SOLUTIONS TO SOME ELECTROMAGNETIC
BOUNDARY VALUE PROBLEMS

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I. INTRODUCTION

The traveling wave slot antenna has been the object of study of many recent investigations.\textsuperscript{1-4} Such an antenna consists of a long slot cut in the wall of a conducting body and excited so as to produce a field traveling along the length of the slot. A common type consists of an axially slotted waveguide opening into space bounded by a cylindrical conducting surface, as illustrated by figure 1. A perturbation technique for the calculation of the complex propagation constant (attenuation and phase) is described in reference 1. If the cylinder is assumed infinite in length, and the tangential electric field in the slot is known, the radiation field can be calculated by known formulas.\textsuperscript{5} A variational method for determining the complex propagation constant of the fields along a slotted waveguide is given herein.

For the analysis, it will be assumed that the slot and cylinder are infinite in length, the walls perfectly conducting, and the dielectric homogeneous everywhere. Consider the waveguide and external cylinder to be coordinate surfaces of two sets of orthogonal cylindrical coordinates, one set applying to the external region and the other to the internal region. No notational distinction between the two
Conducting Cylinder

Waveguide

Long Slot Common To Cylinder And Waveguide

Fig. 1. The traveling wave slot antenna in a cylinder.

Fig. 2. Orthogonal cylindrical coordinates.
coordinate sets will be made at present, since the following is applicable to both regions, and over the slot the two sets coincide.

The orthogonal cylindrical coordinates are defined as the set \( \xi, \eta, z \), for which the given cylinder is the coordinate surface \( \xi = a \).

Unit vectors in the \( \xi, \eta, z \) directions are designated by \( u_1, u_2, k \), respectively, where at the given cylinder \( u_1 \) is perpendicular to the surface and \( u_2 \) is parallel to the surface. The differential element of length is given by

\[
dl = u_1 h_1 d\xi + u_2 h_2 d\eta + k dz. \tag{1}
\]

The coordinate system is illustrated by figure 2.

II. DETERMINATION OF THE FIELDS

Only those fields which are characterized by a single propagation constant in the \( z \) direction and harmonic time variation are to be considered, that is, all field quantities vary as \( e^{\gamma z + j\omega t} \). Such a field may be thought of as being generated by a sheet of currents flowing on an infinite plane perpendicular to the \( z \) axis. An arbitrary field can be represented as the sum of a transverse magnetic (TM) field and a transverse electric (TE) field, such that

\[
E = E^e + E^m, \quad H = H^e + H^m, \tag{2}
\]

where the superscripts \( e \) and \( m \) designate TE and TM components, respectively. The TE field can be expressed in terms of a Hertzian vector \( \mathbf{e}^e \) having only a \( z \) component, and the TM field in terms of \( \mathbf{H}^m \) having
only a $z$ component. Both $\pi^e$ and $\pi^m$ satisfy the scalar wave equation

$$\nabla^2 \pi_z + \beta^2 \pi_z = 0. \quad (3)$$

The electric and magnetic fields in terms of the $\pi$ vectors are given by

$$\mathbf{E} = -\omega \mathbf{A} \times \mathbf{e} + \nabla \times \nabla \times \mathbf{m} \quad \mathbf{H} = \nabla \times \nabla \times \mathbf{m} + j \omega \mathbf{E} \times \mathbf{m}. \quad (4)$$

In orthogonal cylindrical coordinates, the field components are given by

$$
\begin{align*}
E_z &= \kappa^2 \pi_z \\
E_\varphi &= \frac{\gamma \partial \pi_z}{h_1 \partial \xi} \\
E_\eta &= \frac{\gamma \partial \pi_z}{h_2 \partial \eta} \\
H_\varphi &= \frac{j \omega e \partial \pi_z}{h_2 \partial \xi} \\
H_\eta &= \frac{-j \omega e \partial \pi_z}{h_1 \partial \eta} \\
H_z &= 0 \\
e_z &= 0 \\
e_\varphi &= \frac{-j \omega e \partial \pi_z}{h_2 \partial \xi} \\
e_\eta &= \frac{\gamma \partial \pi_z}{h_1 \partial \eta} \\
e_z &= \frac{\gamma \partial \pi_z}{h_1 \partial \xi} \\
e_\varphi &= \frac{j \omega e \partial \pi_z}{h_2 \partial \xi} \\
e_\eta &= \frac{-j \omega e \partial \pi_z}{h_1 \partial \eta} \\
e_z &= \kappa^2 \pi_z.
\end{align*}
(5)$$

where

$$\kappa^2 = \beta^2 + \gamma^2. \quad (6)$$

The fields must satisfy the boundary conditions $E_z = E_\varphi = H_\varphi = 0$ on the surface of the conductors. From $(5)$ it is seen that $E_z = 0$, \ldots
so that the TM field must independently satisfy the condition $E_z = 0$.

The conditions $E_h = H_5 = 0$ on the conductors require

\[
\frac{\nu}{h_2} \frac{\partial E_z}{\partial \eta} + \frac{\omega}{h_1} \frac{\partial H_z}{\partial \xi} = 0 ,
\]

\[
\frac{\omega \mu}{h_2} \frac{\partial H_z}{\partial \eta} + \frac{\gamma}{h_1} \frac{\partial E_z}{\partial \xi} = 0 ,
\]

which can be satisfied only if $E_h = m = H_5 = H_5 = 0$ on the conductors, or $\gamma^2 = -\omega^2 \mu \varepsilon$. The latter possibility implies that the fields are propagated only in the axial direction with the speed of light, that is, $\lambda^2 = 0$ and the fields are TEM (both transverse electric and transverse magnetic). Thus one can conclude that all boundary conditions must be satisfied individually by the TE field and by the TM field, and each can exist independently of the other. There is therefore no loss of generality if consideration is restricted to fields which are either TE or TM.

By an application of Green's theorem, the internal and external fields can be related to the tangential electric field over the slot, and the problem reduces to a two dimensional one. The development for the internal field and that for the external field are almost identical, so only the latter will be given. Define the Green's function $G$ as the scalar potential from an oscillating line source parallel to the $z$ axis at the position $p$, of excitation $e^{\gamma z}$, and satisfying some prescribed boundary condition on the original cylinder. Consider the volume $V$ surrounded by the surface $\Omega$ consisting of the original cylinder $S$, a small circular cylinder $\sigma$ about $p$, a large circular cylinder $\Sigma$ en-
closing the other two, and two transverse planes separated by a distance $\delta$. Figure 3 gives a cross section of this volume.

\[ \nabla^2 G + \beta^2 G = 0 \]  
and varies as $e^{Yz}$. By Green's second identity,

\[ \int_V \left[ \pi_\Sigma (\nabla^2 + \beta^2)G - G(\nabla^2 + \beta^2)\pi_\Sigma \right] dV = \int_S \left[ \gamma_\Sigma \frac{\delta G}{\delta n} - \frac{\delta G}{\delta \Sigma} \right] d\omega, \]

where $\pi_z$ can be either $\pi_z$ or $\pi^m_z$, as previously defined. The volume integral vanishes by (3) and (8), and the surface integral is broken up into its contributions from the two planes and three cylinders as

\[ \int_{\text{top}} + \int_{\text{bottom}} + \int_S + \int_{\Sigma} \left[ \pi_\Sigma \frac{\delta G}{\delta n} - \frac{\delta G}{\delta \Sigma} \right] d\omega = 0. \]

The first two integrals of (10) vanish since $\delta G/\delta n = \gamma G$ and $\delta \pi_z/\delta n = \pi^m_z$.

Fig. 3. Cross section of the volume $V$. 

Everywhere in $V$, Green's function satisfies
γπz over the top and bottom planes. If one now lets \( S \to 0 \), the three remaining integrals of (10) become line integrals, and the exponential \( z \) variation cancels. The integral over \( \Sigma \) is evaluated as follows.

Let the circle \( \Sigma \) be specified by \( \rho = R \) in the polar coordinates \( \rho, \phi \).

External to the given cylinder the field can in general be represented by

\[
\pi_z = \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(\kappa \rho) e^{in\phi},
\]

and the Green's function by

\[
G = \sum_{k=-\infty}^{\infty} b_k H_k^{(2)}(\kappa \rho) e^{jk\phi}.
\]

The integral to be evaluated is

\[
\int_0^{2\pi} \left[ \pi_z \frac{\delta G}{\delta \rho} - G \frac{\delta \pi_z}{\delta \rho} \right] R d\phi.
\]

Substituting for \( \pi_z \) and \( G \) into (13), and integrating, gives

\[
2\pi R \kappa \sum_{n=-\infty}^{\infty} a_n b_n \left[ H_n^{(2)}(\kappa R) H_n^{(2)}(\kappa R) - H_n^{(2)}(\kappa R) H_n^{(2)}(\kappa R) \right],
\]

and the integral over \( \Sigma \) vanishes. In the vicinity of the line source,

\[
G \xrightarrow{r \to 0} - \log r,
\]

where \( r \) is the radial distance from \( p \). The integral over \( \sigma \) becomes

\[
\lim_{r \to 0} \int_0^{2\pi} \left[ \pi_z \frac{1}{r} \frac{\delta \pi_z}{\delta r} \log r \right] R d\phi,
\]

where in the limit, \( \pi_z \to \pi_z(p) \) and \( r \log r \to 0 \). Thus (16) becomes
-2\pi \pi_z(p), and (10) becomes

\[ \pi_z(p) = \frac{1}{2\pi} \oint_{S} \left[ \pi_z \frac{\partial G}{\partial \xi} - G \frac{\partial \pi_z}{\partial \xi} \right] n_1 \, d\gamma \quad \text{ (17)} \]

By specifying boundary conditions on G, the field components at the point \( p(\xi', \eta') \) can be determined from (17) in terms of the tangential electric field over the slot for the TE and TM cases. In particular, for the TE case, specify

\[ \frac{\partial G}{\partial \xi} = 0 \quad \text{for} \quad \xi = a \quad \text{ (18)} \]

that is, the normal derivative of G vanishes on the given cylinder. Then, utilizing (5), equation (17) becomes

\[ H_z(\xi', \eta') = \frac{-\kappa^2}{2\pi j \omega} \int_{\text{slot}} E_\eta \, G \, n_1 \, h_2 \, d\eta \quad \text{ (19)} \]

where the integration need be carried out only over the slot since \( E_\eta = 0 \) elsewhere on S. For the TM case, let

\[ G^* = 0 \quad \text{for} \quad \xi = a \quad \text{ (20)} \]

where the * signifies that the Green's function is that obtained for G vanishing on the cylinder S. By (5) and (17)

\[ H_\eta(\xi', \eta') = \frac{-\kappa e}{2\pi \kappa h_1} \int_{\text{slot}} H_\eta \, \frac{\partial^2 G}{\partial \xi' \partial \xi} \, h_2 \, d\eta \quad \text{ (21)} \]

where again the integration is carried out only over the slot.

The fields inside the waveguide are given by equations identical to (19) and (21), where the point p is now inside the waveguide and G is the potential from a line source at p subject to the same boun-
dary conditions on the waveguide walls as (18) and (20). In the subsequent work, the subscript \( i \) will denote the region internal to the waveguide, and the subscript \( e \) will denote the external region.

III. VARIATIONAL METHOD

The exact determination of \( \kappa \) (or \( \gamma \)) for the fields is a difficult problem entailing the matching of tangential \( H \) for internal and external fields. This can be carried out in a formal manner, but is impractical for obtaining numerical values. Therefore, a variational method will be applied, similar to that used by Schwinger for the treatment of obstacles in waveguides, and by Lucke on corrugated surfaces.

Let \( p \) approach the cylinder \( S \) for the external region and approach the waveguide wall for the internal region, that is, \( \xi_e' \to a \) and \( \xi_i' \to a' \). Then equations (19) and (21) will give tangential \( H \) on the cylinder in terms of tangential \( E \) over the slot. In order to treat both the TE and TM cases simultaneously, the following symbols will be used:

**TE Case**

\[
\begin{align*}
E &= E_n \\
H &= H_{z1} - H_{ze} \\
\psi &= -\kappa^2(g_1 - G_0) \\
h &= h_1 h_2
\end{align*}
\]

**TM Case**

\[
\begin{align*}
E &= E_z \\
H &= H_{n1} - H_{ne} \\
\psi &= -\frac{j \omega e}{2\pi \kappa^2} \left[ \frac{\partial^2 G_0^*}{\partial \xi_e \partial \xi_e} - \frac{\partial^2 G_i^*}{\partial \xi_e \partial \xi_e} \right] \\
h &= h_2
\end{align*}
\]

Consider the integral equation

\[
\int_{\text{slot}} E(\eta') H(\eta') h_1 h_2 \, d\eta' = 0 \quad (22)
\]
This equation is an identity for the true fields, since $H$ is everywhere zero over the slot. It will be shown that if $E$ is chosen to a first order approximation, $\kappa$ obtained from (22) will be correct to the second order. In other words, $\kappa$ determined by (22) is relatively insensitive to variations of $E$ about its correct value.

By substitution of (19) or (21) into (22), one obtains

$$\int_{\text{slot}} dh' h' E(\eta') \int_{\text{slot}} dh h E(\eta) \psi'(\eta, \eta', \kappa) = 0 . \quad (23)$$

Let the assumed $E$ be given by $E + \delta E$, expand $\psi$ about the true $\kappa$, and substitute into (23) giving

$$\int_{\text{slot}} dh' h' (E + \delta E)' \int_{\text{slot}} dh h (E + \delta E) \left[\psi + \delta \kappa \frac{\partial \psi}{\partial \kappa} + \cdots\right] = 0 , \quad (24)$$

Expanding (24) and retaining only first order terms, one obtains

$$\int dh' h' E' \int dh h E \psi + \int dh' h' E' \int dh h \delta E \psi' + \int dh' h' \delta E' \int dh h E \psi' + \delta \kappa \int dh' h' E' \int dh h E \frac{\partial \psi}{\partial \kappa} = 0 . \quad (25)$$

Note that $\psi'(\eta, \eta', \kappa) = \psi'(\eta', \eta, \kappa)$, since by reciprocity the potential at a point $\eta'$ from a line source at $\eta$ is equal to the potential at the point $\eta$ from a line source at $\eta'$. Thus the second and third integrals of (25) are identical and equal to

$$\int_{\text{slot}} \delta E(\eta') H(\eta') h_1 h_2 d\eta' = 0 , \quad (26)$$

since $H(\eta')$ is zero over the slot. Also, the first integral of (25) vanishes by (22) giving
Since the value of the integrals cannot in general be zero, this gives the desired result, \( \delta k = 0 \).

An analogous formulation of the problem could be accomplished in terms of tangential \( H \) over the waveguide and cylinder walls. However, a reasonable assumption of tangential \( H \) for the problems considered, from which to determine \( \kappa \), would be difficult to make, whereas tangential \( E \) can be judiciously chosen for most cases.

In summary, the proposed method consists of: (a) solution of the scalar problem of the potential from a line source in the internal and external regions, subject to prescribed boundary conditions on the waveguide and cylinder walls, and (b) assumption of a reasonable tangential \( E \) over the slot, and calculation of \( \kappa \) by means of (23). The propagation constant \( \gamma \) is then given by (6), where the imaginary part is the phase constant and the real part is the attenuation constant.

IV. GREEN’S FUNCTIONS FOR THE CIRCULAR CYLINDER

In order to apply the preceding theory to the slotted circular cylinder, the appropriate Green’s functions must first be determined. The problem is to determine the potential from a line source subject to boundary conditions on the given cylinder. The \( z \) variation of the fields, which is \( e^{\gamma z} \), is not written explicitly in the following, but is implied. Figure 4 illustrates the geometry and coordinates of
the problem. Considering region e, let \( G_e \) be the sum of an incident field \( G^i_e \) and a reflected field \( G^r_e \), as

\[
G_e = G^i_e + G^r_e .
\]  

(28)

The incident field is simply the field from a line source, of excitation \( e^{\gamma z} \), radiating into free space. This is

\[
G^i_e = \frac{1}{4\pi} J_0(\kappa r) ,
\]  

(29)

where \( J_0(2) \) is the zero order Hankel function. From a consideration of the geometry of figure 4, it is seen that

\[
r = \sqrt{\rho^2 + \rho'^2 - 2 \rho \rho' \cos(\phi - \phi')} .
\]  

(30)

Utilizing the addition theorem for Bessel functions, equation (29) becomes
where $\varepsilon_n$ is Neumann's number. The reflected field is constructed as

$$G_e' = \frac{1}{4j} \sum_{n=0}^{\infty} \varepsilon_n H_n^{(2)}(\kappa \rho') J_n(\kappa \rho) \cos n(\phi - \phi'), \quad \rho < \rho', \quad (31)$$

where the $a_n$ are constants to be determined from boundary conditions. Thus by (31) and (32), equation (28) becomes

$$G_e = \frac{1}{4j} \sum_{n=0}^{\infty} \varepsilon_n \left[ H_n^{(2)}(\kappa \rho') J_n(\kappa \rho) + a_n H_n^{(2)}(\kappa \rho) \right] \cos n(\phi - \phi'), \quad (33)$$

and

$$\frac{\partial G_e}{\partial \rho} = \frac{1}{4j} \sum_{n=0}^{\infty} \varepsilon_n \left[ H_n^{(2)}(\kappa \rho') J_n'(\kappa \rho) + a_n H_n^{(2)}(\kappa \rho) \right] \cos n(\phi - \phi'), \quad (34)$$

where differentiation with respect to the argument of $J_n$ and $H_n^{(2)}$ is denoted by the primes.

For the boundary condition (18), $\frac{\partial G_e}{\partial \rho} = 0$ for $\rho = a$, and each term of the summation must individually be zero due to the orthogonality of the $\phi$ variation over the interval $0$ to $2\pi$. Thus

$$a_n = -\frac{J_n'(ka)}{H_n^{(2)}'(ka)} \frac{H_n(2)(\kappa \rho')}{H_n(2)(\kappa \rho')}, \quad (35)$$

Letting $\rho \to a$, equation (33) becomes

$$G_e = \frac{1}{4j} \sum_{n=0}^{\infty} \varepsilon_n \left[ \frac{J_n(ka)H_n^{(2)}(ka)}{H_n^{(2)}(ka)} - \frac{J_n'(ka)H_n^{(2)}(ka)}{H_n^{(2)}(ka)} \right] H_n^{(2)}(\kappa \rho') \cos n(\phi - \phi'), \quad (36)$$

The numerator of the bracketed quantity of (36) is recognized as $(-j)$ times the Wronskian of Bessel's equation, which is $2/\kappa a$. Letting
\( \rho' \to a \), equation (36) becomes

\[
G_e = \frac{-1}{2\pi a} \sum_{n=0}^{\infty} \epsilon_n \frac{H_n^{(2)}(\kappa a)}{J_n^{(2)}(\kappa a)} \cos n(\phi - \phi') .
\] (37)

For the boundary condition (20), \( G^* = 0 \) for \( \rho = a \), giving from equation (33)

\[
a^*_n = \frac{-J_n(\kappa a)}{J_n^{(2)}(\kappa a)} H_n^{(2)}(\kappa \rho') .
\] (38)

Letting \( \rho \to a \), equation (34) becomes

\[
\frac{\partial G_e^*}{\partial \rho} = \frac{\kappa}{4 \epsilon_0} \sum_{n=0}^{\infty} \epsilon_n \left[ \frac{J_n^{(2)}(\kappa a) H_n^{(2)}(\kappa a) - J_n(\kappa a) H_n^{(2)}(\kappa a)}{H_n^{(2)}(\kappa a)} \right] \frac{H_n^{(2)}(\kappa \rho') \cos n(\phi - \phi')}{J_n^{(2)}(\kappa a)} .
\] (39)

The numerator of the bracketed quantity of (39) is \((j)\) times the Wronskian, and taking the derivative with respect to \( \rho' \) and letting \( \rho' \to a \), one obtains

\[
\frac{\partial^2 G_e^*}{\partial \rho \partial \rho'} = \frac{\kappa}{2 \pi a} \sum_{n=0}^{\infty} \epsilon_n \frac{H_n^{(2)}(\kappa a)}{J_n^{(2)}(\kappa a)} \cos n(\phi - \phi') .
\] (40)

By a similar development, the appropriate Green's functions applicable to the internal region with \( \rho = \rho' = a \) are found to be

\[
G_i = \frac{-1}{2\pi a} \sum_{n=0}^{\infty} \epsilon_n \frac{J_n(\kappa a)}{J_n^{(2)}(\kappa a)} \cos n(\phi - \phi') ,
\] (41)

and

\[
\frac{\partial^2 G_i^*}{\partial \rho \partial \rho'} = \frac{\kappa}{2 \pi a} \sum_{n=0}^{\infty} \epsilon_n \frac{J_n^{(2)}(\kappa a)}{J_n(\kappa a)} \cos n(\phi - \phi') .
\] (42)

Equations (37), (40), (41), and (42) are the Green's function expressions appearing in equations (19) and (21).
V. THE SLOTTED CIRCULAR WAVEGUIDE

The special case of the circular cylinder, for which the external cylinder and waveguide walls coincide, with the waveguide "air filled" will now be treated. The coordinate system is specified by

\[ \xi = \rho \quad h_1 = 1 \quad dl = \rho d\phi, \]  
\[ \eta = \phi \quad h_2 = \rho \]  

(43)

and the cylinder is defined by \( \rho = a \). The appropriate Green's functions were derived in section IV. For the TE case, \( \psi \) (see section III), determined from equations (37) and (41), is given by

\[ \Psi_{TE} = \frac{-1}{2\pi^2a^2\omega}\sum_{n=0}^{\infty} \frac{\varepsilon_n}{J_n'(\kappa a)H_n'^{(2)}(\kappa a)} \cos n(\phi - \phi') \]  

(44)

and for the TM case, from (40) and (42),

\[ \Psi_{TM} = \frac{\omega\varepsilon}{2\pi^2a^2\kappa^2}\sum_{n=0}^{\infty} \frac{\varepsilon_n}{J_n(\kappa a)H_n^{(2)}(\kappa a)} \cos n(\phi - \phi') \]  

(45)

The substitution of (44) into (23) gives, for the TE case,

\[ \sum_{n=0}^{\infty} \frac{\varepsilon_n}{J_n'(\kappa a)H_n'^{(2)}(\kappa a)} \int_{\text{slot}} d\phi' E^m_G(\phi') \int_{\text{slot}} d\phi E^m_G(\phi) \cos n(\phi - \phi') = 0 \]  

(46)

and substitution of (45) into (23) gives for the TM case

\[ \sum_{n=0}^{\infty} \frac{\varepsilon_n}{J_n(\kappa a)H_n^{(2)}(\kappa a)} \int_{\text{slot}} d\phi' E^m_Z(\phi') \int_{\text{slot}} d\phi E^m_Z(\phi) \cos n(\phi - \phi') = 0 \]  

(47)

Equations (46) and (47) are the exact integral equations for the determination of \( \kappa \). They are valid for the case of several slots, as well as for the single slot. These equations, however, do not uniquely determine \( \kappa \), but give an infinite set of solutions. The solutions
represent fields with different propagation constants, similar to the waveguide modes in a closed cylinder. If the slots are reasonably small, these slotted cylinder "modes" will be but slightly perturbed from the waveguide modes. In classifying the modes of the slotted cylinder, the designation used is that of the mode which would exist if the guide were completely closed. Thus, a TE_{11} slotted waveguide would be one excited so as to produce the TE_{11} mode in the closed waveguide, and so on.

The solution for the two lowest order modes with a single slot, the TE_{11} mode with excitation antisymmetric about the diameter through the midpoint of the slot, and the TM_{01} mode, will be considered. The angle subtended by the slot is 2\phi_0, as shown on the cross section in figure 4b. Following the variational method, a reasonable tangential E over the slot is assumed. For the TE_{11} mode, take \( E_\phi \equiv 1 \) over the slot, and for the TM_{01} mode, take \( E_z = \cos \left( \pi \phi / 2\phi_0 \right) \) over the slot. The integrals in equations (46) and (47) are evaluated, respectively, as

\[ \int \]

Fig. 4b. Cross section of the slotted circular waveguide.
\[ I_n = \int_{\phi_0}^{\phi_0} \int_{\phi_0}^{\phi_0} \cos n(\phi - \phi') \, d\phi \, d\phi' \]
\[ = \left[ \frac{2 \sin n\phi_0}{n} \right]^2, \quad (48) \]

and

\[ I_n = \int_{\phi_0}^{\phi_0} \int_{\phi_0}^{\phi_0} \cos \frac{\pi}{\phi_0} \cos \frac{\pi}{\phi_0} \cos n(\phi - \phi') \, d\phi \, d\phi' \]
\[ = \left[ \frac{4 n \phi_0 \cos n\phi_0}{n^2 - (2n\phi_0)^2} \right]^2. \quad (49) \]

Since the desired solution \( \kappa \) is in the vicinity of that for the waveguide mode, denoted by \( \kappa_0 \), let

\[ \kappa a = \kappa_0 a + x, \quad (50) \]

where \( x \) is a small complex number. Expanding (46) and (47) in Taylor series about \( \kappa_0 a \), one obtains equations of the form

\[ \sum_{k=0}^{\infty} a_k(\phi_0) x^k = 0. \quad (51) \]

Since \( |x| \) is small, terms of (51) through the second power are sufficient to determine \( x \) to good accuracy for the slot widths considered.

For the \( TE_{11} \) case, define

\[ b_0 = J_1^{(1)} H_1^{(2)*}, \quad \sum_{n=0}^{\infty} \epsilon_n \frac{I_n}{J_n^{(1)} H_n^{(2)*}}, \]
\[ b_1 = J_1^{(1)} H_1^{(2)*}, \quad \sum_{n=0}^{\infty} \epsilon_n \frac{I_n}{J_n^{(1)} H_n^{(2)*}} \left[ J_n^{(1)*} H_n^{(2)*} + J_n^{(1)*} H_n^{(2)*} \right]. \quad (52) \]

Equation (51) through the second power becomes
\[
\left( \frac{H_{1}^{(2)}}{H_{1}^{(2)}} + \frac{J_{1}^{(1)}}{2J_{1}^{(1)}} \right) b_{0} - b_{1} \right] x^2 + b_{0} x + \frac{6}{I_{1}} = 0 .
\] (53)

The argument of the Bessel and Hankel functions in (52) and (53) is \(\kappa_{0}a = 1.841\). For the \(TM_{01}\) case, define

\[
c_{0} = 2 J_{0}^{(2)} \sum_{n=1}^{\infty} \frac{I_{n}}{J_{n} H_{n}^{(2)}}
\]

\[
c_{1} = 2 J_{0}^{(2)} \sum_{n=1}^{\infty} \frac{J_{n} H_{n}^{(2)}}{[J_{n} H_{n}^{(2)}]^2}
\] (54)

Equation (51) through the second power becomes

\[
\left( \frac{H_{0}^{(2)}}{H_{0}^{(2)}} + \frac{J_{0}^{(1)}}{2J_{0}^{(1)}} \right) c_{0} - c_{1} \right] x^2 + c_{0} x + \frac{6}{I_{0}} = 0 .
\] (55)

The argument of the Bessel and Hankel functions in (54) and (55) is \(\kappa_{0}a = 2.405\). Equations (53) and (55) are those used for numerical calculations.

An interesting property of the solution is that, for a given excitation, only a single plot of \(\kappa a vs \phi_{0}\) is necessary to give complete information on the propagation constant. In previous work \(^{1}\) it was observed that this could have been anticipated since the theoretical problem is formulated in terms of only \(\kappa\) and the boundary surface, specified by \(a\) and \(\phi_{0}\). Then, according to dimensional analysis, the solution will be a function of the two dimensional groups \(\kappa a\) and \(\phi_{0}\).

VI. CALCULATED AND EXPERIMENTAL RESULTS

The theoretical curves of \((\kappa a)^2 vs \phi_{0}\), calculated from (53) and
for the $TE_{11}$ and $TM_{01}$ modes, respectively, are given in figures 5 and 6. Determination of $\gamma = \alpha + j\beta_z$ by equation (6) gives the attenuation constant $\alpha$ and the phase constant $\beta_z$ for any particular wavelength $\lambda = 2\pi/\beta$.

Experimental measurements of $\alpha$ and $\beta_z$ have been obtained to verify the theory. The measurements were taken for the case of a slot of the order of 10 wavelengths in length cut in the wall of a circular waveguide. The phase constant $\beta_z$ was determined from the angle of maximum radiation from the slot, and the attenuation constant $\alpha$ was determined from probing measurements of the waveguide field. A more complete description of the experimental method and the $TM_{01}$ measurements are given in the Antenna Laboratory reports. Experimental and calculated values of the attenuation per wavelength ($\alpha_\lambda = \alpha \lambda$), and the ratio of free-space velocity to that of the slotted guide ($c/v = \beta_z \lambda/2\pi$), are given for the $TE_{11}$ mode in figures 7 and 8, and for the $TM_{01}$ mode in figures 9 and 10.

While it was shown in the theory that, for the case of the infinite slot, if the excitation of the cylinder is TE the entire field is TE, and similarly for the TM case, it is not necessarily true for the finite length slot. Far-field measurements of the experimental models were taken of the two orthogonal spherical field components $E_{\phi}$ and $E_\theta$, where $\theta$ is the angle from the axis of the cylinder and $\phi$ is the azimuth angle. For a TE field $E_\theta = 0$, and for a TM field, $E_\phi = 0$ in the radiation field. These measurements showed that the field was essentially either TE or TM for the finite length slot.
Fig. 5. Plot of $(\kappa a)^2$ for the TE$_{11}$ slotted cylinder.
Fig. 6. Plot of $(\kappa a)^2$ for the TM$_{01}$ slotted cylinder.
Fig. 7. Calculated and experimental values of $c_\lambda$ for the $\text{TE}_{11}$ slotted cylinder.
Fig. 8. Calculated and experimental values of c/v for the TE_{11} slotted cylinder.
Fig. 9. Calculated and experimental values of $c_{x\lambda}$ for the TM$_{01}$ slotted cylinder.
Fig. 10. Calculated and experimental values of $c/v$ for the TM$_{01}$ slotted cylinder.
VII. REFERENCES

1. Interim Engineering Reports 400-1 to 400-11, The Antenna Laboratory, The Ohio State University Research Foundation, prepared under contract with Air Research and Development Command, Wright-Patterson Air Force Base, March 1950 to December 1951.


PART TWO

ON TRANSFORM SOLUTIONS TO INTEGRAL EQUATIONS ARISING FROM BOUNDARY VALUE PROBLEMS

I. INTRODUCTION

In recent years there has been considerable interest in the solution of integral equations obtained from boundary value problems. Notable success has been obtained for a type known as Wiener-Hopf integral equations. The problems for which equations of this type are obtained are characterized by the following. First, the boundary surface is cylindrical, and partly covered by a semi-infinite, perfect conductor. Second, the field variation in the transverse direction is independent of the other two coordinates, so that the problem reduces to a two dimensional one. A typical example of such a problem is that of a semi-infinite circular waveguide, excited in a given mode a large distance from the open end, and radiating into free space. This problem, along with the corresponding acoustical one, has been solved by Levine and Schwinger, and the results have been published by Marcus-vitz. Other problems of different geometry have been solved by Heins, by Lucke, and by Copson.

The method of solution used in the literature is an extension of the solution of Wiener and Hopf. A brief description of the theory is given by Karp, and a more complete one is given in section III of this paper. The solution used in the literature introduces another restriction on the type of problem which can be handled. This is that the excitation must be located in the infinite region, that is, an
incident plane wave or waveguide mode.

A more general procedure for the solution of equations of the Wiener-Hopf type is given in section IV. This method will not only handle, in a more direct manner, those problems treated in the literature, but will allow treatment of problems not solvable by the former method. In particular, it removes the restriction mentioned in the preceding paragraph, that is, it permits the solution of problems containing sources in the finite region.

In the solution, use is made of the theory of complex variables and of Laplace transforms (or generalized Fourier transforms). There are several good texts which cover the necessary complex variable theory. $^8$-10 A Laplace transform pair is given formally by

$$\bar{F}(\gamma) = \int_{-\infty}^{\infty} F(z) e^{-\gamma z} \, dz,$$

$$F(z) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \bar{F}(\gamma) e^{\gamma z} \, d\gamma.$$

The existence of $\bar{F}(\gamma)$ implies certain restrictions on $F(z)$, and on the path of integration in the $\gamma$ plane, which are discussed by a number of authors. $^10$-12 That the transforms arising in the problems considered do exist can be shown from known behavior of solutions to the wave equation.

Section II of this paper gives the general methods of formulating the appropriate transform equations. The next two sections treat the theory, and give examples, of the two afore mentioned methods of solution of the transform equations. In section V the solution to the
problem of a current element adjacent to a semi-infinite plane conductor is given. An attempt is made to present the physical interpretation of the mathematical procedure, which has not been stressed in the literature.

II. FORMULATION OF THE TRANSFORM EQUATION

A. General Method

The integral equation to be considered is of the form

\[ \int_{-\infty}^{0} J(z') G(z-z') \, dz' + F(z) = \begin{cases} E(z), & z > 0, \\ 0, & z < 0, \end{cases} \tag{2} \]

where \( J(z) \) and \( E(z) \) are unknowns. For electromagnetic field problems, usually \( J \) is an equivalent current, \( G \) is a Green's function (or the field from a differential current element), \( F \) is a component of the incident field, and \( E \) is a component of the tangential electric field. As indicated by (2), \( J \) and \( E \) exist over separate intervals on \( z \). The known function \( F \) is not always explicitly written in the integral equation for the Wiener-Hopf formulation, but is implied in the properties of \( J \) and \( E \). This is one of the differences of the two methods of solution presented in this paper, as will be seen later.

Taking the transforms of both sides of (2), making use of real convolution, one obtains

\[ \mathcal{J}(\gamma) \overline{G}(\gamma) + \overline{F}(\gamma) = \overline{E}(\gamma), \tag{3} \]

where \( \mathcal{J}, \overline{G}, \overline{F}, \) and \( \overline{E} \) are the transforms of \( J, G, F, \) and \( E \), respectively. Again, \( \overline{F}(\gamma) \) may or may not appear in the transform equation, depending upon the method of solution used. In order for (3) to be a valid
equation, there should be a common strip of regularity for all terms. If the problem is formulated correctly, such a region does exist for lossy media (β complex), as will be shown in section III. Usually, equation (3) is regular in a strip enclosing the imaginary axis, which shrinks to a line as β becomes real. It is interesting to note that it is not necessary to derive (3) from (2), since the transform equation can be obtained directly.

There are three approaches to the problem whereby the appropriate transform equation may be obtained. These are (1.) the Green's function technique, (2.) the method of equivalent currents, and (3.) construction from the general solution to the wave equation. The first method has been used almost exclusively in the literature, but is not necessarily the most direct approach. The second procedure has never been widely used in the field, probably due to the unavailability of a comprehensive account of the method. It is related to the Green's function technique by reciprocity, and permits a simplification of the work in so far as the application of Green's second identity is no longer necessary. For some cases, the third method appears to be the most direct one, but seems to have been overlooked except in a recent paper by Karp. A general description of these methods is given here, and specific examples are treated in parts B and C.

1. The Green's function technique consists of determining an appropriate solution to the wave equation for a simpler problem, say the field from a point source or a line source. Then Green's second identity is applied to this solution (called the Green's function) and
the unknown solution to the original problem. The Green's function may or may not satisfy boundary conditions on the cylindrical surface under consideration, depending upon the problem. This procedure gives an integral equation relating the field everywhere in a given region to the tangential electric or magnetic field, or both, on the cylindrical surface. In particular, considering the field at the given surface, there results an integral equation of the form of equation (2).

The appropriate Green's functions and their transforms are readily obtainable in many cases. For a point source, the free space scalar Green's function is given by \( e^{-\frac{\beta r}{4\pi r}} \), and its transform by

\[
\overline{G}(\gamma) = \int_{-\infty}^{\infty} \frac{e^{-\frac{\beta r}{4\pi r}} e^{-\gamma z}}{4\pi r} \, dz , \tag{4}
\]

where \( r = \sqrt{(x-x')^2 + (y-y')^2 + z^2} \). Equation (4) is precisely the formula for the potential from an infinite line source of excitation \( e^{-\gamma z} \), evaluated at \( z' = 0 \). This can be shown to be \( e^{-\frac{\beta r}{4\pi r}} \)

\[
\overline{G}(\gamma) = \frac{1}{4J} H_0^{(2)}(\sqrt{\beta^2 + \gamma^2 R}) , \tag{5}
\]

where \( R = \sqrt{(x-x')^2 + (y-y')^2} \). Thus one has the transform of the free space, point-source Green's function for any set of cylindrical coordinates merely by expressing \( R \) in terms of the appropriate transverse coordinates.

For many problems, the field variation in the direction transverse to the z axis is everywhere independent of the other two coordinates. In fact, it is only for such cases that the Wiener-Hopf method has been successfully applied. If this condition is met, it is possible to
choose as the Green's function the field from a line source in the transverse direction, with excitation corresponding to the field variation. This in effect suppresses the variation of the field in this direction, giving the integral equation in terms of only the axial (or z) variable. This procedure can be used only for those cylindrical coordinate systems for which the wave equation is separable.

In particular, for rectangular coordinates, consider a field with no variation in the transverse (or y) direction. Then, following the above procedure, one can choose for the Green's function the field from an infinite line source parallel to the y axis, with uniform excitation. The transform of this Green's function is given by summing the contributions from each differential element of the source, as

$$G(\gamma) = \frac{1}{4j} \int_{-\infty}^{\infty} H_0^{(2)}(\sqrt{\beta^2 + \gamma^2 \sqrt{(x-x')^2 + y^2}}) \, dy \quad (6)$$

The value of this integral of the Hankel function is known. It can also be shown, after some manipulation, to be related to the inverse transform of (5). The result is

$$G(\gamma) = \frac{e^{-j|x-x'|\sqrt{\beta^2 + \gamma^2}}}{2j \sqrt{\beta^2 + \gamma^2}} \quad (7)$$

The y-constant field is the only one which satisfies the condition that the y variation is independent of the other two coordinates, if unbounded half space is considered. However, for a bounded region, other than y-constant fields can be treated by this method, as in the case of the bifurcation of a rectangular waveguide.2

There are a greater number of possibilities in circular cylindrical
coordinates, since the field variation in the \( \phi \) direction is independent of position if it varies as \( \cos n\phi \), \( n = 0, 1, 2, \cdots \). Thus one could choose as a Green's function the field from a circular line source, of radius \( \rho = \rho' \), and with excitation \( \cos n\phi' \). For such a case, summing over the differential elements,

\[
\bar{G}_n(\gamma) = \frac{\rho'_1}{4j} \int_0^{2\pi} \cos n\phi' H_0^{(2)}(\sqrt{\beta^2 + \gamma^2} R) \, d\phi',
\]

where \( R = \sqrt{\rho^2 + \rho'^2 - 2\rho \rho' \cos \phi'} \). This integral is readily evaluated through the use of the addition theorem for Bessel functions,\(^{14}\) giving

\[
\bar{G}_n(\gamma) = \frac{\rho_0}{4j} \frac{\beta}{\gamma} H_n^{(2)}(\sqrt{\beta^2 + \gamma^2} \rho) J_n(\sqrt{\beta^2 + \gamma^2} \rho') , \rho > \rho',
\]

\[
= \frac{\rho_0}{4j} \frac{\beta}{\gamma} J_n(\sqrt{\beta^2 + \gamma^2} \rho) H_n^{(2)}(\sqrt{\beta^2 + \gamma^2} \rho') , \rho < \rho'.
\]

These are the transforms which would apply to circular problems, such as the semi-infinite circular waveguide.

The above derivations treat only the free space Green's functions. In some cases it is desirable to use a Green's function which satisfies boundary conditions on the given cylinder. For rectangular coordinates, this is simply the addition of the transform of an image field to that of the free space field. For circular cylindrical coordinates, the Green's function can be derived by adding a field representing waves originating on the cylinder, with arbitrary coefficients, to the free space field. These coefficients are then determined so that the desired boundary conditions are satisfied. This is the method used in part one, section IV of this thesis.

It is interesting to note that the identical transform equations
result regardless of whether a point source or a line source field is chosen for the Green's function. Mathematically, it amounts to merely an interchange in the order of integrations. All problems could be done using only the point source Green's function.

2. The equivalent current method of formulation differs from the Green's function method chiefly in its philosophy. Schalkunoff\(^{15}\) has shown that the field everywhere in a source-free region is the same as that produced by the equivalent electric and magnetic currents, \(J = \mathbf{n} \times \mathbf{H}, \ K = \mathbf{E} \times \mathbf{n}\), flowing on the boundary. (\(\mathbf{n}\) is the unit normal directed outwards from the volume.) As shown by Professor V. H. Rumsey, the field from these equivalent currents can satisfy arbitrary boundary conditions on the surface. These currents can now be considered to be the sources of the field, giving the field everywhere in a region in terms of the value of tangential \(E\), or tangential \(H\), or both, on the boundary. The result is identical to that obtainable from the Green's function technique. The fields, and their transforms, from a differential element, or line element, of current are those given by equations (4)-(9). The mathematics involved parallel closely those used in the Green's function method.

3. Transform equations of the form of equation (3) can be constructed directly from the general solution to the wave equation in cylindrical coordinates, which is

\[
\psi = \int_{c} f(x, y, \gamma) \, e^{\gamma z} \, d\gamma .
\] (10)

If the contour \(c\) is taken along the imaginary axis of the \(\gamma\) plane,
f is essentially the transform of the field. For those problems for which the field variation in the transverse direction is known and independent of the other two coordinates, satisfying the boundary conditions of the problem gives the appropriate transform equation. Note that equation (10) corresponds to a construction of the solution from cylindrical wave functions, referred to by Sommerfeld as a continuous spectrum of eigenfunctions. Karp suggests that this method be extended to coordinate systems other than cylindrical, but as yet the mechanics of the solution have not been developed.

B. Application to the Circular Cylinder

Two examples of the formulation of the transform equation are given here to illustrate the methods (2.ₐ) and (3.) of the preceding section. For an illustration of method (1.), refer to any of the problems reported in the literature. The transform equations developed in this section are given with \( F(γ) \) omitted and the source implied by the properties of \( \overline{J} \) and \( \overline{E} \), as is usual in the literature. Section C illustrates the inclusion of \( \overline{F(γ)} \) as a separate known function, which is essential for the more general method of solution given in section IV.

The cases considered here are the semi-infinite circular waveguide for (1.) excitation in the \( TM_{01} \) mode, and (2.) excitation in the \( TE_{01} \) mode. The geometry of the problems is illustrated by figure 11. The waveguide is excited in the given mode a large distance from the open end.
1. TM$_{01}$ excitation. By the method of equivalent currents, the field everywhere is given by the free space contributions from the source and from the equivalent electric currents determined from the discontinuity in tangential $H$ at the conductor. No equivalent magnetic currents need be considered, since the electric field is everywhere continuous (zero over the conductor). From symmetry conditions, the field is everywhere TM and independent of $\phi$. The equivalent electric currents are all flowing in the $z$ direction, and are of magnitude $J_z = H_\rho_{\text{external}} - H_\rho_{\text{internal}}$. The field can be expressed in terms of a vector potential $\vec{A}$ having only a $z$ component, the transform of which is given by

$$\vec{A}_z(\rho, \gamma) = \vec{A}_z \text{source} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_z(z') G(\rho, z - z') e^{\gamma z} \, dz' \, dz,$$
\[\bar{A}_z(p,\gamma) = \bar{A}_z\text{source} + \bar{J}_z(\gamma) \ G(p,\gamma) .\quad (11)\]

\[G(p,\gamma) \text{ is the transform of the } z \text{ component of the vector potential from a circular line source of uniform current flowing in the } z \text{ direction, given by equation (9) with } n = 0. \]

From Maxwell's equations, one has the relationship

\[j\omega \bar{E}_z(p,\gamma) = \left(\beta^2 + \gamma^2\right) \bar{A}_z(p,\gamma) .\quad (12)\]

The boundary condition is that \(E_z\) is zero over the conductor, that is, \(E_z(a,z) = 0 \text{ for } z < 0\). Considering the source to be located at \(z = -\infty\), the contribution to \(E_z\), \(z > 0\), from the source is nil since the field varies as \(1/r\). \((r \text{ is the distance from the source to the field point.})\)

Thus, the appropriate transform equation for the problem is

\[\bar{E}_z(a,\gamma) = \frac{j\omega}{2\omega} \bar{J}_z(\gamma) \left(\beta^2 + \gamma^2\right) J_0(\sqrt{\beta^2 + \gamma^2} \ a) H_0^{(2)}(\sqrt{\beta^2 + \gamma^2} \ a),\quad (13)\]

in which \(\bar{E}_z\) and \(\bar{J}_z\) are the unknowns. The characteristics of the source are implied in the properties of \(\bar{J}_z\), as shown in section III. The unknowns \(J_z(z)\) and \(E_z(a,z)\) exist over separate intervals on \(z\), namely, \(J_z = 0 \text{ for } z > 0\), and \(E_z = 0 \text{ for } z < 0\).

2. TE01 excitation. For this case, the field is everywhere TE and independent of \(\phi\). The field is representable in terms of a vector potential \(\mathbf{F}\) having only a \(z\) component, which can be constructed as

\[\mathbf{F}_z(p,z) = \int_{-j\infty}^{j\infty} K(\gamma) \ Z_0(\sqrt{\beta^2 + \gamma^2} \ p) \ e^{\gamma z} \ d\gamma .\quad (14)\]

\(Z_0\) is the appropriate solution to Bessel's equation, which would be
$J_0$ for the region $\rho < a$ and $H_0^{(2)}$ for $\rho > a$. Thus, by inspection, the transform of $F_z$ is

$$F_z(\rho, \gamma) = 2\pi j \psi_1(\gamma) J_0(\sqrt{\beta^2 + \gamma^2} \rho) , \quad \rho < a ,$$

$$= 2\pi j \psi_2(\gamma) H_0^{(2)}(\sqrt{\beta^2 + \gamma^2} \rho) , \quad \rho > a .$$

(15)

The following relations, obtained from Maxwell's equations, hold for the field transforms.

$$-j \omega \mu H_z(\rho, \gamma) = (\beta^2 + \gamma^2) F_z(\rho, \gamma) ,$$

$$E_\phi(\rho, \gamma) = -\frac{1}{\rho} F_z(\rho, \gamma) .$$

(16)

Since $E_\phi$ is continuous at $\rho = a$ for all $\gamma$, and from the uniqueness of the transforms, equating $E_\phi$ internal to $E_\phi$ external gives

$$\psi_1(\gamma) J_1(\sqrt{\beta^2 + \gamma^2} a) = \psi_2(\gamma) H_1^{(2)}(\sqrt{\beta^2 + \gamma^2} a) .$$

(17)

Consider the $\phi$ component of the equivalent electric current due to the discontinuity of tangential $H$ at the conductor, $J_\phi = H_z$ internal $- H_z$ external. The transform of $J_\phi$, obtained from (15), (16), and (17), is

$$\bar{J}_\phi(\gamma) = \frac{2\pi}{\omega \mu} (\beta^2 + \gamma^2) \psi_1(\gamma) \left[ J_1 H_1^{(2)} - J_1 H_0^{(2)} \right] \frac{1}{H_1^{(2)}(\sqrt{\beta^2 + \gamma^2} a)} ,$$

$$= \frac{4\pi}{\omega \mu a} \sqrt{\beta^2 + \gamma^2} \psi_1(\gamma) \frac{1}{H_1^{(2)}(\sqrt{\beta^2 + \gamma^2} a)} .$$

(18)

Now, expressing $\psi_1$ in terms of $E_\phi$ by means of (15) and (16), and substituting into (18), one obtains

$$\bar{E}_\phi(a, \gamma) = \frac{\omega \mu a}{2} J_\phi(\gamma) J_1(\sqrt{\beta^2 + \gamma^2} a) H_1^{(2)}(\sqrt{\beta^2 + \gamma^2} a) .$$

(19)
This is the transform equation to be solved for the unknowns $\overline{J}_\phi$ and $\overline{E}_\phi$. Again, the characteristics of the source are implied in the properties of $\overline{J}_\phi$, and $J_\phi(z) = 0$ for $z > 0$ and $E_\phi(a,z) = 0$ for $z < 0$.

C. Application to a Plane Surface

As an illustration of the formulation of the transform equation for a problem in rectangular coordinates, the case of the diffraction of a plane wave by the edge of a plane conductor is considered. The solution by the Wiener-Hopf method for the case of the magnetic vector parallel to the edge of the conductor has been published by Copson, and shown to be identical to the solution obtained by Sommerfeld by another method.\(^5\) In this section, the transform equation for the case of the electric vector parallel to the conductor edge is derived. The transform equation is derived with the source term, $\overline{F}(\gamma)$, separated out as a distinct known quantity. This is not necessary for the Wiener-Hopf solution, but is required for the general method given in section IV.

Fig. 12. Geometry for problem IIIC.
The geometry of the problem is given in figure 12. The incident plane wave is at the angle $\theta$ to the conductor, which is defined by $x = 0, z < 0$. From symmetry, the field is everywhere independent of $y$, and has only a $y$ component of electric field. Consider the field to be represented as the sum of an incident field, $E^i_y$, and a scattered field, $E^s_y$, as

$$E_y = E^i_y + E^s_y.$$  \hfill (20)

The incident plane wave is specified by

$$E^i_y = e^{j\beta(x \sin \theta + z \cos \theta)}.$$  \hfill (21)

The scattered field can be considered as the field from the equivalent electric currents due to the discontinuity in tangential $H$ over the conductor. Since $H_z$ is the only component of tangential $H$, this current flows entirely in the $y$ direction. The electric field from a differential line element of current is given by

$$G(z-z') = \frac{-\omega}{4} H^2_0(\beta \sqrt{x'^2 + (z-z')^2}).$$  \hfill (22)

Thus, summing over the entire current sheet gives

$$E^s_y = \int_{-\infty}^{\infty} J_y(z') G(z-z') \, dz'.$$  \hfill (23)

The boundary condition over the conductor is that $E_y$ vanishes, from which follows

$$E^s_y = -E^i_y \text{ for } x = 0, \ z < 0.$$  \hfill (24)
Substituting (24) into (23), setting \( x = 0 \), and taking the transform of each side, gives

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_y(z') G(z-z') \, e^{-\gamma z} \, dz' \, dz = -\int_{-\infty}^{\infty} E_y^1 e^{-\gamma z} \, dz + \int_{0}^{\infty} E_y^3 e^{-\gamma z} \, dz . 
\]  

(25)

By convolution, and substitution from (22), the first term in (25) is given by

\[
\int_{-\infty}^{\infty} J_y(z') \, e^{-\gamma z'} \, dz' \int_{-\infty}^{\infty} H_0^{(2)}(\beta |z|) \, e^{-\gamma z} \, dz = \frac{-\omega p}{2 \sqrt{\beta^2 + \gamma^2}} \, \overline{J}(\gamma) . 
\]  

(26)

The second term in (25) is evaluated as

\[
\int_{0}^{\infty} e^{j\beta z} \cos \theta \, e^{-\gamma z} \, dz = \frac{-1}{\gamma - j\beta \cos \theta} . 
\]  

(27)

The third term of (25) is an unknown, \( \overline{E}(\gamma) \). Substitution into (25) gives the appropriate transform equation

\[
\frac{-\omega p}{2 \sqrt{\beta^2 + \gamma^2}} \, \overline{J}(\gamma) = \frac{1}{\gamma - j\beta \cos \theta} + \overline{E}(\gamma) , 
\]  

(28)

where

\[
\overline{J}(\gamma) = \int_{-\infty}^{0} J_y(z) \, e^{-\gamma z} \, dz , 
\]

(29)

\[
\overline{E}(\gamma) = \int_{0}^{\infty} E_y^3(0,z) \, e^{-\gamma z} \, dz . 
\]

Here the source term, \( \overline{F}(\gamma) \), consists of the simple pole \( \gamma = j\beta \cos \theta \).

The inverse transforms of the unknowns \( \overline{J} \) and \( \overline{E} \) exist over separate semi-infinite intervals on \( z \), as indicated by (29). Equation (28) will be used to illustrate the two methods of solution given in sections III and IV.
III. THE METHOD OF SOLUTION USED IN THE LITERATURE

A. Theory

The method of solution used in the literature is an extension of that devised by Wiener and Hopf. However, as mentioned in section I, its use is limited to those problems for which the source appears (or is implied) as a simple pole in the transform equation. The following is a general discussion of the theory of this method.

For the transform equation to be a valid one, there should be a common region of regularity for all terms. This will be shown to be true for dissipative media (β complex). The common domain is generally an infinite strip enclosing (or parallel to) the imaginary axis of the Y plane, which shrinks to a line as β becomes real. It is the usual procedure to consider β complex in the mechanics of the solution, and then let Im(β) → 0 in the result.

First, the regions of regularity of the unknowns will be determined. The two unknowns, J and E, exist over separate, semi-infinite portions of the z variable. To be specific, consider the case where, as in the examples of section II, J(z) is zero for z > 0 and E(z) is zero for z < 0. Certain characteristics of J and E are known, since they are solutions to the wave equation. If the field is not guided, at a large distance from the origin it varies as \(e^{-j\beta r/r^a}\), \(a > 0\), except for possibly a source term (for example, an incoming plane wave). If there are guided modes, there will be terms of the form \(e^{\pm \gamma n z}\), which, except for a source term, decrease exponentially away from the origin. From this information, one can determine the regions
of regularity of the unknowns as follows. Consider the problems of section II B, for which the excitation is an incident guided mode, and there are no guided modes in the region external to the pipe. One then has

\[ \overline{E}(\gamma) = \int_{0}^{\infty} E(z) e^{-\gamma z} \, dz, \]

\[ E(z) \xrightarrow{z \to \infty} \frac{e^{-j\beta z}}{z^\alpha}, \quad \alpha > 0. \quad (30) \]

This implies that \( \overline{E}(\gamma) \) is regular in the region \( \text{Re}(\gamma) > \text{Im}(\beta) \).

For \( J(z) \),

\[ \overline{J}(\gamma) = \int_{-\infty}^{0} J(z) e^{-\gamma z} \, dz \]

\[ J(z) \xrightarrow{z \to \infty} e^{\gamma_0 z} + \sum_{n=0}^{\infty} B_n e^{-\gamma_n z} + \frac{e^{j\beta z}}{z^\alpha}, \quad \alpha > 0, \quad (31) \]

which implies that \( \overline{J}(\gamma) \) has simple poles at \( \gamma = \pm \gamma_0, -\gamma_n, \quad n = 1, 2, \ldots \), and is otherwise regular in the region \( \text{Re}(\gamma) < -\text{Im}(\beta) \). All the poles, except the source pole \( \gamma = \gamma_0 \), lie outside the region of regularity, so that \( \overline{J}(\gamma) \) is regular for \( \text{Re}(\gamma) < -\text{Im}(\beta) \) except for a simple pole at \( \gamma_0 \).

The Green's function can be shown always to be regular in the strip \( |\text{Re}(\gamma)| < -\text{Im}(\beta) \), since

\[ \overline{G}(\gamma) = \int_{-\infty}^{\infty} G(z) e^{-\gamma z} \, dz, \]

\[ G(z) \xrightarrow{|z| \to \infty} \frac{e^{-j\beta |z|}}{|z|^\alpha}, \quad \alpha > 0. \quad (32) \]

This implies that \( \overline{G}(\gamma) \) is regular in the above mentioned strip, as can be readily shown as follows.
The first term on the right of (33) is regular for \( \text{Re}(\gamma) < -\text{Im}(\beta) \) and the second term is regular for \( \text{Re}(\gamma) > \text{Im}(\beta) \), giving the desired result.

In this section, only those problems with the excitation in the infinite region are considered, so that the source term \( \bar{F}(\gamma) \) consists of a simple pole. This source term will not be written explicitly, but implied in the properties of \( \bar{J} \) or \( \bar{E} \), or both, depending upon the problem. (In equations (30) and (31) it was implied in \( \bar{J} \).) Thus it is seen that the common region of regularity of the transform equation is the strip \( |\text{Re}(\gamma)| < -\text{Im}(\beta) \), except for a possible pole, corresponding to the source, which can be excluded.

Now consider the transform equation written in the form

\[
(\gamma - \gamma_0) \bar{J}(\gamma) \bar{G}(\gamma) = (\gamma - \gamma_0) \bar{E}(\gamma),
\]

(34)

which removes the source pole, \( \gamma = \gamma_0 \). Define the functions \( \bar{G}_L(\gamma) \) and \( \bar{G}_R(\gamma) \) such that

\[
\bar{G}(\gamma) = \frac{\bar{G}_L(\gamma)}{\bar{G}_R(\gamma)},
\]

(35)

where \( \bar{G}_L \) is regular and not zero in the left half plane \( \text{Re}(\gamma) < -\text{Im}(\beta) \), and \( \bar{G}_R \) is regular and not zero in the right half plane \( \text{Re}(\gamma) > \text{Im}(\beta) \).

That \( \bar{G} \) can be expressed as in (35) has not been proved in general, but it appears to be always possible. Substituting (35) into (34) gives

\[
(\gamma - \gamma_0) \bar{J}(\gamma) \bar{G}_L(\gamma) = (\gamma - \gamma_0) \bar{E}(\gamma) \bar{G}_R(\gamma)
\]

(36)
The left side of equation (36) is regular in the left half \( \gamma \) plane, and the right side is regular in the right half \( \gamma \) plane. Both sides are regular in the strip \(|\text{Re}(\gamma)| < \text{Im}(\beta)|\). Thus, they are analytic continuations of each other, and together define a single function, \( \lambda(\gamma) \), analytic over the entire \( \gamma \) plane. Furthermore, if the function \( \lambda(\gamma) \) is bounded at infinity, then \( \lambda(\gamma) = C \), a constant. The behavior of \( \overline{E}(\gamma) \) at infinity is determined from that of \( E(z) \) at the origin, according to

\[
\lim_{\gamma \to \infty} \gamma \overline{E}(\gamma) = \lim_{z \to 0^+} E(z) . \tag{37}
\]

If \( E(z) \) is singular at the origin, then for the transform \( \overline{E} \) to exist, that is, to insure integrability about the origin,

\[
\lim_{\gamma \to \infty} \overline{E}(\gamma) = 0 . \tag{38}
\]

Similarly, for \( \overline{J}(\gamma) \),

\[
\lim_{\gamma \to \infty} \gamma \overline{J}(\gamma) = \lim_{z \to 0^-} J(z) . \tag{39}
\]

Once \( G_R \) and \( G_L \) are found, the behavior of \( \lambda(\gamma) \) as \( \gamma \to \infty \) in the right and left half \( \gamma \) planes can be obtained from (36). In general, this integral function can be shown to be bounded at infinity and thus a constant. Equating each side of (36) to a constant gives

\[
\overline{J}(\gamma) = \frac{C}{(\gamma - \gamma_0) G_L(\gamma)} \tag{40}
\]

\[
\overline{E}(\gamma) = \frac{C}{(\gamma - \gamma_0) G_R(\gamma)}
\]

The constant \( C \) can be determined from boundary conditions. Equations
(40) represent the complete solution to the transform equation, the actual field being given through inversion. It is interesting to note that \( \overline{J}(\gamma) \) and \( \overline{E}(\gamma) \) have no zeros in their respective half planes of regularity.

As for the determination of \( G_R(\gamma) \) and \( G_L(\gamma) \), there are two methods which have been used. The more direct one is to express \( G(\gamma) \) in product form, and determine \( G_R \) and \( G_L \) by inspection.\(^2,3,7\) This is not always easy to do. An alternative method is to apply Cauchy's integral formula to the function \( \log G(\gamma) \) in the strip of regularity \( |\text{Re}(\gamma)| < -\text{Im}(\beta) \). When this is done, part of the contour is identifiable with \( G_R(\gamma) \) and part with \( G_L(\gamma) \).\(^1,4\)

B. Interpretation of the Solution

The transform solution is generally of such complexity that the inverse transforms of the field components are not readily obtainable. However, some physical results, such as the far field, the amplitude of waveguide modes, and reflection and transmission coefficients, can be evaluated directly from the transform solution.

As an example, consider the circular waveguide problems treated in section IIB. The amplitude of the waveguide modes in the guide section can be obtained directly from \( \overline{J}(\gamma) \), since it contains simple poles at \( \gamma = \pm \gamma_n, -\gamma_n, n = 1, 2, \cdots \), as given by (31). From transform theory, it is known that the mode amplitudes are simply the residues of \( \overline{J}(\gamma) \) evaluated at the respective poles. The guide is given excitation such that the equivalent current due to the incident mode, that is, tangential H, has unit amplitude. Thus, the constant C in
(40) is determined from the residue of \( \overline{J}(\gamma) \) at \( \gamma = \gamma_0 \), giving

\[ C = \overline{G}_L(\gamma_0) . \]  

(41)

The amplitudes of the reflected waveguide modes are given by

\[ B_n = \left[ \frac{(\gamma + \gamma_n) \overline{G}_L(\gamma_n)}{(\gamma - \gamma_0) \overline{G}_L(\gamma_0)} \right] \gamma = \gamma_n . \]  

(42)

The reflection coefficient for the incident mode is

\[ R = \frac{1}{B_n} = - \frac{\overline{G}_L(-\gamma_0)}{\overline{G}_L(\gamma_0)} . \]  

(43)

The far field for the problem can be given in terms of the tangential \( E \) on the cylinder \( \rho = a \). This result is available in a previous report,\(^{19}\) and it contains the transform of \( E \) evaluated at \( \gamma = -j\beta \cos \theta \).

For example, the far field for problem IIB(1) is given by

\[ E_\theta = \frac{j e^{-j\beta r}}{2\pi \sin \theta} \frac{1}{H_0^{(2)}(\beta a \sin \theta)} \overline{E}_z(-j\beta \cos \theta) . \]  

(44)

Since the transform equation is valid in the strip enclosing the imaginary axis, substituting from (13) into (44) gives

\[ E_\theta = \frac{-j \omega a}{2r} e^{-j\beta r} \sin \theta J_0(\beta a \sin \theta) \overline{J}_z(-j\beta \cos \theta) . \]  

(45)

Note that in (45), for the dissipationless case, \( \overline{J}_z(\gamma) \) has poles on the imaginary axis at \( \gamma = i\gamma_0 \). But the Bessel function, \( J_0(\beta a \sin \theta) \), also has zeros coincident with these poles of \( \overline{J}_z(\gamma) \), resulting in a finite value for \( E_\theta \). Similarly, for problem IIB(2), the far field is given by
The poles of $\bar{J}_0$ on the imaginary axis are coincident with the zeros of $J_1$, in the same manner as in the preceding case.

The above results can be obtained by representing the far fields in terms of the vector potentials $F_z$ or $A_z$, determined from the equivalent currents. Then the field components are found according to $E = \nabla \times F$ or $H = \nabla \times A$. That the far field is essentially the transform of the field evaluated at $\gamma = -j\beta \cos \theta$ can be thought of physically as follows. The external field consists of a continuous spectrum of cylindrical waves of axial propagation constant $\gamma$. A receiver placed a great distance from the aperture will intercept only that component traveling with the propagation constant $\gamma = -j\beta \cos \theta$, that is, emanating from the cylinder at the angle $\theta$.

For a different problem, there might also be a guided wave in the region for which $\overline{E}(z)$ exists, say for the case of a dielectric waveguide. Then $\overline{E}(\gamma)$ would contain simple poles corresponding to transmitted modes in the same manner as $\overline{J}(\gamma)$ had for reflected modes. The amplitudes of the transmitted modes could be determined from the residues of $\overline{E}(\gamma)$ evaluated at the modular propagation constants, and an appropriate set of transmission coefficients could be defined.

C. Scattering by a Half Plane

The solution to the transform equation for the problem of a plane wave scattered by a semi-infinite conducting plane is given now as an illustration of the Wiener-Hopf method. Formulation of the transform
The equation is given in section IIC, and the geometry is illustrated by figure 12. The equation to be considered is (28).

The functions \( G_L \) and \( G_R \) can be readily formed by inspection as

\[
G_L(\gamma) = \frac{-\omega \eta_0}{\sqrt{\gamma^2 - j\beta}},
\]
\[
G_R(\gamma) = 2\sqrt{\gamma^2 + j\beta}.
\]

The regions of regularity for the unknowns are found following the method of section IIIA. All sources for the scattered field (the equivalent current sheet) are in the region \( z < 0 \), so

\[
E(x(z)) \xrightarrow{z \to \infty} e^{-j\beta z}. \tag{48}
\]

This, coupled with (28), implies that \( \overline{E}(\gamma) \) is regular in the half plane \( \text{Re}(\gamma) > \text{Im}(\beta) \). The behavior of \( J(\gamma) \) can be predicted, from a knowledge of tangential \( H \) over the conductor, to be of the form

\[
J(\gamma) \xrightarrow{z \to \infty} e^{j\beta z} \cos \Theta + \frac{e^{j\beta z}}{\sqrt{z}}. \tag{49}
\]

Thus, (49) with (29) implies that \( \overline{J}(\gamma) \) is regular in the region \( \text{Re}(\gamma) < -\text{Im}(\beta) \), except for a simple pole at \( \gamma = j\beta \cos \Theta \).

Now one can rewrite (28) in the form of (36) as

\[
\frac{-\omega (\gamma - j\beta \cos \Theta)}{\sqrt{\gamma^2 - j\beta}} \overline{J}(\gamma) = 2\sqrt{\gamma^2 + j\beta} \left[ (\gamma - j\beta \cos \Theta) \overline{E}(\gamma) + 1 \right]. \tag{50}
\]

The left side of (50) is regular in the left half \( \gamma \) plane, and the right side is regular in the right half \( \gamma \) plane. Both sides are regular in the strip \( |\text{Re}(\gamma)| < -\text{Im}(\beta) \). Thus, by analytic continuation,
both sides taken together define a single integral function, \( \lambda(\gamma) \), over the entire \( \gamma \) plane.

The behavior of the unknown transforms as \( \gamma \to \infty \) is found according to (37) - (39). It is known that \( E_y^0 = -E_y^1 \) at \( z = 0 \), so that \( E_y^0 = -1 \) at the origin. Thus, from (37),

\[
\overline{E}(\gamma) \sim \gamma^{-1} \text{ as } \gamma \to \infty .
\]

(51)

To insure integrability about the origin, it follows from (38) that

\[
\overline{J}(\gamma) \sim \gamma^{-\alpha} , \alpha > 0 , \text{ as } \gamma \to \infty .
\]

(52)

Using these results in (50), it is seen that \( \lambda(\gamma) \sim \gamma^{\frac{1}{2} - \alpha} , \alpha > 0 , \) as \( \gamma \to \infty \) in the left half plane, and \( \lambda(\gamma) \sim \gamma^{\frac{1}{2} - \delta} , \delta > 0 , \) as \( \gamma \to \infty \) in the right half plane. The reason that \( \lambda(\gamma) \) might vary as less than \( \gamma^{\frac{1}{2}} \) in the right half plane is that the right side of (50) contains two terms which could cancel. Thus, the integral function is a polynomial of degree less than \( \frac{1}{2} \), which is a constant, \( C \).

Setting each side of (50) equal to this constant gives

\[
\overline{J}(\gamma) = \frac{-C \sqrt{\gamma - j\beta}}{\omega \mu (\gamma - j\beta \cos \theta)} ,
\]

\[
\overline{E}(\gamma) = \frac{1}{\gamma - j\beta \cos \theta} \left[ \frac{C}{2 \sqrt{\gamma + j\beta}} - 1 \right] ,
\]

(53)

where \( C \) remains to be determined. This can be done from the boundary condition tangential \( E = 0 \) on the conductor. A simpler method is to note, from (24), that the inverse transform of \( \overline{E}(\gamma) \) is zero for \( z < 0 \). Therefore, \( \overline{E}(\gamma) \) must have no singularities in the right half \( \gamma \) plane.
This requires the bracketed quantity in (53) to have a zero at \( Y = j\beta \cos \theta \) to cancel the pole. Thus,

\[
C = 2\sqrt{j2\beta} \cos \theta/2 ,
\]

which, when substituted into (53), completes the solution to the transform equation.

IV. A MORE GENERAL METHOD OF SOLUTION

A. Theory

The method of solution of section III depends upon the construction of a function which, unless it is a constant or a simple pole, is almost impossible to find. This places a serious restriction on the type of problems which can be handled. In this section, a more general method of solution is given, which gives the solution in a straightforward manner with no function synthesis required. It is therefore applicable to a greater variety of problems. This method of solution is similar to that developed by Wiener,\textsuperscript{20} and published by Levinson,\textsuperscript{21} in the theory of prediction and filtering of time functions.

The transform equation must be written with the source term, \( F(\gamma) \), separated from \( J \) or \( \bar{E} \), as in equation (3). The determination of the regions of regularity of the functions, given in section IIIA, is still applicable. Thus, \( J(\gamma) \) is regular in the left half plane \( \text{Re}(\gamma) < -\text{Im}(\beta) \), except for a possible source pole, and \( \bar{E}(\gamma) \) is regular in the right half plane \( \text{Re}(\gamma) > \text{Im}(\beta) \), except for a possible source pole. \( \bar{G}(\gamma) \) is regular in the strip \( |\text{Re}(\gamma)| < -\text{Im}(\beta) \), and \( F(\gamma) \) is a known function. In general,
the transform equation is regular in the strip enclosing the imaginary axis, $|\text{Re}(\gamma)| < -\text{Im}(\beta)$, except for a possible source pole.

As in the previous solution, $\overline{G}(\gamma)$ is again expressed in quotient form, equation (35), with $\overline{G}_L$ regular and not zero in the left half $\gamma$ plane, and $\overline{G}_R$ regular and not zero in the right half $\gamma$ plane. Substituting this representation into the transform equation gives

$$
\overline{E}(\gamma) \overline{G}_R(\gamma) - \overline{J}(\gamma) \overline{G}_L(\gamma) = \overline{F}(\gamma) \overline{G}_R(\gamma) .
$$

The first term on the left of (55) is regular in the right half plane, and the second term is regular in the left half plane. The term on the right is a known function.

The behavior of $\overline{J}(\gamma)$ and $\overline{E}(\gamma)$ as $\gamma \to \infty$ in their respective half planes of regularity is determined as in section IIIA. In general, it is found that $\overline{E}(\gamma) \overline{G}_R(\gamma)$ varies as $\gamma^\alpha$, $\alpha < 1$, as $\gamma \to \infty$ in the right half plane, and $\overline{J}(\gamma) \overline{G}_L(\gamma)$ varies as $\gamma^\delta$, $\delta < 1$, as $\gamma \to \infty$ in the left half plane. Thus, from transform theory, it follows that

$$
\int_{-j\infty}^{j\infty} \overline{E}(\gamma) \overline{G}_R(\gamma) e^{\gamma z} d\gamma = 0 , \quad z < 0 ,
$$

$$
\int_{-j\infty}^{j\infty} \overline{J}(\gamma) \overline{G}_L(\gamma) e^{\gamma z} d\gamma = 0 , \quad z > 0 .
$$

Therefore, through inversion of equation (55), one has

$$
\int_{-j\infty}^{j\infty} \overline{F}(\gamma) \overline{G}_R(\gamma) e^{\gamma z} d\gamma = \begin{cases} 
\int_{j\infty}^{-j\infty} \overline{E}(\gamma) \overline{G}_R(\gamma) e^{\gamma z} d\gamma , & z > 0 , \\
\int_{-j\infty}^{j\infty} \overline{J}(\gamma) \overline{G}_L(\gamma) e^{\gamma z} d\gamma , & z < 0 .
\end{cases}
$$

The function on the left side of (57) is defined over all $z$. On the
the solution, rather than from other considerations. As mentioned previously, this method permits the solution to problems for which the source is in the finite region. There is also the possibility that this method can be extended to problems for which the conductor is of finite length.

B. Scattering by a Half Plane

To illustrate the method described in this section, it will be applied to the same problem as was considered in section IIIC. The transform equation and the geometry of the problem are given in section IIIC.

The first steps of the solution are the same as in section IIIC, giving

\[
\left( \frac{\omega \beta}{\sqrt{\gamma - j\beta}} \right) J(\gamma) + 2 \sqrt{\gamma + j\beta} \overline{E}(\gamma) = -2 \sqrt{\gamma + j\beta} \frac{\gamma}{\gamma - j\beta \cos \theta}
\]  

which is (50) rearranged in the form of (55). The first term of (60) is regular in the half plane to the left of the integration path, and the second term is regular in the half plane to the right of it. Both terms are well behaved as \( \gamma \to \infty \) in their respective half planes of regularity. Thus, substituting into (59), the solution is given by

\[
J(\gamma) = -\frac{\sqrt{\gamma - j\beta}}{2\pi j \omega} \int_{-j\infty}^{j\infty} \frac{2 \sqrt{s + j\beta}}{(s - j\beta \cos \theta)(\gamma - s)} ds, \quad \text{Re}(s) > \text{Re}(\gamma)
\]

\[
\overline{E}(\gamma) = \frac{1}{4\pi j \sqrt{\gamma + j\beta}} \int_{-j\infty}^{j\infty} \frac{2 \sqrt{s + j\beta}}{(s - j\beta \cos \theta)(\gamma - s)} ds, \quad \text{Re}(s) < \text{Re}(\gamma)
\]  

The integration paths are as shown in figure 13. The integrand varies as \( s^{-3/2} \) as \( s \to \infty \), allowing the contour to be closed to the right at
Fig. 13a. Path of integration for $J(\gamma)$, equation (61).

Fig. 13b. Path of integration for $\bar{E}(\gamma)$, equation (61).
infinity. Evaluating the integrals in (61) by the theory of residues gives

\[ J(\gamma) = \frac{-2 \sqrt{12\beta} \cos(\theta/2) \sqrt{\gamma - \beta}}{\omega \mu (\gamma - J^2 \cos \theta)} , \]

\[ E(\gamma) = \frac{1}{\gamma - J^2 \cos \theta} \left[ \frac{\sqrt{12\beta} \cos \theta/2}{\sqrt{\gamma + J^2}} - 1 \right] . \]  

Equations (62) are the solutions to the transform equation, and are identical to those obtained in section IIIC.

V. CURRENT ELEMENT NEAR THE EDGE OF A CONDUCTING HALF PLANE

A. Transform Solution

The problem of a current element adjacent to a semi-infinite plane conductor can be solved by the method of section IV. Consider an infinite line source, defined by \( x = a, z = -b \), in the vicinity of a semi-infinite, perfectly conducting plane, defined by \( x = 0, z < 0 \). The geometry is illustrated by figure 14.

![Figure 14. Current element adjacent to a semi-infinite plane conductor.](image-url)
The line current is of uniform excitation, taken as unity. The field everywhere is independent of the y coordinate, reducing the problem to a two-dimensional one. The field point is specified by the coordinates x and z, or alternatively, by the distance r from the origin and the angle θ measured from the positive z axis. From symmetry considerations, the field is everywhere transverse magnetic to the y axis. It can therefore be represented by

\[ H = \nabla \times A, \]

where \( A = j \psi \), and \( \psi \) is a solution to the scalar wave equation. The boundary conditions for the problem are

\[ E_y = \beta^2 \psi = 0 \text{ at } x = 0, \ z < 0, \]

and the field satisfies the Sommerfeld radiation condition at large distances from the origin.

Let \( \psi \) be expressed in terms of the incident field from the line source, \( \psi^1 \), and a scattered field due to the presence of the plane conductor, \( \psi^3 \), as

\[ \psi(x, z) = \psi^1(x, z) + \psi^3(x, z). \]

The incident field is simply the field from a line source radiating into free space, which is

\[ \psi^1 = \frac{1}{4j} H_0^{(2)}(\beta\sqrt{(x-a)^2 + (z+b)^2}). \]
According to the theory of equivalent currents,\textsuperscript{15} the scattered field can be represented as the radiation from an equivalent current sheet, replacing the plane conductor, with electric current equal to the discontinuity in the tangential magnetic field across the conductor. No equivalent magnetic current sheet is required since the tangential electric field is continuous (equal to zero) across the conductor. Since the only component of tangential magnetic field is $H_z$, the equivalent electric current sheet, $J(z)$, flows in the $y$ direction. The field from $J$ can be expressed as the sum of the contributions from all the differential line elements, as

$$\psi^s = \frac{1}{4j} \int_{-\infty}^{0} J(z') H_0^{(2)}(\beta \sqrt{x^2 + (z-z')^2}) \, dz' . \quad (67)$$

The integration extends only over negative values of $z'$ since $J$ is zero for positive $z'$.

Taking the transform of (65) gives

$$\Psi(x,y) = \Psi^i(x,y) + \Psi^s(x,y) . \quad (68)$$

Both $\Psi^i$ and $\Psi^s$ contain the transform of the Hankel function, which is known.\textsuperscript{3} Thus,

$$\Psi^i = \frac{e^{\gamma b}}{4j} \int_{-\infty}^{\infty} H_0^{(2)}(\beta \sqrt{(x-a)^2 + (z+b)^2}) \, e^{-\gamma(z+b)} \, dz ,$$

$$= \frac{e^{\gamma b} \, e^{-j|x-a|V\gamma^2 + \beta^2}}{2j \sqrt{\gamma^2 + \beta^2} } , \quad (69)$$

and
\[ \Psi = \frac{1}{4j} \int_{-\infty}^{0} J(z') e^{-\gamma z'} dz' \int_{-\infty}^{\infty} H_0^2(2\beta \sqrt{x^2 + (z-z')^2}) e^{\gamma(z-z')} dz \, dz' , \]
\[ = \frac{e^{-j\lambda \sqrt{\gamma^2 + \beta^2}}}{2j \sqrt{\gamma^2 + \beta^2}} \int_{-\infty}^{0} J(z') e^{-\gamma z'} dz' , \]
\[ = \frac{e^{-j\lambda \sqrt{\gamma^2 + \beta^2}}}{2j \sqrt{\gamma^2 + \beta^2}} J(\gamma) . \]  

Once \( \bar{J}(\gamma) \) is determined, substituting equations (69) and (70) into (68), and inverting, gives the field at any point \( x, z \). Letting \( x = 0 \) in (68) gives
\[ 2j \sqrt{\gamma^2 + \beta^2} \Psi(0, \gamma) = e^{\gamma b} e^{-j\alpha \sqrt{\gamma^2 + \beta^2}} + \bar{J}(\gamma) , \]
which is the appropriate transform equation for the problem.

The regions of regularity of the unknowns must now be determined. From the boundary condition, equation (64), it is seen that \( \Psi(0, z) = 0 \) for \( z < 0 \), so that
\[ \Psi(0, \gamma) = \int_{0}^{\infty} \Psi(0, z) e^{-\gamma z} dz . \]

Since \( \Psi \) is a solution to the two dimensional wave equation with the source in the finite region, the behavior of \( \Psi \) for large \( z \) is given by
\[ \Psi(0, z) \xrightarrow{z \to \infty} e^{-j\beta z} \sqrt{\frac{1}{z}} . \]

This implies that \( \Psi(0, \gamma) \) is regular in the region \( \text{Re}(\gamma) > \text{Im}(\beta) \). Since \( J(z) = 0 \) for \( z > 0 \), its transform is given by
\[ \bar{J}(\gamma) = \int_{-\infty}^{0} J(z) e^{-\gamma z} dz . \]
Since $J$ is equal to the discontinuity in $H_z$, which is a solution to the two dimensional wave equation, it is also a solution. Thus,

$$J(z) \xrightarrow{z \to -\infty} \frac{e^{j\beta z}}{\sqrt{-z}},$$

which implies that $\bar{J}(Y)$ is regular in the region $\text{Re}(Y) < -\text{Im}(\beta)$. With these results, it is seen that all terms of equation (71) are regular in the strip enclosing the imaginary axis, $|\text{Re}(Y)| < -\text{Im}(\beta)$.

It is also necessary to know that the unknowns are well behaved, that is, do not increase exponentially, as $Y \to \infty$ in their respective half planes of regularity. That this condition is met is implied in the assumption that the field transforms exist. In order for the fields to be integrable about the $z$ origin, the singularities must be no greater than $1/z$. This implies that both $\bar{J}$ and $\Psi$ do not become infinite faster than the first power of $Y$ as $Y \to \infty$ in their respective half planes of regularity.

The function multiplying $\Psi$ in equation (71) is readily separated into two factors regular and not zero in the right and left half planes, respectively. When this is done, rearranging (71) gives

$$2j \sqrt{Y + j\beta} \Psi(0, \gamma) - \frac{1}{\sqrt{Y - j\beta}} \bar{J}(\gamma) = \frac{e^{\gamma \delta} e^{-j\gamma \sqrt{\gamma^2 + \beta^2}}}{\sqrt{Y - j\beta}},$$

which is of the form of (55). The first term on the left side of (76) is regular in the right half plane and the second term is regular in the left half plane. Thus, the conditions necessary for (59) to hold are met, giving
These are the solutions to the transform equation. Figure 15 gives the path of integration for equation (77). There are branch points at $s = \pm j \beta$, and branch cuts are placed from these points out to infinity, parallel to the imaginary axis. The path of integration must lie to the right of the pole $s = \gamma$. Correspondingly, the path of integration for equation (78) must lie to the left of the pole $s = \gamma$.

B. Calculated and Experimental Results

The solution of part A, through inversion, gives the field at any point in space. However, the complexity of the solution is such that it is impractical to evaluate the inverse transforms. Fortunately, as shown in section IIIB, the far field can be obtained directly from the transform solution, without recourse to inversion.

Equations (65)-(67) give the field as a function of the distance from a current element, $\rho = \sqrt{(x-x')^2 + (z-z')^2}$. For large $\rho$, one can apply the asymptotic form for the Hankel function,\(^{22}\)

$$H_0^{(2)}(\beta \rho) \rightarrow \sqrt{\frac{2}{\pi \rho \beta}} e^{j\frac{\pi}{4} - \beta \rho}. \quad (79)$$

For the far field, the source and each differential element of the equivalent current sheet see the same angle $\theta$ to the field point, $r, \theta$. Considering the geometry of figure 14, for large $r$,
Fig. 15. Path of integration for equation (77).
\[ \rho = r - x \sin \theta - z \cos \theta . \quad (80) \]

Substituting the above into (66) and (67) gives

\[ \psi^i \frac{1}{r \to \infty} \frac{1}{4j} \sqrt{2/\pi r} \quad e^{j(\frac{2}{4} - \beta r)} \quad e^{j(\alpha \sin \theta - b \cos \theta)} , \quad (81) \]

\[ \psi^s \frac{1}{r \to \infty} \frac{1}{4j} \sqrt{2/\pi r} \quad e^{j(\frac{2}{4} - \beta r)} \quad \int_0^\infty J(z) \quad e^{j\beta z \cos \theta} \quad dz . \quad (82) \]

It is noted that the integral in (82) is simply the transform of \( J(z) \) evaluated at \( \gamma = -j\beta \cos \theta \). Therefore, by (65), the far field is given by

\[ \psi = K(r) \left[ e^{j(\alpha \sin \theta - b \cos \theta)} + \overline{J(-j\beta \cos \theta)} \right] , \quad (83) \]

where \( K(r) \) is a proportionality factor dependent only on \( r \), which is kept constant. In equation (77), setting \( \gamma = -j\beta \cos \theta \), and letting \( w = s/\beta \), gives

\[ \overline{J(-j\beta \cos \theta)} = \frac{1 + \cos \theta}{2\pi} \int_{-j\infty}^{j\infty} \frac{e^{-j\beta a V_{1+\omega^2}} e^{\beta bw}}{\sqrt{1 + jw (\cos \theta - jw)}} \quad dw . \quad (84) \]

For \( \beta \) real, the path of integration is given in figure 16. The reduction of the integral in (84) to real integrals, with proper phase of the radicals, is given in part C. Actual integration was carried out by numerical methods.

The experimental measurements were taken at a frequency of 10,000 mc. (\( \lambda = 3.0 \) cm.), using the model illustrated by figure 17. The model utilized a circular image plane, approximately 40\( \lambda \) in diameter, behind which the equipment was placed. Measurements were made by probing
Fig. 16. Path of integration for equation (80).

Fig. 17. Experimental model.
through small holes in the image plane, spaced $10^\circ$ apart and $6\lambda$ in from the edge. The semi-infinite ground plane was approximated by a sheet of 0.017 inch thick copper sheeting, $20\lambda$ in length and $6\lambda$ in width. A $\lambda/4$ stub antenna working against the image plane was used as the current source. That the pattern in the plane perpendicular to the axis of the stub antenna is the same as the pattern from an infinite line source can be seen as follows. The far field from an infinite line source is characterized mathematically by the assumption that each differential element of length sees the same angle to the field point. Thus, by superposition, the pattern is also that from a single differential dipole, in the plane perpendicular to its axis.

The signal source used was a Sperry 2K39 klystron, modulated at 500 cycles. The pick-up probe was a PRD 250 probe, using a type 1N23A crystal detector operated to give approximately square law detection. The detector output was amplified by a high-gain selective amplifier for 500 cycles, developed at the Antenna Laboratory. A Ballantine model 300 electronic voltmeter was used as the indicator.

Figure 18 gives the calculated pattern and experimentally measured points for the case where the source was a distance $\lambda/4$ from the plane reflector and $\lambda/4$ back from the edge, that is, $\beta_a = \lambda/4$, $\beta_b = \lambda/4$. The experimental measurements were accurate to only 20 db. below the maximum, so that some of the back radiation was unmeasurable. However, over the remainder of the range the measured values were essentially in agreement with the calculated pattern. Any discrepancies can be accounted for in the approximations embodied in the experimental model and the
Fig. 18. Calculated and experimental pattern for $\beta_a = \lambda/4$, $\beta_b = \lambda/4$. 
normal errors of measurement.

An interesting check of the solution can be obtained by letting $a = 0, \ b > 0$ in either the transform solution or the far field solution. This is the situation for which the current element lies on the surface of the conductor. Now the branch point at $\gamma = -j\beta$ is no longer present, and the path of integration can be closed at infinity to the left. Evaluation by the theory of residues shows the field to be everywhere zero, which is the expected result.

C. Reduction of the Integral

In order to integrate (84) numerically, it is convenient to break it up into several integrals, corresponding to the paths of different phase of the radicals and to the semi-circles of indentation. The integration path is shown in Figure 16. Consider the integral appearing in (84) to be given by

$$\int_{-\infty}^{\infty} \frac{e^{-j\beta a \sqrt{1 + w^2}} e^{j\beta b w}}{\sqrt{1 + jw (\cos \theta - jw)}} \, dw = \sum_{n=1}^{6} I_n, \quad (85)$$

where the integrals $I_1$ through $I_6$ will be defined later.

As illustrated by Figure 19, one can express any point $w_0$ in terms of the polar coordinates measured from the branch points as

$$w_0 = j + r_1 e^{j\phi_1} = -j + r_2 e^{j\phi_2}. \quad (86)$$

For the radicals to have the correct phase, care must be taken to remain on the proper branch of the function. The values of $\phi_1$ and $\phi_2$ for the several segments of path length are given on Figure 19.
Fig. 19. Determination of the phase of radicals.
By substitution from (86), the radicals are given by

\[ V_1 = \sqrt{r_1 r_2} e^{j(\phi_1 + \phi_2)/2}, \]

\[ V_2 = \sqrt{r_1} e^{j(\phi_1 + \phi_2)/2}, \]

(87)

where \( r_1 = |x-1|, \) \( r_2 = |x+1|, \) \( r_1 r_2 = |x^2-1|, \) on the imaginary axis.

Thus, the integration over the segments of the path as numbered on figure 19 are given by

\[ I_1 = \int_1^\infty \frac{-e^{-\beta x}}{\sqrt{x-1} (x + \cos \theta)} e^{j\beta x} \, dx, \]

\[ I_2 = \int_1^{-1} \frac{e^{-j\beta x}}{\sqrt{1-x} (x + \cos \theta)} e^{j\beta x} \, dx, \]

\[ I_3 = \int_{-\infty}^{-1} \frac{e^{-j\beta x}}{\sqrt{1-x} (x + \cos \theta)} e^{j\beta x} \, dx. \]

(88)

(89)

(90)

The integrals over the semi-circles shown on figure 16 are

\[ I_4 = \oint_{\varepsilon} = 0, \]

\[ I_5 = \int_{j\cos \theta} = \frac{-\pi}{\sqrt{1 + \cos \theta}} e^{-j\beta(a \sin \theta + b \cos \theta)}, \]

\[ I_6 = \int_{-j\alpha} = 0. \]

(91)

(92)

(93)

The above integrals can now readily be divided into their real and imaginary parts, and numerically integrated.
VI. REFERENCES


10. Churchill, R. V., "Modern Operational Mathematics in Engineer-


16. Sommerfeld, A., op. cit., Chapter V.


