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VISCO-PLASTIC CONSTITUTIVE MODEL DEVELOPMENT FOR THERMO-MECHANICAL ANALYSIS OF INTUMESCENT MAT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By
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2001

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ABSTRACT

Intumescent mat material is used to support substrates in catalytic converters and is known to be critical to the durability. The material is highly compressible, rate dependant and expands dramatically due to irreversible vermiculite expansion as temperature increases beyond a critical point.

In this study, several tests are performed to characterize thermo-mechanical properties of the intumescent mat material. Thermo-mechanical properties tested mainly focused on several issues, such as, relation in each stress versus mechanical strain curve at different temperature, permanent vermiculite expansion, rate effects and plastic deformation under different loading and unloading processes. Assumptions are made to simplify mat material property. A constitutive model is developed based on simple isotropic hyper-elastic theory. The theory is extended to include visco-plastic material behavior. A one dimensional plasticity theory in the thickness direction is developed by assuming a special form of the plastic deformation gradient tensor and yield criteria, based on experimental observations. To consider the temperature effect in the constitutive model, the irreversible strain due to vermiculite expansion is defined as a chemical strain. Mechanical property change due to temperature is implemented by the proposed softening model. A stress function is derived from the 2nd law of thermodynamics to consider material softening. A simple relationship among mechanical, thermal and
chemical strain is established by an assumed strain decomposition. Finally, the thermo-
mechanical constitutive modeling is implemented by employing the concept of a master
curve and strain shift function, derived from the softening model and strain
decomposition. The complete spatial elasticity tensor for a new constitutive model is
derived for implicit finite element code and implemented in the ABAQUS™ user
subroutine called UMAT.

The proposed theory is verified by solving simple boundary value problems and
by comparing with measured data. An example field application is discussed for
incompressible anisotropic hyperelastic material under thermo-mechanical loading as a
part of the theory extension. Also, a catalytic converter is analyzed for the canning,
heating, and cooling process. The numerical results are compared with measured data.
Dedicated to
My wife Nayoung, my daughter Yoohyeon
And
My family
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CHAPTER 1

INTRODUCTION

1.1 Research motivation

A catalytic converter is a pollution control device, which converts harmful gases in the combustion exhaust, such as hydrocarbons, CO and NO\textsubscript{x} to non-harmful CO\textsubscript{2}, H\textsubscript{2}O and N\textsubscript{2}. It consists of three major parts; an outer shell, a substrate and a support mat. The outer shell is typically made of stainless steel and houses the substrate. The substrate is made of a ceramic material and converts harmful gases to non-harmful gases. The mat is used to hold the substrate in the outer shell and to provide cushion and insulation.

Generally, the gas temperature passing through the substrate is crucial in determining converter conversion efficiency. Therefore, over the past years, the design concept of the engine room layout has been shifted toward installing a catalytic converter closer to the engine to maximize the performance. As a result, an exhaust mounted converter experiences higher vibration and thermal loading than ever, which makes the durability issues more serious. Severe mechanical and thermal cyclic loads expedite mat deterioration, eventually leading to unacceptable degradation of the substrate.
The intumescent mat material contains aluminosilicate glass fibers, vermiculite mineral particles and an organic binder. The vermiculite, when heated, loses some of the water and increases the mat thickness significantly, as high as 10 times. The vermiculite expansion is an irreversible process. Most of the permanent expansion takes place between 350-550 °C and vermiculite degrades when heated beyond 750 °C. Therefore, when the intumescent mat is exposed to the high temperatures generated by exhaust gas, it expands and provides cushioning effect for the substrate. This expansion and subsequent cushioning is critical to the durability of the substrate as the exhaust system is subjected to various loads throughout its lifetime.

There are typically two types of failure modes for the catalytic converter. One is failure during the canning process and the other is failure during the service life. The canning process is an assembly process that packages a mat, an outer shell and a substrate into one part. During the canning process, the mat is compressed and transfers a compressive force to the substrate. If the mat compressive force is too high, it can break a substrate during the canning process. If mat compressive force is too low, substrate sliding during the engine operation may occur. Therefore, an optimal compressive force should be applied to the mat during the canning process so that mat can retain safe compressive force during engine operation.

To date, most of the catalytic converter has been developed by trial and error by relying on experience and test results. However, it is very difficult to pick out the major design parameters to modify when a particular design does not pass the certification test. Therefore, a predictive simulation capability is absolutely needed in the early design
stage to shorten developing time. An analytic approach to modeling this mat material is needed for the simulation capability.

1.2 Research objective and work scopes

The main objective of this study is developing a constitutive model for the intumescent mat under thermo-mechanical loading to be used as a predictive tool for catalytic converter analysis. To achieve this research goal, the scope of the work are divided into the following topics:

- Material characterization for thermo-mechanical properties from experiments,
- Defining each strain measure and establishing strain relations under thermo-mechanical loading,
- Thermo visco-hyper-elastic model development,
- Development of a plasticity model with rate effects,
- Implementation in ABAQUS™ user subroutine UMAT and verification with test results,
- Application to actual catalytic converter geometry for simulation of canning processes and thermal cyclic loading.
1.3 Dissertation organization

This thesis focuses on the constitutive model development of intumescent mat and its application in catalytic converter analysis. It is composed of 5 chapters. Except Chapter 1 and Chapter 5, each chapter is written as an independent paper which is intended for publication.

Chapter 2, **Study of cyclic loading and unloading phenomena for intumescent mat considering visco-plasticity and material softening**, presents details of the development of the constitutive model for intumescent mat at room temperature. The proposed concept of material softening is established in large deformation theory and a plasticity model with rate effects. Cauchy stress is derived from the 2nd law of thermodynamics for the model with material softening and visco-elasticity, and the complete spatial elasticity tensor is derived for implicit finite element coding.

Chapter 3, **Constitutive model development for anisotropic hyper-elastic material during thermomechanical processes**, presents how to include temperature dependant material properties in the anisotropic hyper-elastic material during thermo-mechanical processes. The theory of material softening developed in Chapter 2 is extended to the general anisotropic temperature dependant material and a new type of Helmholtz free energy function is proposed for anisotropic materials to account for material property change due to temperature. As an example, the softening model is applied in both compressible and incompressible anisotropic hyperelastic model to investigate material behavior at elevated temperature.
Chapter 4, *Constitutive model development of catalytic converter intumescent mat for thermo-mechanical analysis considering viscous and plastic effects*, is devoted to developing the procedure of constitutive modeling for intumescent mat at room and elevated temperature. A new strain is defined to represent irreversible vermiculite expansion and strain relations under thermo-mechanical loading are established by decomposing total strain into mechanical, thermal and chemical strains. Theories developed in Chapter 2 and Chapter 3 are combined and expanded to temperature dependant visco-hyper-elasto-plastic modeling in this chapter. The theory is coded in ABAQUS™ implicit by using the user subroutine UMAT and the model is applied to a catalytic converter to simulate assembly processes and thermal cyclic test conditions.

Chapter 5 summarizes the main conclusions and future work.
CHAPTER 2

STUDY OF CYCLIC LOADING AND UNLOADING PHENOMENA FOR
INTUMESCENT MAT CONSIDERING VISCO-PLASTICITY
AND MATERIAL SOFTENING

ABSTRACT

The mechanical behavior of an intumescent mat material is investigated at room temperature. The temperature effects are studied elsewhere. The subject intumescent mat material is widely used to support ceramic substrates in catalytic converters and behaves very much like a hyper-foam material under a compressive loading. Experiments show that compressive loading curves depend on the ram speed and the number of cycles. The unloading curves show different slopes and paths, depending less on the rate and cycle. The slopes of unloading curves become less stiff as the plastic strain increases, termed as "softening" in this study. The effects of rate, softening and plastic deformation must be considered to investigate the mechanical behavior of intumescent mat material. The theory of finite deformation is applied with a multiplicative decomposition of deformation tensor to characterize the mat material. Developed theory is implemented in an implicit finite element algorithm in the form of UMAT for ABAQUS™/STANDARD.
Needed material parameters are extracted from experiments. Numerical simulation show a good agreement with experiments.

2.1 INTRODUCTION

As a result of their viscous return in impact situations, highly compressible hyper-foam materials are often used as energy absorbing materials. Due to their high capacity to retain elastic strain energy in wide ranges of deformation, foam materials recently began to be used not only as energy absorbers but also as support materials. One application of a hyper-foam support material is intumescent mat in catalytic converters. The intumescent mat is placed between substrate and outer can in catalytic converters. While outer can of a catalytic converter expands and shrinks during thermal cycle or mechanical vibration, highly compressible mat supports the substrate. In this case, predicting mat holding force for both mechanical loading and unloading is very important in catalytic converter design [2.3, 2.4, 2.9, 2.16].

For many years, great progress has been made on constitutive model development for foam material [2.7, 2.8, 2.10, 2.11, 2.15, 2.22]. Depending on the existence of plasticity, formulation models have relied on one of two methods. That is, when there is no plasticity, hyper-elastic theory has been used by assuming the proper elastic strain energy function. Authors, such as Simo [2.19] and Papoulia [2.18] have added visco-hyperelastic theory to account for viscous effects. On the other hand, when there is plasticity, rate dependent plasticity formulations have been employed to represent crushable foam. In this case, plastic strain is obtained incrementally by satisfying
assumed pressure and deviatoric stress dependant yield functions. These studies have been done by Gibson \textit{et al.} [2.24] and Zhang \textit{et al.} [2.23], among others. However, most of these studies have been limited to predicting loading processes only. When the developed constitutive models are applied to unloading processes, they do not provide acceptable results. Study of the unloading phenomenon is a demanding subject. There are several issues that should be considered in developing a constitutive model for a material used as a supporting material and that undergoes cyclic loading and unloading.

Figure 2.1 shows a schematic diagram of a cyclic compression test for a hyper-foam material with no viscous effects. As shown in Figure 2.1, the loading and unloading functions are quite different from each other. The unloading curve softens as compressive strain increases together with an increase in plastic strain. Also, if there are viscous effects in the cyclic compression test, then the loading and unloading curves have very different viscous properties. Therefore, including softening effects on the unloading function and plastic effects for visco-hyper-elastic material is critical to accurate prediction of unloading phenomena. So far, not much research has been done regarding unloading phenomena. Also, the hyper-foam theory cited above typically uses linear elastic or hypo-elastic functions for the elastic response. The elastic response function is assumed to be independent of plastic strain. Therefore, the elastic response function is always assumed to be constant regardless of the plastic strain.

This study is limited to mat type foam that has a small dimension in the thickness direction compared to the lateral direction. In addition, the loading and deformation are assumed to occur only in the thickness direction. In this case, it can also be assumed that
plasticity only occurs in the thickness direction. Therefore, a 1-D plasticity model can be proposed in the foam material. One of the main advantages of introducing the 1-D plastic model to a hyper-elastic formulation is that it can save computational time because it is formulated on the total Lagrangian method and large time steps can be taken in calculating plastic strain.

In this study, to characterize material behavior, several experimental phenomena for intumescent mat are reviewed. A constitutive model is extended from hyper-elasticity to visco-hyper-elasticity. Plastic yield stress is established on the basis of mechanical strain and a new elastic response function is introduced to account for material softening due to mechanical strain. The complete spatial elasticity tensor is derived, and the Newton-Raphson method is employed for the plastic strain calculation. The developed theory is coded in ABAQUS™ implicit through the user subroutine UMAT. To verify the developed theory, experimental results and simulation are compared to each other for several cases.

2.2 EXPERIMENTAL INVESTIGATION

2.2.1 Experimental procedure

The intumescent mat material used in this study has very little tensile strength and is used only under compression. At room temperature, the mechanical behavior is very similar to that of a hyper-foam material. It is highly compressible and has almost zero Possion’s ratio. If heat is applied to intumescent mat, due to chemical reactions, there is a
dramatic permanent thermal expansion in the temperature range between 400°C and
600°C. Therefore, mechanical behavior at elevated temperature is quite different from
room temperature. However, as a first step, in this study, material model development is
limited to room temperature and high temperature mechanical behavior is left as future
work.

All the experiments described in this study use 10 mm thick intumescent mat
specimens. The specimens are punch cut from a large sheet into 29 mm diameter buttons.
The experiments are all performed with compressive loading. They can be divided into
the following two main categories: constant load experiments and compression
experiments.

The constant load experiment is performed by applying constant pressure on the
specimen and recording specimen thickness changes. The constant load experiment
provides strain change information under constant load at room temperature. A custom
load frame was built to perform the constant load experiments. The loading mechanism
has a lever with a 1: 4.6 ratio. Therefore, a small amount of dead weight applied at the
end of the lever can generate a large amount of load on the specimen. From the constant
load experiment, stress versus strain curves without rate effects are obtained by
measuring relaxed mat thicknesses after long periods of time.

Compression tractions are applied through 31.75 mm diameter loading rods which
sandwich the specimen. For the loading process, dead weights are added after the mat is
fully relaxed at each weight. In the same way, for the unloading cases, stacked dead
weights are removed one by one after no additional mat thickness change is found. Thickness is measured in the relaxed state with an LVDT.

The compression experiment procedure consists of compressing the specimen to a certain thickness. Compression experiments are performed to monitor the stress change on the specimen as strain in the specimen changes. Since the compression experiment requires maintaining various gap positions throughout the experiments, it is essential that the loading apparatus have accurate displacement control with a variable command signal. Therefore, a servo-hydraulic load frame was used to do the compression experiments.

Compression experiments are performed by placing the specimen between two 31.75 mm diameter load rods held 10 mm apart. The specimen is compressed to a specified position at a fixed ramp rate. The distance between the two rods is then held constant for a certain holding time. After the hold time, the upper rod is moved upward to the original 10 mm gap position. Specimens are tested at different maximum strain ranges. The maximum strain ranges are 0.3, 0.4, 0.5, and 0.6 engineering strain.

To investigate rate effects, experiments are also performed at various ramping speeds. The cycle is repeated to study cyclic loading effects. Figure 2.2 shows a generic displacement profile for the cyclic compression experiments and Table 2.1 shows corresponding test conditions. See Kim [2.1,2.2] for experimental material response and test details.
2.2.2 Experimental results

Figure 2.3 shows stress versus strain curves for different compression speeds up to 60% compressive engineering strain. The static stress versus strain curve comes from the constant load test and stress versus strain curves with different speeds come from the compression tests. It shows that stress curve becomes stiffer as the compression speed increases. Therefore, mat material is a highly rate dependant material.

Figure 2.4 shows loading and unloading curves during several cycles of a 40% compression test. There is a significant difference between the 1\textsuperscript{st} cycle and 2\textsuperscript{nd} cycle loading curves. Only minor differences are seen between the 1\textsuperscript{st} and 2\textsuperscript{nd} cycle unloading curves. Also, the 1\textsuperscript{st} hysteresis loop from the 1\textsuperscript{st} cycle loading and unloading curve and the 2\textsuperscript{nd} hysteresis loop from the 2\textsuperscript{nd} cycle loading and unloading curve are shown. Generally, hysteresis loops are a material property of visco-elastic material. Therefore, this is evidence that the mat material is a visco-elastic material. The size of hysteresis loop becomes smaller from the 1\textsuperscript{st} to 2\textsuperscript{nd} cycle and there is little further size reduction after the 2\textsuperscript{nd} cycle. The peak stresses from the cyclic 40% compression experiments are plotted in Figure 2.5. Significant stress reduction takes place from the 1\textsuperscript{st} to 2\textsuperscript{nd} cycle and only gradual stress reductions continue after the 3\textsuperscript{rd} cycle.

Figure 2.6 shows a comparison of the monotonic loading and cyclic loading and unloading static stress versus strain curve from the constant load test. During the monotonic loading process, loading is applied continuously without unloading. However, cyclic loading is performed to investigate the existence of a plastic loading function.
Loading is applied to certain stress levels and unloaded and then reloaded again to a higher stress level. The direction of loading and unloading is shown with arrows. The results show that the unloading curves do not follow the monotonic load curve. However, it shows that unloading and reloading are almost the same. Also, as compressive strain increases, it is seen that reloading curve is slightly softer. This softening will be discussed in Section 2.6.1. If the 2\textsuperscript{nd} loading is higher than previous yield stress, then the 2\textsuperscript{nd} loading curve follows the monotonic loading curve. From this result, it is concluded that material behavior of the mat material is very similar to the plastic response of metal.

For different maximum compressive strains from the compression test, Figure 2.7 shows loading and unloading stress versus strain curves during the 1\textsuperscript{st} cycle. Three test results for 30, 40 and 50\% compressive strains are compared to each other under the same compression speed. It shows that the loading curves under 30 and 40\% compression follow the loading curve of the 50\% compression test. Figure 2.8 shows a comparison between the 1\textsuperscript{st} and 2\textsuperscript{nd} cycle loading curve under the same compression speed when different compressive strain is applied. It shows that peak stress is quite different between the 1\textsuperscript{st} and 2\textsuperscript{nd} loading. This indicates that the 1\textsuperscript{st} loading and the 2\textsuperscript{nd} loading curves have different rate dependant material properties.

From the test results, it is concluded that mat is highly nonlinear and has distinct loading and unloading functions. Material behavior is very similar to metal plasticity and the loading and unloading functions have their own rate effects. Also, for different compression strains, there is material damage in both cyclic loading and the 1\textsuperscript{st} loading.
2.3 VISCO-HYPER-ELASTIC MODEL DEVELOPMENT

2.3.1 Hyper-elastic model development

Since the 1940s and 1950s, elastic constitutive modeling for large deformation has been well established [2.5]. Among other work [2.21], Ogden [2.17] has employed the principal values of the stretch ratio of strain in the strain energy function and showed an extremely good representation of the experimental data. Therefore, strain energy which is a function of principal stretches is chosen to develop theory in this study [2.20].

From the virtual work principal, the internal energy variation is expressed as follows:

\[
\int \delta W dV_i = \int S \cdot \delta \varepsilon^e dV_i = \int \tau \cdot \delta \varepsilon^e dV
\]

(2.1)

where \( W \) is a strain energy function, \( S \) is the 2nd Piola Kirchhoff stress, \( \tau \) is Kirchhoff stress, \( \varepsilon^e \) is Lagrangian elastic strain tensor, \( \varepsilon^e \) is Eulerian elastic strain tensor, \( V_i \) is the reference volume, and \( V \) is current volume. From the relation of \( \delta W = S \cdot \delta \varepsilon^e \) in Equation (2.1), \( S \) is obtained as follows:

\[
S = \sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial W}{\partial \lambda_i^e} (N_i^l \otimes N_i^l)
\]

(2.2)

where \( N_i^l \) is a principal direction in the reference configuration, \( \lambda_i^e \) is the principal stretch in each direction, and \( \otimes \) represents the tensor outer product [2.26, 2.27]. \( S \) can be related to the Cauchy stress \( \sigma \) by the deformation gradient tensor \( F \) as follows:
\[ \sigma = \frac{1}{J^e} F^e S F^e \tau \]  \tag{2.3} 

where \( J^e \) is the determinant of \( F^e \). The material elasticity tensor \( D^{(4)} \) is defined as follows:

\[ \dot{S} = D^{(4)} \dot{E}^e. \]  \tag{2.4} 

\( D^{(4)} \) is obtained by differentiating \( S \) in Equation (2.2) with respect to time.

\[
D^{(4)} = \sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial W}{\partial \lambda_i^e} N_{a\alpha}^i + \sum_{i=1}^{3} \sum_{\beta=1}^{3} \frac{1}{\lambda_i^e \lambda_\beta^e} \frac{\partial^2 W}{\partial \lambda_i^e \partial \lambda_\beta^e} N_{a\beta}^i \\
+ \sum_{i,j=12,13,23} \frac{i}{\lambda_i^e - \lambda_j^e} \left( \frac{1}{\lambda_i^e} \frac{\partial W}{\partial \lambda_i^e} - \frac{1}{\lambda_j^e} \frac{\partial W}{\partial \lambda_j^e} \right) \left( N_{ji}^i + N_{ij}^i + N_{ji}^j + N_{ij}^j \right) 
\]  \tag{2.5} 

where \( N_{abcd}^i = N_a^i \otimes N_b^i \otimes N_c^i \otimes N_d^i \). In this study, for convenience, the spatial elasticity tensor is defined by relating the Jaumann rate of Cauchy stress and the rate of stretch tensor \( D^{(4)} \):

\[ \dot{\sigma} = E^{(4)} D^{(4)}. \]  \tag{2.6} 

\( E^{(4)} \) is obtained as follows:

\[
E^{(4)} = -\sigma \otimes I + \frac{1}{J^m} P^{(4)} + H^{(4)} 
\]  \tag{2.7} 

where the 4th order tensors \( H^{(4)} \) and \( P^{(4)} \) are obtained as follows:

\[
H^{(4)}_{ijab} = \frac{1}{2} (\delta_{ia} \sigma_{bj} + \sigma_{ia} \delta_{bj} + \delta_{ib} \sigma_{aj} + \sigma_{ib} \delta_{aj} \)  \tag{2.8} 

\[
P^{(4)}_{ikmn} = F_{ij}^m F_{jk}^m F_{km}^m F_{mn}^m D^{(4)}_{ikmn}. \]  \tag{2.9}
2.3.2 Visco-elastic model development

Visco-elastic models in non-linear large deformation can be derived through simple extension of classic linear visco-elastic modeling. This method was studied by many authors [2.12, 2.13, 2.18, 2.19]. In this case, the Cauchy stress is assumed to be the difference between the instantaneous stress $\sigma_0$ and the internal variable tensor $q$ which is equivalent to stress relaxation:

$$\sigma = \sigma_0 - q = \frac{1}{J^e} F^e \frac{\partial W}{\partial E^e} F^e \tau - q. \quad (2.10)$$

Equation (2.10) can be solved with respect to $q$ by using a linear visco-elastic model with long term stress $\sigma_\infty$ and a Prony series (see Papoulia [2.18] for details):

$$\sigma = \sigma_\infty + \sum_{a=1}^{N} H_a \quad (2.11)$$

where

$$\sigma_\infty = \frac{1}{J^e} F^e \frac{\partial W}{\partial E^e} F^e \tau \quad (2.12a)$$

$$H_a = \int_a^{\infty} e^{-\frac{t}{\tau_a}} \frac{1}{J^e} F^e \frac{\partial W}{\partial E^e} F^e \tau \, dt' \quad (2.12b)$$

Therefore, stress response is divided into static and transient terms. Equation (2.12) is an integral equation with respect to time and, depending on the complexity of energy function, may not necessarily be analytically integrable. Therefore, a numerical integration scheme is required to implement the scheme in the finite element method.

Suppose that time $t$ in Equation (2.12) can be discretized into $n+1$ time steps. Then, numerical integration is possible based on finding a stress at time step $n+1$ with a
given stress and history information at time step \( n \). With this idea, Equation (2.11) can be written in the numerical integration form at time step \( n+1 \) by assuming that \( \sigma_{\omega} \) is linear with respect to time during \( \Delta t \) and applying \( t^{n+1} = t^n + \Delta t \). In this way

\[
\sigma^{n+1} = \sigma^{n+1}_{\omega} + \sigma^{n+1}_{\text{vis}} = \sigma^{n+1}_{\omega} + \sum_{a=1}^{N} H_a^{n+1} \tag{2.13}
\]

with

\[
H_a^{n+1} = e^{\frac{\Delta t}{\tau_a}} H_a^{n} + \frac{(\sigma^{n+1}_{\omega} - \sigma^n_{\omega})}{\Delta t} a_a \tau_a (1 - e^{\frac{\Delta t}{\tau_a}}) \tag{2.14}
\]

In Equation (2.14), \( H_a^{n+1} \) can be calculated for a given \( H^n \) because \( \sigma^{n+1}_{\omega} \) can be determined from the deformation gradient tensor. Notice that except for the assumption of linear stress variation during \( \Delta t \), Equation (2.14) is almost an exact solution. Also, only history information at time step \( n \) is required to calculate stress at time step \( n+1 \). This is a significant advantage in terms of saving computer memory space. Therefore, stress can be calculated at any time step by updating (2.13) with the initial condition \( H_a^0 = 0 \). The corresponding spatial elasticity tensor \( \tilde{E}^{(4)} \) is obtained as follows:

\[
\tilde{E}^{(4)} = \alpha_{\text{vis}} E^{(4)} \tag{2.15}
\]

where

\[
\alpha_{\text{vis}} = 1 + \sum_{a=1}^{N} \frac{1}{\Delta t} a_a \tau_a (1 - e^{\frac{\Delta t}{\tau_a}}) \tag{2.16}
\]
2.4 SOFTENING MODEL DEVELOPMENT FOR THE ELASTIC UNLOADING FUNCTION

2.4.1 Assumptions for the elastic unloading function

Experimental results show that the unloading stress function softens as mechanical compressive strain increases. This phenomenon is called material softening and appears to occur due to material damage. Therefore, the constitutive modeling must account for different unloading functions for different compressive strains. This requires many material constants. However, the unloading function can be simplified by investigating the unloading curve and by assuming that the unloading stress response at different compressive strain is self-similar.

Suppose that the unloading stress function at an arbitrary base state is represented as follows:

\[ \sigma = f(\varepsilon) . \]  

(2.17)

This similarity can be represented by the following two methods. The first method is that the stress versus strain curve can be represented by scaling the stress at constant strain with a scale factor \( h \):

\[ \sigma = h f(\varepsilon) = f'(\varepsilon) . \]  

(2.18)

The second method is that the stress versus strain curve is assumed to be represented by scaling the strain by a scale factor \( h \) while keeping stress constant. In this case, stress is calculated by obtaining equivalent strain \( \varepsilon' \) for given \( \varepsilon \):

\[ \sigma = f(h \varepsilon) = f(\varepsilon'). \]  

(2.19)
Depending on the material, either method or a combination of methods can be used to represent material similarity. In this study, after comparing both methods, the second method was chosen as an appropriate method to represent material similarity.

Therefore, if the unloading stress function at the base state and the softening parameters at a different state are given, then the unloading stress function is always obtained by using Equation (2.19). In this case, the stress function at the base state is called a master curve. In this study, the softening parameter $h$ is assumed to be a function of the mechanical right Cauchy-Green metric tensor $C^m_{33}$ in the thickness direction as follows:

$$h = h(C^m_{33}).$$

Physically, the right Cauchy-Green metric tensor $C^m_{33}$ is a stretch ratio in the material thickness direction and it can be used as a strain measure. Therefore, depending on the strain in the thickness direction, the softening parameter $h$ is determined.

### 2.4.2 Material softening in large deformation theory

Suppose that there exist two different bodies and they deform until they become the same shape and size. In this situation, one body deforms to the final shape by the deformation gradient tensor $F^e$ and the other body deforms to the final shape by the deformation gradient tensor $F'$. If the two different bodies generate the same stress field, then $F'$ and $F^e$ are called equivalent deformation gradient tensors. Material softening phenomenon for unloading functions can be understood by introducing the three different
configurations shown in Figure 2.9. Suppose that $X^1$ is a position vector in the reference configuration $\beta_1$, $X^2$ is the position vector in the intermediate configuration $\beta_2$ and $x$ is a position vector in the current configuration $\beta$. In this case, it is assumed that the reference configuration is a state before material softening and the intermediate configuration is a state after material softening. $F^c$ and $F'$ are the deformation gradient tensors relating each configuration shown in Figure 2.9. $F^h$ is defined as a deformation gradient tensor relating the reference and the intermediate configurations and is called a softening tensor. From the configurations shown in Figure 2.9, the deformation gradient tensor $F^c$ can be defined as follows:

$$F^c = F' F^h. \quad (2.21)$$

It is necessary to establish the relation between $F^c$ and $F'$ to relate material properties at different states. The relation can easily be established in large deformation theory with concepts similar to those used with the infinitesimal elastic functions given in Equation (2.19). In large deformation theory, the mechanical part of the right Cauchy-Green metric tensor $C$ is used to represent the stress function. Therefore, it is assumed that the metric tensor $C'$ is the same as $C'^h$ which is the $h^{th}$ power of $C^c$. In this case, $C'$ is expressed as follows:

$$C' = (C^c)^h. \quad (2.22)$$

$C^c$ and $C'$ can be expressed through the spectral theorem:

$$C^c = \lambda_1^c N_1^c \otimes N_1^c + \lambda_2^c N_2^c \otimes N_2^c + \lambda_3^c N_3^c \otimes N_3^c \quad (2.23)$$

$$C' = \lambda_1' N_1' \otimes N_1' + \lambda_2' N_2' \otimes N_2' + \lambda_3' N_3' \otimes N_3'. \quad (2.24)$$
In this case, the following relations are obtained because $C'$ and $C^e$ have the same principal directions:

$$\lambda' = \lambda^{-h}; \quad N = N'. \quad (2.25a,b)$$

Therefore, Figure 2.9 shows that the material reference configuration expands to the intermediate configuration without changing mechanical properties and stress is calculated based on the deformation gradient tensor $F'$ if the material softens. That is, for a given actual deformation gradient tensor $F^e$, stress becomes smaller as the intermediate configuration gets smaller due to material softening. A similar approach is applied to plasticity theory in large deformation. That is, the intermediate configuration is regarded as a configuration after plastic deformation.

The relation between velocity gradients $L'$ and $L^e$ can be obtained by differentiating Equation (2.21) with respect to time as follows:

$$L' = L^e - G^{(4)}_L L \quad (2.26)$$

where $G^{(4)}_L$ is

$$G^{(4)}_L = \sum_{ijkl} (1-h)n_{ij} + \sum_{ijkl=12,13,23} \lambda_i^e \lambda_j^e - \lambda_i^e \lambda_j^e \left\{ \frac{\lambda_i^e}{\lambda_i^e - \lambda_j^e} \left( \frac{n_{ij} + n_{ji}^e}{\lambda_i^e} \right) + \frac{\lambda_j^e}{\lambda_i^e} \left( \frac{n_{ji} + n_{ij}^e}{\lambda_j^e} \right) \right\}. \quad (2.27)$$

The rate of deformation tensor $D'$ which is a symmetric part of $L'$ can be obtained as follows:

$$D' = D \cdot G^{(4)}_D D \text{ or } D' = T^{(4)}_e D. \quad (2.28)$$

Mathematically, $T^{(4)}_e$ is the 4th order transformation tensor which converts the rate of deformation gradient tensor $D^e$ to the rate of deformation gradient tensor $D'$. 

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2.4.3 A thermodynamics approach to visco-hyper elastic modeling with softening

Suppose that a material point is characterized by the Helmholtz free energy $\psi$, entropy $\eta$, Cauchy stress tensor $\sigma$, temperature gradient $g$, the heat flux vector $q$ for given temperature $\theta$, internal variable tensor $s$, softening parameter $h$ and the right Cauchy-Green metric tensor $C'$ in the intermediate configuration. According to thermodynamic theory [2.25], a general constitutive relation is described as follows:

$$\psi = \hat{\psi}(C', h, s, \theta); \quad \sigma = \hat{\sigma}(C', h, s, \theta)$$  \hspace{1cm} (2.29a,b)

$$q = \hat{q}(C', h, s, g, \theta); \quad \eta = \hat{\eta}(C', h, s, \theta)$$  \hspace{1cm} (2.30a,b)

The 2nd law of thermodynamics, which is called a Clausius-Duhem inequality is described in the spatial description:

$$\rho \eta \geq \rho \frac{r}{\theta} - \text{div}(q)$$  \hspace{1cm} (2.31)

where $r$ is a specific heat supply. The energy conservation equation in a spatial description is expressed as follows:

$$\rho \dot{e} = \sigma \cdot D^e + \rho r - \text{div}q$$  \hspace{1cm} (2.32)

where $e$ is internal energy. Also, internal energy is related to the Helmholtz free energy and entropy by a Legendre transformation:

$$\psi = e - \theta \eta.$$  \hspace{1cm} (2.33)

The inequality in Equation (2.31) reflects energy dissipation and is rewritten by substituting Equation (2.32) and (2.33) into Equation (2.31):
The second term in Equation (2.34) is rewritten as follows by differentiating Equation (2.21):

\[ \hat{\mathbf{a}} \cdot \mathbf{L} = \hat{\mathbf{a}} \cdot \mathbf{L}' + \hat{\mathbf{a}} \cdot F' \mathbf{L} \mathbf{F}^{-1}. \]  

(2.35)

Also, the second term in Equation (2.35) is related to the material softening and simplified as follows:

\[ \hat{\mathbf{a}} \cdot F' \mathbf{L} \mathbf{F}^{-1} = \left( \frac{1}{h} - 1 \right) \hat{\mathbf{a}} \cdot \mathbf{L}' - \left( \frac{1}{h^2} \hat{\mathbf{a}} - \frac{1}{h} \right) \sum_{i=1}^{3} \ln(\lambda_i') \mathbf{n}_i \otimes \mathbf{n}_i \mathbf{h} \mathbf{.} \]  

(2.36)

Therefore, the inequality in Equation (2.34) is rewritten by substituting Equation (2.35) and (2.36) into Equation (2.34):

\[ -\rho \left( \frac{\partial \hat{\mathbf{w}}}{\partial \theta} + \hat{\mathbf{h}} \right) \hat{\mathbf{a}} + \left( \frac{\hat{\mathbf{a}}}{h} - 2\rho F' \frac{\partial \hat{\mathbf{w}}}{\partial \mathbf{C}'} \mathbf{F}' \mathbf{F}' \right) \cdot \mathbf{L}' - \left( \rho \frac{\partial \hat{\mathbf{w}}}{\partial \mathbf{h}} + \frac{1}{h^2} \hat{\mathbf{a}} - \frac{1}{h} \right)^{\sum_{i=1}^{3} \ln(\lambda_i') \mathbf{n}_i \otimes \mathbf{n}_i \mathbf{h} - \rho \frac{\partial \hat{\mathbf{w}}}{\partial \mathbf{s}} \cdot \mathbf{s} - \frac{1}{\theta} \hat{\mathbf{q}} \cdot \mathbf{g} \geq 0 \right. \]  

(2.37)

To satisfy the inequality in Equation (2.37) in all thermo-mechanical processes, the following relation should hold:

\[ \hat{\mathbf{a}} = 2h \rho F' \frac{\partial \hat{\mathbf{w}}}{\partial \mathbf{C}'} \mathbf{F}' \mathbf{F}' ; \quad \hat{\mathbf{h}} = -\frac{\partial \hat{\mathbf{w}}}{\partial \theta} \mathbf{.} \]  

(2.38a,b)

The total dissipative energy power in Equation (2.37) is expressed as follows:

\[ -\left( \rho \frac{\partial \hat{\mathbf{w}}}{\partial \mathbf{h}} + \frac{1}{h^2} \hat{\mathbf{a}} - \frac{1}{h} \sum_{i=1}^{3} \ln(\lambda_i') \mathbf{n}_i \otimes \mathbf{n}_i \right) \mathbf{h} - \rho \frac{\partial \hat{\mathbf{w}}}{\partial \mathbf{s}} \cdot \mathbf{s} - \frac{1}{\theta} \hat{\mathbf{q}} \cdot \mathbf{g} \geq 0 \].  

(2.39)

If dissipative energy power due to material softening and viscosity are decoupled from each other, Equation (2.39) can be separated as follows:
In Equation (2.40), the term in parenthesis can be proved to be always greater than or equal to zero. Therefore, to satisfy the inequality, \( h \) should be less than or equal to zero. This case represents material softening which is very similar to material damage.

To develop the visco-elastic theory, let the Helmholtz free energy \( \psi \) be defined as follows:

\[
\psi = \psi^0(C', h, \theta) - s \cdot C'
\]  

(2.42)

where \( \psi^0 \) is an initial elastic stored energy function and \( s \) is internal variable tensor. Then, the stress can be represented in terms of \( \psi^0 \) and \( s \):

\[
\sigma = 2hpF' \frac{\partial \psi}{\partial C'} F'^T = 2hpF' \frac{\partial \psi^0}{\partial C'} F'^T - 2hpF' s F'^T.
\]  

(2.43)

Let the second term in Equation (2.43) be defined as \( \bar{s} \):

\[
\bar{s}(t) = 2phF'^s F'^T.
\]  

(2.44)

Then, stress is decomposed into initial stress and non-equilibrium stress:

\[
\sigma = \sigma^0 - \bar{s}.
\]  

(2.45)

\( \bar{s} \) is a tensor governed by the following evolution equation:

\[
\frac{d \bar{s}}{dt} + \frac{1}{\tau_R} \bar{s} = \frac{1 - \xi}{\tau_R} \sigma^0, \quad \bar{s} = 0
\]  

(2.46)

where \( \xi \in [0,1] \) is a given parameter. The Clausius-Duhem inequality leads to the following restriction:
\[-\rho \frac{\partial \dot{\psi}}{\partial s} \cdot \dot{s} = -\rho \frac{\partial \dot{\psi}}{\partial s} \cdot \left( \frac{1}{2\rho} \mathbf{F}^{\prime\prime} : \mathbf{F}^{\prime\prime\top} \right) \geq 0. \tag{2.47} \]

The Helmholtz free energy \( \dot{\psi} \), defined as energy per unit mass, can be converted to energy per unit reference volume for convenience and called strain energy \( W' \):

\[ W' = \rho_{R2} \dot{\psi} \tag{2.48} \]

where \( \rho_{R2} \) is a density in the equivalent intermediate configuration \( \beta_2 \) in Figure 2.9.

Therefore, the \( 2 \text{nd} \) Piola Kirchhoff stress is obtained from Equation (2.39) and the strain energy function becomes only a function of \( C' \):

\[ S = 2h \frac{\partial W'(C')}{\partial C'} = h \frac{\partial W'}{\partial E'}. \tag{2.49} \]

If the energy function \( W' \) is defined as

\[ W' = \frac{\dot{W}}{h} \tag{2.50} \]

then the Cauchy stress can be written as follows:

\[ \dot{\sigma} = 2 \frac{1}{J'} \mathbf{F}' \frac{\partial \dot{W}}{\partial C'} \mathbf{F}'^\top. \tag{2.51} \]

Therefore, when there is material softening, Equation (2.51) is used for calculating stress for a given energy function \( \dot{W} \).

2.5 VISCO-HYPER-ELASTO-PLASTIC MODEL DEVELOPMENT

In this section, based on the theory for visco-hyper-elastic material with softening, the detailed procedure for developing the visco-hyper-elasto-plastic model is introduced.
The theoretical development starts with a rate independent plastic model and then expands to a model that can handle rate effects during the loading process.

2.5.1 Assumptions for the rate independent plastic model and its material behavior

If it is assumed that the thickness dimension is much smaller than the lateral direction and most of the deformation takes place in the thickness direction, then a 1-D plasticity model can be introduced in the thickness direction. The loading stress curve can be regarded as the yield stress function and the unloading stress curve can be regarded as the elastic function. Several assumptions are summarized as follows:

- The material has plastic yield only in the thickness direction.
- Yielding only takes under compressive stress.
- The yield stress function $y_-$ is a function of mechanical strain.
- The unloading stress function $\sigma_-$ is a function of elastic strain.

In 1-D plasticity, the yield function is simply defined as follows:

$$ F = e_3 \cdot \sigma_- e_3 - e_3 \cdot y_- e_3 = 0 \quad (2.52) $$

where $e_3$ is a unit directional vector in the current thickness direction and is based on the mechanical deformation gradient tensor. The yield function in Equation (2.52) is designed to account for plasticity in the thickness direction. The physical meaning of the yield function is that stress in the material thickness direction should be the same as the
yield stress when yield occurs. If yield is assumed to occur only when the compressive stress is higher than the compressive yield stress, then the yield criterion can be written as follows:

\[ e_3 \cdot \sigma_y \leq e_3 \cdot \gamma \leq 0. \]  \hspace{1cm} (2.53)

Figure 2.10 shows the 1-D plasticity material behavior in the thickness direction during mechanical loading and unloading. Suppose that a mechanical strain \( e^m \) is applied to the material. The initial elastic unloading function is shown as a black dotted line and the yield stress function is shown as a black solid line. Because of the assumption that the softening parameter \( h \) decreases as mechanical strain increases, compressive strain increases, the elastic function begins to soften, and becomes the gray dotted line.

For a given mechanical strain \( e^m \), stress is evaluated from the gray dotted elastic function which is denoted as the trial stress, and the yield stress is calculated from the yield stress function which is shown as point 1. In this situation, if the magnitude of the trial stress is higher than the magnitude of the yield stress, then this belongs to the yield condition and the trial stress must be modified to satisfy the yield function. Therefore, to satisfy the yield function, the gray dotted line should be shifted to the gray solid line by plastic strain. If the mechanical loading is removed from point 1 after the yield condition is satisfied, then the yield stress becomes higher than the trial stress. In this case, yield does not occur, the trial stress becomes the stress for a given strain, and the gray solid line becomes an unloading curve. Point 2 is a permanent plastic strain when the load is completely removed.
2.5.2 Plastic deformation gradient tensor $F^p$ and yield conditions

In large deformation theory, the mechanical deformation gradient tensor can be multiplicatively decomposed into elastic and plastic parts. This idea was first introduced by Lee [2.14] and can be written as follow:

$$F^m = F^e F^p$$

(2.54)

where $F^m$ is the deformation gradient tensor due to total mechanical strain, $F^e$ is due to elastic strain and $F^p$ is due to plastic strain. It is assumed that material rigid body rotation only comes from the elastic deformation gradient tensor and plastic deformation does not contribute any material rotation during the mechanical deformation. Then, the plastic deformation gradient tensor can be simplified as a right stretch tensor:

$$F^p = R^p U^p = U^p.$$  \hspace{1cm} (2.55)

Since plastic deformation is assumed to take place only in the material thickness direction, there is only one component in the right stretch tensor, and it can be written as follows:

$$U^p = l E_1 \otimes E_1 + l E_2 \otimes E_2 + \lambda^p E_3 \otimes E_3$$  \hspace{1cm} (2.56)

where $\lambda^p$ is a plastic stretch ratio in the thickness direction ($E_3$) and $E_1$, $E_2$ and $E_3$ are the material coordinate directions.

If there is a plastic strain during the deformation, a plastic deformation gradient tensor is created to satisfy the yield function (2.52). In this case, $\lambda^p$ begins to decrease. Therefore, the plastic loading and unloading conditions are defined with $\lambda^p$ as follows:
Loading condition \( \dot{\lambda}^p < 0 \) and \( \lambda^p < 1 \)

Unloading condition \( \dot{\lambda}^p = 0 \).

2.5.3 Definition of the loading stress function \( y_\ast \) and the unloading stress function \( \sigma_\ast \)

Figure 2.11 shows the 4 different configurations, which are needed to define the elastic function and the loading function when there is a material softening and plastic strain. For a given mechanical deformation \( F^m \), the material reference configuration \( \beta_0 \) first changes to the intermediate configuration \( \beta_1 \) due a plastic deformation. Also, if there is a material softening, then the intermediate configuration \( \beta_1 \) changes to the intermediate configuration \( \beta_2 \). Therefore, the unloading stress function can be calculated from \( F^\prime \).

The loading stress \( y_\ast \) is defined between the current configuration \( \beta \) and the reference configuration \( \beta_1 \) because it is a function of mechanical strain. The loading function is assumed to be derived from the strain energy potential for a given deformation gradient tensor \( F^m \) as follows:

\[
y_\ast = \frac{1}{f^m} \sum_{i=1}^{\text{dim}} \lambda_{1i}^m \frac{\partial W_e(\lambda_{1i}^m, \lambda_{2i}^m, \lambda_{3i}^m)}{\partial \lambda_{1i}^m} (n_i \otimes n_i). \tag{2.57}
\]

In this case, material constants can be obtained directly from a uniaxial compression test. The unloading stress function \( \sigma_\ast \) is an elastic function and is defined between the current configuration \( \beta \) and the intermediate configuration \( \beta_2 \). Cauchy stress is obtained by using the principal stretches and principal directions as follows:
2.5.4 The relation between $D^m$ and $D^e$ and the transformation tensor $T_{em}^{(4)}$

The relation between $L^m$ and $L^e$ can be obtained by differentiating Equation (2.54) with respect to time:

$$L^e = L^m - \frac{\dot{\lambda}^p}{\lambda^p} M_L$$  \hspace{1cm} (2.59)

where

$$M_L = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\lambda^e}{\lambda_j} (N_i^1 \cdot E_3)(N_j^1 \cdot E_3)n_i \otimes n_j.$$  \hspace{1cm} (2.60)

The rate of deformation gradient tensor $D^e$, which is a symmetric part of $L^e$, can be obtained as follows:

$$D^e = D^m - \frac{\dot{\lambda}^p}{\lambda^p} M_D$$  \hspace{1cm} (2.61)

where $M_D$ is a symmetric part of $M_L$. However, the rate of plastic stretch $\dot{\lambda}^p$ still remains an unknown variable and must be obtained from the yield condition. Therefore, it is necessary to calculate $\dot{\lambda}^p$ and represent Equation (2.61) with only the known tensor $D^m$.

$\dot{\lambda}^p$ can be obtained by taking a time derivative of the yield function. The time derivative of the yield function results in following equation:

$$\ddot{e}_3 \cdot \sigma_3 e_3 - \ddot{e}_3 \cdot y_3 e_3 = B_1 \dot{\lambda}^p + B_2 = 0$$  \hspace{1cm} (2.62)

where constant $B_1$ and $B_2$ are obtained as:
\[
B_1 = \frac{1}{\lambda^p} \left\{ (T_{\text{ce}}^{(4)} M_0 \cdot I)(e_3 \cdot \sigma_\text{s} e_3) - \frac{1}{J^e} e_3 \cdot A^{(4)} T_{\text{ce}}^{(4)} M_0 e_3 - 2e_3 \cdot (M_L - G_{L}^{(4)} M_L) \sigma_\text{s} e_3 \right\} (2.63)
\]

\[
B_2 = 4e_3 \cdot (\sigma_\text{s} - y_\text{s}) D^m e_3 - (T_{\text{ce}}^{(4)} D^m \cdot I)(e_3 \cdot \sigma_\text{s} e_3) + \frac{1}{J^e} e_3 \cdot A^{(4)} T_{\text{ce}}^{(4)} D^m e_3 + (D^m \cdot I)(e_3 \cdot y_\text{s} e_3) - \frac{1}{J^m} e_3 \cdot A^{(4)} D^m e_3 - 2e_3 \cdot G_{L}^{(4)} D^m \sigma_\text{s} e_3 + 2e_3 \cdot D^m y_\text{s} e_3
\]

with \( A^{(4)} = F^m \bar{Y} F^m \), \( A^{(4)} = F^e \bar{S} F^e \), and \( Y \) as the 2nd Piola Kirchhoff stress for the loading function. Therefore, \( \lambda^p \) can be substituted into Equation (2.61) and \( D^e \) is obtained with respect to \( D^m \):

\[
D^e = D^m - \frac{1}{B_1 \lambda^p} B_2 D^m.
\] (2.65)

To simplify Equation (2.65), the 4th order transformation tensor \( T_{\text{em}}^{(4)} \) can be introduced as follows:

\[
D^e = (I^{(4)} - \frac{1}{B_1 \lambda^p} B_2) D^m = T_{\text{em}}^{(4)} D^m.
\] (2.66)

Mathematically, \( T_{\text{em}}^{(4)} \) is the 4th order transformation tensor which converts the rate of mechanical deformation gradient tensor \( D^m \) to the rate of elastic deformation gradient tensor \( D^e \).

2.5.5 Including rate effects in the plastic model

From the experimental results, it is seen that there are two kinds of rate effects in intumescent mat material. One is a rate effect in the plastic region during the loading process and the other is a rate effect in the elastic region during the unloading process. In
this chapter, to simplify the plastic formulation with rate effects for the 1-D plastic
model, it is assumed that the plastic function does not have rate effects. Only the elastic
function has the two different rate dependant material properties. In the case of the elastic
function, one viscous property is applicable only when plastic deformation occurs and the
other applies only under elastic deformation. Therefore, conditions for different material
properties should be determined from the static yield condition in Equation (2.52). Based
on the yield criterion, two different material properties are used in each strain region.
That is, visco-elastic material properties obtained from the loading function are used
when there is plastic deformation. When there is no plastic deformation, then the visco-
elastic material properties are switched to properties obtained from the unloading
function. Final stress is obtained by adding a rate term into the static term. Table 2.2
summarizes the two cases for the visco-hyper-elasto-plastic model.

Because mat material only works in compression, in applying visco-elastic
material properties, the loading function is assumed to have no rate effects when static
stress in thickness direction is positive.

2.5.6 The spatial elasticity tensor

If stress is defined between the configurations \( \beta \) and \( \beta_2 \) shown in Figure 2.11,
due to material softening, the Jaumman rate of Cauchy stress can be written as follows:

\[
\dot{\sigma} = E^{(4)} \dot{D}^r
\]  

(2.67)
where $E^{(4)}$ is a spatial elasticity tensor. It is necessary to convert the spatial elasticity tensor defined between configuration $\beta$ and $\beta_2$ to the configuration between $\beta$ and $\beta_1$. By using the relation (2.28), Equation (2.67) can be written with respect to $D^e$:

$$
\dot{\sigma} = E^{(4)}T_{\text{re}}^{(4)}D^e.
$$

(2.68)

If the spatial elasticity tensor is defined between configuration $\beta$ and $\beta_0$, the following relation is obtained through Equation (2.66):

$$
\dot{\sigma} = E^{(4)}T_{\text{re}}^{(4)}T_{\text{em}}^{(4)}D^m.
$$

(2.69)

Also, if there is a rate effect, then by using Equation (2.15), $E^{(4)}_{\text{total}}$ becomes the following:

$$
E^{(4)}_{\text{total}} = \alpha \gamma E^{(4)}T_{\text{re}}^{(4)}T_{\text{em}}^{(4)}.
$$

(2.70)

If there is no plastic strain, then $\dot{\lambda}^p$ is zero and $T_{\text{em}}^{(4)}$ becomes the Identity tensor in Equation (2.70).

2.5.7 The Numerical procedure to calculate the plastic stretch ratio and the stress update

When the mechanical deformation gradient tensor is given at each time increment, it is necessary to decompose the mechanical strain into elastic strain and plastic strain by satisfying the yield function. The plastic stretch ratio $\lambda^p$ can be calculated by the Newton Raphson method with the condition $D^m = 0$ at each time increment. Suppose that, an approximation solution $\Delta \lambda^p$ has been obtained after a time step $n$. Let $\Delta \lambda^p$ be the
difference between this solution and the exact solution at time step $n+1$. Then the following function must be satisfied at time step $n+1$:

$$F(\lambda^n + \Delta\lambda^{n+1}) = 0. \quad (2.71)$$

Performing a Taylor series expansion of Equation (2.71) and neglecting high order terms results in the following relation:

$$\Delta\lambda^{n+1} = \frac{-F}{\frac{\partial F}{\partial \lambda^p}}. \quad (2.72)$$

Therefore, $\lambda^p$ at time step $n+1$ can be updated in the following way until $F$ goes zero:

$$\lambda^{p,n+1} = \lambda^n + \Delta\lambda^{n+1}. \quad (2.73)$$

The elastic trial stress $\sigma_\varepsilon$ at time step $n+1$ can be calculated from the elastic deformation gradient tensor by using the previous plastic deformation gradient tensor at time step $n$. The procedure is as follows:

i) Assume the elastic deformation gradient tensor to be $\hat{F}^{e,n+1} = F^{m,n+1} F^{p-1,n}$ with $\lambda^p$ at the previous time step $n$.

ii) Calculate the elastic trial stress $\sigma_\varepsilon$ by Equation (2.58).

iii) Calculate the yield stress $\sigma_\text{y}$ by Equation (2.57).

iv) Check the yield condition by Equation (2.52).

If yield occurs,

i) Update $F^p$ by the Newton Raphson method until the yield condition Equation (2.52) is satisfied.
ii) Calculate $\sigma_p$ by Equation (2.58).

iii) Calculate $\sigma_{vis}^{p}$ by using Equation (2.14) with $a = a_p$ and $\tau = \tau_p$.

iv) Write the stress as $\sigma = \sigma_p + \sigma_{vis}^p$.

If no yield occurs,

i) Calculate $\sigma_{vis}^e$ by using Equation (2.14) with $a = a^e$ and $\tau = \tau^e$.

ii) Write the stress as $\sigma = \sigma^e + \sigma_{vis}^e$.

2.6 MODEL PARAMETERS AND COMPARISONS WITH EXPERIMENTS

In this chapter, the Ogden strain energy function [2.6] is used as a base function for both the loading and unloading functions. The loading function, $W_L$ in Equation (2.57) and the unloading function, $W_U$ in Equation (2.58) are defined as follows.

$$W_L(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu L}{a_L^2} \left[ \lambda_1^{\alpha u} + \lambda_2^{\alpha u} + \lambda_3^{\alpha u} - 3 + \frac{1}{\beta_L} (J^{\alpha u} - 1) \right] \quad (2.74)$$

$$W_U(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu U}{a_u^2} \left[ \lambda_1^{\alpha u} + \lambda_2^{\alpha u} + \lambda_3^{\alpha u} - 3 + \frac{1}{\beta_U} (J^{\alpha u} - 1) \right] \quad (2.75)$$

where $J$ is Jacobean of the deformation gradient tensor, $\lambda$ are the principal stretches, and $\alpha$, $\mu$, and $\beta$ are material parameters to be determined from experiments. The constitutive model is coded in the ABAQUS™ user subroutine called UMAT. Results from simulations are compared with several experimental results. A single 8 node solid
element is used to simulate the constitutive model for comparison with the single compression test and cyclic compression test. Figure 2.12 shows element node connectivity and the material thickness direction. The size of element is 10mm*10mm*10mm. In the case of a single compression test for different compression speeds, bottom surface nodes of the element are fixed in all direction and compression is applied to the top surface nodes of the element in the Z-direction. Different compression speeds are used to compare with experimental results.

In the case of cyclic compression testing, if the mat response is very slow compared to the machine response and if there is plastic deformation during the loading process, the mat can be detached from the loading rod during the displacement unloading. This effect must be included in the simulations by employing a contact surface between loading rod and top surface of the mat. Figure 2.13 shows a schematic diagram for the cyclic compression simulation. Cyclic displacement is applied to the rigid plate in the Z direction. Several cycles of displacement loading and unloading are applied to the rigid plate and simulation parameters are shown in Figure 2.2 and Table 2.1.

2.6.1 Determination of material parameters

In the proposed visco-hyper-elasto-plastic model, two different loading and unloading functions are defined. As explained in Section 2.5.7, the loading function has only a static term and the unloading function has a static and a transient term. There are
two material parameters in the static term and two material parameters to define the transient term.

In the case of the loading function, material parameters are determined from the static stress versus strain curve shown in Figure 2.3. Material parameter $\alpha_L$ and $\mu_L$ are obtained from non-linear curve fitting of the static curve. The subscript L indicates loading function.

In the case of the unloading function, each static unloading or the 2nd loading function at different mechanical strains has a different strain at zero stress and a different slope as shown in Figure 2.6. Therefore, to obtain material parameters for the unloading function, the static unloading function should be defined first from Figure 2.6. From the idea of material similarity for each unloading function, all unloading curves can be represented as one master curve by changing the softening parameter $h$ and by assuming zero strain point at zero stress. Figure 2.14 shows a master curve and the unloading comes from different mechanical strains with material softening removed. Table 2.3 shows softening parameters for several mechanical strains. As shown in Table 2.3, the softening parameter $h$ decreases as mechanical strain increases. From the master curve shown in Figure 2.14, the static material parameter $\alpha_U$ and $\mu_U$ are determined. Subscript $U$ indicates unloading function. In the elastic region, transient material parameters $a^e$ and $\tau^e$ are obtained from both the relaxation curve at the end of the 1st loading and the rate dependant 2nd loading curves shown in Figure 2.8. Superscript $e$ indicates elastic region. However, in the case of the plastic deformation region, transient material parameter $a^p$
and $\tau_s^p$ are obtained from the transient stress versus strain curve in Figure 2.6 with the static loading function. Superscript $p$ indicates plastic region.

2.6.2 Simulation results

From the material parameters obtained in Section 2.6.1, simulations are compared with the compression experimental results for several cycles. Figure 2.15 shows a comparison of the rate effects between the experiment and simulation during the 1st loading process up to 60% compression. It is seen that there is good agreement between experiment and simulation at each compression speed in all strain ranges except the small strain region. In the small strain region, the simulations show a higher stress than the experimental results, especially as ram speed increases. It appears that the mat material has different rate effects at different compressive strain levels.

Figure 2.16 shows a comparison between the static cyclic compression test shown in Figure 2.6 and the simulation. Simulation results show good agreement for different unloading curves. Figure 2.17 shows a comparison between experiment and simulation for the cyclic loading and unloading curves when 60% compression is applied with 1.6mm/s ram speed during two cycles. The loading and unloading experimental curves for the 1st and 2nd cycle have good agreement with the simulation results. Figure 2.18 shows a simulation result for 50% compression with 1.6mm/s ram speed. During the 1st loading, stress from the simulation is a little higher than the experiments. However, in most strain ranges, satisfactory results are obtained for the 1st and 2nd cycle.
Corresponding to the conditions listed in Table 2.1, Figure 2.19 shows stress variation with respect to time for 60% compression with 1.6mm/s ram speed. For two cycles, the figure shows the stress jump during loading and relaxation during unloading. The peak value at each cycle and the stress relaxation with respect to time are very well represented by the simulation. However, stress relaxation from the 2nd cycle shows some discrepancy between experiment and analysis. Figure 2.20 shows a comparison of peak stresses versus number of cycles from compression test when 50% and 60% strain is applied under 1.6mm/s ram speed. For both cases, good agreement is seen through the 2nd cycle but some deviation is shown from the 3rd cycle. Simulation results show a constant stress level after the 3rd cycle but experimental results show gradual stress decay. This may be due to cyclic material damage after plastic deformation. Therefore, incorporation of cyclic material damage is believed to be necessary for better simulation of longer term cycling.

2.7 CONCLUDING REMARKS

This chapter presents detail development of a constitutive model for intumescent mat. By using hyper-elastic theory, the model incorporates rate effects, plasticity and material softening during cyclic loading process. Research is limited to situation, which have a small dimension in the thickness direction compared to the lateral direction and the dominant deformation is in the thickness direction at room temperature. Several experiments for intumescent mat are performed and results are reviewed. A new material softening model is proposed from the assumption of the material similarity. The softening
parameter is chosen as a function of mechanical strain in the thickness direction. In the case of the 1-D plastic model, new plastic deformation gradient tensor and yield criteria in the thickness direction are proposed. Also, the plastic yield stress function is established on the basis of mechanical strain. Two different rate effects in the loading and unloading functions are implemented by employing different viscosity parameter for the plastic and elastic unloading regions.

The complete spatial elastic tensor in the Eulerian frame is derived for implicit finite element code and the Newton-Raphson method is applied to obtain plastic strain. The developed theory is coded in ABAQUS™ implicit by using the user subroutine UMAT. Experimental results and simulation are compared to each other for several test cases and satisfactory results are achieved with the new model for up to two cycles.

This research shows that further study is still required to predict accurate loading and unloading processes for many cycles. It is believed that cyclic damage effects after the 2nd loading and different viscous effects at different compressive strain should be considered in the future. In addition, future versions of the model need to be able to handle mechanical behavior at elevated temperatures.

REFERENCES


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<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
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<tr>
<td>$D_1/T_1$</td>
<td>0.005 mm/s</td>
<td>0.4 mm/s</td>
<td>1.6 mm/s</td>
</tr>
<tr>
<td>$T_2$</td>
<td>125.0 s</td>
<td>125.0 s</td>
<td>125.0 s</td>
</tr>
<tr>
<td>$T_3$</td>
<td>10.0 s</td>
<td>10.0 s</td>
<td>10.0 s</td>
</tr>
<tr>
<td>$D_1$</td>
<td>3,4,5,6,7 mm</td>
<td>3,4,5,6,7 mm</td>
<td>3,4,5,6,7 mm</td>
</tr>
</tbody>
</table>

Table 2.1: Test conditions for cyclic compression experiments.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Plastic region</th>
<th>Elastic region</th>
</tr>
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<tbody>
<tr>
<td>$\dot{\lambda}^p &lt; 0$ and $\lambda^p &lt; 1$</td>
<td>$\dot{\lambda}^p = 0$</td>
<td></td>
</tr>
<tr>
<td>Viscoelastic material property</td>
<td>$a = a_a^p$ and $\tau = \tau_a^p$</td>
<td>$a = a_a^e$ and $\tau = \tau_a^e$</td>
</tr>
<tr>
<td>Stress</td>
<td>$\sigma = \sigma_w + \sigma_{vis}$</td>
<td>$\sigma = \sigma_w + \sigma_{vis}$</td>
</tr>
</tbody>
</table>

Table 2.2: The Stress calculation procedure for the visco-hyper-elasto-plastic model at room temperature.

<table>
<thead>
<tr>
<th>Mechanical strain</th>
<th>0.8</th>
<th>0.9</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Softening parameter $h$</td>
<td>1</td>
<td>0.98</td>
<td>0.92</td>
</tr>
<tr>
<td>Zero strain</td>
<td>0.532</td>
<td>0.6</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Table 2.3: Softening parameter $h$ for several applied mechanical strains for unloading curves.
Figure 2.1: A schematic diagram of static cyclic compression at room temperature.

Figure 2.2: The displacement loading profile for cyclic compression experiments.
Figure 2.3: Stress versus strain curves for different compression speed from compression experiments at room temperature (60% compression).

Figure 2.4: Cyclic compression experimental results at room temperature for 40% compression ($v=1.6\text{mm/s}$).
Figure 2.5: Peak stress variations versus the number of cycles for different compression speeds at room temperature (40% compression).

Figure 2.6: Comparison of monotonic static loading and cyclic loading from constant load experiments at room temperature.
Figure 2.7: 1st cycle compression results for different maximum compressive strains at room temperature.

Figure 2.8: Comparison of the 1st and the 2nd loading curves for different maximum compressive strains at room temperature (v=1.6mm/s).
Current Configuration $\beta$
Position vector $x$

Reference Configuration $\beta_1$
Position vector $x^1$

Intermediate Configuration $\beta_2$
Position vector $x^2$

Figure 2.9: Configuration change due to material softening.

Figure 2.10: Material behavior of 1-D plasticity at room temperature.
Figure 2.11: Configuration change due to plastic strain and material softening at room temperature.

Figure 2.12: The one-element model for constitutive model verification.
Figure 2.13: A schematic diagram of the cyclic compression simulation.

Figure 2.14: The master curve and unloading test data at different compressive stresses with material softening removed.
Figure 2.15: Comparison of simulation (analysis) results with test data at different compression speeds for the 1st loading process (60% compression).

Figure 2.16: Comparison of the simulation and experimental static cyclic loading and unloading curve.
Figure 2.17: Simulation (analysis) comparison with the 60% cyclic compression test during 2 cycles at room temperature (v=1.6mm/s).

Figure 2.18: Comparison of the 50% cyclic compression test during 2 cycles at room temperature (v=1.6mm/s).
Figure 2.19: Comparison of stress versus time from the 60% compression test during 2 cycles (v=1.6mm/s).

Figure 2.20: Comparison of peak stress versus number of cycles (v=1.6mm/s).
CHAPTER 3

CONSTITUTIVE MODEL DEVELOPMENT FOR A THERMO-MECHANICALLY LOADED ANISOTROPIC HYPER-ELASTIC MATERIAL

ABSTRACT

This chapter presents constitutive model development for an anisotropic hyperelastic material undergoing thermo-mechanical process. From the basic idea that the material response at different temperatures is self-similar, the concept of material softening, which can handle material property changes due to elevated temperatures, is established for a linear elastic material. The material softening concept is then extended to large deformation theory by employing a proposed equivalent strain and through the stiffness method. To account for material property changes due to temperature, a new softening model is proposed for anisotropic hyperelastic material. For implementation in implicit finite element code, the complete stress function and the spatial elasticity tensor in the Eulerian frame are derived based on the 2nd law of the thermodynamics. This chapter proposes a new temperature dependent strain energy function for isotropic hyperfoam material. Also, the softening model is applied to an incompressible anisotropic hyperelastic material under thermo-mechanical loading.
3.1 INTRODUCTION

Cyclic mechanical loading or thermo-mechanical loading are very common boundary conditions in many engineering applications. The thermal loading aspects usually result in a body with non-isothermal temperature distributions due to either internal heat generation or heat conduction from outside. Generally, material properties of bodies are affected by body temperature. In some special cases, small temperature changes in a body lead to the significant changes in material properties and mechanical response. Therefore, it is very important to develop a constitutive model which can handle thermo-mechanical loading and to accurately predict stress response.

Different kinds of material behavior can be categorized in developing a constitutive model for thermo-mechanical loading. That is, isotropic material at room temperature can either be isotropic or anisotropic at elevated temperature. Anisotropic material at room temperature can have either the same or different material anisotropic directions at elevated temperature. If the material directions do not change after the thermal loading, the situation is limited to isotropic material softening. If the material directions do change after the heating, the situation is called anisotropic material softening. In this chapter, study is limited to only isotropic material softening for isotropic and anisotropic materials.

Theories for isothermal non-linear constitutive model development have been well established since the 1950s [3.6, 3.11, 3.12, 3.13]. In cases of isotropic hyper-elastic models, Mooney-Rivlin [3.11] proposed a general strain energy function, in which strain
invariants and integer power series in the strain invariants were used. The Mooney-Rivin strain energy function is well known and widely used. However, the function requires many material constants to represent various types of non-linear stress functions. Ogden [3.9] has employed the principal values of strain in a strain energy functions and showed very good representation of experimental data with fewer material constants. For anisotropic hyper-elastic models, research has been done by Weiss [3.16], Bonet [3.17], Park et al. [3.10], Hoger [3.15] and so on. Most of the energy functions for anisotropic hyperelastic models have been based on strain invariants.

To date, constitutive models with different approaches were proposed to deal with temperature dependant material behavior under thermo-mechanical loading. However, not much research on thermo-mechanical modeling of non-linear materials has been done. One approach applies a linear interpolation method to calculate the unknown stress versus strain curve at arbitrary temperature from given stress curves at different temperatures. In this case, material properties at different temperature must be provided for the isothermal constitutive model. This method is applied in many commercial codes. However, if there are not enough curve sets at different temperatures, accurate result cannot be expected from this method. The second approach is to extrapolate the known isothermal constitutive model at certain temperature to the unknown constitutive model at arbitrary temperature by applying the laws of thermodynamics. This method was proposed by Holzapfel [3.4] and is very similar to a Taylor expansion of the constitutive model with respect to temperature. However, unless the thermodynamics variable is defined properly, this approach may not be suitable for wide temperature ranges. The
third approach develops the temperature dependent constitutive model directly by using both strain and temperature as independent variables. In this case, based on the idea that the material responses at different temperatures are self-similar, a stress function is generally assumed to be a simple multiplication of a strain dependant function and a temperature dependant function. The strain dependant function is obtained from a master stress versus strain curve at a reference temperature and the temperature dependant function is a scale factor. This method is called the equivalent stiffness method. This approach has been well applied to many material models, such as those for metals. The main advantage for this method is that sufficient solution accuracy can be guaranteed with few material constants. However, due to its simple formulation, this method has been used for only limited applications.

In this study, to overcome limited applications of the equivalent stiffness method, from the linear elastic material, a new concept of material softening is introduced. Then, the concept is extended to general large deformation theory by combining the proposed equivalent strain method and the equivalent stiffness method. In this modeling, by scaling both the stress and strain with one master stress versus strain curve at room temperature and a softening parameter as a function of temperature, it is possible to provide the complete stress function for all temperature ranges.

The study is limited to the development of the isotropic softening model for anisotropic materials during the thermo-mechanical processes. A new isotropic softening model and Helmholtz free energy function are proposed, and the stress function and the elasticity tensor are derived from the 2nd law of thermodynamics. As an example, to
investigate material behavior at elevated temperature, the isotropic softening model is applied to an incompressible anisotropic hyperelastic material.

3.2 CONTINUUM MECHANICS PRELIMINARIES

3.2.1 Basic kinematics definitions

Suppose that the position vector $x$ is defined in the current configuration $\beta$ and the position vector $X^0$ is defined in the reference configuration $\beta_0$. The line vector $dx$ in the configuration $\beta$ has a length $ds$ and a direction $n$. The line vector $dX^0$ in $\beta_0$ has a length $dS^0$ and a direction $N^0$. If $F$ is defined as a transformation tensor relating $dx$ and $dX^0$, then the following relation is obtained:

$$dx = FdX^0.$$  \hspace{1cm} (3.1)

When $dx$ and $dX^0$ are replaced with length and direction vectors, Equation (3.1) becomes:

$$dsn = FdS^0 N^0.$$  \hspace{1cm} (3.2)

The relation between the principal stretch $\lambda$ and $F$ is obtained as:

$$FN^0 = \frac{ds}{dS^0} n = \lambda n.$$  \hspace{1cm} (3.3)

The right Cauchy-Green metric tensor $C$ and the left Cauchy-Green metric tensor $B$ are defined as follows:

$$C = F^T F; \quad B = FF^T.$$  \hspace{1cm} (3.4a,b)
From the spectral theorem, \( C \) and \( B \) can be rewritten in terms of principal stretches and directions as follows:

\[
C = \sum_{i=1}^{3} \lambda_i^2 N_i^0 \otimes N_i^0; \quad B = \sum_{i=1}^{3} \lambda_i^2 n_i \otimes n_i.
\]  

(3.5a,b)

The Lagrangian strain tensor \( \varepsilon \) and the Eulerian strain tensor \( e \) are defined from the metric tensor \( C \) and \( B \) respectively:

\[
\varepsilon = \frac{1}{2} (C - I); \quad e = \frac{1}{2} (I - B^{-1}).
\]  

(3.6a,b)

The velocity gradient tensor \( L \) is defined as the gradient of the spatial description of the velocity \( v \) between the two configurations:

\[
L = \frac{\partial \mathbf{x}}{\partial \mathbf{x}}.
\]  

(3.7)

The time derivative of \( F \) can be related to \( L \) and \( F \) as follows:

\[
\dot{F} = \frac{\partial \mathbf{x}}{\partial x^q} = \frac{\partial \mathbf{x}}{\partial x} \frac{\partial x}{\partial x^q} = LF.
\]  

(3.8)

Also, the time derivatives of \( C \) can be described with respect to \( D \), which is the symmetric part of \( L \):

\[
\dot{C} = 2 \dot{\varepsilon} = 2 F^T DF.
\]  

(3.9)
3.2.2 Stress and the corresponding material and spatial elasticity tensors

In large deformation theory, the 2nd Piola Kirchhoff stress $S$ is defined in the reference configuration, and the Cauchy stress $\sigma$ is defined with respect to the current configuration. They are related to each other by the deformation gradient tensor $F$:

$$\tau = J \sigma = FSF^T$$

(3.10)

where $\tau$ is the Kirchhoff stress, and $J$ is the determinant of $F$. The Jaumann rate of Kirchhoff stress $\dot{\tau}$ is defined as follows:

$$\dot{\tau} = F\dot{S}F^T + \tau D + D\tau.$$  \hspace{1cm} (3.11)

The material elasticity tensor $D^{(4)}$ is defined in the reference configuration by differentiating the 2nd Piola Kirchhoff stress $S$ with respect to $E$ as follows:

$$D^{(4)} = \frac{\partial S}{\partial E}.$$  \hspace{1cm} (3.12)

In this study, for convenience, the spatial elasticity tensor $E^{(4)}$ is defined by relating the Jaumann rate of Cauchy stress and the rate of stretch tensor $D$:

$$\dot{\sigma} = E^{(4)}D.$$  \hspace{1cm} (3.13)

The elasticity tensor $E^{(4)}$ and $D^{(4)}$ can be related as follows:

$$E^{(4)} = -\sigma \otimes \frac{1}{J} P^{(4)} + H^{(4)}$$  \hspace{1cm} (3.14)

where $\otimes$ is the tensor outer product and the 4th order tensors $H^{(4)}$ and $P^{(4)}$ are:
3.2.3 Basic hyper-elastic model development for isothermal conditions

For isotropic materials, strain energy functions are commonly developed based on the principal stretch of strain or strain invariants. On the other hand, most of the strain energy functions for anisotropic materials are proposed based only on the strain invariants. In this section, stress is derived from energy functions with principal stretches of strain and/or strain invariants.

From the principal of virtual work, the internal energy variation is expressed as follows:

\[ \int \delta W \, dV_0 = \int S \cdot \delta \epsilon \, dV_0 = \int \tau \cdot \delta \epsilon \, dV \]  \hspace{1cm} (3.17)

where \( W \) is a strain energy function, \( V_0 \) is the reference volume and \( V \) is the current volume. If the strain energy function is given with principal stretches, the variation of the energy function \( W \) is expressed as follows:

\[ \delta W = \frac{\partial W}{\partial \lambda_1} \delta \lambda_1 + \frac{\partial W}{\partial \lambda_2} \delta \lambda_2 + \frac{\partial W}{\partial \lambda_3} \delta \lambda_3. \]  \hspace{1cm} (3.18)

The magnitude of the principal stretch \( \lambda \) is obtained from Equation (3.5):

\[ \lambda_i^2 = N_i^0 \cdot CN_i^0, \ i = 1,2,3 \]  \hspace{1cm} (3.19)

with \( \delta_{ij} \) as the Kronecker delta.
where \( N^0 \) is a principal direction in the reference configuration. Therefore, the variation of \( \lambda \) is obtained from Equation (3.19):

\[
\delta \lambda_i = \frac{1}{\lambda_i} N_i^0 \cdot \delta \varepsilon N_i^0, \quad i = 1, 2, 3.
\]  

(3.20)

From Equation (3.17), the 2nd Piola Kirchhoff stress and the Cauchy stress are derived as follows, respectively:

\[
S = \frac{1}{J} \sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i} (N_i^0 \otimes N_i^0); \quad \sigma = \frac{1}{J} \sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i} (n_i \otimes n_i).
\]

(3.21a,b)

The material elasticity tensor \( D^{(4)} \) defined in Equation (3.12) is obtained by differentiating \( S \) in Equation (3.21) with respect to time:

\[
D^{(4)} = \frac{1}{J} \sum_{a=1}^{3} \frac{1}{\lambda_a} \frac{\partial W}{\partial \lambda_a} N_{a\alpha}^0 + \frac{1}{J} \sum_{b=1}^{3} \frac{1}{\lambda_b} \frac{\partial W}{\partial \lambda_b} N_{b\beta}^0 + \sum_{i=12,13,23} \frac{1}{\lambda_i^2 - \lambda_j^2} \left( \frac{1}{\lambda_i} \frac{\partial W}{\partial \lambda_i} - \frac{1}{\lambda_j} \frac{\partial W}{\partial \lambda_j} \right) (N_{ij}^0 + N_{ij}^0 + N_{ij}^0 + N_{ij}^0)
\]

(3.22)

where \( N_{ijkl}^0 = N_i^0 \otimes N_j^0 \otimes N_k^0 \otimes N_l^0 \). The 4th order tensor \( P^{(4)} \) defined in Equation (3.16) is obtained by using Equation (3.22):

\[
P^{(4)} = \sum_{a=1}^{3} \lambda_a \frac{\partial W}{\partial \lambda_a} n_{a\alpha} + \sum_{b=1}^{3} \lambda_b \frac{\partial W}{\partial \lambda_b} n_{b\beta} + \sum_{i=12,13,23} \frac{\lambda_i^2 \lambda_j^2}{\lambda_i^2 - \lambda_j^2} \left( \frac{1}{\lambda_i} \frac{\partial W}{\partial \lambda_i} - \frac{1}{\lambda_j} \frac{\partial W}{\partial \lambda_j} \right) (n_{ij} + n_{ij} + n_{ij} + n_{ij})
\]

(3.23)

where \( n_{ijkl} = n_i \otimes n_j \otimes n_k \otimes n_l \).

If a strain energy function is represented with respect to strain invariants of the right Cauchy-Green metric tensor \( C \), the strain energy function for isotropic material is described by \( I_1, I_2 \) and \( I_3 \). Also, for anisotropic material, the strain energy function is
depends not only on the invariants $I_1$, $I_2$ and $I_3$ but also on the material direction vector $a$. In the case of transversely isotropic materials, the direction vector $a$ can be represented through of $I_4$ and $I_5$:

$$W = W(I_1, I_2, I_3, I_4, I_5)$$

(3.24)

where the invariants $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$ are defined as follows:

$$I_1 = \text{tr}(C); \quad I_2 = \frac{1}{2}((\text{tr}C)^2 - \text{tr}C^2); \quad I_3 = \text{det} C$$

(3.25a-c)

$$I_4 = a \cdot Ca; \quad I_5 = a \cdot C^2 a.$$  

(3.26a,b)

The $2^{nd}$ Piola Kirchhoff stress $S$ for transversely isotropic materials is obtained by differentiating the energy function with respect to $C$:

$$S = 2 \left\{ \frac{\partial W}{\partial I_1} I - \frac{\partial W}{\partial I_2} I - \frac{\partial W}{\partial I_3} I^{-1} C^{-1} + \frac{\partial W}{\partial I_4} a \otimes a + \frac{\partial W}{\partial I_5} (a \otimes Ca + aC \otimes a) \right\}. $$

(3.27)

The material elasticity tensor $D^{(4)}$ can be derived through the following equation:

$$S = 4 \frac{\partial^2 W}{\partial C^2} \bar{\varepsilon} = D^{(4)} \bar{\varepsilon}. $$

(3.28)

3.3 SOFTENING MODEL DEVELOPMENT FOR TEMPERATURE DEPENDANT MATERIALS

3.3.1 Motivation from linear elastic material models

In the case of a linear elastic material, the constitutive equation is represented by the Hooke's law:
\[ \sigma = \mathbf{E}^{(4)} \varepsilon \]  \hspace{1cm} (3.29)

where \( \sigma \) is stress, \( \varepsilon \) is strain and \( \mathbf{E}^{(4)} \) is the 4th order elastic constants tensor. If material properties change due to temperature, then the constitutive equation at elevated temperature can be simply represented as follows by using Equation (3.29):

\[ \sigma = h \mathbf{E}^{(4)} \varepsilon . \]  \hspace{1cm} (3.30)

Equation (3.30) is a simply multiplication of Equation (3.29) by a scale factor \( h \). In this case, Equation (3.29) provides the master curve which is a stress versus strain curve at the reference temperature, and \( h \) is called a softening parameter. Equation (3.30) provides a softening model. If \( h \) is given at specific temperatures, the stress versus strain curve at any temperature can be obtained through Equation (3.30). Therefore, one master curve with softening parameters at different temperatures can represent stress versus strain curves at different temperatures.

For the development of a softening model for non-linear materials, it is necessary to investigate two different aspects of Equation (3.30). One aspect related to the equivalent stiffness \( \mathbf{E}^{(4)} = h \mathbf{E}^{(4)} \) at elevated temperature. That is, the elastic modulus is modified to account for material property changes as temperature changes. This method is called an equivalent stiffness method. On the other hand, the equivalent strain \( \varepsilon' = h \varepsilon \) at elevated temperature can also be defined. In this case, the elastic modulus tensor \( \mathbf{E}^{(4)} \) does not change with temperature but strain is changed to the equivalent strain to give the same stress value. Therefore, it is only necessary to calculate the equivalent strain \( \varepsilon' \) at elevated temperature from the given actual strain \( \varepsilon \) at room temperature to obtain the same stress level. This method is an "equivalent strain method". In linear elasticity, there
is no difference between the two methods. However, there is a significant difference for non-linear elastic materials, such as, hyper-foam materials.

3.3.2 The assumption of material similarity for the non-linear stress function

It is assumed that, in most materials, the material response at different temperatures are self-similar, and its similarity is represented by either the equivalent strain or the equivalent stiffness method or their combination of the methods. Figure 3.1 and 3.2 show the two different types of material softening models introduced in Chapter 3.3.1 for hyper-foam material. A solid line shows a stress versus strain curve at room temperature and a dotted line shows a stress versus strain curve at elevated temperature. Suppose that a stress versus strain function at room temperature can be written as follows:

\[ \sigma = f(\varepsilon). \quad (3.31) \]

If the equivalent stiffness method is applied to Equation (3.31), a stress versus strain curve at elevated temperature is represented by scaling the stress with a scale factor \( h \) at constant strain:

\[ \sigma = hf(\varepsilon) = f'(\varepsilon). \quad (3.32) \]

The resulting equation is a simple scalar multiplication of Equation (3.31) by the parameter \( h \). However, if the equivalent strain method is applied to Equation (3.31), a stress versus strain curve at elevated temperature is obtained by scaling the strain with a scale factor \( h \):
To date, the author is unaware of any research which employs the equivalent strain method.

3.3.3 The equivalent strain method in large deformation theory

Suppose that there exist two different bodies and they deform until they become the same shape and size. In this situation, one body deforms to the final shape through a deformation gradient tensor \( F^m \) and the other body deforms to the final shape through a deformation gradient tensor \( F' \). If the two different bodies generate the same stress field, then \( F' \) and \( F^m \) are called equivalent deformation gradient tensors. The deformation gradient tensor \( F^m \) at room temperature can be regarded as the deformation gradient tensor \( F' \) at elevated temperature to create the same stress level when material properties change. This conceptually the same Equation (3.33)

Figure 3.3 shows the three different configurations which relate \( F^m \) and \( F' \). Suppose that \( X^1 \) is a position vector in the reference configuration \( \beta_1 \), which is regarded as a configuration at room temperature, \( X^2 \) is a position vector in the intermediate configuration \( \beta_2 \), which is regarded as an intermediate configuration at elevated temperature, and \( x \) is a position vector in the current configuration \( \beta \). In these three configurations, the deformation gradient tensors \( F' \), \( F^h \) and \( F^m \) are defined as follows:

\[
F' = \frac{\partial x}{\partial X^2} ; \quad F^h = \frac{\partial X^2}{\partial x} ; \quad F^m = \frac{\partial x}{\partial X^1}
\]  

(3.34-36)
where $F^h$ is defined as a softening deformation gradient tensor, which is only generated when temperature changes. From the configurations shown in Figure 3.3, the deformation gradient tensor $F^m$ can be related with $F'$ and $F^h$ as follows.

$$F^m = F' F^h$$  \hspace{1cm} (3.37)

It is necessary to establish the relationship between $F^m$ and $F'$ to relate material properties at room and elevated temperatures. The relation can easily be established in large deformation theory with concepts similar to those used with the infinitesimal elastic function shown in Equation (3.32). In large deformation theory, the mechanical part of the right Cauchy-Green metric tensor $C$ is used to represent a stress function. Therefore, it is assumed that the metric tensor $C'$ is the same as $(C^m)^h$, where $h$ is the scale factor.

This is isotropic material softening, and it belongs to the situation when an isotropic material at room temperature becomes isotropic at elevated temperature or when an anisotropic material does not change anisotropic material direction after temperature increases.

3.4 THERMO-MECHANICAL CONSTITUTIVE MODEL DEVELOPMENT

3.4.1 Thermal deformation gradient and material softening tensor

It was assumed by Simo [3.4] and Holzapfel [3.5] that the total deformation gradient tensor under thermo-mechanical loading in a large deformation theory can be
multiplicatively decomposed into a thermal and a mechanical deformation gradient
tensors as follows:

\[ F = F^m F^\theta \]  \hspace{1cm} (3.38)

where \( F^m \) is a deformation gradient tensor due to mechanical strain, \( F^\theta \) is a deformation
gradient tensor due to thermal expansion, and \( F \) is a total deformation gradient tensor.

Figure 3.4 shows the three different configurations for thermo-mechanical loading.

Suppose that \( X^0 \) is the position vector in the reference configuration \( \beta_0 \), \( X^1 \) is the position
vector in the intermediate configuration \( \beta_1 \), and \( x \) is the vector in the current
configuration \( \beta \). Conceptually, Equation (3.38) illustrates that deformation first proceeds
to the intermediate configuration \( \beta_1 \) by thermal expansion, and to the final (current)
configuration \( \beta \) by mechanical strain.

In the case of orthotropic thermal expansion, \( F^\theta \) is written as follows:

\[ F^\theta = \lambda_1^\theta E_1 \otimes E_1 + \lambda_2^\theta E_2 \otimes E_2 + \lambda_3^\theta E_3 \otimes E_3 \]  \hspace{1cm} (3.39)

If thermal strain is defined on the basis of true strain, then \( \lambda_i^\theta \) is represented as follows:

\[ \lambda_i^\theta = e^{\alpha_i \Delta \theta} \]  \hspace{1cm} (4.40)

where \( \alpha_i \) is a linear thermal expansion coefficient. Figure 3.5 shows the four different
configurations when temperature affects material softening. During the thermo-
mechanical process, the configuration \( \beta_0 \) changes to \( \beta_1 \) by thermal expansion, changes
to \( \beta_2 \) by material softening, and finally changes to the current configuration \( \beta \) by
mechanical deformation. In this case, the total deformation gradient tensor \( F \) is written as
follows:
\[
F = F^r F^0 = F^b F^0. 
\]

Also, the relation between \( C \) and \( C' \) and the relation between \( C^m \) and \( C \) are obtained as follows:

\[
C' = (F^b)^{T} C^m (F^b)^{-1} = (F^h)^{T} (F^0)^{T} C(F^0)^{-1} (F^h)^{T} \tag{3.42}
\]

\[
C^m = (F^0)^{T} C(F^0)^{-1}. \tag{3.43}
\]

\( C^m \) and \( C' \) are expressed as follows through the spectral theorem:

\[
C^m = \lambda^m_1 N_1^i \otimes N^i_1 + \lambda^m_2 N_2^i \otimes N^i_2 + \lambda^m_3 N_3^i \otimes N^i_3 \tag{3.44}
\]

\[
C' = \lambda'_{1} N_1^i \otimes N^i_1 + \lambda'_{2} N_2^i \otimes N^i_2 + \lambda'_{3} N_3^i \otimes N^i_3. \tag{3.45}
\]

In this case, the following relations are obtained because \( C' \) and \( C^m \) has the same principal directions:

\[
\lambda' = \lambda^m; \quad N = N'. \tag{3.46-47}
\]

Also, the softening deformation gradient tensor \( F^h \) is expressed as follows:

\[
F^h = \lambda^{m-h}_1 N_1 \otimes N_1 + \lambda^{m-h}_2 N_2 \otimes N_2 + \lambda^{m-h}_3 N_3 \otimes N_3. \tag{3.48}
\]

3.4.2 A thermodynamics approach to implementation of the softening model

Suppose that a material point is characterized by the Helmholtz free energy \( \psi \), entropy \( \eta \), Cauchy stress tensor \( \sigma \), the heat flux vector \( \mathbf{q} \) for a given temperature \( \theta \), temperature gradient vector \( \mathbf{g} \), softening parameter \( h \), and equivalent right Cauchy-Green...
metric tensor $C'$. According to thermodynamics theory [3.7], a general constitutive relation is described as follows:

$$\psi = \psi(C,\theta,h,g); \quad \sigma = \sigma(C,\theta,h,g) \quad (3.49a,b)$$

$$q = q(C,\theta,h,g); \quad \eta = \eta(C,\theta,h,g). \quad (3.50a,b)$$

The 2\textsuperscript{nd} law of thermodynamics, which is called the Clausius-Duhem inequality, is described in a spatial description:

$$\rho \eta \geq \rho \frac{\partial}{\partial \theta} \text{div} \left( \frac{q}{\theta} \right) \quad (3.51)$$

where $r$ is a specific heat supply. The energy conservation equation in a spatial description is expressed as follows:

$$\rho e = \sigma \cdot D + \rho r - \text{div} q \quad (3.52)$$

where $e$ is internal energy. Also, internal energy is related to the Helmholtz free energy and entropy by a Legendre transformation:

$$\psi = e - \theta \eta. \quad (3.53)$$

The inequality Equation (3.51) reflects energy dissipation, and is rewritten by substituting Equation (3.52) and (3.53) into Equation (3.51):

$$-\rho \left( \frac{\partial \psi}{\partial \theta} + \eta \right) \partial + \hat{\sigma} \cdot L^m - 2\rho F \frac{\partial \psi}{\partial C} F^\top \cdot L' - \rho \frac{\partial \psi}{\partial h} \hat{h} - \rho \frac{\partial \psi}{\partial g} \hat{g} - \frac{1}{\theta} \hat{q} \cdot g \geq 0. \quad (3.54)$$

In Equation (3.54), $L^m$ is decomposed as follows by using Equation (3.37):

$$L^m = L' + F^\top L F^{-1}. \quad (3.55)$$

The second term in Equation (3.54) is rewritten as follows:

$$\hat{\sigma} \cdot L^m = \hat{\sigma} \cdot L' + \hat{\sigma} \cdot F^\top L F^{-1}. \quad (3.56)$$
Also, the second term in Equation (3.56) is related to the material softening and simplified as follows:

\[ \mathbf{\hat{e}} \cdot \mathbf{F}' \mathbf{L}^b \mathbf{F}'^{-1} = \left( \frac{1}{h^2} \mathbf{\hat{\zeta}} \cdot \mathbf{\hat{L}}' \cdot \frac{1}{h^2} \mathbf{\hat{\zeta}} \cdot \sum_{i=1}^{3} \ln(\lambda_i') \mathbf{n}_i \otimes \mathbf{n}_i \mathbf{h} . \] (3.57)

Therefore, the inequality Equation (3.54) is rewritten by substituting Equation (3.56) and (3.57) into Equation (3.54):

\[ -\rho \left( \frac{\partial \psi}{\partial \theta} + \tilde{\eta} \right) \mathbf{\hat{\zeta}} + \left( \mathbf{\hat{\zeta}} \cdot \left( \mathbf{F}' \cdot \frac{\partial \mathbf{F}'}{\partial \mathbf{C}} \cdot \mathbf{F}' \right) \right) \cdot \mathbf{L}' - \\
\left( \rho \frac{\partial \psi}{\partial h} + \frac{1}{h^2} \mathbf{\hat{\zeta}} \cdot \sum_{i=1}^{3} \ln(\lambda_i') \mathbf{n}_i \otimes \mathbf{n}_i \right) \mathbf{h} - \rho \frac{\partial \psi}{\partial \mathbf{q}} \cdot \mathbf{g} - \frac{1}{\theta} \hat{q} \cdot \mathbf{g} \geq 0 . \] (3.58)

To satisfy the inequality in Equation (3.58) in all thermo-mechanical processes, the following relations should hold:

\[ \mathbf{\hat{\zeta}} = 2 \rho \mathbf{F}'_r \frac{\partial \psi}{\partial \mathbf{C}} ; \quad \tilde{\eta} = -\frac{\partial \psi}{\partial \theta} ; \quad \frac{\partial \psi}{\partial \mathbf{g}} = 0 \] (3.59a-c)

\[ -\left( \rho \frac{\partial \psi}{\partial h} + \frac{1}{h^2} \mathbf{\hat{\zeta}} \cdot \sum_{i=1}^{3} \ln(\lambda_i') \mathbf{n}_i \otimes \mathbf{n}_i \right) \mathbf{h} \geq 0 \] (3.60)

\[ \frac{\partial \psi}{\partial \mathbf{g}} = 0 ; \quad \hat{q} \cdot \mathbf{g} \leq 0 . \] (3.61a,b)

In the dissipative power due to material softening (Equation (3.60)), the term in the parenthesis can be proved to be always greater than or equal to zero. To satisfy the inequality equation, \( \mathbf{h} \) should be less than or equal to zero. Therefore, to satisfy the 2nd law of thermodynamics, material must soften as temperature increases.

The Helmholtz free energy \( \psi \), defined as energy per unit mass, can be converted to energy per unit reference volume for convenience, and called strain energy \( W' \):

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\[ W' = \rho_{R2} \hat{\psi} \]  

(3.62)

where \( \rho_{R2} \) is a density in the intermediate configuration \( \beta_2 \) in Figure 3.3. Therefore, the 2\textsuperscript{nd} Piola Kirchhoff stress is obtained from Equation (3.59):

\[ S = 2h \frac{\partial W'(C', \theta)}{\partial C'} = h \frac{\partial W'}{\partial E'} . \]  

(3.63)

Furthermore, the Helmholtz free energy \( \hat{\psi} \) at temperature \( \theta \) can be generalized by introducing the temperature dependant function \( L(\theta) \) and the anisotropic strain energy function \( \hat{W}(C') \) as follows:

\[ \hat{\psi} = \frac{W'}{\rho_{R2}} = \frac{L(\theta) \hat{W}(C')}{h \rho_{R2}} . \]  

(3.64)

In this case, Cauchy stress is written as follows:

\[ \sigma = 2L(\theta) \frac{1}{J} F \frac{\partial \hat{W}}{\partial C'} F^{T} . \]  

(3.65)

In Equation (3.65), \( L(\theta) \) is simply multiplied by a stress function and this concept is the same as the equivalent stiffness method. Therefore, the stress function in Equation (3.65) is a combined model of the equivalent strain method and the equivalent stiffness method.

If \( L(\theta) \) is equal to 1 and when the strain measure is chosen as true strain in principal directions, then the same strain relation is obtained for linear elasticity and large deformation theory:

\[ h = \frac{\varepsilon'}{\varepsilon'^{m}} = \frac{\ln(\lambda')}{\ln(\lambda'^{m})} . \]  

(3.66)

If the material property changes isotropically as temperature changes, Equation (3.65) holds for any anisotropic material for a given \( h \) and a known room temperature strain.
energy function $\hat{W}(C)$. In the case of a transversely isotropic material, a stress function at elevated temperatures can be obtained by modifying the strain invariants with respect to the configuration $\beta_2$ as follows:

$$S = 2L(\theta) \frac{\partial \hat{W}(I'_1, I'_2, I'_3, I'_4, I'_5)}{\partial C'} \quad (3.67)$$

where $I'_1$, $I'_2$, $I'_3$, $I'_4$ and $I'_5$ are redefined as follows:

$$I'_1 = \lambda_1^{2h} + \lambda_2^{2h} + \lambda_3^{2h}; \quad I'_2 = (\lambda_1 \lambda_2)^{2h} + (\lambda_2 \lambda_3)^{2h} + (\lambda_3 \lambda_1)^{2h}; \quad I'_3 = (\lambda_1 \lambda_2 \lambda_3)^{2h} \quad (3.68a-c)$$

$$I'_4 = \lambda_1^{2h} (a \cdot N_1)^2 + \lambda_2^{2h} (a \cdot N_2)^2 + \lambda_3^{2h} (a \cdot N_3)^2 \quad (3.69)$$

$$I'_5 = \lambda_1^{4h} (a \cdot N_1)^2 + \lambda_2^{4h} (a \cdot N_2)^2 + \lambda_3^{4h} (a \cdot N_3)^2. \quad (3.70)$$

The equivalent strain method can also be applied to incompressible hyperelastic materials. For such material, deformation gradient tensor is decoupled into a deviatoric and a dilational response as follows [3.16]:

$$F = F_{vol} \tilde{F}; \quad F_{vol} = J^{1/3} I; \quad \tilde{F} = J^{-1/3} F \quad (3.71a-c)$$

where $F_{vol}$ is a volumetric tensor, $\tilde{F}$ is a deviatoric tensor, and $J$ is the determinant of $F$.

In the case of an incompressible material, the following relation is obtained for $F_{vol}$ and $\tilde{F}$:

$$\det(F_{vol}) = J; \quad \det(\tilde{F}) = 1. \quad (3.72a,b)$$

Also, the energy function is assumed to be decoupled into volumetric and deviatoric components as follows:

$$W(C) = U(J) + \hat{W}(\tilde{C}) \quad (3.73)$$
where \( U \) is a volumetric strain energy function and \( \tilde{W} \) is a deviatoric strain energy function. Therefore, the following strain energy function for elevated temperature is proposed:

\[
\tilde{W}(C') = \hat{U}(J') + \tilde{W}(\tilde{C}')
\]  

(3.74)

where \( J' \) is the determinant of \( \mathbf{F}' \). In Equation (3.74), if there is no volumetric change, then \( J' \) becomes 1. The corresponding stress can be obtained from the strain energy function through Equation (3.65).

3.4.3 The velocity gradients \( \mathbf{L}' \) and \( \mathbf{L}^m \) in isotropic softening model

The velocity gradients \( \mathbf{L}' \) and \( \mathbf{L}^m \) are defined between two adjacent configurations in Figure 3.3:

\[
\mathbf{L}' = \frac{\partial \mathbf{x}(t, \mathbf{x}')}{\partial \mathbf{x}(t, \mathbf{x})} \quad \text{for the fixed configuration } \beta_2
\]  

(3.75)

\[
\mathbf{L}^m = \frac{\partial \mathbf{x}(t, \mathbf{x}')}{\partial \mathbf{x}(t, \mathbf{x})} \quad \text{for the fixed configuration } \beta_1.
\]  

(3.76)

The relationship between the velocity gradients \( \mathbf{L}' \) and \( \mathbf{L}^m \) can be obtained by differentiating Equation (3.38) with respect to time as follows:

\[
\mathbf{L}' = \mathbf{L}^m - G_{L'}^{(4)} \mathbf{L}^m
\]  

(3.77)

where

\[
G_{L'}^{(4)} = \sum_{i=1}^{3} (1-h) n_{iii} + \sum_{i,j=1,2,3} \frac{\lambda_i \lambda_j}{\lambda_i^2 - \lambda_j^2} \left\{ \frac{\lambda_i}{\lambda_i} (n_{iji} + n_{jii}) + \frac{\lambda_j}{\lambda_j} (n_{jiji} + n_{ji}) \right\}
\]  

(3.78)
The rate of deformation tensor which is the symmetric part of Equation (3.77) can be obtained as follows:

\[
D' = D^m - G''_0 D
\]  
(3.79)

where \( G''_0 \) is the symmetric part of \( G'' \). Equation (3.79) can be simply rewritten as:

\[
D' = T''(4) D^m.
\]  
(3.80)

Mathematically, \( T''(4) \) is the 4th order transformation tensor which converts the rate of deformation gradient tensor \( D \) to the rate of deformation gradient tensor \( D' \).

3.4.4 The spatial elasticity tensor for thermal-mechanical loading

The spatial elasticity tensor between two arbitrary configurations is defined in Equation (3.13). If stress function is defined between the configurations \( \beta \) and \( \beta_2 \), due to the material softening as shown in Figure 3.5, through Equation (3.13), the Jaumann rate of the Cauchy stress can be represented with respect to the rate of deformation gradient tensor \( D' \) as follows:

\[
\dot{\sigma} = E''(4) D'
\]  
(3.81)

where \( E''(4) \) is written as follows:

\[
E''(4) = -\sigma \otimes I + \frac{1}{J'} P''(4) + H''(4).
\]  
(3.82)

To represent the elasticity tensor with respect to \( D^m \), it is necessary to convert the elasticity tensor \( E''(4) \) defined between \( \beta \) and \( \beta_2 \) to the elasticity tensor between \( \beta \) and
\[ \mathbf{\dot{\sigma}} = \mathbf{E}^{(4)} \mathbf{T}^{(4)} \mathbf{D}^m, \]  
\[ (3.83) \]

and the spatial elasticity tensor is written as follows:

\[ \mathbf{\dot{\mathbf{E}}}^{(4)} = \mathbf{E}^{(4)} \mathbf{T}^{(4)} . \]  
\[ (3.84) \]

When there is a thermal expansion, the rate of stretch tensor \( \mathbf{D} \), which is defined between \( \beta \) and \( \beta_0 \), must be decomposed into the rate of stretch tensor \( \mathbf{D}^m \) by mechanical loading and the rate of stretch tensor due to thermal expansion. The velocity gradient tensor \( \mathbf{L} \) is obtained by differentiating Equation (3.38) with respect to time and using Equation (3.39) as follows:

\[ \mathbf{L} = \mathbf{L}^m + \mathbf{M}^\theta \hat{\theta} \]  
\[ (3.85) \]

where \( \mathbf{M}^\theta \) is a 2\(^{nd}\) order tensor and is defined as follows:

\[ \mathbf{M}^\theta = \sum_{i=1}^{3} \frac{1}{\lambda_i^\theta} \frac{\partial \lambda_i^\theta}{\partial \theta} (\mathbf{F}^m \mathbf{E}_i \otimes \mathbf{F}^{m-T} \mathbf{E}_i) . \]  
\[ (3.86) \]

By plugging the symmetric part of \( \mathbf{L} \) in Equation (3.85) into Equation (3.83), the Jaumann rate of Cauchy stress can be obtained in terms of the rate of stretch tensor \( \mathbf{D} \) and the temperature change \( \hat{\theta} \):

\[ \mathbf{\dot{\sigma}} = \mathbf{\dot{\mathbf{E}}}^{(4)} \mathbf{D} - \mathbf{\dot{\mathbf{E}}}^{(4)} \mathbf{M}^\theta \hat{\theta} . \]  
\[ (3.87) \]
3.5 NUMERICAL IMPLEMENTATION OF THE THERMO-MECHANICAL MODEL

3.5.1 Numerical procedure for stress calculation

Suppose that a master curve, softening parameters at different temperatures, and thermal expansion coefficients are known. In this case, the stress versus strain curve at elevated temperatures during the thermo-mechanical process can be easily obtained by the following procedures:

i) For a given temperature \( \theta \), \( F^0 \) is obtained through Equation (3.39).

ii) From a given total deformation gradient tensor \( F \) and a thermal deformation gradient tensor \( F^0 \) at elevated temperature, a mechanical deformation gradient tensor \( F^m \) is obtained through Equation (3.38).

iii) From the deformation gradient tensor \( F^m \), the principal stretch \( \lambda^m \) and direction vectors are obtained by Equation (3.5).

iv) The principal stretch \( \lambda^m \) can be converted to \( \lambda' \) by the softening model in Equation (3.46).

v) Stress is evaluated with \( F' \) by Equation (3.65).
3.5.2 The proposed temperature dependent strain energy function for hyper-foam materials

The following Ogden type strain energy function is used by ABAQUS™ [3.8] to represent isotropic hyper-foam materials at room temperature:

\[ W(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu}{\alpha^2} \left[ \lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3 + \frac{1}{\beta} \left( f^{\theta - \theta_0} - 1 \right) \right] \] (3.88)

where \( J \) is the Jacobean of the deformation gradient tensor, \( \lambda \) is the principal stretch, and \( \alpha, \mu, \) and \( \beta \) are material parameters to be determined from experiment. If Equation (3.88) is a function at room temperature, then the energy function at elevated temperature can be obtained as follows by changing the configuration \( \beta_1 \) to the configuration \( \beta_2 \) (shown in Figure 3.5) and by using Equation (3.65):

\[ \hat{W}(\lambda'_1, \lambda'_2, \lambda'_3, \theta) = L(\theta) \frac{2\mu}{\alpha^2} \left[ \lambda'_1^\alpha + \lambda'_2^\alpha + \lambda'_3^\alpha - 3 + \frac{1}{\beta} \left( f^{\theta - \theta_0} - 1 \right) \right] \] (3.89)

where \( L(\theta) \) is a temperature dependant function. If temperature is equal to \( \theta_0 \), then Equation (3.89) is the same as Equation (3.88). In some cases, the Williams-Landell-Ferry (WLF) form is used for \( L(\theta) \) and applied by many authors [3.14]:

\[ L(\theta) = \exp\left( - \frac{C_1 (\theta - \theta_0)}{C_2 + \theta - \theta_0} \right) \] (3.90)

where \( C_1 \) and \( C_2 \) are material constants.
3.5.3 Comparison between the equivalent stiffness method and the equivalent strain method for hyper-foam materials

In this section, the equivalent stiffness and equivalent strain isotropic softening models are compared to each other. A highly compressible intumescent mat, which is used to support substrate in catalytic converters, is chosen as an example. Figure 3.6 shows stress versus strain curves at different temperatures from compression experiments. As shown in the figure, the material properties of the mat soften as the temperature increases. For more details, see [3.1, 3.2, 3.3].

For the two different softening models, the softening parameters at each temperature can be obtained from Figure 3.6. Through each softening model, all the stress versus strain curves at different temperatures can be converted to the stress versus strain curves with material softening removed. The equivalent stiffness method is applied, and the resulting Figure 3.7 shows stress versus strain curves at different temperatures with material softening removed. Deviations between each curve are seen especially in low strain ranges. Therefore, the equivalent stiffness method is not a suitable method for representing temperature dependent material behaviors of intumescent mat. In Figure 3.8, the stress versus strain curves at different temperatures with material softening removed result from application of the equivalent strain method. Figure 3.8 shows that most of the stress versus strain curves at different temperature lies on the same curve. From these curves, a master curve can be obtained. In this case, stress versus strain curves at different temperatures can be obtained from the master curve and
the softening parameter $h$. One can use one of the methods or a combination of methods. Although the equivalent strain method works well for intumescent, it may not be applicable to other materials.

3.5.4 Application of the softening model to anisotropic incompressible materials

In this section, the isotropic softening model is applied to an incompressible anisotropic hyperelastic material. As an example, transversely isotropic biological soft tissues are chosen. This material was studied by Weiss et al. [3.16] and the following transversely isotropic energy function at room temperature was proposed:

$$W = c_1(I_1 - 3) + c_2(I_2 - 3) + c_3(\exp(I_4) - I_4) \quad \text{and} \quad J = 1. \quad (3.91)$$

When the equivalent strain method is applied to the strain energy function in Equation (3.91), the strain energy function is modified as follows:

$$\hat{W} = c_1(I_1' - 3) + c_2(I_2' - 3) + c_3(\exp(I_4') - I_4') \quad \text{and} \quad J' = 1 \quad (3.92)$$

where $\hat{W}$ is the deviatoric part of a strain energy function in the intermediate configuration, which is defined in Equation (3.74). Therefore, from the energy function in Equation (3.92), Cauchy stress can be derived as follows (see details in [3.16]):

$$\sigma = \mathbf{p} I + \frac{2}{J'} \left\{ (\hat{W}_1 + \hat{W}_2 \mathbf{I}) \mathbf{B}' \cdot \hat{W}_2 \mathbf{B}'^2 + \hat{W}_4 \mathbf{I}' \mathbf{a} \otimes \mathbf{a} \cdot \frac{1}{3} (\hat{W}_1 \mathbf{I}' + 2 \hat{W}_2 \mathbf{I}' + \hat{W}_4 \mathbf{I}') \right\} \quad (3.93)$$

where $\hat{W}_i = \frac{\partial \hat{W}}{\partial I_i}$, $\mathbf{a}$ is a transversely isotropic direction vector, and $\mathbf{p}$ is pressure. To investigate the effect of the softening parameter $h$, simulations are performed for uniaxial...
and equibiaxial loading cases. One solid element is used, and the direction vector $a$ is defined as the $z$-direction in the simulations.

Figure 3.9 shows stress versus strain curves from the uniaxial simulations in the $z$ direction. Figure 3.10 shows stress versus strain curves from the uniaxial simulations in the $x$ direction. Figure 3.11 shows stress versus strain curves in each direction under the equibiaxial simulations. All of the stress versus strain curves depend on $h$ and it is seen that stress function softens as the softening parameter $h$ decreases. Therefore, the equivalent strain method is applicable to incompressible hyperelastic materials under thermal loading.

3.6 CONCLUSION

In this study, detailed development of a constitutive model for anisotropic materials under thermo-mechanical loading is presented. Based on the idea that material responses at different temperatures are self-similar, two kinds of material softening are proposed: the equivalent stiffness method and the equivalent strain method. The theory of the equivalent strain method is extended to large deformation, and a new isotropic softening model is proposed for anisotropic hyperelastic materials. The isotropic softening model is extended to the combined model of the equivalent strain and stiffness method by assuming a generalized Helmholtz free energy function. For implementation in implicit finite element codes, through the 2nd law of thermodynamics, the complete stress function and the spatial elasticity tensor in the Eulerian frame are derived for the
Ogden type strain energy. The equivalent stiffness and the equivalent strain method are compared to each other for highly compressible intumescent mat. It is concluded that, either method or their combination can be chosen depending on the material behavior in actual application because it may not be applicable to other materials although the equivalent strain method works well for intumescent mat. Also, the isotropic softening model is applicable to incompressible transversely isotropic hyperelastic materials. This model would be appropriate for rubber materials under thermal loading.

Up to this point, anisotropic material softening has not been considered in the constitutive model development. Therefore, future work in this field includes implementation of an anisotropic softening model.

REFERENCES


Figure 3.1: The equivalent stiffness method for non-linear elasticity.

Figure 3.2: The equivalent strain method for non-linear elasticity.
Current Configuration $\beta$
Position vector $\mathbf{x}$

Reference Configuration $\beta_1$
Position vector $\mathbf{X}^1$

Intermediate Configuration $\beta_2$
Position vector $\mathbf{X}^2$

$F^m$
$F^h$

Figure 3.3: Various configurations and their relationships in the equivalent strain method.

Current Configuration $\beta$
Position vector $\mathbf{x}$

Reference Configuration $\beta_0$
Position vector $\mathbf{X}^0$

Virtual Intermediate Configuration $\beta_1$
Position vector $\mathbf{X}^1$

$F$
$F^m$

$F^0$

Figure 3.4: Various configurations and their relationships under thermal loading.
Figure 3.5: Various configurations and their relationships under thermal loading and with material hardening.
Figure 3.6: Stress versus mechanical strain curves at different temperatures.

Figure 3.7: Modified stress versus strain curves at different temperatures from the equivalent stiffness method.
Figure 3.8: Modified stress versus strain curves at different temperatures from the equivalent stiffness method.

Figure 3.9: Stress versus strain curves for 2 values of $h$ for transversely isotropic hyperelastic materials under uniaxial loading in the $z$-direction.
Figure 3.10: Stress versus strain curves for 2 values of h for transversely isotropic hyperelastic materials under uniaxial loading in the x-direction.

Figure 3.11: Stress versus strain curves for 2 values of h for transversely isotropic hyperelastic materials under equibiaxial loading in z and x-direction.
CHAPTER 4

CONSTITUTIVE MODEL DEVELOPMENT OF CATALYTIC CONVERTER
INTUMESCENT MAT FOR THERMO-MECHANICAL ANALYSIS
CONSIDERING VISCOUS AND PLASTIC EFFECTS

ABSTRACT

Intumescent mat is used to support substrates in catalytic converters and is subjected to thermo-mechanical loading. This chapter presents the procedure used to develop the constitutive model for Intumescent mat under thermo-mechanical loading. Several experiments focusing on rate effects and plastic deformation in large strain ranges are performed at room and elevated temperature to characterize thermo-mechanical behavior. The constitutive model formulation is developed based on a hyperelastic theory by including viscous and plastic effects and material softening due to temperature and mechanical strain. A new material softening model is proposed from the assumption of material similarity. A master curve and a strain shift function are proposed to relate material property changes at different temperatures. A plasticity model is introduced by defining a plastic deformation gradient tensor and a yield criterion. The theory is coded in ABAQUS™ implicit through the user subroutine UMAT. The
constitutive model is verified by comparing with experimental results. Finally, the constitutive model is applied to a catalytic converter to simulate an assembly process and a thermal cyclic test. The simulation results are compared with the experimental data.

4.1 INTRODUCTION

A catalytic converter is a pollution control device, which converts the harmful gases in combustion exhaust (e.g., hydrocarbons, CO and NOx) to the non-harmful CO2, H2O and N2. It consists of three major parts: an outer shell, a substrate, and an intumescent (swelling) mat. The substrate is a coated ceramic material that chemically converts the gases. The mat is used to support the substrate in the outer shell and to provide cushion and insulation. Figure 4.1 shows a sectional view of a catalytic converter. When the mat is exposed to high temperatures generated by the converter and the exhaust gas, the mat expands to provide cushioning for the substrate even while the outer shell expands due to thermal expansion. As the exhaust system is subjected to various loads throughout the life of a vehicle, this expansion and the subsequent cushioning are critical to the durability of the substrate. Additionally, during the initial temperature cycle, the behavior of the mat is quite non-linear as it retains some residual strain at room temperature. These effects have been experimentally known for some time. However, there are known attempts to develop a model specifically aimed at understanding intumescent mat material. To reduce the number of costly, destructive
catalytic converter experiment, and to improve designs in the early stages of catalytic converter development, an analytic approach to model the mat material is required.

The mat material contains aluminosilicate glass fibers, vermiculite mineral particles, and organic binders. The vermiculite is a micaceous hydrated magnesium aluminum-silicate mineral which, when heated, loses water and thereby significantly increases the mat layer thickness (as high as 10 times [4.15]). This characteristic of the vermiculite leads to the intumescent expansion characteristic of the mat, as shown schematically in Figure 4.2. Along two directions (lateral and thickness direction), the expansion ratios to the original volume are different from each other. Once the mat experiences the vermiculite expansion during the heating process, after the cool-down process, the mat does not recover its original shape. Most of the permanent expansion takes place between 400-600 °C, and the vermiculite degrades when heated above 750 °C.

Several attempts have been made to model the intumescent mat material. The first rather approach appears to be the one taken by Chen [4.3]. For the analysis of the canning process, he used a gap element to model the mat material in ABAQUS™. The gap element, which is a kind of a non-linear spring, was limited to one dimensional stress analysis in the thickness direction at room temperature. Muju et al. [4.12] modeled the mat with a hyper-foam material, which is a highly compressible and non-linear elastic material. He used different stress versus strain curves at different temperatures to analyze the canning process and thermal cycling after assembly. In his analysis, he employed an equivalent isotropic thermal expansion coefficient to account for the vermiculite expansion at elevated temperature. In this case, the vermiculite expands isotropically both
in the thickness and the lateral directions, overestimating the lateral deformation during the heating cycle. Also, he did not consider the permanent vermiculite expansion after the thermal unloading. To overcome these issues, Taylor et al. [4.21] and Fuene et al. [4.5] proposed a modified isotropic hyper-elastic formulation which employed anisotropic thermal and vermiculite expansion at elevated temperature. The vermiculite expansion was defined as a chemical strain. Thermo-mechanical behavior was implemented by a master curve and a shift function. However, due to limitations of the proposed simple master curve and shift function, it did not represent material properties at different temperatures quite well. Also, the chemical strain was not clearly explained on their work.

In this study, a new constitutive model for the intumescent mat material under thermo-mechanical loading is proposed. The experiments focused on stress versus mechanical strain curves at different temperatures, the permanent vermiculite expansion, rate effects, and plastic deformation. The experimental results are used to characterize the thermo-mechanical properties. Several assumptions are made to simplify the observed mechanical response. A new strain is defined to represent the vermiculite expansion, and strain relations under the thermo-mechanical loading are established by decomposing total strain into each involved strain. From a simple visco-hyper-elastic model, material softening due to temperature and mechanical strain together with plasticity are included in the constitutive model. The constitutive model is implemented in the ABAQUS™ user subroutine called UMAT. Experimental results are used to define model parameters and
for independent verification. Finally, the model is applied to a catalytic converter model to simulate the assembly process and thermal cycling.

4.2 EXPERIMENTAL INVESTIGATIONS OF INTUMESCENT MAT

4.2.1 Types of experiments

Mat material experiments focus on stress versus mechanical strain curves at different temperatures, viscous effects, the characteristics of the vermiculite expansion, differences between loading and unloading curves, and the permanent deformation after the thermo-mechanical unloading. These issues are assumed to include the fundamental material properties which must be presented in the constitutive model.

10 mm thick specimens are used in all experiments. The specimens are punch cut from a large sheet into 29 mm diameter buttons. The experiments are all performed under compressive loading. They can be divided into the following two main categories: constant load experiments and compression experiments. For elevated temperature experiments, a clamshell type infrared heating chamber is used. The heating chamber has four 1000W-Quartz lamps to heat the specimen. A controller with a K-Type thermocouple reads the temperature at the specimen/load rod interfaces and enables feedback control. Since the nickel-base alloy Hastelloy has good physical properties under high temperature, it is used for the load rods for elevated temperature experiments. Cooling of the heating chamber and the Hastelloy rods is accomplished by a water
distribution manifold and rod cooling adapters which have spiral grooves to force the water to circulate from the inlet on the top to the outlet on the bottom.

The constant load experiment is performed to investigate a steady state stress response by applying a constant pressure on the specimen at room and elevated temperatures. Thus the constant load experiment provides strain change information under constant load. At room temperature, for the loading process, dead weights are added after the mat is fully relaxed at each weight. In the same way, for the unloading cases, the stacked dead weights are removed one by one after no additional mat thickness change is found. Thickness is measured in the relaxed state with an LVDT. For the constant load experiments at elevated temperature, the specimen is placed between two circular rods along with the thermocouples at room temperature, and aluminum foil is wrapped around the outer radial surface of the specimen to prevent the specimen from instantly burning upon exposure to direct infrared radiation. Constant load is applied to the rod after the data acquisition is initiated. After a five minute hold, the heater is turned on to increase temperature up to the desired maximum temperature at a certain heating rate and the maximum temperature is held for 5 minutes. Cooling begins after the 5 minute hold.

The compression experiment procedure consists of compressing the specimen to a certain thickness. It is performed to monitor the stress change on the specimen as the strain in the specimen changes. Since the compression experiment requires maintaining various gap positions throughout the experiments, the loading apparatus must have accurate displacement control with a variable command signal. Therefore, a servo-
hydraulic load frame is used to do the compression experiments. Room temperature compression experiments are performed by placing the specimen between two 31.75 mm diameter load rods held 10 mm apart. The specimen is compressed to a specified position at a fixed ramp rate. The distance between the two rods is then held constant for a two-minute holding time. After the two-minute hold time, the upper rod is moved upward to the original 10 mm gap position. The next cycle is performed after a 10 second wait from the previous unloading. Specimens are tested to different maximum strain ranges. The maximum strain ranges for the room temperature experiments are 0.3, 0.4, 0.5, and 0.6 engineering strain. To investigate rate effects, experiments are also performed at various ramping speeds. The cycle is repeated to study cyclic loading effects. See Kim [4.8] for more details on the experimental approaches and results. Due to high thermal expansion of the Hastelloy rod, without a high temperature extensometer, it is difficult to maintain the gap distance during the experiment. The results in this chapter do not attempt to adjust for the initial gap.

4.2.2 Review of experimental results

Figure 4.3 shows stress versus strain curves at room temperature for different compression speeds to 60% compressive engineering strain. The static stress versus strain curve comes from a constant load experiment and the stress versus strain curves with different speeds come from the cyclic compression experiments. It shows that the stress
versus strain curve becomes stiffer as the compression speed increases. Therefore, the mat material is a highly rate dependant material.

Figure 4.4 shows loading and unloading curves during several cycles of a 40% compression test at room temperature. There are significant differences between the 1\textsuperscript{st} cycle and the 2\textsuperscript{nd} cycle loading curves. Only minor differences are seen between the 1\textsuperscript{st} and the 2\textsuperscript{nd} cycle unloading curve. Also, the 1\textsuperscript{st} hysteresis loop from the 1\textsuperscript{st} cycle loading and unloading curve and the 2\textsuperscript{nd} hysteresis loop from the 2\textsuperscript{nd} cycle loading and unloading curve are shown. Generally, hysteresis loops are a material property of a visco-elastic material. Therefore, this is evidence that the mat material is a visco-elastic material. The size of hysteresis loop becomes smaller from the 1\textsuperscript{st} to the 2\textsuperscript{nd} cycle and there is little further size reduction after the 2\textsuperscript{nd} cycle. Therefore, this material response appears to be a combination of plastic damage and visco-elasticity.

Figure 4.5 shows a comparison of the monotonic loading and the cyclic loading and unloading static stress versus strain curves from the constant load experiments at room temperature. During the monotonic loading process, loading is applied continuously without unloading. However, the cyclic loading is performed to investigate the existence of a plastic loading function. Loading is applied to certain stress levels and unloaded and then reloaded again to a higher stress level. The direction of the loading and the unloading is shown with arrows. The results show that the unloading curves do not follow the monotonic load curve. However, it shows that unloading and reloading are almost the same. Also, as compressive strain increases, it is seen that the reloading curve is slightly softer. If the 2\textsuperscript{nd} loading is higher than the previous yield stress, then the 2\textsuperscript{nd}
loading curve follows the monotonic loading curve. From this result, it is concluded that the material behavior of the mat material is very similar to the plastic response of metal.

Figure 4.6 shows a comparison between the room temperature 1st and the 2nd cycle loading curves under the same compression speed when different compressive strain is applied. It shows that the peak stress is quite different between the 1st and the 2nd loading to the same strain, which indicates plasticity and/or material damage has occurred. It is expected that the plasticity or material damage incurred during the 1st loading would change the rate dependencies of subsequent loading.

Figure 4.7 shows relative expansion versus temperature curves from the constant load experiments at elevated temperatures. Different specimens were used for each pressure. The relative expansion R at the pressure P is defined as the following ratio of deformed thicknesses:

\[ R = \frac{t_T}{t_0} \]  \hspace{1cm} (4.1)

where \( t_T \) is the deformed thickness at temperature \( T \) under the pressure \( P \), and \( t_0 \) is the deformed thickness at room temperature under the pressure \( P \). The total true strain \( \varepsilon^t \) at temperature \( T \) is a function of both temperature and pressure and is defined as follows:

\[ \varepsilon^t = \ln\left(\frac{t_T}{t_i}\right) = \ln\left(\frac{R}{t_0/t_i}\right) = \ln(R) + \varepsilon^0 \]  \hspace{1cm} (4.2)

where \( t_i \) is the original mat thickness at room temperature before the thermal loading and \( \varepsilon^0 \) is the mechanical strain under the pressure \( P \) at room temperature. From the relation in Equation (4.2), the relative expansion \( R \) can be regarded as a thickness change due to temperature. Figure 4.7 shows that the relative expansion begins to decrease until 300°C and then increases significantly until 700°C. This sudden expansion appears to occur as a
result of the vermiculite expansion. Also, the relative expansion at elevated temperatures decreases as the loading pressure increases. Therefore, the elevated temperature constant load test at low pressure will result in very high relative expansion. This makes the reference length of the material at elevated temperature under zero pressure highly variable. This phenomenon prevents performing the mechanical strain based compression test at elevated temperature. When the mat is cooled down, the relative expansion follows a different unloading path as compared to the thermal loading path. Therefore, the material has permanent deformation due to the thermal loading and unloading.

Static stress versus total strain curves at different temperature during the thermal loading and unloading can be obtained from Figure 4.7. Figure 4.8 shows stress versus total strain curves during the thermal loading. Figure 4.9 shows stress versus total strain curves during the thermal unloading after heated to 800°C. During the thermal loading, it is seen that the stress versus total strain curves soften as temperature increases. However, once the material softens, material softening remains after the thermal unloading.

From the experimental results, it is concluded that the mat is a visco-elastic material with different loading and unloading curves. In addition, there is a plastic strain after the mechanical unloading process. During the thermal loading, there is a irreversible thermal expansion due to the vermiculite expansion. The mat has different stress versus strain curves at different temperatures and material softens as temperature increases.
4.3 STRAIN DECOMPOSITION AND MATERIAL SIMILARITY UNDER THERMAL LOADING

Stress versus mechanical strain curves are needed for the development of constitutive model. However, for the intumescent mat material, the constant load experiments at elevated temperature only provide total strain results. Therefore, several assumptions are made in order to establish relations between the total and the mechanical strain.

4.3.1 Material similarity and the concept of a master curve

If material responses at each temperature are self-similar, then it is assumed that material similarity is represented by a simple function. Suppose that a stress versus strain curve at room temperature is represented as follows:

\[ \sigma = f(\varepsilon^m). \]  \hfill (4.3)

Then, stress versus strain curves at different temperatures for the intumescent mat are simply represented with a base function and a scale factor \( h \) by introducing two different types of material softening models. One is an equivalent stiffness method and the other is an equivalent strain method. If the stress versus strain curve at elevated temperature can be represented as follows:

\[ \sigma = hf(\varepsilon^m) = f'(\varepsilon^m) \]  \hfill (4.4)
then, it is called a equivalent stiffness method. The resulting equation is just simple scalar multiplication of Equation (4.3) at constant strain with a scale factor $h$. On the other hand, if the equivalent strain method is applied to Equation (4.3), the stress versus strain curve at elevated temperature is assumed to be represented as follows by scaling the strain with a scale factor $h$ for a constant stress level:

$$\sigma = f(h\varepsilon'') = f(h\varepsilon'''). \quad (4.5)$$

The scale factor $h$ at an arbitrary stress level can be calculated from a strain $\varepsilon_2$ at room temperature and a strain $\varepsilon_1$ at elevated temperature:

$$\sigma = f(\varepsilon_2) = f(h\varepsilon_1) \quad (4.6)$$

$$h = \frac{\varepsilon_1}{\varepsilon_2}. \quad (4.7)$$

The base function in Equation (4.3) at room temperature is called a master curve and $h$ is called a softening parameter. Also, the functions in Equation (4.4) and (4.5) which represent the material similarity are called softening models. In actual application, depending on the material behavior, either the equivalent stiffness method or equivalent strain method, or a combination of methods can be used to relate material behavior at different temperatures. Therefore, if a master curve and softening parameters at different temperatures are known, then all the stress versus mechanical strain curves at different temperature can be represented by these softening models. If $h$ is less than 1, it is called material softening. If $h$ is greater than 1, then it is called material hardening.
There are two sources of material softening in this chapter. One is the material softening due to temperature and the other is due to mechanical strain. These will be discussed later in section 4.3.3.

4.3.2 Definitions of thermal and chemical strains and strain decomposition

The experimental results show that the thermal cyclic loading is an irreversible process. Therefore, the strain induced by temperature must be decomposed into recoverable and unrecoverable strains. Unrecoverable strain is a permanent strain which is not recovered after the thermal unloading. Unrecoverable strain appears to occur due to the vermiculite expansion which undergoes a chemical reaction. The amount of the permanent thermal strain is defined as a chemical strain. On the other hand, the recoverable strain is defined as a thermal strain which disappears after the thermal unloading. Recoverable strain appears to occur as a result of simple linear thermal expansion. For intumescent mat material, the chemical reaction only occurs if the mat experiences high enough temperatures, and once the reaction occurs, no further expansion is observed. Therefore, the thermal strain is simply obtained by measuring the thermal expansion coefficient $\alpha$ from test specimens which already underwent several thermal loading cycles. The thermal expansion coefficient $\alpha$ is computed from the $5^{th}$ cycle expansion curve directly by the following definition of the thermal strain $\varepsilon^\theta$:

$$\varepsilon^\theta = \alpha \Delta T . \quad (4.8)$$
It is assumed that during a thermo-mechanical process, the total strain $\varepsilon^t$ can be decomposed into a chemical $\varepsilon^c$, a thermal $\varepsilon^\theta$, and a mechanical $\varepsilon^m$ strain as follows:

$$\varepsilon^t = \varepsilon^m + \varepsilon^c + \varepsilon^\theta. \quad (4.9)$$

Furthermore, if the thermal and the chemical strains are assumed to be only a function of temperature, the sum of the thermal and the chemical strain is defined as a shift strain $\varepsilon^{\text{shift}}$. Then, for isothermal condition, a shift strain is constant regardless the stress level, and the total strain is simply written as follows:

$$\varepsilon^t(P, T) = \varepsilon^m(P, T) + \varepsilon^{\text{shift}}(T). \quad (4.10)$$

Since the mechanical strain under zero pressure is always zero, in theory, the shift strain can be obtained from the constant load test under zero pressure by measuring the total strain as temperature increases. Therefore, if the shift strain is obtained from the experiment, the stress versus total strain curve can be converted to a stress versus mechanical strain curve by using Equation (4.10). Once the shift strain is obtained, the chemical strain at each temperature can be obtained by subtracting the thermal strain from the shift strain during the thermal loading. Since the chemical strain is permanent, it is represented as a maximum value during the temperature history:

$$\varepsilon^c = \text{Max}(\varepsilon^{\text{shift}} - \varepsilon^\theta). \quad (4.11)$$
4.3.3 Master curves for the loading and the unloading functions

For the loading function, stress versus total strain curves at each temperature measured from the experiments can be converted to stress versus mechanical strain curves by using the shift strain. Because of the unstable vermiculite expansion at zero pressure, experimentally, it is very difficult to measure the shift strain from the constant load test without applying any load. However, numerically, the shift strain and the softening parameter $h_L$ for the loading function can be obtained by using the assumption that stress versus mechanical strain curves at room temperature can be modified to the stress versus total strain curves at elevated temperature. That is, the shift strain and the softening parameter $h_L$ are assumed and modified iteratively until the curve at room temperature becomes very close to the curve at high temperature. Then, this procedure is repeated for the next temperature.

In the same way, from the obtained shift strain and the softening parameter $h_L$ at each temperature, all the stress versus total strain curves at each temperature can also be converted to the master curve which is the stress versus mechanical strain curve at room temperature before material softening. This can be done by subtracting the shift strain from the total strain and multiplying the mechanical strain by $1/h_L$. Theoretically, the obtained master curve from the curve at each temperature should be exactly same. If the obtained master curve is not same, then it material similarity does not exist for the material and the softening model cannot be used. Figure 4.10 shows stress versus mechanical strain loading curves with material softening removed, these curves were
converted from stress versus total strain curves at different temperatures. It is clearly seen that the most of the stresses versus mechanical strain at different temperature lie on one curve. The thick solid line is the master curve for the loading function and is obtained through curve fitting. Figure 4.11 shows the corresponding softening parameter and the shift strain as a function of temperature. The material softening curve shows that the softening parameter $h_L$ decreases as temperature increases. Also, the shift strain curve shows that material expands significantly after $400^\circ$C.

In the case of the unloading function, static unloading functions are measured at different mechanical strain at room temperature. As shown in Figure 4.5, each static unloading curve has a different strain at zero stress and a different slope. Therefore, to obtain a stress versus mechanical strain curve, the static unloading function must be defined first from Figure 4.5. By applying the material similarity concept used for the loading function, it is assumed that, by changing the softening parameter $h_U$ and assuming zero strain point at zero stress, each unloading curve can be represented with a master curve. Figure 4.12 shows the master curve for the unloading function and each unloading function with material softening removed. Table 4.1 shows softening parameter versus applied mechanical strain. As shown in Table 4.1, the softening parameter $h_U$ decreases as the mechanical strain increases. So far, only the material softening due to mechanical deformation are discussed for the unloading function. The temperature induced softening has not mentioned. For the results in this chapter, it is assumed that material softening due to elevated temperature for the unloading function is
the same as elevated temperature material softening determined from the loading function.

4.4 MATERIAL CHARACTERIZATION FOR CONSTITUTIVE MODEL

4.4.1 Material simplification

Based on the experimental results, various assumptions are employed in the development of the mat material model. The assumptions are summarized as follows:

- The mat is a highly compressible isotropic visco-elastic material with Poisson's ratio $= 0$.
- The mat only works under compressive stress.
- The thermal and chemical strains are orthotropic in the material direction. The chemical strain is considered to be permanent, but thermal strain is reversible.
- Viscous effects at elevated temperature assumed to be negligible compared to room temperature.
- The mat has different loading and unloading stress functions. The loading function is assumed to be a yield stress and the unloading function is an elastic function.
- Material similarity exists for both the loading and unloading functions.
• The loading function has a permanent material softening due to temperature and the unloading function has permanent material softening due to both mechanical strain and temperature.

4.4.2 Rate independent 1-D plasticity and its behavior during the thermomechanical process

When a thickness dimension is much smaller than a lateral direction, and dominant deformation takes place in the thickness direction, then a 1-D plasticity model can be introduced in the thickness direction. In the 1-D plastic theory, the yield stress is defined on the basis of the mechanical strain and yielding only occurs under compressive stress.

Figure 4.13 shows the assumed 1-D plasticity material behavior in the thickness direction during the thermomechanical loading and unloading process. Suppose that a mechanical strain $e^m$ is applied at temperature $T$. At room temperature, the elastic function and the yield stress function are initially distinct functions respectively. The initial elastic function is shown as a thin dotted line and the yield stress function is shown as a thin solid line. As the temperature and the compressive strain increase, both the loading and unloading curves based on total strain must be mapped into a mechanical strain domain by removing the thermal strain. In this case, the elastic function begins to soften and becomes the thick dotted line. Also, the yield stress function begins to soften and becomes the thick solid line. Then, stress is evaluated from the thick dotted elastic...
function which is denoted as the trial stress, and the yield stress is calculated from the yield stress function which is shown as point 1 for a given mechanical strain $\varepsilon^m$. In this situation, if the magnitude of the trial stress is higher than the magnitude of the yield stress, then this belongs to a yield condition and the trial stress is modified by increasing the plastic strain. Therefore, the thick dotted line should be shifted to the gray solid line to satisfy the yield function. If the mechanical loading is removed from point 1 from the unloading process, then the yield stress becomes higher than the trial stress. In this case, yielding does not occur, the trial stress becomes a stress for the given mechanical strain, and the thick solid gray line becomes the unloading curve. Finally, point 2 is the permanent plastic strain when loading is completely removed.

4.5 CONSTITUTIVE MODEL DEVELOPMENT

4.5.1 The thermal and chemical deformation gradient tensor

It was assumed by Simo [4.18] and Holzapfel [4.6] that the total deformation gradient tensor under the thermo-mechanical loading in large deformation theory can be multiplicatively decomposed into thermal and mechanical deformation gradient tensors. In this chapter, based on this idea, the total deformation gradient tensor is assumed to be multiplicatively decomposed into thermal, chemical, and mechanical deformation gradient tensors as follows:

$$F^t = F^m F^\circ F^\varepsilon$$ (4.12)
where $F^m$ is a deformation gradient tensor due to a mechanical strain, $F^\theta$ is due to a thermal expansion, $F^\epsilon$ is a total deformation gradient tensor, and $F^\xi$ is due to a chemical strain. Suppose that there are 4 different configurations under the thermo-mechanical loading. The configurations are shown in Figure 4.14. $F^\epsilon$ is defined between the current configuration $\beta$ and the reference configuration $\beta_0$. $F^\xi$ is defined between the reference configuration $\beta_0$ and the intermediate configuration $\beta_1$. $F^\theta$ is defined between the intermediate configuration $\beta_1$ and the intermediate configuration $\beta_2$. $F^m$ is defined between the current configuration $\beta$ and the intermediate configuration $\beta_2$. Conceptually, Equation (4.12) illustrates that the reference configuration $\beta_0$ first proceeds to the intermediate configuration $\beta_1$ through chemical straining, to the intermediate configuration $\beta_2$ by thermal expansion, and then to the final current configuration $\beta$ through mechanical straining.

If thermal expansion and chemical strain are assumed to be orthotropic, they can be obtained from experiments and are be defined as follows:

$$F^\xi = \lambda^\xi_i E_i \otimes E_i + \lambda^\xi_2 E_2 \otimes E_2 + \lambda^\xi_3 E_3 \otimes E_3$$  \hspace{1cm} (4.13)

where $\otimes$ represents the tensor outer product and $\lambda^\xi_i$ is a stretch due to chemical strain which is defined as follows:

$$\lambda^\xi_i = e^{\epsilon_i}.$$  \hspace{1cm} (4.14)

In the case of the thermal expansion, $F^\theta$ is written as follows:

$$F^\theta = \lambda^\theta_1 E_1 \otimes E_1 + \lambda^\theta_2 E_2 \otimes E_2 + \lambda^\theta_3 E_3 \otimes E_3$$  \hspace{1cm} (4.15)

where $\lambda^\theta_i$ is stretch due to temperature.
4.5.2 Visco-hyper-elastic model development

Since the 1940s, elastic constitutive modeling for large deformation has been studied by many authors \[4.2, 4.11, 4.13, 4.19, 4.20, 4.22\]. Oden type strain energy, which is a function of principle stretches, has been widely used in many applications. In the case of the Oden strain energy function, Cauchy stress is obtained as follows:

\[
\sigma = \frac{1}{J^e} \sum_{i=1}^{3} \lambda_i^e \frac{\partial W(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} (n_i \otimes n_i)
\]  

(4.16)

where \(\lambda_i^e\) are principle stretches and \(n_i\) are principle directions in the current configuration.

Visco-elastic modeling in non-linear large deformation can be derived with a simple extension to classic linear visco-elastic modeling. This method has been studied by many authors \[4.7, 4.9, 4.14, 4.16, 4.17\]. In this case, Cauchy stress is assumed to be the difference between the instantaneous stress \(\sigma_0\) and the internal variable tensor \(q\) which is equivalent to stress relaxation:

\[
\sigma = \sigma_0 - q = \frac{1}{J^e} F^e \frac{\partial W_0}{\partial \varepsilon^e} F^{e^T} q - q.
\]  

(4.17)

Equation (4.17) can be solved with respect to \(q\) by using a linear visco-elastic model with a long term stress \(\sigma_-\) and a Prony series (see Papoulia \[4.14\] for details):

\[
\sigma = \sigma_- + \sum_{n=1}^{N} H_n
\]  

(4.18)

where

\[
\sigma_- = \frac{1}{J^e} F^e \frac{\partial W_-}{\partial \varepsilon^e} F^{e^T}
\]  

(4.19a)
\[ H_a = \int_0^t a_\alpha e^{\frac{1}{T}t(\tau-\tau)} \frac{d}{dt} \left( \frac{1}{2} F^\alpha \frac{\partial W}{\partial E^\alpha} F^{\alpha^T} \right) dt' \] (4.19b)

where \( a_\alpha \) and \( \tau_\alpha \) are material constants. Therefore, stress response is divided into a static and a transient term. Equation (4.19b) is an integral equation with respect to time and, depending on the complexity of the energy function, may not necessarily be analytically integrable. Therefore, a numerical integration scheme is required to implement the scheme in the finite element method.

Suppose that time \( t \) in Equation (4.19b) can be discretized into \( n+1 \) time steps. Then, the numerical integration is possible based on finding a stress at time step \( n+1 \) with given stress and history information at time step \( n \). With this idea, Equation (4.19b) can be written in the numerical integration form at time step \( n+1 \) by assuming that \( \sigma^a_\alpha \) is linear with respect to time during \( \Delta t \) and applying \( t^{n+1} = t^n + \Delta t \). In this way

\[ \sigma^{n+1} = \sigma^{n+1}_a + \sigma^{n+1}_{vis} = \sigma^{n+1}_a + \sum_{\alpha=1}^N H^{n+1}_a \] (4.20)

with

\[ H^{n+1}_a = e^{\frac{\Delta t}{T_a}} H^n_a + \left( \frac{\sigma^{n+1}_a - \sigma^n_a}{\Delta t} \right) a_\alpha \tau_\alpha (1 - e^{-\frac{\Delta t}{T_a}}). \] (4.21)

In Equation (4.21), \( H^{n+1}_a \) can be calculated for a given \( H^n \) because \( \sigma^{n+1}_a \) can be determined from the deformation gradient tensor. Therefore, stress can be calculated at any time step by updating Equation (4.20) with the initial condition \( H^0_a = 0 \).

For the visco-elasticity model at elevated temperature, it is assumed that the intumescent mat has very small rate effects at elevated temperature compared to room temperature. In most cases, the material coefficients \( a_\alpha \) and \( \tau_\alpha \) in Equation (4.22) are
different at different temperatures. Therefore, the material parameters should be measured at each temperature. However, in this study, to simplify the viscoelastic model, it is assumed that the time-dependent material properties do not change but the magnitude of the transient stress changes as a function of temperature. If Equation (4.20) is written in the simple form as follows:

$$\sigma(\varepsilon, t, T) = \sigma_\infty(\varepsilon, T) + \sigma_{\text{vis}}(\varepsilon, t, T)$$  \hspace{1cm} (4.22)$$
then the second term in Equation (4.22) can be multiplicatively decomposed into the temperature dependant function $\phi(T)$ and a temperature independent function:

$$\sigma(\varepsilon, t, T) = \sigma_\infty(\varepsilon, T) + \phi(T) \times \sigma_{\text{vis}}(\varepsilon, t).$$ \hspace{1cm} (4.23)$$

The function $\phi(T)$ is assumed to be a simple linear function which is 1 at room temperature and linearly decreases to zero at high temperature.

4.5.3 The plastic deformation gradient tensor $F^p$ and the yield condition

It was proposed by Lee [4.10] that the mechanical deformation gradient tensor under the mechanical loading in large deformation theory can be multiplicatively decomposed into an elastic and a plastic part:

$$F^m = F^e F^p$$ \hspace{1cm} (4.24)$$

where $F^m$ is the mechanical deformation gradient tensor, $F^e$ is the elastic deformation gradient tensor, and $F^p$ is the plastic deformation gradient tensor. Also, if it is assumed that material rigid body rotation only comes from the elastic deformation gradient tensor, and the plastic deformation does not contribute to any material rotation during the
mechanical deformation, then the plastic deformation gradient tensor can be simplified to a right stretch tensor. For thin materials like sheet or paper mat, it is assumed that plastic deformation only takes place in the thickness direction. Then, the plastic deformation gradient tensor is simply written as follows:

\[ \mathbf{F}^p = \mathbf{1} \mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{1} \mathbf{E}_2 \otimes \mathbf{E}_2 + \lambda^p \mathbf{E}_3 \otimes \mathbf{E}_3 \]  

(4.25)

where \( \lambda^p \) is a plastic stretch ratio in the thickness direction, \( \mathbf{E}_1, \mathbf{E}_2 \) and \( \mathbf{E}_3 \) are the material coordinate directions in the reference configuration, and \( \mathbf{E}_3 \) is the unit vector in the thickness direction. When yielding only occurs in the material thickness direction, a yield function can be simply defined in 1-D plasticity as follows:

\[ \mathbf{F} = \mathbf{e}_3 \cdot \mathbf{e}_3 - \lambda^p \mathbf{e}_3 \cdot \mathbf{y} = 0 \]  

(4.26)

where \( \mathbf{e}_3 \) is defined as a unit vector in the thickness direction in the current configuration. Physically, Equation (4.26) indicates that stress in the current thickness direction is the same as yield stress when yielding occurs. If there is a plastic strain during the deformation, the plastic deformation gradient tensor will satisfy the yield function (4.26). In this case, \( \lambda^p \) begins to decrease. Therefore, the plastic loading and unloading conditions are defined with \( \lambda^p \) as follows:

\[ \text{Loading condition} \quad \dot{\lambda}^p < 0 \text{ and } \lambda^p < 1 \]

\[ \text{Unloading condition} \quad \dot{\lambda}^p = 0 . \]
4.5.4 Stress due to material softening in the large deformation theory

Suppose that there exist two different bodies and they deform until they become the same shape and size. In this situation, one body deforms to the final shape by the deformation gradient tensor $F^e$ and the other body deforms to the final shape by the deformation gradient tensor $F'$. If the two different bodies generate the same stress field, then $F'$ and $F^e$ are called equivalent deformation gradient tensors. Material softening phenomenon can be understood by introducing the three different configurations as shown in Figure 4.15. Suppose that a reference configuration $\beta_3$ is the state before material softening and an intermediate configuration $\beta_4$ is the state after material softening. $F^e$ is defined as a deformation gradient tensor relating the current configuration $\beta$ and the reference configuration $\beta_3$. $F'$ is a deformation gradient tensor relating the current configuration $\beta$ and the intermediate configuration $\beta_4$. $F^h$ is defined as the deformation gradient tensor relating the reference $\beta_3$ and the intermediate configuration $\beta_4$ and is called a softening tensor. Then, the deformation gradient tensor $F^e$ is defined as follows:

$$F^e = F' F^h. \tag{4.27}$$

It is necessary to establish the relation between $F^e$ and $F'$ to relate material properties at different states. In large deformation theory, a mechanical part of the right Cauchy-Green metric tensor is used to represent a stress function. Therefore, it is assumed that the metric tensor $C'$ is the same as $C^e$ raised to the power $h$ (the scaling parameter). In this case, $C'$ is expressed as follows:
with $\mathbf{C}'$ and $\mathbf{C}^e$ having the same principal directions. Equation (4.27) explains that the material reference state $\beta_3$ expands to the intermediate state $\beta_4$ without changing mechanical properties, and stress is calculated based on the deformation gradient tensor $\mathbf{F}'$ if the material softens. That is, for a given actual deformation gradient tensor $\mathbf{F}^e$ due to material softening, stress decreases as the intermediate configuration gets smaller.

When the material softens, the corresponding Cauchy stress is obtained by applying the 2\textsuperscript{nd} law of thermodynamics theory. Suppose that the material point is characterized by the Helmholtz free energy $\psi$ as follows:

$$\psi = \hat{\psi}(\mathbf{C}', h, \theta)$$

(4.29)

where $\theta$ is temperature, $h$ is a softening parameter, and $\mathbf{C}'$ is a right Cauchy-Green metric tensor in the intermediate configuration. According to thermodynamic theory [4.4], by using an energy conservation equation and a Legendre transformation, the Clausius-Duhem inequality is written in the spatial description as follows:

$$-\rho \left( \frac{\partial \hat{\psi}}{\partial \theta} + \dot{\theta} \right) \dot{\theta} + \left( \frac{\dot{\sigma}}{h} - 2 \rho \mathbf{F}' \frac{\partial \hat{\psi}}{\partial \mathbf{C}'} \mathbf{F}'^T \right) \cdot \mathbf{L}' - $$

$$\left( \rho \frac{\partial \hat{\psi}}{\partial h} + \frac{1}{h^2} \dot{\theta} \cdot \sum_{i=1}^{3} \ln(\lambda_i) \mathbf{n}_i \otimes \mathbf{n}_i \right) \dot{h} - \frac{1}{\theta} \mathbf{q} \cdot \mathbf{g} \geq 0$$

(4.30)

where $\eta$ is entropy, $\sigma$ is the Cauchy stress tensor, $\mathbf{g}$ is the temperature gradient and $\mathbf{q}$ is the heat flux vector. To satisfy the inequality in Equation (4.30) for all thermomechanical processes, the following relation should hold:

$$\dot{\sigma} = 2 \rho h \mathbf{F}' \frac{\partial \hat{\psi}}{\partial \mathbf{C}'} \mathbf{F}'^T.$$  

(4.31)
For convenience, Helmholtz free energy $\tilde{\psi}$, defined as energy per unit mass, can be converted to energy per unit reference volume, and called strain energy $\hat{W}'$. For further simplification of Equation (4.31), $\hat{W}(C')$ is defined as follows:

$$\tilde{\psi} = \frac{W'}{\rho_{R4}} = \frac{\hat{W}(C')}{\hat{W}}$$

(4.32)

where $\rho_{R4}$ is the density in the intermediate configuration $\beta_4$ (see Figure 4.15). Cauchy stress with material softening is obtained as follows:

$$\hat{\sigma} = 2\frac{1}{J'}F'\frac{\partial \hat{W}}{\partial C'}F'^T.$$  

(4.33)

4.5.5 The static loading function $y_\sigma$ and the unloading function $\sigma_\sigma$ in plasticity

When there is no thermal and chemical strain, the stress function with material softening is derived in Equation (4.33). It is necessary to define the loading and the unloading functions with thermal, chemical strain and plastic strains.

Figure 4.14 shows the 5 different configurations for the loading function. For a given thermo-mechanical deformation, the material reference configuration $\beta_0$ changes to $\beta_1$ through chemical straining, changes to $\beta_2$ through thermal straining, and changes to $\beta_3$ through material softening. Finally, the loading function is evaluated between the current configuration $\beta$ and the intermediate configuration $\beta_5$. Therefore, for the loading stress function $y_\sigma$, the strain energy function is written as follows:

$$\hat{W}_L = \hat{W}_L(\lambda'^m, \lambda'^m, \lambda'^m), \quad \lambda'^m = \lambda^{m_L}.$$ 

(4.34a,b)
Also, the softening parameter \( h_L \) is a temperature dependent variable and is defined as follows:

\[
h_L = \text{Min}[h_L(\theta)]. \tag{4.35}
\]

The softening parameter \( h_L \) is a kind of damage parameter, and it always maintains a minimum value. Equation (4.34) implies that, when the stretch ratio in Equation (4.34b) is represented as a true strain in principle axis, the stress function can soften by stretching along the original strain axis by \( h_L \). Therefore, the strain energy function evolves to another function by changing to \( h_L \) from 1. Since, as shown in Figure 4.14, the loading stress \( y^- \) is defined between the current configuration \( \beta \) and the reference configuration \( \beta_0 \), the loading function is assumed to be derived from the strain energy potential for a given deformation gradient tensor \( F^m \) as follows:

\[
y^- = \frac{1}{J^m} \sum_{i=1}^{3} \lambda_i^m \frac{\partial \hat{W}(\lambda_1^m, \lambda_2^m, \lambda_3^m)}{\partial \lambda_i^m} (n_i \otimes n_i). \tag{4.36}
\]

In this case, the material constants can be obtained directly from the uniaxial compression test.

Figure 4.15 shows the 6 different configurations that are needed to define the unloading function under thermo-mechanical loading when there is material softening and plastic strain. Since the unloading function is a function of elastic strain, a plastic strain should be removed from the mechanical strain during the thermo-mechanical process. Therefore, the material configuration \( \beta_0 \) changes to \( \beta_1 \) through chemical straining, changes to \( \beta_2 \) by thermal straining, changes to \( \beta_3 \) through plastic strain and changes to \( \beta_4 \) by the material softening. Finally, the unloading function is evaluated
between the current configuration $\beta$ and the intermediate configuration $\beta_4$. For the unloading stress function $\sigma_\alpha$, the material softening due to mechanical strain is implemented by defining an energy function as follows:

$$W_u = \tilde{W}_u(\lambda_1^e, \lambda_2^e, \lambda_3^e), \quad \lambda^e = \lambda^{hu}. \quad (4.37)$$

The softening parameter $h_u$ is both a function of mechanical strain and temperature and is defined as follows:

$$h_u = \text{Min}[h_u(e^m)h_L(\theta)]. \quad (4.38)$$

The softening parameter $h_u$ is a kind of damage parameter and always maintains a minimum value. The unloading stress function $\sigma_\alpha$ is an elastic function and is defined between the current configuration $\beta$ and the intermediate configuration $\beta_4$ as shown in Figure 4.15. Cauchy stress is obtained by using principal stretches and directions as follows:

$$\sigma_\alpha = \frac{1}{\gamma^e} \sum_{i=1}^{3} \lambda_i^e \frac{\partial \tilde{W}_u(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} (n_i \otimes n_i). \quad (4.39)$$

### 4.5.6 Inclusion of rate effect in the plastic model

From the experimental results, it is seen that there are two kinds of rate effects in the intumescent mat material. One is a rate effect in the plastic region during the loading, and the other is a rate effect in the elastic region during the unloading. In this chapter, to simplify the 1-D plastic formulation with rate effects, it is assumed that the plastic function does not have any rate effects. Only the elastic function has the two different
rate dependant material properties. In the case of the elastic function, one viscous property is applicable only when plastic deformation occurs and the other applies only under elastic deformation. Therefore, conditions for different material properties will be determined from the static yield condition in Equation (4.26). Based on the yield criterion, two different material properties are used in each strain region. That is, the visco-elastic material properties obtained from the loading function are used when there is plastic deformation. When there is no plastic deformation on the mat material, then the visco-elastic material properties are switched to the ones that are obtained from the unloading function. Final stress is obtained by adding a rate term into the static term. Table 4.2 summarizes the different regions for visco-hyper-elasto-plastic model. Because the mat material only applies in compression, the loading function is assumed to have no rate effects when the static stress in the thickness direction is positive.

4.5.7 Numerical procedure for the stress calculation

Suppose that the total deformation gradient tensor $F'$ and temperature $\theta$ are given at each time increment. Since the chemical and the thermal deformation gradient tensors are only a function of temperature, the total deformation gradient tensor can be decoupled into the mechanical, the chemical, and the thermal deformation gradient tensors. Also, the strain energy functions for the loading and the unloading can be updated from the given mechanical strain and temperature. At each time increment, the elastic trial stress $\tilde{\sigma}_-$ and the yield stress $\sigma_-$ are calculated. The trial and yield stresses are compared to each other.
to check the yield condition. When yielding occurs during the thermo-mechanical process, the plastic stretch ratio \( \lambda^p \) in \( F^p \) can be calculated by the Newton-Raphson method for the given \( F^m \), the loading, and the unloading function. Initially, \( F^g \), \( F^0 \) and \( F^p \) are set to the identity tensor \( I \) at zero time increment. The procedure for calculating stress during the thermo-mechanical loading is outlined below:

For a given temperature and total deformation gradient tensor at each increment,

i) Calculate \( F^g \), \( F^0 \) and \( \phi(T) \) for given temperature through Equation (4.13) and (4.14).

ii) Calculate \( F^m \) from \( F^e \), \( F^g \) and \( F^0 \) through Equation (4.12).

iii) Update the loading and the unloading strain energy function in Equation (4.34) and (4.37) from the softening parameter \( h_L \) and \( h_U \).

iv) Calculate the elastic deformation gradient tensor by \( \tilde{F}^{e^{n+1}} = F^{m^{n+1}}F^{p-1} \) with \( F^p \) at time step \( n \).

v) Calculate the elastic trial stress \( \tilde{\sigma}_e \) and the yield stress \( y_\infty \) by Equation (4.39) and (4.36).

vi) Check the yield condition by Equation (4.26).

If yield occurs,

i) Update \( F^0 \) by the Newton Raphson method until the yield condition is satisfied.

ii) Calculate \( \sigma_- \) by Equation (4.39).
iii) Calculate $\sigma^p_{\text{us}}$ by using Equation (4.20) with $a = a^p_a$ and $\tau = \tau^p_a$.

iv) Write the stress as $\sigma = \sigma_\text{us} + \phi(\theta)\sigma^p_{\text{us}}$.

If yield does not occur,

i) Calculate $\sigma^p_{\text{us}}$ by using Equation (4.20) with $a = a^e_a$ and $\tau = \tau^e_a$.

ii) Write the stress as $\sigma = \tilde{\sigma}_\text{us} + \phi(\theta)\sigma^p_{\text{us}}$.

4.6 VERIFICATION OF THE CONSTITUTIVE MODEL

An Ogden type strain energy function [4.1] is used for both the loading and unloading functions. The loading function $\hat{W}_L$ in Equation (4.37) and the unloading function $\hat{W}_U$ in Equation (4.40) are defined as follows (subscript $L$ indicates loading and $U$ indicates unloading):

\[
\hat{W}_L(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu_L}{a_L^2} \left[ \lambda_1^{\text{eq}} + \lambda_2^{\text{eq}} + \lambda_3^{\text{eq}} - 3 + \frac{1}{\beta_L}(1^{-\alpha_L\beta_L} - 1) \right] \tag{4.40}
\]

\[
\hat{W}_U(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu_U}{a_U^2} \left[ \lambda_1^{\text{eq}} + \lambda_2^{\text{eq}} + \lambda_3^{\text{eq}} - 3 + \frac{1}{\beta_U}(1^{-\alpha_U\beta_U} - 1) \right] \tag{4.41}
\]

where $\alpha_L$, $\mu_L$ and $\beta_L$ are the material parameters for the loading curve, and $\alpha_U$, $\mu_U$ and $\beta_U$ are the material parameters for the unloading curve. The constitutive model is implemented in the ABAQUS™ user subroutine called UMAT.
A solid element is used to verify the constitutive model. The size of element is 10mm*10mm*10mm. At room and elevated temperature, the experimental results are compared with simulations for the loading and unloading during the single or multiple cycles. In the case of the single compression test at different compression speeds, the bottom surface nodes of the element are fixed in all directions, and compression is applied in the thickness direction to the top surface nodes of element. In the case of the cyclic compression testing, if the mat response is very slow compared to the machine response, and if there is plastic deformation during the loading, the mat will detach from the loading rod during the displacement unloading. Therefore, to avoid inadvertently applying tension to the mat, the simulations employ a contact surface between the loading bar and the top surface of the mat. The contact surface is defined to represent the relation between the loading machine and the mat specimen. The bottom surface is fixed in all directions. Several cycles of displacement loading and unloading are applied to the rigid plate in the Z direction.

Figure 4.16 shows a comparison of rate effects between the experiment and simulation during the 1st loading process up to 60% compression. It shows a good agreement between the experiment and the simulation at each compression speed in all strain ranges except the small strain region. In the small strain region, the simulations show a higher stress than the experiment results, especially as ram speed increases. It appears that the mat material has different rate effects at different compression strain levels.
Figure 4.17 shows a comparison of the cyclic loading and unloading curves when 60% compression is applied with 1.6mm/s ram speed during two cycles at room temperature. The experimental loading and unloading curves for the 1st and 2nd cycle are in good agreement with the simulations.

Figure 4.18 shows a comparison between the simulations and the constant load experiment at elevated temperature. In the simulation, temperature is first applied to all nodes so that the mat will experience the thermal and chemical expansions. Then pressure is applied to the top surface of the mat to obtain a stress versus strain at the specified temperature. As shown in Figure 4.18, good agreement is seen between the simulation and the experimental results in all temperature ranges. Figure 4.19 shows a stress versus strain comparison during the thermal unloading process. It shows good agreement between the experiment and the simulation results.

4.7 F.E. ANALYSIS OF A CATALYTIC CONVERTER

By using the newly developed constitutive model for the mat material, the canning, heating and cooling processes for a catalytic converter are modeled. Figure 4.20 shows the canning process for catalytic converter. The canning process is an assembly process that packages a mat, an outer shell and a substrate into one part. After the outer shell is closed, the flange is welded to prevent can opening. To simulate the heating and cooling experiment, temperature is imposed on the mat and the outer shell after the canning. Simulation results are compared with experimental data.
4.7.1 Modeling assumptions and boundary conditions

Catalytic converter analysis requires models of the mat, the substrate and the outer shell. However, if the stiffness of the substrate is very high compared to the mat and the outer shell, and the stress distribution in the substrate is not important, then the substrate can be treated as a rigid surface. Also, if the substrate has negligible thermal expansion compared to the other materials, then the rigid surface can be simplified by using restraint boundary condition. In this situation, the rigid surface is excluded from the F.E. model. Figure 4.21 shows a quarter model for the catalytic converter. The outer shell is modeled with 2012 shell elements and the mat material is modeled with 712 solid elements by using one layer element through the thickness direction.

For the simulation of the actual canning process, contact elements are defined between the mat and the outer shell because each component is separate before assembly and may not penetrate each other during the simulation. Also, finite sliding should be allowed in the contact element between the mat and the outer shell to complete the canning process. However, this finite sliding contact causes a long computational time for the canning analysis. Also, because it creates excessive distortion of the solid element around flange, there is a high chance of numerical difficulties caused by the contact element. Therefore, to simplify the canning analysis, shrink fit contact is used between the outer shell and the mat. Figure 4.22 shows a simplified canning process by using shrink fit contact. In the shrink fit contact, the upper and the lower outer shell are welded first. The undeformed mats are placed in the right assembly position without the contact
element in an initial setup. In this case, the mat and the outer shell penetrate each other before the simulation begins. During the simulation, the contact is defined and the mat begins to shrink little by little to interact with the outer shell until no further penetration is present. Finally, the mat and the outer shell reach to an equilibrium position, and the deformed shape and stress are obtained from the F.E. model.

Figure 4.23 shows applied temperature and restraint boundary conditions for the F.E. quarter model. There are two symmetric planes for the quarter model. Symmetry boundary conditions are imposed on the symmetry planes. In the case of the nodes on the welding line, the nodes must be allowed to move around on the welding plane without detaching from the welding plane. Therefore, the symmetric boundary conditions are applied to those nodes on the welding line. The front-most nodes of the outer shell are restrained to prevent a rigid body motion in the X-direction. The inner surface of the mat is fully restrained in all directions during the canning process.

After the canning analysis, the restraint boundary conditions of the nodes on the inner surface of the mat are changed to allow thermal expansion of the mat and the outer shell in the X-direction. Then, the temperature boundary condition is applied to all the nodes on the catalytic converter to simulate the heating process. Temperature for the outer shell is measured experimentally at ArvinMeritor (Columbus, IN). The temperature for the mat is obtained from CFD (STAR-CD) and simulation is exported to the F.E. nodes. Finally, the cooling simulation is performed by imposing room temperature on all nodes after the heating analysis is done.
4.7.2 Comparison of simulations and experiments

To verify the F.E. results, actual heating and cooling tests are performed for the catalytic converter after the canning process. The room temperature Z-deflection, defined as a converter height difference between the state before experiment and the state after experiment, is measured. Figure 4.24 shows several lines along which key results will be shown. Z-displacements and contact pressures are examined along these lines.

Figure 4.25 shows the comparison of the Z-deflection between the experiment and the simulation along line 1 shown in Figure 4.24. Although the Z-deflection trend is very similar, the simulations differ from the experiment by up to 0.2mm. However, since the total displacement during the heating is greater than 2.5mm in the Z-direction, the simulations are considered to be in good agreement with the experimental result. The Z-deflection is closely coupled to the vermiculite expansion, which is determined by temperature. This implies that accurate mat temperature distributions are very important for predicting accurate deflection. Figure 4.26 shows the Z-displacement along line 1 before and after the experiment. Figure 4.27 shows Z-displacement distributions during the canning, heating and cool-down processes. As temperature increases, the Z-displacement reaches up to 2.5mm in the middle zone. Due to the permanent vermiculite expansion, this Z-displacement remains after the cool-down process.

Figure 4.28 shows the contact pressure distributions (von Mises stress) for the mat material during the canning, heating and cool-down processes. Most of the high stress is distributed along line 2. The contact pressure is the highest after canning. In spite of the
mat expansion, the contact pressure is lowest during heating due to the outer shell expansion. Figure 4.29 shows the contact pressure along line 3. Figure 4.30 shows the contact pressure along line 1 and 2 during the heating. There are many contact pressure fluctuations due to the ribs, and the contact pressure is low in the middle due to the high Z-displacement.

From the simulation results, the contact pressure between the mat and the outer shell and the Z-deflection on the shell are predicted with reasonable accuracy. Therefore, it is expected that the numerical model for mat can be used as a predictive tool to reduce time and cost in the development of catalytic converters.

4.8. CONCLUSION

This chapter has presented the development of a constitutive model for intumescent mat under thermo-mechanical loading. To characterize the mechanical behavior of the intumescent mat, constant load and cyclic compression experiment were performed at room and high temperature. The experimental results show that the mat is a highly compressible and time dependant material. It expands significantly due to the vermiculite expansion with increasing temperature. Also, the mat has different loading and unloading material behavior. The material behavior is quite similar to metal plasticity with softening as temperature and mechanical strain increase. The total strain under the thermo-mechanical loading is decomposed into a mechanical, a chemical, and a thermal strain. The chemical and the thermal strains are assumed to be a function of temperature.
and are defined through a shift strain. From the assumption of material similarity and shift strain, stress versus mechanical strain relations at different temperatures are obtained from stress versus total strain experimental data. To relate different stress curves at different temperatures, the concept of a master curve is proposed in the constitutive equation. The 1-D plasticity model is introduced for the case when the mat has a small dimension in the thickness direction compared to the lateral direction and has dominant deformation in the thickness direction. In the plasticity model, a new plastic deformation gradient tensor and yield criteria are proposed. Material softening due to temperature and mechanical strain are implemented to the proposed plasticity model. The rate effects are considered in both the loading and unloading functions. The material formulation is developed based on a visco-hyperelastic theory and is coded in ABAQUS™ implicit through the user subroutine UMAT. Experimental and the simulation results are compared to each other for several cases to verify the constitutive model. For wide ranges of strain, good agreement is found for the rate dependent loading and unloading curves at room and elevated temperature. The constitutive model is applied to the canning, heating and cooling of a catalytic converter. The simulations are compared with the experiments, and satisfactory results are achieved. Therefore, the constitutive model is expected to be applicable as a predictive tool in the early design stages for catalytic converters.

Material softening due to elevated temperatures and rate effects at elevated temperatures have not been thoroughly investigated experimentally. With more high temperature experimental data, it is possible that modifications to the constitutive model would be needed. This is a potential area for future work.
REFERENCES


Table 4.1: Softening parameter $h_U$ versus applied mechanical strain for unloading function.

<table>
<thead>
<tr>
<th>Mechanical strain</th>
<th>0.8</th>
<th>0.9</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Softening parameter $h_U$</td>
<td>1</td>
<td>0.98</td>
<td>0.92</td>
</tr>
<tr>
<td>Zero strain</td>
<td>0.532</td>
<td>0.6</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Table 4.2: Stress calculation procedure for thermo-visco-hyper-elasto-plastic model.

<table>
<thead>
<tr>
<th>Plastic region</th>
<th>Elastic region</th>
</tr>
</thead>
<tbody>
<tr>
<td>Condition</td>
<td>$\dot{\lambda}^p &lt; 0$ and $\lambda^p &lt; 1$</td>
</tr>
<tr>
<td>Visco material property</td>
<td>$a = a^p_a$ and $\tau = \tau^p_a$</td>
</tr>
<tr>
<td>Stress at room T.</td>
<td>$\sigma = \sigma_m + \sigma^p_{vis}$</td>
</tr>
<tr>
<td>Stress at elevated T.</td>
<td>$\sigma = \sigma_m + \phi(\theta)\sigma^p_{vis}$</td>
</tr>
</tbody>
</table>
Figure 4.1: A sectional view of a catalytic converter.

Figure 4.2: A schematic representation of vermiculite expansion due to high temperature.
Figure 4.3: Stress versus strain curves at different compression speeds from compression experiments at room temperature (60% compression).

Figure 4.4: Typical cyclic compression results for 40% compression at room temperature (v=1.6mm/s).
Figure 4.5: Comparison of monotonic loading and cyclic loading from constant load experiments at room temperature and under static conditions.

Figure 4.6: Comparison of 1st and 2nd loading curves for different compressive strains at room temperature (v=1.6mm/s).
Figure 4.7: Relative expansion results for the 800°C constant load experiments.

Figure 4.8: Stress versus total strain curves during loading at different temperatures.
Figure 4.9: Stress versus total strain curves during unloading at different temperatures.

Figure 4.10: The master curve together with loading test data at different temperatures with material softening removed.
Figure 4.11: The softening parameter and shift strain at different temperatures for the loading function.

Figure 4.12: The master curve together with room temperature unloading test data at different compressive stresses with material softening removed.
Figure 4.13: Material behavior through 1-D plasticity under thermo-mechanical loading.

Figure 4.14: Configurations for the loading function and their relationships during thermo-mechanical loading and material softening.
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Figure 4.16: Comparison of simulation (analysis) results with test data at different compression speeds for the 1st loading process (60% compression).
Figure 4.17: Simulation (analysis) comparison with the 60% cyclic compression test during 2 cycles at room temperature (v=1.6mm/s).

Figure 4.18: Comparison of simulation (analysis) and experimental loading stress versus strain curves at different temperature.
Figure 4.19: Comparison of simulation (analysis) and experimental unloading stress versus strain curves at different temperature.

Figure 4.20: A schematic of the cross section of a clamshell design catalytic converter and the canning process.
Figure 4.21: One quarter of the catalytic converter F.E. model and the contact surfaces.

Figure 4.22: The simplified canning process by "shrink fit contact".
Figure 4.23: Restraint and temperature boundary conditions for the F.E. model.
Figure 4.24: Definitions of lines for which comparison data is output.
Figure 4.25: Comparisons of experimental and simulation (analysis) results from the canning process. Results are shown along line 1 defined in Figure 4.24.

Figure 4.26: The Z-displacements comparison along line 1.
Figure 4.27: Z-displacement contours during canning, heating and cool-down processes.
Figure 4.28: von Mises stress distributions during canning, heating and cool-down process.
Figure 4.29: The contact pressure distribution along line 3.

Figure 4.30: The contact pressure distributions along line 1 and 2 after the heating process.
CHAPTER 5

CONCLUDING REMARKS AND FUTURE WORKS

This dissertation has detailed the procedure for developing a constitutive model for intumescent mat which is used to support catalytic converter during the thermo-mechanical loading. To characterize the thermo-mechanical material behavior of intumescent mat, constant loading and cyclic compression tests were designed and performed at room and elevated temperature on the basis of total strain. Test results are summarized as follows.

- Mat is highly compressible transversely isotropic visco-elastic material with Poisson's ratio = 0.
- Mat only works under compressive stress and has very small resistance in tension.
- Mat is highly time dependant at room temperature.
- Mat has different loading and unloading stress function. Each function has different viscous effects and has material behavior similar to that of metal plasticity.
• The loading function has permanent material softening due to temperature and the unloading function has permanent material softening due to both mechanical strain and temperature.

• As temperature increases, mat expands dramatically due to vermiculite expansion.

• The thermal and chemical strains are orthotropic in material direction, and the chemical strain is irreversible.

From the test results, several assumptions are made to simplify the material model and the constitutive model is developed based on hyperelastic theory. Mat is defined as a highly compressible isotropic foam material which has different loading and unloading functions. The loading function is assumed to be yield stress and the unloading function is as an elastic function. Two functions are related by defining a plasticity model which only exists in the thickness direction for the limited case when mat has a small dimension in the thickness direction compared to the lateral direction and has a dominant deformation in the thickness direction. In the plasticity model, a new plastic deformation gradient tensor and yield criteria in the thickness direction are proposed. Material similarity is introduced for both the loading and the unloading function to take into account mechanical property changes due to temperature and mechanical strain. They are represented with one master curve and a softening parameter at different temperatures and mechanical strains. Viscous effects are included to the time independent stress-strain function by extending infinitesimal visco-elastic theory to large deformation theory. Rate effects are considered in both the loading and the unloading function. During the thermo-
mechanical loading process, total strain is decomposed into mechanical, chemical and thermal strain. Chemical and thermal strains are assumed to be only a function of temperature and defined with a shift strain. From the assumption of material similarity and shift strain, stress versus mechanical strain relations at different temperature is obtained from stress versus total strain test data. Rate effects at elevated temperature are implemented by introducing a temperature dependant scale function. Material formulation is developed based on an Ogden type strain energy function and coded in ABAQUS™ implicit by using the user subroutine UMAT. Experimental results and simulation are compared to each other for several cases for verification of the constitutive model. The constitutive model is applied to canning, heating, and cooling analyses for catalytic converter. Analysis results are compared with test results and satisfactory results are achieved.

From this dissertation, several academic contributions are achieved and are summarized as follows:

- Experimental method are designed and performed to investigate material behavior of intumescent mat.
- Cyclic loading and unloading phenomenon are studied for foam material in view of visco-elasticity and plasticity theory.
- A new isotropic material softening model is proposed in large deformation theory to account for material property change due to temperature and mechanical strain.
• The general form of the Helmholtz free energy function is proposed for anisotropic material to account for material property change due to temperature.

• Complete stress function and spatial elasticity tensors are derived based on the 2nd law of thermodynamics when there is material softening for implicit finite element code.

• An isotropic material softening model is applied to the incompressible anisotropic hyperelastic material.

• The constitutive model is developed as a predictive tool in the early design stage for catalytic converters.

Though great progress has been made on the constitutive model development of intumescent mat, several experimental and modeling issues remain unclear. Therefore, study of the following items is suggested as future study.

• Two sources of material softening are introduced for loading and unloading functions. However, material softening due to temperature for the unloading function is not clearly defined and is assumed to be same as the loading function. Also, rate effects at elevated temperature are not completely determined from compression experiments. Therefore, more experiments at high temperature are needed and must be implemented in future models.

• Anisotropic material softening is not considered in the constitutive model development. Only isotropic material softening is considered for both isotropic
material and anisotropic material to consider material property changes. Therefore, to generalize the material softening model for anisotropic materials, anisotropic softening model development is required.

- Further study is required to predict accurate loading and unloading processes after several cycles. Good results are shown up to the 2\textsuperscript{nd} cycle by employing plasticity and visco-elasticity modeling, but continuing cyclic damage is found after 2\textsuperscript{nd} cycle. Therefore, inclusion of damage modeling is required together with different time dependant effects at different compressive strains.
BIBLIOGRAPHY


