly from pre-test to posttest in overcoming the height/slope misconception.

The experimental groups, semi-linked and linked, performed better than the control group in the category of graph as picture misconception, and the improvement of the linked group was significant.

Although all students were able to provide accurate answers for the questions in the posttest, experimental group students were able to explain their answers more resourcefully than the control group students.

The software was helpful in providing an environment that offered resources and constraints for students to experience assimilation and accommodation through a series of equilibrium-disequilibrium states.

Linkage was helpful for some students in constructing mathematical ideas. Students who have already constructed those ideas and trust them may not need the linkage.

Semi-linked software forced students to be more active in their learning process, to make use of their existing knowledge and the information provided by the software and to construct new mathematical ideas or connections among the representations with the help of information provided by the software.

Since more information was provided to students in the linked version than in the semi-linked version, linked students trusted the technology and did not question it as opposed to semi-linked students who question the software sometimes.

Most of the students mentioned that they found VideoPoint helpful in learning mathematics and that easy access to all representations at once was helpful in comparing different representations or checking their answers.

Personal preferences and previous experience/knowledge were the main themes emerging from the data as reasons for preferring a representation, and there were specific reasons for choosing a particular representation, such as being able to find an exact answer with an equation, the visual advantages of graphs, or the organized information provided by tables.

Table 6.3. Conclusions of the Present Study

Implications for Teaching

The use of technology in mathematics education as a tool for learning has been growing for decades. There are many researchers who state the benefits of technology in teaching and learning mathematics. The main question is whether technology is an integral part of everyday mathematics classrooms. The National Center for Education Statistics (1999) indicates that 75.5% of students in grades 7-12 used computers at schools in 1997. Moreover, in 1999, the Third International Mathematics and Science
Study (TIMSS-Repeat or TIMSS-R) was readministered at the eighth grade level (http://timss.bc.edu/timss1999i/math_achievement_report.html). TIMSS-R indicated that there were shortages or inadequacies in equipment and materials for mathematics instruction that affected schools' capacity to provide instruction in mathematics some or a lot. Forty-seven percent of the students suffered a shortage of computer hardware for mathematics instruction and forty-eight percent of the students had a shortage of computer software. On the other hand, the same study reported that ninety-seven percent of the students had at least some access to computers, with fewer than 15 students per computer. So, computers are in schools, and mathematics teachers and students are utilizing them. In this study, students were pleased to use computers, too. However, they knew that the use was part of a study and not an integral part of their mathematics classroom. Ideally, it would have been more effective if computers were introduced as part of a well-planned, tightly integrated part of their mathematics course.

In this age of technology, computers are in the students' life before their school life starts. So, there is no need for teachers to worry about the time they will spend to teach students how to use computers. The time needed for students to use the software may be lessened with well-explained activity sheets. In this study, students started to use the software by themselves the first week without any problem.

Many students asked where they were going to need the mathematics they were learning in the classroom. Since the computer software used in this study presented real-life phenomena with a movie and depicted the mathematical aspects of it through multiple representations, it helped students to build connections between real-life and mathematics, as supported by much research and advocated by NCTM (2000).
Moreover, in addition to the movies of the real phenomena, VideoPoint presented multiple mathematical representations at once. This had a couple of advantages. First of all, students were able to manipulate different types of representations easily which they would not have preferred in working with paper and pencil. Participants in this study reported that they preferred tables and equations in general, but they liked to use graphs and tables when working with the VideoPoint. Furthermore, having access to multiple representations helped students with different learning styles to choose the types of representation they were most comfortable with. Another advantage of technological multiple representations was that more focus was spent on the relationships among representations and the mathematical content instead of calculating or drawing. Drudgery-free access to multiple representations allowed students to concentrate on the mathematical ideas. Finally, the software was able to offer an environment with resources and constraints for students to construct new schemes or change their existing ones through a series of equilibrium-disequilibrium states.

VideoPoint offered a link among those representations. Connections among representations were even highlighted by the linked version of VideoPoint. Linkage offers an interesting potential for the teaching and learning of mathematics. However, the results of this study illustrate that the mathematics education community can benefit from both versions, linked and semi-linked. Having two versions of the software would be more helpful than having either one of them alone. Being able to switch between the linked and semi-linked versions would be invaluable because the linked and semi-linked versions have their own different advantages. Another related but very crucial outcome is the importance of the teacher's role in the classroom. No matter how powerful is the
technology, it can never replace the teacher. Technology is a very effective tool in the process of teaching and learning of mathematics. However, there are many important decisions to be made by the teacher, such as when to use linked or semi-linked versions and with whom as the results of the present study suggest.

The best example of a significant benefit of the linkage was helping students to overcome the graph as picture misconception. As mentioned above, in this study the experimental groups, semi-linked and linked, performed better than the control group in the category of graph as picture misconception. With linkage, students were able to see corresponding highlighted points in the movie and on the graphical representation. As expected, the improvement of the linked group in separating the graph from the real-life picture was significant. If a student has weak cognitive schemes about the connections among representations, the linked version could be very helpful.

However, students who have already constructed the relationships among representations mentally might be better off with the semi-linked version. Some advantages of the semi-linked software were enabling students to be more active in their learning process, to make use of their existing knowledge and the information provided by the software, and to construct new mathematical ideas or connections among the representations with the help of new information.

Mathematics teachers might prefer linked or semi-linked versions of software for different age groups or grade levels. The most beneficial usage could come from using a linked version to introduce a mathematical idea and help students construct their schemes. Once accomplished, the linkage could be removed and the semi-linked version could be turned on in order to make students use their newly constructed schemes.
Another implication for teaching was the importance of the utilization of multiple representations in mathematics instruction. Students have different preferences for different representations. Students in this study indicated that their personal preferences and previous experience/knowledge were the main influences in choosing a representation to work with. Therefore, presenting multiple representations for them to work with could increase the efficiency of the mathematical learning in the classroom.

**Recommendations for Future Research**

Based on conclusions from this study, it is evident that more studies are needed to investigate various aspects of the electronic linkage provided by some software. Some aspects of the linked and semi-linked versions of the software have been explored by this study, while many other possible areas of research have been identified for further investigation. Moreover, there were some limitations of this study, which could be overcome with new studies. One limitation was time. A longer study in which students have more experience with the software could be more informative. Due to time limitations, students encountered the software on only one topic—linear relationships. The study of more advanced topics such as quadratic relationships or exponential relationships with VideoPoint could reveal how the linked or semi-linked version might help students to understand more sophisticated ideas.

There were limitations to the software. The versions of semi-linked and linked software were created by the developer of VideoPoint. The semi-linked version did not provide linkages among the movie, the graph, and the table windows at all but only a
linkage between the graph and the algebraic form. It would be very interesting to develop another version of the software that provides a linkage among all the representations not automatically but upon request.

Studying with only one class was another limitation. More participants from different age groups or grade levels could bring different issues to bear. Although this study benefited from both quantitative and qualitative research methods, purely qualitative case studies could afford more insight into the advantages or disadvantages of both linked and semi-linked software.

The following areas are additional recommendations to the reader for further research:

- Design a study of VideoPoint software in a setting where the linked version was used prior to the semi-linked, or vice versa. What are the influences of the order of the utilization of the software on students' understandings?
- Design a study that explores the use of the semi-linked and linked versions of VideoPoint with a single group of students who have the authority to turn the linkage on and off. How does utilization of such software impact students' understandings about representations? Which version is more used by the students and why?
- Design a study that investigates students' ability to construct a representation when a representation is removed completely from the screen, after they have had a chance to work with all representations.
Design a study which examines students’ reactions to software that provides incorrect linkages among the representations, after students have had experience with a consistent version long enough to construct efficient schemes about the relationships among the representations. See if students understand the correspondence between representations well enough to believe the software could be incorrect.

The aim of this study was to understand the effects of automatically linked and semi-linked educational software on students’ understanding of several mathematical topics. It aimed to help educators to see the effects of linked and semi-linked software and appreciate the advantages of each kind of software for student learning. Technology can be very effective if used appropriately. It is not a solution to all problems in the mathematics classrooms but it is a tool for instructional purposes. We, as educators can help our students benefit from technology without becoming uncritically dependent on it.
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VideoPoint 2.1.2 (Modified by the developer for this study) [Computer Software]. (2000). Lenox, MA: Lenox Softworks.


Dear Title Name:

This letter is to request permission to do research in your school. My dissertation study, under the direction of Dr. Sigrid Wagner, Professor of Mathematics Education at The Ohio State University, involves an analysis of the mathematical understanding of high school students with the availability of educational software. In particular, I am interested in finding out the effects of computer-based multiple representations on students' learning of mathematical concepts.

I would like to observe a freshmen Algebra class and have a computer lab hours with the students. Two groups of students will be formed in order to study with two different versions of the same software emphasizing different linking properties. I would request permission of parents and teachers beforehand. I would like to interview a total of 10 students from the lab sessions about their understanding of a mathematical concept with the availability of the computer software. The interviews will be audio recorded by the computer. Computer will also be saving the computer screen which will include students' actions. There will be no video recording in this study. The work of the students will remain strictly confidential. No names or other identifying information will be used in any research reports. In addition to student interviews, I would like to talk to class teacher and use classroom materials including tests and homework to get a better idea of the type of mathematical tasks typically done by the students in your particular school.

Results from this study may help improve our understanding of how children learn mathematics. Of course, a copy of the final report will be available to you and any other interested parties upon request. In return for your participation in this study, I would like to volunteer time to discuss results of this study or any other issues of learning and instruction in mathematics with any of your teachers.

Thank you for considering this request. I will call to schedule an appointment with you to discuss my research proposal.

Sincerely,

S. Asli Ozgun-Koca  
Doctoral Candidate  
The Ohio State University

Sigrid Wagner  
Professor, Mathematics Education  
The Ohio State University
Dear Teacher,

I have received permission from your principal, Mr., to do research in your school. My dissertation study, under the direction of Dr. Sigrid Wagner, Professor of Mathematics Education at The Ohio State University, involves an analysis of the mathematical understanding of high school students with the availability of educational software. In particular, I am interested in finding out the effects of computer-based multiple representations on students’ learning of mathematical concepts.

I would like to observe your freshmen Algebra class and have a computer lab hours with the students. Two groups of students will be formed in order to study with two different versions of the same software emphasizing different linking properties. I would request permission of parents beforehand. I would like to interview a total of 10 students from the lab sessions about their understanding of a mathematical concept with the availability of the computer software. The interviews will be audio recorded by the computer. Computer will also be saving the computer screen which will include students’ actions. There will be no video recording in this study. The work of the students will remain strictly confidential. No names or other identifying information will be used in any research reports. In addition to student interviews, I would like to have an interview with you and use classroom materials including tests and homework to get a better idea of the type of mathematical tasks typically done by the students in your classroom. In addition, I would appreciate your help in distributing and collecting permission forms from parents.

Thank you for your help. If you have any questions, please feel free to contact me at the phone number below. My advisor, Sigrid Wagner, can also be reached at 292-8058. Of course, a copy of the final report will be available to you and any other interested parties upon request. I would also be happy to discuss this research or any other issues of mathematics teaching and learning with you.

Sincerely,

S. Asli Ozgun-Koca
Doctoral Candidate
The Ohio State University

Sigrid Wagner
Professor, Mathematics Education
The Ohio State University
Dear Parent/Guardian:

Your child’s school has been selected to participate in a study under the direction of Dr. Sigrid Wagner, Professor of Mathematics Education at The Ohio State University. The study involves an analysis of the students’ understanding of mathematical concepts through the use of educational software. In particular, I am interested in finding out the effects of computer-based multiple representations on students’ learning; what will students’ understanding and ease of use of graphing, numerical and algebraic techniques for representing concepts in the algebra curriculum?

I will observe your child’s mathematics class. As part of the class students will attend computer lab sessions designed to enhance their mathematical understanding. The class will be divided into two groups in the computer lab. Students will be using two different versions of the same computer software; the main difference between them is their ability to automatically represent concepts in different forms/representations.

Beyond the observations in the classroom, I would like to interview a total of 10 students about their understanding of a mathematical concept while they are working with the computer software. Both their interactions with the computer and the audio portion of the interview will be simultaneously recorded by the computer as the students work. There will be no video recording in this study. The work of your child will remain strictly confidential—no names or other identifying information will be associated with your child’s work in any research reports or discussions of the data. Participation is absolutely voluntary and your child may withdraw at any time without consequences. I will be offering free tutoring to students participating in the interviews, particularly to prepare for exams, assignments, and further clarification of concepts.

If you agree to allow this interview, simply complete the form on the next page and return one copy to your child’s classroom mathematics teacher.

Thank you very much for considering this request. The broader the participation in this study, the better picture we will have of students’ mathematical understanding. Of course, a copy of the final report will be available to you and any other interested parties upon request. If you have any questions, please call me or my advisor at the numbers provided below.

Sincerely,

S. Asli Ozgun-Koca
Doctoral Candidate
The Ohio State University

Sigrid Wagner
Professor, Mathematics Education
The Ohio State University
CONSENT FOR PARTICIPATION IN SOCIAL AND BEHAVIORAL RESEARCH

I consent to my child's participation in the research entitled:

The Effects of Computer-Based Multiple Representations on Students' Learning of Mathematical Concepts

I acknowledge that Dr. Sigrid Wagner or her authorized representative, S. Asli Ozgun-Koca, has provided information about the purpose of the study, the procedures to be followed, and the expected duration of my child's participation. Possible benefits of the study have been described.

I acknowledge that I may contact Dr. Wagner at 292-8058 or S. Asli Ozgun-Koca at 451-3066 to obtain additional information regarding the study and that any questions I have raised have been answered to my full satisfaction. Furthermore, I understand that I am free to withdraw consent at any time and to discontinue participation in the study without prejudice to me or to my child.

Finally, I acknowledge that I have read and fully understood the consent form. I sign it freely and voluntarily. A copy has been given to me.

Date: ______________________ Time ______ AM PM  Child's name: ___________________

Signed: _____________________________________________________________________
(Parent or guardian)

To be completed by the student:

I agree to participate in the study and understand what will occur.

Signed: ___________________________ Date: ___________________________
(Student)

I certify that I have provided information regarding this study to the subject or his/her representative before requesting the subject or his/her representative to sign this permission form.

Date: _______________ Signed: _____________________________________________________________________
(Principal Investigator or his/her authorized representative)
APPENDIX B

Lesson Plan and Worksheets for Computer Labs
Computer Lab Structure

Date:

Time: 78 minutes

Before the Class:

• Software with linked property will be installed into computer.

The First Session (35 min):

• Students come into the computer lab and handout will be distributed.
• Students complete the task by following the instruction on the handout.
• I wander and answer possible questions.

Break (8 min):

• I save the students work and install the semi-linked version of the software.

The Second Session (35 min):

• Second group of students come into the computer lab and handout will be distributed.
• Students complete the task by following the instruction on the handout.
• I wander and answer possible questions.

After the Class:

• I save the students work, uninstall the program form the computers, and try to take some filed notes from the sessions.
For this investigation, we will work on the two fish which swim with constant rate. Watch the movie.

- How does the distance between two fish change as they swim?

**Section 1**
Predict what graphs of distance from the left-hand side of the screen versus time look like for both fish. Please explain your graph.

Now let's check the graph produced by the computer. In order to see the graphs of distances of these fish from the left-hand side of the screen against time click on the \( \text{graph button on the vertical menu bar on your left. Choose "x" from upper right-hand side menu and "Position" from the lower menu AND CLICK "ADD" and then choose "Point S1" from the upper left-hand side menu and click OK.}

- How does this compare with your prediction?
Looking at the graph on the computer, identify and describe the point which represents where the two fish meet. Explain your reasoning.

Section 2
Let's consider a table of representation of the data. Describe how the numerical values of these positions will behave if we put them in the table? Are they going to increase all the time or decrease? Explain your answer.

Please fill out the following table with your predictions. Please explain how you came up with the numbers?

<table>
<thead>
<tr>
<th>Time (Seconds)</th>
<th>Grey Fish’s Distance</th>
<th>Striped Fish’s Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>150 pixels</td>
<td></td>
<td>150 pixels</td>
</tr>
<tr>
<td>130 pixels</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 second</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Estimate the point which represents where the two fish meet would be according to this small table.

To see the values and open the table window by clicking on the left-hand vertical menu.

Check your values. How does this compare with values that you filled in, above?
• Identify the point which represents where the two fish meet would be according to the computer produced table.

**Section 3**

What do you think about the equations of these relationships?
Are they going to be linear equation or quadratic?

Do they have a negative or positive slopes? Estimate the slopes.

Do they have negative or positive \( y \)-intercepts? Estimate the \( y \)-intercepts.

In order to make your prediction about the equation of this event/data, first pay attention to the axes and their values in the computer-produced-graph then click on the \( M \) button in the graph window. It will give you choices on the family of the graph (type of the graph) and select the appropriate one. Enter your predictions about coefficients of the variables and you will see a blue line in the graph window which is the graph of your equation. Please write down your final prediction and answer the following questions:

**Grey Fish.** While studying the equation of the grey fish, choose Point S2 from the upper menu.

**Equation:**

How did you decide what to choose as coefficients?
Striped Fish. While studying the equation of the striped fish, choose Point S1 from the upper menu.

Equation:
How did you decide what to choose as coefficients?

- How do these equations compare with your predictions above?

- Interpret the differences in the equations of the two fish?

- How could you find out the point which represents where the two fish meet by using the equations?

Section 4
Identify the distance between the two fish at the beginning/middle/end.

What did you use in order to find out the answer? Verify your answer with two other representations other than the one you used. For example if you used the graph, check the table and equation.

- Describe what you learned from this computer lab task. Pose questions or unclear issues that you would like to have answered or to explore in this task.
For this investigation, we will work on two fish which swim with constant rate. Watch the movie.

- How does the distance between the two fish change as they travel?

**Section 1**

Predict what graphs of distance from the left-hand side of the screen versus time look like for both fish. Please explain your graph.

Now let’s consider the graph produced by the computer. In order to see the graphs of distances of these fish from the left-hand side of the screen against time click on the graph button on the vertical menu bar on your left. Choose “x” from upper right-hand side menu and “Position” from the lower menu AND CLICK “ADD” and then choose “Point S1” from the upper left-hand side menu and click OK.

- How does this compare with your prediction?
• Looking at the graph on the computer, identify and describe the point which represents where the two fish meet is. [Double click on the point on the graph that you consider to see the movie at that time]

Section 2
Let’s consider a table of representation of the data. Describe how the numerical values of these positions will behave if we put them in the table? Are they going to increase all the time or decrease? Explain your answer.

• Please fill out the following table with your predictions. Please explain how you came up with the numbers.

<table>
<thead>
<tr>
<th>Time (Seconds)</th>
<th>Grey Fish’s Distance</th>
<th>Striped Fish’s Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>150 pixels</td>
<td></td>
</tr>
<tr>
<td>1 second</td>
<td>130 pixels</td>
<td></td>
</tr>
</tbody>
</table>

• Estimate the point which represents where the two fish meet would be according to this small table.

To see the values and open the table window by clicking on the left-hand vertical menu.
• Check your values and your answer. How does this compare with values that you filled in, above?
• Identify the point which represents where the two fish meet would be according to the computer produced table.

Section 3
What do you think about the equations of these relationships?
Are they going to be linear equation or quadratic?

Do they have a negative or positive slopes? Estimate the slopes.

Do they have negative or positive \( y \)-intercepts? Estimate the \( y \)-intercepts.

In order to view the equation, click on the F button in each graph window.
For the equation of the grey fish, choose Point S2 from the upper menu.
Equation of the Grey Fish:
Was it what you expected? Why or why not?

For the equation of the striped fish, choose Point S1 from the upper menu.
Equation of the Striped Fish:
Was it what you expected? Why or why not?

• How do these equations compare with your predictions above?
• Interpret the differences in the **equations** of the two fish?

• How could you find out the point which represents where the two fish meet by using the **equations**?

**Section 4**
Identify the distance between the two fish at the beginning/middle/end.

What did you use in order to find out the answer? Verify your answer with two other representations other than the one you used. For example if you used the graph, check the table and equation.

• Describe what you learned from this computer lab task. Pose questions or unclear issues that you would like to have answered or to explore in this task.
APPENDIX C

The Sources and Scoring of the Questions in the Pre- and Posttest
<table>
<thead>
<tr>
<th><strong>Original Question</strong></th>
<th><strong>Adapted Version</strong></th>
<th><strong>Source and the Type of the Question</strong></th>
<th><strong>Scoring</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>In the graph shown, which of the following could be the number of bicycles built in 1975?</td>
<td>In the graph shown, what is the number of bicycles built in 1975? During which time period did bicycle construction increase the most?</td>
<td>SAT Question</td>
<td>Incorrect: 0 Correct: 1</td>
</tr>
<tr>
<td><img src="image" alt="Graph of Millions of Bicycles Built Per Year in Country ABC" /></td>
<td></td>
<td><a href="http://testprep.cpm/satmenu.html">http://testprep.cpm/satmenu.html</a></td>
<td></td>
</tr>
<tr>
<td>Based on the table, what is the cost of a 20 minute call from city X to Z?</td>
<td>Same Question</td>
<td>SAT Question</td>
<td>Incorrect: 0 Correct: 1</td>
</tr>
<tr>
<td><strong>Telephone Call Costs</strong></td>
<td></td>
<td><a href="http://testprep.cpm/satmenu.html">http://testprep.cpm/satmenu.html</a></td>
<td></td>
</tr>
<tr>
<td>From City X to:</td>
<td>First 3 minutes:</td>
<td>Each additional minute:</td>
<td></td>
</tr>
<tr>
<td>City Y</td>
<td>60¢</td>
<td>20¢</td>
<td></td>
</tr>
<tr>
<td>City Z</td>
<td>20¢</td>
<td>5¢</td>
<td></td>
</tr>
<tr>
<td>A car has a fuel tank that holds 35 L of fuel. The car consumes 7.5 L of fuel for each 100 km driven. A trip of 250 km was started with a full tank of fuel. How much fuel remained in the tank at the end of the trip? A. 16.25 L B. 17.65 L C. 18.75 L D. 23.75 L</td>
<td>A water tank is full with 1845 gallons of water. If we drain the water at the rate of 15 gallons per minute, estimate how long it will take to clear the tank.</td>
<td>TIMSS Question</td>
<td>Incorrect: 0 Correct: 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td><a href="http://timss.bc.edu/TIMSS1/TIMSSPDF/c_items.pdf">http://timss.bc.edu/TIMSS1/TIMSSPDF/c_items.pdf</a></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Reading Tables</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Word Problems</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**continued**
Table C.1 continue

<table>
<thead>
<tr>
<th>Time-Card-Name:</th>
<th>Total-Daily-Earnings($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mon 10:00am—3:00pm</td>
<td>27.50</td>
</tr>
<tr>
<td>Tues 9:00am—4:00pm</td>
<td>38.50</td>
</tr>
<tr>
<td>Wed 8:00am—4:00pm</td>
<td></td>
</tr>
<tr>
<td>Thurs 2:00pm—8:00pm</td>
<td>33.00</td>
</tr>
<tr>
<td>Fri 3:00pm—5:00pm</td>
<td>11.00</td>
</tr>
</tbody>
</table>

According to the information in the table, what is the average hourly wage for Thursday's earnings if the total earnings for the five days were $153.50? The hourly wage rate changes during the day. At what time does the hourly wage rate change.

A certain machine produces 300 nails per minute. At this rate, how long will it take the machine to produce enough nails to fill 5 boxes of nails if each box will contain 250 nails?
A. 4 min  
B. 4 min 6 sec
C. 4 min 10 sec
D. 4 min 50 sec
E. 5 min

A certain machine produces 75 labels per minute. At this rate, how long will it take the machine to produce enough labels to fill 15 pages of labels if each page has 20 labels?

NAEP Question
http://nces.ed.gov/nationsreportcard/ITMRLS/ITMRLS.HTM
Block: 1996-12M12 No:2
Incorrect: 0
Correct: 1

Constructing Tables

Word Problems

continued
A plumber charges customers $48 for each hour worked plus additional $9 for travel. If \( h \) represents the number of hours worked, which of the following expressions could be used to calculate the plumber's total charge in dollars?

A. \(48 + 9 + h\)  
B. \(48 
 
C. \(48 + (9 \cdot h)\)  
D. \((48 \cdot 9) + h\)  
E. \((48 \cdot h) + 9\)

Bill is 350 miles away from home and travels with his car at a constant rate of 50 miles/hour towards his home. If \( h \) represents the number of hours traveled and \( d \) represents the distance from home, write the equation for distance in terms of hours. How far away from home is Bill after he traveled 3 hours?

Constructing Equations

If \(12x = 4(x + 5)\), then \(x\) equals  
a) \(\frac{1}{12}\)  
b) \(\frac{5}{8}\)  
c) 1.25  
d) \(2.5\)

If \(12x = 4(x + 5)\), determine the value of \(x\)  

Solving Equations
The figure below represents the distances traveled by car A and car B in 6 hours. Which car is going faster and by how much? Explain how you arrived at your answer.

New York Mathematics A Regents Assessment Test, Sample Draft, Spring 1998

Incorrect: 0
Correct answer but height/slope misconception: 0.25

Correct answer for the first of the questions but not for the second part: 0.5
Correct: 1

continued
Two video rental clubs offer two different rental fee plans: Club A charges $12 for membership and $2 for each rented video. Club B has a $4 membership fee and charges $4 for each rented video. The graph below represents the total cost of renting videos from Club A.

a) On the same set of xy-axes, draw a line to represent the total cost of renting videos from Club B. Please explain your graph.

b) For what number of video rentals is it less expensive to belong to Club A? Explain how you arrived at your answer.

---

New York Mathematics A Regents Assessment Test, Sample Draft, Spring 1998

Point/Interval Misconception

Height/Slope Misconception

Incorrect: 0
Correct: 1

In 1997:

Incorrect: 0
Correct: 1

In 1998:

Incorrect: 0
Correct: 1

continued
<table>
<thead>
<tr>
<th>Original Questions</th>
<th>Type of the Questions</th>
<th>Scoring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write an equation of the relationship that is expressed in the graph below.</td>
<td>Constructing Equations</td>
<td>Incorrect: 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Correct: 1</td>
</tr>
<tr>
<td><img src="image" alt="Graph" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If \( y = 2x + 3 \) is changed to \( y = 1x + 3 \), sketch the graph of the new equation and discuss how the graph of the new equation is different from the old one.

<table>
<thead>
<tr>
<th></th>
<th>Height/Slope Misconception</th>
<th>Incorrect: 0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Correct sketch of the graph but wrong discussion: 0.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Correct: 1</td>
</tr>
</tbody>
</table>
Two cable companies are offering plans for pay-per-view service as presented below in the table. 
a) After how many movies Company B's plan becomes cheaper? 
Write the equations for both plans offered by two companies to estimate how much one will pay if he watched 10 movies to the two companies.

<table>
<thead>
<tr>
<th>Number of Movies</th>
<th>Company A ($)</th>
<th>Company B ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>29</td>
<td>22</td>
</tr>
</tbody>
</table>

Reading Tables
Constructing Equations
Incorrect: 0
Correct answer but height/slope misconception: 0.25
Correct answer for the first of the questions but not for the second part: 0.5
Correct: 1

269

continued
### Table C.1 continue

<table>
<thead>
<tr>
<th>Questions Adapted from the Research Articles</th>
<th>Sources &amp; Type</th>
<th>Scoring</th>
</tr>
</thead>
</table>

![Graphs](image)

| **If this graph below represents a wide jar being filled with water, draw a graph for narrower jar being filled with water on the same coordinate axes.** | Janvier (1978) cited in Leinhardt, G.; Zaslavsky, O. and Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of Educational Research, 60(1), 1-64 | Incorrect: 0 Correct: 1 |

![Graphs](image)

| **Height/Slope Misconception** | | |

continued
Table C.1. The Sources and Scoring of the Questions in the Pre- and Posttest

Correct: 1 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Yards</td>
<td>Height/Slope Misconception</td>
<td>---</td>
</tr>
<tr>
<td>Bill</td>
<td>Mark</td>
<td>---</td>
</tr>
</tbody>
</table>
APPENDIX D

Mathematics Posttest
Mathematics Posttest

1. A water tank is full with 1845 gallons of water. If we drain the water at the rate of 15 gallons per minute, estimate how long it will take to clear the tank.

2. According to the information in the table, what is the total daily earning for Wednesday?

<table>
<thead>
<tr>
<th>Time Card Name:</th>
<th>Total Daily Earnings($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>J Jasmine</td>
<td></td>
</tr>
<tr>
<td>Mon 10:00am – 3:00pm</td>
<td>27.50</td>
</tr>
<tr>
<td>Tues 9:00am – 4:00pm</td>
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<td></td>
</tr>
<tr>
<td>Thurs 2:00pm – 8:00pm</td>
<td>33.00</td>
</tr>
<tr>
<td>Fri 3:00pm – 5:00pm</td>
<td>11.00</td>
</tr>
</tbody>
</table>

a) Bill is 350 miles away from home and travels with his car at a constant rate of 50 miles/hour towards his home.

a) If \( h \) represents the number of hours traveled and \( d \) represents the distance from home, write the equation for distance in terms of hours.

b) How far away from home is Bill after he traveled 3 hours?

4. a) In the graph shown, what is the number of bicycles built in 1965?

b) During which time period did bicycle construction increase the least?
5. a) If \( y = 2x + 3 \) is changed to \( y = lx + 3 \), sketch the graph of the new equation and discuss how the graph of the new equation is different from the old one.

b) If \( y = 2x + 3 \) is changed to \( y = 2x + 5 \), sketch the graph of the new equation and discuss how the graph of the new equation is different from the old one.

6. The figure below represents the distances traveled by car A and car B in 6 hours. Which car is going faster and by how much? Explain how you arrived at your answer.
7. Write an equation of the relationship that is expressed in the graph below.

8. If \(2x = 18 - 3(x + 1)\), determine the value of \(x\).

9. If this below graph represents a wide jar being filed with water, draw a graph for narrower jar being filled with water on the same coordinate axes.

10. a) Two video rental clubs offer two different rental fee plans. For what number of video rentals is it less expensive to belong to Club A?

b) Which plan is cheaper per a movie rental?
10. Mark is walking and Bill is riding his bicycle. At 2 minutes, who travels faster?

![Graph showing distance vs. time for Mark and Bill]

Yards

12. Which graph shows the distance from the beginning the person has walked with a constant rate from A to B? Why?

![Graphs showing different distance-time relationships]

13. Based on the table below, what is the cost of a 25 minute call from city X to Y?

<table>
<thead>
<tr>
<th>Telephone Call Costs</th>
<th>From City X to</th>
<th>First 3 minutes</th>
<th>Each additional minute</th>
</tr>
</thead>
<tbody>
<tr>
<td>City Y</td>
<td>60¢</td>
<td>20¢</td>
<td>20¢</td>
</tr>
<tr>
<td>City Z</td>
<td>20¢</td>
<td>5¢</td>
<td>5¢</td>
</tr>
</tbody>
</table>
14. A certain machine produces 75 labels per minute. At this rate, how long will it take the machine to produce enough labels to fill 15 pages of labels if each page has 20 labels?

15. Two cable companies are offering plans for pay-per-view service as presented below in the table.
a) After how many movies Company B’s plan becomes cheaper?

<table>
<thead>
<tr>
<th>Number of Movies</th>
<th>Company A ($)</th>
<th>Company B ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>29</td>
<td>22</td>
</tr>
</tbody>
</table>

b) Write the equations for both plans offered by two companies to estimate how much one will pay if he watched 10 movies to the two companies.
APPENDIX E

Oral Script
I’d like to talk with you for about 30 minutes today about mathematics. I want to ask you some questions that you may or may not have seen in your classroom and in your computer lab hours. I would like you to use the computer and the software while solving the problem. This interview won’t affect your grades and your teacher and parents won’t know how you answered the questions. Your answers are only to help me to understand how students think about math and the use of computers in education.

I would like to record the interview through the use of computer, if that’s OK with you. Computer will record our voices and your actions on the screen. When I write up my report, I won’t use your real name. You don’t have to answer any questions that you don’t want to, and you can ask to stop the interview at any time. Would you be willing to participate?
APPENDIX F

Teacher Interview Protocol
1. Please, tell me about your class.
2. What kinds of pedagogical methods do you use in this class?
3. What do you think about the use of multiple representations in mathematics classrooms? Why?
4. What are the students’ attitudes towards multiple representations?
5. What is dominant representation used by students? Why?
6. Do students use more than one representation? In what situation and why?
7. What can technology bring in this issue?
8. What do you think which is better: Linked multiple representations vs. Semi-linked multiple representations?
9. What do you think about if technology is a part of instruction, do students change their attitudes towards the use of representations? Why?
APPENDIX G

The Survey
The Survey

Directions:

Your answers to the following questions will help me understand your attitudes, opinions, and experiences in your mathematics class. It is important to remember that there are no “right” or “wrong” answers here!

1. Gender □ Female □ Male
2. Age ______
3. Computer Experience □ Home Years ______ □ School Years ______ □ Other Years ______

Please indicate your personal opinion about each statement by circling the appropriate number.

<table>
<thead>
<tr>
<th>Statement</th>
<th>SA</th>
<th>A</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics makes me feel uneasy and confused.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Mathematics is a very worthwhile subject for every person.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Mathematics is enjoyable and stimulating to me.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Mathematics has been my worst subject.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Mathematics helps develop a person’s mind and teaches him/her to think logically.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Mathematics problems can be solved in various ways by using different representations such as tables, graphs, and equations.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>I like using more than one representation such as graphs, tables, and equations to solve mathematics problems.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Given a mathematical problem, I find it easier to focus on one representation than to deal with many representations.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>When a mathematics problem is presented with more than one representation, it means that there are as many questions as representations.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
<tr>
<td>Solving a mathematics problem with different representations such as graphs, tables, and equations results in totally different answers.</td>
<td>SA</td>
<td>A</td>
<td>D</td>
<td>SD</td>
</tr>
</tbody>
</table>
After this point, please choose one of the options that indicates your opinion the most.

14. I like using □ Equations, □ Tables, □ Graphs the most when solving math problems. 
   Because

15. □ Equations, □ Tables, □ Graphs make mathematical topics the easiest for me to understand. Because

16. If □ Equations, □ Tables, □ Graphs were the only option I had to solve math problems then I would have the most difficult time doing the problem. Because

17. I find it the hardest to construct □ Equations, □ Tables, □ Graphs when solving problems using paper and pencil. Because

18. I will usually start solving mathematics problems with □ Equations, □ Tables, □ Graphs. Because

19. I find □ Equations, □ Tables, □ Graphs confusing when working on a mathematics problem. Because
20. Using software, VideoPoint, in our class was □ Confusing, □ Helpful in understanding mathematical representations such as graphs, tables, and equations. Because

21. While using VideoPoint, I liked using □ Equations, □ Tables, □ Graphs the most. Because

22. VideoPoint made it the easiest to use □ Equations, □ Tables, □ Graphs when solving math problems. Because

23. Using the computer was helpful in understanding the relationships between tables, graphs, and equations. □ Yes □ No. Because

24. I liked being able to see all representations at the same time on the computer screen. □ Yes □ No. Because

25. Graphs are useful for understanding mathematical concepts. □ Yes □ No. Because
26. Tables are useful for understanding mathematical concepts. □ Yes □ No.  
Because

27. Equations are useful for understanding mathematical concepts. □ Yes □ No.  
Because

28. What do you like the most about using VideoPoint?

29. What do you like the least about using VideoPoint?

30. Is there anything influencing your choice of or preference for a representations, such as graph, table, or equation in solving mathematics problems? If yes, what and why?