A STUDY OF THE CHARACTERISTICS
OF DASHPOTS:
SOME DESIGN CRITERIA FOR HYDRAULIC
SHOCK ABSORBERS

DISSERTATION
Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State
University

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****

The Ohio State University
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1. Introduction

(a) Statement of the Problem

A vibrating system of one degree-of-freedom consists of a mass, a spring element, and a damping element. The spring and damping elements can be either linear or nonlinear. An example of a nonlinear spring is an air column, and an example of a nonlinear damping device is a hydraulic dashpot having an orifice for the damping of the fluid flow.

The general problem in the design of such a vibrating system consists of two parts: (1) to determine the exact nature of the nonlinearities, and if that is not possible, to ascertain the forms such nonlinearities would take and to express them as mathematical formulae. (2) to seek a logical method for the design of the various parts of the system to satisfy certain prescribed conditions.

Several other investigations done at The Ohio State University studied the first part of the general problem. A more detailed account of these investigations will be given in § 1 (b). This investigation studies the second part of the general problem which seeks a method for the design of a hydraulic shock absorber having an air-spring and a hydraulic dashpot to satisfy prescribed con-
ditions. The conditions are either (1) a maximum allowable force to be transmitted to the mass is specified; or (2) a time for the mass to settle within a specified distance from the static-equilibrium position is specified; or both.

(b) Investigations of the problem at The Ohio State University

One of the early investigators of dashpot characteristics was J. B. Peterson\(^{(1)}\)\(^*\) of the National Bureau of Standards. The dashpot he investigated was essentially a linear one as it consisted of a cylinder, a piston and some fluid, but not an orifice.

The first of this series of investigations done at The Ohio State University was made by R. E. Dine\(^{(2)}\) in 1948. He designed a dashpot testing machine equipped with an orifice whose size could be varied by a metering rod. The apparatus consists of a vibrator, a spring, the vibrating mass, and the dashpot functioning as a damper. The vibrator is an induction motor driving two pairs of bevel gears in opposite directions with out-of-balance weights adjusted to be in phase at top and bottom positions only. The horizontal components of the centrifugal force cancel each other out. The out-of-balance weights cause a sinusoidal oscillation in the up-and-down direction. The spring consists of a steel beam onto which the motor is attached. Different beams can be substituted to vary the spring stiffness. The vibrating mass consists of the motor, gears, pistons, yoke, the oil in the dashpot, and a portion of the weight of the beam. The

\* Superior numbers refer to bibliography.
dashpot consists of two pistons connected by a yoke moving inside a cylinder and separated by a plate having the orifice. The dashpot is mounted on a solid base beneath the clamped beam, and the yoke is connected to the mid-point of the beam. The steady-state amplitude of vibrations for the system is measured by electrical circuitry with the output shown on an oscilloscope. The experiment covers various input frequencies, sizes of orifice, sizes of beam, and oils of various viscosity.

In 1949 J. E. Michaels\(^{(3)}\) obtained the first experimental data on the machine using a 2-inch diameter piston and S.A.E. 20 motor oil. He postulated that the damping force is proportional to the square of the piston velocity, and inversely proportional to the square of the orifice discharge coefficient. He then converted the hydraulic damping coefficient into an equivalent viscous damping coefficient for the case of the system at resonance, and solved for the steady-state amplitude by linear vibrations theory. His predicted and experimental results agreed fairly well not only at resonance but also at other frequencies, in which cases the concept of equivalent viscous damping is not strictly valid. This shows that the assumption of the damping force being proportional to the square of the velocity is a fairly good approximation.

In 1950 R. L. Kastor\(^{(4)}\) refined the problem by reasoning that in addition to the hydraulic damping of the orifice, there is viscous damping between the piston and the cylinder as in Peterson's dashpot. The combination of these two effects is no doubt one cause for the fact that the damping is between viscous and hydraulic.
In 1950 M. Gurpinar(5) also postulated that the damping is between viscous and hydraulic by another way of reasoning. He went along with Michaels' assumption that the damping force is proportional to the square of the piston velocity and inversely proportional to the square of the orifice discharge coefficient. However, the value of the orifice discharge coefficient is a function of the Reynold's number which is in turn a function of the velocity. Hence the total effect would be between viscous and hydraulic. Gurpinar found that some dry friction as well as viscous friction is also present in the system.

It seems to be established then that the hydraulic damping is a good approximation and a damping between viscous and hydraulic is a better refinement. This conclusion is essentially substantiated by other investigators whose work will be discussed later. In view of this fact, a refinement of Michaels' equivalent viscous damping coefficient for the range between viscous and hydraulic damping is given in Appendix I of this study.

It is now clear that the previous investigations essentially deal with a system with one degree of freedom, having a linear spring (the beam) and nonlinear damping. The system is excited by a sinusoidal forcing function.

This study proposes to investigate a slightly more general system in that the spring is an air column whose characteristic is also nonlinear. Instead of being excited sinusoidally the system is forced by a step-function.
The two ways of approach correspond to the two standard studies of servomechanisms. The former corresponds to the frequency-response test. The latter corresponds to the transient-response test.

Physically our system represents an oleo-pneumatic shock-absorber having a mass dropped onto it, or the shock-absorber and the mass dropped together as in the case of a landing airplane.

In addition to obtaining the transient-response of the shock-absorber, the study also seeks a rational method for dimensioning the various parts of the shock-absorber. The method answers, for instance, the question: What size of orifice and what length of air column should the designer prescribe (a) if no force of more than F lb should be transmitted to the body to be stopped, or (b) if the body should settle down to no appreciable displacement from the static equilibrium position t seconds after the impact?

The same method of approach can be applied to a shock-absorber with a linear spring under impact.

2. The orifice damping force

Some evidence as presented in §1 shows that the damping force D is proportional to the velocity of the piston to some power s where $1 \leq s \leq 2$. Here the reasoning will be propounded still further.

If the fluid flow through the orifice were purely laminar, then the damping force obeys the Hagen-Poiseuille law, or

$$D = \frac{8 \pi \mu h u_{av}}{}$$

(1)
where, $\mu$ is the dynamic viscosity of the fluid in reyns,
$h$ is the length of the orifice in inches,
and $u_{av}$ is the average velocity of the fluid in in. per sec.

Applying the condition of continuity,

$$ a u_{av} = A v $$

(2)

where, $a$ is the cross-sectional area of the orifice in sq. in.
$A$ is the cross-sectional area of the piston in sq. in.
and $v = \dot{y}$ is the velocity of the piston in in. per sec.

This gives

$$ D = \frac{8 \pi \mu h A v}{a} $$

(3)

Then we have $s = 1$ in the case that the flow is laminar.

If the fluid flow were purely turbulent, the viscosity of the fluid will not affect the damping force. Here Bernoulli's theorem can be applied, or

$$ \frac{\Delta p}{\rho} = \frac{(u_{av})^2}{2} $$

(4)

where, $\Delta p$ is the pressure difference for the two sides of the orifice in psi
and $\rho$ is the mass density of the fluid in lb-sec$^2$/in.\(^4\)

Applying the same condition of continuity into (4) gives

$$ \Delta p = \frac{\rho}{2} \left( \frac{A v}{a} \right)^2 $$

(5)
from which \[ D = \Delta \rho \cdot A = \frac{\rho A^3 v^2}{2 a^2} \] (6)

Hence we have \( s = 2 \) in the case the flow is purely turbulent.

Equation (6), which is very similar to the equation used by Michaels, is very widely used for the study of hydraulic shock-absorbers. Investigators whom we can cite include Keller,\(^6\) Callerico,\(^7\) Burger,\(^8\) Brown,\(^9\) and Yorgiadis.\(^10\)

If the condition of continuity, Equation (2), is modified by the orifice discharge coefficient \( C_d \), then

\[ C_d \cdot u_{av} = A \cdot v \] (2a)

and Equation (6) becomes

\[ D = \frac{\rho A^3 v^2}{2 C_d^2 a^2} \] (6a)

Both (6) and (6a) are based on the hypothesis that the flow through the orifice is purely turbulent which, in actuality, is probably not strictly true. Still, what value of \( C_d \) is suitable for the approximate representation of the actual damping force?

Hurtz\(^11\) used Equation (6a) and an oil of specific gravity 0.834, thus

\[ \frac{\rho}{2} = \frac{0.834 \times 62.5}{2 \times 3.86 \times 1728} = 3.9 \times 10^{-5} \]

and

\[ D = \frac{\rho A^3 v^2}{2 C_d^2 a^2} = \frac{3.9 \times 10^{-5} A^3 v^2}{C_d^2 a^2} \] (7)

He then tried to determine the value of \( C_d \) from actual drop-test records of the shock-absorber. He wrote: "The determination of \( C_d \)
from the records is more difficult and the values found show considerable scatter. It is found convenient to use an orifice pressure coefficient, $C_p$.

$C_p$ is defined by

$$\Delta p = \frac{F}{2 \cdot \frac{A \cdot v}{a}} = C_p \left(\frac{A \cdot v}{a}\right)^2$$

Hence

$$C_p = \frac{3.9 \times 10^{-5}}{C_d^2}$$

(8)

The pressure drop may be found if the damping force is known. In Hurzt's analysis of the records, the force was assumed to be the difference between the total compressive force on the piston and that due to air pressure. This neglects the effects of friction which should be taken into account at this point. Attempts to evaluate the frictional force were not successful because the data were not complete. However, near the end of the compressive stroke where the damping force approaches zero, friction was found to be unimportant, as indicated by the fact that the measured force due to air pressure was nearly equal, in all cases, to the total force. The computed pressure drops were plotted vs. fluid velocity through the orifice. An average value of $C_p = 3.5 \times 10^{-5}$ was found.

Thus

$$C_d = \sqrt{\frac{3.9}{3.5}} = 1.05$$

(9)

Of course, $C_d$ theoretically cannot be greater than unity. Hence $C_d = 1$ is a fair approximation.

In addition to the reasons advanced by Kastor and Gurpinar to explain the fact that the overall effect of orifice damping is between viscous and hydraulic, it may also be reasoned that undoubtedly
during a portion of the cycle the flow is laminar, during another portion it is in the transition region, and during another portion it is probably turbulent. In any case the obscurity of the exact nature of the friction does not prevent a logical approach for design. In designing parts of a machine, one often has to idealize the influencing factors in order that the behavior of the parts can be mathematically predicted. Subsequent experiments or redesigns may be necessary to make them meet specifications.

Since \( C_d \approx 1 \), Equation (6) will be used as an approximate expression of the orifice damping force for preliminary design. A more realistic expression will be given by the equation,

\[
D = K (u_{av})^s
\]  

(10)

where \( K \) and \( s \) are both constants. Applying the condition of continuity to (10) gives

\[
D = K (\frac{A}{a})^s = K_d (\frac{v}{a})^s
\]  

(11)

with \( 1 \leq s \leq 2 \)

Both the constants \( K_d \) and \( s \) are experimentally determined for the hydraulic shock-absorber whose final design is being sought. The method for determining these constants, as proposed by Durand and Rosenberg, \(^{(12)} \) will be given in \( \S 6 \).

3. The air spring

The air column under compression obeys the polytropic gas law,
pV^n = constant. If the compression process is isentropic, n = C_p/C_v = 1.4. On the other hand if the process is isothermal, then n = 1.0. Different investigators give different values of n. (Whose experiment is reliable?) Two British engineers, Burger(13) and Dirac(14) use n = 1.3 without presenting any experimental evidence. Similarly, Keller,(6) Yordiadis,(10) and Cicala(15) use n = 1.2. Hertz(11) experimentally determines n to be 1.1 in his investigation. Durand and Rosenberg(12) experimentally obtain n = 1.0.

The fact that n is not 1.4 implies that the air column is not being compressed adiabatically, but rather, some heat transfer must have taken place through the cylinder wall, or as more likely is the case, the heat is lost into the oil spray during impact.

Since only the values of 1.1 and 1.0 are experimentally substantiated, we shall use n = 1.1 for design to indicate that the compression process may not be exactly isothermal.

Let \( l \) (in) = length of air column when the strut is extended such that the pressure is atmospheric,

\[ p_a \text{ (psi)} \]

\( A \) (sq. in.) = cross-section of cylinder

\( p \) (psi) = pressure when the air column is compressed a distance \( y \) (in.)

Applying the polytropic gas law gives

\[ p = p_a \left( \frac{e}{e-y} \right)^n \]
The spring force, \( f \), is

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} \left[ (1 - \gamma \dot{\alpha})^{n-1} \right]
\]  

(12)

4. The idealized system and the equation of motion

The shock-absorbing system is idealized as a system with one degree of freedom (Fig. 1). In the shock-absorbing system for airplane landing, our idealization can be justified if the weight of the landing gears, including the oleo struts, is small compared with that of the plane, and if the effect of the tires is neglected. In the case of a weight dropping onto a shock-absorber, the idealization fits the actual system even better.

The equation of motion of the system under impact is

\[
\frac{W}{g} \ddot{y} + D + f = W
\]  

(13)

Both the displacement \( y \) and the time \( t \) are measured initially at the instance of impact.

Substituting equations (6) and (12) into equation (13) gives

\[
\frac{W}{g} \ddot{y} + \frac{PA^3}{2} \left( \frac{\dot{y}}{a} \right)^2 + PaA \left[ (1 - \gamma \dot{\alpha})^{n-1} \right] = W
\]  

(14)

And the substitution of (11) and (12) into (3) gives the more accurate equation of motion,

\[
\frac{W}{g} \ddot{y} + \kappa d\left( \frac{\dot{y}}{a} \right)^3 + PaA \left[ (1 - \gamma \dot{\alpha})^{n-1} \right] = W
\]  

(15)
FIG. 1 SCHEMATIC REPRESENTATION OF SHOCK-ABSORBING SYSTEM

FIG. 2 SHOCK-ABSORBING SYSTEM WITH TIRE
5. Outlines for the design of hydraulic shock-absorbers

(a) By the use of past drop-test records

The dynamical design of a hydraulic shock-absorber involves the solution of a nonlinear differential equation such as (14) or (15). In the case that the system has two degrees of freedom (e.g., the effects of the mass of the airplane undercarriage and the elasticity of the tires are included) two simultaneous nonlinear differential equations have to be solved. This is ordinarily a difficult and tedious task.

If the maximum allowable transmitted force were the only design criterion and the record of past satisfactory designs available, the determination of the orifice size can be obtained without first solving the differential equations. The method of Nightingale (16) (which is very similar to that of Dirac (14)) applying to a shock-absorbing system with tire (Fig. 2) is given in Appendix II.

(b) Design for a specified, allowable transmitted force

The equation of motion is given by equation (15). Here $W$ and $A$ will have been decided by the designer. Also $p_a = 14.7$ psi, $g = 386$ in. per sec.$^2$, and $n = 1.1$. $K_d$ and $s$ for the type of shock-absorber concerned will have been determined by a pre-design experiment as is described in §6. The designer then proceeds to design the various parts in the following manner:

(i) The determination of the air-column length, $l$,

or the static-equilibrium displacement, $y_s$

For static equilibrium, equation (15) has the follow-
boundary conditions:

\[ \dot{y} = y = 0, \quad \text{and} \quad y = y_s \]

(15) then becomes

\[ f_e A \left[ (1 - \frac{y_s}{e})^{-n} - 1 \right] = W \]

(16)

If the designer wishes to specify \( y_s \), then \( e \), the fully-extended air column can be calculated from (16), or vice versa. Hadkeke (12) recommends that for airplane struts, the following proportions should be maintained:

\[ \frac{1}{2} \leq \frac{y_s}{e} \leq \frac{5}{A} \]

(17)

Some similar proportions found to be satisfactory from experience can be used as guides for the designer in checking his design.

(ii) The determination of the area of the orifice, \( a \)

The solution of the nonlinear differential equation (15) either by numerical (\$ 9 and 10) or by analog method (\$ 11 and 12) shows that the maximum retardation of the mass occurs at the instant of impact. At this instant, the displacement \( y = 0 \), and \( \dot{y} = v \), the design impact-velocity (in the case of an airplane, \( v \) is the vertical component of the landing velocity). The retardation is \( \ddot{y} = \dot{y}_{\text{max}} \) which is related to the maximum allowable transmitted force, \( F_e \) by the equation,

\[ F_e = W \left( 1 - \frac{\dot{y}_{\text{max}}}{g} \right) \]

(18)

In the case of the airplane \( F_e \) is the design load of the landing
gear column. Applying the boundary conditions to (15) gives

$$F_e = \kappa_d \left( \frac{\nu}{a} \right)^s$$

(19)

The area of the orifice, $a$, can then be determined from equation (19).

A specific example using $F_e = 300$ lb is numerically worked out in § 9 and 10.

(c) Design for a specified "settling" time

To design a shock-absorber for a specified "settling" time involves the complete solution of the nonlinear differential equation. Since the equation has no solution in forms of elementary functions and since design is the reversed process of analysis, the only workable method would be to solve a series of such equations with arbitrarily chosen orifice area, $a$, and air-column length, $l$. The combination that gives the specified "settling" time is one possible design.

The solution of the differential equation, even numerically, is a very tedious job. The only practical way for design is to employ the aid of an analog computer. A specific example requiring the body to settle down to no appreciable displacement from the static-equilibrium position $0.4$ seconds after the impact is given in § 11 and 12.

(d) Design for minimum transmitted-force

The method of Durand and Rosenberg (13) for the design of a minimum transmitted-force shock-absorber employing a metering pin of
the correct shape is given in Appendix III.

6. Procedure for the experimental determination of the orifice damping constants $k$ and $\alpha$ and polytropic constant $n$

If a designer starts to design a hydraulic shock-absorber from the beginning, without any previous drop-test records to guide him, he may well start with the idealized equation (14) as a first approximation. If the performance requirement is not too exact, this first design might suffice. On the other hand, a model of the shock-absorber might be built based on the preliminary design, and experiments of Durand and Rosenberg$^{(13)}$ as described below performed on it to determine more accurately the parameters of the more exact equation (15). The final design of the shock-absorber will be based on the solution of this equation.

The general scheme of testing consists of locating the shock-absorber beneath a fixture which permits the dropping of a known weight from a predetermined height on the top of the extended shock-absorber. A sketch of this experimental setup is shown in Fig. 3.

A method of measuring displacements and velocities is provided by a sliding-wire rheostat attached to the shock-absorber in such a manner that displacement of the piston pushes the rider of the rheostat along, thus varying a voltage in proportion to the displacement. The velocity is obtained by electrical differentiation of the voltage which is controlled by the rheostat. Displacement, velocity and time are recorded simultaneously with a recording oscillograph.

A practical method of measuring the orifice damping, and air
FIG. 3
DURAND & ROSENBERG'S EXPERIMENTAL DROP-TEST ARRANGEMENT
forces is as follows:

If a weight of magnitude \( W_1 \) is dropped onto the shock-absorber, the resisting force is

\[
F_e = W_t \left( 1 - \frac{\ddot{y}}{g} \right) \tag{20}
\]

From recording the velocity and time, the acceleration is computed as \( \ddot{y} = \frac{\Delta v}{\Delta t} \), which completely defines the force \( F_e \) whose magnitude is to be determined.

If the air chamber is sealed and oil is provided in the oil chamber, but the orifice is removed \( F_e \) is the air-spring force.

If the air chamber is opened and the strut is otherwise completely assembled, \( F_e \) is the oil damping force.

The experimental results of the air-spring force \( f \) are then plotted against the piston displacement \( y \). (The curve will be of the shape shown in Fig. 4.) By fitting the experimental curve with equation (12) using various values of \( n \), the constant \( n \) can then be determined.

For determining \( k_d \) and \( s \) in the fluid damping force, the shock-absorber is equipped with an orifice. The total force required to obtain the measured accelerations is computed. From this force the inertia and air spring forces are subtracted leaving

\[
D = k_d \frac{v_s^5(y)}{a^s}
\]

where the net orifice area, \( a \), is now constant. The quantity \( D \) is plotted against the velocity on log-log paper (see Fig. 5). The
slope of a straight line which fairs through the experimental points is the value of the coefficient \( s \). The intercept of this line with the ordinate \( v = 1 \) permits the computation of the constant \( K_d \).

Durand and Rosenberg devised this method and applied it to an oleopneumatic shock-absorber (Fig. 7) with a cylinder diameter of 3 inches and an orifice equivalent to a 1/2 inch hole. (They used a metering pin.) They obtained the following results (in lb., in., sec., units):

\[
\begin{align*}
    n &= 1.0 \\
    s &= 1.46 \\
    K_d &= 0.143
\end{align*}
\]

7. A specific design problem and its realistic equation of motion

Since this method of experimentally determining the constants appears logical and realistic, the resulting equation from (15) will be a good description of this dynamical system.

Durand and Rosenberg's data will be used, with certain modifications, to demonstrate the design approach. \( n \) is taken to be 1.1 instead of 1.0 as explained in §3. The constant \( s \) is taken to be 1.5 instead of 1.46 because a function to the power 1.5 is much easier generated in an analog computer circuit, and the experimental result might not be accurate to the second place of decimals.

Using the modifications the design data become:

\[
\text{Diameter of piston} = 3 \text{ in.}
\]
FIG. 4 DETERMINATION OF \( n \)

\[ f = p_a A \left[ (1 - \frac{x}{L})^n - 1 \right] \]

FIG. 5 DETERMINATION OF \( K_d \) & \( S \)

\[ \log_{10} D = \log_{10} K_d + s \log_{10} \left[ \frac{V}{A} \right] \]

FIG. 6 APPROXIMATE SLOPE OF A CURVE.
FIG. 7

DETAIL OF SHOCK-ABSORBER

"O" RINGS

MILLED SLOT

ORIFICE

METERING PIN
A = 7.07 sq. in.

$k_d = 0.143$

Dropped weight, $W = 48.5$ lb

Impact velocity, $v(0) = v_0 = 48$ in./sec.

Rewriting equation (15) in a more suitable form, and taking into account the fact that $y$ might be positive or negative, gives

$$
\dddot{y} = \mp \frac{9k_d}{W\alpha^5} |\dot{y}|^5 - \frac{p_0 A g}{W} (1-\beta) y^n + \left(\frac{p_0 A g}{W} + q\right)
$$

(21)

where, $\beta = \frac{1}{\ell}$

Now,

$$
\frac{9k_d}{W} = \frac{386 \times 0.143}{48.5} = 1.14
$$

$$
\frac{p_0 A g}{W} = \frac{14.7 \times 7.07 \times 386}{48.5} = 828
$$

$$
\frac{p_0 A g}{W} + q = 1214
$$

Equation (21) then becomes

$$
\dddot{y} = \mp \alpha |\dot{y}|^{1.5} - 828 (1-\beta) y^{1.5} + 1214
$$

(22)

where, $\alpha = \frac{1.14}{\alpha^{1.5}}$

with the initial conditions $y(0) = 0,$

$$
\dot{y}(0) = 48
$$

In order to solve the design problems mentioned in §5, it is required that the nonlinear equation (22) be solved.
8. Methods for the solution of the nonlinear differential equation and their relative merits. (See Duncan^{17})

(a) Classification of methods

The first and broadest division is into purely numerical or digital methods and those which are not entirely numerical, though capable of giving numerical results. (See the classification chart which follows.)

The methods which are not purely digital can be divided into analogic and graphical. Analogic methods depend on the use of a physical or kinematic analogy, and include the use of such instruments as analog computers. Graphical methods are, strictly, analogic and this is indicated by the dotted line in the chart.

The digital methods can be divided broadly into progressive and holic or unitary. In the progressive methods, the process of solution begins at one point and passes to other points in regular sequence; in some the whole process may be repeated until the required accuracy is obtained. The characteristic of the holic methods is that the range of integration is treated as a whole throughout the process of integration.

Progressive methods may be classified as regular, when the process of integration is the same throughout the range, and irregular. The irregular methods usually start with a very accurate computation of the solution over a short initial part of the range and then proceed step-by-step. The ordinary step-by-step and iterative methods are regular.
Classification Chart for Methods

Methods of Numerical Integration

Purely Numerical or Digital

Not Purely Numerical

Graphical

Analogic

Integraphs

Differential analysers

Holic or unitary

Progressive

Series

Taylor

Frobenius

Fourier

Regular

Irregular

Adams

Milne

Levy and Baggott

Falkner

Linear Combination of Assigned Functions

Lagrange

Minimal

Galerkin

Collocation

Rayleigh

Ritz

Step-by-step

Euler

Runge

Kutta

Part analytical

Iterative

Picard

Chart No. 1
The holistic methods depend either on approximate analytical solutions (series, etc.) or representation of the solution as a linear combination of known functions which satisfies the boundary conditions for all values of the parameters. The latter are so determined that an approximate solution is obtained.

(b) Brief review of methods

Graphical Methods. — These may be of value in rough work or in preliminary explorations. The attainable accuracy is low.

Analogic Methods. — These are probably of most value for rather complicated equations or systems and when the range of integration is great. The attainable accuracy is not high.

Digital or Numerical Methods. — Some of these methods are very elegant with high attainable accuracy. However, the amount of work involved is tremendous. If the methods were set up with the actual calculations done by a digital computer, very satisfactory results could be obtained.

Yorgiadis\(^{(10)}\) and Edman\(^{(18)}\) attacked this problem through the method of graphical analysis. Their methods are not followed here because of their low accuracy. Their method is based on the assumption that the system has quadratic damping.

Schlaefke\(^{(19)}\) used very elementary numerical integration on a momentum equation instead of on a force equation.

Here the method of Bickley\(^{(20)}\) is chosen. It is one of the progressive, regular, step-by-step integration methods. It provides considerable security against arithmetical mistakes as a check is
provided at every step. It is not as accurate as some of the more sophisticated methods, but, on the other hand, not as tedious. Even then, many hours of calculation went into the following results with the aid of a desk calculator. As far as is known the application of the method of Bickley to the study of this problem has not been attempted elsewhere.

9. Numerical solution of the problem

First an arbitrary case is taken where the orifice is of \( \frac{3}{4} \) in. diameter, and the air column is of 9 in. length.

Here \( d = \frac{3}{4} \) in., \( a = 0.4416 \) sq. in.

\[ a^{1.5} = 0.293 \quad \alpha = \frac{1.14}{a^{1.5}} = 3.89 \]

\[ \beta = \frac{1}{\ell} = \frac{1}{7} = 0.143 \]

Equation (22) then becomes

\[ \begin{align*}
\dot{y}^4 & = 3.89 \dot{y}^{1.5} - 828(1 - 0.111 \dot{y}) + 1214 \\
y(0) & = 0, \quad \dot{y}(0) = 48
\end{align*} \]

Equation (23)

By MacLaurin’s theorem

\[ y(t) = y(0) + t \dot{y}(0) + \frac{t^2}{2!} \ddot{y}(0) + \frac{t^3}{3!} \dot{y}''(0) + \cdots \] (24)

Substituting the initial conditions \( y(0) = 0, \dot{y}(0) = 48 \) into equation (23) gives

\[ \ddot{y}(0) = 1214 - 3.89(48)^{1.5} - 828 = -907.66 \] (25)
Differentiating equation (23) with respect to $t$ gives

$$y'' + 5.835(y)^{0.5}y'' + 101.1(1 - 0.111y)^{-2.1}y = 0 \quad (26)$$

from which

$$y''(0) = 3.1839 \times 10^4 \quad (27)$$

Further differentiation gives

$$y^{(4)} + 5.835(y)^{0.5}y^{(4)} + 2.9715(y)^3(y)^{-2.1}y'' + 101.1y''(1 - 0.111y)^{-2.1} + 23.566(y)^2(1 - 0.111y)^{-3.1} = 0 \quad (28)$$

from which

$$y^{(4)}(0) = -1.5965 \times 10^6 \quad (29)$$

Also

$$y^{(5)} + 5.835(y)^{0.5}y^{(5)} + 8.7525(y)^2(y)^{-0.5}y'' - 1.4588(y)^{-1.5}(y)^3 + 23.566y''y''(1 - 0.111y)^{-3.1} + 101.1y''(1 - 0.111y)^{-2.1} + 8.11(y)^3(1 - 0.111y)^{-4.1} + 47.132y''(1 - 0.111y)^{-3.1} = 0 \quad (30)$$

from which

$$y^{(5)}(0) = 9.673 \times 10^7 \quad (31)$$

Substituting from equations (25), (27), (29), (31) into (24) leads to the approximate solution,

$$y(t) = 48t - 453.83t^2 + 5.3065 \times 10^3 t^3 - 6.652 \times 10^4 t^4 + 8.061 \times 10^5 t^5 \quad (32)$$

Differentiating (32) gives

$$\dot{y}(t) = 48 - 907.66t + 1.592 \times 10^4 t^2 - 2.661 \times 10^5 t^3 + 4.031 \times 10^6 t^4 \quad (33)$$
\( y, \dot{y}, \ddot{y} \) are calculated for \( t = 0, 0.01, 0.02 \) sec., using equations (32), (33) and (23). This gives the first three rows of Table I, and it is important that they should be accurate, because errors once introduced are liable to be cumulative.

When \( t = 0 \)
\[
y = 0, \quad \dot{y} = 48, \quad \ddot{y} = 907.66
\]

When \( t = 0.01 \), equation (32) gives
\[
y = 0.48 - 0.04538 + 0.00531 - 0.00067 + 0.00008 = 0.4393
\]
Equation (33) gives
\[
\dot{y} = 48 - 9.0766 + 1.592 - 0.2661 + 0.0403 = 40.290
\]
Equation (23) gives
\[
\ddot{y} = -3.89 (40.290)^{1.5} - 828 (1 - 0.111 x 0.4393)^{-1.1} + 1214 = -672.24
\]

When \( t = 0.02 \), similar calculations give
\[
y = 0.8129, \quad \dot{y} = 34.731, \quad \ddot{y} = 500.81
\]
Referring to Fig. 6 it is seen that the approximate slope at \( P \) is
\[
\dot{y}_h = \frac{\dot{y}_{h+1} - \dot{y}_{h-1}}{2h}
\]
so
\[
\dot{y}_{h+1} = 2h \dot{y}_h + \dot{y}_{h-1} \quad (34)
\]
Also by Simpson's rule
\[
y_{h+1} - y_{h-1} = \frac{1}{3} h (\dot{y}_{h+1} + 4\dot{y}_h + \dot{y}_{h-1})
\]
which gives
\[
y_{h+1} = \frac{1}{3} h (\dot{y}_{h+1} + 4\dot{y}_h + \dot{y}_{h-1}) + y_{h-1} \quad (35)
\]
\( y_{r+1} \) is then calculated by equation (23). The value of \( y \) is checked as the computation proceeds, by aid of the Simpson's rule formula

\[
y_{r+1} = \frac{1}{3} h (y_r^{'''} + 4y_r^{''} + y_r^{'''}_1) + y_{r-1}
\]

(36)

To obtain row 4 in Table I, put \( h = 0.01, r = 2 \) in (34), and use \( y_2, y_1 \) from the table. Then

\[
y_3 = 2h y_2^{''} + y_1 = 2 \times 10^{-2} (-500.81) + 40.290 = 30.274
\]

(37)

From (35)

\[
y_3 = \frac{1}{3} h (y_3^{'''} + 4y_2^{''} + y_1^{''}) + y_1
\]

\[
= \frac{10^{-2}}{3} (30.274 + 4 \times 34.731 + 40.290) + 0.4393
\]

\[
= 1.1376
\]

(38)

From (23)

\[
y_3^{'''} = -3.89(30.274)^{1.5} - 8.28(1 - 0.111 \times 1.1376)^{-1.5} + 40.290 = 394.44
\]

(39)

Checking \( y_3 \) by (36),

\[
y_3 = \frac{h}{3} (y_3^{'''} + 4y_2^{''} + y_1^{''}) + y_1
\]

\[
= - \frac{10^{-2}}{3} (394.44 + 4 \times 500.81 + 672.24) + 40.290
\]

\[
= 30.057
\]

(40)

The results of Table I are plotted graphically in Figs. 8, 9, and 10.

To find the position of static equilibrium we let \( \dot{y} = y'' = 0 \) in equation (23).
Hence

\[ 828 (1 - 0.111 y_s)^{-1.1} = 1214 \]  \hspace{1cm} (41) \]

\[ 1 - 0.111 y_s = 0.7059 \]

\[ y_s = 2.6495 \text{ in.} \]  \hspace{1cm} (42)
TABLE I

\( h = 0.01 \) in \( t \)

<table>
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<tr>
<th>( r )</th>
<th>( t )</th>
<th>( y' )</th>
<th>( y )</th>
<th>( y'' )</th>
<th>check for ( y' )</th>
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<td></td>
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TABLE I (cont'd.)

\( h = 0.01 \text{ in } t \)

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<th>( \dot{y} )</th>
<th>( y )</th>
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</table>
FIG. 8 DISPLACEMENT-TIME CURVE

STATIC EQUILIBRIUM POSITION

\[ y_e = 2.6495'' \]
FIG. 9 VELOCITY-TIME CURVE
FIG. 10 ACCELERATION—TIME CURVE
10. Implications of the result

(a) Fig. 8 shows that the displacement of the system oscillates with decreasing amplitude about the static equilibrium position as should be the case.

(b) Fig. 10 shows that the maximum deceleration occurs at the instant of impact. This finding agrees with the result of Yorgiadis\(^{(10)}\).

This last finding is of great importance for the design of the size of the orifice for a specified dynamic load. We can now solve problem (b) in §5 explicitly. By D'Alembert's principle, the transmitted force to the bottom of the weight is

\[ F_c = W \left(1 - \frac{\ddot{y}}{g}\right) \]

which is to be 300 lb. So

\[ \ddot{y} = g \left(1 - \frac{F_c}{W}\right) = 386 \left(1 - \frac{300}{48.5}\right) \]

\[ = -2,000 \text{ in. per sec.}^2 \quad (43) \]

Hence at the time of impact, \(y = 0\), \(\dot{y} = 48\), \(\ddot{y} = -2000\)

Substituting into equation (22) gives

\[ K_2 \left(\frac{\dot{y}}{a}\right)^5 = W \left(1 - \frac{\ddot{y}}{g}\right) = F_c \]

or

\[ \frac{0.143(48)^{1.5}}{a^{1.5}} = 300 \]

\[ a^{1.5} = 0.158 \]

\[ a = 0.292 \text{ sq. in.} \quad (44) \]
Hence the diameter of the orifice is
\[ d = 0.610 \text{ in.} \quad (45) \]

It is previously mentioned that \( K_d \) and \( s \) are determined experimentally to ascertain more precisely the orifice damping. However, before the shock-absorber is built and experiments performed on it, these constants cannot be predetermined. In which case the designer will use the idealized equation (14) for the approximate design. At the time of impact,

\[ F = \frac{\rho A^2 (\dot{y})^2}{2} = W(1 - \frac{y''}{g}) = F_e \quad (46) \]

Here it is assumed that the specific gravity of the oil is 0.834 as that used by Hurtz, for which \( f/2 = 3.9 \times 10^{-5} \).

From (46)

\[ \frac{3.9 \times 10^{-5} \cdot (7.07)^3 \cdot (48)^2}{a^2} = 300 \]

\[ a^2 = 0.106 \]
\[ a = 0.325 \text{ sq. in.} \quad (47) \]
\[ d = 0.644 \text{ in.} \quad (48) \]

11. **Displacement-time design of shock-absorber and the analog computer**

The design of a shock-absorber to meet the specified condition of problem (c) in \( \S 5 \) cannot be performed directly. The only obvious way is to determine the dynamical characteristics of a large number of shock-absorbers having various sizes of the orifice and
various lengths of the air column. The only practical approach, other than building a large number of shock-absorbers and testing them, is to set up equation (22) in an analog computer, to vary the parameters $\alpha$ and $\beta$, and to obtain the dynamical characteristics from a recording oscillograph.

A brief introduction to analog computers will be given here (21).

The analog devices operate with physical variables such as shaft rotations or electrical voltages. There are two types of analogy as shown in the chart below.

Classification of Analog Computers

![Diagram of Classification of Analog Computers]

The chart shows the classification of analog computers into indirect, direct, and dual types. The indirect category includes mechanical and fluid models, while the direct category includes electrical models with equivalent circuits and network analyzers. Specific examples include mechanical scale models, fluid model dams, and electrical operational amplifiers and type differential analyzers.

Chart No. 2
The direct analogy is characterized by those cases where problem variables and problem parameters are represented directly by variables and parameters on the machine. An example is the direct analogy which exists between the energy storage in a mechanical spring and the energy storage in an electrical capacitor. By application of the principle of duality it is also possible to have a computer which operates as the dual of the problem. The mechanical direct analog computers are most generally scale models such as wind-tunnel models. The electrical direct analogs include such instruments as the network analyzers and equivalent circuits. In the fluid analog are found such devices as model dams and stream beds.

The indirect analog computers are of a type which can carry out or assist in the solution of algebraic or differential equations. The electrical (or electronic) indirect analog computer is probably the most common of the indirect type. This type normally employs high gain amplifiers which, when applied in appropriate feedback loops, will perform many mathematical operations. The precision and general availability in pure form of such components as resistors and capacitors is exploited to perform the mathematical operations.

The major components in most analog computers are the following: (a) amplifiers, (b) integrators, (c) frequency-dependent functions, (d) gain potentiometers, (e) limiters and comparators, (f) servo units, and (g) recorders. The theory and operation of these units are discussed in the following sections, and the symbols
representing them are given (21).

(a) **Amplifier units.** The basic elements of the amplifier (Fig. 11) consists of input and feedback resistors and a high gain, phase inverting, d-c amplifier. The feedback resistor $R_F$ is connected between the plate of the output stage and the grid of the input stage of the d-c amplifier. The circuit is such that the grid draws practically no current. Therefore $R_1$ and $R_F$ act as a voltage divider between $e_i$ and $e_o$. Since the high-gain amplifier is of the phase-inverting type, the polarity of $e_o$ is always opposite to that of $e_i$. The relationship between $e_o$ and $e_i$ is

$$e_o = -\frac{R_F}{R_1} e_i$$  \hspace{1cm} (49)

In practice, several input resistors are connected to the input grid (Fig. 12), and the output may be represented by the relation

$$e_o = -\frac{R_F}{R_1} e_i - \frac{R_F}{R_2} e_i - \frac{R_F}{R_3} e_i - \frac{R_F}{R_4} e_i$$  \hspace{1cm} (50)

These units are usually represented by the symbol shown in Fig. 13. Here $K_1$ is equal to the ratio $R_F/R_1$.

(b) **Integrator units.** The operation of the integrator may be seen by considering Fig. 14. Assume a voltage $e_i$ is applied. Since practically no current flows through the grid of the high-gain amplifier, all the current flows through $R_1$ and $C$. Also, since $e_g$ tends to become zero. Hence

$$e_i = R_1 i, \quad e_o = \frac{1}{C} \int i \, dt$$

$$e_o = -\frac{1}{R_1 C} \int e_i \, dt$$  \hspace{1cm} (51)
The symbol for a computer summing integrator is shown in Fig. 15. Here $K_1$ is a gain factor and is equal to $1/R_1 C$. The equation describing the output of the summing integrator is

$$e_o = -\int e_1 dt - K_2 \int e_2 dt - K_3 \int e_3 dt - K_4 \int e_4 dt + E_0$$

(52)

where $E_0$ is the initial value of $e_o$. Provision is included in most computers for charging the capacitor $C$ at the beginning of an operation to a desired magnitude so that $e_o$ may have an initial value.

(c) Frequency-dependent functions. In a manner analogous to the summing amplifier and summing integrator units, frequency-dependent functions may be obtained. Fig. 16 shows the basic elements. The equation describing the output voltage is

$$e_o = -\frac{Z_F}{Z_I} e_I$$

(53)

If $Z_I$ and $Z_F$ are both parallel resistance-capacitance functions, it will be

$$e_o = \frac{-R_2 (1 + R_1 C_f s)}{R_1 (1 + R_2 C_f s)} e_I$$

(54)

where,

$$s = j\omega$$

The symbol used to represent frequency-dependent functions is shown in Fig. 17.

(d) Gain potentiometers. In order to be able to set gains and coefficients at any value, potentiometers are used. Fig. 18 shows the wiring schematic of a potentiometer, and Fig. 19 shows the representing symbol. The equation for the output voltage is
FIG. 11 COMPUTER AMPLIFIER

FIG. 12 SUMMING AMPLIFIER

FIG. 13 SUMMING AMPLIFIER SYMBOL

FIG. 14 COMPUTER SUMMING INTEGRATOR

FIG. 15 COMPUTER SUMMING INTEGRATOR SYMBOL

FIG. 16 FREQUENCY-DEPENDENT FUNCTION

FIG. 17 FREQUENCY-DEPENDENT FUNCTION SYMBOL
where $K$ is approximately $R_1/R_2$.

(e) **Limiters and comparators.** Units called "limiters" which limit or compare magnitudes of voltage signals are used to simulate nonlinear effects such as signal saturations. The schematic diagram of a typical limiter is shown in Fig. 20. Appropriate values of bias voltages determine the limits. The symbol for a limiter is shown in Fig. 21.

(f) **Servo units.** An analog computer servo unit is a positioning device in which a shaft is rotated through an angle proportional to an input signal. These units are useful in solving equations which involve trigonometric functions, linear differential equations with variable coefficients, and the multiplication and division of two variables. Fig. 22 shows the basic elements of a servo unit. A voltage $e_1$ is applied to a summing network and amplifier. This in turn causes the operation of the motor, changing the position of the potentiometer wiper arm until the wiper arm voltage cancels the applied voltage. If $e_1$ is measured in volts, and if $+100$ and $-100$ volts are applied at either end of the potentiometer, an equilibrium point occurs when the wiper arm has turned through $-e_1/100$ of the total distance from the center to one end. Fig. 23 shows the symbol for a servo unit. The solid line indicates a voltage and the broken line indicates a shaft position.

(g) **Recorders.** The symbol of a cathode ray oscillograph pen
The oscillograph recorder is shown in Fig. 24.

With the symbols explained, our design problem (equation 22) can be represented by the analogous computer circuit in Fig. 25. The damping function, $f \alpha |\dot{y}|^{1.5}$ is generated by servo unit number 1, and the function, $(1 - \beta |y|)^{-1}$ is generated by servo unit number 2. The value of $n = 1$ instead of 1.1 is used because of the difficulty in generating such a function in the servo unit. Also the value of the initial velocity, $\dot{y}(0) = 40$ is used. Hence the design equation becomes

$$\begin{align*}
\ddot{y} &= f \alpha |\dot{y}|^{1.5} - 828(1 - \beta |y|) + 1214 \\
y(0) &= 0, \quad \dot{y}(0) = 40
\end{align*} \tag{56}$$

Now if various sizes of orifice (parameter $a$) and various lengths of air column (parameter $\beta$) are substituted into equation (56), and the dynamic responses of the system are obtained from the analog computer solutions, the designer can then choose with confidence the hydraulic shock-absorber to meet the specifications. The values of $y$, $-\dot{y}$, and $\ddot{y}$ vs. $t$ are recorded by the oscillograph recorder.

The setting-up of the computer circuit and the operations for obtaining the numerical solutions of this problem are due to Dr. C. E. Warren, professor of electrical engineering and Mr. Robert E. Fenton, a graduate student in electrical engineering, both of The Ohio State University. The total time for setting up the equipment and solving 30 equations took two days as compared to three weeks of calculation in partially solving one equation by numerical integration.
FIG. 25 ANALOG COMPUTER CIRCUIT
12. Results of computer solution.

First, the above numerical solution is checked against the computer solution, bearing in mind that the impact velocity and the polytropic constant are slightly different. The computer-solution results are shown in Table II, and also graphically in Figs. 26, 27, and 28. The shapes of the curves and the values of the quantities are very much similar to those in Figs. 8, 9, and 10.

Secondly, the computer solutions are obtained for three sizes of orifices \( d = 3/4 \text{ in.}, 1/2 \text{ in.}, 3/8 \text{ in.} \), all of them with an air column of 8 in. The results are shown in Tables III, IV, and V; and also graphically in Figs. 29, 30, and 31. Observation of Fig. 29 will show that the displacement-time curve for \( d = 1/2 \text{ in.} \) will be very nearly a critically-damped system if the vibrations were linear.

Thirdly, computer solutions are obtained for \( d = 3/4, 5/8, 1/2, 7/16, 3/8 \text{ in.}, \text{ and } \ell = 10, 9, 8.5, 8, 7.5, 7 \text{ in.}; \) the combinations give us 30 shock-absorbers. We record in Table VI for each shock-absorber the type of oscillation and the "settling-down" time, \( t_s \), which is time taken for the weight to assume no appreciable displacement from the static equilibrium position.

Now to return to the problem (b), that of designing a shock-absorber such that the weight will have no appreciable displacement after 0.4 sec. Observation from Table VI will show an orifice of 7/16 in. diameter and an air column of 8 1/2 in. giving \( t_s = 0.38 \text{ sec.} \) and such dimensions might be an appropriate design.

If the designer specifies a "tolerance" of say \( \pm 1/4 \text{ in.} \) of
### TABLE II

**COMPUTER SOLUTION**

\[ l = 9 \text{ in.}, \beta = 0.111; d = 3/4 \text{ in.}, \alpha = 3.89; \dot{y}(0) = 40 \]

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FIG. 26 DISPLACEMENT—TIME CURVE

\[ v = 40 \text{ in/sec} \]
\[ l = 9 \text{ in.} \]
\[ d = 3/4" \]
Fig. 27 VELOCITY-TIME CURVE

V = 40 IN/SEC
l = 9 IN.
d = 3/4 IN.
FIG. 28 ACCELERATION – TIME CURVE

\[ v = 40 \text{ in/sec} \]

\[ l = 9 \text{ in.} \]

\[ d = \frac{3}{4} \text{ in.} \]
Table III

Computer Solution

\( \ell = 8 \text{ in.}, \beta = 0.125; d = 3/4 \text{ in.}, a = 3.89 \)

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<th>( y ) (in.)</th>
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<th>( \ddot{y} ) (in./sec.²)</th>
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<th>( y ) (in.)</th>
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**COMPUTER SOLUTION**

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TABLE V

**COMPUTER SOLUTION**

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<th>y (in.)</th>
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TABLE V (cont'd.)

COMPUTER SOLUTION

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<th>( y ) (in.)</th>
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FIG. 29  DISPLACEMENT—TIME CURVES
FIG. 30 VELOCITY - TIME CURVES
FIG. 31 ACCELERATION - TIME CURVES
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* Uncertain because of short run
### TABLE VI (cont'd.)

"SETTLING-DOWN" TIME, $t_s$

#### $l = 8''$

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#### $l = 7.5''$

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</table>

#### $l = 7''$

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<tr>
<td>3/8</td>
<td>0.65</td>
<td>Aperiodic</td>
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</table>
+ 5% of \( y_s \), a "band of tolerance" can be drawn across the displacement-time curve. In which case the "settling-down" time \( t_s \) can be determined with even greater precision.

13. Conclusion

This discourse on shock-absorber emphasizes the synthesis or design of its components rather than the analysis of the performance of the shock-absorbing system, even though the latter is a necessary step to reach the former. Fairly complete information on methods of design has been presented, some parts of which are the results of other investigators, and the other parts are original. They are grouped together in this discourse as an aid to designers of shock-absorbers.

Nightingale's method will give an explicit determination of the size of the orifice if past performance records of similar designs are available. (See Appendix II.)

The numerical integration of the nonlinear differential equation presents one method of analyzing the performance of the shock-absorber system.

For a one-degree system with no "tire effect", the maximum deceleration occurs at the instant of impact. The size of the orifice for any permissible transmitted load can then be easily determined as shown in § 9 and 10.

To design the shock-absorber such that the system will "settle-down" to a specified range of displacement from the static equilibrium position at a specified time, the only practical method is to
make use of an analog computer. The method is demonstrated in § 11 and 12.

The method of Durand and Rosenberg of designing a minimum transmitted-force shock-absorber is described in Appendix III, and an example is worked out in detail.
APPENDIX I

Equivalent viscous damping for \((1 \leq s \leq 2)\)

If the same form of displacement-time function for the case of nonlinear damping as for linear damping is assumed, then the function will be

\[
\chi = X_0 \sin(\omega t - \alpha)
\]

where, \(X_0\) = maximum amplitude \(\approx\) resonance amplitude

\(\omega\) = forcing frequency

\(\alpha\) = phase angle

The energy absorbed by the damping force for one cycle of oscillation is

\[
E = \int_0^\tau \left| B \chi^{(s+1)} \right| dt = 2 B X_0 \omega \int_0^{\pi/2} \left| \cos^{(s+1)}(\omega t - \alpha) \right| dt
\]

where, \(B = \) actual damping coefficient \(= k_d/\alpha^s\) in equation (11)

and \(\tau =\) period of oscillation

Let \((\omega t - \alpha) = y\), then

\[
E = 2 B X_0 \omega \int_{-\alpha}^{\pi/2} \left| \cos^{(s+1)} y \right| dy = 4 B X_0 \omega \int_0^{\pi/2} \cos^{(s+1)} y dy
\]  

(57)

Now, the integral in terms of Gamma function is

\[
\int_0^{\pi/2} \cos^{(s+1)} y dy = \frac{\sqrt{\pi} \Gamma\left(\frac{s+1}{2}\right)}{2 \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)} \quad (58)
\]
Equating this energy with that of an equivalent viscous system gives
\[ \frac{\pi}{2} \omega X_0^2 C_{eq} = \frac{2/\pi}{\Gamma(1)} \beta (X_0 \omega)^{(s-1)} \]

where, \( C_{eq} \) = the equivalent viscous damping coefficient

Hence
\[ C_{eq} = \frac{2/\pi}{\Gamma(1)} \beta (X_0 \omega)^{(s-1)} \]  \hspace{1cm} (59)

The correctness of this expression can be verified from the special cases of \( s = 1 \) and \( s = 2 \).

For \( s = 1 \), equation (59) gives
\[ C_{eq} = \frac{2}{\sqrt{\pi}} \beta \frac{B}{X_0 \omega} = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{\Gamma(1/2)}{\Gamma(1)} = B \]  \hspace{1cm} (60)

For \( s = 2 \),
\[ C_{eq} = \frac{2}{\sqrt{\pi}} \beta \frac{B X_0 \omega}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{\Gamma(1/2)}{\Gamma(1)} = \frac{3}{\sqrt{\pi}} B \]  \hspace{1cm} (61)

Equation (59) can be written as
\[ C_{eq} = K(s) B (X_0 \omega)^{(s-1)} \]  \hspace{1cm} (62)

where,
\[ K(s) = \frac{2/\pi}{\Gamma(1)} \frac{\Gamma(3/2 + 1)}{\Gamma(1)} = \frac{4}{\sqrt{\pi}} \frac{\Gamma(3/2 + 1)}{\Gamma(1)} \]  \hspace{1cm} (63)

The results of \( K(s) \) and \( C_{eq}/B \) for values of \( s = 1, 1.1, 1.2, \ldots \) 2.0 are given in Table VII.
<table>
<thead>
<tr>
<th>s</th>
<th>K(s)</th>
<th>$C_2/\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>1.1</td>
<td>0.981</td>
<td>0.981 ($X_0 \omega^{0.1}$)</td>
</tr>
<tr>
<td>1.2</td>
<td>0.963</td>
<td>0.963 ($X_0 \omega^{0.2}$)</td>
</tr>
<tr>
<td>1.3</td>
<td>0.947</td>
<td>0.947 ($X_0 \omega^{0.3}$)</td>
</tr>
<tr>
<td>1.4</td>
<td>0.931</td>
<td>0.931 ($X_0 \omega^{0.4}$)</td>
</tr>
<tr>
<td>1.5</td>
<td>0.915</td>
<td>0.915 ($X_0 \omega^{0.5}$)</td>
</tr>
<tr>
<td>1.6</td>
<td>0.901</td>
<td>0.901 ($X_0 \omega^{0.6}$)</td>
</tr>
<tr>
<td>1.7</td>
<td>0.887</td>
<td>0.887 ($X_0 \omega^{0.7}$)</td>
</tr>
<tr>
<td>1.8</td>
<td>0.873</td>
<td>0.873 ($X_0 \omega^{0.8}$)</td>
</tr>
<tr>
<td>1.9</td>
<td>0.861</td>
<td>0.861 ($X_0 \omega^{0.9}$)</td>
</tr>
<tr>
<td>2.0</td>
<td>0.849</td>
<td>$X_0 \omega$</td>
</tr>
</tbody>
</table>
Explicit determination of the orifice size by the use of drop-test records.

Nightingale's shock-absorbing system is that of a landing airplane with the tire effect --- a linear spring (6)(14)(16) --- taken into account (Fig. 2). The weight of the oleo strut, \( w \), is small compared with that of the airplane, \( W \). The representation of the tires by a linear spring of stiffness \( k \) is justifiable, since most tire load-deflection curves are nearly straight lines. The equations of motion for the dynamical system of Fig. 2 are

\[
\frac{W}{g} (\ddot{x} + \dot{x}_t) + D + f = 0
\]

\[ (64) \]

\[
\frac{w}{g} \ddot{x}_t - D - f + k \dot{x}_t = 0
\]

\[ (65) \]

where, \( x_t \) is the compression of the tire, and \( x + x_t \) is the displacement of the main weight; both measured from the equilibrium positions. Here the external force caused by gravity (i.e., the weight of the body) is balanced by the initial displacement of the air spring. Hence the equations are homogeneous instead of non-homogeneous.

Since \( w \) is negligible compared with \( W \), and

\[
|D + f| = \left| \frac{W}{g} (\ddot{x} + \dot{x}_t) \right| \Rightarrow \left| \frac{w}{g} \ddot{x}_t \right|
\]

-67-
then the dynamical reaction at the ground (from 65) is

\[ R = k x_t \equiv D + f \]  \hspace{1cm} (66)

Substituting into (16) gives

\[ - \frac{g R}{W} = \ddot{x} + x_t = \dddot{x} + \frac{1}{k} \frac{d^2 R}{d t^2} \]  \hspace{1cm} (67)

But here

\[ \frac{d^2 R}{d t^2} = \dddot{x} \frac{dR}{dx} + \ddot{x} \frac{d^2 R}{d x^2} \]

Therefore

\[ - \frac{g R}{W} = \dddot{x} \left( 1 + \frac{1}{k} \frac{dR}{dx} \right) + \frac{\ddot{x}^2}{k} \frac{d^2 R}{d x^2} \]  \hspace{1cm} (68)

Now the fluid damping force is given by (6) to be

\[ D = \frac{\rho A^3 v^2}{2 a^2} = (\frac{v}{\psi})^2 \]  \hspace{1cm} (69)

where,

\[ \psi = \frac{a}{A^{3/2}} \sqrt{\frac{2}{\rho}} \]

is called the orifice parameter.

Equations (69) and (66) give

\[ \dot{x}^2 = v^2 = \psi^2 D = \psi^2 (R - f) \]  \hspace{1cm} (70)

It may be written

\[ \dddot{x} = \frac{1}{2} \frac{d \dot{x}^2}{d x} = \frac{\psi^2}{2} \frac{d}{d x} (R - f) \]  \hspace{1cm} (71)
Substituting (70), (71) into (68) yields

\[-\frac{gR}{W} = \frac{\Psi^2}{2} \frac{d}{d\alpha} (R-f) + \frac{\Psi^2}{R} \frac{dR}{d\alpha} \frac{d}{d\alpha} (R-f) + \frac{\Psi^2}{R} (R-f) \frac{d^2R}{d\alpha^2}\]

or

\[-\frac{gR}{W} = \frac{\Psi^2}{2} \frac{d}{d\alpha} (R-f) + \frac{\Psi^2}{R} \left[ (R-f) \frac{dR}{d\alpha} \right] - \frac{\Psi^2}{2} \frac{dR}{d\alpha} \frac{d}{d\alpha} (R-f) \]

Integrating over the total stroke used, \(x_1\) gives

\[-\frac{gR}{W} \int_0^{x_1} R d\alpha = \frac{\Psi^2}{2} \int_0^{x_1} (R-f) d\alpha + \frac{\Psi^2}{R} \int_0^{x_1} \left[ (R-f) \frac{dR}{d\alpha} \right] d\alpha - \frac{\Psi^2}{2} \int_0^{x_1} \frac{dR}{d\alpha} \frac{d}{d\alpha} (R-f) d\alpha\]

At the beginning and at the end of the stroke, both the velocity, \(v\) and the damping force, \(D\) are zero, or \(R = f\).

Hence

\[-\frac{gR}{W} \int_0^{x_1} R d\alpha = \frac{\Psi^2}{2} \int_0^{x_1} \frac{dR}{d\alpha} \frac{d}{d\alpha} (R-f) d\alpha\]

The expression for the orifice parameter is therefore

\[\Psi^2 = \frac{2}{\kappa g} \frac{\int_0^{x_1} R d\alpha}{\int_0^{x_1} \frac{dR}{d\alpha} \frac{d}{d\alpha} (R-f) d\alpha}\]  

(73)

Here \(x_1\) is the designed stroke for the maximum allowable dynamical reaction, \(R_m\). Now if \(\ell\) is the available stroke, \(x_1\) can be specified as a fraction of \(\ell\). The fact that \(\ell > x_1\), takes care of any overload. Hadekel\(^{(12)}\) mentioned that the British (Messier) practice is to have \(x_1/\ell = 0.75\) in designing airplane struts.

The \(R\) vs. \(x\) curve can be determined by experiments on past designs. The \(f\) vs. \(x\) curve can be calculated from the polytropic gas
law. Hence (73) can be used to determine the orifice parameter.

Nightingale's experiments show that $\psi^2$ can be expressed in the following simple form:

$$
\psi^2 = \frac{2kg}{W} \left( \frac{C l^2}{R_m} \right) = \frac{2C k g}{\lambda} \left( \frac{l}{W} \right)^2
$$

where, $C = 0.27$, approximately

$$
R_m = \lambda W = \text{maximum dynamical reaction allowable to be transmitted to the structure of the body.}
$$

$\lambda = \text{load factor.}$

$\lambda = 3$ is recommended by Burger (8) for airplane struts.

From (69) and (74) we have

$$
\psi = \frac{a}{A^{3/2}} \sqrt{\frac{2}{f}} = 14.5 \frac{e}{W} \sqrt{\frac{R}{\lambda}}
$$

Hence the size of the orifice, $a$, can be designed directly from (75).
Design of a minimum transmitted-force shock-absorber having a metering pin.

In the designing of the shock-absorber for a permissible transmitted force, as described in § 9 and 10, the maximum retardation, which occurs at the instant of impact, is used. The orifice is of uniform diameter throughout the stroke. On the other hand, if the size of the orifice can be varied by a metering pin, and the retardation is maintained more or less uniform throughout the stroke, a much smaller retardation and hence a much smaller transmitted force will result.

The following method of design is due to Durand and Rosenberg\(^\text{(13)}\). The design criterion is that the selected time history of the motion will be made to satisfy the requirement that the integral of the square of the acceleration of the piston motion with respect to the stroke be a minimum. The integrand is the second power of the acceleration rather than the first because the square of the acceleration is non-negative, but the acceleration is not.

The problem is to find a \( \ddot{y} = \ddot{y}(y) \) such that

\[
I = \int_0^y (\dot{y}')^2 \, dy
\]

becomes a minimum.
Let

\[ \nu = \frac{dy}{dt}, \quad \nu' = \frac{dv}{dy} \]

An alternative statement of the condition is that the variation of \( I \) is zero.

\[ \delta I = \delta \int_{y_0}^{y_1} (\nu \nu')^2 dy = 0 \]  

(77)

We apply Euler–Lagrange differential equation of the calculus of variations which states that for

\[ I = \int_{y_0}^{y_1} f(\nu, \nu, \nu') dy \]

to be an extremum, the necessary and sufficient condition is that

\[ \frac{\partial f}{\partial \nu} - \frac{\partial}{\partial \nu} \left( \frac{\partial f}{\partial \nu'} \right) = 0 \]  

(78)

Here

\[
\begin{align*}
  f &= (\nu \nu')^2 \\
  \frac{\partial f}{\partial \nu} &= 2 \nu \nu' \nu' \\
  \frac{\partial f}{\partial \nu'} &= 2 \nu^2 \nu'
\end{align*}
\]

(79)

Substitution of (79) in (78) gives

\[ \frac{d^2\nu}{dy^2} + \frac{1}{\nu} \left( \frac{dv}{dy} \right)^2 = 0 \]  

(80)

or

\[ \nu' \frac{dv'}{d\nu} + \frac{1}{\nu} (v')^2 = 0 \]
Therefore

\[
\frac{d v'}{d v} + \frac{v'}{v} = 0
\]  

(81)

The solution of (81) is

\[
\frac{d v}{d y} = v' = \frac{c_i}{v}
\]  

(82)

Hence

\[
v = \pm \sqrt{c_i y + c_2}
\]  

(83)

Differentiation of (83) gives

\[
v' = \pm \frac{c_i}{2 \sqrt{c_i y + c_2}}
\]  

(84)

The constant of integration are evaluated as follows:

At the instant of impact, \( t = y = 0 \); the boundary conditions as functions of \( y \) are

\[
\begin{align*}
v(0) &= v_0 \\
v(y_1) &= 0
\end{align*}
\]  

(85)

Substituting these conditions in (83) and (84) gives

\[
v(y) = v_0 \sqrt{1 - y/y_1}
\]  

(86)

where the positive sign of (83) is chosen thus defining a downward velocity as positive. Equation (84) becomes

\[
\frac{d v}{d y} = \frac{-v_0/y_1}{2 \sqrt{1 - y/y_1}}
\]  

(87)
From (86) and (87), the acceleration is found to be

\[ \ddot{y} = v \frac{dv}{dy} = -\frac{v_0^2}{2y_1} = \text{constant} \]  

(88)

The result requiring the acceleration to be constant is rather common in shock-absorber design. Integrating (88) gives

\[ \dot{y}(t) = v = v_0 - \frac{v_0^2}{2y_1} t \]  

(89)

and

\[ y(t) = v_0 t - \frac{v_0^2}{4y_1} t^2 \]  

(90)

Note here that this displacement-time function is the particular integral to which the differential equation, (13), is to be fitted, rather than a function found as the result of solving the differential equation.

Equation (13) can be written as

\[ \frac{W}{f} \dot{y} + Ka \left[ \frac{\dot{y}(y)}{a(y)} \right]^5 + \rho A \left[ (1 - \rho y)^{-n} - 1 \right] = W \]  

(91)

Hence the net orifice area is given by

\[ a(y) = \sqrt{\frac{Ka \dot{y}^5(y)}{\frac{W}{f} \left[ (1 - \frac{y}{y_1}) - \rho A \left[ (1 - \rho y)^{-n} - 1 \right] \right]}} \]  

(92)
In equation (92), \( \ddot{y}(y) \) is obtained from equation (86) and \( \dot{y} \) from (88). The shape of the metering pin, \( r(y) \), can be calculated from

\[
a(y) = \pi \left[ r_o^2 - r^2(y) \right] \tag{93}
\]

where, \( r_o \) = the radius of the orifice hole 
and \( r(y) \) = the metering pin radius at \( y \)

As a specific example we again take the experimental shock-absorber.

The design parameters are chosen to be:

\[
\begin{align*}
A &= 7.07 \text{ sq. in.} \\
p_a &= 14.7 \text{ psi} \\
K_d &= 0.143 \\
r_o &= 0.500 \text{ in.} \\
W &= 48.5 \text{ lb} \\
n &= 1.0 \\
\dot{y}(0) &= 48 \text{ in./sec.} \\
s &= 1.5 \\
\ddot{y} &= -27 \text{ ft./sec.}^2 \\
\beta &= 1/\xi = 1/8.5 = 0.118 \\
\beta &= -324 \text{ in./sec.}^2 \\
\end{align*}
\]

To find the stroke of the piston, we use equation (88),

\[
\dot{y} = -\frac{v_o^2}{2\ddot{y}} = \frac{-(48)^2}{2(-324)} = 3.56 \text{ in.} \tag{94}
\]

Substitution of the numerical values renders (92) to become

\[
a(y) = \sqrt[15]{\frac{0.143 \ \ddot{y}^{15}}{89.4 - 104[(1-0.118y)^{-1} - 1]}} \tag{95}
\]

and (86) to become

\[
\dot{y} = 48 \sqrt{1 - 0.281y} \tag{96}
\]
Upon the solution of equations (95), (96) and (93) for various values of \( y \), the shape of the metering pin will be determined. The results are shown in Table VIII and Fig. 32. The position of the metering pin in relation to the rest of the shock-absorber is shown in dotted lines in Fig. 7.

**TABLE VIII**

<table>
<thead>
<tr>
<th>( y ) (in.)</th>
<th>( \dot{y} ) (in./sec.)</th>
<th>( a ) (sq.in.)</th>
<th>( r ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>48.0</td>
<td>0.656</td>
<td>0.204</td>
</tr>
<tr>
<td>1.0</td>
<td>40.0</td>
<td>0.626</td>
<td>0.226</td>
</tr>
<tr>
<td>2.0</td>
<td>31.8</td>
<td>0.584</td>
<td>0.254</td>
</tr>
<tr>
<td>3.0</td>
<td>19.0</td>
<td>0.505</td>
<td>0.299</td>
</tr>
<tr>
<td>3.2</td>
<td>15.2</td>
<td>0.475</td>
<td>0.314</td>
</tr>
<tr>
<td>3.4</td>
<td>9.2</td>
<td>0.347</td>
<td>0.374</td>
</tr>
<tr>
<td>3.56</td>
<td>0</td>
<td>0</td>
<td>0.500</td>
</tr>
</tbody>
</table>

The cross-section of the metering pin near the end of the stroke increases rather rapidly, and at the end of the stroke it is of the same size as that of the orifice. To prevent injury to the orifice due to higher impact velocity or heavier weight than those for which the system is designed, the shape of the pin near the end of the stroke might be modified. The acceleration will then vary from the constant value for a small fraction of the stroke.
FIG. 32  PIN RADIUS AS A FUNCTION
OF DISPLACEMENT
BIBLIOGRAPHY


AUTOBIOGRAPHY

I, Ching-u Ip, was born in Hong Kong (B.C.G.) January 26, 1920. I received my secondary school education at St. Stephen's in Hong Kong. My undergraduate training was obtained at the University of Hong Kong, from which I received the degree Bachelor of Science in Mechanical Engineering in 1941. From California Institute of Technology, I received the degree Master of Science in Mechanical Engineering in 1942. After five years of working in industry in an engineer's capacity, I joined the Department of Mechanical Engineering of Michigan State University in 1947, and am now an Assistant Professor. During the academic year of 1951-52, I went to The Ohio State University on a leave of absence from Michigan State University to pursue advanced studies in Mechanical Engineering. I continued to work on the requirements for the degree Doctor of Philosophy under the provision of off-campus research at Michigan State University.