ON THE SYMMETRIC STRUCTURE OF UNCONDITIONED
POINT SETS AND REAL FUNCTIONS

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ON THE SYMMETRIC STRUCTURE OF UNCONDITIONED
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INTRODUCTION

The general aim of this dissertation is that of finding new geometric properties of any point set, subject to no other condition than that it is a point set, and likewise for the unconditioned real function. Most of the results obtained give information on the local symmetric structure of unconditioned point sets and real functions. The dissertation consists of new results and systematic derivations, extensions, and generalizations of known results. Nearly all of our results concern planar point sets and functions \( f(P) \) of a planar point \( P \), i.e., functions of two real variables. Some of the results are parallels of the corresponding results in linear space, but some are of considerable more complexity.

A rigorous development of the real number system by Weierstrass, Cantor, Dedekind, and others, and a development of the general theory of sets by Cantor and others, both in the last quarter of the nineteenth century, paved the way for fruitful investigations into the symmetric
structure of (euclidean) point sets. During the same period the meaning of function was clarified and the real significance of Dirichlet's definition of a general function began to be grasped: A function is a mathematical system consisting of a set A and a set B and a correspondence which mates with every element of A one or more elements of B.

It seems that one of the earliest results on the symmetric structure of a real function \( f(x) \) of one variable is the following one given by W. H. Young in 1908 (see [35] and [34]): If \( f(x) \) is an arbitrary real function, then at every point \( x \) not belonging to a countable set, the set of limit values of \( f(x) \) from the right is identical to the set of limit values of \( f(x) \) from the left. The first sentence of a paper by G. C. Young [33] published in 1916 reads as follows: "Theorems relating to perfectly general functions are rare." A similar statement concerning unconditioned point sets would probably have been valid at that time. But it seems clear that this remark of G. C. Young no longer applies today, for during the last forty years a variety of results on arbitrary sets and functions have been added to the mathematical literature. Among these are results of W. H.

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1Numbers in brackets refer to the bibliography at the end of the paper.
We obtain in Chapter I some results on the decomposition of an unconditioned point set into two disjoint components, one of the components being of a symmetric nature in the sense that it possesses the same property at all of its points. The results are obtained by the process of generalization, and a rather detailed exposition of the steps is given. The final results yield an easy way of securing a great many theorems which comprise and multiply the literature results in question.

In Chapter II new characterizations of several set properties are given in terms of (conditioned) region properties; a type of antisymmetry of a linear interval property is defined and discussed; and a number of properties of an unconditioned planar set are derived.

Chapter III concerns the (symmetric) approach of arbitrary sets to the points of a straight line. A number of results on the contingent (as defined by G. Bouligand [8]) of an arbitrary set at the points of a straight line are obtained and generalized. Some of these generalizations are given in terms of pairs of sector properties.
Although the observations and results of Chapter IV are fairly self-evident separately, they belong to a chain of implications which constitute a (considerably) simplified derivation of certain fundamental theorems on the contingent of a planar set. These theorems, which generalize the famous result of Denjoy [15] on Dini derivatives of a function, are not new for the most part, papers concerning them having been published by W. H. Young, A. Kolmogorov and J. Verčenko, U. S. Haslam-Jones, Saks, and F. Roger. Several new generalizations of these results are given; and some of the results in Chapter III on sector properties are extended to the plane and applied to sets.

We give in Chapter V two general procedures for carrying theorems on point sets over to obtain related theorems on functions. Using results on sets from the earlier chapters and applying these two procedures, we obtain a number of results on the symmetric structure of an unconditioned, one- or many-valued, real function \( f(P) \) defined in the plane.
CHAPTER I

CERTAIN DECOMPOSITIONS OF POINT SETS

§1. A Direct Generalization

1.1. The present chapter gives a detailed illustration of the process of methodical generalization of theorems, i.e., the derivation of a general theorem from a particular one by means of steps which may be regarded as belonging to specifiable method. We obtain a common source, or genesis - or rather, several modes of genesis - of diverse analogues of a particular decomposition theorem (Theorem T). By a change of form, our original problem on decomposing point sets becomes one of securing a genesis and characterization of all unconditionally additive set properties. This characterization is the main contribution of the present chapter.

1.2. In 1885 Cantor [12, §2] extended to arbitrary sets the following classical decomposition theorem of Cantor-Bendixson (see [10, p. 575], [11, p. 470], and [1]): A closed set is the disjoint sum of a perfect set and a countable set. The following simplified form of this
extension is the origin of the present generalization. (For simplicity, we shall consider our fundamental space to be the plane or the line.)

**Theorem T (particular case).** Every set $E$ is the sum of a dense-in-itself set and a scattered set.

**Definition.** A set is scattered if it contains no non-empty subset which is dense-in-itself. This concept was introduced by Cantor [11, p. 471], who has proved [12, p. 115] that a scattered set is necessarily countable. (In this and the following theorems, the constituent sets in the decomposition are disjoint, i.e., mutually without any common point, and permissibly empty.)

**Proof.** Let $S$ be the sum of all the dense-in-itself subsets of $E$. Each point of $S$, being a limit point of a subset of $S$, is a limit point of $S$. Hence, $S$, too, is dense-in-itself, and is the "maximal" dense-in-itself subset of $E$. $E - S$ contains no non-empty dense-in-itself subset, and is therefore scattered.

This familiar, obvious proof (see, for example, [20, p. 226]) depends on the fact that the set property dense-in-itself is unconditionally additive.

**Definition.** A set property $\varphi$ is a property such that for every set $E$ (of the space in question), either $E$ has or has not property $\varphi$; $\varphi$ is unconditionally
additive if the sum of an arbitrary number of sets of property \( A \) is again of property \( A \).

Obviously, for an arbitrary unconditionally additive set property \( A \), we may define the "maximal" subset of property \( A \) of \( E \) -- namely, as the sum of all the subsets of \( E \) having property \( A \). We are thus led to a generalized theorem:

**Theorem G (generalization).** If \( A \) is an unconditionally additive set property, every set is the sum of a set of property \( A \) and a set containing no non-empty subset of property \( A \).

§2. Genesis of Unconditionally Additive Set Properties

1.3. To utilize Theorem G, we need to find a source of unconditionally additive set properties. To this end we examine more closely the particular set property dense-in-itself, denoting it by \( A_0 \). This property goes back to the notion of limit point. That a point \( A \) is a limit point of a set \( E \) is a particular instance of a set-point property or relationship which \( E \) has with respect to \( A \). We denote it by \( B_0 \) so that

\[(E,A)_{B_0} := A \text{ is a limit point of } E.\]

**Definition.** A set-point property \( B \) is a property such that for every set \( E \) and every point \( A \), either \( E \) has or has not property \( B \) with respect to \( A \). We use the
notation \( E^\alpha \) and \((E,A)^\beta\) to mean respectively that \( E \) has property \( \alpha \) and \( E \) has property \( \beta \) with respect to \( A \). The symbol \(:=\) signifies "means" or "if and only if." (The concept "set-point property" is, of course, not new. For a discussion of six of the more familiar examples of ascending (see below) set-point properties, see Morris Hendrickson [21].)

Every point of a dense-in-itself set is a limit point of the set; hence, the set property \( d_0 \) may be defined in terms of \( \beta_0 \) as follows:

\[ E^d_0 := (E,A)^{\beta_0} \text{ for every } A \in E. \]

Toward generalization, it is consequently suggested that we replace the particular set-point property \( \beta_0 \) by an unconditioned set-point property \( \beta \), and associate a set property \( \alpha \) with \( \beta \) as \( d_0 \) is associated with \( \beta_0 \). We thus obtain the following definition for \( \alpha \) (\( = \alpha(\beta) \)):

\[ E^\alpha := (E,A)^\beta \text{ for all } A \in E. \]

Since \( \alpha \) is a natural generalization of dense-in-itself (\( = d_0 \)), we inquire: Is \( \alpha \) unconditionally additive for every \( \beta \)? The trivial example

\[ (E,A)^\beta := \text{ A is an isolated point of E} \]

shows that this is not so. We are led to further condition \( \beta \). Now the particular set-point property \( \beta_0 \) is such that if \( (E,A)^{\beta_0} \), then \( \beta_0 \) subsists between \( E_1 \) and \( A \) for every superset \( E_1 \) of \( E \). It is thus suggested that
we require $\beta$ to be ascending, i.e., if $(E,A)^\beta$ and $E_1 \supseteq E$, then $(E_1,A)^\beta$. We have the following theorem:

**Theorem.** If $\beta$ is an ascending set-point property, and the set property $\alpha (\beta)$ is defined by

$$E^\alpha := (E,A)^\beta \text{ for all } A \in E$$

then $\alpha$ is unconditionally additive. Conversely, if $\alpha$ is an unconditionally additive set property, there exists an ascending set-point property $\beta$ such that

$$E^\alpha := (E,A)^\beta \text{ for all } A \in E.$$ 

**Proof.** The direct part of the theorem is obvious.

To prove the converse: If $\alpha$ is an unconditionally additive set property, and $E$ a set, let $\mathcal{S} = S_E$ be the maximal subset of $E$ of property $\alpha$. Let $(E,A)^\beta := A \in S$. Then $\beta$ is an ascending set-point property, and

$$E^\alpha := (E,A)^\beta \text{ for all } A \in E.$$ 

1.4. We shall call $A$ a $\beta$-point of a set $E$ if $(E,A)^\beta$, and $E$ a $\beta$-set if $(E,A)^\beta$ for every $A \in E$, i.e., if $E^\alpha$. In generalization of the set property scattered we shall say $E$ is a pervasive $\bar{\beta}$-set when no non-empty subset of $E$ is a $\beta$-set.

By the theorem of 1.3 and Theorem G, we have the

**Corollary.** If $\beta$ is an ascending set-point property, every set is the sum of a $\beta$-set and a pervasive $\bar{\beta}$-set.

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2 Decimal numbers in the text refer to articles.
1.5. We now seek a source of ascending set-point properties. Returning to the particular set property dense-in-itself, and the ascending set-point property $\rho_0$ in terms of which it is defined, we notice that $\rho_0$, in turn, goes back to a particular set-interval property.

**Definition.** A set-interval property $\gamma$ is a property of a set with respect to an open interval $I$; if the basic space is the plane, an interval is understood to be an oriented rectangle; $(E, I)^\gamma$ will mean that $E$ has property $\gamma$ with respect to $I$.

That $A$ is a limit point of $E$ means that for every interval $I \supset (A)$, $E_I$ is infinite. Hence if we denote by $\gamma_0$ the set-interval property defined by

$$(E, I)^{\gamma_0} := E_I \text{ is infinite}$$

then the set-point property $\rho_0$ is expressed in terms of $\gamma_0$ as follows:

$$(E, A)^{\rho_0} := (E, I)^{\gamma_0} \text{ for every } I \supset (A).$$

We thus have the cue for similarly associating a set-point property $\beta$ with every set-interval property. If $\gamma$ is a set-interval property, we define $\beta$ as follows:

$$(1) \quad (E, A)^{\beta} := (E, I)^{\gamma} \text{ for every } I \supset (A).$$

To render $\beta$ ascending, we require $\gamma$ to be ascending, i.e., if $(E, I)^{\gamma}$, and $E_I \supset E$, then $(E_I, I)^{\gamma}$. 
Alternatively, we may define set-point properties in terms of an ascending set-interval property as follows:

(2) \((E,A)_{\beta} :=: \text{there exists an interval } I \supseteq (A) \text{ such that } (E,I) \not\in \)

(3) \((E,A)_{\beta} :=: \text{there exists an infinitesimal interval } I_n, n = 1, 2, \ldots, \text{ which approaches } A \text{ (i.e., if } P_n \text{ is contained in } I_n, P_n \text{ necessarily approaches } A) \text{ and is such that } (E,I_n) \not\in \text{ for every } n.

Corresponding to a given ascending set-interval property \(\not\in\), we shall denote by \(\beta_1, \beta_2, \text{ and } \beta_3\) the set-point properties as defined, respectively, in (1), (2), and (3) above. We have the immediate result:

**Theorem.** If \(\not\in\) is an ascending set-interval property, the set-point properties \(\beta_1, \beta_2, \text{ and } \beta_3\) are ascending.

Applying the corollary of 1.4, we have the

**Corollary.** If \(\not\in\) is an ascending set-interval property, every set is the sum of a \(\beta_n\)-set and a pervasive \(\bar{\beta}_n\)-set, \(n = 1, 2, \text{ and } 3\).

1.6. We note that with a given set-interval property \(\not\in\) we may associate another such property \(\not\in_{\xi}\) as follows:

\((E,I)_{\not\in_{\xi}} :=: (E,J) \not\in \text{ for every interval } J \subset I.

For example, if \((E,I) \not\in :=: EI \text{ is non-denumerable, then } (E,I)_{\not\in_{\xi}} :=: EJ \text{ is non-denumerable for every subinterval}
J of I. In the latter case we say that \( E \) is **homogeneously non-denumerable** in I.

We observe further that the ascending set-interval property \( J_0 \) is defined in terms of a particular ascending set property \( J'_0 \), where \( E^{J'_0} := E \) is infinite. (That the set property \( J \) is ascending means, of course, that if \( E^J \) and \( E_1 \supset E \) then \( E_1^J \).) We may similarly define a set-interval property in terms of any ascending set property and thereby obtain the following result:

**Theorem.** If \( J \) is an ascending set property and the set-interval property \( J \) is defined by the relation

\[
(E, I)^J := E^J \text{ has property } J
\]

or alternatively by the relation

\[
(E, I)^J := E^J \text{ is of property } J \text{ relative to } I
\]

(i.e., \( E^J \) is the product of \( I \) and a set of property \( J \)) then \( J \) is ascending.

It is evident that the set-interval property \( J \) associated as in this theorem with an ascending set property \( J \) is localized in the sense that if \( (E, I)^J \) then \( (E^J, I)^J \). For such a set-interval property we may specify more exactly the character of the associated pervasive \( \bar{\beta}_2 \)-set:

If \( J \) is an ascending, localized set-interval property, a necessary and sufficient condition for a set to be a pervasive \( \bar{\beta}_2 \)-set is that it contain no \( \beta_2 \)-point.
1.7. We may go one step further. The following statement whose proof is obvious gives a genesis of all ascending set properties in terms of unconditioned set properties.

If $d$ is a set property and the set property $d_1$ is defined by the relation

$$E^{d_1} := E \text{ contains a subset of property } d$$

then $d_1$ is ascending. Conversely, if $d_1$ is an ascending set property, there exists a set property $d$ such that

$$E^{d_1} := E \text{ contains a subset of property } d.$$ 

§3. Illustrative Examples and Remarks

1.8. Let $(E,A)^\beta := E$ has interior metric density 1 at $A$. Then a $\beta$-set has interior metric density 1 at each of its points; we call such a set m-open. Since $\beta$ is ascending, we have by the corollary of 1.4:

**Every set is the sum of an m-open set and a set containing no non-empty m-open set.**

Now the unconditionally additive property "m-open" has the following close relation with the countably additive property "measurable": A set is measurable if and only if it is the sum of an m-open set and a set of measure 0. Calling a set pervasively non-measurable if it has no measurable subset of positive measure, we obtain from the above decomposition a corollary of a result of H. Blumberg [7]:

Every set is the sum of a measurable set and a pervasively non-measurable set.

1.9. Let \( (E,I) \rightarrow \): EI is non-denumerable. Then \( \rightarrow \) is ascending and a \( \beta_1 \)-set is a condensation set, i.e., every point of it is a point of condensation. By the corollary of 1.5 we have the result: Every set is the sum of a condensation set and a set containing no non-empty condensation set as subset. It follows from this that every set is the sum of a condensation set and a countable set ("countable" :=: "finite or denumerable").

Let \( (E,I) \rightsquigarrow :=: E \) is homogeneously non-denumerable in \( I \). In the terminology of H. Blumberg the set-point properties \( \beta_2, \beta_3 \) are then defined, respectively, as follows:

1. A is a point of concentrated non-denumerability of the set \( E \) if there exists an interval \( I \supset (A) \) such that \( (E,I) \rightsquigarrow \).

2. A is a point of strong non-denumerability of the set \( E \) if there exists an infinitesimal interval \( \{I_n\} \) which approaches \( A \) and such that \( (E,I_n) \rightsquigarrow \) for every \( n \).

We define in a similar way analogous terms such as a point of concentrated (strong) inexhaustibility of \( E \) ("exhaustible" :=: "of first category of Baire"), a point of concentrated (strong) approach of \( E \) with respect to positive exterior measure, etc.
By the corollary of 1.5 we have the following particular decomposition theorems:

Every set $E$ is the sum of sets $S$ and $H$ such that each point of $S$ is a point of concentrated (strong) non-denumerability of $S$, of concentrated (strong) inexhaustibility of $S$, of concentrated (strong) approach with respect to positive exterior measure of $S$, and $H$ has no non-empty subset having, respectively, the stated property of $S$.

A set containing no point of concentrated inexhaustibility is exhaustible. Consequently, one of the latter decomposition theorems may be restated as follows:

Every set is the sum of a set which approaches each of its points in the sense of concentrated inexhaustibility and an exhaustible set.

1.10. Let $(E,I)^\dagger := (i) \ E I$ is dense in $I$; (ii) $EI$ is residual in $I$, i.e., $I - E$ is exhaustible. For $\dagger$ as defined in (i), a $\beta_2$-point $A$ of a set $E$ is densely approached by $E$, i.e., there exists an interval $I \supset (A)$ such that $EI$ is dense in $I$. Since in this case a pervasive $\beta_2$-set contains no $\beta_2$-point, it is non-dense, and by the corollary of 1.5 we have the result:

Every set is the sum of a set which approaches each of its points densely and a non-dense set.
If \( f \) is defined as in (ii), we say a \( \beta_2 \)-point of \( E \) is residually approached by \( E \). By the corollary of 1.5 we conclude:

Every set is the sum of a set which approaches each of its points residually and a set which approaches none of its points residually.

1.11. The following theorem indicates the scope of Theorem G.

**Theorem.** The set \( U \) of set properties which are unconditionally additive but not ascending has the same cardinal number as the set \( T \) of all set properties.

(If the set property \( \mathcal{A} \) is ascending, Theorem G gives a trivial decomposition.)

**Proof.** The cardinal of \( T \) is \( 2^{2^c} \) since there are \( 2^c \) different sets, \( c \) denoting the cardinal number of the real continuum.

The cardinal of \( U \) is at least (and therefore is) \( 2^{2^c} \). For let \( E \) be a proper subset of the continuum of cardinal \( c \). Since \( cc = c \), we can decompose \( E \) into \( c \) disjoint sets \( E_1 \) each of cardinal \( c \). Let \( S \) consist of all distinct sets that can be constructed by taking one element from each set \( E_1 \). \( S \) consists of \( c^c = 2^c \) sets and is such that the sum of two or more sets of \( S \) is not identical to any set of \( S \). To define the set property \( \mathcal{A} \), we invest each element of \( S \), independently and arbitrarily,
with the property $d$ or $\overline{d}$ (≠ not $d$). Let the sum of any number of sets of $S$ of property $d$ also have property $d$, and all other sets have property $\overline{d}$. Then $d$ is unconditionally additive but not ascending (except for the single vacuous case when no set is of property $d$) and since there are $2^{2^c}$ possible distributions of $d$ and $\overline{d}$ among the elements of $S$, there are $2^{2^c}$ distinct set properties $d$ thus defined.

1.12. A related result indicates the scope of the theorem of 1.6:

**The set $U$ of ascending set properties has the same cardinal number as the set $T$ of all set properties.**

**Proof.** Decompose the whole continuum $E$ into $c$ disjoint sets $E_1$ each of cardinal $c$. Let $S$ consist of all distinct sets that can be constructed by taking one element from each set $E_1$. $S$ consists of $c^c = 2^c$ sets and is such that no set of $S$ is a subset or a superset of another set of $S$. To define the set property $d$, we invest each element of $S$, independently and arbitrarily, with the property $d$ or $\overline{d}$. Let every superset of a set of $S$ of property $d$ also have property $d$, and all other sets have property $\overline{d}$. Then $d$ is ascending, and, as in 1.11, there are $2^{2^c}$ distinct set properties thus defined.
CHAPTER II

REGION AND INTERVAL PROPERTIES

§4. Region Properties

2.1. In this chapter, we derive certain results on region properties and interval properties, and then apply them to obtain theorems on sets. Unless otherwise stated, we shall consider planar sets, although many of the results are valid for n-space.

2.2. We define a region to mean a planar open set, and the two terms will be used interchangeably. A region property \( \mu \) is a property such that for every region \( G \), either \( G \) has property \( \mu (G^\mu) \) or \( G \) does not have property \( \mu (G^{\overline{\mu}}) \).

We shall denote by \( E' \), \( cE \), and \( \overline{E} \) respectively the derivative, the closure, and the complement (usually with respect to the planar continuum) of \( E \). A frontier point of \( E \) is a point of both \( cE \) and \( c\overline{E} \). The frontier \( Fr(E) \) of \( E :=: \) the set of all frontier points of \( E \), i.e., \( Fr(E) = cEc\overline{E} \). The interior \( E^0 \) of \( E :=: E - Fr(E) \).

A neighborhood of a point \( P \) is a region containing \( P \). A partial neighborhood of \( P \) is a region \( G \) of which \( P \)
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is a frontier point. (This latter term has been used by H. Blumberg [6, §1] in the different sense that P is either an interior point or a frontier point of G.)

2.3. The following theorem gives a characterization of a non-dense set in terms of a region property $\mu$. $\mu$ is ascending (descending) := $G^\mu$ and $G_1 \supset G$ ($G_1 \subset G$) imply $G_1^\mu$.

**Theorem.** If $\mu$ is a region property, let $H_\mu$ denote the set of all points of the plane which satisfy the two conditions:

(a) for every region $G$ containing $P$, $G$ has property $\mu$;
(b) there exists a partial neighborhood $G$ of $P$ with $G$ of property $\bar{\mu}$.

For every region property $\mu$, $H_\mu$ is non-dense, and, conversely, every non-dense set $N$ is the set $H_\mu$ associated with some ascending region property $\mu$.

**Proof.** No point of a region $G$ of property $\bar{\mu}$ is in $H_\mu$. Hence, by property (b), $H_\mu$ is non-dense.

To prove the converse, first choose a non-dense set $N$. Define an ascending region property $\mu$ as follows: $G^\mu :=$ either $GN$ is non-empty or $G$ has a frontier point in $(cN - N)$. Let $P \in N$. By the definition of $\mu$, if $G \supset P$, then $G^\mu$. But there exists a partial neighborhood $G$ of $P$ with $GN = 0$ and whose frontier has the single
point \( P \) in common with \( cN \). For let \( G = \sum R_n, n = 1, 2, \ldots \), where \( R_n \) is an open rectangle whose closure contains no point of \( cN \) and which is contained in \( C(P, l/n) \), the circle of center \( P \) and radius \( l/n \). Thus, since \( G \) has no frontier point in \( (cN - N) \), \( G \) has property \( \overline{\mu} \). Consequently, \( P \in H^\mu \), i.e., \( H^\mu \supset N \).

Now let \( P \in H^\mu \). Then for every \( G \supset P \), \( G^\mu \), i.e., either \( GN \neq 0 \) or \( G \) has a frontier point in \( (cN - N) \). If there exists an infinitesimal neighborhood \( G \) of \( P \) with \( GN \neq 0 \), then \( P \in cN \). Otherwise, for \( G \supset P \) and sufficiently small in diameter, \( G \) has a frontier point in \( (cN - N) \). In this case, \( P \) is a limit point of \( cN \), and, since \( cN \) is closed, \( P \in cN \). But also, for \( P \in H^\mu \), there exists a partial neighborhood \( G \) of \( P \) with \( G \) of property \( \overline{\mu} \). Thus, in particular, \( G \) has no frontier point in \( (cN - N) \). Therefore \( P \not\in (cN - N) \), and since \( P \in cN \), it follows that \( P \in N \), i.e., \( N \supset H^\mu \). The theorem follows.

We state the following

**Corollary.** A set \( N \) is non-dense if and only if it is the set \( H^\mu \) for some region property \( \mu \).

2.4. We shall now show that a characterization of a non-dense, closed set may be given -- using the same definition of \( H^\mu \) -- in terms of a conditioned region property \( \mu \). The region property \( \mu \) is additive \( \Rightarrow \) the sum of any number of regions of property \( \mu \) also has property \( \mu \).
(There is no distinction between a countably additive and an unconditionally additive region property since the sum of any number of regions is the sum of a properly selected countable number of them.) In the following theorem, \( H_\mu \) is defined as in 2.3.

**Theorem.** For every region property \( \mu \) such that \( \overline{\mu} \) is additive, \( H_\mu \) is non-dense and closed, and, conversely, every non-dense, closed set is the set \( H_\mu \) associated with some ascending region property \( \mu \) where \( \overline{\mu} \) is additive.

**Proof.** Choose a region property \( \mu \) such that \( \overline{\mu} \) is additive. By 2.3, \( H_\mu \) is non-dense. Suppose that \( P \) is a limit point of a sequence \( \{P_n\} \) of distinct points contained in \( H_\mu \). Every \( G \) containing \( P \) also contains some \( P_n \) and therefore \( G^\mu \). Also, for every \( n \), there is a partial neighborhood \( G_n \) of \( P_n \) which has property \( \overline{\mu} \). Hence \( P \notin G_n \). It follows that \( G = \sum G_n, n = 1, 2, \ldots \), is a partial neighborhood of \( P \). Further, since \( \overline{\mu} \) is additive, \( G \) has property \( \overline{\mu} \). Thus \( P \in H_\mu \), i.e., \( H_\mu \) is closed.

To prove the converse, choose a non-dense, closed set \( N \). Define a region property \( \mu \) as follows:

\[ G^\mu := \text{GN is non-empty.} \]

It is evident that \( \overline{\mu} \) is additive and \( \mu \) is ascending.

Let \( P \in N \). Then, for every \( G \supseteq P, G^\mu \). Since \( N \) is non-dense, there exists a partial neighborhood \( G \) of \( P \) with \( GN = 0 \), i.e., with \( G \) of property \( \overline{\mu} \). Thus \( H_\mu \supset N \).
Let $P \in H_\mu$. Then for every $G \supset P$, $G^\mu$, i.e.,
$GN \neq 0$. Therefore, either $P \notin N$ or $P$ is a limit point
of $N$. In either case, since $N$ is closed, $P \in N$, i.e.,
$N \supset H_\mu$. Thus $N = H_\mu$, and the theorem follows.

We state the following

Corollary. A set is closed and non-dense if and only
if it is the set $H_\mu$ for some additive region property $\mu$.

2.5. As corollaries of the last two results, we
obtain characterizations of an exhaustible set and an
exhaustible, $F_\sigma$ set in terms of a sequence of region
properties. $H_\mu$ is defined as in 2.3.

(1) For every sequence $\{\mu_n\}$ of region properties,
$H = \sum H_\mu$ is exhaustible, and, conversely, every exhaust-
ible set is the set $H = \sum H_\mu$ associated with some
sequence $\{\mu_n\}$ of (ascending) region properties.

Proof. This is immediate from 2.3.

(2) For every sequence $\{\mu_n\}$ of additive region
properties, $H = \sum H_\mu$ is an exhaustible, $F_\sigma$ set, and,
conversely, every exhaustible, $F_\sigma$ set is the set $H = \sum H_\mu$
associated with some sequence $\{\mu_n\}$ of (ascending) region
properties for which $\mu_n$ is additive.

Proof. This follows from 2.4 and the result: Every
exhaustible, $F_\sigma$ is the sum of a sequence of sets $E_n$ each
of which is both closed and non-dense.
2.6. We shall now make precise our use of certain terminology used in this and later sections. The upper (exterior) metric density \( m^*(E, P) \) of the set \( E \) at the point \( P := \lim \sup \frac{m(E \cap R)}{m(R)} \) for an infinitesimal oriented rectangle \( R \supset P \). (Here, \( m(E) \) (\( m(E) \)) denotes the Lebesgue exterior measure (measure) of an arbitrary (measurable) set \( E \).) Similarly, the lower (exterior) metric density \( m^*_-(E, P) \) of \( E \) at \( P \) is defined. If \( m^*(E, P) = m^*-(E, P) \), this common value, denoted by \( m^*_0(E, P) \), is called the (exterior) metric density of \( E \) at \( P \). If \( E \) is measurable, the subscripts, denoting exterior measure, may be omitted. If \( E \) is linear, in the above definitions we may replace the infinitesimal interval \( I \supset x \) with an infinitesimal interval \( I \) with \( x \) as right, left end-point to obtain the concepts of upper, lower metric density of \( E \) at \( x \) from the left, right.

2.7. Using the terminology of H. Blumberg [6, §3], a region \( G \) is said to be non-vanishing at \( P \) if the metric density of \( G \) at \( P \) is not 0, and a partial neighborhood \( G \) of \( P \) is called non-vanishing if \( G \) is non-vanishing at \( P \).

Theorem. If \( \mu \) is a region property, let \( K_\mu \) denote the set of points \( P \) of the plane which satisfy the conditions:

(a) for every region \( G \) containing \( P \), \( G \) has property \( \mu \);
(b) there exists a non-vanishing partial neighborhood $G$ of $P$ with $G$ of property $\overline{\mu}$.

For every region property $\mu$, $K_\mu$ is non-dense and of measure 0.

Proof. $K_\mu$ is non-dense by 2.3. Also, $K_\mu$ is of measure 0. For, let $P \in K_\mu$. Since there is a non-vanishing partial neighborhood of $P$ of property $\overline{\mu}$ containing no point of $K_\mu$, $K_\mu$ is not of metric density 1 at $P$. Applying the density theorem of Lebesgue, the result follows.

2.8. The class of sets $K_\mu$ defined as in 2.7 -- even when $\mu$ is restricted to be ascending and such that $\overline{\mu}$ is additive -- contains as a (proper) subset the class of all sets $N$ which are closed, non-dense, and of measure 0. For if $N$ is so conditioned and if $G^\mu := GN \neq 0$, then $K_\mu = N$.

However, it is readily seen that the converse of the theorem of 2.7 is not valid, i.e., a set which is non-dense and of measure 0 is not necessarily a $K_\mu$. To obtain an example of such a set, take a closed rectangle $R$ and let $R = G + F$, where $G$ is a region which is everywhere dense on $R$, $m(G) < m(R)$, and $F = R - G$. Then clearly there exists a subset $N$ of $F$ such that (1) $N$ is non-dense and of measure 0 and (2) $N$ is dense on $F$ and contains points at which $N^0$ has metric density 0. However, since $G^\mu$
implies that $G_\mu$ contains no point of $K_\mu$, it follows by (b) of the definition that $K_\mu^0$ must be non-vanishing at each point of $K_\mu$. Thus $N$ is not a $K_\mu$.

2.9. Letting $K_\mu$ be defined as in 2.7, our remarks of 2.8 suggest the following result:

For every region property $\mu$, $K_\mu$ is (1) non-dense, (2) of measure 0, and (3) such that $K_\mu^0$ is non-vanishing at each point of $K_\mu$. Conversely, every set $N$ which has properties (1), (2), and (3) is the associated set $K_\mu$ for some (ascending) region property $\mu$.

Proof. The direct part of the result follows from 2.7 and a remark of 2.8. The converse may be proved in a similar way to that of 2.3. If $N$ satisfies the properties (1), (2), and (3), define a region property $\mu$ as in 2.3: $G^\mu :=$ either $G_N \neq 0$ or $\operatorname{Fr}(G)(cN - N) \neq 0$. If $P$ is a point in $N$, there exists an infinitesimal sequence $\{R_n\}$ of (oriented) rectangles containing $P$ such that the limit of the relative measure of $N^0$ in $R_n$ is $p > 0$. Then let $G = \sum G_n$, $n = 1, 2, \ldots$, where $G_n$ is a region defined as follows: (a) $G_n \subset R_n$; (b) $cG_n cN = 0$; (3) $m(G_n) > m(N_{\mu} R_n)/2$. It then follows that $\limsup m(G_n)/m(R_n) \geq \limsup m(G_n)/m(R_n) \geq p/2$. Therefore $G$ is a non-vanishing partial neighborhood of $P$, $G_N = 0$, and $\operatorname{Fr}(G)(cN = (P)$. Hence, $\operatorname{Fr}(G)(cN - N) = 0$, and it follows that $G$
has property $\overline{P}$. Consequently, $P \subseteq K_\mu$, i.e., $K_\mu \supseteq N$.

By the proof of 2.3, $N \supseteq H_\mu$. Obviously, $H_\mu \supseteq K_\mu$.

Hence $N \supseteq K_\mu$. The result follows.

§5. Linear Interval Properties

2.10. A region property $\mu$ may be said to possess a certain type of antisymmetry at the points of $H_\mu$ and $K_\mu$ defined respectively in 2.3 and 2.7. In this section, we consider a property $\nu$ of an open linear interval $I: (a, b)$, i.e., for every interval $I$, either $I^\nu$ or $I^{\overline{\nu}}$, and we derive certain results on the symmetry of a linear set. In a later chapter, some of the definitions and results of this section will be extended to the plane.

2.11. A linear set $E$ is said to be voidal at the point $\xi$ if there exists an open interval with $\xi$ as endpoint and containing no point of $E$. A set is termed voidal if it is voidal at each of its points. This set property, which does not seem to have been given a name in the literature, is extended to the plane in Chapter IV.

We make the following definitions for an interval property $\nu$:

(1) $\nu$ is ascending (descending) := $I^\nu$ and $J \supset I$ ($J \subset I$) imply $J^\nu$.

(2) $\nu$ is hereditary := $(a, b)^\nu$ and $a < c < b$ imply that either $(a, c)^\nu$ or $(c, b)^\nu$. 
We omit the proofs of the following elementary properties of a linear voidal set:

1. Every linear voidal set is both countable and non-dense. However, it is not necessarily scattered, and, hence, its closure may contain a non-empty perfect set.

2. A necessary and sufficient condition that a linear set be voidal is that it be a subset of the set of end-points of the open intervals constituting an open set \( G \).

2.12. We now give a characterization of a linear voidal set in terms of an interval property \( v \).

Theorem. If \( v \) is an interval property, let \( V_v \) denote the set of all points \( \xi \) which satisfy the conditions

(a) for every interval \( I \supset (\xi) \), \( I \) has property \( v \);

(b) there exists an \( I \) of property \( \bar{v} \) with an end-point at \( \xi \).

For every interval property \( v \), \( V_v \) is voidal. Conversely, every voidal set is the associated set \( V_v \) for some (ascending and hereditary) interval property \( v \).

Proof. No point of an interval \( I \) of property \( \bar{v} \) is in \( V_v \). Hence, by property (b), \( V_v \) is voidal.

To prove the converse, let \( E \) be a voidal set. If \( \xi \) is a point of \( E \), choose an interval \( I_\xi \) with \( \xi \) as end-point and with \( I_\xi E = 0 \), and let \( J_\xi \) denote the open half of \( I_\xi \) with an end-point at \( \xi \). If \( I: (a,b) \) has both end-points in \( J_\xi \) or if \( I \) has one end-point at \( \xi \) and one in \( J_\xi \), let \( I \)
have property $\overline{v}$. Assume, as we may, that for every $\xi$ in $E$ such a set of intervals of property $\overline{v}$ is defined. Let all other intervals have property $v$. It is readily seen that $v$ is both ascending and hereditary, and that $V_v = E$.

2.13. An interval property $v$ may be thought of as lacking a certain symmetry at the points of $V_v$. A related type of antisymmetry has been defined by H. Blumberg [4] as follows: $\xi$ is a point of antisymmetry of (an interval property) $v$ with $v$ valid on the right $i.e.$ if the positive number $h$ is sufficiently small then $(\xi, \xi + h)^v$ and $(\xi - h, \xi)\overline{v}$. Denote the set of all such points by $D_v$. Then $D_v$ denotes the set of points of antisymmetry of $v$ with $v$ valid on the left. We call

$$D(v) = D_v + D_{\overline{v}}$$

the set of points of antisymmetry of $v$.

The direct part of the following result is due to H. Blumberg [4]; the converse was proved by C. Goffman [16].

If $v$ is an interval property, $D_v$ (and therefore $D(v)$) is countable; conversely, every countable set is the associated set $D_v$ for some interval property $v$.

2.14. We may now give a second characterization of a voidal set in terms of $D(v)$ and a conditioned interval property $v$. 
If \( v \) is an ascending interval property, then \( D(v) \) is voidal; conversely, every voidal set is the set \( D(v) \) for some ascending (and hereditary) interval property \( v \).

Proof. Since \( v \) is ascending, \( \nabla v \) is descending, and therefore no point of an interval of property \( \nabla v \) is in \( D(v) \). Hence, \( D(v) \) is voidal.

To prove the converse, we define \( v \) just as in 2.12.

2.15. The following statements on the comparison of the sets \( D(v) \) and \( V_v \) are easily derived.

(1) For an arbitrary interval property \( v \), neither of the relations \( D(v) \supset V_v, V_v \supset D(v) \) necessarily holds.

(2) If \( v \) is ascending, then \( V_v \supset D(v) \), but \( D(v) \) does not necessarily contain \( V_v \).

(3) If \( v \) is both ascending and hereditary, then \( V_v = D(v) \).

§6. Some Theorems on Sets

2.16. Applying the theorems of the last two sections, we derive in this section certain theorems on sets. Unless stated otherwise, the sets will be planar. In Chapter V we shall use a general procedure to carry some of these theorems over to real functions.

2.17. The theorems of the last two sections associate with region and interval properties (possibly conditioned) certain conditioned sets. There are many ways
to define region and interval properties in terms of sets and functions. We now give one easily accessible way to define a region property in terms of an arbitrary set and a set property.

Let, then, $E$ be an arbitrary set, and $\mathcal{A}$ a set property. Define the region property $\mu$ as follows:

$$G^\mu := \text{GE has property } \mathcal{A}.$$ 

Then $\mu$ is respectively ascending, additive if $\mathcal{A}$ is ascending, countably additive.

We shall use the following terminology:

$P$ is $\mathcal{A}$-approached by $E :=$ for every region $G \supset P$, $GE$ has property $\mathcal{A}$.

$P$ is $\mathcal{A}$-approached (homogeneously) by $E :=$: there exists a region $G \supset P$ such that for every sub-region $G_1$ of $G$, $G_1E$ has property $\mathcal{A}$.

2.18. We state an immediate lemma:

If $\mathcal{A}$ is descending and countably additive, then $P$ is not $\mathcal{A}$-approached by a set $E$ if and only if there is a partial neighborhood $G$ of $P$ with $GE$ of property $\mathcal{A}$.

Letting $G^\mathcal{A} := GE$ is of property $\mathcal{A}$, by 2.4 we have:

(a) If $E$ is a set and $\mathcal{A}$ a descending and countably additive set property, the set of all points which are $\mathcal{A}$-approached by $E$ but not $h-\mathcal{A}$-approached by $E$ is non-dense and closed.
Likewise we have by 2.7:

(b) If $E$ is a set and $\mathcal{I}$ a set property, the set of all points $P$ which are $\overline{\mathcal{I}}$-approached by $E$ but such that there exists a non-vanishing partial neighborhood $G$ of $P$ with $GE$ of property $\mathcal{I}$ is non-dense and of measure 0.

2.19. Let $E_1^\mathcal{I} := E_1$ is empty. Then applying (a) of 2.18 to $E$ we obtain the elementary result:

If $E$ is a set, the set of all points which are not interior points of $E$ and at which there is a partial neighborhood consisting exclusively of points of $E$ is non-dense and closed. In particular, the frontier of an open or closed set is non-dense and closed.

2.20. Let $E_1^\mathcal{I} := E_1$ is exhaustible. By (a) of 2.18 we see that the exceptional set in the following result is not only non-dense (see [2, p. 673]) but also closed:

If $E$ is a (planar) set, the set of points (of the plane) which are inexhaustibly approached by $E$ but not h-inexhaustibly approached by $E$ is non-dense and closed.

2.21. Let $E_1^\mathcal{I} := E_1$ is respectively finite, countable, exhaustible, of measure 0. To signify that $P$ is $\overline{\mathcal{I}}$-approached by $E$ we say respectively that $P$ is approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive (exterior) measure by $E$. By (b) of 2.18, we obtain:
If $E$ is a (planar) set, the set of points $P$ (of the plane) which are approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive measure by $E$, but such that there exists a non-vanishing partial neighborhood $G$ of $P$ with $GE$ respectively finite, countable, exhaustible, of measure 0 is non-dense and of measure 0.

2.22. If $E$ is a (planar) set, the set of points (of the plane) at which $E$ has positive upper metric density but not metric density 1 is of measure 0.

Proof. This theorem follows by applying the density theorem of Lebesgue (for measurable sets) to a "measurable envelope" of $E$.

2.23. If $E$ is a linear set, the set of points of the line at which the lower metric density of $E$ on one side is more than the upper metric density of $E$ on the other side is countable.

Proof. It is sufficient to prove that the set of points $x$ at which the lower metric density of $E$ on the right is more than the upper metric density of $E$ on the left is countable. Denote this set by $H$. Then letting $H_h :=$ the set of points $x$ at which the lower metric density of $E$ on the right is more than $h$, the upper metric density of $E$ on the left is less than $h$, it follows that
H = \sum \mathbb{H}_h, \text{ where } h \text{ ranges over the rational numbers in } (0,1).

Letting $I^\nu := \frac{m_\nu(E \cap I)}{m(I)} > h$, it follows that every point $x$ of $\mathbb{H}_h$ is a point of antisymmetry (see 2.13) of the interval property $\nu$ with $\nu$ valid on the right. By 2.13 it is seen that $\mathbb{H}_h$ is countable for every $h$, and therefore $H$ is countable.

2.24. The following two results are immediate corollaries of 2.23:

(1) If $E$ is a linear set, the set of points of the line at which the lower metric density of $E$ is positive on one side and the (upper) metric density of $E$ is 0 on the other side is countable.

(2) If $E$ is a linear set, the set of points of the line at which the metric density of $E$ is 0 on one side and 1 on the other side is countable.
CHAPTER III
ON APPROACH OF ARBITRARY SETS TO A STRAIGHT LINE

§7. Notation and Types of Approach

3.1. The literature contains a variety of results on the approach of an unconditioned function of two variables to the points of a straight line (see, for example, H. Blumberg [3], [4], M. F. Schmeiser [30], [31], and V. Jarnik [22]). Taking the characteristic function \( C(A) \) of a planar set \( E \) (\( C(A) = 1 \) if \( A \in E \), \( C(A) = 0 \) if \( A \notin E \)), one obtains as corollaries of these results on functions, results on the approach of a planar set to a straight line. But a simpler method of attack seems to be that of first deriving results on sets, and then carrying these set theorems over to functions by some general procedure such as we give in Chapter V.

A typical result of this chapter states that if \( L \) is a straight line, and \( E \) a set in one of the half-planes into which \( L \) divides the plane, then \( E \) approaches \( L \) in a certain way at every point \( A \) of \( L \) not belonging to an exceptional set \( H \), which is meager in some sense. As we shall see in Chapter IV, many of these results may be
extended to obtain theorems on the approach of a planar set to the points of a plane. Such an extension cannot be made, of course, by merely decomposing the plane into a set of parallel lines. For the set $H$ may be countable, or of measure 0 for each line, say, and yet the sum of the sets $H$ may be respectively non-denumerable, of positive exterior planar measure.

3.2. If $\theta$ is a real number, the direction $\theta$ (in a given plane $K$) will mean the direction obtained by a rotation of $\theta$ radians from a given fixed 0-direction -- the rotation being counterclockwise or clockwise according as $\theta$ is positive or negative. We say directions $\theta$ and $\phi$ are equal if and only if $\theta$ is congruent to $\phi$ modulo $2\pi$. $\bar{\theta}$ ($= \theta + \pi$) denotes the direction opposite to $\theta$, and the direction $\theta$ will be called rational whenever $\theta/\pi$ is rational. If $\theta$, $\phi$ are two distinct directions in $K$, we denote by $\theta\phi$ or $\angle\theta\phi$ the angle ($< 2\pi$) obtained by a counterclockwise rotation from $\theta$ to $\phi$. If the direction $d$ is in the open angle $\theta\phi$, we write $d \in \angle\theta\phi$. $S(A;\theta\phi h)$, where $A$ is a point in $K$ and $h > 0$, denotes the open, circular sector of vertex $A$, angle $\theta\phi$, and radius $h$; $S(A;\theta\phi)$ denotes the corresponding sector of infinite radius. For brevity, we may denote by $\sigma$ the triple $\theta\phi h$, and say $\sigma$ is rational if $\theta$, $\phi$, and $h$ are rational. If $\sigma$ is rational, we say the sector $S(A;\sigma)$ has rational dimensions. $S(A;\theta)$
denotes the variable open sector of vertex $A$ containing the direction $\theta$, i.e., $S(A;\theta) = S(A;\text{cdh})$, where $\theta \in \angle \text{cd}$ and $h > 0$.

3.3. Let $E$ be a planar set, $A$ a limit point of $E$, and let $(A,\theta)$ denote the ray (= half-line) of origin $A$ in direction $\theta$. $(A,\theta)$ will be called a tangent ray of $E$ if and only if there exists a sequence of points $A_n \notin A$ of $E$ which converges to $A$, and such that the ray $AA_n$ converges to $(A,\theta)$, i.e., the angle which $AA_n$ makes with $(A,\theta)$ goes to 0. In other words, $(A,\theta)$ is a tangent ray of $E$ if every (open) sector $S(A;\theta)$ contains at least one point of $E$. We then say that $E$ approaches $A$ via direction $\theta$. Following G. Bouligand [8], [9, p. 66], the set of all tangent rays of $E$ at $A$ will be called the contingent of $E$ at $A$. It will be denoted by $T(A,E)$ — or $T(A)$, for short.

Evidently, for every set $E$ and point $A$, $T(A,E)$ is closed, i.e., $(A,\theta) \in T(A,E)$ if $(A,\theta)$ is a limit (ray) of rays in $T(A,E)$. The contingent $T(A,E)$ of the planar set $E$ is said to be the whole plane if it contains every ray $(A,\theta)$ at $A$; a half-plane, if there exists a line $L$ through $A$ such that $T(A,E)$ consists exclusively of the rays $(A,\theta)$ on one side of $L$ (necessarily including the two rays of $L$ originating at $A$, since $T(A,E)$ is closed);
and a single line, if it consists exclusively of two opposite rays of origin A constituting a line through A.

A half-plane K on one side of a line L is said to be closed or open according as it contains or does not contain the points of L. Thus, if K is a closed half-plane, $K^0$, the interior of K, is the corresponding open half-plane. For $A \in L$, the ray $(A, \theta)$ and the direction $\theta$ are said to be in K, $K^0$ according as the ray $(A, \theta)$ lies wholly in K, $K^0$.

3.4. In 3.3 we defined the equivalent concepts:

1. $(A, \theta)$ is a tangent ray of E,
2. E approaches A via direction $\theta$,
3. $(A, \theta) \in T(A, E)$, the contingent of G. Bouligand. Results on this and various other types of approach of a set to a point in a given direction will be obtained in the sequel.

We proceed to define a more restricted type of approach. $(A, \theta)$ will be called a strong tangent ray of E at A if and only if A is the limit of a sequence of points $A_n$ of E lying in the ray $(A, \theta)$, open at A. We then say that E approaches A (or A is approachable by E) along direction $\theta$. (We would emphasize here the distinction in our use of the terms via and along.) The set of all strong tangent rays of E at A will be called the strong contingent of E at A and will be denoted by $T'(A, E)$ -- or $T'(A)$, for short. Clearly, $T(A, E)$ contains $T'(A, E)$. 
§8. \( \lambda \)-approach

3.5. We now define a generalized type of approach, which we shall then consider in some detail, before considering the types of approach defined in 3.3 and 3.4.

Let \( E \) be a planar set, \( A \) a point in the plane, and \( \lambda \) a set property. \((A,\theta)\) will be called an \( \lambda \)-tangent ray of \( E \) (and we shall say that \( E \) \( \lambda \)-approaches \( A \) via direction \( \theta \)) if and only if for every sector \( S = S(A;\theta) \), \( SE \) is of property \( \lambda \). \((A,\theta)\) will be called an \( \lambda(1) \)-tangent ray of \( E \) if and only if for every sector \( S(A;\theta) \) which is sufficiently small — in angle and radius — \( SE \) is of property \( \lambda \). \((A,\theta)\) will be called an \( \lambda(2) \)-tangent ray of \( E \) if and only if there exists an infinitesimal sector \( S(A,\theta) \) — infinitesimal in angle and radius — such that \( SE \) is of property \( \lambda \). Obviously, \((A,\theta)\) is an \( \lambda(2) \)-tangent ray of \( E \) if and only if \((A,\theta)\) is not a \( \lambda(1) \)-tangent ray of \( E \) (\( \lambda = \) not \( \lambda \)). The set of all \( \lambda \)-tangent rays of \( E \) at \( A \) will be called the \( \lambda \)-contingent of \( E \) at \( A \) and will be denoted by \( T_\lambda(A,E) \) or \( T_\lambda(A) \), for short. It is evident that \( T_\lambda(A,E) \) is closed for every \( \lambda \).

If for every region \( G \supseteq A, GE \) has property \( \lambda \), then \( A \) is \( \lambda \)-approached by \( E \) (see 2.17). The set of all points that are \( \lambda \)-approached by \( E \) will be called the \( \lambda \)-derivative of \( E \) and will be denoted by \( E_\lambda \). For every set property \( \lambda \), \( E_\lambda \) is a closed set.
3.6. The following theorem on sets is related to a theorem of M. F. Schmeiser [31, §1, Theorem 2'] on the approach of a function to the points of a straight line. However, she restricts the approach to two fixed directions, and does not mention applications to sets.

Theorem. If $L$ is a straight line, $K$ one of the closed half-planes into which $L$ divides the plane, $\mathcal{A}$ a set property such that the sum of two sets of property $\mathcal{A}$ fills no region, and $E$ a set in $K$, then the set of points $A$ of $L$ for which there is a pair of rays $(A, \theta)$, $(A, \emptyset)$ in $K$ which are respectively $\mathcal{A}$-tangent rays of $E$, $E' (= K - E)$ is countable.

Proof. Assume that, for $A \in L$, both sectors $S(A; \sigma_1)$ = $S(A; abh)$ and $S(A; \sigma_2) = S(A; cdk)$ contain interior points of $K$. Let $H_{\sigma_1\sigma_2}$ denote the set of points $A \in L$ for which both $S(A; \sigma_1)E$ and $S(A; \sigma_2)E'$ are of property $\mathcal{A}$ . For two points $A$, $B$ in $L$ and sufficiently close together, either the pair $S(A; \sigma_1)$, $S(B; \sigma_2)$ or the pair $S(A; \sigma_2)$, $S(B; \sigma_1)$ have as product a region. Thus, since the sum of two sets of property $\mathcal{A}$ fills no region, at least one of the points $A$, $B$ does not belong to $H_{\sigma_1\sigma_2}$. $H_{\sigma_1\sigma_2}$ is therefore an isolated set and hence countable. Let $H = \sum H_{\sigma_1\sigma_2}$, where in the summation $\sigma_1$, $\sigma_2$ range over all rational subscripts of the type above. Hence, $H$ is also countable. If $A$ is a point of the exceptional set of the theorem,
there exist rational subscripts $\sigma_1, \sigma_2$ for which $A \in H_{\sigma_1\sigma_2}$.

The theorem follows.

3.7. If the set property $\lambda$ is descending, $(A, \vartheta)$ is an $\lambda(1)$-tangent ray of $E$ if and only if there exists a sector $S(A; \vartheta)$ with $SE$ of property $\lambda$; thus, in this case, $(A, \vartheta)$ is not an $\lambda(1)$-tangent ray of $E : = (A, \vartheta)$ is an $\lambda$-tangent ray of $E$. Now let $\lambda$ be both descending and such that the sum of two sets of property $\lambda$ fills no region. In this case, a set of property $\lambda$ may be regarded as "meager" in a certain general sense. Consequently, if there exists a sector $S(A; \vartheta)$ with $SE$ of property $\lambda$, we shall say that $E$ has an $\lambda$-void at $A$ in direction $\vartheta$. Thus the direct part of the following result is a corollary of 3.6.

If $L$ is a straight line, $K$ one of the closed half-planes into which $L$ divides the plane, $\lambda$ a set property which is descending and such that the sum of two sets of property $\lambda$ fills no region, and $E$ a set in $K$, then the set $H (= H(E, \lambda))$ of points $A$ in $L$ for which there is a pair of directions $\vartheta, \varphi$ in $K$ such that $E$ and $E$ have $\lambda$-voids at $A$ respectively in directions $\vartheta, \varphi$ is countable.

Conversely, if $H$ is a countable set in $L$, and $\lambda$ is a set property such that in every region there is a subset of property $\lambda$ and no set of property $\lambda$ fills a region, then there exists a set $E$ in $K$ such that $H(E, \lambda) = H$.
Proof of converse. At each point \( A \in H \), construct a circle \( C(A) \) tangent to \( L \) at \( A \) and lying in \( K \), and such that no two circles \( C(A) \) overlap. For each \( A \in H \), construct two non-overlapping sectors \( S_1 \) and \( S_2 \) with vertices at \( A \) and lying in \( C(A) \). In each sector \( S_1 \), choose a set \( E_1 \) of property \( \mathcal{A} \), and in each \( S_2 \), choose a set \( E_2 \) so that \( E_2 = S_2 - E_2 \) has property \( \mathcal{B} \). Let \( E \) consist of the sum of all the sets \( E_1 \) and \( E_2 \) plus all the points of \( K \) which are not in any sector \( S_1 \) or \( S_2 \). Obviously, if \( A \in H \), \( A \in H(E, \mathcal{A}) \). If \( A \not\in H \), and if \( \Theta \) is in \( K \), then every \( S(A; \Theta) \) contains a region every point of which is in \( E \). Hence \( SE \) is not of property \( \mathcal{A} \). Therefore for no \( \Theta \) in \( K \) does \( E \) have an \( \mathcal{A} \)-void at \( A \) in direction \( \Theta \).

3.8. We may restate the direct part of the result of 3.7 as follows:

If \( L \) is a straight line, \( K \) one of the closed half-planes into which \( L \) divides the plane, \( \mathcal{A} \) a set property which is descending and such that the sum of two sets of property \( \mathcal{A} \) fills no region, and \( E \) a set in \( K \), then for every point \( A \) of \( L \) not belonging to an exceptional, countable set, the \( \mathcal{A} \)-contingent of either \( E \) or \( \overline{E} \) is the half-plane \( K \).

Special cases of this result are obtained by letting \( E^\mathcal{A} := E \) is (a) exhaustible, (b) of measure 0, and (c) of cardinal number \( < c \), the cardinal of the continuum.
§9. A Generalization: Pairs of Sector Properties

3.9. We notice that, for a set property \( \lambda \), an \( \lambda \)-tangent ray of \( E \) is defined in terms of the following sector property \( \mu \):

\[ S^\mu := \text{SE is of property } \lambda . \]

Specifically, \((A, \theta)\) is an \( \lambda \)-tangent ray of \( E := \) every sector \( S(A; \theta) \) has property \( \mu \).

Toward generalization, let \( \mu \) be an arbitrary sector property. In analogy with the definitions of 3.5, we define the following properties of a ray \((A, \theta)\) in terms of a sector property \( \mu \):

1. \((A, \theta)\) is a \( \mu \)-ray :=: every sector \( S(A; \theta) \) has property \( \mu \);

2. \((A, \theta)\) is a \( \mu(1) \)-ray :=: for all sectors \( S(A; \theta) \) which are sufficiently small (both in angle and radius), \( S(A; \theta) \) has property \( \mu \);

3. \((A, \theta)\) is a \( \mu(2) \)-ray :=: there exists a sector \( S(A; \theta) \) which is infinitesimal (both in angle and radius) and which has property \( \mu \).

\( \bar{\mu} \) will denote "not \( \mu \)." It is evident that \((A, \theta)\) is a \( \bar{\mu}(2) \)-ray (\( \bar{\mu}(1) \)-ray) if and only if \((A, \theta)\) is not a \( \mu(1) \)-ray (\( \mu(2) \)-ray). The set of all \( \mu \)-rays at \( A \) will be called the \( \mu \)-contingent at \( A \) and will be denoted by \( T_\mu(A) \). It follows immediately that for every sector property \( \mu \) and point \( A \), \( T_\mu(A) \) is closed.
3.10. The exceptional points of the result of 3.6 were defined in terms of the pair of sector properties \( \mu, \nu \):

\[ S^\mu := SE \text{ is of property } \mu \]

and

\[ S^\nu := SE \text{ is of property } \nu \]

It is suggested that we replace these special sector properties by a pair of arbitrary sector properties \( \mu, \nu \).

(For our present considerations, we may restrict the range of the sectors \( S \) to those having vertex on a line \( L \).) Let \( L \) be a straight line, and let \( K \) be one of the closed half-planes into which \( L \) divides the plane. Let \( A \in L \) be a point for which there is a pair of rays \( (A, \theta), (A, \varphi) \) in \( K \) which are respectively a \( \mu(1) \)-ray, \( \nu(1) \)-ray. Again, as in the proof of 3.6, \( A \in H_{\sigma_1 \sigma_2} \) for some \( \sigma_1 \) and \( \sigma_2 \) which are rational and contain directions in \( K^0 \), where \( H_{\sigma_1 \sigma_2} \) means the set of points \( A \in L \) for which \( S(A; \sigma_1), S(A; \sigma_2) \) are respectively of property \( \mu, \nu \).

Without some restriction on \( \mu, \nu \), it is obvious that \( H_{\sigma_1 \sigma_2} \) is unconditioned. Consequently, we require the pair \( \mu, \nu \), like the special properties of 3.6, to be mutually non-overlapping in the sense that if \( S_1^\mu \) and \( S_2^\nu \) then \( S_1 \) and \( S_2 \) are non-overlapping. With this condition on the pair \( \mu, \nu \), it is evident that \( H_{\sigma_1 \sigma_2} \) is isolated and therefore countable. Again, \( H = \sum H_{\sigma_1 \sigma_2} \), where in
the summation the subscripts are rational and contain
directions in $K^0$, is countable. We have thus proved the
direct part of the theorem:

**Theorem (generalization).** If $L$ is a straight line,
$K$ one of the closed half-planes into which $L$ divides the
plane, and $\mu, \nu$ a pair of sector properties which are
mutually non-overlapping, then the set $H (= H(\mu, \nu))$ of
points $A$ of $L$ for which there are two rays $(A, \theta), (A, \phi)$
in $K$ which are respectively a $\mu(1)$-ray, $\nu(1)$-ray is count­
able. Conversely, if $H$ is a countable set in $L$, there
exist two mutually non-overlapping sector properties $\mu, \nu$
such that $H(\mu, \nu) = H$.

**Proof of converse.** The converse may be verified by
means of an example. At each point $A \in H$, draw a circle
$C(A)$ in $K$ tangent to $L$ at $A$, and such that no two circles
overlap. For each $A \in H$, choose two non-overlapping sec­
tors $S_1$ and $S_2$ with vertices at $A$ and lying in $C(A)$.
Assign exclusively to those sectors $S(A; \sigma)$ contained in
$S_1$ the property $\mu$ and exclusively to those sectors $S(A; \sigma)$
contained in $S_2$ the property $\nu$. For $A \notin H$, let every
sector $S(A; \sigma)$ have both property $\overline{\mu}$ and $\overline{\nu}$. Obviously, the
two properties $\mu, \nu$ are mutually non-overlapping, and
$H = H(\mu, \nu)$.

3.11. Since $(A, \theta)$ is a $\overline{\mu}(2)$-ray if it is not a $\mu(1)$-
ray, we may restate the direct part of 3.10 as follows:
If $L$ is a straight line, $K$ one of the closed half-planes into which $L$ divides the plane, and $\mu, \nu$ a pair of mutually non-overlapping sector properties, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either every ray $(A, \theta)$ in $K$ is a $\overline{\mu}(2)$-ray or every ray $(A, \theta)$ in $K$ is a $\overline{\nu}(2)$-ray.

3.12. We say the sector property $\mu$ is descending if $S_1^{\mu}$ and $S_2 \subseteq S_1$ imply $S_2^{\mu}$. $\mu$ is descending if and only if $\overline{\mu}$ is ascending, the meaning of the latter term being evident. Hence, if $\mu$ is descending, $(A, \theta)$ is a $\overline{\mu}(2)$-ray if and only if it is a $\overline{\mu}$-ray. Thus if $\mu$ and $\nu$ are both descending, in the result of 3.11 we may replace the terms $\overline{\mu}(2)$-ray, $\overline{\nu}(2)$-ray respectively by the terms $\overline{\mu}$-ray, $\overline{\nu}$-ray.

3.13. As we have seen (see 3.9, 3.10), particular pairs of mutually non-overlapping sector properties give rise to the results 3.6, 3.7. We now list some additional pairs $\mu, \nu$ of mutually non-overlapping sector properties defined in terms of a set $E$, each of which gives rise to a special case of 3.10 or 3.11.

(a) Let $\alpha_1$, $\alpha_2$ be two set properties such that a set of property $\alpha_1$ plus a set of property $\alpha_2$ fills no region. $S^{\mu} := SE$ is of property $\alpha_1$; $S^{\nu} := SE$ is of property $\alpha_2$.

(b) In (a), add the condition that both $\alpha_1$ and $\alpha_2$ are descending.
(c) $S^\mu := E$ is dense in $S$; $S^\nu := SE$ is non-dense.

(d) $S^\mu := SE$ is of measure 0; $S^\nu := SE$ is non-dense.

(e) $S^\mu := SE$ is measurable; $S^\nu := SE$ is non-measurable.

(f) $S^\mu := E$ is homogeneously inexhaustible in $S$; $S^\nu := SE$ is exhaustible.

3.14. By applying 3.11 to the pair of sector properties defined in (f), 3.13, we obtain the following particular result. (We denote by "e" the set property "exhaustible."

If $L$ is a straight line, $K$ one of the closed half-planes into which $L$ divides the plane, and $E$ a set in $K$, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either the $e$-contingent of $E$ at $A$ is the half-plane $K$, or else every sector $S(A; \theta)$ with $\theta$ in $K$ contains a region $G$ on which $\overline{E}$ is residual.

3.15. A variant of the theorem of 3.10 may be stated in terms of two sector properties $\mu, \nu$ which are mutually non-intersecting in the sense that $S_1^\mu$ and $S_2^\nu$ imply that $S_1$ and $S_2$ are non-intersecting. $S_1$ and $S_2$ are intersecting means that both sides of $S_1$ intersect both sides of $S_2$. The proof of the theorem is essentially the same as that of 3.10 and is omitted; we must, however, restrict $K$ to be open and $\theta, \emptyset$ to be distinct.
Theorem. If $L$ is a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, and $\mu$, $\nu$ two mutually non-intersecting sector properties, then the set of points $A$ of $L$ for which there are two distinct rays $(A, \theta)$, $(A, \phi)$ in $K^0$ which are respectively a $\mu(1)$-ray, $\nu(1)$-ray is countable.

3.16. The result of 3.15 may be stated in the following alternative form:

If $L$ is a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, and $\mu$, $\nu$ two mutually non-intersecting sector properties, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either (1) for every direction $\theta$ in $K^0$, $(A, \theta)$ is a $\mu(2)$-ray, or (2) for every direction $\theta$ in $K^0$, $(A, \theta)$ is a $\nu(2)$-ray, or (3) for every direction $\theta$ in $K^0$ -- with the possible exception of one particular direction $\theta^*_A$, variable with $A$ -- $(A, \theta)$ is both a $\mu(2)$-ray and a $\nu(2)$-ray.

Proof. (We actually show the equivalence of this result and the result of 3.15.) First, it is easily verified that for every point $A$ of $L$, either at least one of the cases (1), (2), (3) of the above result holds, or else we have case (4): there are two distinct rays $(A, \theta)$, $(A, \phi)$ in $K^0$ which are respectively a $\mu(1)$-ray, $\nu(1)$-ray. Thus 3.15 implies 3.16. Second, case (4) has no point in common with any of the cases (1), (2), (3). Using these
two facts (or since a subset of a countable set is countable, this second fact alone), 3.16 implies 3.15.

Remark 1. If \( \mu \) and \( \nu \) are such that no ray is both a \( \mu(1) \)-ray and a \( \nu(1) \)-ray, case (3) of the above result may be omitted.

Remark 2. The above results involve certain properties of a ray \((A, \theta)\) defined in terms of a sector property. At times it may be convenient to speak of a general ray property: \((A, \theta)\) is a \( U \)-ray (\( U \)-ray) \( := \) \((A, \theta)\) has property \( U \) (\( U = \text{not } U \)). The idea of the equivalence of 3.15 and 3.16 can be generalized as follows:

If \( L \) is a straight line, \( K^o \) one of the open half-planes into which \( L \) divides the plane, and \( U, V \) two ray properties, for each point \( A \) of \( L \) exactly one of the following conditions I, II holds:

I. There are two distinct rays \((A, \theta), (A, \emptyset)\) in \( K^o \) which are respectively a \( U \)-ray, \( V \)-ray.

II. Either (1) for every direction \( \theta \) in \( K^o \), \((A, \theta)\) is a \( U \)-ray, or (2) for every direction \( \theta \) in \( K^o \), \((A, \theta)\) is a \( V \)-ray, or (3) for every direction \( \theta \) in \( K^o \) -- with the possible exception of one particular direction \( \theta_A \), variable with \( A \) -- \((A, \theta)\) is both a \( U \)-ray and a \( V \)-ray.

Again, if \( U \) and \( V \) are such that no ray is both a \( U \)-ray and a \( V \)-ray, case (3) of II may be omitted.
3.17. We list some examples of pairs \( \mu, \nu \) of mutually non-intersecting sector properties defined in terms of a set \( E \), each of which gives a special case of 3.15 or 3.16.

(a) \( S^H := \) at least one radial line of \( S \) consists exclusively of points of \( E \); \( \nu \) is similarly defined with respect to \( \overline{E} \).

(b) \( S^L := \) \( SE \) contains an arcwise connected set whose frontier contains the vertex of \( S \) and at least one point on the circular boundary of \( S \); \( \nu \) is similarly defined with respect to \( \overline{E} \).

(c) \( S^K := \) \( EcS \) contains a continuum, i.e., a compact, connected set, which contains the vertex and a point on the circular boundary of \( S \); \( \nu \) is similarly defined with respect to \( \overline{E} \).

Suppose that the \( \mu, \nu \) of (c) are not mutually non-intersecting, i.e., that two intersecting sectors \( S_1, S_2 \) exist which have the properties \( \mu, \nu \) respectively, and let \( K_1, K_2 \) denote the continua which are contained in \( E, \overline{E} \). Using the fact that for every \( \varepsilon > 0 \) any two points of a connected set can be joined by an \( \varepsilon \)-chain, there may be defined a convergent sequence \( \{A_n\} \) in \( K_1 \) and a sequence \( \{B_n\} \) in \( K_2 \) such that \( d(A_n, B_n) < 1/n \). Denote by \( A \) the common limit of these two sequences. Since \( K_1 \)
and $K_2$ are compact, it follows that $A$ is contained in both $K_1$ and $K_2$ and hence in both $E$ and $\bar{E}$, a contradiction.

**Remark.** It may be shown by means of an example that if in (c) the term "continuum" is replaced by "connected set," the resulting pair of sector properties $\mu$, $\nu$ need not be mutually non-intersecting.

(d) Let $\mathcal{A}_1$, $\mathcal{A}_2$ be set properties such that a set of property $\mathcal{A}_1$ plus a set of property $\mathcal{A}_2$ fills no region. $S^\mu :=$ there is a domain (= connected, open set) $G$ in $S$ whose frontier contains the vertex of $S$ and at least one point on the circular boundary of $S$ and is such that $GE$ is of property $\mathcal{A}_1$; $\nu$ is similarly defined with respect to $\mathcal{A}_2$ and $E$.

To prove that the $\mu$, $\nu$ of (d) are mutually non-intersecting, it will suffice to note that if two intersecting sectors have properties $\mu$, $\nu$ respectively, and $G_1$, $G_2$ are the specified domains in $S_1$, $S_2$, then $G_1G_2$ is non-empty and thus contains a region. This follows from the theorem that a domain in the plane is arcwise connected.

3.18. We make the definition: $(A, \theta)$ is a U-ray (V-ray) := an open, initial segment of $(A, \theta)$ is contained in $E$ ($\bar{E}$). (An open, initial segment of $(A, \theta)$ is an open interval of $(A, \theta)$ with $A$ as one end-point.) Letting the sector properties $\mu$, $\nu$ be defined as in (a), 3.17, it is
evident that every U-ray (V-ray) is a μ(1)-ray (ν(1)-ray). Applying 3.15, we obtain:

If L is a straight line, K^0 one of the open half-planes into which L divides the plane, and E a set in K^0, then the set of points A of L for which there are two distinct rays (A,θ), (A,φ) in K^0 which are respectively a U-ray, V-ray is countable.

Since no ray is both a U-ray and a V-ray, we can apply Remark 2 of 3.16 to obtain the following equivalent result of H. Blumberg [4, Corollary VII]:

If L is a straight line, K^0 one of the open half-planes into which L divides the plane, and E a set in K^0, then every point A of L not belonging to an exceptional, countable set is approachable by E along every ray (A,θ) in K^0, or A is so approachable by E, i.e., (using the notation of 3.14) either T'(A,E) or T'(A,E) equals K^0.

3.19. Omitting the statement of other special cases of 3.15 or 3.16, we state the basic lemma needed for a detailed proof of 3.15, and which will be used below.

If L is a straight line, K^0 one of the open half-planes into which L divides the plane, and μ, ν two mutually non-intersecting sector properties, then the set of points A of L for which there exist two non-overlapping, non-adjacent sectors S(A;μ_1), S(A;μ_2) in K^0 with rational
dimensions and which are respectively of property $\mu$, $\nu$ is countable.

3.20. Let $L$ be a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, and $E$ a set in $K^0$. Let $H$ denote the set of points $A$ of $L$ which have the following double character: $A$ is the common terminal point of two open, simple arcs in $K^0$ of determinate directions $\theta$, $\phi$ in $K^0$ at $A$, composed, the one exclusively of points of $E$, the other exclusively of points of $\overline{E}$. It then follows that there exist two sectors $S(A;\theta)$, $S(A;\phi)$ of rational dimensions in $K^0$, non-overlapping and non-adjacent, and respectively of property $\mu$, $\nu$ where $\mu$, $\nu$ are defined as in (b), 3.17. Applying 3.19, it follows that $H$ is countable. Further, applying 3.16, Remark 2, we obtain a result which is somewhat stronger than one of H. Blumberg [4, Theorem IV].

If $L$ is a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, and $E$ a set in $K^0$, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either (1) every simple arc in $K^0$ with determinate direction at $A$ in $K^0$ contains points of $E$, or (2) every such arc contains points of $\overline{E}$, or (3) every such arc -- with the possible exception of arcs having one particular direction $\theta_A$, variable with $A$ -- contains points of both $E$ and $\overline{E}$. 
3.21. Toward generalizing the result of 3.20, again let \( \mu, \nu \) denote the sector properties of (b), 3.17. We note that if \( A \in L \) is a terminal point of a simple arc contained in \( E \) of determinate direction \( \theta \) at \( A \) in \( K^\circ \), then \( (A,\theta) \) is a \( \mu(2) \)-ray. This, in turn, implies that there exists an infinitesimal sector of rational dimensions \( S(A;\theta) \) in \( K^\circ \) of property \( \mu \), since \( \mu \) has the two following properties: (1) \( \mu \) is **angularly ascending** in the sense that \( S(A;ab) \) has property \( \mu \) and \( \angle cd \supset \angle ab \) imply that \( S(A;cd\theta) \) has property \( \mu \); (2) \( \mu \) is **radially descending** in the sense that \( S(A;ab\theta) \) has property \( \mu \) and \( k < h \) imply that \( S(A;abk) \) has property \( \mu \).

These remarks indicate how 3.19 may be applied to obtain the following result, which may be considered as a direct generalization of 3.20, and, like 3.15, as a variant of 3.10.

**Theorem.** If \( L \) is a straight line, \( K^\circ \) one of the open half-planes into which \( L \) divides the plane, and \( \mu, \nu \) two mutually non-intersecting sector properties each of which is angularly ascending and radially descending, then the set of points \( A \) of \( L \) at which there are two distinct rays \( (A,\theta), (A,\phi) \) in \( K^\circ \) which are respectively a \( \mu(2) \)-ray, \( \nu(2) \)-ray is countable.

3.22. The result of 3.21 may be stated in the following alternative and equivalent form (cf. 3.16):
If L is a straight line, $K^0$ one of the open half-planes into which L divides the plane, and $\mu$, $\nu$ two mutually non-intersecting sector properties each of which is angularly ascending and radially descending, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either (1) for every direction $\theta$ in $K^0$, $(A, \theta)$ is a $\mu(1)$-ray, or (2) for every direction $\theta$ in $K^0$, $(A, \theta)$ is a $\nu(1)$-ray, or (3) for every direction $\theta$ in $K^0$ -- with the possible exception of one particular direction $\theta^*$, variable with $A$ -- $(A, \theta)$ is both a $\mu(1)$-ray and a $\nu(1)$-ray.

Remark. If $\mu$, $\nu$ are defined as in (b), 3.17, then whenever $(A, \theta)$ is a $\mu(1)$-ray ($\nu(1)$-ray), it follows that every simple arc in $K^0$ with determinate direction $\theta$ at $A$ contains points of $E$ ($E$). Thus for this choice of $\mu$, $\nu$, 3.20 is a direct corollary of 3.22, and 3.22 may be considered a direct generalization of 3.20.

3.23. Examples of pairs of mutually non-intersecting sector properties which are angularly ascending and radially descending are given in (a), (b), and (c), 3.17.

As in (d), 3.17, let $\mathcal{A}_1$, $\mathcal{A}_2$ be set properties such that a set of property $\mathcal{A}_1$ plus a set of property $\mathcal{A}_2$ fills no region. Let $S^\mu :=$ there is a domain $G$ in $S$ whose frontier contains the vertex of $S$ and at least one point on the circular boundary of $S$ and is such that $GE$ is a subset of a set of property $\mathcal{A}_1$; let $\nu$ be defined
similarly with respect to $\lambda_2$ and $\mathcal{E}$. It follows that $\mu$, $\nu$ are mutually non-intersecting and angularly ascending and radially descending. Now suppose that $A \in \mathcal{L}$ is the common terminal point of two open, simple arcs in $K^0$ of determinate and distinct directions $\theta$, $\emptyset$ at $A$ in $K^0$, the one contained in a region $G$ with $GE$ a subset of a set of property $\wp_1$, the other contained in a region $G$ with $GE$ a subset of a set of property $\wp_2$. It follows that $(A,\theta)$, $(A,\emptyset)$ are respectively a $\mu(2)$-ray, $\nu(2)$-ray. We apply 3.22 to obtain the result:

If $L$ is a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, $\lambda_1$, $\lambda_2$ two set properties such that the sum of a set of property $\lambda_1$ and a set of property $\lambda_2$ fills no region, and $E$ a set in $K^0$, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either (1) for every region $G$ containing a simple arc in $K^0$ with determinate direction at $A$ in $K^0$, $GE$ is of property $\lambda_1$, or (2) for every such region, $GE$ is of property $\lambda_2$, or (3) for every such region -- with the possible exception of regions containing arcs having one particular direction $\theta_A$, variable with $A$ -- $GE$ is of property $\lambda_1$ and $GE$ is of property $\lambda_2$. 


§10. A Type of Metric Approach

3.24. Definition. E has (exterior) metric density $k$ at $A$ via direction $\theta := \lim m_g[S(A;abh)E]/m(S(A;abh)]$ exists and equals $k$ as $h \to 0$ and $\angle ab \to 0$, $\theta \in \angle ab$, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $\theta \in \angle ab < \delta$ and $h < \delta$, $\left| m_g[S(A;abh)E]/m(S(A;abh)] - k \right| < \varepsilon$.

If in the above definition the term "lim" is replaced by "lim sup" and "lim inf" we say respectively that $E$ has upper, lower (exterior) metric density $k$ at $A$ via direction $\theta$. Using the notation of 3.5, we say that $(A, \theta)$ is a $Z(1)$-tangent ray ($Z = "of measure 0") if for every sufficiently small sector $S = S(A;\theta)$, $SE$ is of measure 0.

3.25. The following result is related to a theorem of Schmeiser [31, §1, Theorem 2"] on functions. Her theorem, however, is restricted to two fixed directions of approach.

Theorem. If $L$ is a given straight line, $K$ one of the half-planes into which $L$ divides the plane, and $E$ a set in $K^0$, then the set $H$ of points $A$ of $L$ for which there exist two directions $\theta, \phi$ in $K^0$ such that $E$ has metric density 1 at $A$ via direction $\theta$ and $(A, \phi)$ is a $Z(1)$-tangent ray is countable.

Proof. Let $A \in H$, where $\theta$ and $\phi$ have the properties stated in the theorem. Assume, say, that $K^0 = S(A;0/\pi)$
and $\theta < \phi$. Then there exists a sector $S(A;\sigma_2) = S(A;cdk)$ in $K^\circ$, where $\sigma_2 = cdk$ is rational, $\phi \in \angle cd$, $\theta \notin \angle cd$, and $m[S(A;\sigma_2)E] = 0$. If $P$ is to the right of $A$, the rays $(A,\theta)$ and $(P,c)$ will intersect in a point $R$ and $(A,\theta)$ and $(P,d)$ in a point $R'$. Let $p' = (AR'/AR)^2$, and let $p$ denote a rational number such that $1 > p > (1 + p')/2$. Then $\sigma_1 = abh$ may be chosen rational with $\theta \in \angle ab$ and such that $\angle ab$ and $h$ are so small that (1) for every sub-sector $S(A;\theta)$ of $S(A;\sigma_1) = S(A;abh)$ the relative exterior measure of $E > p$ and (2) if $P$ is close enough to $A$ and on the right of $A$, if $Q$ is the quadrilateral of intersection of $S(A;\sigma_1)$ and $S(P;\sigma_2)$, and if $h_1$ is the smallest number for which $S(A;abh_1)$ contains $Q$, then $m(Q)/m[S(A;abh_1)] > (1 - p')/2 > 1 - p$, i.e., if $Q$ is the complement of $Q$ with respect to $S(A;abh_1)$, the ratio $m(Q)/m[S(A;abh_1)] < (1 + p')/2 < p$. For, if $\angle ab$ is fixed, $m(Q)/m[S(A;abh_1)]$ is independent of $P$, and this ratio approaches $1 - p'$ as $\angle ab$ containing $\theta$ goes to 0.

$A$ is therefore a point of $H_{\sigma_1\sigma_2p} = H_{abhcdkp}$, by which we mean specifically that (1) $S(A;\sigma_1)$ and $S(A;\sigma_2)$, both in $K^\circ$, are non-overlapping, (2) $0 < p < 1$, (3) $m[S(A;\sigma_2)E] = 0$, (4) if $h_1 \leq h$, the relative exterior measure of $E$ in $S(A;abh_1) > p$, and (5) if $P$ is close enough to $A$ and on the proper side of $A$ so that $S(A;abh)$ and $S(P;cdk)$ intersect in a quadrilateral $Q$, and if
$S(A_{abhp})$, differing from $S(A_{abh})$ only in its radius, is the smallest subsector of $S(A_{abh})$ containing $Q$, then $m(Q)/m[S(A_{abhp})] > 1 - p$. As we have seen,

$$H \subseteq \sum H_{abhp},$$

the seven subscripts ranging independently over the admissible rational values as described above.

It thus suffices to prove that every set $H_{o\theta^2p}$ = $H_{abhp}$ is voidal and therefore countable. But this follows immediately since no point $A$ of $H_{o\theta^2p}$ can be a limit point of this set from both sides. For if $P \in H_{o\theta^2p}$ is close enough to $A$ and on the proper side of $A$, $m(QE) = 0$, and hence there exists an $h_1 < h$ such that the relative exterior measure of $E$ in $S(A_{abh})$ is $> p$ and also $< p$, a contradiction.

3.26. Remark. If the (exterior) metric density of $E$ at $A$ exists and is $0, 1$, then if the (exterior) metric density of $E$ at $A$ via direction $\theta$ exists, it must be respectively $0, 1$. A partial converse of this remark also holds:

If for every direction $\theta$ not belonging to a set of directions of measure 0, the metric density of $E$ at $A$ via direction $\theta$ is 0, then the restricted metric density of $E$ at $A$ is 0, i.e., if $C(A;h)$ is a circle of center $A$ and radius $h$, then $m_\theta[C(A;h)E]/m[C(A;h)] \rightarrow 0$ as $h \rightarrow 0$. 
Proof. Choose $\varepsilon > 0$. Then for every direction $\theta$ in $I$: $0 < \theta < 2\pi$ not contained in a set $Z$ of measure 0, there exists a sector $S(A;\alpha\beta h)$ with $\theta \in \alpha\beta$ which is such that the relative measure of $E$ in every subsector $S(A;\theta)$ is less than $\varepsilon$. Thus, for every such $\theta$, there exists a fixed $h$ and an infinitesimal $\alpha\beta$ containing $\theta$ such that the relative measure of $E$ in the corresponding sector $< \varepsilon$.

By the Vitali covering theorem, there exists a finite set of these sectors $S(A;\alpha\beta h)$ which are non-overlapping and for the corresponding set $T$ of angles $\alpha\beta$ considered as intervals $(a,b)$, we have $m_\varepsilon(T) < \varepsilon$. Let $h_1 > 0$ be the minimum of the numbers $h$ corresponding to this finite set of sectors. Replacing $h$ with $h_1$, the relative exterior measure of $E$ in each of these sectors $S(A;\alpha\beta h_1) < \varepsilon$.

The last statement also holds if $h_1$ is replaced by a smaller positive number. Thus if $h_1 > h_2 > 0$, and if $C(A;h_2)$ is a circle of center $A$ and radius $h_2$, the relative measure of $E$ in $C < 2\varepsilon$. The theorem follows.

§11. On the Contingent and Approach to a Line

3.27. If $L$ is a given straight line, $K$ one of the closed half-planes into which $L$ divides the plane, and $E$ a set in $K^0$, then for every point $A$ of $L$ not belonging to an exceptional, exhaustible set $H$, either $A \notin E'$ or the contingent $T(A,E)$ of $E$ at $A$ is the half-plane $K$. 
Proof. If \( A \) is a point of the exceptional set \( H \), \( A \) is a limit point of \( E \) and there exists in \( K^0 \) a sector \( S(A;\sigma) = S(A;\emptyset h) \) which is void of points of \( E \). It is evident that \( \sigma \) may be chosen rational. For \( \sigma \) fixed, let \( H_\sigma \) denote the set of points \( A \) of the double character just described. Since \( H = \sum H_\sigma \), where in the summation \( \sigma \) is rational and \( S(A;\sigma) \) is in \( K^0 \), it will suffice to prove that, for \( \sigma \) fixed, \( H_\sigma \) is non-dense.

Suppose otherwise that there exists a point \( A \in H_\sigma \) which is densely approached by \( H_\sigma \). Then there exists in \( L \) an open interval \( I \) of center \( A \) and a countable set \( R \) dense on \( I \) such that, for every point \( P \in R \), \( S(P;\sigma) \) is void of points of \( E \). Now, as is readily seen, \( \sum S(P;\sigma) \), where in the summation \( P \) varies over \( R \), contains an open parallelogram \( Q \) in \( K^0 \) of base \( I \) and with sides adjacent to \( I \) of length \( h \) and in direction \( \emptyset \). Thus \( Q \) contains no point of \( E \), and therefore \( A \) is not a limit point of \( E \), a contradiction.

3.28. If \( L \) is a given straight line, \( K \) one of the closed half-planes into which \( L \) divides the plane, and \( E \) a set in \( K \), then for every point \( A \) of \( L \) not belonging to an exceptional, voidal set, either \( A \neq E' \) or \( T(A,E) \) contains both directions of the line \( L \).

Proof. The exceptional set \( H \) of the theorem may be taken to consist of the points \( A \) of \( E'L \) for which \( T(A,E) \)
does not contain at least one of the directions of \( L \).

Then if \( A \in H \), there exists an open sector \( S(A;\theta) \) containing no point of \( E \), \( \theta \) being one of the directions of \( L \). It follows that \( S \) contains no point of \( E' \) and hence no point of \( H \). Thus \( H \) is voidal (with respect to \( L \)).

3.29. Theorem. If \( L \) is a given straight line, \( K \) one of the closed half-planes into which \( L \) divides the plane, and \( E \) a set in \( K^0 \), then for every point \( A \) of \( L \) not belonging to an exceptional set \( H \) which is exhaustible and of measure 0, either \( T(A, E) \) contains every direction in \( K \) or else \( T(A, E) \) contains no direction in \( K^0 \).

Proof. By 3.27 it remains only to prove that the exceptional set \( H \) is of measure 0. If \( A \in H \), it is evident that there exist in \( K^0 \) two non-overlapping sectors \( S(A;\sigma_1) = S(A;abh) \) and \( S(A;\sigma_2) = S(A;cdh) \), the first containing a subset of \( E \) which has \( A \) as limit point and the second being void of \( E \). It is evident, too, that \( \sigma_1 \) and \( \sigma_2 \) may both be chosen rational. Conversely, if two such sectors of vertex \( A \in L \) exist, it follows that \( A \in H \).

If for \( \sigma_1 \) and \( \sigma_2 \) fixed, \( H_{\sigma_1\sigma_2} \) denotes the set of points \( A \) of \( L \) for which the above two sectors have respectively the specified properties, it follows that \( H = \sum H_{\sigma_1\sigma_2} \), where in the summation \( \sigma_1 \) and \( \sigma_2 \) are rational and the corresponding sectors are non-overlapping and in \( K^0 \).
Thus, it will suffice to prove that $H^{0\alpha_0\alpha_2}$, for fixed subscripts, is of measure 0.

Suppose, first, that $H^{0\alpha_0\alpha_2}$ is not of measure 0, i.e., that it is of positive exterior measure, and let $A$ be a point of it of exterior metric density 1. We may assume, say, that one direction of $L$ is the 0-direction, $K^0$ is $S(A;0\Pi)$, and $0 < a < b < c < d < \pi$. For $P \in H^{0\alpha_0\alpha_2}$ and sufficiently close to $A$ and to the right of $A$, i.e., of 0-direction from $A$, both sides of $S(A;ab)$ will intersect both sides of $S(P;cd)$, say in points $A_{ac}$, $A_{bc}$, $A_{bd}$, and $A_{ad}$, where $A_{ac}$ denotes the point of intersection of $(A,a)$ and $(P,c)$, and so forth. The quadrilateral $A_{ac}A_{bc}A_{bd}A_{ad}$, denoted by $Q$, is void of $E$. Let $B$ be the point of intersection of the ray $(P,c)$ and the line through $A_{ad}$ which is parallel to $L$. Now $A_{ad}B/AB = 2k > 0$, and, by similar polygons, this ratio is independent of $P$. (The open segment from $A$ to $B$ will be denoted by $(A,B)$ and its length by $AB$. Considering $A$ and $B$ as position vectors, $A + B$ and $A - B$ have the usual meanings.) Now let $P$ have the additional property that for every point $P'$ in $(A,P)$, the relative exterior measure of $H^{0\alpha_0\alpha_2}$ in $(A,P')$ is more than $1 - k$. Now choose, as we may, a point $P_1$ of $H^{0\alpha_0\alpha_2}$ in $(P - 2k(P - A), P - k(P - A))$. Like $Q$ was associated with $P$, there is a quadrilateral $Q_1$ associated with $P_1$ which completely overlaps with $Q$, i.e., $Q + Q_1$ is a quadrilateral.
two sides of which are segments of the radial lines of \( S(A;\sigma_1) \).

In the same way, we may choose \( P_2 \) with respect to \( P_1 \) and associate with \( P_2 \) a quadrilateral \( Q_2 \). In general, we may choose a infinite sequence \( \{P_n\} \) in \((A,P)\) monotonically approaching \( A \) and such that, for every \( n \), \( Q_n \) completely overlaps with \( Q_{n-1} \). In fact, as is readily seen, \( AP_n < (1 - k)^n AP \). It follows that the sum of the quadrilaterals \( Q_n \), each of which is void of \( E \), contains all the points of \( S(A;\sigma_1) \) in a neighborhood of \( A \). \( S(A;\sigma_1) \) is thus void of \( E \) in a neighborhood of \( A \), a contradiction.

**Remark.** The fact that the exceptional set \( H \) of this result is of measure 0 follows from a known fundamental theorem on the contingent of a planar set. In Chapter IV, we use the present result as a lemma in deriving this fundamental theorem.

3.30. Applying 3.28 and 3.29, we obtain the result:

**Theorem.** If \( L \) is a given straight line, \( K \) one of the closed half-planes into which \( L \) divides the plane, and \( E \) a set in \( K \), then for every point \( A \) of \( L \) not belonging to an exceptional set which is exhaustible and of measure 0, either (1) \( A \not\in E' \), or (2) \( T(A,E) \) is the line \( L \), or (3) \( T(A,E) \) is the half-plane \( K \).

3.31. Although we have not obtained a complete answer to the converse questions relating to the results
of 3.27-3.30, we give three partial converses. Let \( K \) denote one of the closed half-planes into which a line \( L \) divides the plane.

1. For every countable set \( H \) in \( L \), there exists a set \( E \) in \( K^0 \) such that, for every \( A \in H \), the contingent \( T(A,E) \) contains one and only one of the directions in the open half-plane \( K^0 \), and, for \( A \in H = L - H \), \( T(A,E) \) is the half-plane \( K \).

Proof. At each point \( A \in H \), construct a circle \( C(A) \) in \( K \) and tangent to \( L \) at \( A \), and such that no two circles overlap. Let \( E \) consist of the set sum of all the points in \( K^0 \) contained in no open circle \( C(A) \) together with the points in one open chord of origin \( A \) in each circle \( C(A) \). Then, for each point \( A \in H \), \( E \) approaches \( A \) via one and only one direction in \( K^0 \), and, for each point \( A \in H \), \( E \) approaches \( A \) via every direction in \( K \).

Letting \( E \) consist exclusively of the points in \( K^0 \) which are contained in no open circle \( C(A) \), we may state the following example.

For every countable set \( H \) in \( L \), there exists a set \( E \) in \( K^0 \) such that, for every \( A \in H \), \( T(A,E) \) is \( L \), and, for \( A \in H \), \( T(A,E) \) is \( K \).

2. There exists a non-denumerable set \( H \) in \( L \) and a set \( E \) in \( K^0 \) such that, for every \( A \in H \), \( T(A,E) \) contains
at least one but does not contain every direction in $K^o$, and, for $A \in \bar{H}$, $T(A,E)$ is the half-plane $K$.

Proof. Mark off the line $L$ into unit intervals, and let the part of $H$ in each such interval be the Cantor-ternary set. $H$ is thus a perfect set of cardinal $\aleph_0$. For each complementary interval $I$ of $H$, construct a solid, open equilateral triangle $T_I$ in $K$ with $I$ as base. Let $E$ consist of the set sum of the points in the triangles $T_I$. For $A \in H$, the open infinite sector of vertex $A$, angle of $\pi/3$ radians, and center direction perpendicular to $L$, contains no point of $E$, while $T(A,E)$ contains every direction $\theta$ in $K$ which makes with one of the opposite directions of $L$ an angle not more than $\pi/6$ radians. For $A \in H$, it is obvious that $T(A,E)$ is $K$.

(3) Let $I$ be a unit interval in $L$, and $\varepsilon$ a positive number less than 1. Then there exist a set $H$ in $I$ of measure more than $1 - \varepsilon$ and a set $E$ in the open half-plane $K^o$ such that for $A \in H$, $T(A,E)$ contains $L$ but does not contain every direction in $K^o$, and, for $A \in \bar{H} = I - H$, $T(A,E)$ is $K$. Further, for almost every point $A$ of $H$, $T(A,E)$ is $L$.

Proof. First construct a non-dense, perfect set $H$ in $I$ whose measure is more than $1 - \varepsilon$. For each complementary interval $J$ of $H$, construct a solid, open equilateral triangle $T_J$ in $K$ with $J$ as base, and let $E$ consist of the set sum of all of these triangles. For $A \in H$, it follows -- as
in the construction in (2) -- that there is a sector in $K$ of vertex $A$ and angle of at least $\pi/3$ radians which contains no point of $E$, while $T(A,E)$ contains $L$, i.e., the two opposite directions in $L$. Of course, by 3.29, for every point $A \in H$ not belonging to a subset of $H$ which is of measure 0 and exhaustible, $T(A,E)$ is $L$. For $A \in \bar{H}$, $T(A,E)$ is $K$.

**Remark.** It is thus seen by (2) that the exceptional sets of 3.29 and 3.30 need not be countable, and by (3) that the exceptional set of 3.27 need not be of measure 0.

§12. A Generalization: Region Properties

3.32. If $\mu$ is a region property, the point $A$ will be called a $\mu$-point if and only if for every region $G$ containing $A$, $G^H$, i.e., $G$ has property $\mu$. The set of all $\mu$-points will be called the $\mu$-derivative and will be denoted by $D(\mu)$. As has been remarked by H. Blumberg [5, p. 811], for every region property $\mu$, $D(\mu)$ is a closed set. Since we have considered only open sectors, a region property $\mu$ defines a unique sector property $\mu$. Thus in the discussion below, we use the notation of 3.9.

3.33. The following three lemmas will be used to obtain a generalization of 3.30. The first has been noted by H. Blumberg [4, p. 23], the second follows from the first, and the third is easily derived.
(1) If \( \mu \) is a descending, additive region property and \( G \) is a region of property \( \bar{\mu} \), then there exists at least one \( \bar{\mu} \)-point in \( G \).

(2) If \( \mu \) is a descending, additive region property and \( A \) a point, then the \( \bar{\mu} \)-contingent at \( A \) is the contingent of \( D(\bar{\mu}) \) at \( A \), i.e., \( T_{\bar{\mu}}(A) = T(A, D(\bar{\mu})) \).

(3) If \( \mu \) is a region property, then at every \( \bar{\mu} \)-point \( A \) not belonging to an exceptional, countable set, \( A \in D(\bar{\mu}) \) and the \( \bar{\mu} \)-contingent is not empty. If \( \mu \) is a descending region property and the \( \bar{\mu} \)-contingent at \( A \) is not empty, then \( A \) is a \( \bar{\mu} \)-point.

3.34. By (2) and (3) of 3.33 and by applying 3.30 to the set \( E = D(\bar{\mu}) \), we obtain the following generalization of 3.30:

Theorem. If \( L \) is a given straight line, \( K \) one of the closed half-planes into which \( L \) divides the plane, and \( \mu \) a descending, additive region property such that every region containing no point of \( K \) has property \( \mu \), then for every point \( A \) of \( L \) not belonging to an exceptional set which is exhaustible and of measure 0, exactly one of the following cases holds: (1) \( A \not\in D(\bar{\mu}) \); (2) \( T_{\bar{\mu}}(A) \) is the line \( L \); (3) \( T_{\bar{\mu}}(A) \) is the half-plane \( K \).

The other results of §11 may likewise be generalized to region properties. This last result could have been proved mutatis mutandis by the method of proof of 3.30.
3.35. We list several examples of descending, additive region properties $\mu$ such that every region containing no point of the closed half-plane $K$ has property $\mu$, each being defined in terms of an arbitrary set in $K$.

(a) $G^{\mu} := \emptyset E$ is empty. For this property $\mu$, 3.34 becomes the special case 3.30.

(b) Let $\mathcal{C}$ be a descending and countably additive set property, and let $G^{\mu} := \emptyset E$ is of property $\mathcal{C}$.

(c) Let $\mathcal{C}$ be a countably additive set property, and let $G^{\mu} := \emptyset E$ is either for every subregion $G_1$ of $G$, $G_1 E$ has property $\mathcal{C}$ or else $G \subseteq K$.

3.36. Let $G^{\mu}$ be defined as in (b), 3.35. Using the notation of 3.5 and applying 3.34, we obtain the result:

If $L$ is a given straight line, $K$ one of the closed half-planes into which $L$ divides the plane, $\mathcal{C}$ a set property which is descending and countably additive, and $E$ a set in $K$, then for every point $A$ of $L$ not belonging to an exceptional set which is exhaustible and of measure 0, either (1) $A \notin E^L$, or (2) $T^L(A, E)$ is the line $L$, or (3) $T^L(A, E)$ is the half-plane $K$.

3.37. To state a corollary of a result obtainable by applying 3.34 to (c) of 3.35, we supplement the definitions of 3.5 with the following: $E$ h-$\mathcal{C}$-approaches $A$ via direction $\theta$ :=: there exists a sector $S(A, \theta)$ such that, for every region $G$ in $S$, $GE$ has property $\mathcal{C}$.
If L is a straight line, K one of the closed half-planes into which L divides the plane, \( \mathcal{A} \) a countably additive set property, and \( E \) a set in K, then for every point A of L not belonging to an exceptional set which is exhaustible and of measure 0, A is \( h-\mathcal{A} \)-approachable by E either via every direction in \( K^0 \) or via no direction in \( K^0 \).

3.38. Now let \( \mathcal{A} \) be a set property which is descending, countably additive, and such that no set of property \( \mathcal{A} \) contains a region, and let \( E \) be a set in the closed half-plane K. If (i) \( E (= K - E) \) \( h-\mathcal{A} \)-approaches A via \( \emptyset \), then (ii) \( E \) \( h-\overline{\mathcal{A}} \)-approaches A via \( \emptyset \), and (ii), in turn, implies that (iii) \( E \) \( \overline{\mathcal{A}} \)-approaches A via \( \emptyset \). Further, if (ii) holds for every \( \emptyset \in K^0 \), then (iii) holds for every \( \emptyset \in K \). Using 3.36 for both \( E \) and \( E \) and 3.37 for the ascending (and hence countably additive) set property \( \overline{\mathcal{A}} \), we obtain the result:

If L is a straight line, K one of the closed half-planes into which L divides the plane, \( \mathcal{A} \) a set property which is descending, countably additive, and such that no set of property \( \mathcal{A} \) contains a region, and \( E \) a set in K, then for every point A of L not belonging to an exceptional set which is exhaustible and of measure 0, exactly one of the following cases holds:
(1) A is not \( \mathcal{J} \)-approachable by \( E \);
(2) the \( \mathcal{J} \)-contingent of \( E \) at \( A \) is \( L \);
(3) the \( \mathcal{J} \)-contingent of \( E \) at \( A \) is \( K \) but \( A \) is \( h \)-\( \mathcal{J} \)-approachable by \( E \) via no direction;
(4) \( A \) is \( h \)-\( \mathcal{J} \)-approachable by \( E \) via every direction in \( K_0 \) but \( A \) is \( h \)-\( \alpha \)-approachable by \( E \) via no direction;
(5) \( A \) is \( h \)-\( \alpha \)-approachable by \( E \) via every direction in \( K_0 \).

Remarks. This result emphasizes the symmetry of the \( \mathcal{J} \)-approach of \( E \) to the points of \( L \). Case (5) is the strongest kind of \( \mathcal{J} \)-approach of \( E \) at \( A \), and case (4) is stronger than case (3).

Let, for example, \( \alpha \) denote the set property "exhaustible." Then \( h \)-\( \mathcal{J} \)-approach of \( E \) means \( h \)-inexhaustible approach of \( E \), and \( h \)-\( \alpha \)-approach of \( E \) means residual approach of \( E \).

§13. Two Other Theorems

3.39. We now derive two other theorems on the approach of a planar set to a straight line, the application of which would give respectively two theorems that are related to results of Schmeiser (cf. [30, §2] and [31, Theorem 4]) on the approach of a function. The method of proof given here is different from that of Schmeiser, being an adaptation of earlier proofs in this chapter.
For these two theorems we shall assume that the linear set property \( \mathcal{L} \) is descending, countably additive, and such that the empty set is not the only set of property \( \mathcal{L} \). In the first, second theorem respectively \( \mathcal{L} \) is assumed to be invariant under parallel, central projection. It follows that a countable set has property \( \mathcal{L} \). If all the points of a linear set of property \( \mathcal{L} \) are joined to a point not on the line, we say, following Schmeiser, that the pencil of lines thus obtained is of type \( \mathcal{L} \). If \( \mathcal{L} \) is invariant under central projection, any section of a pencil of type \( \mathcal{L} \) by a transversal is a linear set of property \( \mathcal{L} \).

We shall need the following lemma:

If \( \mathcal{L} \) is descending and countably additive, and \( E \) is a (linear) set of property \( \overline{\mathcal{L}} \), then every point of \( E \) not belonging to a set of property \( \mathcal{L} \) is \( \overline{\mathcal{L}} \)-approached by \( E \). If countable sets are of property \( \mathcal{L} \), the set of points of \( E \) \( \overline{\mathcal{L}} \)-approached by \( E \) is non-denumerable.

(1) Theorem. If \( L \) is a straight line, \( K^0 \) one of the open half-planes into which \( L \) divides the plane, \( \theta, \phi \) fixed directions in \( K^0 \), and \( E \) a set in \( K^0 \), then for every point \( A \) of \( L \) not belonging to an exceptional set of property \( \mathcal{L} \), either \( E \) \( \overline{\mathcal{L}} \)-approaches \( A \) along direction \( \theta \) or else \( \overline{E} (= K^0 - E) \) approaches \( A \) along direction \( \phi \).
Proof. Suppose $\emptyset \neq \emptyset$ and let $R$ denote the set of points $A$ of $L$ such that $\overline{E}$ does not approach $A$ along direction $\emptyset$. Then for $A \in R$ there is an initial segment of the ray $(A, \emptyset)$ contained in $E$. Let $R_n$ denote those points of $R$ for which this segment may be taken of length $\geq 1/n$. By the above lemma, every point $A$ of $R_n$ not belonging to a set of property $\mathcal{L}$ is $\mathcal{L}$-approached by $R_n$. By 2.12, with the exception of a countable set, these $\mathcal{L}$-points are $\mathcal{L}$-approached from both sides. Therefore, since a countable set has property $\mathcal{L}$, every point $A$ of $R_n$ not belonging to an exceptional set $H_n$ of property $\mathcal{L}$ is $\mathcal{L}$-approached by $R_n$ from both sides. Since $\mathcal{L}$ is invariant under parallel projection, it follows that if $A$ is in $R_n - H_n$, then $E$ $\mathcal{L}$-approaches $A$ along direction $\emptyset$.

Thus the exceptional set of the theorem is contained in $\sum H_n$, which is a set of property $\mathcal{L}$.

If $\emptyset = \emptyset$, it is easily shown that either, for every point $A$ of $L$, $\overline{E}$ approaches $A$ along direction $\emptyset$, or else every interval, and therefore the whole linear continuum, has property $\mathcal{L}$. Hence, the theorem is also true in this trivial case.

(2) Theorem. If $L$ is a straight line, $K^\circ$ one of the open half-planes into which $L$ divides the plane, and $E$ a set in $K^\circ$, then for every point $A$ of $L$ not belonging to an exceptional, countable set, either $E$ approaches $A$
along every direction in $K^0$ not in a pencil of type $\mathcal{L}$, variable with $A$, or $\overline{E} (= K^0 - E)$ $\mathcal{J}$-approaches $A$ along every direction in $K^0$.

Proof. Let $R$ denote the set of points $A$ of $L$ such that, for some $\theta \in K^0$, $E$ does not $\mathcal{J}$-approach $A$ along $(A, \theta)$. Let $R_{\sigma_1}$, where $\sigma_1 = abh$ and $a, b \in K^0$, denote the set of points $A$ of $R$ for which at least one ray of $S(A; \sigma_1)$ is made up, with the possible exception of a set of property $\mathcal{L}$, of points of $E$. Since $\mathcal{L}$ is descending, $R = \sum R_{\sigma_1}$, where $\sigma_1$ varies over all rational values for which $a, b \in K^0$, $h > 0$. With the possible exception of a countable set $H_{\sigma_1}$, every point of $R_{\sigma_1}$ is approached by $R_{\sigma_1}$ from both sides. Hence, $H = \sum H_{\sigma_1}$ is countable. Let $\angle cd$ be an angle with $c, d \in K^0$ which neither overlaps nor abuts $\angle ab$. If $A \in (R_{\sigma_1} - H_{\sigma_1})$, there is a sequence $\{B_m\}$ of points of $R_{\sigma_1}$ approaching $A$ from the proper side so that, if $m$ is large enough, $S(B_m; \sigma_1)$ will intersect $S(A; cd)$. Thus with the exception of a pencil $T_m$ of type $\mathcal{L}$, every ray of $S(A; cd)$ contains a point of $E S(B_m; \sigma_1)$. Then $T = \sum T_m$ is also a pencil of type $\mathcal{L}$, and, for every direction $\theta$ of $\angle cd$ not in $T$, $E$ approaches $A$ along $\theta$.

If $A$ is an exceptional point of the theorem, then (1) there is a direction $\theta \in K^0$ along which $\overline{E}$ does not $\mathcal{J}$-approach $A$, and (2) there is in $K^0$ a pencil of rays of type $\mathcal{L}$ at $A$ along each of which $E$ does not approach
A. By the above lemma, there is a direction \( \phi \) in \( K^o \) different from \( \theta \) which is \( \mathcal{I} \)-approached by this pencil of rays. Thus, for some rational \( \sigma_1 = abh \) with \( \theta \in \mathcal{I}_{ab} \) and \( a, b \in K^o \), there exists an \( \mathcal{I}_{cd} \) with \( \phi \in \mathcal{I}_{cd} \) and \( c, d \in K^o \) which neither overlaps nor abuts \( \mathcal{I}_{ab} \) and such that (1) \( A \in R_{\sigma_1} \) and (2) \( S(A; cd) \) contains a pencil of rays of type \( \mathcal{I} \) along each of which \( E \) does not approach \( A \). Hence \( A \in H_{\sigma_1} \subseteq H \), and the theorem follows.
CHAPTER IV

ON APPROACH OF ARBITRARY SETS TO THE POINTS OF A PLANE

§14. Preliminary Results

4.1. In this chapter we extend some of the results of Chapter III to approach of arbitrary sets to the points of a plane. We give a simplified derivation of certain fundamental theorems on the contingent of a planar set, and generalize these theorems in various ways.

4.2. Lemma. If $E$ is a planar set such that for every point $A \in E$, $S(A;\sigma)$ ($\sigma$ fixed) contains no point of $E$, then $S(A;\overline{\sigma})$ contains no point of $E$, for every $A \in E$.

Proof. Let $A \in E$ and $P \in S(A;\sigma)$. (If $\sigma = \Theta \phi h$, then $\overline{\sigma} = \Theta \phi h$.) Then $S(P;\sigma)$ contains $A$, and consequently $P \not\in E$.

4.3. Following H. Blumberg [4, p. 11], we shall say a planar set $E$ is sparse if and only if it lies on a countable number of single-valued curves, each monotone in some oblique cartesian coordinate system. This concept is due to W. H. Young, who has designated it by the French word ride' (see [34]).
Lemma. If for every point $A$ in the planar set $E$, $S(A;\sigma)$ ($\sigma$ fixed) contains no point of $E$, then $E$ is sparse.

Proof. Let $\sigma = \theta \phi h$, where we may assume, without loss of generality, that $\theta \phi < \pi$. Divide the plane, chessboard fashion, into a set of congruent rhombi with sides of length $h/2$ by drawing equally spaced parallel lines in directions $\theta$ and $\phi$ respectively. Let $R$ be a particular rhombus of this set. It will suffice to prove that $RE$ is sparse, since the sum of a countable number of sparse sets is again sparse. Form an oblique cartesian coordinate system with $x$-axis, $y$-axis respectively in directions $\theta$, $\phi$. Let $A:(\xi,\gamma)$ be in $RE$. $S(A;\sigma)$ contains no point of $E$ and, a fortiori, no point of $RE$. By 4.2, the same statement applies to $S(A;\bar{\sigma})$. Therefore if $P:(x,y) \in RE$, with $x > \xi$ ($x < \xi$), then direction $AP$ lies in $\theta \phi$ ($\phi \bar{\theta}$). Now if a new oblique cartesian coordinate system is formed with $X$-axis, $Y$-axis respectively in directions $\theta$, $\phi_1$, with $\theta < \phi_1 < \phi$, $A$ is the only point of $RE$ on the line through $A$ and parallel to the $Y$-axis, i.e., in direction $\phi_1$. Further, since $\theta \phi$, $\phi \bar{\theta}$ are contained in $\phi_1 \theta$, $\phi_1 \bar{\theta}$ respectively, direction $AP$ lies in $\phi_1 \theta$ or $\phi_1 \bar{\theta}$. Thus in the $XY$-system, $RE$ lies on the graph of a monotone non-increasing, single-valued function of $X$. The lemma follows.
4.5. Consider now an oblique cartesian coordinate system with x-axis, y-axis respectively in directions $\theta = 0$, and $\phi$, with $0 < \phi < \pi$. Now suppose that the curve $C: y = f(x)$ is monotone non-increasing (and single-valued) with respect to the xy-system, and let $P_1: (x_1, y_1)$, $P_2: (x_2, y_2)$ be points of $C$ with $x_2 > x_1$. Take $0 < \phi_1 < \phi$, and form a new oblique coordinate system with X-axis, Y-axis respectively in directions $0$, $\phi_1$. Let $P_1 = (X_1, Y_1)$ and $P_2 = (X_2, Y_2)$ with respect to the new system. It then readily follows that if $X_1 \neq X_2$

$$|Y_2 - Y_1| < \frac{\sin \phi}{\sin (\phi - \phi_1)} |X_2 - X_1|.$$ 

Thus with respect to the XY-system, $C: Y = F(X)$ is monotone non-increasing and satisfies a Lipschitz condition, i.e., there exists a positive $k = \frac{\sin \phi}{\sin (\phi - \phi_1)}$ such that for every pair $X_1 \neq X_2$,

$$|f(X_2) - f(X_1)| < k |X_2 - X_1|.$$

(If the X-axis, Y-axis are taken in directions $\theta_1$, $\phi_1$ where $0 < \theta_1 < \phi_1 < \phi$, then

$$k = \frac{\sin (\phi - \theta_1)}{\sin (\phi - \phi_1)}$$

and $Y = F(X)$ has the additional property of being monotone decreasing in the strict sense.)

If $C$ is monotone non-decreasing with respect to the xy-system, $\phi < \phi_1 < \pi$, and $X_1 \neq X_2$, then

$$|X_2 - X_1| < \frac{\sin \phi}{\sin (\phi_1 - \phi)} |X_2 - X_1|.$$ 

We have thus proved:
(1) If $C$ is a monotone non-increasing (monotone non-decreasing) curve with respect to an oblique coordinate system, then for a suitably chosen oblique coordinate system, $C$ is monotone decreasing (monotone increasing) and satisfies a Lipschitz condition.

We state an immediate corollary of (1).

(2) A planar set is sparse if and only if it lies on a countable number of single-valued curves, each monotone and satisfying a Lipschitz condition with respect to some oblique cartesian coordinate system.

4.6. A planar set $E$ has a sectorial void at the point $A$ if there exists a sector $S(A;\sigma)$ with $SE = 0$. In this case, $S(A;\sigma)$ is said to be void of $E$. We may now obtain a theorem of W. H. and G. C. Young [36]:

Theorem (Young). If $E$ has a sectorial void at each of its points, then $E$ is non-dense and sparse.

Proof. Obviously $E$ is non-dense. For fixed $\sigma$, let $H_\sigma$ denote the set of points $A$ of $E$ for which $S(A;\sigma)$ contains no point of $E$. A fortiori, $S(A;\sigma)$ contains no point of $H_\sigma$. By 4.4, $H_\sigma$ is sparse. Moreover, $E = \sum H_\sigma$, where in the summation $\sigma$ is rational. Since the sum of a countable number of sparse sets is sparse, the theorem follows.

4.7. In analogy with 2.11, we shall say a planar set $E$ is voidal at $A$ if $E$ has a sectorial void at $A$. We shall call a set voidal if it is voidal at each of its
points. We may therefore restate the result 4.6 of Young as follows:

A voidal planar set is non-dense and sparse.

The converse of this result is easily seen to be false. For, let $E$ denote the sum of a sequence of concentric circles of center $A$ and radius $1/n$, plus the point $A$. $E$ is non-dense and sparse, but is not voidal at $A$. Apropos 4.2, we also remark that a set may be voidal and yet contain points $A$ such that for every $\sigma$ either $S(A;\sigma)$ or $S(A;\bar{\sigma})$ contains points of the set. $E_1 = E - SE$ is such a set if $E$ is defined as above and $S = S(A;\sigma)$ has an acute angle.

4.8. Corollary. If $E$ is a planar set, then the subset $H$ of $E$ made up of points at which $E$ has a sectorial void is non-dense and sparse.

Proof. This is an immediate consequence of 4.6 since a sector containing no point of $E$ a fortiori contains no point of $H$.

4.9. Theorem. If $E$ is a planar set, the set $H$ of points of $E'$ at which there is a sectorial void with respect to $E$ is sparse and an $F_\sigma$.

Proof. If $S$ is an open sector, $SE' = 0$ if $SE = 0$. Applying 4.8 to the set $E'$, it follows a fortiori that $H$ is sparse. This has been proved by W. H. Young [34] in a different way.
Let $H$ denote the set of points $A$ of $E'$ for which $S(A;\sigma)$ contains no point of $E$. Now $H = \sum H_{\sigma}$, where in the summation $\sigma$ is rational. Let $A$ be a limit point of $H_{\sigma}$. Then $ES(A;\sigma) = 0$. For otherwise $S(A;\sigma)$ contains a point $B$ of $E$, and for any point $P$ sufficiently close to $A$, $S(P;\sigma)$ contains $B$. This contradicts the fact that $A$ is a limit point of $H_{\sigma}$. Since $E'$ is closed, it follows that $H_{\sigma}$ is closed and $H$ is an $F_\sigma$.

We may restate this theorem in the following form:

If $E$ is a planar set, then the set of points of $E'$ at which the contingent of $E$ is not the whole plane is a sparse, $F_\sigma$ set.

1.10. Since a voidal set is sparse and the property sparse is countably additive, the sum of a countable number of voidal sets is sparse. Furthermore, since a set lying on a single curve which is monotone with respect to an oblique coordinate system is voidal, every sparse set is the sum of a countable number of voidal sets. We may state:

A planar set is sparse if and only if it is the sum of a countable number of voidal sets.

1.11. We now give a characterization (and genesis) of a voidal set in terms of a sector property (cf. 2.12).

Theorem. If $\mu$ is a given sector property, let $H_\mu$ consist of all points $A$ of the plane which satisfy the
conditions
(a) for every (open) sector \( S \supset A, S^H \);
(b) there exists a sector \( S \) of vertex at \( A \) with \( S^\mu \).

Then \( H_\mu \) is voidal, and, conversely, every voidal planar set is the associated set \( H_\mu \) for some ascending sector property \( \mu \).

Proof. No point of a sector of property \( \mu \) is in \( H_\mu \). Hence by the condition (a), \( H_\mu \) is voidal.

To prove the converse, first choose a voidal set \( E \).
If \( H_\mu \) is to contain \( E \), then, by (a), necessarily \( S^H \) if \( SE \neq 0 \). But if \( S^H := SE \neq 0, H_\mu \) would be the set of points of \( cE \) at which \( E \) is voidal. To prevent the points of \( cE - E \) from being in \( H_\mu \), we may let every sector of vertex in \( cE - E \) have property \( \mu \). We therefore define \( \mu \) as follows: \( S^H := \) either \( SE \neq 0 \) or the vertex of \( S \) is in \( cE - E \).

Let \( A \in E \). If \( S \supset A \), then \( S^H \). Since \( E \) is voidal, there exists a sector \( S \) with \( SE = 0 \) and with vertex at \( A \notin (cE - E) \). Thus \( S^\mu \). It follows that \( A \in H_\mu \), i.e., \( H_\mu \supset E \).

Let \( A \in H_\mu \). Then for every sector \( S \supset A, S^H \).
Hence \( A \in cE \). But there exists a sector \( S \) with vertex at \( A \) and with \( S^\mu \). Hence \( A \notin (cE - E) \). Therefore \( A \in E \), i.e., \( E \supset H_\mu \). Since \( \mu \) is ascending, the theorem follows.
This theorem and 4.10 imply that a sparse set may be characterized in terms of a sequence of sector properties.

4.12. Augmenting the definitions of 3.5, A is $\mathcal{A}$-approached by $\mathcal{E}$ via $\angle \emptyset$ :=: for every sector $S(A; \emptyset \emptyset h)$, $SE$ has property $\mathcal{A}$. We give a simple application of the direct part of 4.11:

If $\mathcal{E}$ is a planar set and $\mathcal{A}$ is a descending set property, then with the exception of the points of a voidal set $H$, every point $A$ of the plane which is $\mathcal{A}$-approached by $\mathcal{E}$ is $\mathcal{A}$-approached by $\mathcal{E}$ via every angle.

Proof. Let $S^\mu :=: SE$ has property $\mathcal{A}$. Then $H = H_\mu$ is voidal.

4.13. By applying 4.12 we obtain a variant of the following known decomposition theorem: If $\mathcal{A}$ is a descending, countably additive set property and $\mathcal{E}$ a set, then $\mathcal{E} = H + R$, where $H$ is of property $\mathcal{A}$ and $R$ $\mathcal{A}$-approaches each of its points. Let $\mathcal{E} = H + R$ be the decomposition of this theorem. By 4.12, $R = R_1 + R_2$, where $R_1$ is voidal and for every sector $S$ whose vertex is a point of $R_2$, $SR$ is of property $\mathcal{A}$. If we assume further that a voidal set is of property $\mathcal{A}$, it follows that $SR_2$ is of property $\mathcal{A}$. Hence, $\mathcal{E} = (H + R_1) + R_2$, where the set $H + R_1$ is of property $\mathcal{A}$ and $R_2$ $\mathcal{A}$-approaches each of its points via every angle. We have thus proved:
If \( \mathcal{A} \) is a descending, countably additive set property such that every voidal set is of property \( \mathcal{A} \), and \( E \) is a set, we have the decomposition \( E = E_1 + E_2 \), where \( E_1 \) is of property \( \mathcal{A} \) and \( E_2 \) \( \mathcal{A} \)-approaches each of its points via every angle.

4.14. Lemma. If \( \angle \theta \theta > \pi \) and \( E \) is a planar set such that for every point \( A \in E \), \( S(A;\sigma) = S(A;\theta \phi h) \) contains no point of \( E \), then \( E \) is isolated and therefore countable.

Proof. By 4.2, for every \( A \in E \), neither \( S(A;\sigma) \) nor \( S(A;\overline{\theta}h) \) contains a point of \( E \). Since these sectors overlap, the circle \( C(A;h) \) of center \( A \) and radius \( h \) contains no point of \( E \) and the lemma follows.

4.15. We now easily deduce a result of A. Denjoy [15, p. 147]:

Theorem. If \( E \) is a planar set, the set \( H \) of points of \( E \) (of \( E' \)) at which there is a sectorial void with respect to \( E \) of angular magnitude \( > \pi \) is countable.

Proof. Let \( H_\sigma \) consist of the points \( A \) of \( E \) for which \( S(A;\sigma) \) contains no point of \( E \), \( \sigma \) being fixed. It follows that \( H = \sum H_\sigma \), where in the summation \( \sigma = \theta \phi h \) is rational and \( \angle \theta \phi > \pi \). By 4.14, \( H_\sigma \) is countable for fixed \( \sigma \), and the theorem follows.

The second part of the theorem follows by replacing \( E \) by \( E' \) and noting that \( SE' = 0 \) if \( SE = 0 \).
4.16. We now give a variant of the characterization of 4.11; using 4.15, its proof follows mutatis mutandis from that of 4.11:

If \( \mu \) is a sector property, let \( H_\mu \) consist of all points \( A \) of the plane which satisfy the conditions

(a) for every sector \( S \supseteq A, S^\mu \);
(b) there exists a sector \( S \) of vertex \( A \) and angular magnitude \( > \pi \) with \( S^\mu \).

Then \( H_\mu \) has at each of its points a sectorial void of angular magnitude \( > \pi \) and is countable. Conversely, every planar set which has at each of its points a sectorial void of angular magnitude \( > \pi \) is the associated set \( H_\mu \) for some ascending sector property \( \mu \).

§15. The Fundamental Theorem and Generalizations

4.17. If \( A \) and \( B \) are points in the plane, we denote by \( d(A,B) \) the distance between them. If \( E \) is a planar set, we denote by \( d(E) \) the diameter of \( E \), i.e., \( d(E) := \max d(A,B) \) for all pairs of points \( A, B \) in \( E \); we denote by \( X(E) \) the set of \( x \)-coordinates of the points of \( E \). If a real function \( y = f(x) \) is defined on the linear set \( E \), we denote respectively by \( T(f;E) \) and \( Y(f;E) \) the set of points \( (x,f(x)) \) and the values \( f(x) \) for \( x \in E \).

4.18. C. Carathéodory [13] has defined as follows the concept of outer length (or exterior linear measure)
of a planar point set. If \( E \) is a planar set and \( \varepsilon > 0 \), consider all decompositions of \( E \) into a countable number of sets \( E_n \) each of diameter < \( \varepsilon \). Then let \( \lambda(E, \varepsilon) \) denote the minimum (finite or infinite) of \( \sigma = \sum d(E_n) \) for all such decompositions. As \( \varepsilon \) decreases, \( \lambda(E, \varepsilon) \) does not decrease. Therefore

\[
\lambda_\varepsilon(E) := \lim_{\varepsilon \to 0} \lambda(E, \varepsilon)
\]

exists and is called the outer length of \( E \). (\( \lambda_\varepsilon(E) \) may be \(+\infty\).) \( \lambda_\varepsilon(E) \) is an example of an outer Caratheodory measure (see [1], pp. 238-239]). The planar set \( E \) has outer length 0, i.e., \( \lambda_\varepsilon(E) = 0 \), if and only if \( E \) is the sum of a countable number of sets of arbitrarily small diameter sum.

4.19. Lemma. If, in an oblique cartesian coordinate system \( xy \), \( y = f(x) \) is defined on a linear set \( E \) and satisfies a Lipschitz condition for the constant \( k > 0 \), then

\[
\lambda_\varepsilon[T(f;E)] \leq (k + 1) m_\varepsilon(E).
\]

Therefore, if \( m(E) = 0 \), we have \( \lambda_\varepsilon[T(f;E)] = 0 \).

Proof. This result follows from the elementary facts:

1. \( E \) can be covered with a sequence of open intervals whose length sum is arbitrarily close to \( m_\varepsilon(E) \) and such that every interval \( I \) of the sequence is of arbitrarily small length; (2) for every \( I \) of the sequence, there exists a parallelogram with sides respectively parallel to the
x-axis and y-axis of dimensions m(I) and k m(I) containing all the points of T(f;EI). Consequently, d[T(f;EI)] ≤ (k + 1) m(I).

4.20. Since the diameter of an orthogonal projection of a (planar) set does not exceed the diameter of the set, the outer length of an orthogonal projection of a set does not exceed the outer length of the set. For an oblique coordinate system xy, consider the orthogonal projection of a set E on a line perpendicular to the y-axis. If θ is the angle between the positive coordinate axes, then m_e[X(E)] ≤ csc θ λ_e(E). Therefore, if λ_e(E) = 0, then m_e[X(E)] = 0. Combining this result with 4.19, we may conclude:

**Lemma.** If y = f(x) is defined on a set E and satisfies a Lipschitz condition with respect to an oblique cartesian coordinate system xy, then \( \lambda_e[T(f;E)] = 0 \) if and only if m(E) = 0.

4.21. We state without proof the well-known theorem of Lebesgue [26, p. 128]: Every real function y = f(x) of bounded variation (with respect to an oblique coordinate system) has a finite derivative for almost all x. By 4.20, this theorem of Lebesgue yields:

**If the curve C: y = f(x) satisfies a Lipschitz condition with respect to an oblique coordinate system, then for every point (x,f(x)) not belonging to an exceptional**
set of outer length 0, the tangent line of the curve exists and is not parallel to the y-axis.

4.22. By 4.20, 4.21, and the density theorem of Lebesgue, we have a stronger result:

If \( H \) is a set lying on a curve \( C: y = f(x) \) which satisfies a Lipschitz condition with respect to an oblique coordinate system, then for every point \((x, f(x))\) of \( H \) not belonging to an exceptional set of outer length 0, the following conditions hold: (1) \( C \) has a tangent line at \((x, f(x))\) which is not parallel to the y-axis, and (2) \( X(H) \) has exterior metric density 1 at \( x \).

4.23. Augmenting the definition of 4.6, the infinite sector \( S(A; \emptyset \emptyset) \) will be called a maximal sectorial void of \( E \) if \((A, \emptyset)\) and \((A, \emptyset)\) are tangent rays of \( E \) and no ray \((A, a)\) with \( a \in \emptyset \emptyset \) is a tangent ray of \( E \) (see [18]).

4.24. The following two theorems (on the contingent) are contained in results deduced by A. Kolmogoroff and J. Verčenko [25] from a theorem (stated without proof) on functions. Some equivalent results have been derived by Saks [29, Chapter IX, §3], and an extension to n-space has been given by F. Roger [27]. The present proof of the following theorem is a modification of one of U. S. Haslam-Jones [18], but is based on various lemmas and auxiliary results which we have earlier specifically stated and proved.
Theorem. If $E$ is a planar set, the set $H$ of points of the plane at which $E$ has a maximal sectorial void of magnitude less than $\Pi$ is a set of outer length 0.

Proof. By decomposition, $H = \sum E_\gamma$, where (1) $E_\gamma = E_{\sigma_1 \sigma_2 \sigma_3}$ is the set of points $A$ of the plane such that each of the sectors $S(A;\sigma_1) = S(A;ab)$ and $S(A;\sigma_2) = S(A;cd)$ contains a subset of $E$ which has $A$ as a limit point and such that $S(A;\sigma_3) = S(A;pq)$ is void of $E$, and (2) in the summation, $\sigma_1$, $\sigma_2$, and $\sigma_3$ are rational with $a < b < p < q < c < d$ and $\angle ad < \Pi$.

It will suffice to prove that $E_\gamma = E_{\sigma_1 \sigma_2 \sigma_3}$, for fixed subscripts, is of outer length 0; this, in turn, will follow if we prove that for an arbitrary rhombus $R$ with sides of length $h/2$ and in directions $p$, $q$, respectively, $RE_\gamma$ has outer length 0. Choose $p'$, $q'$, with $p < p' < q' < q$. By 4.4 and 4.5, the points of $RE_\gamma$ lie on a monotone decreasing curve $C$ which satisfies a Lipschitz condition with respect to oblique $xy$-axes respectively in directions $p'$, $q'$. It follows by 4.22 that $RE_\gamma$ is of outer length 0 unless there is a point $A$ in $RE_\gamma$ at which (1) $C$ has a tangent at $A$ and (2) $X(RE_\gamma)$ has exterior metric density 1 at $X(A)$.

Suppose such a point $A$ exists. Let $\theta$ be the direction of the tangent of the curve at $A$. Either $\theta < a < q < \theta + \Pi$ or $\theta < p < d < \theta + \Pi$. Suppose that the
former holds, say, and choose $\delta$ with $0 < \delta < \text{the minimum of } a - \theta, q - q'$.

Let $S = S(A; (\theta - \delta)(\theta + \delta)h)$. Then the projection of $S \cap E = K$ onto the $x$-axis: $(A, p')$ and thus onto $L: (A, \theta + \delta)$ by directions $q'$ (parallel to the $y$-axis) is of metric density $1$ at $A$ from the right. If $B \in K$, let $(B, q')$ intersect $(A, \theta + \delta)$ at $P$. $S(B; p'q'h)$ contains no point of $E$ and therefore $S(P; p'q'h)$ contains no point of $RE$. By the argument of 3.29, $A$ cannot be a limit point of $ES(A; \sigma_1)$. The theorem follows.

4.25. The following theorem, which may be called the fundamental theorem on the contingent of a planar set, is a consolidation of the results 4.15 and 4.24.

**Theorem.** If $E$ is a planar set, then at every point of $E'$ not belonging to an exceptional set of outer length $0$ the contingent of $E$ is either (1) the whole plane, or (2) a half-plane, or (3) a single line.

**Proof.** Since a countable set plus a set of outer length $0$ is of outer length $0$, it follows by 4.15 and 4.24 that at every point $A$ of $E'$ not belonging to a set which is of outer length $0$, every maximal sectorial void of $E$ is of magnitude $\pi$. Thus, according as there are none, one, or two such maximal sectorial voids of $E$ at $A$, the contingent of $E$ at $A$ is respectively the whole plane, a half-plane, a single line.
4.26. We now give generalizations of 4.9, 4.15, and 4.25 in terms of a conditioned region property \( \bar{\mu} \), using the notation of 3.9 and 3.32. The theorems, being parallels of 3.34, follow immediately by making use of (2) and (3) of 3.33 and by applying respectively 4.9, 4.15, and 4.25 to the set \( D(\bar{\mu}) \).

1. If \( \mu \) is a descending, additive region property, then the set of points of \( D(\bar{\mu}) \) at which the \( \bar{\mu} \)-contingent is not the whole plane is a sparse, \( F_\sigma \) set.

2. If \( \mu \) is a descending, additive region property, then the set of points of \( D(\bar{\mu}) \) at which the \( \bar{\mu} \)-contingent is contained in an angle of magnitude less than \( \pi \) is countable.

3. If \( \mu \) is a descending, additive region property, then at every point \( A \) of \( D(\bar{\mu}) \) not belonging to an exceptional set of outer length 0 the \( \bar{\mu} \)-contingent is either (1) the whole plane, or (2) a half-plane, or (3) a single line.

4.27. To indicate the scope of 4.26, we state two related results. Both are variants of a theorem of H. Blumberg [4, p. 24].

1. A region property \( \mu \) is descending and additive if and only if a region \( G \) contains a point of \( D(\bar{\mu}) \) whenever \( G \) has property \( \bar{\mu} \).
(2) With every descending, additive region property \( \mu \), we may associate a closed point set \( F \) such that a region \( G \) is of property \( \mu \) or \( \mu \) according as \( G \) contains or does not contain a point of \( F \), and conversely.

4.28. Let \( E \) be a planar set, \( \mathcal{A} \) a descending, countably additive set property, and let \( G^\mathcal{A} := GE \) is of property \( \mathcal{A} \). For this \( \mu \) we get the following special cases of 4.26, using the notation of 3.5.

(1) If \( E \) is a planar set and \( \mathcal{A} \) a descending, countably additive set property, then the set of points of \( E^\mathcal{A} \) at which the \( \mathcal{A} \)-contingent of \( E \) is not the whole plane is a sparse, \( F_\sigma \) set.

(2) If \( E \) is a planar set and \( \mathcal{A} \) a descending, countably additive set property, then the set of points of \( E^\mathcal{A} \) at which the \( \mathcal{A} \)-contingent of \( E \) is contained in an angle of magnitude less than \( \pi \) is countable.

(3) If \( E \) is a planar set and \( \mathcal{A} \) a descending, countably additive set property, then at every point of \( E^\mathcal{A} \) not belonging to an exceptional set of outer length 0 the \( \mathcal{A} \)-contingent of \( E \) is either (1) the whole plane, or (2) a half-plane, or (3) a single line.

Using the fact that the set of isolated points of a set is countable, the special cases of these results obtained by letting \( E^\mathcal{A} := E \) is empty are equivalent respectively to 4.9, 4.15, and 4.25.
4.29. We now explain under what conditions we may say a planar set $E$ has a linear property $\alpha$ with respect to a given line $L$ in the plane, which we may identify as the $x$-axis. Given a planar set $E$ and a linear set property $\alpha$, we say $E$ has property $\alpha$ with respect to $L$ if and only if $X(E)$ (= the perpendicular projection of $E$ on $L$) is of property $\alpha$. We shall say a point $A$ of the plane is $\alpha$-approached by $E$ with respect to $L$ if for every region $G \supset A$, $GE$ has property $\alpha$ with respect to $L$. Similarly, we may make relative all of the definitions of 3.5, obtaining such concepts as (1) the $\alpha$-derivative $E'_{\alpha}$ of $E$ with respect to $L$, (2) $(A, \theta)$ is an $\alpha$-tangent ray of $E$ with respect to $L$, and (3) the $\alpha$-contingent $T_{\alpha}(A, E)$ of $E$ at $A$ with respect to $L$.

If the linear property $\alpha$ is descending and countably additive, then the (planar) property that $E$ has property $\alpha$ with respect to $L$ is descending and countably additive. We may thus obtain corollaries (1), (2), and (3) of the corresponding results of 4.28 by stipulating that $\alpha$ is a descending, countably additive linear set property and that $E'_{\alpha}$ is the $\overline{\alpha}$-derivative of $E$ with respect to $L$, and by taking the $\overline{\alpha}$-contingent of $E$ with respect to $L$.

4.30. An immediate corollary of a theorem of A. J. Ward [32] is the following:
If $E$ is a planar set, then at every point of $E'$ not belonging to an exceptional, countable set the contingent of $E$ contains at least one line (both directions).

Making use of (2) and (3) of 3.33 and applying this corollary of Ward's theorem to the set $D(\mu)$, we obtain the following generalization:

**Theorem.** If $\mu$ is a descending, additive region property, then at every point $A$ of $D(\mu)$ not belonging to an exceptional, countable set the $\mu$-contingent contains at least one line.

**Corollary.** If $E$ is a planar set and $\mathcal{A}$ a descending, countably additive planar (linear) set property, then at every point of $E_{\mathcal{A}}$ (with respect to $L$) not belonging to an exceptional, countable set the $\mathcal{A}$-contingent of $E$ (with respect to $L$) contains at least one line.

If a planar set has property $\mathcal{A}$ if and only if it is countable, this corollary becomes the theorem of Ward referred to above.

4.31. We give an extension of the following theorem of Lusin: If $f(x)$ is a real function and $E$ is the set of points $x$ at which $f'(x)$ exists and equals 0, then $Y(f;E)$ is of measure 0. (For the notation, see 4.17.)

**Theorem.** If $y = f(x)$ is a given real function (defined in an oblique coordinate system $xy$), $\theta$ a fixed direction not parallel to the $y$-axis, $E$ the set of points $x$
at which \( f(x) \) has a tangent line in direction \( \theta \), and \( L \) a line orthogonal to \( \theta \), then the orthogonal projection of \( T(f;E) \) on \( L \) is of measure 0.

**Proof.** We may assume that \( f(x) \) is defined on an interval \( I \). For every \( x \in I \), let \( y = L_x(t) \) be the equation of the line through \( (x,f(x)) \) in direction \( \theta \). Choose \( \varepsilon > 0 \). For an arbitrary positive integer \( n \), let \( E_n \) denote the set of points \( x \) of \( E \) for which \( |f(t) - L_x(t)| < \varepsilon |t - x| \) whenever \( |t - x| < 1/n \). Evidently \( E_n \) is monotone non-decreasing and \( E = \lim E_n \).

We can cover \( E_n \) with a sequence of open intervals \( I_k^n, k = 1, 2, \ldots \), each of length < \( 1/n \), and such that

\[
\sum_k m(I_k^n) \leq m_\varepsilon(E_n) + \varepsilon.
\]

Choose \( x_1 \in E_n I_k^n \). Then for every \( x_2 \in E_n I_k^n \),

\[
|f(x_2) - L_{x_1}(x_2)| \leq \varepsilon |x_2 - x_1| < \varepsilon m(I_k^n),
\]

and therefore, letting \( T_\theta(f;E_n I_k^n) \) be the orthogonal projection of \( T(f;E_n I_k^n) \) on a line \( L \) orthogonal to \( \theta \),

\[
m_\varepsilon[T_\theta(f;E_n I_k^n)] \leq 2\varepsilon m(I_k^n).
\]

Hence, for every \( n \),

\[
m_\varepsilon[T_\theta(f;E)] \leq \sum_k m_\varepsilon[T_\theta(f;E_n I_k^n)] \leq 2\varepsilon \sum_k m(I_k^n) \leq 2\varepsilon [m_\varepsilon(E_n) + \varepsilon].
\]

Letting \( n \to \infty \),

\[
m_\varepsilon[T_\theta(f;E)] \leq 2\varepsilon [m_\varepsilon(E) + \varepsilon].
\]

Since \( \varepsilon > 0 \) may be chosen arbitrarily, \( m[T_\theta(f;E)] = 0 \).

**4.32.** We give a simplified proof of a theorem of Haslam-Jones [18], basing our proof on the fundamental results on the contingent of a set at a point. (In a
footnote of this paper [18], a remark of a referee is cited that such a simplification is possible.)

**Theorem.** If $E$ is a planar set, $\theta$ a fixed direction, $R$ the set of points $A$ of the plane at which $E$ has a maximal sectorial void with one extreme direction equal to $\theta$, and $L$ a line orthogonal to $\theta$, then the orthogonal projection of $R$ on $L$ is of measure 0.

**Proof.** By 4.6 and (2) of 4.5, $R = \sum H_n$, where each set $H_n$ lies on a curve $C_n$ which is monotone and satisfies a Lipschitz condition with respect to some oblique coordinate system. By 4.22, it is clear that for every point $A$ of $H_n$ not belonging to a set $Z_n$ of outer length 0, $C_n$ has a tangent line in a direction $\theta$ not parallel to the y-axis where both $(A, \theta)$ and $(A, \overline{\theta})$ are tangent rays of $H_n$ and therefore of $E$. By 4.20, the orthogonal projection of $Z_n$ on $L$ is of measure 0.

Let $K_n = H_n - Z_n$, and let $M_n$ denote the set of points of $K_n$ for which either $\theta = \emptyset$ or $\theta = \overline{\theta}$. By 4.31, the orthogonal projection of $M_n$ on $L$ is of measure 0. Let $N_n = K_n - M_n$. For $A \in N_n$, the three directions $\theta$, $\emptyset$, and $\overline{\theta}$ are distinct. Since $(A, \emptyset)$, $(A, \emptyset)$, and $(A, \overline{\theta})$ are tangent rays of $E$, the maximal sectorial void of $E$ at $A$, having one extreme direction equal to $\theta$, is of angular magnitude less than $\pi$. Hence by 4.24, $N_n$ is of outer length 0, and therefore its orthogonal projection on $L$ is of measure 0.
It follows that the orthogonal projection of $H_n$ and therefore of $R$ on $L$ is of measure 0.

4.33. If $\mu$ is a sector property, $S(A;\Theta\emptyset)$ will be called a maximal sectorial void of $\mu$ if $(A,\Theta)$ and $(A,\emptyset)$ are $\mu$-rays but no ray $(A,a)$ with $a \in \Theta\emptyset$ is a $\mu$-ray (cf. 4.23). If $\sigma$ is a planar (linear) set property, $S(A;\Theta\emptyset)$ will be called a maximal $\sigma$-sectorial void of $E$ (with respect to $L$) if $(A,\Theta)$ and $(A,\emptyset)$ are $\sigma$-tangent rays of $E$ (with respect to $L$) but no ray $(A,a)$ with $a \in \Theta\emptyset$ is an $\sigma$-tangent ray of $E$ (with respect to $L$).

By (2) of 3.33 and 4.32, we obtain a generalization of 4.32.

(1) If $\mu$ is a descending, additive region property, $\Theta$ a fixed direction, $R$ the set of points $A$ of the plane at which $\mu$ has a maximal sectorial void with one extreme direction equal to $\Theta$, and $L_1$ a line orthogonal to $\Theta$, then the orthogonal projection of $R$ on $L_1$ is of measure 0.

We state a special case of (1):

(2) If $E$ is a planar point set and $\sigma$ a descending, countably additive planar (linear) set property, $\Theta$ a fixed direction, $R$ the set of points $A$ of the plane at which $E$ has a maximal $\sigma$-sectorial void (with respect to $L$) with one extreme direction equal to $\Theta$, and $L_1$ a line orthogonal to $\Theta$, then the orthogonal projection of $R$ on $L_1$ is of measure 0.
§16. Extension of Some Results on Sector Properties

4.34. Just as 4.25 may be considered as a (partial) extension of 3.29 to the plane, certain other results of Chapter III on approach of a set to a straight line may be extended to approach of a set to the points of a plane. More generally, some of the results on sector properties in Chapter III may be extended to the plane.

For example, let \( \mu, \nu \) denote a pair of mutually non-overlapping sector properties (defined in 3.10), and let \( H = H(\mu, \nu) \) denote the set of points \( A \) of the plane for which there exist two sectors with \( A \) as vertex of property \( \mu, \nu \) respectively. If a sector \( S \) has either property \( \mu \) or \( \nu \), it contains no point of \( H \). Hence \( H \) is voidal, and we may state the result, an extension of 3.10:

**Theorem.** If \( \mu, \nu \) are mutually non-overlapping sector properties, the set of points \( A \) of the plane for which there is a pair of sectors \( S(A; \sigma_1), S(A; \sigma_2) \) respectively of property \( \mu, \nu \) is voidal.

A partial converse of this result also holds:

If \( H_1 \) is a countable, voidal set, there exist two mutually non-overlapping sector properties \( \mu, \nu \) such that \( H(\mu, \nu) = H_1 \).

**Proof.** The proof here is about the same as the proof of the converse of 3.10. At each point \( A \) of \( H_1 \), there is a sector \( S(A; \sigma) \) void of \( H_1 \). Choose two non-overlapping
sectors $S_1$ and $S_2$ with vertices at $A$ and lying in $S$. The proof continues as in 3.10.

4.35. Using the notation of 3.9, we may restate the direct part of 4.34 as follows:

If $\mu$, $\nu$ are two mutually non-overlapping sector properties, then for every point $A$ of the plane not belonging to an exceptional voidal set, either the $\mu$-contingent is the whole plane or else the $\nu$-contingent is the whole plane.

Letting $S^\mu$ ($S^\nu$) mean that $SE$ ($SE$) is empty, we have the special case on the contingent of a set:

If $E$ is a planar set, then at every point $A$ of the plane not belonging to an exceptional, voidal set, either the contingent of $E$ or the contingent of $E^c$ is the whole plane.

4.36. Now let $\mu$, $\nu$ be a pair of mutually non-intersecting sector properties (see 3.15). Assume the non-overlapping, non-adjacent sectors $S_1 = S(A;\sigma_1)$ and $S_2 = S(A;\sigma_2)$ lie in an angle at $A$ of magnitude less than $\Pi$. Then the set $H_{\text{olo}2}$ of points $A$ of the plane for which $S_1$, $S_2$ are respectively of property $\mu$, $\nu$ is voidal. For if $A \in H_{\text{olo}2}$, there exists a sector $S$ of vertex $A$ and containing no point of $H_{\text{olo}2}$. In fact, if $\sigma_1 = abh$ and $\sigma_2 = cdk$ with $a < b < c < d$, such a sector is $S = S(A;\delta ap)$ if $p > 0$ is chosen small enough. The set $H = \sum H_{\text{olo}2}$.
σ₁ and σ₂ ranging over all rational subscripts of the type above, is therefore sparse. We have thus proved:

If μ, ν are two mutually non-intersecting sector properties, then the set of points A of the plane for which there exist two non-overlapping, non-adjacent sectors S(A;σ₁), S(A;σ₂) with rational dimensions, both lying in an angle of magnitude less than π and respectively of property μ, ν is sparse.

4.37. If for the point A there exist two rays (A,θ), (A,φ), neither the same nor opposite, which are respectively a μ(1)-ray, ν(1)-ray (see 3.9), there also exist two sectors S(A;θ), S(A;φ) with rational dimensions -- non-overlapping and non-adjacent and both lying in an angle at A of magnitude less than π -- which are respectively of property μ, ν. Thus, by applying 4.36 we may state the following result, an extension of 3.15:

Theorem. If μ, ν are two mutually non-intersecting sector properties, the set of points A of the plane for which there are two rays (A,θ), (A,φ), neither the same nor opposite, which are respectively a μ(1)-ray, ν(1)-ray is sparse.

4.38. The theorem of 4.37 may be stated in the following alternative form:

Theorem. If μ, ν are two mutually non-intersecting sector properties, then for every point A of the plane not
belonging to an exceptional, sparse set, either (1) every ray \((A, \theta)\) is a \(\bar{\mu}(2)\)-ray, or (2) every ray \((A, \theta)\) is a \(\bar{\nu}(2)\)-ray, or (3) every ray \((A, \theta)\) with the possible exception of the rays of one particular line \(L_A\) through \(A\), its direction variable with \(A\) is both a \(\bar{\mu}(2)\)-ray and a \(\bar{\nu}(2)\)-ray.

The proof of the equivalence of 4.37 and this result is similar to that of 3.16 and is omitted.

Remark. If \(\mu\) and \(\nu\) are such that no ray is both a \(\mu(1)\)-ray and \(\nu(1)\)-ray and if case (3) alone holds for \(A\), then one of the rays of \(L_A\) has properties \(\mu(1)\) and \(\bar{\nu}(2)\) while the other has properties \(\nu(1)\) and \(\bar{\mu}(2)\).

4.39. Let \(\mu, \nu\) be a pair of mutually non-intersecting sector properties which are both angularly ascending and radially descending (see 3.21). Omitting the proof, we state the following extension of 3.22:

If \(\mu, \nu\) are two mutually non-intersecting sector properties each of which is angularly ascending and radially descending, then for every point \(A\) of the plane not belonging to an exceptional, sparse set, either (1) every ray \((A, \theta)\) is a \(\bar{\mu}(1)\)-ray, or (2) every ray \((A, \theta)\) is a \(\bar{\nu}(1)\)-ray, or (3) every ray \((A, \theta)\) with the possible exception of the rays of one particular line \(L_A\) through \(A\), its direction variable with \(A\) is both a \(\bar{\mu}(1)\)-ray and a \(\bar{\nu}(1)\)-ray.
4.40. If $E$ is a planar set, let $S^\mu := SE$ contains a simple arc whose closure contains the vertex of $S$ and a point on the circular boundary of $S$; $v$ is similarly defined with respect to $E$. Then $\mu, v$ are mutually non-intersecting and each is angularly ascending and radially descending. Whenever $(A, \theta)$ is a $\mu(1)$-ray ($\nu(1)$-ray) it readily follows that every simple arc with determinate direction $\theta$ at $A$ contains points of $E (E)$. Applying 4.39 we obtain a result which is stronger than one of Blumberg [4, Theorem III']. (We say $A$ is approachable by $E$ along (see 3.14) the arc $C$ if $A$ is the limit of a sequence of points of $CE$.)

Theorem. If $E$ is a planar set, then for every point $A$ of the plane not belonging to an exceptional, sparse set, either (1) $A$ is approachable by $E$ along every simple arc of determinate direction at $A$, or (2) $A$ is so approachable by $E$, or (3) $A$ is approachable by both $E$ and $E$ along every simple arc of determinate direction at $A$ not tangent to a singular line $L_A$ at $A$ (which may or may not exist).

4.41. As a corollary of 4.40, we obtain a result on the strong contingent (see 3.4), an extension of 3.18.

If $E$ is a planar set, then for every point $A$ of the plane not belonging to an exceptional, sparse set, either (1) $T'(A, E)$ is the whole plane, or (2) $T'(A, E)$ is the whole plane, or (3) both $T'(A, E)$ and $T'(A, E)$ contain the whole
plane with the possible exception of the rays of a singular line $L_A$ through $A$.

If case (3) is the only case that holds, it follows that $T'(A,E)$ does not contain one of the rays of $L$, and $T'(A,E)$ does not contain the other ray of $L_A$. We may thus restate the present result in the following more complete form which shows a remarkable symmetry in the structure of every planar set. (For our use of "along," see 3.4.)

Corollary. If $E$ is a planar set, then for every point $A$ of the plane not belonging to an exceptional, sparse set, either (1) $A$ is approachable by $E$ along every ray $(A,\theta)$, or (2) $A$ is approachable by $E$ along every ray $(A,\theta)$, or (3) $A$ is approachable by both $E$ and $\overline{E}$ along every ray $(A,\theta)$ not in a singular line $L_A$, along one direction of which there is exclusive approach by $E$ and along the other exclusive approach by $\overline{E}$.

4.42. Other results of Chapter III, such as 3.23, may be extended to the plane, but we omit their statements.

We now ask: how are the contingent and strong contingent related? Obviously, whenever $(A,\theta) \in T'(A,E)$, then also $(A,\theta) \in T(A,E)$. On the other hand, in example (2) below, we define a set $E$ so that for every point $A$ of the plane, $T(A,E)$ is the whole plane and $T'(A,E)$ is empty. We first describe in (1) the rather interesting distribution of the
strong contingent for the familiar set of rational points
in the plane.

(1) We call a point P: (r, s) in the plane rational
if both of the coordinates r, s are rational numbers. Let
E denote the set of all rational points in the plane. It
follows immediately that if a line L contains at least two
rational points then it contains a set of rational points
which is everywhere dense on it. Let T denote the set of
all lines each of which contains at least two rational
points, and let H denote the set of all points constitut­
ing the lines of T. Since the set E of rational points
is denumerable, T is a denumerable set of lines. It fol­
lows that H is sparse.

Since E is everywhere dense on the planar continuum,
it is clear that for every point A of the plane, the con­
tingent T(A, E) of E at A is the whole plane. If A ∈ H,
a ray (A, θ) contains at most one point of E. Hence, for
such a point, the strong contingent T'(A, E) of E at A is
empty. If A ∈ (H - E), then A lies on a single line of
T, and T'(A, E) consists of the two opposite rays constit­
tuting this line. This is true since if A lay on two
distinct lines of T, it would be the point of intersec­
tion of these two lines whose equations have rational
coefficients, and would therefore be rational. If A ∈ E,
$T'(A,E)$ is a countable set of rays whose directions are everywhere dense, constituting the lines of $T$ through $A$.  

(2) We now define a set $E$, which is countable and everywhere dense like the set in (1), and is such that no line contains more than two points of it. Consider the set of squares in the plane of dimensions $1/n$ and whose lower left-hand corner is $(r,s)$, $n = 1, 2, \ldots$, and $(r,s)$ varying over the set of all rational points in the plane. This set of squares can clearly be written out in a sequence: $Q_1, Q_2, \ldots, Q_n, \ldots$. Choose any point $A_1$ in $Q_1$ and a different point $A_2$ in $Q_2$. Choose any point $A_3$ in $Q_3$ which does not lie on the line through $A_1$ and $A_2$. Assuming $A_{n-1}$ has been defined, choose a point $A_n$ in $Q_n$ which does not lie on any line joining two of the points $A_1, A_2, \ldots, A_{n-1}$. Let the set $E$ consist of the points of the infinite sequence $\{A_n\}$ which is thus defined by finite induction.

It is obvious that $E$ is everywhere dense on the plane, and therefore for every point $A$ of the plane, the contingent $T(A,E)$ is the whole plane. Since no line contains more than two points of $E$, a ray $(A, \theta)$ contains at most two points of $E$. It follows that for every point $A$ in the plane, the strong contingent $T'(A,E)$ is empty.
We remark that \( C(P) \), the characteristic function of the set \( E \), is everywhere discontinuous with respect to \( P : (x, y) \), but for every straight line \( L \), it is continuous on \( L \) with respect to \( L \), exception being made of at most two points.
CHAPTER V

THE EXTENSION OF SET THEOREMS TO FUNCTIONS

§17. Two General Theorems on Functions

5.1. In this section, we give two generalized procedures for passing from certain theorems on sets to related theorems on functions. These theorems apply to unconditioned as well as conditioned functions; and their special cases may be interpreted as giving information on the symmetric structure of the functions to which they apply. As remarked in 3.1, this procedure of going from set theorems to function theorems seems to be simpler, at least for certain classes of theorems, than proving the function theorems directly; also, because of the simpler nature of sets than (general) functions, it appears that a better insight of the proofs may thus be attained.

The first theorem and a corollary of it are extensions of some of the concepts and results of Chapter I (on ascending set-point properties) to functions. The second theorem gives a formulation for carrying a different class of theorems on sets over to functions.

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5.2. Let \( z = f(P) = f(x,y) \) be a given real function of the two real variables \( x \) and \( y \), which is defined, say, for every point of the xy-plane. For convenience, we shall often use the notation \( P = (x,y) \), \( A = (\xi, \eta) \); \( f(P) \), \( f(A) \) respectively for \( f(x,y) \), \( f(\xi, \eta) \); and \( B = (A; \zeta) \), \( (P; f(P)) \) respectively for \( (\xi, \eta, \zeta) \), \( (x,y,f(x,y)) \). A concept used in the following general results is that of function-point property.

**Definition.** A function-point property \( \beta \) is a property such that for every real function \( z = f(P) = f(x,y) \) and for every point \( B = (A; \zeta) = (\xi, \eta, \zeta) \) of euclidean 3-space, either \( f(P) \) has or has not property \( \beta \) at \( B \) (in notation, either \( (f(P),B)^\beta \) or \( (f(P),B)^{\overline{\beta}} \)).

5.3. We shall restrict our considerations to function-point properties which are defined in terms of set-point properties. Given a function \( f(P) \), and \( h < k \), let

\[
E_{hk} := E_p[h < f(P) < k],
\]

i.e., let \( E_{hk} \) consist of the set of points \( P \) for which \( f(P) \) -- or, in case \( f(P) \) is many-valued, at least one value of \( f(P) \) -- lies in the open interval \((h,k)\). Let \( \beta \) be a given set-point property (see 1.3). As in 1.4, we say that \( A \) is a \( \beta \)-point of a set \( E \) if \((E,A)^\beta \). In terms of the set-point property \( \beta \) we define the three following function-point properties.
(1) \((f(P), B) \beta\) or \(B = (A; S)\) is a \(\beta\)-point of \(f(P)\) iff for every \(h < S < k\), \(A\) is a \(\beta\)-point of \(E_{hk}\).

(2) \((f(P), B) \beta(1)\) or \(B = (A; S)\) is a \(\beta(1)\)-point of \(f(P)\) iff for every \(h < S < k\) with \(S - h\) and \(k - S\) sufficiently small, \(A\) is a \(\beta\)-point of \(E_{hk}\).

(3) \((f(P), B) \beta(2)\) or \(B = (A; S)\) is a \(\beta(2)\)-point of \(f(P)\) iff there exists an infinitesimal open interval \((h, k)\) containing \(S\) such that \(A\) is a \(\beta\)-point of \(E_{hk}\).

Remarks. (1) The idea of \(\beta\)-point of a real function \(f(x)\) has been used by M. Hendrickson [21], who considers, in particular, the function whose graph consists of all \(\beta\)-points of \(f\). (2) It is evident that \((A; S)\) is a \(\beta(1)\)-point of \(f(P)\) if and only if it is not a \(\beta(2)\)-point of \(f(P)\).

5.4. For the first theorem, we are especially concerned with the points \((A; f(A))\) of the surface \(z = f(P)\). Now, \((A; f(A))\) is not a \(\beta\)-point of \(f(P)\) iff there exist real numbers \(h, k, h < f(A) < k\), with \((E_{hk}, A) \bar{\beta}\), i.e., \(A \in H_{hk}\) where \(H_{hk} := E_p [P \in E_{hk}, (E_{hk}, P) \bar{\beta}]\). Now if the set-point property \(\beta\) is assumed to be ascending (see 1.3), then \(\bar{\beta}\) is descending, and \(A \in H_{hk}\) implies \(A \in H_{rs}\) for \(h < r < f(A) < s < k\). Thus \(\sum H_{hk} = \sum H_{rs}\), where in the summations \(h < k\) and \(r < s\) range respectively
over the set of real numbers and the set of rational numbers. Call this common set $H$. Hence if $H_{hk}$ is of some property $\alpha$ for every $h < k$, then $H$ is a subset of a set of property $\alpha_\sigma$, where $E$ has property $\alpha_\sigma$ if $E$ is the sum of a countable number of sets of property $\alpha$. A sufficient condition for $H$ to be a subset of a set of property $\alpha_\sigma$ is, of course, that for every planar set $E$, the set $E_p[F \in E, (E,P)^B]$ has property $\alpha$. We may therefore state the following theorem which generalizes a result of H. Blumberg [4, Theorem VIII, part 2]:

**Theorem.** Let $\beta$ be an ascending set-point property such that for every planar set $E$ the set of points of $E$ which are not $\beta$-points of $E$ is of property $\alpha$. If $z = f(P)$ is an unconditioned, one- or many-valued, real function, then for all points $A$ of the plane not belonging to a set of property $\alpha_\sigma$, every point $(A; f(A))$ of the surface $f(P)$ is a $\beta$-point of $f$ in the sense that for every $h, k, h < f(A) < k$, $A$ is a $\beta$-point of $E_{hk}$.

5.5. We make the following remarks with regard to the theorem of 5.4:

1. It is evident that the theorem, with slight adjustments, applies to a function defined on an arbitrary planar set.

2. If $\beta$ is an ascending set-point property such that for every set $E$ of property $\alpha'$, the set of points
in \( E \) which are not \( \beta \)-points of \( E \) is of property \( \alpha \),
then the theorem holds for every function \( f(P) \) which satisfies the condition that for every \( h < k \), \( E_{hk} \) is of property \( \alpha' \).

(3) If in the formulation of the theorem it is further assumed that for every set \( E \) the set of points of the plane which are not \( \beta \)-points of \( E \) is of property \( \alpha \), we may conclude that for all points \( A \) of the plane not belonging to a subset of a set of property \( \alpha' \), and for every \( \gamma \), \( (f(P),(A; \gamma))^{\beta} \).

5.6. If \( \beta \) is an ascending set-point property, a pervasive \( \beta'_\sigma \)-set := the sum of a countable number of pervasive \( \beta' \)-sets (see 1.4).

Corollary. Let \( \beta \) be an ascending set-point property and \( z = f(P) \) be a one- or many-valued real function. Then for all points \( A \) of the plane not belonging to a pervasive \( \beta'_\sigma \)-set, every point \( (A;f(A)) \) of the surface \( f(P) \) is a \( \beta \)-point of \( f(P) \).

Proof. It follows by 1.4 that for every \( h < k \), we may decompose \( E_{hk} \) into a maximal \( \beta \)-set and a pervasive \( \beta' \)-set \( E_{hk} \). Now if \( (A;f(A)) \) is not a \( \beta \)-point of \( f(P) \), there exist numbers \( h < f(A) < k \) such that \( A \) is not a \( \beta \)-point of \( E_{hk} \). It follows that \( A \in E_{hk} \), and we may apply 5.4 to obtain the result.
5.7. We now proceed to derive the second theorem mentioned in 5.1. Let $\beta$ be a set-point property such that for every planar set $E$ the set of points of the plane which are $\beta$-points of $E$ is of property $\alpha$. If $B = (A; \mathcal{F})$ is a $\beta(1)$-point of $f(P)$, then for every $h < \mathcal{F} < k$ with $\mathcal{F} - h$ and $k - \mathcal{F}$ sufficiently small, $A$ is a $\beta$-point of $E_{hk}$. Clearly $h < k$ may be chosen rational. Let $H_{hk} :=$ the set of points of the plane which are $\beta$-points of $E_{hk}$ and let $H = \sum H_{rs}$, with $r, s, r < s$, ranging over the rational numbers. If for some $\mathcal{F}$, $(A; \mathcal{F})$ is a $\beta(1)$-point of $f(P)$, then $A \in H$, a set of property $\mathcal{L}_\sigma$. We obtain a generalization of a result of H. Blumberg [4, Theorem VIII, part 1]:

Theorem. Let $\beta$ be a set-point property such that for every planar set $E$ the set of points of the plane which are $\beta$-points of $E$ is of property $\mathcal{L}$. If $z = f(P)$ is an unconditioned, one- or many-valued, real function, the set of points $A$ of the plane for which there is a $\mathcal{F}$ such that $(A; \mathcal{F})$ is a $\beta(1)$-point of $f(P)$ is contained in a set of property $\mathcal{L}_\sigma$.

5.8. Using the notation and the second remark of 5.3, we may state the theorem of 5.7 in the following alternative form:

Theorem. Let $\beta$ be a set-point property such that for every planar set $E$ the set of points in the plane which
are not \( \beta \)-points of \( E \) is of property \( \lambda \). If \( z = f(P) \) is an unconditioned, one- or many-valued, real function, then for all points \( A \) of the plane not belonging to a set of property \( \lambda \), and for every real number \( j \), \( B = (A; j) \) is a \( \beta(2) \)-point of \( f(P) \).

Remarks similar to those of 5.5 apply here also. Actually 5.4 is a corollary of a revised form of 5.8. For if \( \beta \) is ascending, \( B = (A; j) \) is a \( \beta(2) \)-point of \( f(P) \) if and only if it is a \( \beta \)-point of \( f(P) \). The two forms as stated, however, seem to be best for the present applications.

§18. Extension of Some Results of Chapters I and II

5.9. Let \((E, A)^\beta := A\) is a point of concentrated inexhaustibility of \( E \) (see 1.9). \( \beta \) is ascending, and by 1.9, for every set \( E \), the set of points of \( E \) which are not points of concentrated inexhaustibility of \( E \) is exhaustive. We shall call a \( \beta \)-point of \( f(P) \) a point of concentrated inexhaustibility of \( f(P) \). Applying 5.4, we obtain a result due to H. Blumberg [2, Theorem II]:

If \( z = f(P) \) is an unconditioned, one- or many-valued, real function, then for every point \( A \) not belonging to an exhaustive set, \((A, f(A))\) is a point of concentrated inexhaustibility of \( f(P) \).

5.10. Let \( \lambda \) be a descending and countably additive set property. Let \((E, A)^\beta := A\) is \( \lambda \)-approached by \( E \)
(see 2.17). \( \beta \) is ascending. It is readily shown that for every set \( E \) the set of points of \( E \) which are not \( \bar{I} \)-approached by \( E \) is of property \( \nu_\sigma \). Since \( \nu \) is countably additive, a set of property \( \nu_\sigma \) is of property \( \nu \). We say \( B = (A; S) \) is \( \bar{I} \)-approached by \( f(P) \) if and only if \( B \) is a \( \beta \)-point of \( f(P) \). Applying 5.14, we obtain the result:

If \( \nu \) is a descending and countably additive set property and \( z = f(P) \) is an unconditioned, one- or many-valued, real function, then for all points \( A \) of the plane not belonging to a set of property \( \nu \), \( (A, f(A)) \) is \( \bar{I} \)-approached by \( f(P) \).

5.11. Again, let \( \nu \) be a descending and countably additive set property. Let \( (E, A) \beta := \) there exists a (two dimensional) open interval \( I \) containing \( A \) with \( \overline{\text{IE}} \) of property \( \nu \). Then \( \beta \) is ascending. Essentially following E. H. Hanson [17], we use the following notation. \( A \) is an \( \nu \)-interior point of \( E := (E, A) \beta \). It follows that an \( \nu \)-open set, i.e., a set made up of \( \nu \)-interior points, is of the form \( G - H \), where \( G \) is open and \( H \). A set \( E \) is said to be \( \nu \)-almost \( \nu \)-open if it is the sum of an \( \nu \)-open set and a set of property \( \nu \), i.e., if \( E = G - H + K \), with \( G \) open and \( H \) and \( K \) of property \( \nu \). \( f(P) \) is said to be \( \nu \)-regular if for every \( h < k \), \( E_{hk} \) is
\( \alpha \)-almost \( \alpha \)-open. \( f(P) \) is said to be \( \alpha \)-continuous at \( A \) if \((A, f(A)) \) is a \( \beta \)-point of \( f(P) \).

If \( E \) is \( \alpha \)-almost \( \alpha \)-open, the set of points of \( E \) which are not \( \alpha \)-interior points of \( E \) is of property \( \alpha \).

Applying the theorem of 5.4, as modified by Remark (2) of 5.5, we obtain the direct part of the following result which, except for a change of form, is a generalization of a result of Hanson [17].

If \( \alpha \) is a descending and countably additive set property, every real, \( \alpha \)-regular function \( z = f(P) \) is \( \alpha \)-continuous at all points not belonging to a set \( S \) of property \( \alpha \), and conversely.

Proof of converse. Assume that \( f(P) \) is \( \alpha \)-continuous at all points not belonging to a set \( S \) of property \( \alpha \), and let \( h < k \). Every point of \( E_{hk} \) is an \( \alpha \)-interior point of \( S \). Thus \( E_{hk} \) is \( \alpha \)-open and \( E_{hk} \) is \( \alpha \)-almost \( \alpha \)-open.

5.12. Using the terminology of H. Blumberg [7, §2] and that of 2.6, \( B = (A; \mathcal{F}) \) is positively approached (fully approached) by the function \( z = f(P) \) := for every \( h < \mathcal{F} < k \), \( E_{hk} \) has positive upper metric density at \( A \) (has metric density 1 at \( A \)); \( (A; \mathcal{F}) \) is vanishingly approached by \( f(P) \) := a pair of numbers \( h < \mathcal{F} < k \) exists such that \( E_{hk} \) has metric density 0 at \( A \).
If \( y = f(x) \) is a function of a single variable, then
\[ U^+(\xi) = \text{the minimum (greatest lower bound) of all numbers } k \text{ such that } E_x = E_{x}^k \{ f(x) > k \} \text{ is of metric density 0 at } \xi \text{ from the right (left)}. \]

Similarly, \( L^+(\xi) \) is defined as the maximum of all numbers \( k \) such that \( E_x = E_{x}^k \{ f(x) < k \} \) is of metric density 0 at \( \xi \) from the right (left).

5.13. Let \( (E, A)^B := E \) has positive upper metric density but not metric density 1 at \( A \). Then by 2.22, for every set \( E \), the set of \( B \)-points of \( E \) (in the plane) is of measure 0. A \( B(1) \)-point of \( f(P) \) is a point \( (A; S) \) which is positively approached but not fully approached by \( f(P) \). Applying 5.7, we obtain a new proof of a result of Blumberg [7, Theorem III]:

If \( z = f(P) \) is an arbitrary, one- or many-valued, real function, the set of points \( A \) for which a \( S \) exists such that \( (A; S) \) is positively but not fully approached by \( f(P) \) is of measure 0.

5.14. We omit the proof of the following corollary of 5.13:

For every real function \( y = f(x), U^+(x) = U^-(x) \) and \( L^+(x) = L^-(x) \) almost everywhere.

5.15. If \( E \) is a linear set, let \( (E, \xi)^B := \) the lower metric density of \( E \) is positive on one side and the metric density of \( E \) is 0 on the other side of \( \xi \).
By (1) of 2.24, the set of points of the line which are \( \beta \)-points of \( E \) is countable. A \( \beta(1) \)-point of \( f(x) \) is positively approached from one side but vanishingly approached from the other side by \( f(x) \). Applying the linear analogue of 5.7, we obtain the result:

If \( y = f(x) \) is an arbitrary, one- or many-valued, real function, the set of abscissas \( \xi \) for which an \( \eta \) exists such that \((\xi, \eta)\) is positively approached from one side but vanishingly approached from the other side by \( f(x) \) is countable.

5.16. We now give a new proof of a theorem of S. Kempisty [23]. (For another proof, see Blumberg [3, Example 5]). Although this passage from set theorem to function theorem does not come directly under the formulation of 5.7, the ideas involved are the same.

**Theorem.** For every real function \( y = f(x) \), the set \( E_x[\alpha^+(x) < \alpha^-(x)] \) is countable.

**Proof.** Let \( \xi \in H = E_x[\alpha^+(x) < \alpha^-(x)] \). Then there exists a rational number \( r \) with \( U^+(\xi) < r < L^-(\xi) \), and therefore \( E_r = E_x[f(x) > r] \) has metric density 0 at \( \xi \) on the right and metric density 1 at \( \xi \) on the left, i.e., \( \xi \in H_r \), the set of all points of the line so related to \( E_r \). By (2) of 2.24, \( H_r \) is countable and therefore \( H = \sum H_r \), \( r \) ranging over all rational numbers, is countable. The theorem follows.
5.17. The following definition is equivalent to one of S. Kempisty [24, §2]: \( f(P) \) is quasi-continuous at \( A := \) for every \( h < f(A) < k \), \( E_{hk} \) contains a partial neighborhood of \( A \) (see 2.2). Let \( (E,A)^B := E \) contains a partial neighborhood but does not contain a neighborhood of \( A \). By 2.19, the set of points of the plane which are \( \beta \)-points of \( E \) is non-dense. \( (A,f(A)) \) is a \( \beta(1) \)-point of a single-valued function \( f(P) \) if and only if \( f(P) \) is quasi-continuous but not continuous at \( A \). Applying 5.7, we conclude:

The set of points \( A \) of the plane at which an arbitrary, single-valued, real function \( z = f(P) \) is quasi-continuous but not continuous is exhaustible.

Remark. The concepts of continuity and quasi-continuity and the above result may be readily extended to many-valued functions.

5.18. An immediate corollary of 5.17 is the following:

A real function which is everywhere quasi-continuous is continuous on a residual set.

5.19. Extending the terminology of 2.21 to functions, the point \( (A; \bar{U}) \) is approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive (exterior) measure by \( f(P) \) (by \( f(P) \) via the partial neighborhood \( G \) of \( A \) ) := for every \( h, k, h < \bar{U} < k, A \)
is respectively approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive (exterior) measure by $E_{nk}$ ($GE_{nk}$).

Let $(E, A)^\beta := A$ is approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive measure by $E$, but there exists a non-vanishing partial neighborhood $G$ of $A$ with $GE$ respectively finite, countable, exhaustible, of measure 0. By 2.21, for each of these set-point properties, the set of all $\beta$-points of $E$ is non-dense and of measure 0. Applying 5.7, we obtain the theorem:

If $z = f(P)$ is an arbitrary, one- or many-valued, real function, then for all points $A$ (of the plane) not belonging to a set which is exhaustible and of measure 0, every point $(A; \gamma)$ which is approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive measure by $f(P)$ is also, respectively, approached, non-denumerably approached, inexhaustibly approached, approached in the sense of positive measure by $f(P)$ via every non-vanishing partial neighborhood $G$ of $A$.

5.20. Using special cases of 5.10 with $E_1^\alpha$ meaning respectively that $E_1$ is countable, of measure 0, together with the second and fourth parts of 5.19, we
obtain the following result, the second part of which is due to Blumberg [6, Theorem VIII]:

If \( z = f(P) \) is an arbitrary, one- or many-valued, real function, then for all points \( A \) (of the plane) not belonging to a set which is (1) exhaustible and of measure 0, (2) of measure 0, \((A, f(A))\) -- for all values of \( f(A) \) -- is respectively (1) non-denumerably approached, (2) approached in the sense of positive measure by \( f(P) \) via every non-vanishing partial neighborhood \( G \) of \( A \).

§19. Extension of Some Results of Chapter III

5.21. Many of the results of Chapter III may be carried over to functions. General formulations of the type 5.4 and 5.7 may be derived for this purpose, but we shall merely give some illustrative examples.

5.22. As a first example, we shall carry over the result 3.7 to functions. Let, then, \( L \) be a straight line, \( K \) one of the closed half-planes into which \( L \) divides the plane, and \( \mathcal{C} \) a set property which is descending and such that the sum of two sets of property \( \mathcal{C} \) fills no region.

We shall use the abbreviations max, min respectively for maximum, minimum of a set consisting of real numbers and possibly one or more of the ideal points \(+\infty, -\infty\). Let \( z = f(P) \) be a one- or many-valued, real (and finite) function defined on the half-plane \( K \). If \( S \) is an open
sector, we denote by $U_{\mathcal{A}}(f,S)$ ($L_{\mathcal{A}}(f,S)$) the min (max), for all sets $T$ of property $\mathcal{A}$, of the max (min) of $f(P)$ on $ST$ ($T = K - T$). We denote by $U_{\mathcal{A}}(f,A,\theta)$ ($L_{\mathcal{A}}(f,A,\theta)$) the min (max) of $U_{\mathcal{A}}(f,S)$ ($L_{\mathcal{A}}(f,S)$) for all sectors $S(A;\theta)$ (see 3.2). ($U_{\mathcal{A}}(f,A,\theta)$ ($L_{\mathcal{A}}(f,A,\theta)$) may be called the $\mathcal{A}$-maximum ($\mathcal{A}$-minimum) of $f(P)$ at $A$ in direction $\theta$.) Let $I_{\mathcal{A}}(f,A,\theta)$ denote the interval $[L_{\mathcal{A}}(f,A,\theta),U_{\mathcal{A}}(f,A,\theta)]$.

Because of the restrictions on the set property $\mathcal{A}$, it follows that (1) necessarily $L_{\mathcal{A}}(f,A,\theta) \leq U_{\mathcal{A}}(f,A,\theta)$, and (2) for every $k > U_{\mathcal{A}}(f,A,\theta)$ ($k < L_{\mathcal{A}}(f,A,\theta)$), the set $E[f(P) > k] = E_p[f(P) > k$ for at least one value $f(P)]$ ($E[f(P) < k]$) has an $\mathcal{A}$-void at $A$ in direction $\theta$, i.e., $(A,\theta)$ is an $\mathcal{A}(1)$-tangent ray of $E$ (see 3.7).

Let $E_k$ denote the exceptional, countable set of the result of 3.7 when the set $E$ of this result is identified with $E_k = E[f(P) > k]$, and let $H = \sum H_k, k$ ranging over the set of rational numbers. We readily see that if $A$ is in $L - H$, and $(A,\theta), (A,\emptyset)$ are two rays in $K$, then $I_{\mathcal{A}}(f,A,\theta)$ overlaps or abuts $I_{\mathcal{A}}(f,A,\emptyset)$. For suppose, say, that $U_{\mathcal{A}}(f,A,\theta) < L_{\mathcal{A}}(f,A,\emptyset)$. Then, taking a rational $k$ between these two numbers, it follows that $E_k$, $E_k$ respectively have $\mathcal{A}$-voids at $A$ in directions $\theta, \emptyset$, i.e., $A \in E_k \subset H$. We thus obtain a result related to one of M. F. Schmeiser [31, §1, Theorem 2''], the result
of Schmeiser being restricted to two fixed directions of approach.

Theorem. If $L$ is a given straight line, $K$ one of the closed half-planes into which $L$ divides the plane, $z = f(P)$ a one- or many-valued, real function defined in $K$, and $\mathcal{A}$ a set property which is descending and such that the sum of two sets of property $\mathcal{A}$ fills no region, then for every point $A$ of $L$ not belonging to an exceptional, countable set, $I_{\mathcal{A}}(f,A,\emptyset)$ and $I_{\mathcal{A}}(f,A,\emptyset)$ either overlap or abut for every pair of directions $\Theta$, $\emptyset$ in $K$.

5.23. As another example, we obtain a new and simpler proof of a theorem of Jarnik [22], being an improvement of a theorem of Schmeiser [30]. The theorem of Schmeiser, in turn, is an improvement of a theorem of Blumberg [3]. By an application of a theorem on sets, Blumberg [4] has given a new proof of this theorem of Schmeiser. We use the same set theorem in our application below.

Theorem (Blumberg, Schmeiser, and Jarnik). If $L$ is a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, and $z = f(x,y) = f(P)$ a one- or many-valued, real function defined in $K^0$, then for every point $A$ of $L$ not belonging to an exceptional, countable set, and every pair of directions $\Theta$, $\emptyset$ in $K^0$, the set $E(A,\Theta)$ of all limit values of $f(P)$ obtainable as $P$
approaches $A$ along $(A, \emptyset)$ has at least one point in common with $E(A, \emptyset)$, similarly defined.

Remarks. (1) The ideal points $+\infty$ and $-\infty$ are allowable elements of $E(A, \emptyset)$ and $E(A, \emptyset)$. (2) The statement of the theorem of Schmeiser is obtained if in the above statement the sets $E(A, \emptyset)$, $E(A, \emptyset)$ are respectively replaced by the smallest closed intervals containing them. The theorem of Blumberg is then obtained if the two directions are fixed and do not vary with $A$.

Definitions. Letting $\mathcal{C}$ denote the linear continuum, we shall mean by $\mathcal{C}'$ the set $\mathcal{C}$ plus the ideal points $+\infty$ and $-\infty$. A set $F$ will be called $\mathcal{C}'$-closed (closed with respect to $\mathcal{C}'$) if it is closed and contains $+\infty$, $-\infty$ respectively whenever $FC$ is not bounded from above, below. A set will be called a restricted $\mathcal{C}'$-closed set if it is the sum of a finite number of closed intervals with endpoints in $\mathcal{C}'$. The term "R-restricted" will be used in place of "restricted" if the finite end-points of these intervals are rational numbers.

Lemma (1). If $F_1$ and $F_2$ are disjoint $\mathcal{C}'$-closed sets, then there exists an R-restricted $\mathcal{C}'$-closed set $F$ for which $d(F, F_2) > 0$ and $d(F, F_1) > 0$.

Proof. At least one of the sets $F_1$, $F_2$ is bounded, or else one is bounded from above and the other from below. In any case, there is a positive distance between
the two sets, say \( d(F_1, F_2) > 2\delta > 0 \). Choose \( m < M \) so that each of the intervals \([-\infty, m] \) and \([M, +\infty] \) contains a point of at most one of the two given sets. Choose \([a, b]\) so that \( b - a \) is a multiple of \( \delta \) and \( a + \delta < m, M + \delta < b \). Let the subdivision \( \sigma \) divide \([a, b]\) into closed subintervals of length \( \delta \), and let \( T \) consist of the intervals of \( \sigma \) together with \([-\infty, a] \) and \([b, +\infty]\).

Call each interval of \( T \) black, red, or white according as it contains a point of \( F_1 \), a point of \( F_2 \), or no point of either set. No interval of \( T \) is both black and red, nor can a black interval abut a red interval. Grouping abutting black intervals of \( T \) into a single interval, call these black intervals \( I_i, i = 1, 2, \ldots, n \). Each finite end-point of every \( I_i \) is also an end-point of a white interval of \( T \). For such an end-point, enlarge \( I_i \) by choosing a rational point in the middle third of this white interval. We thus obtain a closed interval \( J_i \supset I_i \) whose end-points are either rational or infinite. Then \( F = \sum J_i, i = 1, 2, \ldots, n \), is an \( R \)-restricted \( C^1 \)-closed set, \( d(F, F_2) \geq \delta/3 \), and \( d(F, F_1) \geq \delta/3 \).

Lemma (2). The set of all \( R \)-restricted \( C^1 \)-closed sets is denumerable.

Proof of theorem. Let \( H \) denote the exceptional set of points \( A \) of \( L \) as described in the theorem. If \( A \) is in \( H \), there exist two directions \( \theta \) and \( \phi \) in \( K^0 \) for which
\(E(A,\theta)E(A,\emptyset) = 0\). It is readily seen that both \(E(A,\theta)\) and \(E(A,\emptyset)\) are \(C'\)-closed sets. By Lemma (1), there exists an \(R\)-restricted \(C'\)-closed set \(F\) for which \(d[F,E(A,\emptyset)] > 0\) and \(d[F,E(A,\theta)] > 0\). Let \(E_F = E_P[f(P) \in F]\), and let \(H_F\) denote the exceptional, countable set associated with \(E_F\) by the set theorem 3.18 of H. Blumberg. Then the point \(A\) is in \(H_F\), for \(A\) is not approachable by \(E_F\) along the ray \((A,\emptyset)\) and \(A\) is not approachable by \(E_F\) (= \(K^o - E_F\)) along \((A,\theta)\). Hence \(H \subseteq \sum H_F\), where in the summation \(F\) ranges over all \(R\)-restricted \(C'\)-closed sets. By Lemma (2) the class of all such sets \(F\) is denumerable, and it follows that \(H\) is countable. The theorem follows.

5.24. Let \(L\) be a straight line, \(K^o\) one of the open half-planes into which \(L\) divides the plane, and \(z = f(P)\) a one- or many-valued, real function defined in \(K^o\). If \(\theta\) is a direction in the xy-plane, we say that \(\theta\) is a (planar) direction of approach (or \((A,\theta)\) is in the planar contingent) of \(f(P)\) at \((A; \zeta)\) if there exists a sequence \(\{A_n\}\) which approaches \(A\) via direction \(\theta\) and such that \(\{(A_n;f(A_n))\}\) approaches \((A; \zeta)\), \(f(A_n)\) being a properly chosen value of \(f\) at \(A_n\). Then \(\theta\) is a direction of approach of \(f(P)\) at \((A; \zeta)\) if and only if for every \(h < \zeta < k\), \(E_{hk} = E_P[h < f(P) < k]\) approaches \(A\) via direction \(\theta\), i.e., \((A,\theta) \in T(A,E_{hk})\), the contingent of \(E_{hk}\) at \(A\).
Suppose that \( A \in L \), and \((A, \theta), (A, \emptyset)\) are two rays in \( K^0 \) such that \( f(P) \) approaches \((A; \xi)\) via direction \( \theta \) but not via direction \( \emptyset \). Hence there exist rational numbers \( r < \xi < s \) such that \( E_{rs} \) approaches \( A \) via direction \( \theta \) but not via direction \( \emptyset \). Thus \( A \in H_{rs} \), the exceptional set associated with the set \( E_{rs} \) according to the result 3.29. \( H_{rs} \) is exhaustible and of measure 0 and so is the set \( H = \sum H_{rs} \), where in the summation \( r < s \) range over all rational numbers. Likewise, every point \( A \in L \) for which there is a \( \xi \) such that \((A; \xi)\) is approached via one direction in \( K^0 \) but not via all directions in \( K^0 \), is contained in \( H \). We may conclude:

**Theorem.** If \( L \) is a given straight line, \( K^0 \) one of the open half-planes into which \( L \) divides the plane, and \( z = f(P) \) a one- or many-valued, real function defined in \( K^0 \), then for all points \( A \) of \( L \) not belonging to an exceptional set \( H \) which is exhaustible and of measure 0, the set of limit values of \( f(P) \) obtainable by approaching \( A \) via any one direction in \( K^0 \) is the same as the set of limit values obtainable by approaching \( A \) via all directions in \( K^0 \).

5.25. If \( \alpha \) is a set property, we shall say that \((A; \xi)\) is \( \alpha \)-approached by \( f(P) \) via the direction \( \theta \) in the \( xy \)-plane (or \( \xi \) is an \( \alpha \)-limit value of \( f(P) \) that is obtainable by approaching \( A \) via direction \( \theta \)) if for
every $h < \zeta < k$, $E_{hk} = E_{P}[h < f(P) < k]$ $\mathcal{A}$-approaches $A$ via direction $\theta$ (see 3.5). We may apply 3.36 to obtain the following result. The details, being like those of 5.24, are omitted.

If $L$ is a straight line, $K^0$ one of the open half-planes into which $L$ divides the plane, $z = f(P)$ a one- or many-valued, real function defined in $K^0$, and $\mathcal{A}$ a set property which is descending and countably additive, then for every point $A$ of $L$ not belonging to an exceptional set $H$ which is exhaustible and of measure 0, the set of all $\mathcal{A}$-limit values of $f(P)$ obtainable by approaching $A$ via any one direction in $K^0$ is the same as the set of $\mathcal{A}$-limit values obtainable by approaching $A$ via all directions in $K^0$.

§20. Extension of Some Results of Chapter IV

5.26. Adding to the notation of 3.5 and 5.25, the planar $\mathcal{A}$-contingent of $f(P)$ at $B = (A; \zeta)$, denoted by $T_{\mathcal{A}}(B,f)$, will consist of all rays $(A, \theta)$ in the xy-plane for which $B$ is $\mathcal{A}$-approached by $f(P)$ via the direction $\theta$. The following lemma is immediate. As usual, we use the notation $E_{hk} = E_{P}[h < f(P) < k]$.

Lemma. If $z = f(P)$ is a one- or many-valued, real function, $\mathcal{A}$ a descending and countably additive set property, and $B = (A; \zeta)$ a point in 3-space, then
\[ T_{\mathcal{J}}(B,f) = \bigcap_{n} T_{\mathcal{J}}(A,E_{h_n,k_n}) \]

for every infinitesimal open interval \((h_n,k_n)\) containing \(J\). Consequently, \(T_{\mathcal{J}}(B,f)\) is closed.

5.27. If \(E\) is a planar set, \(\mathcal{A}\) a descending and countably additive set property, and \(A\) a point in the plane, let \((E,A)^{\beta} := \) the \(\overline{\mathcal{A}}\)-contingent \(T_{\mathcal{J}}(A,E)\) of \(E\) at \(A\) is either empty or the whole plane (see 3.5). By (3) of 3.33 and (1) of 4.28, if \(E\) is a planar set, every point \(A\) of the plane not belonging to a sparse set is a \(\beta\)-point of \(E\).

By 5.26, it is clear that if \(B = (A;J)\) is a \(\beta(2)\)-point of \(f(P)\), then \(T_{\mathcal{J}}(B,f)\) is either empty or the whole plane. And by 5.8, for all points \(B = (A;J)\) in 3-space such that \(A\) does not belong to an exceptional sparse set, \(B\) is a \(\beta(2)\)-point of \(f(P)\). We have thus derived the following result:

**Theorem.** If \(z = f(P)\) is a one- or many-valued, real function, and \(\mathcal{A}\) a descending and countably additive set property, then for all points \(A\) of the plane not belonging to an exceptional sparse set, the set of \(\overline{\mathcal{A}}\)-limit values of \(f(P)\) obtainable by approaching \(A\) via any one direction in the \(xy\)-plane consists of all the \(\overline{\mathcal{A}}\)-limit values of \(f(P)\) at \(A\).

5.28. For the remainder of this section, we shall let \(\mathcal{A}\) denote a descending and countably additive set
property, and let \((E,A)^\beta := T_\mathcal{J}(A,E)\) is either (1) empty, or (2) a single line, or (3) a half-plane, or (4) the whole plane. By (3) of 4.28 we consequently have the result:

If \(E\) is a planar set, the set of points of the plane which are not \(\beta\)-points of \(E\) is of outer length 0.

5.29. If \(f(P)\) is a function and \(B = (A; \zeta)\) is a \(\beta(2)\)-point of \(f(P)\), there exists an infinitesimal interval \((h_n, k_n)\), \(h_n < \zeta < k_n\), with \((E_n, k_n, A)^\beta\) for every \(n\). Since \(T_\mathcal{J}(A, E_n)\) is contained in \(T_\mathcal{J}(A, E)\) whenever \(E_n\) is contained in \(E\), it follows by the lemma of 5.26 that \(T_\mathcal{J}(B, f)\) is (1), (2), (3), or (4) according as for all but a finite number of the \(n\), \(T_\mathcal{J}(A, E_{nk})\) is (1), (2), (3), or (4), respectively. We thus have the result:

If \(B = (A; \zeta)\) is a \(\beta(2)\)-point of \(f(P)\), then (I) \(T_\mathcal{J}(B, f)\) is either (1) empty, or (2) a single line, or (3) a half-plane, or (4) the whole plane, and (II) according as \(T_\mathcal{J}(B, f)\) is (1), (2), (3), (4), then \(T_\mathcal{J}(A, E_{hk})\) is respectively (1), (2), (3), (4) for \(h < \zeta < k\) and \(\zeta - h, k - \zeta\) sufficiently small.

5.30. If \(z = f(P)\) is a one- or many-valued, real function defined in the plane, then for all points \(B = (A; \zeta)\) in 3-space such that \(A\) does not belong to an exceptional planar set of outer length 0, \(B\) is a \(\beta(2)\)-point of \(f(P)\).
Proof. Using 5.28, this result is a direct corollary of 5.8.

5.31. If $T_j(B,f)$ (or $T_j(A,E)$) is a single line or a half-plane (cases (2) and (3)), it has what we shall call an extreme direction, namely either of the directions of the single line or the boundary line of the half-plane. If either of the cases (1), (4) holds, then any direction may be considered as an extreme direction.

5.32. If $A \notin H$, the exceptional set of 5.30, then by (I) of 5.29, for every $\xi$, $T_j[(A; \xi), f]$ is either (1), (2), (3), or (4). It appears, however, that the extreme direction of $T_j[(A; \xi), f]$ may vary with $\xi$ if $H$ is defined as in 5.8. We shall therefore enlarge $H$. Let $E_{rstuv} = E_{rs} + E_{tu}$, and let $H_{rstuv}$ denote the set of points of the plane which are not $\beta$-points of $E_{rstuv}$. Now let $H_1 = \sum H_{rstuv}$, where in the summation the subscripts $r < s$ and $t < u$ range over all rational numbers. Then $H_1 \supset H$ but $H_1$ is still of outer length 0. Let $A \notin H_1$, and suppose the extreme directions of $T_j[(A; \xi_1), f]$ and $T_j[(A; \xi_2), f]$ are different. It follows by (II) of 5.29 that there exist rational intervals $(r, s)$, $(t, u)$ respectively containing $\xi_1$, $\xi_2$, and such that $A$ is not a $\beta$-point of $E_{rstuv}$, i.e., $A \notin H_{rstuv} \subset H_1$, a contradiction. Thus, if $A \notin H_1$, the extreme direction of $T_j[(A; \xi), f]$ does not vary with $\xi$, and we have the following
Theorem. If \( z = f(P) \) is a one- or many-valued, real function defined in the plane and \( \alpha \) a descending and countably additive set property, then for every point \( A \) of the plane not belonging to an exceptional set of outer length 0, there is a line \( L \) (possibly arbitrary) through \( A \) of direction \( \theta \) such that every \( L \)-limit value of \( f(P) \) at \( A \) is obtainable by approaching \( A \) via either of the directions \( \theta, \bar{\theta} \), and if \( \theta_1, \theta_2 \) are in a fixed open half-plane determined by \( L \), the sets of \( L \)-limit values of \( f(P) \) obtainable by approaching \( A \) respectively via directions \( \theta_1, \theta_2 \) are identical.

5.33. Let \( E_1 \) be empty. For this particular descending, countably additive set property, \( L \)-approach of a set or function becomes (ordinary) approach, and an \( L \)-limit value of \( f(P) \) becomes an (ordinary) limit value. For this \( L \), 5.27 becomes a theorem of W. H. Young [34, §8], and is an interesting application of the general formulation 5.8 for carrying set theorems over to functions.

Also, for this same particular \( L \), 5.32 becomes a result of U. S. Haslam-Jones [19, Theorem 5]. Haslam-Jones bases his proof on certain theorems on "non-oscillatory curves," and on certain auxiliary results on the upper and lower limit functions of the given function \( f(P) \). His principal auxiliary result, an immediate corollary of the end result, has been stated without proof.
by Kolmogorov and Verčenko [25]. Our proof of the Haslam-Jones end result (which may be obtained mutatis mutandis from the proof of the generalized theorem 5.32) seems to be considerably simpler than his, for it is obtained by a direct carry over to functions of the consolidated result 4.25 on sets.

5.34. Just as we applied the set theorem 3.18 to obtain the function theorem 5.23, we may apply the set theorem 4.41 to obtain a theorem on functions. We use the notation of 5.23.

Theorem. If \( z = f(P) \) is a one- or many-valued, real function defined in the plane, then for every point \( A \) of the plane not belonging to an exceptional, sparse set and for every pair of non-opposite directions \( \theta, \phi \) in the \( xy \)-plane, the set \( E(A,\theta) \) of all limit values of \( f(P) \) obtainable as \( P \) approaches \( A \) along \( (A,\theta) \) has at least one point in common with \( E(A,\phi) \), similarly defined.

Proof. Let \( H \) denote the exceptional set of points \( A \) as described in the theorem. If \( A \) is in \( H \), there exist two non-opposite directions \( \theta, \phi \) for which \( E(A,\theta) \cap E(A,\phi) = \emptyset \). Since both \( E(A,\theta) \) and \( E(A,\phi) \) are \( C^1 \)-closed sets, it follows by Lemma (1) of 5.23 that there exists an \( R \)-restricted \( C^1 \)-closed set \( F \) for which \( d[F,E(A,\phi)] > 0 \) and \( d[F,E(A,\theta)] > 0 \). Let \( E_F = E_F[f(P) \in F] \) and let \( H_F \) denote the exceptional, sparse set associated with \( E_F \) by 4.41.
Then $A \in H_F$, for $A$ is not approachable by $E_F$ along the ray $(A, \emptyset)$ and $A$ is not approachable by $E_F$ along $(A, \emptyset)$. Hence $H \subset \sum H_F$, where in the summation $F$ ranges over all $R$-restricted $C'$-closed sets. By Lemma (2) of 5.23, it follows that $H$ is contained in the sum of a denumerable number of sparse sets and is therefore sparse. The theorem follows.

§21. On Generalized Dini Derivatives

5.35. Throughout this section, let $y = f(x)$ be defined on an arbitrarily chosen linear point set $R$, assuming at each point of $R$ one or more values, and let $\alpha$ denote a linear set property such that $\alpha$ is descending and countably additive. $\xi$ is an $\alpha$-point of $E$ (from the right, left) :=: for every open interval $I$ containing $\xi$ (with $\xi$ as left, right end-point), $IE$ has property $\alpha$.

As usual, let $E_{hk}$ consist of the points $x$ such that at least one value of $f(x)$ lies in $(h, k)$. We say that $(\xi, \eta)$ is an $\alpha$-point of $f$ (from the right, left) if for every $h < \eta < k$, $\xi$ is an $\alpha$-point of $E_{hk}$ (from the right, left). $(\xi, +\infty)$ $[((\xi, -\infty))$ is an $\alpha$-point of $f$ (from the right, left) :=: for every real $k$, $\xi$ is an $\alpha$-point of $E_{x}[f(x) > k] [E_{x}[f(x) < k]]$ (from the right, left).

The upper (lower) $\alpha$-limit of $f(x)$ at $\xi$, which we denote by $U_{\alpha} (\xi, f)$ $(L_{\alpha} (\xi, f))$, is the max (min) of all
for which \( A: (\xi, \eta) \) is an \( \alpha \)-point of \( f \). The upper, right and left, and the lower, right and left, \( \alpha \)-limits of \( f \) are similarly defined in terms of the right- and left-hand \( \alpha \)-points \( (\xi, \eta) \) of \( f \) and are denoted respectively by \( U_\alpha^+ (\xi, f) \), \( U_\alpha^- (\xi, f) \), \( L_\alpha^+ (\xi, f) \), \( L_\alpha^- (\xi, f) \).

Let \( R_1 \) denote the set of all points \( \xi \) for which the line \( x = \xi \) contains at least one \( \alpha \)-point of \( f \). Then there exists a countable set \( D \) so that whenever \( \xi \) is in \( R_1 - D \), it follows that \( A: (\xi, \eta) \) (possibly infinite) either possesses none or all of the properties: (1) \( A \) is an \( \alpha \)-point of \( f \), (2) \( A \) is a right-hand \( \alpha \)-point of \( f \), (3) \( A \) is a left-hand \( \alpha \)-point of \( f \). If \( \xi \in (R_1 - D) \), then \( U_\alpha (\xi, f) = U_\alpha^+ (\xi, f) = U_\alpha^- (\xi, f) \) and \( L_\alpha (\xi, f) = L_\alpha^+ (\xi, f) = L_\alpha^- (\xi, f) \), each point of any of these functions possessing the three properties (1), (2), and (3).

5.36. \( S(A; \theta) \) (\( S^\#(A; \theta) \)) is a variable sector of vertex \( A \), with \( \theta \) as an interior direction, and of finite (infinite) radius. \( S(A; \theta^+) \), \( S(A; \theta^-) \), \( S^\#(A; \theta^+) \), \( S^\#(A; \theta^-) \) denote variable sectors of the forms \( S(A; \theta \phi h) \), \( S(A; \theta \phi h) \), \( S(A; \theta \phi) \), \( S(A; \theta \phi) \), respectively, the latter two being of infinite radius (see 3.2). \( (A, \theta) \) is called an \( \alpha \)-derivate ray (Dini- \( \alpha \)-derivate ray) of \( f \) if for every sector \( S = S(A; \theta) \) \( (S = S^\#(A; \theta)) \), \( E_\xi = E_\xi [(x, f(x)) \in S \text{ for at least one value of } f(x)] \) has \( \xi \) as an \( \alpha \)-point. An \( \alpha \)-derivate ray or Dini- \( \alpha \)-derivate ray of \( f \) of finite
slope is said to be right-hand (left-hand) if it is in
the right half-plane \((\pi/2, \pi)\) (the left half-plane
\((\pi, 3\pi/2)\)). \((A, \pi/2)\) is a right-hand (left-hand)
\(\alpha\)-derivate ray [Dini- \(\alpha\)-derivate ray] of \(f\) if \(\xi\) is an
\(\alpha\)-point of \(E_\alpha\) for every sector \(S(A; \pi/2 +)\) \((S(A; \pi/2 -))\)
\([S\delta(A; \pi/2 +)\) \((S\delta(A; \pi/2 -))\]. Similar definitions are
made for \((A, 3\pi/2)\).

The upper and lower \(\alpha\)-derivatives [Dini- \(\alpha\)-derivatives] on the right (left) of \(f(x)\) at \(A:(\xi, \eta)\) are the
max and min, respectively, of the slopes \(m(A, \theta)\) of all
right-hand (left-hand) \(\alpha\)-derivate rays [Dini- \(\alpha\)-derivate rays] \((A, \theta)\) of \(f\) at \(A\). We shall denote them by
\(D^+_{\alpha}(A, f)\), \(D^\alpha_{+}(A, f)\), \((D^\alpha_{-}(A, f), D^-_{\alpha}(A, f))\), \([D^\delta_{\alpha}(A, f),\]
\(D^\delta_{+}(A, f),\) \((D^\delta_{\alpha}(A, f), D^\delta_{-}(A, f))\)] respectively. If the
four extreme \(\alpha\)-derivatives [Dini- \(\alpha\)-derivatives] of \(f\)
are all equal at \(A\), \(f(x)\) will be said to have an \(\alpha\)-
derivative [Dini- \(\alpha\)-derivative] at \(A\) equal to this com-
mon value and denoted by \(D^\alpha_{\alpha}(A, f)\) \([D^\delta_{\alpha}(A, f)]\).

5.37. If \(\xi \in (R_1 - D)\), defined as in 5.35, then it
is readily seen that the four extreme \(\alpha\)-derivatives
(Dini- \(\alpha\)-derivatives) are defined at every finite \(\alpha\)-
point (point) on \(x = \xi\). Letting \(Q\) denote the set of all
points \((x, f(x))\) \((Q = T(f; R)\) by the notation of 4.17), it
is evident that the set of \(\alpha\)-derivate rays of \(f\) at \(A\)
is simply the \( \alpha \)-contingent \( T_{\alpha}(A,Q) \) of \( Q \) at \( A \) with respect to the \( x \)-axis (see \( 4.29 \)).

**Theorem.** If \( y = f(x) \) is a one- or many-valued, real function defined on an arbitrary set on the line, \( \alpha \) a set property such that \( \alpha \) is descending and countably additive, then (I) the set of \( \alpha \)-points of \( f \) at each of which the set of \( \alpha \)-derivate rays of \( f \) is contained in an angle of magnitude less than \( \pi \) is countable, and (II) at every \( \alpha \)-point \( A = (\xi, \eta) \) of \( f \) for which \( \xi \) does not belong to an exceptional set \( Z \) of measure 0, the set of \( \alpha \)-derivate rays of \( f \) at \( A \) constitute either (1) a single line of finite slope, or (2) a half-plane whose boundary line is of finite slope, or (3) the whole plane.

**Proof.** Part (I) follows from Corollary (2) of 4.29. Part (II) follows from Corollary (3) of 4.29 and (2) of 4.33. It is evident how this result may be stated in terms of the extreme \( \alpha \)-derivatives.

**5.38.** Every right-hand (left-hand) \( \alpha \)-derivate ray of \( f \) at \( A \) is a right-hand (left-hand) Dini-\( \alpha \)-derivate ray of \( f \) at \( A \), but one of the latter of infinite slope is not necessarily one of the former. If \( D_{\alpha}^{+}(A,f) \) is finite, then \( D_{\alpha}^{+}(A,f) = +\infty \) if and only if the line \( x = \xi \) contains a right-hand \( \alpha \)-point of \( f \) above \( A \). Making use of this and other similar statements about the other extreme \( \alpha \)-derivatives, together with 5.35 and
and 5.37, we obtain the following theorem on the extreme Dini-$\mathcal{A}$-derivatives.

Theorem. If $y = f(x)$ is a one- or many-valued, real function defined on an arbitrary set on the line, $\mathcal{A}$ a set property such that $\mathcal{A}$ is descending and countably additive, $R_1$ the set of points $\xi$ for which the line $x = \xi$ contains at least one $\mathcal{A}$-point of $f$, and $\mathcal{J}$ a finite number, then

(I) there exists a countable set $D$ such that if $\xi$ is in $(R_1 - D)$ and $L_{\mathcal{A}}(\xi, f) \leq \mathcal{J} \leq U_{\mathcal{A}}(\xi, f)$, then $D_{\mathcal{A}}^+(A, f) > D_{\mathcal{A}}^-(A, f)$ and $D_{\mathcal{A}}^+(A, f) > D_{\mathcal{A}}^-(A, f)$ where $A = (\xi, \mathcal{J})$;

(II) there exists a set $Z$ of measure 0 such that if $\xi \in (R_1 - Z)$ and $L_{\mathcal{A}}(\xi, f) \leq \mathcal{J} \leq U_{\mathcal{A}}(\xi, f)$, and letting $A$ denote $(\xi, \mathcal{J})$, then either

1) $f(x)$ has a finite Dini-$\mathcal{A}$-derivative at $A$, or

2) the upper Dini-$\mathcal{A}$-derivative of $f$ at $A$ on one side is $+\infty$, the lower Dini-$\mathcal{A}$-derivative of $f$ at $A$ on the other side is $-\infty$, and the other extreme Dini-$\mathcal{A}$-derivatives of $f$ at $A$ are finite and equal, or

3) the two upper Dini-$\mathcal{A}$-derivatives of $f$ at $A$ are $+\infty$, and the two lower Dini-$\mathcal{A}$-derivatives of $f$ at $A$ are $-\infty$.

5.39. If $y = f(x)$ is restricted to be a single-valued real function defined on the line and if $E^{\mathcal{A}} := E$ is non-empty, 5.38 becomes the classical theorem of Denjoy [15],
Young [33], and Saks [28] on Dini derivatives. U. S. Haslam-Jones [18] has obtained this special case by applying results equivalent to 4.15, 4.24, and 4.32. Again for a single-valued function $f(x)$, a corollary of 5.38 is a result of Hanson [17] on the Dini-$\alpha$-derivatives, which he derived in a different way. The set property $\alpha$ of Hanson is more restricted than ours, satisfying two additional properties; in particular, his result does not contain the classical theorem on Dini derivatives as a special case.
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I, Herbert Charles Parrish, was born in Jacksboro, Texas, October 8, 1919. I received my secondary school education in the public schools of Jermyn, Texas. My undergraduate training was obtained at North Texas State College, from which I received the degree of Bachelor of Science in 1939. I also received the degree of Master of Science from North Texas State College in 1941. While in residence at The Ohio State University during the years of 1941-43 and 1946-48, I acted in the capacity of Assistant in the Department of Mathematics. I held the position of Instructor in Mathematics at The Ohio State University during the year 1948-49. I have been teaching mathematics at North Texas State College since September, 1949.