Dynamic Analysis of Fractionally-Damped Elastomeric and Hydraulic Vibration Isolators

DISSERTATION

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By

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Abstract

This research focuses on the development of new analytical and semi-analytical models for a class of elastomeric and hydraulic bushings. Historically, such vibration isolators have been studied in the context of the linear system theory using simplified viscoelastic elements with focus on uniaxial behavior at lower frequencies. Relatively few researchers have utilized advanced viscoelastic models or nonlinear, excitation-amplitude sensitive models. To overcome these voids, the primary goals of this work are to develop isolator models using a fractional calculus viscoelastic formulation and to accurately predict amplitude dependent dynamic stiffness of fluid-filled devices.

First, the fractional calculus is utilized to describe the damping properties of elastomeric bushings in both time and frequency domains. For the frequency domain, a new spectral element approach is proposed to determine the multi-axis dynamic stiffness terms of elastomeric isolators with fractional damping over a broad range of frequencies using the continuous system theory (in terms of homogeneous rods or Timoshenko beams). The transfer matrix type dynamic stiffness expressions are developed from analytical harmonic solutions given translational or rotational displacement excitations. Predictions match well with available measurements up to 600 Hz. Second, a lumped parameter system model with fractional damping is developed to explore the dynamic stiffness behavior over the lower frequency range. The inverse Laplace transform of the dynamic stiffness spectrum is taken via the Residue Theorem to produce impulse
response functions. Since the fractional calculus based solution is given in terms of problematic integrals, a new time-frequency domain estimation technique is proposed which approximates time-domain responses for a class of transient excitation functions. The approximation error is found to be reasonably small, and tractable closed-form transient response functions are generated.

Third, refined quasi-linear and nonlinear hydraulic bushing models are proposed with focus on the sinusoidal excitation amplitude sensitivity in the dynamic stiffness magnitude and loss angle spectra (up to 50 Hz). Nonlinear chamber compliance and track resistance elements are incorporated in order to improve amplitude sensitive predictions. New solution approximations for the governing equations are constructed using the multi-term harmonic balance term method. Finally, fractional calculus and friction type damping elements are added to the chamber compliance. Improved quasi-linear models are proposed at four amplitudes, demonstrating amplitude sensitivity in model parameters. Refined nonlinear models are validated using measurements at multiple amplitudes. The sensitivity of the fractional and frictional damping parameters to shaping the dynamic stiffness spectra is qualitatively evaluated.

This dissertation makes three distinct contributions to the scientific literature. First, fractional calculus based viscoelasticity better captures the damping properties of elastomeric and hydraulic isolators. Next, a novel estimation procedure allows for the calculation of transient responses of fractionally damped systems to steady-state and transient excitations, revealing dynamic properties that are usually masked in the frequency-domain. Finally, new nonlinear fluid system models clarify the underlying physics that governs the amplitude sensitive dynamic behavior of hydraulic bushings.
Dedicated to my father, a great and humble man.
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- second harmonic;  
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## List of Symbols

### Chapter 2 Symbols:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>cross-sectional area (m(^2))</td>
</tr>
<tr>
<td>$a$</td>
<td>left-side node (at $x = 0$)</td>
</tr>
<tr>
<td>$B$</td>
<td>translational solution coefficient</td>
</tr>
<tr>
<td>$b$</td>
<td>right-side node (at $x = L$)</td>
</tr>
<tr>
<td>$C$</td>
<td>rotational solution coefficient</td>
</tr>
<tr>
<td>$c$</td>
<td>viscous damping coefficient</td>
</tr>
<tr>
<td>$\mathcal{D}$</td>
<td>generalized differential operator</td>
</tr>
<tr>
<td>$D$</td>
<td>diameter (m)</td>
</tr>
<tr>
<td>$E$</td>
<td>elastic modulus (Pa)</td>
</tr>
<tr>
<td>$F$</td>
<td>force (N)</td>
</tr>
<tr>
<td>$f$</td>
<td>frequency (Hz)</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus (Pa)</td>
</tr>
<tr>
<td>$H$</td>
<td>shear equation coefficients</td>
</tr>
<tr>
<td>$h$</td>
<td>loss factor</td>
</tr>
<tr>
<td>$I$</td>
<td>second moment of area (m(^4))</td>
</tr>
<tr>
<td>$K$</td>
<td>dynamic stiffness (N/m)</td>
</tr>
<tr>
<td>$L$</td>
<td>length (m)</td>
</tr>
<tr>
<td>$M$</td>
<td>moment (N·m)</td>
</tr>
</tbody>
</table>
$N$ number of terms

$Q$ generalized force

$q$ generalized displacement

$t$ time (s)

$u$ translational displacement (m)

$x, y, z$ coordinates or directions (m)

$\alpha$ fractional damping order

$\beta$ complex-valued wavenumber

$\gamma$ $B$ to $C$ conversion factor

$\varepsilon$ error

$\eta$ fractional damping coefficient

$\theta$ rotational displacement (rad)

$\kappa$ Timoshenko shear coefficient

$\nu$ Poisson’s ratio

$\rho$ density (kg/m$^3$)

$\omega$ circular frequency (rad/s)

**Subscripts**

0 static

e excitation amplitude

$m, n$ matrix indices (row, column)

max maximum bandwidth

$\lambda$ natural frequency
Abbreviations

DOF  degree(s) of freedom
FE    finite element
SEM   spectral element method

Chapter 3 Symbols:

A   real part of quadrant II residue
a   real part of quadrant II singular point
B   imaginary part of quadrant II residue
b   imaginary part of quadrant II singular point
C   Bromwich contour
c   fractional damper coefficient
D   derivative operator
E   two-parameter Mittag-Leffler function
h   impulse response
j   index
K   dynamic stiffness (N/m)
k   spring stiffness coefficient (N/m)
ℓ   integration variable
L   Laplace transform operator
m   effective internal mass (kg)
N   maximum number of terms
n   common denominator of fractional orders
p   scaled Laplace variable
frac$_q$ fractional order numerator

$R$ outer Bromwich contour radius

$r$ relaxation response

Res simple residue

$s$ Laplace transform variable (rad/s)

$T$ time constant for step-like excitation (s)

$t$ time (s)

$v$ vibratory response

$X$ step height (m)

$x$ displacement of boundary (m)

$y$ displacement of internal mass (m)

$Z$ dynamic stiffness denominator

$\alpha$ fractional damper order

$\beta$ estimation root

$\eta$ estimation exponent

$\varepsilon$ mean squared Laplace-domain error

$\Gamma(\cdot)$ Gamma function

$\Gamma(\cdot,\cdot)$ upper incomplete Gamma function

$\gamma(\cdot)$ lower incomplete gamma function

$\theta$ phase offset of vibratory response (rad)

$\psi$ step-like response phase offset (rad)

$\xi$ positive real offset
\( \lambda \)  singular point

\( \rho \)  inner Bromwich radius

\( \tau \)  integration time (s)

\textit{Subscripts}

0  static

1, 2  coordinate of inner/outer sleeve

\textit{Superscripts}

*  complex conjugate

\(^{\wedge} \)  estimate

\textbf{Chapter 4 Symbols:}

\( A \)  effective pumping area (m\(^2\))

\( C \)  fluid compliance (m\(^5\)/N)

\( D \)  Fourier domain derivative

\( D \)  diameter (m)

\( F \)  force (N)

\( \mathcal{F} \)  discrete Fourier transform

\( I \)  fluid inertance (N\cdot s\(^2\)/m\(^5\))

\( K \)  stiffness (N/m)

\( p \)  pressure (Pa)

\( l \)  length of flow passage (m)

\( M \)  ensemble size

\( m \)  mean component (dc value)
\( N \) number of harmonics considered
\( q \) volume flow rate \((m^3/s)\)
\( R \) fluid resistance \((N\cdot s/ m^5)\)
\( S \) scaling
\( t \) time \((s)\)
\( V \) volume \((m^3)\)
\( x \) displacement \((m)\)
\( z \) state variable
\( \beta \) nonlinear model parameter
\( \theta \) normalized time
\( \phi \) loss angle \((\text{rad})\)
\( \varepsilon \) approximation error
\( \Gamma \) residual in Fourier domain
\( \gamma \) residual in time domain
\( \Lambda \) eigenvalue matrix
\( \lambda \) eigenvalue
\( \mu \) dynamic viscosity \((N\cdot s/ m^2)\)
\( \rho \) fluid density \((\text{kg/m}^3)\)
\( \Omega \) excitation frequency \((\text{rad/s})\)

Subscripts
\( a \) alternating component
\( b \) baseline, or reference value
c compliance chamber
d measured data
f fluid system
i inertia track
j index
k index
L left chamber
R right chamber
r rubber path
T transmitted
\( \alpha \) transfer function coefficient
_ vector
= matrix

Superscripts

~ complex valued
^ normalized
* complex conjugate

Abbreviations

DFT discrete Fourier transform
HBM single-term harmonic balance method
LTI linear-time-invariant
MHBM multi-term harmonic balance method
Chapter 5 Symbols:

A  effective pumping area (m$^2$)
B  transfer function coefficient
C  fluid compliance (m$^5$/N)
D  derivative operator
F  force (N)
G  elastic model parameter
g  damping forcing function (N·s/m)
h  time step (s)
I  fluid inertance (N·s$^2$/m$^5$)
j  fractional derivative history term index
K  dynamic stiffness (N/m)
k  elastic spring coefficient (N/m)
Ł  Laplace transform
m  effective mass (kg)
N  number of terms
n  time step index
p  pressure (Pa)
q  volume flow rate (m$^3$/s)
R  fluid resistance (N·s/m$^5$)
s  Laplace domain variable (1/s)
t  time (s)
\( V \)  
volume \( (m^3) \)

\( X \)  
displacement of inner sleeve in the Laplace domain \( (m) \)

\( x \)  
displacement of inner sleeve in the time domain \( (m) \)

\( y \)  
effective displacement of pumping chamber wall \( (m) \)

\( Z \)  
arbitrary displacement in Laplace domain \( (m) \)

\( z \)  
arbitrary displacement in time domain \( (m) \)

\( \alpha \)  
fractional derivative order

\( \beta \)  
model parameter

\( \Gamma \)  
the gamma function

\( \varepsilon \)  
root mean squared error

\( \eta \)  
fractional/ viscous damping coefficient

\( \kappa \)  
Runge-Kutta increment

\( \mu \)  
friction coefficient \( (N) \)

\( \xi \)  
empirical correction factor

\( \sigma \)  
smoothing factor \( (1/s) \)

\( \tau \)  
variable of integration \( (s) \)

\( \phi \)  
loss angle \( (rad) \)

\( \Omega \)  
excitation frequency \( (rad/s) \)

\( [\ ] \)  
discrete variable indexing

Subscripts

0  
initial or static value

1, 2  
pumping chambers 1 and 2
amplitude (peak-to-peak)
Pumping chamber
Fluid path
Inertia track
Rubber path
Total transmitted force

Superscripts

\(-\) history
\(~\) complex-valued variable

Abbreviations

DFT discrete Fourier transform
GL Grünwald-Letnikov fractional derivative
NL nonlinear
RK4 4\textsuperscript{th} order Runge-Kutta algorithm
RMS root mean squared
QL quasi-linear
Chapter 1. Introduction

1.1 Background and Motivation

The dynamic modeling and characterization of elastomeric [1.1-1.12] and fluid-filled [1.13-1.28] vibration isolators have been scientifically studied for many years. Nevertheless, these topics still present rich research opportunities because of the complexities of viscoelastic materials and the intricate designs which are sometimes employed, particularly in the case of hydraulic devices. Elastomeric damping effects exhibit distinct behavior foreign to common engineering materials (such as metals) and play a significant role in influencing their dynamic properties [1.1-1.3]. Likewise, the presence of an internal fluid system enhances the dynamic tunability of an isolator, while adding considerable complexity via interacting design features which are generally sensitive to the amplitude and spectral contents of the excitation [1.13-1.14,1.26-1.28]. Although the scientific literature on viscoelastic isolators is extensive [1.1-1.12], voids still remain in terms of dynamic modeling and characterization. Finite element models are often used in practice to estimate the dynamic properties of vibration isolators, but with more complex designs, such models become computationally expensive—particularly when embedded in larger system models. Therefore analytical or lower order, physics-based models are often preferred to guide the computational studies and to analyze competing designs.
Historically, viscoelastic behavior has been represented in terms of spring-damper networks, leading to classical models such as the Kelvin-Voigt, Maxwell, and standard-linear-solid models [1.32]. Advanced viscoelastic models have been developed based on fractional calculus [1.1-1.3,1.8,1.12,1.29-1.34], providing a generalization of the conventional integer-order calculus formulation and improving viscoelastic characterization while also reducing the needed number of parameters. In general, fractional calculus models should offer superior results but may complicate the analysis. Integrating fractional calculus into existing analytical models or time-domain simulations is generally non-trivial at best, and may be tailored to specific applications.

The linear time-invariant system theory is typically used to characterize such devices with spring-dashpot networks to represent the viscoelasticity and mechanical analog systems for the fluid subsystems [1.19-1.23]. While such a simple approach offers generic utility, it would tend to yield a large number of parameters, each lack physical meaning. Additionally, elastomeric and hydraulic isolators exhibit significant sensitivity to excitation amplitude, which cannot be captured by linear time-invariant models. Refined modeling approaches must relax the assumption of linearity, but this requires a more precise physical characterization of certain isolator design features. Experimental studies (such as sinusoidal tests) are needed to characterize these nonlinear features, particularly for hydraulic devices, which tend to be more complicated than purely elastomeric isolators. To address some of the above mentioned issues, this dissertation intends to study three specific example cases which are shown in Figure 1.1: voided elastomeric bushings, annular elastomeric bushings, and hydraulic bushings. Viscoelastic behavior and nonlinear physics will be addressed specifically for each problem.
1.2 Literature Review and Unresolved Issues

Although in practice, most analysis of elastomeric isolators uses linear system principles and viscoelastic properties based on integer-order calculus, more recently, researchers have devoted considerable effort to understanding fractional calculus based viscoelasticity and applying it to engineering systems. Rossikhin and Shitikova have offered an extensive review of fractional calculus methods and applications [1.32], which tend to break down into two categories. Analytical studies with fractional viscoelastic elements [1.8,1.12,1.33] rely on the Cauchy integral definitions, which are particularly suitable for frequency-domain studies. The Grünwald-Letnikov definition of the fractional derivative is generally more useful for nonlinear studies in the time domain [1.2,1.3,1.34]. The numerical approach is equivalent (with sufficient resolution), but
offers limited physical insight into the relationship between time and frequency domain responses. Analytical, closed-form solutions for the time and frequency domain responses of the same device would offer substantial insight into the dynamic properties of elastomeric isolators.

The frequency-domain dynamic properties of elastomeric isolators are typically characterized using non-resonant elastomer test machines, though their operational bandwidth is often limited; even high frequency test machines are typically limited to around 500 to 1000 Hz in the uniaxial measurement mode [1.5, 1.35]. However, the frequency range of interest may be much higher in many applications given vibration isolation, impedance mismatch, or acoustic comfort requirements [1.6, 1.11, 1.36]. Therefore, it is of considerable interest to understand the isolator dynamics and its vibration transmission properties over a broad range of frequencies. The spectral element method [1.37] offers analytical predictions of dynamic properties of certain classical structures at very high frequencies using the linear, continuous system theory. Very few studies have used the continuous system approach for elastomeric isolators [1.7, 1.8]. These have produced very high frequency results, but are quite limited in terms of isolator geometry and loading directions.

A substantial body of literature exists on hydraulic mounts [1.13-1.18], while relatively few researchers have characterized hydraulic bushings [19-28], particularly with nonlinear models [25-28]. Since these devices have many features and many possible configurations which significantly affect the dynamic properties [1.20-1.22], the field is rich with research opportunities. For example, very limited focus has been given to the dynamic behavior of the compliant chambers which contain the fluid and pump it
through an inertia track. In particular, the fluid chambers of Figure 1.1(c) exhibit both amplitude-sensitive fluid compliance and damping behavior. These features strongly influence the dynamic properties and must therefore be captured in the model.

1.3 Problem Formulation

The scope of this dissertation is limited to a class of elastomeric isolators as depicted in Figure 1.1. First, a frequency-domain study of a voided bushing will be undertaken using the continuous system theory. This work assumes steady-state harmonic excitation and a linear system, but would introduce a fractional calculus based damping formulation. The continuous system theory with all 6 degrees of freedom (including coupling directions) will be compared with finite element predictions well beyond the measurable frequency range (say up to 5 kHz). Second, the example case is generalized to an annular bushing (as in Figure 1(b)), and a lumped parameter modeling framework will be used to estimate the response of this device to a step-like input based on the dynamic stiffness prediction. This objective assumes a linear system under uniaxial loading, but will begin to unify the time and frequency domains with analytical fractional calculus formulations [1.8,1.12,1.33].

Next, the hydraulic bushing (Figure 1.1(c)) is considered, and a lumped-parameter grey-box modeling approach is used to develop quasi-linear and nonlinear models which capture the amplitude-sensitive dynamic stiffness behavior over a smaller frequency range (up to 50 Hz) with steady-state harmonic excitation in the pumping direction of the bushing. Theoretical and experimentally characterized nonlinear elements are employed to quantify the behavior of the fluid system. Finally, elements of viscoelasticity (studied
in Chapters 2 and 3) will be combined amplitude sensitive nonlinearities (in Chapter 4),
leading to the characterization of a hydraulic bushing with a nonlinear, fractionally
damped fluid system.

The four principal objectives of this research are as follows, corresponding to
Chapters 2-5 of the dissertation. Figure 1.2 displays the systems examined with each
objective.

Figure 1.2 Systems examine in (a) Chapter 2, (b) Chapter 3, (c) Chapter 4, and (d)
Chapter 5 of this dissertation. For symbols, see appropriate chapter.
**Objective 1**: Propose an analytical frequency-domain model of an elastomeric vibration isolator with fidelity at very high frequencies (up to 5 kHz) in 6 degrees of freedom, including coupling directions.

1a. Derive the spectral element model for fractionally damped isolator and construct multi-axis dynamic stiffness matrices.

1b. Verify various dynamic stiffness terms of an elastomeric cylinder using finite element models, establishing limitations in terms of aspect ratio and frequency range (up to 5 kHz).

1c. Offer experimental validation for the spectral element model of an isolator up to 600 Hz excitation [1.35].

**Objective 2**: Develop a lumped-parameter model for an elastomeric isolator which captures both resonant and low-frequency damping-controlled dynamic stiffness behavior and predicts the dynamic response to transient excitations.

2a. Develop a uniaxial minimal order model which can effectively simulate the dynamic stiffness of a production bushing, capturing both low-frequency and broadband behavior covering the first resonance (0-1 kHz).

2b. Propose a new estimation technique to yield time-domain solution approximations for transient excitation using fractional calculus.

**Objective 3**: Develop tractable lumped parameter models to yield reasonable predictions of the sensitivity of a hydraulic bushing’s dynamic stiffness to excitation amplitude without using fractional damping.
3a. Propose and experimentally validate quasi-linear and nonlinear bushing models (with nonlinear resistance and compliance elements) which capture amplitude sensitivity in the example case of a production bushing with a single inertia track.

3b. Refine and utilize the semi-analytical multi-term harmonic balance method to construct dynamic stiffness spectra and gain physical insight into the role and interaction of nonlinearities in the model.

Objective 4: Extend the model from objective 3 by introducing fractional calculus based viscoelasticity and additional nonlinear damping elements to enhance modeling capabilities and to better understand the underlying physics of hydraulic bushings.

4a. Develop quasi-linear models which capture pumping chamber damping effects leading to amplitude-dependent dynamic stiffness behavior of a production bushing.

4b. Improve the nonlinear models of objective 3 by including fractional calculus and dry-friction type damping in the pumping chambers.

4c. Validate the predictions of the new models against dynamic stiffness measurements.

Table 1.1 provides an overview of the scope of each study. Since this dissertation considers several types of isolator in the time and frequency domain using both linear and nonlinear models, each chapter is organized to be self-sufficient. Therefore each chapter states its own specific assumptions and limitations, and has a dedicated list of references and symbols. Furthermore, each chapter corresponds to a publication or submission to a peer-reviewed journal.
The dynamic stiffness type transfer function is utilized to characterize elastomeric or hydraulic vibration isolators in each objective of this dissertation. This approach is chosen to be consistent with non-resonant experimental characterization procedures [1.38], as well as the literature [1.2-1.5,1.13-1.28]. Typically, elastomeric test machines excite rubber materials or components sinusoidally with a given displacement amplitude (under either force or displacement control) under a specified preload. Harmonic

<table>
<thead>
<tr>
<th>Research Issue</th>
<th>Objective</th>
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<tr>
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<tr>
<td>Type of Isolator</td>
<td>Elastomeric</td>
</tr>
<tr>
<td>Linear / Nonlinear</td>
<td>Linear</td>
</tr>
<tr>
<td>Modeling Framework</td>
<td>Continuous (Distributed Parameter) System</td>
</tr>
<tr>
<td>Domain</td>
<td>Frequency</td>
</tr>
<tr>
<td>Upper Frequency Limit</td>
<td>5 kHz</td>
</tr>
<tr>
<td>Energy Dissipation Formulations</td>
<td>Fractional Damping</td>
</tr>
</tbody>
</table>
distortion is ignored, and the modulus and loss angle of dynamic stiffness is extracted at any frequency of interest. While this process typically provides spectral characterization for a wide variety of devices or materials, several limitations apply. First, at higher frequencies, significant force may be required to maintain displacement amplitudes which are easy to achieve at lower frequencies. This results in increased error in dynamic stiffness measurements at higher frequencies and an upper frequency limit (say from 500 to 1000 Hz) must be placed. Secondly, as the test machine steps through frequencies, the built-in controller may make adjustments to maintain the excitation displacement amplitude; also, alternate force and displacement sensors might be used. These might appear in measured stiffness data as abrupt changes in the dynamic stiffness magnitude, or more likely in the phase.
Chapter 1 References


Chapter 2. High Frequency, Multi-Axis Dynamic Stiffness Analysis of a Fractionally Damped Elastomeric Isolator Using Continuous System Theory

2.1. Introduction

Elastomeric devices including vibration isolators are widely used in machines, equipment, and vehicles for their relatively low stiffness properties (with desired damping depending on the composition) at a relatively low cost. Prediction of their dynamic properties from basic principles is often difficult, since elastomeric materials often exhibit anisotropy, temperature- and age-dependence, as well as amplitude sensitive behavior [2.1-2.3]. As a result, dynamic stiffness at selected frequencies (and amplitudes of excitation under a given preload) are often measured using non-resonant elastomer test machines, though their operational bandwidth is often limited; even high frequency test machines are typically limited to around 500 to 1000 Hz in the uniaxial measurement mode [2.4-2.5]. However, the frequency range of interest may be much higher in many applications given vibration isolation, impedance mismatch, or acoustic comfort requirements [2.4-2.11]. Therefore, it is of vital importance to understand the isolator dynamics and its vibration transmission properties over a broad range of frequencies. Given the inherent limitations of elastomer test machines, few researchers have suggested indirect methods. In particular, Noll et al. [2.6] estimated the stiffness properties of elastomeric isolators with three degrees of freedom (DOF) embedded in an elastic beam system from the modal properties up to around 1 kHz; however, the isolator stiffness was
assumed to be spectrally invariant. Kim and Singh [2.7] used mobility synthesis to estimate the multi-axis dynamic properties of an isolator between two known mass elements by measuring the frequency responses of the assembled structure up to 2 kHz. Finally, Meggitt et al. [2.8] proposed a similar procedure to determine in situ isolator properties up to 2 kHz. In each case, the frequency range is restricted by experimental limitations and the underlying estimation methods.

Typically, the static and dynamic analyses in industry are conducted with commercial finite element (FE) codes. This approach has some disadvantages since many elements are required to yield accurate predictions of dynamic properties, particularly at higher frequencies. Component-level analysis of an elastomeric joint may not accurately capture its influence on a larger system, but embedding multiple high-order, FE joint type models into a larger system model can escalate the model size to an unreasonable extent. Furthermore, the usable frequency range of modal or dynamic stiffness analyses of such structures is limited by the element size, parameter uncertainty, and natural frequency spacing, suggesting the need for an analytical approach that may guide computationally intensive exercise in a more rational manner. For instance, minimal order lumped parameter models have been considered to address these difficulties with partial success over the lower frequency range [2.3, 2.7, 2.9]. However, the inertial and elastic properties are more distributed at higher frequencies or smaller wavelengths [2.10]. This suggests the use of continuous system methods though only certain types of solutions are tractable [2.11-2.12]. In particular, Kim and Singh [2.11] applied the continuous theory to flexural and longitudinal motion of an isolator, predicting the dynamic properties of elastomeric paths up to 4 kHz with the structural damping assumption. Likewise, Östberg and Kari
[2.12] proposed a wave-guide approach for fractionally damped cylindrical isolators which achieved accurate dynamic stiffness predictions as long as the isolator length is much larger than its diameter.

This chapter seeks to address some of the above mentioned limitations by employing the spectral element method (SEM) to develop a coupled, 6-DOF dynamic stiffness matrix of a fractionally damped viscoelastic isolator (with a relatively short aspect ratio) up to 5 kHz, that would extend beyond the typical measurement range. The proposed method intends to offer analytical solutions that should supplement computationally expensive high-fidelity FE predictions.

2.2. Problem Formulation

The spectral element method [2.13] in some ways combines features of both continuous system and minimal order models. Minimal order models have the advantage of dramatically decreased computation time, enhanced physical insight (from more physically meaningful parameters), and simplified analytical characterization when compared to FE models. The continuous system approach typically assumes an ideal geometry (e.g. beam, plate, or shell) and obtains an exact solution to the governing equations in that structure. An exact dynamic stiffness matrix is calculated from the continuum model of a structure, but that structure may then be treated as a discrete element and even assembled within other formulations [2.13-2.19]. Such an approach requires that a structure be decomposed into basic elements suitable for continuous system analysis, but it yields accurate dynamic stiffness predictions over a very large frequency range. This approach has been used for fluid systems [2.14], geophysical
applications [2.15], etc., but relatively few authors [2.16-2.19] have employed SEM to mechanical structures or elastomeric components that are common to the automotive, appliance, and aerospace industries. These often involve repetitive structures, which are particularly well suited to the SEM.

This chapter intends to employ linear partial differential equations to obtain a frequency-domain characterization of an isolator, thus amplitude-, preload-, and temperature-dependence are beyond the scope of this study. Such a simplification is reasonable since large amplitudes would be physically unrealistic over the medium and high frequency regimes anyway. Three damping formulations will be considered in this chapter since structural and viscous damping have been previously used in the SEM to compare with FE predictions. Of particular note is the fractional damping, which is regarded as a more general formulation for elastomeric materials; in fact, it has been found to be more accurate than viscous or structural damping [2.3,2.20-2.21]. Isolator elements will be modeled as homogeneous rods in the longitudinal and torsional direction, and as Timoshenko beams [2.11] for the shear and flexure directions. Specific objectives include: 1. Derive the spectral element model for fractionally damped isolator and construct multi-axis dynamic stiffness matrices, 2. Verify various dynamic stiffness terms of an elastomeric cylinder using FE models, establishing limitations in terms of aspect ratio and frequency range (up to 5 kHz), and 3. Offer experimental validation for the spectral element model of an isolator up to 600 Hz excitation [2.4].

The model developed in this chapter considers a circular rubber cylinder depicted in Figure 2.1 as a basic element, from which more complicated structures may be constructed. Six degrees of freedom \((u_x, u_y, u_z, \theta_x, \theta_y, \theta_z)\) are considered for each node,
such as \((a, b)\) on each side of the elastomeric material, which are assumed to be perfectly bonded to rigid surfaces. To simplify the nomenclature, generalized displacement \((\mathbf{q})\) and force \((\mathbf{Q})\) coordinates are used,

\[
\mathbf{q} = \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\} = \{q_1, q_2, q_3, q_4, q_5, q_6\},
\]

(2.1)

\[
\mathbf{Q} = \{F_x, F_y, F_z, M_x, M_y, M_z\} = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}.
\]

(2.2)

Two-parameter fractional damping (Kelvin-Voigt type) is introduced, and it is embedded within the elastic modulus as,

\[
E = E_0 \left(1 + \eta \mathcal{D}_t^\alpha \right),
\]

(2.3)

where \(E_0\) is the static elastic modulus of the material, \(\eta\) is the fractional damping coefficient, and \(\alpha\) is the fractional order. Here, \(\mathcal{D}_t^\alpha\) is a generalized fractional order differential operator,

\[
\mathcal{D}_t^\alpha = \frac{d^\alpha}{dt^\alpha}.
\]

(2.4)

Additionally, the overhead tilde notation corresponds to a complex-valued quantity in the frequency domain. For instance, the elastic modulus assumes the following form,

\[
\tilde{E}(\omega) = E_0 \left(1 + \eta (i \omega)^\alpha \right),
\]

(2.5)

applying the definition of fractional calculus in the frequency domain [2.20-2.21]. Structural damping may be viewed as a special case of the fractional damping when \(\alpha = 0, \eta = ih\) where \(h\) is the loss factor. Viscous damping may also be considered a special case of the fractional damping when \(\eta = c\) and \(\alpha = 1\), where \(c\) is the viscous damping coefficient.
Figure 2.1. Cylindrical elastomeric isolator used as the primary example. The viscoelastic material is bonded to metal end-caps, which are assumed to be rigid compared to the elastomer. Here, \( a \) and \( b \) represent the end points at \( x = 0 \) and \( x = L \) points (nodes on each end), \( \tilde{E}(\omega) \) represents the complex-valued elastic modulus, \( \nu \) is the Poisson’s ratio, \( \rho \) is the density of elastomeric material, \( D \) is the diameter, and \( L \) is the length.

2.3. Analytical Formulation of the Spectral Element Model

2.3.1 Combined multi-axis dynamic stiffness matrix

The full 6-DOF dynamic stiffness matrix \( K \), as defined below, describes the dynamic properties of the isolator in all translational and rotational directions, including both driving-point and cross-point terms according to the transfer matrix or four pole method [2.22],
such that

\[ \begin{bmatrix} Q_a \\ Q_b \end{bmatrix} = K(\omega) \begin{bmatrix} q_x \\ q_y \end{bmatrix} \]  

(2.7)

Generally, each 6×6 submatrix may be fully populated where each motion or force is coupled with all others. However, if the coordinates are judiciously chosen, then each \( K_{m,n} \) may be rather sparse and would contain off-diagonal terms only when coupling is directly implied by the physics, such as with shear/flexure coupling in an elastic beam. Each uncoupled direction or set of coupled directions is first considered separately, and the resulting dynamic stiffness terms are then combined into the overall \( K(\omega) \) matrix of equation (2.6).

2.3.2 Axial dynamic stiffness

In the axial direction \((x\) in Figure 2.1), the longitudinal wave equation is employed in the rubber section,

\[ \frac{\partial}{\partial x} \left( EA \frac{\partial q_1}{\partial x} \right) = \rho A \frac{\partial^2 q_1}{\partial t^2}, \]  

(2.8)

where \( q_1 \) is the axial displacement, \( A \) is the cross-sectional area, and \( \rho \) is the mass density. Assuming homogeneous material properties and uniform cross-sectional geometry yields,

\[ E \frac{\partial^2 q_1}{\partial x^2} = \rho \frac{\partial^2 q_1}{\partial t^2}. \]  

(2.9)
Assuming harmonic excitation and response (and dropping the ubiquitous $e^{i\omega t}$ term), the spatial solution and complex-valued axial wavenumber $\tilde{\beta}_x$ are defined in the frequency domain as follows,

$$\tilde{q}_1(x) = (\tilde{B}_1e^{\tilde{\beta}_x x} + \tilde{B}_2e^{-\tilde{\beta}_x x}), \quad (2.10-a)$$

$$\tilde{\beta}_x(\omega) = i\omega \sqrt{\frac{\rho}{k(\omega)}}. \quad (2.10-b)$$

The elastic axial force, assumed to be uniform throughout the cross-section, is defined,

$$F_x = Q_1 = EA \frac{\partial \tilde{q}_1}{\partial x}, \quad (2.11)$$

or in the frequency domain,

$$\tilde{Q}_1(\omega) = \tilde{E}A\tilde{\beta}_x \left( \tilde{B}_1e^{\tilde{\beta}_x x} - \tilde{B}_2e^{-\tilde{\beta}_x x} \right). \quad (2.12)$$

Using a four-pole type formulation, the dynamic stiffness matrix at any frequency ($\omega$) for this element can be written,

$$\begin{bmatrix} \tilde{Q}_{1a} \\ \tilde{Q}_{1b} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{1a,1a} & \tilde{K}_{1a,1b} \\ \tilde{K}_{1b,1a} & \tilde{K}_{1b,1b} \end{bmatrix} \begin{bmatrix} \tilde{q}_{1a} \\ \tilde{q}_{1b} \end{bmatrix}, \quad (2.13)$$

where the “a” subscript corresponds to the location $x=0$, “b” is at $x=L$, and the stiffness elements are determined by applying the appropriate boundary conditions as described below,

$$\tilde{K}_{1a,1a} = \frac{\tilde{Q}_{1a}}{\tilde{q}_{1a}} \bigg|_{\tilde{q}_{1a}=0}, \quad \tilde{K}_{1a,1b} = \frac{\tilde{Q}_{1a}}{\tilde{q}_{1a}} \bigg|_{\tilde{q}_{1a}=0}, \quad \tilde{K}_{1b,1a} = \frac{\tilde{Q}_{1b}}{\tilde{q}_{1a}} \bigg|_{\tilde{q}_{1a}=0}, \quad \tilde{K}_{1b,1b} = \frac{\tilde{Q}_{1b}}{\tilde{q}_{1a}} \bigg|_{\tilde{q}_{1a}=0}. \quad (2.14-a)$$

$$\tilde{K}_{1a,1b} = \frac{\tilde{Q}_{1b}}{\tilde{q}_{1a}} \bigg|_{\tilde{q}_{1a}=0}, \quad \tilde{K}_{1b,1b} = \frac{Q_{1b}}{q_{1b}} \bigg|_{\tilde{q}_{1a}=0}. \quad (2.14-b)$$
Due to the symmetry of the structural element as shown in Figure 2.1, \( \tilde{K}_{1a,1b} = \tilde{K}_{1b,1a} \), and 
\( \tilde{K}_{1a,1a} = \tilde{K}_{1b,1b} \). This symmetry would vanish if the cross-sectional area \( A \) were to vary 
with \( x \). In this case, the differential equation (2.8) would no longer have constant 
coefficients (with respect to \( x \)), and a new type of solution would be needed. Applying 
boundary conditions for the cross-point dynamic stiffness in the frequency domain,

\[
\tilde{q}_1(0) = 0 = (\tilde{B}_1 + \tilde{B}_2),
\]

(2.15-a)

\[
\tilde{q}_1(L) = q_e = (\tilde{B}_1 e^{i\beta L} + \tilde{B}_2 e^{-i\beta L}),
\]

(2.15-b)

and solving for the coefficients yields,

\[
\tilde{B}_1 = q_e \left( e^{i\beta L} - e^{-i\beta L} \right)^{-1}, \tilde{B}_2 = -\tilde{B}_1.
\]

(2.16)

The cross-point and driving-point dynamic stiffness terms are then derived as,

\[
\tilde{K}_{1b,1a} = \tilde{K}_{1a,1b} = \frac{\tilde{Q}_{1a}}{\tilde{q}_{1b}} \bigg|_{q_{1a}=0} = \frac{\tilde{E}(\omega) A \tilde{\beta}_x}{\sinh(\tilde{\beta}_x L)},
\]

(2.17-a)

\[
\tilde{K}_{1a,1a} = \tilde{K}_{1b,1b} = \frac{\tilde{Q}_{1b}}{\tilde{q}_{1b}} \bigg|_{q_{1a}=0} = \frac{\tilde{E}(\omega) A \tilde{\beta}_x}{\tanh(\tilde{\beta}_x L)}.
\]

(2.17-b)

2.3.3 **Torsional dynamic stiffness**

The torsional behavior of the element is similar to the axial behavior, and its 
coupling with other directions is assumed to be negligible. The torsional wave equation is 
given in terms of the torsional displacement \( q_4 = \theta_x(x,t) \),

\[
G \frac{\partial^2 q_4}{\partial x^2} = \rho \frac{\partial^2 q_4}{\partial t^2},
\]

(2.18)
and is recognized to be very similar to the axial wave equation (2.8), where $G$ is the shear modulus, related to $\tilde{E}(\omega)$ and Poisson’s ratio ($\nu$) as,

$$\tilde{E}(\omega) = 2(1+\nu)\tilde{G}(\omega).$$  

(2.19)

The same harmonic solution can be used, leading to a torsional wavenumber,

$$\tilde{\beta}_{0}\left(\omega\right) = i\omega \sqrt{\frac{\rho}{\tilde{G}(\omega)}} = \sqrt{2(1+\nu)}\tilde{\beta}_{t}.$$  

(2.20)

Elastic moment is calculated,

$$M_{x} = Q_{4} = GI_{xx} \frac{\partial q_{a}}{\partial x},$$  

(2.21)

where $I_{xx}$ is the polar second moment of area. Boundary conditions for cross- and driving-point calculations are equivalent to the axial direction, symmetry is once again assumed, and the corresponding torsional dynamic stiffness terms are found to be,

$$\tilde{K}_{4b,4a} = \tilde{K}_{4a,4b} = \frac{\tilde{Q}_{4b}}{\tilde{q}_{4b}} \bigg|_{q_{x}=0} = \frac{\tilde{G}(\omega)I_{xx}\tilde{\beta}_{0}}{\sinh(\tilde{\beta}_{0}L)},$$  

(2.22-a)

$$\tilde{K}_{4a,4a} = \tilde{K}_{4b,4b} = \frac{\tilde{Q}_{4a}}{\tilde{q}_{4b}} \bigg|_{q_{x}=0} = \frac{\tilde{G}(\omega)I_{xx}\tilde{\beta}_{0}}{\tanh(\tilde{\beta}_{0}L)}.$$  

(2.22-b)

2.3.4 Shear and flexural dynamic stiffness terms

While a no-coupling assumption is reasonable for small-amplitude, axial and torsional vibrations of the last two subsections, shear and bending motions must be analyzed together given strong coupling. The Timoshenko beam formulation [2.11] offers a robust framework for this analysis with “short” beam elements by including shear
deformation and rotational inertia. Two coupled wave equations are given for a
Timoshenko beam with homogeneous material properties and a uniform cross section,

\[ \kappa AG \frac{\partial}{\partial x} \left[ \frac{\partial^2 q_2}{\partial x^2} - q_6 \right] = \rho A \frac{\partial^2 q_2}{\partial t^2}, \quad (2.23) \]

\[ EI \frac{\partial^2 q_6}{\partial x^2} + \kappa AG \left[ \frac{\partial q_2}{\partial x} - q_6 \right] = \rho \frac{\partial^2 q_6}{\partial t^2}, \quad (2.24) \]

for shear deformation in the \( y \) direction and bending about the \( z \) direction. Here, the
graphy dependent shear coefficient \( \kappa \) accounts for the shear stress being not uniformly
distributed on the cross section [2.23]. These can be combined into a single equation in
terms of just one variable,

\[ EI \frac{\partial^4 q_2}{\partial x^4} + \rho A \frac{\partial^2 q_2}{\partial t^2} - \rho I \left( 1 + \frac{2(1+\nu)}{\kappa} \right) \frac{\partial^4 q_2}{\partial x^4} + \frac{\rho^2 AI}{\kappa AG} \frac{\partial^4 q_2}{\partial t^4} = 0, \quad (2.25) \]

which has the general solution,

\[ \bar{q}_2(x) = (\tilde{B}_{y1} e^{\tilde{\beta}_{y1} x} + \tilde{B}_{y2} e^{\tilde{\beta}_{y2} x} + \tilde{B}_{y3} e^{\tilde{\beta}_{y3} x} + \tilde{B}_{y4} e^{\tilde{\beta}_{y4} x}). \quad (2.26) \]

For the sake of compactness, equation (2.24) is rewritten as,

\[ \frac{\partial^4 \bar{q}_2}{\partial x^4} + 2\tilde{H}_{y2}(\omega) \frac{\partial^2 \bar{q}_2}{\partial x^2} + \tilde{H}_0(\omega) \bar{q}_2 = 0, \quad (2.27) \]

where,

\[ \tilde{H}_{y2}(\omega) = \frac{\rho \omega^2}{E(\omega)} \left( \frac{1}{2} + \frac{\omega}{\kappa} \right), \quad (2.28-a) \]

\[ \tilde{H}_0(\omega) = \frac{\rho \omega^2}{E(\omega)} \left( \frac{\rho \omega^2}{\kappa G(\omega)} - \frac{A}{I_{zz}} \right). \quad (2.28-b) \]
The following two complex-valued wavenumbers emerge from the application of the quadratic formula to equation (2.26),

\[ \beta_{y1}^2, \beta_{y2}^2 = -\bar{H}_{y2} \pm \sqrt{\bar{H}_{y2}^2 - \bar{H}_{y0}}. \]  

(2.29)

The rotational displacement is determined from equation (2.23),

\[ \ddot{q}_6(x) = \frac{\ddot{q}_2}{\partial x} + \frac{\rho \omega^2}{\kappa G(\omega)} \int \ddot{q}_2 \, dx + \left( \tilde{C}_{y1} e^{\beta_{y1} x} + \tilde{C}_{y2} e^{-\beta_{y1} x} + \tilde{C}_{y3} e^{\beta_{y2} x} + \tilde{C}_{y4} e^{-\beta_{y2} x} \right). \]  

(2.30-a)

\[ \tilde{C}_{y1} = \left( \tilde{\beta}_{y1} + \frac{\rho \omega^2}{\kappa G(\omega) \beta_{y1}} \right) \tilde{B}_{y1} = \tilde{y}_{y1} \tilde{B}_{y1}, \]  

(2.30-b)

\[ \tilde{C}_{y2} = -\left( \tilde{\beta}_{y1} + \frac{\rho \omega^2}{\kappa G(\omega) \beta_{y1}} \right) \tilde{B}_{y2} = -\tilde{y}_{y1} \tilde{B}_{y2}, \]  

(2.30-c)

\[ \tilde{C}_{y3} = \left( \tilde{\beta}_{y2} + \frac{\rho \omega^2}{\kappa G(\omega) \beta_{y2}} \right) \tilde{B}_{y3} = \tilde{y}_{y2} \tilde{B}_{y3}, \]  

(2.30-d)

\[ \tilde{C}_{y4} = -\left( \tilde{\beta}_{y2} + \frac{\rho \omega^2}{\kappa G(\omega) \beta_{y2}} \right) \tilde{B}_{y4} = -\tilde{y}_{y2} \tilde{B}_{y4}. \]  

(2.30-e)

Shear force and bending moment are defined,

\[ F_y = Q_z = \kappa AG \left( \frac{\ddot{q}_2}{\partial x} - q_6 \right), \]  

(2.31)

\[ M_z = Q_6 = -EI \frac{\ddot{q}_6}{\partial x}. \]  

(2.32)

Since there are two degrees of freedom at each node, the dynamic stiffness matrix must be of dimension 4. Three fixed displacements and one applied force boundary condition are applied for each dynamic stiffness term. For a translational harmonic displacement input,
\( \ddot{q}_2(0) = 0 = \ddot{\bar{B}}_{y_1} + \ddot{\bar{B}}_{y_2} + \ddot{\bar{B}}_{y_3} + \ddot{\bar{B}}_{y_4}, \)  
(2.33-a)

\[ \ddot{q}_2(L) = q_e = \left( \bar{B}_{y_1}e^{\beta_{1L}} + \bar{B}_{y_2}e^{-\beta_{1L}} + \bar{B}_{y_3}e^{\beta_{1L}} + \bar{B}_{y_4}e^{-\beta_{1L}} \right), \]  
(2.33-b)

\[ \ddot{q}_6(0) = 0 = \ddot{\bar{C}}_{y_1} + \ddot{\bar{C}}_{y_2} + \ddot{\bar{C}}_{y_3} + \ddot{\bar{C}}_{y_4}, \]  
(2.33-c)

\[ \ddot{q}_6(L) = 0 = \ddot{\bar{C}}_{y_1}e^{\beta_{1L}} + \ddot{\bar{C}}_{y_2}e^{-\beta_{1L}} + \ddot{\bar{C}}_{y_3}e^{\beta_{1L}} + \ddot{\bar{C}}_{y_4}e^{-\beta_{1L}}, \]  
(2.33-d)

or in matrix form,

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
e^{\beta_{1L}} & e^{-\beta_{1L}} & e^{\beta_{2L}} & e^{-\beta_{2L}} \\
\tilde{q}_{y_1} & -\tilde{q}_{y_1} & \tilde{q}_{y_2} & -\tilde{q}_{y_2} \\
\tilde{q}_{y_1}e^{\beta_{1L}} & -\tilde{q}_{y_1}e^{-\beta_{1L}} & \tilde{q}_{y_2}e^{\beta_{2L}} & -\tilde{q}_{y_2}e^{-\beta_{2L}}
\end{bmatrix}
\begin{bmatrix}
\ddot{\bar{B}}_{y_1} \\
\ddot{\bar{B}}_{y_2} \\
\ddot{\bar{B}}_{y_3} \\
\ddot{\bar{B}}_{y_4}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]  
(2.34)

The \( \tilde{B} \) coefficients may be explicitly calculated from this equation, but for brevity’s sake, this is left as an exercise for the reader. The dynamic stiffness terms are found as,

\[
\tilde{K}_{2a,2a} = \tilde{K}_{2b,2b} = \frac{\dddot{Q}_{2b}}{\dddot{q}_{2b}} \bigg|_{\dddot{q}_{2a}, \dddot{q}_{4a}, \dddot{q}_{4a} = 0} = \kappa A\tilde{G} \left( \begin{array}{c}
\left( \ddot{\beta}_{y_1} - \ddot{\gamma}_{y_1} \right) \left( \ddot{\bar{B}}_{y_1}e^{\beta_{1L}} - \ddot{\bar{B}}_{y_2}e^{-\beta_{1L}} \right) \\
\left( \ddot{\beta}_{y_2} - \ddot{\gamma}_{y_2} \right) \left( \ddot{\bar{B}}_{y_3}e^{\beta_{2L}} - \ddot{\bar{B}}_{y_4}e^{-\beta_{2L}} \right)
\end{array} \right),
\]  
(2.35-a)

\[
\tilde{K}_{2b,2a} = \tilde{K}_{2a,2b} = \frac{\dddot{Q}_{2a}}{\dddot{q}_{2a}} \bigg|_{\dddot{q}_{2b}, \dddot{q}_{4a}, \dddot{q}_{4a} = 0} = \kappa A\tilde{G} \left( \begin{array}{c}
\left( \ddot{\beta}_{y_1} - \ddot{\gamma}_{y_1} \right) \left( \ddot{\bar{B}}_{y_1} - \ddot{\bar{B}}_{y_2} \right) \\
\left( \ddot{\beta}_{y_2} - \ddot{\gamma}_{y_2} \right) \left( \ddot{\bar{B}}_{y_3} - \ddot{\bar{B}}_{y_4} \right)
\end{array} \right),
\]  
(2.35-b)

\[
\tilde{K}_{6a,2a} = \tilde{K}_{6b,2b} = \frac{\dddot{Q}_{6b}}{\dddot{q}_{6b}} \bigg|_{\dddot{q}_{2a}, \dddot{q}_{4a}, \dddot{q}_{4a} = 0} = \frac{-\mathcal{L}I}{\dddot{q}_{2b}} \left( \ddot{\bar{B}}_{y_1} \ddot{\gamma}_{y_1} \dddot{B}_{y_4}e^{\beta_{1L}} + \ddot{\bar{B}}_{y_1} \dddot{\gamma}_{y_1} \dddot{B}_{y_2}e^{-\beta_{1L}} \\
+ \ddot{\bar{B}}_{y_2} \ddot{\gamma}_{y_2} \dddot{B}_{y_3}e^{\beta_{2L}} + \ddot{\bar{B}}_{y_2} \dddot{\gamma}_{y_2} \dddot{B}_{y_4}e^{-\beta_{2L}} \right),
\]  
(2.35-c)
\[
\tilde{K}_{6b,2a} = \tilde{K}_{6a,2b} = \frac{\tilde{Q}_{6a}}{\tilde{q}_{2b}} \bigg|_{\tilde{q}_{2a}, \tilde{q}_{2b}, \tilde{q}_{6a} = 0} = \frac{-\ddot{E}I}{zz} \left( \beta_{y1}\tilde{y}_{y1}\tilde{B}_{y1} + \beta_{y1}\tilde{y}_{y1}\tilde{B}_{y2} + \beta_{y2}\tilde{y}_{y2}\tilde{B}_{y2} + \beta_{y2}\tilde{y}_{y2}\tilde{B}_{y4} \right) \right) \nonumber.
\]

(2.35-d)

For a harmonic rotational displacement input,

\[
\tilde{q}_2(0) = 0 = \tilde{B}_{y1} + \tilde{B}_{y2} + \tilde{B}_{y3} + \tilde{B}_{y4},
\]

(2.36-a)

\[
\tilde{q}_2(L) = 0 = \tilde{B}_{y1}e^{\beta_{y1}L} + \tilde{B}_{y2}e^{-\beta_{y1}L} + \tilde{B}_{y3}e^{\beta_{y2}L} + \tilde{B}_{y4}e^{-\beta_{y2}L},
\]

(2.36-b)

\[
\tilde{q}_6(0) = 0 = \tilde{C}_{y1} + \tilde{C}_{y2} + \tilde{C}_{y3} + \tilde{C}_{y4},
\]

(2.36-c)

\[
\tilde{q}_6(L) = q_e = \left( \tilde{C}_{y1}e^{\beta_{y1}L} + \tilde{C}_{y2}e^{-\beta_{y1}L} + \tilde{C}_{y3}e^{\beta_{y2}L} + \tilde{C}_{y4}e^{-\beta_{y2}L} \right),
\]

(2.36-d)

or again in matrix form,

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & e^{\beta_{y1}L} & e^{-\beta_{y1}L} & e^{\beta_{y2}L} & e^{-\beta_{y2}L} \\
\tilde{y}_{y1} & -\tilde{y}_{y1} & \tilde{y}_{y2} & -\tilde{y}_{y2} & \tilde{y}_{y1}e^{\beta_{y1}L} & -\tilde{y}_{y1}e^{-\beta_{y1}L} & \tilde{y}_{y2}e^{\beta_{y2}L} & -\tilde{y}_{y2}e^{-\beta_{y2}L}
\end{bmatrix}
\begin{bmatrix}
\tilde{B}_{y1} \\
\tilde{B}_{y2} \\
\tilde{B}_{y3} \\
\tilde{B}_{y4}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

(2.37)

The dynamic stiffness terms are then determined as,

\[
\tilde{K}_{6a,6a} = \tilde{K}_{6b,6b} = \frac{\tilde{Q}_{6b}}{\tilde{q}_{6b}} \bigg|_{\tilde{q}_{2a}, \tilde{q}_{2b}, \tilde{q}_{6a} = 0} = \frac{-\ddot{E}I}{zz} \left( \beta_{y1}\tilde{C}_{y1}e^{\beta_{y1}L} - \beta_{y1}\tilde{C}_{y2}e^{-\beta_{y1}L} + \beta_{y2}\tilde{C}_{y3}e^{\beta_{y2}L} - \beta_{y2}\tilde{C}_{y4}e^{-\beta_{y2}L} \right) \right) \nonumber.
\]

(2.38-a)

\[
\tilde{K}_{6b,6a} = \tilde{K}_{6a,6b} = \frac{\tilde{Q}_{6b}}{\tilde{q}_{6b}} \bigg|_{\tilde{q}_{2a}, \tilde{q}_{2b}, \tilde{q}_{6a} = 0} = \frac{-\ddot{E}I}{zz} \left( \beta_{y1}\tilde{C}_{y1} - \beta_{y1}\tilde{C}_{y2} + \beta_{y2}\tilde{C}_{y3} - \beta_{y2}\tilde{C}_{y4} \right) \right) \nonumber.
\]

(2.38-b)
For shear in the $z$ direction and coupled bending about the $y$ axis, an analogous procedure is used, but the sense of $q_5 = \theta_y$ is reversed since the coordinate is defined using the right hand rule. The solutions are of the same exponential form,

$$
\tilde{q}_3(x) = \left( \tilde{B}_{21} e^{\tilde{\beta}_{1x}} + \tilde{B}_{23} e^{-\tilde{\beta}_{3x}} + \tilde{B}_{33} e^{\tilde{\beta}_{3x}} + \tilde{B}_{44} e^{-\tilde{\beta}_{4x}} \right),
$$

(2.39-a)

$$
\tilde{q}_5(x) = \left( \tilde{C}_{21} e^{\tilde{\beta}_{1x}} + \tilde{C}_{23} e^{-\tilde{\beta}_{3x}} + \tilde{C}_{33} e^{\tilde{\beta}_{3x}} + \tilde{C}_{44} e^{-\tilde{\beta}_{4x}} \right).
$$

(2.39-b)

The wavenumbers are defined, also as before,

$$
\tilde{H}_{z2}(\omega) = \frac{\omega^2 \rho}{E(\omega)} \left( \frac{1}{2} + \frac{(1 + \nu)}{\kappa} \right),
$$

(2.40-a)

$$
\tilde{H}_{z0}(\omega) = \frac{\rho}{E(\omega)} \left( \frac{\omega^2 \rho}{\kappa G(\omega)} - \frac{\omega^2 A}{I_{yy}} \right),
$$

(2.40-b)

$$
\tilde{\beta}_{z1}^2, \tilde{\beta}_{z2}^2 = -\tilde{H}_{z2} \pm \sqrt{\tilde{H}_{z2}^2 - \tilde{H}_{z0}^2},
$$

(2.41)

and since the angle has the opposite sense, the $\tilde{C}$ coefficients are defined,

$$
\tilde{C}_{z1} = -\left( \tilde{\beta}_{z1} + \frac{\rho \omega^2}{G(\omega) \tilde{\beta}_{z1}} \right) \tilde{B}_{z1} = \tilde{\gamma}_{z1} \tilde{B}_{z1},
$$

(2.42-a)
\[
\tilde{C}_{z2} = \left( \tilde{\beta}_{z1} + \frac{\rho \omega^2}{G(\omega)} \beta_{z1} \right) \tilde{B}_{z2} = -\tilde{\gamma}_{z1} \tilde{B}_{z2},
\]  
\tag{2.42-b}

\[
\tilde{C}_{z3} = -\left( \tilde{\beta}_{z2} + \frac{\rho \omega^2}{G(\omega)} \beta_{z2} \right) \tilde{B}_{z3} = \tilde{\gamma}_{z2} \tilde{B}_{z3},
\]  
\tag{2.42-c}

\[
\tilde{C}_{z4} = -\left( \tilde{\beta}_{z2} + \frac{\rho \omega^2}{G(\omega)} \beta_{z2} \right) \tilde{B}_{z4} = -\tilde{\gamma}_{z2} \tilde{B}_{z4}.
\]  
\tag{2.42-d}

Analogous boundary conditions are applied, beginning with a translational displacement,

\[
\begin{pmatrix}
\begin{bmatrix} 1 & 1 \\ e^{\beta_{z1} t} & e^{-\beta_{z1} t} \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ e^{\beta_{z2} t} & e^{-\beta_{z2} t} \end{bmatrix} \\
\tilde{\gamma}_{z1} & -\tilde{\gamma}_{z1} \\
\tilde{\gamma}_{z2} & -\tilde{\gamma}_{z2}
\end{bmatrix}
\begin{bmatrix} \tilde{B}_{z1} \\ \tilde{B}_{z2} \end{bmatrix}
= \begin{bmatrix} 0 \\ q_e \end{bmatrix},
\]  
\tag{2.43}

leading to the following dynamic stiffness terms,

\[
\tilde{K}_{3a,3a} = \tilde{K}_{3b,3b} = \frac{\tilde{Q}_{3b}}{\tilde{q}_{3b}} \left. \frac{\partial \tilde{K}}{\partial \tilde{q}} \right|_{\tilde{q}_{1a}, \tilde{q}_{1b}, \tilde{q}_{3s}=0} = \kappa A \tilde{G} \begin{pmatrix} \left( \tilde{B}_{z1} - \tilde{\gamma}_{z1} \right) \left( \tilde{B}_{z1} e^{\beta_{z1} t} - \tilde{B}_{z2} e^{-\beta_{z1} t} \right) \\ +\left( \tilde{B}_{z2} - \tilde{\gamma}_{z2} \right) \left( \tilde{B}_{z2} e^{\beta_{z2} t} - \tilde{B}_{z4} e^{-\beta_{z2} t} \right) \end{pmatrix} ,
\]  
\tag{2.44-a}

\[
\tilde{K}_{3b,3a} = \tilde{K}_{3a,3b} = \frac{\tilde{Q}_{3a}}{\tilde{q}_{3b}} \left. \frac{\partial \tilde{K}}{\partial \tilde{q}} \right|_{\tilde{q}_{1a}, \tilde{q}_{1b}, \tilde{q}_{3s}=0} = \kappa A \tilde{G} \begin{pmatrix} \tilde{B}_{z1} \left( \tilde{B}_{z1} - \tilde{\gamma}_{z1} \right) - \tilde{B}_{z2} \left( \tilde{B}_{z1} - \tilde{\gamma}_{z1} \right) \\ +\tilde{B}_{z2} \left( \tilde{B}_{z2} - \tilde{\gamma}_{z2} \right) - \tilde{B}_{z4} \left( \tilde{B}_{z2} - \tilde{\gamma}_{z2} \right) \end{pmatrix} ,
\]  
\tag{2.44-b}

\[
\tilde{K}_{5a,3a} = \tilde{K}_{5b,3b} = \frac{\tilde{Q}_{5b}}{\tilde{q}_{3b}} \left. \frac{\partial \tilde{K}}{\partial \tilde{q}} \right|_{\tilde{q}_{1a}, \tilde{q}_{1b}, \tilde{q}_{3s}=0} = \tilde{E} I_y \begin{pmatrix} \tilde{\beta}_{z1} \tilde{\gamma}_{z1} \tilde{B}_{z1} e^{\beta_{z1} t} + \tilde{\beta}_{z2} \tilde{\gamma}_{z1} \tilde{B}_{z2} e^{-\beta_{z1} t} \\ +\tilde{\beta}_{z2} \tilde{\gamma}_{z2} \tilde{B}_{z2} e^{\beta_{z2} t} + \tilde{\beta}_{z2} \tilde{\gamma}_{z2} \tilde{B}_{z4} e^{-\beta_{z2} t} \end{pmatrix} / \left( \tilde{B}_{z1} e^{\beta_{z1} t} + \tilde{B}_{z2} e^{-\beta_{z1} t} + \tilde{B}_{z3} e^{\beta_{z2} t} + \tilde{B}_{z4} e^{-\beta_{z2} t} \right) ,
\]  
\tag{2.44-c}
\[ \tilde{K}_{5b,3a} = \tilde{K}_{5a,3b} = \frac{\tilde{Q}_{5b}}{q_{3b}} \bigg|_{\tilde{q}_{3a}, \tilde{q}_{3b}, \tilde{q}_{3c} = 0} = \frac{-\tilde{E}I_{yy}}{q_{3b}} \left( \tilde{B}_{21}e^{\tilde{\beta}_{1L}}L + \tilde{B}_{22}e^{\tilde{\beta}_{2L}}L + \tilde{B}_{23}e^{\tilde{\beta}_{3L}}L + \tilde{B}_{24}e^{\tilde{\beta}_{4L}}L \right). \]  

(2.44-d)

Under a rotational displacement excitation, the following boundary conditions are applied,

\[ \begin{bmatrix}
1 & 1 & 1 & 1 & e^{\tilde{\beta}_{1L}} & e^{\tilde{\beta}_{2L}} & e^{\tilde{\beta}_{3L}} \\
\tilde{\gamma}_{z1} & -\tilde{\gamma}_{z1} & \tilde{\gamma}_{z2} & -\tilde{\gamma}_{z2} & e^{\tilde{\beta}_{1L}} & e^{\tilde{\beta}_{2L}} & e^{\tilde{\beta}_{3L}}
\end{bmatrix} \begin{bmatrix}
\tilde{B}_{21} \\
\tilde{B}_{22} \\
\tilde{B}_{23} \\
\tilde{B}_{24}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
q_{c}
\end{bmatrix}, \]  

(2.45)

leading to the last set of dynamic stiffness terms,

\[ \tilde{K}_{5a,5a} = \tilde{K}_{5b,5b} = \frac{\tilde{Q}_{5b}}{q_{5b}} \bigg|_{q_{5a}, \tilde{q}_{5b}, \tilde{q}_{5c} = 0} = \frac{-\tilde{E}I_{yy}}{q_{5b}} \left( \tilde{C}_{21}e^{\tilde{\beta}_{1L}}L - \tilde{C}_{22}e^{\tilde{\beta}_{2L}}L + \tilde{C}_{23}e^{\tilde{\beta}_{3L}}L + \tilde{C}_{24}e^{\tilde{\beta}_{4L}}L \right), \]  

(2.46-a)

\[ \tilde{K}_{5b,5a} = \tilde{K}_{5a,5b} = \frac{\tilde{Q}_{5b}}{q_{5b}} \bigg|_{q_{5a}, \tilde{q}_{5b}, \tilde{q}_{5c} = 0} = \frac{-\tilde{E}I_{yy}}{q_{5b}} \left( \tilde{C}_{21}e^{\tilde{\beta}_{1L}}L - \tilde{C}_{22}e^{\tilde{\beta}_{2L}}L + \tilde{C}_{23}e^{\tilde{\beta}_{3L}}L + \tilde{C}_{24}e^{\tilde{\beta}_{4L}}L \right), \]  

(2.46-b)

\[ \tilde{K}_{1a,5a} = \tilde{K}_{3b,5a} = \frac{\tilde{Q}_{5b}}{q_{5b}} \bigg|_{q_{5a}, \tilde{q}_{3b}, \tilde{q}_{5c} = 0} = \kappa A\tilde{G} \left( \tilde{B}_{21}e^{\tilde{\beta}_{1L}} + \tilde{B}_{22}e^{\tilde{\beta}_{2L}} + \tilde{B}_{23}e^{\tilde{\beta}_{3L}} + \tilde{B}_{24}e^{\tilde{\beta}_{4L}} \right), \]  

(2.46-c)

\[ \tilde{K}_{3b,5a} = \tilde{K}_{5b,5a} = \frac{\tilde{Q}_{5b}}{q_{5b}} \bigg|_{q_{5a}, \tilde{q}_{3b}, \tilde{q}_{5c} = 0} = \kappa A\tilde{G} \left( \tilde{C}_{21}e^{\tilde{\beta}_{1L}} + \tilde{C}_{22}e^{\tilde{\beta}_{2L}} + \tilde{C}_{23}e^{\tilde{\beta}_{3L}} + \tilde{C}_{24}e^{\tilde{\beta}_{4L}} \right). \]  

(2.46-d)

2.4. Verification of Dynamic Stiffness Expressions Using Finite Element Models

A finite element study is carried out as verification for the spectral element model. A cylindrical rubber isolator similar to the one depicted in Figure 2.1 is taken with two
aspect ratios ($L/D$) as example cases and their properties are given in Table 2.1. Both ends (at $x=a$ and $x=b$) are rigidly constrained using kinematic coupling, effectively simulating bonded boundary conditions. A FE study is undertaken with the parameters of Table 2.2 to produce computational stiffness spectra up to 5 kHz. Typical dynamic stiffness magnitudes in axial, torsional, shear, and flexure are given in Figures 2.2-2.6, respectively. Due to the circular cross-section, the shear-bending properties in the $y-\theta_z$ direction are equivalent in the $z-\theta_y$ direction. Further, the FE and SEM shear/flexure coupling dynamic stiffness spectra for the $L/D=1$ case are depicted in terms of both magnitude and phase in Figure 2.7. Since the aspect ratio changes so significantly, some consideration of the relationship between aspect ratio and frequency range is needed to present appropriate comparison. For example, in the uncoupled directions, the natural frequency ($\omega_\lambda$) of the continuous system follows $\omega_\lambda \propto 1/L$. Thus, the frequency range of interest relative to the internal modes shifts with the aspect ratio, and so the bandwidth is limited to $\omega \leq L\omega_{max}/D$. This causes the 1/4 aspect ratio to require a 10 kHz frequency range, while higher aspect ratios include a comparable amount of resonance and anti-resonance behavior.

It is clear from the dynamic stiffness plots that excellent agreement is achieved up to 5 kHz for both the torsional and shear directions. The axial, bending, and shear/bending coupling direction all show good agreement up to near the first resonance, and moderate agreement thereafter. One reason for this is that the bonded boundary condition distorts the modeshepes near each end of the elastomeric material—an effect not captured by the spectral element model. Errors magnified in the short aspect ratio are
Table 2.1. Physical properties for finite element and spectral element studies of the isolator of Figure 2.1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0$</td>
<td>8 MPa</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.48</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1.1 kg/m$^3$</td>
</tr>
<tr>
<td>$h$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Figure 2.2. Axial dynamic stiffness spectra for the isolator of Figure 2.1. with values listed in Table 2.1. (a) $K_{1a1a}$ with unity aspect ratio ($L/D = 1$), (b) $K_{1a1a}$ with short aspect ratio ($L/D = 1/2$), (c) $K_{1b1b}$ with $L/D = 1$, (a) $K_{1b1b}$ with $L/D = 1/2$. Key: 
- spectral element model;  - finite element model.
Table 2.2. Parameters used for the finite element study of the isolator of Figure 2.1.

<table>
<thead>
<tr>
<th>Aspect Ratio $(L/D)$</th>
<th>$D$ (mm)</th>
<th>$L$ (mm)</th>
<th>Number of Elements Employed</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>30</td>
<td>120</td>
<td>4480</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>60</td>
<td>2320</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>30</td>
<td>4080</td>
</tr>
<tr>
<td>3/4</td>
<td>30</td>
<td>22.5</td>
<td>1970</td>
</tr>
<tr>
<td>1/2</td>
<td>30</td>
<td>15</td>
<td>4820</td>
</tr>
<tr>
<td>1/4</td>
<td>30</td>
<td>7.5</td>
<td>16180</td>
</tr>
</tbody>
</table>

Figure 2.3. Torsional dynamic stiffness spectra for the isolator of Figure 2.1. with values listed in Table 2.1. $(a) K_{4a4a}$ with $L/D=1$, $(b) K_{4a4a}$ with $L/D=1/2$, $(c) K_{4a4b}$ with $L/D=1$, $(d) K_{4a4b}$ with $L/D=1/2$. Key: — spectral element model; ••• — finite element model.
more likely to be related to end-effects. Additionally, the spectral element model assumes slender rod-like behavior in the axial direction. This assumes that displacement is uniform in the cross-section. At high frequencies, longitudinal modes which violate this assumption begin to participate, introducing additional error. Similar effects and errors are present in the bending direction. Finally, many elements are required to preserve accuracy at very high frequencies, and thus some high-frequency errors may be due to insufficient spatial resolution; also, numerical errors are known to exist in the FE method at higher frequencies. Nevertheless, in general, the spectral element model produces reasonable dynamic stiffness predictions compared with finite element predictions.

Figure 2.4. Shear dynamic stiffness spectra for the isolator of Figure 2.1. with values listed in Table 2.1. \((a)K_{2a2a}\) with \(L/D=1\), \((b)K_{2a2a}\) with \(L/D=1/2\), \((c)K_{2a2b}\) with \(L/D=1\), \((d)K_{2a2b}\) with \(L/D=1/2\). Key:  – spectral element model;  – finite element model.
The discrepancies between FE and SEM magnitudes are quantified using the following percent-difference style error index where the SEM is a benchmark,

\[ \varepsilon = \frac{(100\%) \sum [|\tilde{K}_{FE}| - |\tilde{K}_{SEM}|]}{N |\tilde{K}_{SEM}|} \]  

(2.47)

The results of this study are given in Table 2.3, and several insights emerge. First, certain directions have similar errors at all aspect ratios, including the torsional and shear directions, which vary only minimally. The average error across all aspect ratios is also at a minimum in these two directions, confirming that the SEM assumptions are most

![Figure 2.5](image)

Figure 2.5. Shear/flexure coupling dynamic stiffness spectra for the isolator of Figure 2.1.

With values listed in Table 2.1. (a) \( K_{2a6a} = K_{6a2a} \) with \( L/D = 1 \), (b) \( K_{2a6a} = K_{6a2a} \) with \( L/D = 1/2 \), (c) \( K_{2a6b} = K_{6a2b} \) with \( L/D = 1 \), (d) \( K_{2a6b} = K_{6a2b} \) with \( L/D = 1/2 \). Key:

- spectral element model; 
- finite element model.

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reasonable in these directions and thus the predictions are strongest. In the axial, bending, and shear/bending coupling directions, significant dependence on the aspect ratio is observed. The SEM assumptions in these directions become less reasonable as the aspect ratio is decreased, primarily because the FE model has bonded boundary conditions whereas the SEM assumes free deformation in the cross-sectional plane.

Figure 2.6. Flexural dynamic stiffness spectra for the isolator of Figure 2.1. With values listed in Table 2.1. (a) $K_{aba}$ with $L/D = 1$, (b) $K_{aba}$ with $L/D = 1/2$, (c) $K_{aba}$ with $L/D = 1$, (a) $K_{aba}$ with $L/D = 1/2$. Key: — spectral element model; —— finite element model.
Figure 2.7. Comparison between SEM and FE predicted dynamic stiffness spectra in terms of (a) magnitude and (b) phase for the isolator of Figure 2.1. With values listed in Table 2.1. The driving-point shear/flexure coupling term with $L/D = 1$ is used as an illustrative example. Key: — spectral element model; — finite element model.

Table 2.3. Error in the dynamic stiffness magnitudes between finite element (FE) and spectral element method (SEM) predictions as defined by Eq. (2.47).

<table>
<thead>
<tr>
<th>Aspect Ratio ($L/D$)</th>
<th>Bandwidth (kHz)</th>
<th>$K_{11}$</th>
<th>$K_{44}$</th>
<th>$K_{22}$</th>
<th>$K_{26}$ or $K_{62}$</th>
<th>$K_{66}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.625</td>
<td>8.1%</td>
<td>8.4%</td>
<td>19.0%</td>
<td>17.8%</td>
<td>16.2%</td>
</tr>
<tr>
<td>2</td>
<td>1.25</td>
<td>16.1%</td>
<td>7.0%</td>
<td>22.9%</td>
<td>20.3%</td>
<td>16.6%</td>
</tr>
<tr>
<td>1</td>
<td>2.5</td>
<td>50.5%</td>
<td>16.0%</td>
<td>18.6%</td>
<td>24.8%</td>
<td>54.8%</td>
</tr>
<tr>
<td>3/4</td>
<td>3.33</td>
<td>51.9%</td>
<td>10.3%</td>
<td>26.6%</td>
<td>35.0%</td>
<td>54.5%</td>
</tr>
<tr>
<td>1/2</td>
<td>5</td>
<td>70.8%</td>
<td>15.9%</td>
<td>24.9%</td>
<td>69.0%</td>
<td>56.5%</td>
</tr>
<tr>
<td>1/4</td>
<td>10</td>
<td>74.3%</td>
<td>12.3%</td>
<td>27.1%</td>
<td>60.6%</td>
<td>57.2%</td>
</tr>
<tr>
<td>Average Error</td>
<td></td>
<td>45.3%</td>
<td>11.7%</td>
<td>23.2%</td>
<td>37.9%</td>
<td>42.6%</td>
</tr>
</tbody>
</table>
2.5. Experimental Validation

The spectral element approach proposed in this chapter is particularly useful for a class of isolators which is shown in Figure 2.8. Using geometric and other material parameters (as reported in Table 2.4), the elastic and damping properties are judiciously chosen to simulate the measured dynamic properties of the device as provided in an earlier paper [2.4]. Since the excitation is purely in the shear direction, the experiment corresponds to the shear direction in the SEM. Thus, $\bar{K}_{SEM} = 2\bar{K}_{2a,2b}$, which is comparable to the measured response.

Figure 2.8. Schematic of the isolator used for experimental validation of spectral element method. Two elastomeric cylindrical elements (as depicted in more detail in Figure 2.1) are configured in parallel to ensure that the isolator works in the shear mode. Refer to Table 2.4 for properties and damping formulation parameters.
Table 2.4. Properties and parameters used for spectral element model of the isolator of Figure 2.8.

<table>
<thead>
<tr>
<th>Property/Parameter</th>
<th>Damping Formulation</th>
<th>Fractional</th>
<th>Structural</th>
<th>Viscous</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>25.4 mm</td>
<td>25.4 mm</td>
<td>25.4 mm</td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td>25.4 mm</td>
<td>25.4 mm</td>
<td>25.4 mm</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>1 kg/m$^3$</td>
<td>1 kg/m$^3$</td>
<td>1 kg/m$^3$</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>$E_0$</td>
<td>6.2 MPa</td>
<td>12 MPa</td>
<td>12 MPa</td>
<td></td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.2</td>
<td>0.13i</td>
<td>1.9×10^{-5}</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.17</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Over the measured frequency range (up to 600 Hz), the SEM predicted cross-point dynamic stiffness achieves excellent agreement with measurements in terms of both magnitude and phase as demonstrated in Figure 2.9. In particular, observe the low frequency trend (as the frequency decreases towards zero), where the magnitudes of measured and spectral element predicted dynamic stiffness converge to the static stiffness, while the structurally damped FE over-predicts the stiffness at lower frequencies. For higher frequencies, measurement of dynamic stiffness becomes increasingly difficult [2.8], and FE models become computationally expensive. The spectral element model predicts dynamics beyond the measured range as shown in Figure 2.10 in a log-scale, illustrating both low- and high-frequency prediction capabilities of the SEM. Additional experimental studies would be necessary to confirm the accuracy at higher frequencies, but this example case provides sufficient validation for the proposed SEM-based approach.
To highlight the benefits of the fractional calculus based damping included in the SEM, the effects of three damping formulations are compared in Figure 2.11. A viscous damping mechanism tuned to the first resonance dramatically decays the higher modes and fails to match the measured stiffness magnitude or loss angle at lower frequencies. Structural damping offers reasonable predictions of many modes, but likewise fails to capture the low-frequency behavior.

Figure 2.9. Experimental validation of the spectral element method for the isolator of Figure 2.8. Dynamic stiffness spectra up to 600 Hz are shown in terms of (a) magnitude and (b) phase. Key: + – measurement [2.4]; – finite element model [2.4]; – spectral element model with parameters of Table 2.4.
Figure 2.10. Experimental validation of the broadband dynamic stiffness for the isolator of Figure 2.8 in terms of (a) magnitude and (b) phase. Key: ★ – measurement [2.4]; — dashed line – finite element model [2.4]; — solid line – spectral element model with parameters of Table 2.4.

2.6. Conclusion

This principal contribution of this chapter is the development of an analytical spectral element approach to characterize the dynamic properties of viscoelastic vibration isolators up to very high frequencies. It improves the prior literature by extending the stiffness spectra with a fractional calculus based damping mechanism, capturing all 6-DOF including coupling terms, and allowing the isolator to resemble a short beam with \( L/D < 1 \). The theory is developed from the wave equation assuming longitudinal and torsional waves in a rod alongside the Timoshenko beam theory for shear and flexure, although the method is adaptable to more complicated geometries.
A second contribution of this chapter is the computational verification and experimental validation of the proposed SEM. Good comparative agreement is achieved while verifying the SEM with FE simulations over a very large frequency range (up to 5 kHz). Since the two methods agree on an order of magnitude basis, the proposed SEM is an improvement in analytical modeling capabilities for cylindrical isolators with short aspect ratios [2.12]. Compared with experimental characterization of a laboratory isolator, the SEM predicted dynamic stiffness spectra achieve excellent qualitative and quantitative accuracy in terms of both magnitude and phase. In particular, the fractional damping overcomes the inherent limitations of both the viscous and structural damping.

Figure 2.11. Experimental validation of the fractional damping formulation when utilized in the spectral element model for the isolator of Figure 2.8 in terms of (a) low-frequency dynamic stiffness magnitude, (b) low-frequency loss angle, and (c) broadband dynamic stiffness magnitude. Key: * – measurement [2.4]; ----- fractional damping; ---- structural damping; ....... viscous damping.
formulations while capturing the physical phenomena observed in the measured dynamic stiffness in terms of both magnitude and phase.

Finally, this chapter adds to the body of knowledge by offering physical insight and providing a compact and elegant method to characterize viscoelastic isolators. And while many elastomeric materials exhibit sensitivity to excitation amplitude, preload, temperature, and material orientation, all of these may be approximated by the linear theory as the starting point since the small amplitude assumption is reasonable over a large frequency bandwidth.
Chapter 2 References


## Chapter 2 Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>cross-sectional area ($m^2$)</td>
</tr>
<tr>
<td>$a$</td>
<td>left-side node (at $x = 0$)</td>
</tr>
<tr>
<td>$B$</td>
<td>translational solution coefficient</td>
</tr>
<tr>
<td>$b$</td>
<td>right-side node (at $x = L$)</td>
</tr>
<tr>
<td>$C$</td>
<td>rotational solution coefficient</td>
</tr>
<tr>
<td>$c$</td>
<td>viscous damping coefficient</td>
</tr>
<tr>
<td>$D$</td>
<td>generalized differential operator</td>
</tr>
<tr>
<td>$D$</td>
<td>diameter (m)</td>
</tr>
<tr>
<td>$E$</td>
<td>elastic modulus (Pa)</td>
</tr>
<tr>
<td>$F$</td>
<td>force (N)</td>
</tr>
<tr>
<td>$f$</td>
<td>frequency (Hz)</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus (Pa)</td>
</tr>
<tr>
<td>$H$</td>
<td>shear equation coefficients</td>
</tr>
<tr>
<td>$h$</td>
<td>loss factor</td>
</tr>
<tr>
<td>$I$</td>
<td>second moment of area ($m^4$)</td>
</tr>
<tr>
<td>$K$</td>
<td>dynamic stiffness (N/m)</td>
</tr>
<tr>
<td>$L$</td>
<td>length (m)</td>
</tr>
<tr>
<td>$M$</td>
<td>moment (N·m)</td>
</tr>
<tr>
<td>$N$</td>
<td>number of terms</td>
</tr>
<tr>
<td>$Q$</td>
<td>generalized force</td>
</tr>
<tr>
<td>$q$</td>
<td>generalized displacement</td>
</tr>
</tbody>
</table>
$t$  time (s)
$u$ translational displacement (m)
$x, y, z$ coordinates or directions (m)
$\alpha$ fractional damping order
$\beta$ complex-valued wavenumber
$\gamma$ $B$ to $C$ conversion factor
$\varepsilon$ error
$\eta$ fractional damping coefficient
$\theta$ rotational displacement (rad)
$\kappa$ Timoshenko shear coefficient
$\nu$ Poisson’s ratio
$\rho$ density (kg/m$^3$)
$\omega$ circular frequency (rad/s)

Subscripts
0 static
$e$ excitation amplitude
$m, n$ matrix indices (row, column)
$max$ maximum bandwidth
$\lambda$ natural frequency

Abbreviations
DOF degree(s) of freedom
FE finite element

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SEM   spectral element method
Chapter 3. Estimation of the Transient Response of a Tuned, Fractionally Damped Elastomeric Isolator

3.1. Introduction

Elastomeric isolators, mounts, and bushings are widely used in machines, vehicles, and buildings. In automotive suspension, for instance, bushings are used extensively to ensure vibration isolation and impedance mismatch between critical subsystems. Such isolators have been analytically studied with continuous system theory [3.1-3.3] as well as lumped parameter models on both the component [3.4-3.6] and system [3.7] level, often with many simplifications. In many cases, isolators exhibit nonlinear behavior [3.3-3.5,3.8] which may be more readily apparent from transient responses [3.4, 3.5]. In recent decades, increased attention has been given to the use of fractional order derivatives to describe the viscoelastic behavior of elastomeric materials [9-14]. Fractional calculus theory has many physical applications [3.9-3.10] and has been applied to viscoelasticity in the context of pure constitutive relations [3.10-3.13], material models suitable for finite element analysis [3.12], and lumped system modeling [3.14-3.15]. Despite the many approaches to using fractional calculus for viscoelasticity [3.10], it is still not widely used in practice, presumably due to the difficulty of the mathematical sophistication which is often required to produce useful results. Prior approaches to employ fractional calculus can be divided into two categories: analytical transformations of fractional calculus formulations based on Cauchy integrals [3.9-3.11, 3.13-3.14] and
those numerical estimation of fractional derivatives based on the variations of the Grünwald-Letnikov definition [3.9-3.10, 3.12, 3.15]. Each methodology presents unique strengths and weaknesses, but this chapter aims to adopt and build on the analytical approach by introducing an estimation procedure which may simplify the difficult time-domain transformation calculations [3.13-3.14], extending the method to a new class of system. The chief goal is to develop semi-analytical, time-domain response estimates to a class of transient excitation signals when applied to a tuned elastomeric isolator (with inertial effects) in the context of a minimal order model.

3.2. Problem Formulation

Elastomeric isolation components used in automotive suspension exhibit significant frequency-dependent stiffness behavior over a wide range of frequencies. Such “tuned” properties result from inertial and damping effects in the material, each being dominant in a different frequency regime. Capturing the relevant physics in a reduced-order, lumped-parameter model which is effective over a broad frequency range is challenging. Recently, Noll et al. developed a lumped parameter model for an elastomeric joint which clarified the frequency-dependent stiffness by capturing internal mass effects [3.6]. Their model offers good broadband dynamic stiffness predictions, but significant error is found in the lower-frequency regime (say, up to 100 Hz). This error is linked to the damping mechanism assumed in the model [3.6]. Neither structural nor viscous damping is able to produce the low-frequency behavior with a minimal-order model [3.15], although large, empirical viscoelastic networks may reproduce the effect on
an ad hoc basis. A damping mechanism based on fractional calculus is expected to yield superior predictions as assumed in this chapter.

The objectives of this chapter are as follows: 1. Develop a uniaxial minimal order model which can effectively simulate the dynamic stiffness of a production bushing, capturing both low-frequency and broadband behavior covering the first resonance (0-1 kHz), and 2. Propose a new estimation technique to yield time-domain solution approximations for transient excitation using fractional calculus. Spectral characterization of such viscoelastic isolators is useful but insufficient, as the assumption of purely harmonic excitation is unrealistic and may mask properties of the elastomeric system that are relevant under aperiodic or transient excitation. Studying transient responses requires a time-domain representation of the system, such as an impulse response function. Fractional dampers preclude conventional inverse Laplace transform calculations to obtain the impulse response; however, the transform may be obtained using the Residue Theorem and multi-domain estimation techniques. The step response is useful in the analysis of elastomeric isolators since it includes an abrupt shift in the operating loads which may excite interesting behavior from any amplitude-dependent nonlinearities in the system [3.3, 3.5, 3.10, 3.12, 3.15]. Determination of such amplitude-dependent behavior is beyond the scope of this work; nevertheless, a “step-like” input will be used as an example for a realistic transient excitation. Additional complexities may be observed in elastomeric materials such as temperature dependence, aging effects, and anisotropy [3.16]. Since dynamic behavior of viscoelastic materials often exhibit limited sensitivity to these effects for small perturbations about an operating point, these are beyond the scope of this work.
3.3. Spectral Characterization and Minimal Order Model

Elastomeric isolators are typically characterized in terms of dynamic stiffness spectra [3.1-3.3, 3.6-3.7]. A comparison between the measured cross-point stiffness magnitude spectrum and finite element predictions of a laboratory bushing (very similar to production devices [3.6]) is given in Figure 3.1. Details and parameters of the finite element model are already reported in [3.6]. Good broadband accuracy is achieved by the finite element model; however, at low frequencies (below 100 Hz), the measurement reveals a damping mechanism which is not incorporated in the model. Conversely, the measured dynamic stiffness spectrum only goes up to 600 Hz (due to the limitations of the dynamic elastomer test machine), whereas finite element models can extend to larger bandwidths to capture stiffness peaks due to the internal mass. Figure 3.2 depicts the simulated dynamic stiffness spectra for a production bushing [3.6], including both driving-point and cross-point stiffness curves. A model framework which can predict the broadband dynamics while capturing the measured low-frequency behavior observed in Figure 3.1 would be quite useful.

A minimal order model of a rubber bushing, as shown schematically in Figure 3.3(a), is proposed to capture the stiffness, mass, and damping properties. Here, the 1 and 2 coordinates represent the inner and outer metal sleeves the bushing, assumed to be rigid compared to the elastomeric material. Figure 3.3(b) shows the minimal order model using the four-pole type representation with arbitrary boundaries [3.17]. The governing equations (with fractional damping elements) are given by,

\[
m\ddot{y} + (k_1 + c_1 D_{l_1}^{\alpha_1} + k_2 + c_2 D_{l_2}^{\alpha_2}) y(t) = (k_1 + c_1 D_{l_1}^{\alpha_1}) x_1(t) + (k_2 + c_2 D_{l_2}^{\alpha_2}) x_2(t), \quad (3.1-a)
\]
Figure 3.1. Dynamic stiffness predictions of an elastomeric isolator showing insufficiency of structural damping mechanism available in finite element model [3.6].

Key: ■ - Measurement; - Finite element model with structural damping.

Figure 3.2. Dynamic stiffness terms of equation (3.2) using a finite element model [3.6] of a production bushing with structural damping. Key: --- $K_{11}$; - $K_{12}$; - $K_{22}$.
Figure 3.3. Example case for the isolator study: (a) schematic of the elastomeric bushing with inner (1) and outer (2) sleeves and viscoelastic material shown in dark color. Only the axial direction is considered. (b) The proposed minimal order model. Here, $x_1$, $x_2$, and $F_1$, $F_2$ are the displacements and forces of the inner and outer sleeves, respectively. The effective internal mass of the elastomeric material is $m$, and $y$ is its displacement. The spring constants $k_1$ and $k_2$ denote the stiffness of the elastic elements while $c_1, c_2$ and $\alpha_1, \alpha_2$ denote the fractional damper coefficients and order parameters, respectively.
where \(x_1, x_2, F_1,\) and \(F_2\) are the displacements and forces of the inner and outer sleeves, respectively. The \(m\) parameter is the effective internal mass of the elastomeric material, representing the effective inertia that participates in the first vibration mode (analogous to a modal mass). The corresponding displacement coordinate is given as \(y\). Fractional Kelvin-Voigt elements characterize the viscoelastic behavior. The spring constants \(k_1\) and \(k_2\) denote the stiffness of the elastic elements connecting the internal mass to each boundary. The constitutive equations for the two fractional dampers relate dissipative path force to a fractional derivative of displacement, \(F = cD_t^\alpha x\), where \(D_t^\alpha\) is a derivative operator of order \(\alpha\) with respect to \(t\). Therefore, the fractional dampers have four parameters in all: coefficients \((c_1, c_2)\) and order parameters \((\alpha_1, \alpha_2)\).

Assuming harmonic excitation, driving point \((K_{11}, K_{22})\) and cross-point \((K_{12}, K_{21})\) dynamic stiffness terms for the system of Figure 3.3(b) are defined in the Laplace domain as,

\[
K_{11}(s) = \left. \frac{F_1}{x_1} \right|_{x_2=0} = \frac{\left( k_1 + c_1s^{\alpha_1} \right)^2}{ms^2 + c_1s^{\alpha_1} + c_2s^{\alpha_2} + k_1 + k_2} - 1, \quad (3.2-a)
\]

\[
K_{22}(s) = \left. \frac{F_2}{x_2} \right|_{x_1=0} = \frac{\left( k_2 + c_2s^{\alpha_2} \right)^2}{ms^2 + c_1s^{\alpha_1} + c_2s^{\alpha_2} + k_1 + k_2} - 1, \quad (3.2-b)
\]
Figure 3.4 shows plots of the dynamic stiffness terms using assumed, yet realistic model parameters derived from curve-fitting typical measured stiffness spectra with equations (2-a,b,c), where the dynamic magnitude is normalized by the corresponding static stiffness, \( k_0 = k_1 k_2 / (k_1 + k_2) \). The minimal order model predictions qualitatively agree with finite element predictions (of [3.6], in Figure 3.4) in terms of resonance behavior; however, whereas finite element predictions fail to capture the low-frequency trend, the proposed model produces low-frequency behavior that is similar to the measurements (as shown in Figure 3.5) by using fractional dampers instead of structural damping (in the form of a loss factor).

\[
K_{12}(s) = K_{21}(s) = \frac{F_2}{x_1} \bigg|_{s=s_2} = \frac{(k_1 + c_1 s^\alpha_1)(k_2 + c_2 s^\alpha_2)}{m s^2 + c_1 s^\alpha_1 + c_2 s^\alpha_2 + k_1 + k_2}. \tag{3.2-c}
\]

Figure 3.4. Dynamic stiffness spectra predicted from the (a) minimal order model and (b) finite element model [3.6]. Two frequency regimes emerge below and above 100 Hz. The two models agree qualitatively in terms of broadband behavior. Key: \( \cdots \cdots \cdots \) - \( K_{11} \); \( \cdots \cdots \cdots \) - \( K_{12} \); \( \cdots \cdots \cdots \cdots \) - \( K_{22} \).
Figure 3.5. Low-frequency behavior of minimal order model, capturing the damping-controlled convergence to static stiffness while the finite element predictions fail to capture the behavior. Key: - Minimal order model; - Finite element model [3.6].

3.4. Time Domain Characterization

An impulse response function can be calculated from any of the dynamic stiffness expressions,

$$h(t) = \mathcal{L}^{-1}\{K(s)\}, \quad (3.3)$$

using the inverse Laplace transform,

$$\mathcal{L}^{-1}\{K(s)\} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\varepsilon - iR}^{\varepsilon + iR} K(s) e^{st} ds, \quad (3.4)$$
which is a contour integral in the complex plane, and $\zeta$ is a positive, real offset. The Residue Theorem relates a contour integral along a closed loop ($C$) to the residues of all singular points ($\lambda_j$) of the function enclosed by the loop,

$$\int_C K(s) e^{st} ds = 2\pi i \sum_j \text{Res}\left[K(s) e^{st}, \lambda_j\right], \quad (3.5-a)$$

$$\text{Res}\left[K(s) e^{st}, \lambda_j\right] = \lim_{s \to \lambda_j} (s - \lambda_j) K(s) e^{st}. \quad (3.5-b)$$

If $C$ encloses any branch points of $K(s)$, then the integrand’s discontinuity will violate the conditions of the theorem. Fractional damping elements generate a branch point at the origin, so the contour is judiciously chosen to avoid it as seen in Figure 3.6(a). Jordan’s lemma demonstrates that as $R \to \infty$ and $\rho \to 0$, the integrals along the contours $C_{2a}$, $C_{2b}$, and $C_4$ vanish, leaving three remaining terms as shown in Figure 3.6(b),

$$\int_C K(s) e^{st} ds = \int_{C_1} K(s) e^{st} ds + \int_{C_{3a}} K(s) e^{st} ds + \int_{C_{3b}} K(s) e^{st} ds. \quad (3.6)$$

Since the contour $C_1$ is equivalent to the transform integral from equation (3.4), it follows from equations (3.4), (3.5-a), and (3.6) that

$$h(t) = \sum_j \text{Res}\left[K(s) e^{st}, \lambda_j\right] - \frac{1}{2\pi i} \left(\int_{C_{3a}} K(s) e^{st} ds + \int_{C_{3b}} K(s) e^{st} ds\right). \quad (3.7)$$

Observe that the two terms of $h(t)$ are quite different, and thus they are defined individually as,

$$h(t) = h_v(t) + h_r(t), \quad (3.8)$$

where $h_v$ is the sum of the residues and $h_r$ is the remaining integral term. Although it is not obvious from equation (3.7), the two terms are named in this manner because the
residues yield a vibratory response while the integrals produce a relaxation effect as illustrated in the next two sections.

3.5. **Vibratory Response**

The vibratory response is given as the sum of the residues of $K(s)e^{\alpha}$ in equation (3.7) which requires that any singular points contained in the region bounded by $C$ are located. Singular points occur at the zeros of the $K(s)$ denominator $Z(s) = ms^2 + c_1s^{\alpha_1} + c_2s^{\alpha_2} + k_1 + k_2$, which (unlike the simple harmonic oscillator systems with viscous damping) is a fractional polynomial. Given complications in root-finding

![Diagram](image)

Figure 3.6. Application of the Residue Theorem showing (a) the Bromwich contour used to evaluate the inverse Laplace transform and (b) division of the contour into zero and nonzero segments. Key: - Bromwich contour; - Integrals sum to zero; - Inverse Laplace transform segment; - Remaining segments.
calculations, several approaches may be used to locate these roots. First, assume that both fractional order parameters are rational (\( \alpha_1 = q_1 / n \), \( \alpha_2 = q_2 / n \), where \( q_1 \), \( q_2 \), and \( n \) are integers) and defining a second Laplace variable, \( p = s^{1/n} \). \( Z \) is now converted to a more conventional polynomial,

\[
Z = ms^2 + c_1s^{\alpha_1} + c_2s^{\alpha_2} + k_1 + k_2 = mp^{2n} + c_1p^{\beta_1} + c_2p^{\beta_2} + k_1 + k_2. \quad (3.9)
\]

The roots may be calculated by generating a Frobenius companion matrix [3.18] and calculating its eigenvalues. This yields the \( 2n \) roots of the right hand side of equation (3.9), which should collapse to the two roots of the left hand side, \( \lambda_s = \lambda_p^n \). However, an unacceptable degree of error may creep into the calculation of companion matrix eigenvalues for a large \( n \) (which may become quite large for arbitrary fractional order parameters), and thus the roots \( \lambda_p \) may fail to collapse to just two roots \( \lambda_s \). Due to the inherent extreme behavior of \( K(s) \) near these roots, the residues may be sensitive to the error in singular point values, and therefore an alternate technique is proposed below.

The Newton-Raphson Method is a classical approach for both root-finding and minimization problems. Although it is an iterative approach, this method can locate a function’s roots in the complex plane with high precision after only a few steps given a well-behaved function and a good initial guess. As it happens, the example case in this chapter offers both. For \( 0 < \alpha_1, \alpha_2 < 1 \), there are always two complex-conjugate roots in the left half-plane. Selection of an initial guess in quadrant II (e.g. \( s_0 = -1 + i \)) leads to a quick convergence to the singular point in that quadrant with a very high degree of precision, as shown in Figure 3.7.
Figure 3.7. Minimization procedure using the Newton-Raphson Method to locate singular points in the transfer function. Key: □ - Approximate singular point location; - Path of Newton-Raphson Method.

Once the singular points $\lambda_s, \lambda^*$ are calculated, the residues may be evaluated. For the cross-point dynamic stiffness, the residue from equation (3.2-c) is,

$$\text{Res} \left[ K(s) e^{\omega t}, \lambda_s \right] = \lim_{s \to \lambda_s} \frac{(s - \lambda_s)(k_1 + c_1 s^{\alpha_1})(k_2 + c_2 s^{\alpha_2})}{ms^2 + c_1 s^{\alpha_1} + c_2 s^{\alpha_2} + k_1 + k_2} e^{\omega t},$$  \hspace{1cm} (3.10)

which yields an indeterminate form. Application of L’Hopital’s rule leads to a simplified solution as,

$$\text{Res} \left[ K(s) e^{\omega t}, \lambda_s \right] = \frac{(k_1 + c_1 \lambda_s^{\alpha_1})(k_2 + c_2 \lambda_s^{\alpha_2})}{2ms + \alpha_1 c_1 \lambda_s^{\alpha_1-1} + \alpha_2 c_2 \lambda_s^{\alpha_2-1}} e^{\omega t} = (A+iB)e^{(a+ib)t},$$  \hspace{1cm} (3.11-a)

$$\text{Res} \left[ K(s) e^{\omega t}, \lambda^* \right] = (A-iB)e^{(a-ib)t},$$  \hspace{1cm} (3.11-b)
where \( a, b \) are the real and imaginary parts of the quadrant III singular point and \( A, B \) are the real and imaginary parts of its residue, respectively. The vibratory impulse response \( h_v(t) \) is now calculated,
\[
h_v(t) = (A + iB)e^{(a+ib)t} + (A - iB)e^{(a-ib)t},
\]
and rewritten as,
\[
h_v(t) = 2\sqrt{A^2 + B^2}e^{at}\sin(bt + \theta),
\]
\[
\theta = \tan^{-1}\left(\frac{A}{-B}\right).
\]

### 3.6. Relaxation Response

The second term of the impulse response function in equation (3.7) is somewhat more difficult to calculate. The integrals can be simplified by letting \( s = \ell e^{i\pi} \) on \( C_{3a} \) and \( s = \ell e^{-i\pi} \) on \( C_{3b} \),
\[
h_r(t) = -\frac{1}{2\pi i} \lim_{\rho \to 0} \lim_{R \to \infty} \int_{\rho}^{R} \left( K(\ell e^{i\pi}) - K(\ell e^{-i\pi})\right)e^{-\ell t} d\ell.
\]
Using equation (3.2-c) in equation (3.14) yields,
\[
h_r(t) = -\frac{1}{\pi i} \int_{0}^{\infty} \text{Im}[K(\ell e^{i\pi})]e^{-\ell t} d\ell,
\]
which cannot be evaluated by conventional integration techniques. While it is always possible that a solution to this integral exists, any significant modifications to \( K(s) \) would likely require new analytical solutions, so an alternate approach is suggested. Decompose \( K(s) \) into vibratory and relaxation terms by taking the Laplace transform of equation (3.8) as shown in Figure 3.8,
\[ K_r(s) = K(s) - \mathcal{L}\{h_v(t)\}. \] (3.16)

The plot of Figure 3.8 illustrates that \( K_r(s) \) is itself a relatively crude approximation of \( K(s) \), while the effects of \( K_r(s) \) are an order of magnitude smaller except at the very low frequency end. Nevertheless, capturing such low-frequency behavior is among the benefits of using fractional damping in a reduced order model. The algebraic expression for \( K_r(s) \) is quite complicated and would not always have a convenient inverse Laplace transform. As such, the following two-parameter functional approximation is proposed to yield similar behavior with greatly simplified calculations,

\[
\hat{K}_r(s) = \frac{h_{20}}{(s - \beta)^\eta}, \quad (3.17-a)
\]

\[
h_{20} = (-\beta)^\eta \text{Re}[K_r(0)]. \quad (3.17-b)
\]

The two parameters \((\beta, \eta)\) are chosen to approximate the magnitude, \(|\hat{K}_r(s)| \approx |K_r(s)|\), and a good agreement in magnitude is achieved as shown in Figure 3.9. However, significant phase error exists, but the corresponding amplitude is quite small so it is ultimately negligible when combined with \( K_v(s) \). The mean-squared error of the estimate \( \varepsilon \) may be quantified as follows for typical, realistic model parameters,

\[
\varepsilon = \frac{1}{Nk_0^2} \sum_j (\hat{K}_j - K_j) (\hat{K}_j^* - K_j^*). \quad (3.18)
\]

Figure 3.10 compares the \( \hat{K}(s) \) estimate with the original \( K(s) \) expression, obtaining \( \varepsilon < 1\% \) and showing substantial improvement over a viscously damped approximation as
exemplified by $K_r(s)$. Recall that the functional form of $\hat{K}_r(s)$ is chosen in equation (3.17-a) in part for its inverse Laplace transform which is given by,

$$\hat{h}_r(t) = L^{-1}\left\{\frac{h_{20}}{(s - \beta)^{\eta}}\right\} = \frac{h_{20}}{\Gamma(\eta)} t^{\eta-1} e^{\beta t},$$  \hspace{1cm} (3.19)$$

where $\Gamma(\cdot)$ is the Gamma function. This leads to the estimate form of equation (3.8),

$$\hat{h}(t) = h_v(t) + \hat{h}_r(t),$$

which is illustrated in Figure 3.11. The plot illustrates expected behavior of $h_v(t)$ as equivalent to a viscously damped second-order system response, but the effects of $\hat{h}_r(t)$ are less apparent from the impulse response. Mathematically,

$$\hat{h}_r(t) \rightarrow -\infty \text{ as } t \rightarrow 0^+, \text{ but the physical meaning and consequences will become clear when the response to a realistic transient excitation is studied in the next section.}$$

![Figure 3.8](image)

Figure 3.8. Spectral components of the dynamic stiffness where the vibratory stiffness term may serve as a rough estimate of the total dynamic stiffness. Key: - Total dynamic stiffness, $K$; - Vibratory dynamic stiffness, $K_v$; - Relaxation dynamic stiffness, $K_r$. 

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3.7. Response to a Transient Excitation

The following function represents a realistic displacement profile of a dynamic elastomer test machine [3.19] simulating a “step-like” excitation,

\[ x(t) = X\left(1 - e^{-t/T}\right), \tag{3.20} \]

which is equivalent to a first-order system response where \( T \) is the time constant. Calculating the “step-like” response to this input requires an evaluation of the following convolution integral,

\[ F(t) = \int_0^t \hat{h}(\tau)x(t-\tau)d\tau, \tag{3.21} \]

Figure 3.9. Estimation of the relaxation stiffness component in terms of (a) magnitude and (b) phase. Key: \(- K - K_v; \) \(- \hat{K}_r.\)

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from which the vibratory and relaxation transient responses may be separated as,

\[ F_v(t) = 2X\sqrt{A^2 + B^2} \int_0^t e^{\alpha \tau} \sin(b\tau + \theta) \left(1 - e^{-(t-\tau)/\tau_\theta} \right) d\tau, \]  
(3.22-a)

\[ \hat{F}_r(t) = \frac{k_{20}X}{\Gamma(\eta)} \int_0^t \tau^{\eta-1} e^{\beta \tau} \left(1 - e^{-(t-\tau)/\tau_\eta} \right) d\tau. \]  
(3.22-b)

Equation (3.22-a) may be expanded using trigonometric identities,

\[ F_v(t) = 2X\sqrt{A^2 + B^2} \left[ \cos(\theta) \left( \int_0^t e^{\alpha \tau} \sin(b\tau) d\tau - e^{-i\tau_\eta} \int_0^t e^{(\alpha + i\tau_\eta)\tau} \sin(b\tau) d\tau \right) \right. \]
\[ + \left. \sin(\theta) \left( \int_0^t e^{\alpha \tau} \cos(b\tau) d\tau - e^{-i\tau_\eta} \int_0^t e^{(\alpha + i\tau_\eta)\tau} \cos(b\tau) d\tau \right) \right] \]  
(3.23)

Figure 3.10. Dynamic stiffness estimation in terms of (a) magnitude and (b) phase. Key:

- Vibratory dynamic stiffness, \( K_1 \);  
- Total dynamic stiffness estimate, \( \hat{K}_r \);  
- True dynamic stiffness, \( K_r \).
and then integrated,

\[ F_v(t) = 2X \sqrt{A^2 + B^2} \left( \frac{e^{at} \sin(bt + \psi) - \sin(\psi) - e^{at} \sin(bt + \psi_T) - e^{-t/T} \sin(\psi_T)}{\sqrt{a^2 + b^2} \sqrt{(a + 1/T)^2 + b^2}} \right) \]  

(3.24-a)

\[ \psi = \theta - \tan^{-1}\left( \frac{b}{a} \right), \quad \psi_T = \theta - \tan^{-1}\left( \frac{b}{a + 1/T} \right). \]  

(3.24-b,c)

Next, the relaxation response is divided in a similar manner,

\[ \hat{F}_r(t) = \frac{h_{20}X}{\Gamma(\eta)} \left[ \int_0^t \tau^{\eta-1} e^{\beta \tau} d\tau - e^{-t/T} \int_0^t \tau^{\eta-1} e^{(\beta + 1/T)\tau} d\tau \right], \]  

(3.25)

and solved in terms of the incomplete Gamma function,

![Figure 3.11. Impulse response function, showing both vibratory and relaxation components. Key: - Total impulse response, \( h(t) \); - Relaxation impulse response term, \( \hat{h}_r(t) \); - Vibratory impulse response term, \( h_v(t) \).](image)

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This is further simplified in terms of the lower incomplete Gamma function [3.20],

\[ \hat{F}_r(t) = \frac{-h_{20}X}{\Gamma(\eta)} \left[ (-\beta)^{-\eta} \Gamma(\eta, -\beta t) - e^{-t/T} \left[ (-\beta - \frac{1}{T})^{-\eta} \Gamma(\eta, -\left(\beta + \frac{1}{T}\right)t) \right]_0^t \right]. \] (3.26)

The lower incomplete gamma function may be expressed as a power series [3.20],

\[ \gamma(a, b) = \Gamma(a) b^n e^{-b} \sum_{j=0}^{\infty} \frac{b^j}{\Gamma(a + j + 1)}. \] (3.28-a)

leading to the simplification,

\[ \hat{F}_r(t) = h_{20}X t^\eta e^{\beta t} \sum_{j=0}^{\infty} \frac{(-t)^j \left( (\beta) - (\beta + 1/T)^j \right)}{\Gamma(\eta + j + 1)}. \] (3.28-b)

Observe two-parameter Mittag-Leffler functions \( E_{\alpha,\beta}(z) \), as reported in [3.9, 3.21]) in equation (3.27-a), and thus it is written more compactly as,

\[ \hat{F}_r(t) = h_{20}X t^\eta e^{\beta t} \left[ E_{\eta, 1}(-\beta t) - E_{\eta, 1}(-\left(\beta + \frac{1}{T}\right)t) \right]. \] (3.29)

### 3.8. Results and Discussion of the Transient Response

The overall transient response is compared with each term as well as the response of a model with zeroth-order system dynamics \( K(s) = k_0 \) in Figure 3.12, and the significance of the relaxation response (for which it is named) emerges. Two time scales are apparent. One time scale is on the order of the natural period of the internal mass (1.8 ms for this system) and governs the oscillations in the vibratory response. Conversely, the second time scale represents a slower process (about 40 ms to saturation) whereby the
force transmitted from the transient event relaxes to the static equilibrium. The vibratory response behaves as a second-order system (with a pair of complex roots), while the relaxation response in some ways resembles a first-order system (with one real valued, negative root). This suggests that the fractionally damped isolator model (with a mass element) qualitatively behaves like a third-order dynamic system, even though the differential equation (3.1-a) is only of the second order.

Figure 3.12. Transient response of the isolator of Figure 3.3(b) to “step-like” inputs with time constants of (a) $T = 5$ ms and (b) $T = 2$ ms. Key:  
- Vibratory response, $F_v(t)$;  
- Relaxation response, $\hat{F}_r(t)$;  
- Total response, $\hat{F}(t)$;  
- Zeroth-order response, $k_0x(t)$. 

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The attributes of a fractionally damped system have significant physical ramifications for the design of vibration isolation or energy absorption components. While the spectral stiffness properties remain relevant to the design process, many applications would involve harsh transient excitation which may not reflect the frequency-domain behavior. For instance, the effect of the step time constant $T$ on this particular system is significant. As one would expect, a quicker rise time excites the system more abruptly, producing increased oscillation and a more pronounced relaxation. These two behaviors are also strongly affected by the $\alpha_1, \alpha_2$ parameters, as seen in Figure 3.13. An increased in the fractional damping order would decrease the settling time in terms of both ring-down and relaxing to the static equilibrium, but it increases the overshoot of the non-oscillatory response.

![Figure 3.13](image)

Figure 3.13. Sensitivity of “step-like” responses to fractional order parameters: (a) $\alpha_1 = \alpha_2 = 0.075$, (b) $\alpha_1 = \alpha_2 = 0.15$, and (c) $\alpha_1 = \alpha_2 = 0.30$. Key: - Vibratory response, $F_v(t)$; - Relaxation response, $\hat{F}_r(t)$; - Total response, $\hat{F}(t)$; - Zeroth-order response, $k_0x(t)$. 

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It must be noted that the transient responses discussed in this chapter rely on the accurate evaluation of the Mittag-Leffler functions in equation (3.29). Due to computational limitations, these functions converge only within a finite time range, after which the error becomes excessively large. Figure 3.14 illustrates this problem. The time range for which the solution converges is sensitive to both the model and its excitation parameters, most notably $\alpha_1, \alpha_2$ and $T$, respectively. Using a recursive definition of the Mittag-Leffler function in the context of the power series partially mitigates this problem. Since the initial transient behavior resulting from realistic excitations are within the convergent time range, an examination of more advanced methods for evaluating Mittag-Leffler functions [3.21] is beyond the scope of this chapter.

Figure 3.14. Transient response showing numerical divergence using $T = 1 \text{ ms}$.  

$\alpha_1 = \alpha_2 = 2\alpha$. Key: - Vibratory response, $F_v(t)$; - Relaxation response, $\hat{F}_r(t)$; - Total response, $\hat{F}(t)$; - Zeroth-order response, $k_0x(t)$. 

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3.9. Conclusion

The major contribution of this chapter is the successful development of a minimal order system model of tuned, fractionally damped vibration isolator where the limitations of a finite element model with structural damping at the low-frequency end are overcome. This benchmark model should provide a convenient vehicle for further studies in this area. The second major contribution is the new time-frequency domain estimation procedure to calculate time-domain responses of a class of tuned, fractionally damped elastomeric joint models to realistic transient excitations. While direct analytical solutions are possible for some fractionally damped systems, the multi-domain approach proposed in this chapter may yield good solution estimates especially in those cases where direct solutions are infeasible. Difficulties related to the evaluation of complex, improper integrals which arise using the Residue Theorem for inverse Laplace transformation [3.9-3.11, 3.13-3.14] are avoided by approximating troublesome terms with similar functions having a more well-behaved analytical behavior. This approximation achieves excellent accuracy, introducing only a minimal mean-squared error. The proposed method highlights interesting physical characteristics of a fractionally damped system, including parallel vibratory and relaxation behavior qualitatively similar to a third-order system despite starting with a second-order differential equation. Finally, this chapter makes a contribution to the fractional calculus based applications [3.9-3.15].

The methods proposed in this chapter have certain limitations resulting from the problem formulation in terms of a single degree of freedom (implying uniaxial motion) and assumptions which permit a tractable solution. Linearity is implicit to the Laplace
transform, and thus the proposed analytical framework is valid for small perturbations about operating point in terms of amplitude, preload, temperature, material orientation and aging, etc. [3.16]. The proposed solutions provide meaningful physical insight in the context of engineering analyses of automotive suspensions and other physical systems which employ viscoelastic isolators.
Chapter 3 References


### Chapter 3 Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>real part of quadrant II singular point</td>
</tr>
<tr>
<td>$A$</td>
<td>real part of quadrant II residue</td>
</tr>
<tr>
<td>$b$</td>
<td>imaginary part of quadrant II singular point</td>
</tr>
<tr>
<td>$B$</td>
<td>imaginary part of quadrant II residue</td>
</tr>
<tr>
<td>$c$</td>
<td>fractional damper coefficient</td>
</tr>
<tr>
<td>$C$</td>
<td>Bromwich contour</td>
</tr>
<tr>
<td>$D$</td>
<td>derivative operator</td>
</tr>
<tr>
<td>$E$</td>
<td>two-parameter Mittag-Leffler function</td>
</tr>
<tr>
<td>$h$</td>
<td>impulse response</td>
</tr>
<tr>
<td>$j$</td>
<td>index</td>
</tr>
<tr>
<td>$k$</td>
<td>spring stiffness coefficient</td>
</tr>
<tr>
<td>$K$</td>
<td>dynamic stiffness</td>
</tr>
<tr>
<td>$\ell$</td>
<td>integration variable</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>Laplace transform operator</td>
</tr>
<tr>
<td>$m$</td>
<td>effective internal mass</td>
</tr>
<tr>
<td>$n$</td>
<td>common denominator of fractional orders</td>
</tr>
<tr>
<td>$N$</td>
<td>maximum number of terms</td>
</tr>
<tr>
<td>$p$</td>
<td>scaled Laplace variable</td>
</tr>
<tr>
<td>$q$</td>
<td>fractional order numerator</td>
</tr>
<tr>
<td>$r$</td>
<td>relaxation response</td>
</tr>
<tr>
<td>$R$</td>
<td>outer Bromwich contour radius</td>
</tr>
</tbody>
</table>
Res: simple residue

$s$: Laplace transform variable

$t$: time

$T$: time constant for step-like excitation

$\nu$: vibratory response

$x$: displacement of boundary

$X$: step height

$y$: displacement of internal mass

$Z$: dynamic stiffness denominator

$\alpha$: fractional damper order

$\beta$: estimation root

$\eta$: estimation exponent

$\varepsilon$: mean squared Laplace-domain error

$\gamma(\cdot)$: lower incomplete gamma function

$\Gamma(\cdot)$: Gamma function

$\Gamma(\cdot,\cdot)$: upper incomplete Gamma function

$\theta$: phase offset of vibratory response

$\psi$: step-like response phase offset

$\xi$: positive real offset

$\lambda$: singular point

$\rho$: inner Bromwich radius
$\tau$ integration time

*Subscripts*

0 static

1, 2 coordinate of inner/outer sleeve

*Superscripts*

* complex conjugate

^ estimate
Chapter 4. Harmonic Amplitude Dependent Dynamic Stiffness of Hydraulic Bushings: Alternate Nonlinear Models and Experimental Validation

4.1 Introduction

Hydraulic elastomeric devices are often employed in automotive powertrain and suspension systems because of their unique dynamic properties, leading to both vibration isolation and motion control [4.1-4.18]. These properties are achieved by an internal fluid system working in tandem with the elastomeric structure of a bushing. Despite some similarities with hydraulic engine mounts that have been extensively studied [4.1-4.9], the behavior of hydraulic bushings is quite different and merits its own in-depth studies [4.10-4.18]. Prior, though limited, investigations of these devices have largely focused on simpler transfer function type formulations based on the linear time-invariant (LTI) system theory [4.10-4.16]. For instance, Arzanpour and Golnaraghi [4.12] developed a reduced order linear system model for a hydraulic bushing which attempts to capture some aspects of the physics, such as fluid resistance, compliance, and inertance. Chai, et al. [4.13-4.16] further developed the linear models for frequency and time domain characteristics for a laboratory bushing device and even introduced a nonlinear fluid resistance term [4.17]. Fredette, et al. [4.18] recently developed a new laboratory experiment to measure the nonlinear fluid compliance of hydraulic bushing pumping chambers.
Under harmonic excitation, hydraulic bushings exhibit significant amplitude dependent behavior [4.10, 4.11, 4.14-4.18], which cannot be described by the linear time-invariant system theory. Both the mechanical (rubber path) stiffness and fluid compliance elements of the bushings arise from the molded interfacial elastomeric material. Since most elastomeric materials exhibit inelastic behavior, nonlinear fluid compliance behavior has been suggested, [4.4, 4.5, 4.7-4.9, 4.17, 4.18] but the amplitude sensitivity has never been mathematically described. The chief goal of this chapter is therefore to propose new or improved quasi-linear and nonlinear reduced-order hydraulic bushing models, predict amplitude sensitivity characterization under harmonic loading, and experimentally validate alternate nonlinear models. Additionally, the prior work [4.17] on the multi-term harmonic balance method (MHBM) will be extended and refined to explain the underlying physics.

4.2 Problem Formulation

Modeling of hydraulic bushings is challenging due to their complexity that arises due to the interacting nonlinear design features, nonlinear materials, and variation in production bushing designs. Hydraulic bushings are typically constructed of an elastomeric material constrained by a metal inner and outer sleeve, as shown in Figure 4.1. Fluid-filled internal chambers deform when the bushing is displaced, pumping the fluid through a long passage between the chambers. A Kelvin-Voigt (linear system) model is assumed for the elastomeric structural path, while this chapter focuses on nonlinear fluid elements.
The lumped parameter modeling method is suitable for the fluid system contained within hydraulic bushings at low frequencies (up to 50 Hz), where the corresponding wavelength is much larger than the bushing dimensions. For the example case, model parameters include fluid compliance, \( C \), and effective pumping area, \( A \), for each chamber, the fluid resistance, \( \mathcal{R} \), and inertance, \( I \), of the inertia track, and the stiffness, \( K \), and viscous damping coefficient, \( c \), of the rubber structure within the device. The state variables of the system include the absolute pressure, \( p \), in each chamber (left \((L)\) and right \((R)\)) as well as the volume flow rate of fluid between the chambers, \( q \). The outer sleeve is considered to be constrained, so the system is excited by the displacement of the inner sleeve, \( x(t) \); the force transmitted to the outer sleeve, \( F_r \), is the response.

Tractable lumped parameter models are needed to yield reasonable predictions of amplitude sensitive dynamic stiffness, which is a useful metric for design and diagnostic purposes, and to provide physical insight. Accordingly, the specific objectives of this chapter are as follows. (1) Propose and experimentally validate quasi-linear and nonlinear bushing models (with nonlinear resistance and compliance elements) which capture amplitude sensitivity in the example case of a production bushing with a single inertia track as displayed in Figure 4.1. (2) Refine and utilize the semi-analytical multi-term harmonic balance method to construct dynamic stiffness spectra and gain physical insight into the role and interaction of nonlinearities in the model. The scope of the work is limited to sinusoidal excitation only with peak-to-peak displacement amplitudes of 0.1 mm, 0.5 mm, 1.0 mm, and 2.0 mm over a frequency range from 1 to 50 Hz to capture the amplitude sensitivity of the tuned dynamic properties.
Figure 4.1. Hydraulic bushing model. $F_T$ is the transmitted force, $x(t)$ is the displacement excitation, $K_r$ is the stiffness and $c_r$ is the damping of the rubber path. For the pumping chambers, $C$ is the fluid compliance, $A$ is the effective pumping area, and $p$ is the absolute pressure in each chamber, right ($R$) and left ($L$). In the inertia track, $I_i$ is the fluid inertance, $R_i$ is the fluid resistance and $q_i$ is the volume flow rate.
The bushing is characterized by a cross-point dynamic stiffness, assuming a sinusoidal excitation at angular frequency $\Omega$, where $x_m$ is the mean component and $x_a$ is the peak-to-peak excitation amplitude,

$$x(t) = x_m + \frac{x_a}{2}\sin(\Omega t). \quad (4.1)$$

The force transmitted to a rigid base is calculated by summing the contributions from parallel structural and fluid paths. The force through the elastomeric structural path may be directly calculated, $F_r = K_r x + c_r \dot{x}$, while the force through the fluid path requires a solution to the following governing fluid system equations,

$$\begin{align*}
C_L \dot{p}_L &= A_L \dot{x} - q_i, \quad (4.2-a) \\
C_R \dot{p}_R &= -A_R \dot{x} + q_i, \quad (4.2-b) \\
I_i \dot{q}_i &= p_L - p_R - R_i q_i. \quad (4.2-c)
\end{align*}$$

Note that both pressure $p_L$ and $p_R$ time histories are needed for the calculation, since the fluid path force is: $F_f = A_L p_L - A_R p_R$. The equations can be expressed in the following canonical state form allowing direct calculation of the total transmitted force from the fluid state; here, $u = \begin{bmatrix} x & \dot{x} \end{bmatrix}^T$ is the excitation vector and $z$ is the fluid state vector,

$$z = \begin{bmatrix} p_L \\ p_R \\ q_i \end{bmatrix}, \quad (4.3)$$
When the system is linear time-invariant (as shown in Figure 4.1), the total transmitted force will be sinusoidal where \( F_m \) is the mean component, \( F_a \) is the peak-to-peak force amplitude, and \( \phi \) is the phase (or loss angle),

\[
F_T (t) = F_m + \frac{F_a}{2} \sin (\Omega t - \phi).
\]

In this case, the harmonic excitation and response allows for a precise definition of complex-valued dynamic stiffness, \( \tilde{K} \), at \( \Omega \), given by magnitude and angle respectively,

\[
|\tilde{K} (\Omega)| = \frac{F_a}{x_a}\bigg|_{\Omega}, \quad \angle \tilde{K} (\Omega) = \phi(\Omega).
\]

### 4.3 Development of Quasi-linear Model

#### 4.3.1 Linear time-invariant formulation

A transfer function representation of dynamic stiffness is easily obtained by applying the Laplace Transform to the differential equations which govern the fluid system. This allows for an algebraic solution of the pressures, \( p_L \) and \( p_R \) in Laplace (s)

\[
\begin{bmatrix}
\dot{z} \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & -\frac{1}{C_L} \\
0 & 0 & \frac{1}{C_L} \\
\frac{1}{I_R} & -\frac{1}{I_R} & -\frac{R}{I_R}
\end{bmatrix}\begin{bmatrix}
z \\
\end{bmatrix} + \begin{bmatrix}
0 & \frac{A_L}{C_R} \\
0 & -\frac{A_R}{C_R} \\
0 & 0
\end{bmatrix}\begin{bmatrix}
u_s \\
\end{bmatrix},
\]

\[
F_T = [A_L - A_R 0]z + \begin{bmatrix}
K_r & c_r
\end{bmatrix}u.
\]

\[(4.4-a)\]

\[(4.4-b)\]
domain, and thus a direct calculation of the dynamic stiffness transfer function (in $s$ domain) as,

$$
K(s) = \frac{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}{s^2 + \alpha_5 s + \alpha_4},
$$

(4.7-a)

where the coefficients are defined as follows,

$$
\alpha_0 = \frac{K_r}{I_i} \left( \frac{1}{C_L} + \frac{1}{C_R} \right) + \frac{1}{I_i} \left( \frac{A_L - A_R}{C_L C_R} \right)^2,
$$

(4.7-b)

$$
\alpha_1 = K_r \frac{R_i}{I_i} + c_r \left( \frac{1}{C_L} + \frac{1}{C_R} \right) + \frac{R_i}{I_i} \left( \frac{A_L^2}{C_L} + \frac{A_R^2}{C_R} \right),
$$

(4.7-c)

$$
\alpha_2 = K_r + c_r \frac{R_i}{I_i} + \left( \frac{A_L^2}{C_L} + \frac{A_R^2}{C_R} \right),
$$

(4.7-d)

$$
\alpha_3 = c_r,
$$

(4.7-e)

$$
\alpha_4 = \frac{1}{I_i} \left( \frac{1}{C_L} + \frac{1}{C_R} \right),
$$

(4.7-f)

$$
\alpha_5 = \frac{R_i}{I_i}.
$$

(4.7-g)

Unlike the transfer function method based on the LTI system theory, significant amplitude dependence is observed in the measured dynamic stiffness $\tilde{K}_d$, of a production bushing. It is displayed in Figure 4.2 for both $|\tilde{K}_d|$ and $\phi$. Here, frequency is normalized by the frequency at which the loss angle is maximum, $\tilde{\Omega} = \Omega / \Omega_b$, and the dynamic stiffness magnitude is normalized by the static stiffness, $|\tilde{K}_d| = |\tilde{K}_d| / K_s$. 

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Figure 4.2. Measured dynamic stiffness magnitude (top) and loss angle (bottom) of a production bushing showing significant sensitivity to excitation amplitude. Key: \( x_a = 0.1 \) mm; \( x_a = 0.5 \) mm; \( x_a = 1.0 \) mm; \( x_a = 2.0 \) mm peak to peak excitation amplitude.

4.3.2 Quasi-linear formulation (Model I)

Since the LTI model is not capable of capturing the effects in Figure 4.2, an extension to a quasi-linear model is necessary. Allowing model coefficients \( \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \) and \( \alpha_5 \) or physical parameters \( I_i, R_i, C, A, K_r, \) and \( c_r \) to be influenced by the state variables of the fluid system may better characterize the physical behavior observed in hydraulic bushings [4.2] and thus capture the amplitude dependent
effects. The parameters most sensitive to changes in \( x \) are identified by an empirical parameter estimation method. This is accomplished by approximating \( \tilde{K}_d(\Omega, x) \) for a production bushing (such as the results of Figure 4.2) with an assumed form,

\[
\tilde{K}_f(\Omega, x) = \frac{\left(i\Omega_j\right)^3 \alpha_5(x) + \left(i\Omega_j\right)^2 \alpha_2(x) + \left(i\Omega_j\right) \alpha_1(x) + \alpha_0(x)}{\left(i\Omega_j\right)^2 + \left(i\Omega_j\right) \alpha_5(x) + \alpha_4(x)}, \quad (4.8)
\]

where \( s \rightarrow i\Omega \) from equation (4.7-a). A nonlinear constrained optimization with an interior-point algorithm [4.21] is used to optimize the \( \alpha \) coefficients to minimize the difference between the measured \( \tilde{K}_d \) and the quasi-linear model, \( \tilde{K}_f \), at a given \( x \). The optimization is achieved with the numerator and denominator \( \alpha \) coefficients instead of the physical parameters \( (I, R, C, A, K_r, \text{and} \varepsilon) \) because of the numerical difficulties resulting from an extreme dynamic range displayed by the physical parameters. An objective function \( (\varepsilon_a) \) is defined as follows, where \( \tilde{K}_d(\Omega, x) \) is the measured dynamic stiffness of the production bushing at a given amplitude \( x \) which has been measured at frequency, \( \Omega_j \),

\[
\varepsilon_a(x) = \sum_j \left( \left| \tilde{K}_f(\Omega_j, x) - \tilde{K}_d(\Omega_j, x) \right| \right)^2. \quad (4.9)
\]

Physical parameters may be calculated from the optimized values of \( \alpha_j \) using equations (4.7-b,c,e,f,g). Since the bushing is geometrically symmetric (barring any manufacturing variations), the assumptions \( C_L = C_R = C \) and \( A_L = A_R = A \) reasonably reduce the number of physical parameters. With these assumptions, the number of physical parameters reduces to six, which is equal to the number of transfer function
coefficients. However, the set of equations is not linearly independent. An assumed value for fluid inertance (based on geometry and density calculations) allows for the direct estimation of the other parameters, $K_r, c_r, \mathcal{R}_r, C$, from equations (4.7-b,c,d,e,f,g):

\[ K_r = \frac{\alpha_0}{\alpha_4} \]

(4.10-a)

\[ c_r = \alpha_3, \]

(4.10-b)

\[ \mathcal{R}_r = \alpha_3 I_r, \]

(4.10-c)

\[ C = \frac{2}{\alpha_4 I_i}. \]

(4.10-d)

This leaves two equations with a single unknown, $A$. Running yet another (though similar) optimization using pumping area $A$ (assuming symmetry) as the only parameter reveals that the effective value is equal to the one calculated from equation (4.7-d),

\[ A = \frac{C}{2} \left( \alpha_2 - K_r - \frac{c_r \mathcal{R}_r}{I_i} \right). \]

(4.10-e)

The coefficients $\alpha_j$ are then recalculated from the new physical parameters, producing the empirical $\tilde{K}_j$. The results of this process for each measured amplitude are plotted in Figures 4.3 and 4.4 showing reasonable accuracy in dynamic stiffness amplitude and loss angle spectra, respectively. The identified parameters are compared in Table 4.1, where each parameter is normalized by its value with a 0.1 mm amplitude peak-to-peak excitation, such that $\tilde{\mathcal{R}}_{r,j} = \mathcal{R}_{r,j} / \mathcal{R}_{r,1}$, etc. The approximation error of this model will be quantified later.
Figure 4.3. Comparison of quasi-linear models and measured dynamic stiffness magnitudes at various peak to peak excitations: (a) $x_a = 0.1\text{ mm}$, (b) $x_a = 0.5\text{ mm}$, (c) $x_a = 1.0\text{ mm}$, (d) $x_a = 2.0\text{ mm}$. Key: Measured; transfer function coefficient curve-fit; physical parameter curve-fit.
Figure 4.4. Comparison of quasi-linear models and measured loss angles of the production bushing at various peak to peak excitations: 

(a) $x_a = 0.1\, \text{mm}$, (b) $x_a = 0.5\, \text{mm}$, 

(c) $x_a = 1.0\, \text{mm}$, (d) $x_a = 2.0\, \text{mm}$. Key: 

- **Measured:**
- **Transfer function coefficient curve-fit:**
- **Physical parameter curve-fit:**
Table 4.1. Effect of displacement excitation amplitude ($x_a$) on the parameters of quasi-linear model (I). Parameters are normalized by their baseline values (at 0.1mm peak-to-peak excitation amplitude).

<table>
<thead>
<tr>
<th>Amplitude (mm, p-p)</th>
<th>$I_i$</th>
<th>$R_i$</th>
<th>$C$</th>
<th>$A$</th>
<th>$k_r$</th>
<th>$c_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.5</td>
<td>1.00</td>
<td>1.03</td>
<td>0.64</td>
<td>0.41</td>
<td>0.65</td>
<td>0.69</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>1.00</td>
<td>0.50</td>
<td>0.24</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>2.0</td>
<td>1.00</td>
<td>0.90</td>
<td>0.40</td>
<td>0.13</td>
<td>0.34</td>
<td>0.14</td>
</tr>
</tbody>
</table>

These results suggest that many of the model parameters are amplitude sensitive. Quasi-linear modeling is a reasonable method of estimating the amplitude sensitivity, but it requires sinusoidal measurements on a physical device at each amplitude of interest and offers no physical insight.

### 4.4 Development of Nonlinear Models

#### 4.4.1 Nonlinear resistance element (Model II)

Nonlinear models take a different mathematical approach than a quasi-linear model, basing the governing parameters directly on theory (in terms of differential equations) with some augmentation from limited laboratory experiment(s) rather than empirical curve-fitting of dynamic stiffness. The mathematical formulation of a nonlinear element may then implicitly introduce amplitude sensitivity to the model. For instance, in previous work [4.17], a nonlinear fluid resistance element was introduced to the fluid system to produce amplitude dependent damping. This element, which is also used in the current chapter, is based on a turbulent flow formulation [4.22],
\[ R_i(q_i) = \frac{0.242 l \mu^{0.25} \rho^{0.75}}{D_i^{0.75}} |q_i|^{0.75} = \beta_i |q_i|^{0.75}, \]

where \( D_i \) and \( l_i \) are the hydraulic diameter and length of the inertia track respectively, \( \mu \) and \( \rho \) are the dynamic viscosity and density of the pumping fluid respectively, and \( q_i \) is the volume flow rate through the track. Thus, \( \beta_i \) quantifies the effect of fluid properties and flow passage geometry on \( R_i \); typical values (in consistent units) from this study are on the order of \( 10^{12} \). The large value is due to the small diameter of the flow passage.

Some amplitude sensitivity can be achieved here, since the amplitude of \( q_i \) increases with \( x_a \). However, significant error still exists, which will be quantified later and compared with other models. To improve amplitude sensitivity characterization, an additional nonlinear element is needed.

### 4.4.2 Nonlinear compliance element (Model III)

Compliance in the fluid system results from several sources, including the compressibility of the pumping fluid, the much larger compressibility of any small amounts of entrained gas in the fluid, and the flexibility of the chamber walls (denoted by the subscript \( c \)) [4.23]. Assuming there is a very small amount of gas and the fluid is nearly incompressible, the chamber compliance is much higher than the other sources, so the compliance may be simplified as, \( C \approx C_c \). Chamber compliance is related to deformation of the elastomeric chamber volume \( V_c \) under pressure, \( p_c \).

The pumping volume for each chamber (\( L, R \), as shown in Figure 4.1) depends on the displacement of the inner sleeve and the pressure in the chamber, \( V_c = V_c(p_c, x) \).
Some coupling is also present, wherein a non-zero static displacement alters the pressure dependence, and the pressure in the chamber affects the volume vs. displacement relation. These unknown inter-dependencies are simplified by introducing the compliance ($C$) and effective pumping area ($A$) parameters. Each parameter is related to the volume by a linearization of the volume function about an operating point as follows,

$$C(p) = \frac{\partial V}{\partial p} \bigg|_{x=0}$$

$$A = \frac{\partial V}{\partial x} \bigg|_{p=0}$$

where $p$ is the mean pressure in each chamber. In order to experimentally obtain the nonlinear compliance parameter, the relation between volume and pressure must be measured.

In prior work, Kim et al. [4.5] characterized the compliance of hydraulic mounts by filling the pumping chambers with water and attaching a graduated column to the fill port so that the height of the fluid column may be measured. The gas above the fluid column is then pressurized so that both the volume and the pressure in the fluid chambers can be measured. If any gas is left inside the chambers (which is difficult to verify) then the additional compliance of that gas introduces error into the measurement. To resolve this difficulty, Fredette et. al. [4.18] proposed a new experiment to measure the fluid compliance of hydraulic bushing pumping chambers using pressurized air so that only a single fluid is present in the chambers. Finite element models are utilized in other studies [4.7-4.9, 4.17] to computationally determine the fluid compliance. Confidence in the
computational result relies on accurate characterization of material properties, which is
difficult to verify for the elastomeric materials used in production bushings.

Using the current author’s experiment [4.18] with the production bushing yields
the $V_c(p_c)$ relation of the pumping chambers, illustrated in Figure 4.5. A least-squares
curve fit is applied, leading to the following polynomial form, where $\beta_c$ are empirical
parameters,

$$V_c(p_c) = \frac{\beta_{c,3}}{3} p_c^3 + \frac{\beta_{c,2}}{2} p_c^2 + \beta_{c,1} p_c + \beta_{c,0}. \quad (4.14)$$

The compliance of the chambers is calculated from equations (4.12) and (4.14) as,

$$C_c(p_c) = \beta_{c,3} p_c^2 + \beta_{c,2} p_c^2 + \beta_{c,1}. \quad (4.15)$$

Figure 4.5. Measured pressure dependent volume of the pumping chambers on a
production bushing, and least squares cubic curve fit. Key: Measured data from bench
experiment; least squares curve fit.

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Although the above mentioned cubic polynomial is chosen to fit the $V_c(p_c)$ measurements, curve-fitting with other functional forms is certainly possible. However, chamber pressures under dynamic loading (particularly at high excitation amplitudes) exceed the measured pressure range of the compliance characterization experiment [4.18]. Therefore, the compliance curve must be extrapolated beyond the measured regime during simulation. Quadratic, logarithmic, and power functions have been investigated as alternatives, but the cubic model of equation (4.14) yields the greatest accuracy in dynamic stiffness predictions.

The compliance measured by this experiment describes the two chambers of the bushing acting together. The parameters of Figure 4.1 ($C_L$ and $C_R$) describe the compliance of only a single chamber. Since the inertia track is much stiffer than the pumping chambers, the effective fluid compliance parameter is given by $C_L = C_R = 0.5C_c(\sigma_c)$. Here, $\sigma_c$ is an empirical correction factor which accounts for the dynamic stiffness amplification of the elastomeric structural material (under harmonic loading since the compliance is measured under static conditions). The correction factor also resolves any tuning error resulting from inaccurate fluid inertance estimation, given the complex flow passages of the inertia track. The correction factor is calculated by comparing the peak frequency in the dynamic stiffness loss angle spectrum from a 0.1 mm amplitude measurement with the simulated peak frequency using the nonlinear compliance with $\sigma_c = 1$. The peaks are brought together by supplying a new value of $\sigma_c$, which is roughly equivalent to the square of the ratio of simulated to measured peak frequencies. A typical correction factor value from this study is found to be on the order
of 0.1. While this value is relatively low, further improvement requires additional studies focused on the many experimental and physical considerations which affect it; this is beyond the scope of this chapter.

This nonlinear compliance is normalized by the value of the 0.1 mm amplitude quasi-linear compliance parameter and then plotted against this parameter in Figure 4.6. It is clear from this plot that the nonlinear compliance will introduce further amplitude sensitivity into the model, since the magnitude of chamber pressures increases with excitation amplitude, $x_a$.

4.4.3 Nonlinear resistance and compliance elements used concurrently (Model IV)

Inclusion of both the nonlinear R and C elements (from Models II and III, respectively) should incorporate interactions between nonlinearities and thus generate additional amplitude sensitivity. Comparison of these models requires a solution method, discussed in the next section. The model error will be quantified later.

4.5 Numerical Solution of Nonlinear Models (II, III, and IV)

Unlike the frequency-domain representation of dynamic stiffness which assumes an explicit form in the quasi-linear model, a numerical solution is needed for models containing nonlinear elements. The governing ordinary differential equations are solved by assuming a set of initial condition given computational parameters like time steps and tolerances. The time-domain solution is generated using a Runge-Kutta integrator [4.21]. The early part of solution is dominated by the initial conditions, but the response rapidly converges to steady-state given significant damping in the system. The final period of the simulated steady-state response is used to calculate the dynamic stiffness as follows.
Since models II, III, and IV include nonlinear elements, the transmitted force can no longer be assumed to be purely sinusoidal and some harmonic distortion will occur. In this case, a sinusoidal approximation may be curve-fit to the transmitted force, providing the basis for an effective dynamic stiffness [4.24]. A constrained nonlinear optimization with an interior-point algorithm [4.21] is utilized to calculate the force amplitude, $F_a$, and phase shift, $\phi$, which minimize the approximation error $\varepsilon_F$, defined as follows,

$$
\varepsilon_F = \sum_{j=1}^{M} \left( F_{T,j} - F_m - \frac{F_a}{2} \sin \left( \Omega t_j - \phi \right) \right),
$$

(4.16)

where $M$ is the number of data points in one period of the simulated $F_T$ time series excited at frequency, $\Omega$. The mean force component is calculated and subtracted prior to the optimization.

Figure 4.6. Nonlinear compliance $C(p)$ compared with linear compliance $C$ identified from empirical quasi-linear modeling with a 0.1 mm amplitude. Parameters are normalized by the linear compliance. Key: Nonlinear $C(p)$; Linear $C$. 

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Figure 4.7 shows the predictions of model IV, showing similar amplitude dependence trends as the measured stiffness spectra. Figures 4.8 and 4.9 compare the simulated dynamic stiffness magnitudes and loss angles of models II, III, and IV with measured spectra from the production bushing. The agreement between measurement and the nonlinear models is not as close as it is with the quasi-linear model (with amplitude dependent parameters). Observe that neither $\tilde{K}_i$ nor $\tilde{K}_m$ fully capture the amplitude sensitivity of $\tilde{K}_d$ but the interaction between the two nonlinearities in model IV improves the approximation, though significant error still exists, particularly for low excitation amplitudes. Some of the error may be due to excessive extrapolation of the compliance relationship. Measurement of pressures up to about 200 kPa are considered due to experimental limitations [4.18]; however, dynamic pressures up to 800 kPa are found in simulation. Nevertheless, the amplitude dependent accuracy of this model (despite the extrapolation) indicates that the proposed model (IV) is satisfactory since it captures much of the underlying physics.

4.6 Construction of Solution using the Harmonic Balance Method

Unlike the numerical integration method, more insight may be gained by using the multi-term harmonic balance method [4.17, 4.21, 4.22] to generate a semi-analytical solution for model IV. To begin, time ($t$) is scaled by the excitation frequency, $\theta = \Omega t$, where $\theta$ is the scaled time and $\Omega$ is the excitation frequency in rad/s. The time derivative is related to the scaled time derivative, $\dot{\theta} = \Omega \dot{t}$, where the dot notation refers to derivative with respect to time, and the prime notation refers to derivation with respect to scaled time. The state variables are also scaled as,
\[ p_L = p_b \hat{p}_L, \]
\[ p_R = p_b \hat{p}_R, \]
\[ q_i = q_b \hat{q}_i. \]  

Here, the subscript \( b \) denotes a baseline or reference value, and the over-hat symbol denotes normalization. The system state vector \( z \) is normalized, where \( \dot{S}_z \) is the magnitude scaling matrix,

\[
\dot{z} = \dot{S}_z \hat{z} = \begin{bmatrix} p_b & 0 & 0 \\ 0 & p_b & 0 \\ 0 & 0 & q_b \end{bmatrix} \begin{bmatrix} \dot{p}_L \\ \dot{p}_R \\ \dot{q}_i \end{bmatrix}. \]  

Figure 4.7. Predicted stiffness spectra of Model IV. Key: \( x_a = 0.1 \) mm; \( x_a = 0.5 \) mm; \( x_a = 1.0 \) mm; \( x_a = 2.0 \) mm.
Figure 4.8. Comparison between amplitude dependent measured and predicted dynamic stiffness magnitudes: (a) $x_a = 0.1$ mm, (b) $x_a = 0.5$ mm, (c) $x_a = 1.0$ mm, (d) $x_a = 2.0$ mm.

Key: \[ \hat{\mathbf{K}}_1; \quad \hat{\mathbf{K}}_2; \quad \hat{\mathbf{K}}_3; \quad \hat{\mathbf{K}}_4. \]
Figure 4.9. Comparison between measured and predicted loss angles at various excitation amplitudes: (a) $x_a = 0.1 \text{ mm}$, (b) $x_a = 0.5 \text{ mm}$, (c) $x_a = 1.0 \text{ mm}$, (d) $x_a = 2.0 \text{ mm}$. Key:

- $\phi_{II}$;
- $\phi_{III}$;
- $\phi_{IV}$;
- $\phi_1$. 

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When a linear compliance parameter along with a nonlinear resistance term is used, the system equations are effectively reduced into the following single governing equation as utilized previously by Chai et al. [4.17]:

$$I_i \ddot{q}_i + \mathcal{R}_i(q_i) \dot{q}_i + \left( \frac{1}{C_L} + \frac{1}{C_R} \right) q_i = \left( \frac{A_L}{C_L} + \frac{A_R}{C_R} \right) \dot{x}$$  \hspace{1cm} (4.19)

where $\mathcal{R}_i(q_i)$ is a nonlinear parameter. A semi-analytical solution approximation was then generated for $q_i(t)$ by Chai, et al. [4.17] based on equation (4.19). Conversely, when the nonlinear compliance and nonlinear resistance elements are concurrently employed, the above mentioned reduction to a single equation is no longer possible. The MHBM in procedure of prior work [4.17, 4.21] must therefore be modified to work with a system of equations containing nonlinear compliance parameters. The system equations are converted from the form of equation (4.4) into residual ($\gamma$) form,

$$\gamma = \begin{cases} C(p_L) \dot{p}_L' + q_i - A_L \dot{x}' \\ C(p_R) \dot{p}_R' - q_i + A_R \dot{x}' \\ I_i \dot{q}_i' + \mathcal{R}_i(q_i) q_i + p_R - p_L \end{cases},$$  \hspace{1cm} (4.20)

where $C(p)$ is the experimentally determined compliance function in equation (4.15), $\mathcal{R}_i(q_i)$ is the theoretical fluid resistance in the inertia track based on a turbulent flow formulation. Additionally, $\gamma$ is normalized using the residual scaling matrix,

$$\gamma = \tilde{\gamma} = \begin{bmatrix} q_b & 0 & 0 \\ 0 & q_b & 0 \\ 0 & 0 & p_b \end{bmatrix} \dot{\gamma}.$$  \hspace{1cm} (4.21)

This yields the normalized residual function $\gamma(\tilde{\gamma})$ in terms of normalized state variables,
Application of the harmonic balance method in time-domain is possible, but computationally expensive and inaccurate. This analysis functions better in the frequency domain, for which a Discrete Fourier Transform (DFT) is required [4.17]. In this case, it is easier to define the inverse DFT matrix,

$$\mathcal{F}^{-1} = \begin{bmatrix} 1 & \sin(\theta) & \cos(\theta) & \sin(2\theta) & \cos(2\theta) & \cdots \end{bmatrix}, \quad (4.23)$$

where $\theta$ is the normalized time vector. A derivative in time domain corresponds to multiplication by the derivative matrix,

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -2 & \\ 0 & 0 & 0 & 2 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \quad (4.24)$$

Using these operator matrices, the relationship between state variables in time domain, $z_j$, and in frequency domain, $a_j$, is defined,

$$\hat{z}_j = \mathcal{F}^{-1} a_j, \quad (4.25)$$

where $a$ is the vector of Fourier coefficients. The derivative of the state vector is also related,

$$\hat{z}'_j = \frac{d}{d\theta} \mathcal{F}^{-1} \frac{d}{d\theta} a_j. \quad (4.26)$$
These equations allow the transformation of the residual into Fourier domain, \( \Gamma_j = \mathcal{F} \hat{\phi}_j \), where the optimization algorithm may now be applied. Further, Newton’s method is selected as the algorithm to minimize the residual. This requires the calculation of the Jacobian \( (\mathcal{J}) \) of the residual in Fourier domain,

\[
\mathcal{J}_{j,k} = \frac{\partial \Gamma_j}{\partial a_k} = \mathcal{F} \left[ \frac{\partial \hat{\phi}_j}{\partial \hat{z}_k} \frac{\partial \hat{z}_k}{\partial a_k} + \frac{\partial \hat{\phi}_j}{\partial \hat{z}'_k} \frac{\partial \hat{z}'_k}{\partial a_k} \right].
\]

(4.27)

Differentiating equations (4.25) and (4.26) and substituting into equation (4.27) yields,

\[
\mathcal{J}_{j,k} = \mathcal{F} \left[ \frac{\partial \hat{\phi}_j}{\partial \hat{z}_k} \mathcal{F}^{-1} + \frac{\partial \hat{\phi}_j}{\partial \hat{z}'_k} \mathcal{F}^{-1} \mathcal{D} \right],
\]

(4.28)

leaving the partial derivatives of the residual to be calculated in terms of submatrices,

\[
\begin{bmatrix}
\frac{\Omega p_b}{q_b} \left[ \frac{\partial C(p_L)}{\partial p_L} \hat{p}'_L \right] & 0 & \frac{1}{\mathcal{D}_{\text{diag}}} \\
0 & \frac{\Omega p_b}{q_b} \left[ \frac{\partial C(p_R)}{\partial p_R} \hat{p}'_R \right] & -\frac{1}{\mathcal{D}_{\text{diag}}} \\
\frac{1}{\mathcal{D}_{\text{diag}}} & -\frac{1}{\mathcal{D}_{\text{diag}}} & \frac{q_b}{p_b} \left[ \frac{\partial R_i(\hat{q}_i)}{\partial \hat{q}_i} \hat{q}_i \right]_\text{diag} + \mathcal{R}_i(\hat{q}_i) \\
\end{bmatrix}
\]

(4.29)

\[
\begin{bmatrix}
\frac{\Omega p_b}{q_b} \left[ C(\hat{p}_L) \right]_\text{diag} & 0 & 0 \\
0 & \frac{\Omega p_b}{q_b} \left[ C(\hat{p}_R) \right]_\text{diag} & 0 \\
0 & 0 & \frac{I_r q_b}{p_b} \frac{1}{\mathcal{D}_{\text{diag}}} \\
\end{bmatrix}
\]

(4.30)
where the $diag$ subscript indicates a diagonal matrix, in this case of dimension $M$, where $M$ is the number of points in one period of $\theta$ and where the state variables are evaluated at each sampled point in normalized time.

An initial guess of Fourier coefficients for each state variable is needed to run the algorithm. Iteratively choosing small, normally distributed random numbers for each yields different results each time. Since pressures inside the pumping chambers of production bushings cannot be easily measured, Figure 4.10 shows a representative comparison between the semi-analytical approximation (using a single harmonic term) with the numerical solution of the nonlinear model. Significant error exists in the mean component of each pressure term, which can be traced to a singular Jacobian matrix which has a complex-valued eigenvalue matrix $\tilde{\Lambda}_{diag}$, with distinct complex-conjugate pairs of eigenvalues, $\tilde{\lambda}_j$,

$$
\tilde{\Lambda}_{diag} = diag\left[\tilde{\lambda}_1, \tilde{\lambda}_1^*, \tilde{\lambda}_2, \tilde{\lambda}_2^*, 0, \tilde{\lambda}_3, \tilde{\lambda}_3^*\right].
$$

The zero eigenvalue acts as a blind spot in the algorithm. The mean components of both $p_L$ and $p_R$ have significant modal participation in the corresponding eigenvector, suggesting that the optimization is not sensitive to these terms. The dependence of the nonlinear compliance on chamber pressure indicates that the system is sensitive to mean pressure. An initial mean chamber pressure must be supplied in the numerical simulation; likewise, the initial guess of mean pressure is an external constraint rather than part of the dynamic system response. Alternative guesses of the initial condition or average pressure from simulation yield improved results, as illustrated in Figure 4.11.
Substantial inaccuracy is still present in the MHBM approximation since insufficient harmonics are included to match the non-sinusoidal pressure trace. An increase in the number of harmonics improves the accuracy, as shown in Figure 4.12. Three harmonics produce excellent agreement with the numerically integrated pressures.

The presence of super-harmonics illustrates some of the limitations of conventional dynamic stiffness characterization methods widely used for hydraulic bushings. In sinusoidal testing procedures [4.24], harmonic distortion (caused by super or

![Figure 4.10. Representative comparison between nonlinear simulation results and solution approximation generated with single-term harmonic balance method (HBM) and normally distributed random values for the initial guess of Fourier coefficients. Very poor accuracy is achieved. Key: Simulated $p_L$; Simulated $p_R$; HBM $p_L$; HBM $p_R$.](image)

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subharmonics) is generally eliminated by frequency-domain filtering or sine regression techniques, reducing the dynamic behavior to a seemingly linear response at a given amplitude. Analyzing the superharmonic content of measured transmitted force signals (if

![Graph](image)

Figure 4.11. Representative comparison between nonlinear simulation results and solution approximation generated with the single-term harmonic balance method (HBM).

(a) Initial guess for mean pressure taken from initial condition in numerical simulation.

(b) Initial guess for mean pressure taken by averaging simulated pressure traces.
Marginal accuracy is achieved. Key: \( \text{Simulated } p_L \); \( \text{Simulated } p_R \); \( \text{HBM} \).

\[ p_L; \text{ HBM } p_R. \]
available) may improve experimental insight, particularly when compared to the signatures produced by reduced order models and the MHBM.

Finally, note that the MHBM is able to produce a solution approximation, yielding frequency responses and harmonic decomposition even in unstable and multi-valued regimes. This requires much less computational time when compared to “brute

Figure 4.12. Representative comparison between simulation results and solution approximation generated with the multi-term harmonic balance method using multiple harmonics. (a) Two harmonics. Improved accuracy is achieved. (b) Three harmonics. Excellent accuracy is achieved. Key: Simulated $p_L$; Simulated $p_R$; MHBM $p_L$; MHBM $p_R$. 

$p_L$; $p_R$. 

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force” numerical methods. While unstable and multi-valued regimes do not seem to be present in this study, another device with different parameters may demonstrate such behavior. Therefore, the semi-analytical technique adds robustness to the proposed methods as well as providing additional insight.

4.7 Comparative Evaluation of 4 Models

Four hydraulic bushing models are considered to address the amplitude sensitivity of $K(\Omega)$ in both magnitude and loss angle spectra: quasi-linear model (I) and nonlinear models, specifically models containing only a nonlinear resistance parameter (II), only a nonlinear compliance parameter (III), and a model containing both nonlinear resistance and compliance elements (IV). The error in each model $\epsilon_k$ is quantified, where $M_{\Omega}$ is the number of frequencies considered,

$$
\epsilon_{K_k} = \frac{1}{K_r M_{\Omega}} \sqrt{\frac{M_{\Omega}}{\sum_{j=1}^{M_{\Omega}} \left\| \vec{K}_d (\Omega_j) - \vec{K}_d (\Omega_j) \right\|^2},}
$$

(4.32)

taking model II as an example. The error is normalized by the static (rubber path) and the accuracy of each model is compared in terms of this error in Table 4.2.

The quasi-linear bushing model produces reasonable results capturing amplitude sensitivity, but they lack the ability to predict the trends in amplitude sensitivity. Nonlinear models remove the need for dynamic stiffness measurements at many amplitudes. The nonlinear resistance model (II) presents significantly high error, heavily weighted towards the lower excitation amplitudes. Model III (with nonlinear compliance) yields somewhat reduced error, particularly at lower amplitudes because the linear resistance parameter was taken from Model I with $x_a = 0.1$ mm. The interaction between
the two nonlinear elements in model IV generally reduces the error across all amplitudes, suggesting that the inclusion of nonlinear compliance is necessary but not sufficient to capture the physics of this device. Further extension to characterize a nonlinear effective pumping area parameter, \( A(x, p) \), would be a logical step since the excitation to the fluid system is proportional to \( A \).

Use of the multi-term harmonic balance method offers some insight into the physics of the system, since each of the harmonic terms may be examined. The method produces coefficients for sine and cosine terms at each harmonic, which are easily converted to sinusoidal magnitudes and phases. Figure 4.13 shows the amplitudes of the first harmonic pressure term at each excitation amplitude. The shape of these pressure spectra in some ways resembles the dynamic stiffness plots in Figure 4.7 because the fluid force path contribution becomes most significant especially when the fluid system nears its effective resonance. Although this is a nonlinear system, the corresponding linearized fluid system has a single-valued resonance, so the response amplitudes increase significantly near that frequency.

The amplitude sensitivity of superharmonic pressure magnitudes in model IV is depicted in Figure 4.14, while the phases are compared in Figure 4.15. Two peaks are visible in the magnitude of the third harmonic at 0.1 mm, but the second peak begins to flatten with increasing amplitude while the first peak seems unaffected. The second peak occurs at \( \hat{\Omega} = 1.25 \), which corresponds to the peak in the first harmonic and represents the resonant frequency of the linearized fluid system. The first peak occurs at \( \hat{\Omega} = 0.44 \),
which is the closest point to \( \hat{\Omega}/3 \). To sort out which nonlinearity is causing these effects, the harmonic terms in models II and III are compared.

Table 4.2. Quantification of error, \( \varepsilon_k \), associated with quasi-linear (I) and nonlinear (II, III, and IV) models.

<table>
<thead>
<tr>
<th>Model</th>
<th>Description</th>
<th>Amplitude (mm, p-p)</th>
<th>0.1</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
<th>Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Quasi-Linear</td>
<td></td>
<td>0.203</td>
<td>0.183</td>
<td>0.186</td>
<td>0.204</td>
<td>0.194</td>
</tr>
<tr>
<td>II</td>
<td>Only ( R(q) ) Nonlinearity</td>
<td></td>
<td>0.672</td>
<td>0.476</td>
<td>0.373</td>
<td>0.246</td>
<td>0.442</td>
</tr>
<tr>
<td>III</td>
<td>Only ( C(p) ) Nonlinearity</td>
<td></td>
<td>0.343</td>
<td>0.361</td>
<td>0.356</td>
<td>0.290</td>
<td>0.338</td>
</tr>
<tr>
<td>IV</td>
<td>Both ( R(q) ) and ( C(p) ) Nonlinearity</td>
<td></td>
<td>0.467</td>
<td>0.281</td>
<td>0.191</td>
<td>0.128</td>
<td>0.267</td>
</tr>
</tbody>
</table>

Figure 4.13. Amplitude of the fundamental (first) pressure harmonic of Model IV. Key:

- \( x_a = 0.1 \) mm;
- \( x_a = 0.5 \) mm;
- \( x_a = 1.0 \) mm;
- \( x_a = 2.0 \) mm.

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Figure 4.14. Existence and relative magnitudes of super harmonics as illustrated by harmonic pressure terms in the nonlinear dynamic responses. Sine and cosine coefficients are calculated by the multi-term harmonic balance method, converted to magnitude form, and normalized by excitation amplitude. (a) $x_a = 0.1$ mm, (b) $x_a = 0.5$ mm, (c) $x_a = 1.0$ mm, and (d) $x_a = 2.0$ mm. Key: — First harmonic; —— second harmonic; ——- third harmonic.

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Figure 4.15. Existence and relative phases of super harmonics as illustrated by harmonic pressure terms in the nonlinear dynamic responses. Sine and cosine coefficients are calculated by the multi-term harmonic balance method and converted to phase form at several excitation amplitude, (a) $x_a = 0.1$ mm, (b) $x_a = 0.5$ mm, (c) $x_a = 1.0$ mm, and (d) $x_a = 2.0$ mm. Key:  
- First harmonic;  
- second harmonic;  
- third harmonic.
The first three pressure harmonics for models II, III, and IV at 0.1 mm excitation amplitude are compared in Figure 4.16. The magnitude of the second pressure harmonic in Model II is several orders of magnitude smaller than the first and third harmonics, so effectively, only a third harmonic exists. At first glance, it appears that the superharmonic terms in model IV are simply a superposition of the superharmonics from models II and III, though the superposition principle should not apply to nonlinear systems. Instead, the observed effect represents an interaction between the $R_i(q_i)$ and $C_p$ nonlinearities in the system. This result highlights the insufficiency of Model II, which only produces a third harmonic. This is most likely because the exponent on the state variable $q_i$ is 1.75, which is very close to $\sqrt{3} \approx 1.73$, producing a third harmonic in the response. Also, neither model II nor III captures the regime change near $\hat{\Omega} = 0.6$ where a shift in the dominant harmonic suggests that a different physical effect controls the system response. Below this frequency, the third harmonic is much higher than the second, indicating that $R_i(q_i)$ is the dominant nonlinearity. Beyond this frequency, the second harmonic is larger, suggesting that $C(p)$ becomes the dominant effect, somewhat analogous to the mass or stiffness controlled regimes of a simple mechanical oscillator. The absence of a peak in the second harmonic near $\hat{\Omega} = 0.44$ is explained because $C(p)$ contains linear, quadratic, and cubic terms, which prevents significant resonant excitation at exactly one half or one third of the fluid system’s linearized resonant frequency.
Figure 4.16. Relative magnitudes of super harmonics as illustrated by harmonic pressure terms in the nonlinear dynamic responses. Sine and cosine coefficients are calculated by the multi-term harmonic balance method, converted to magnitude form, and normalized by excitation amplitude. (a) Model II, (b) Model III, and (c) Model IV. Key: — First harmonic; — second harmonic; —— third harmonic.
4.8 Conclusion

This chapter compares two modeling approaches to predict and explain the amplitude dependent dynamic stiffness spectra of a hydraulic bushing. Unlike the prior work [4.17] on bushing models that focused on a laboratory device and used only a nonlinear resistance parameter, this chapter describes a production bushing, which makes the parameter estimation a more difficult process. A quasi-linear model is first proposed which is based on a curve-fit of measured dynamic stiffness using an assumed transfer function form of the governing system. The effect of amplitude on each physical parameter is observed, but the trends offer minimal physical insight. Since dynamic stiffness measurements are inherently required at each amplitude of interest, the utility of the quasi-linear approach is limited, especially when a new device is to be designed. Consequently, alternate nonlinear models must be considered and implemented for up-front design and diagnostics work to meet increasingly stringent vehicle requirements. Nonlinear models better capture the trends in amplitude sensitivity, provide physical insight, and eliminate the need for dynamic testing at all amplitudes since the model’s amplitude sensitivity is not based on empirical stiffness data. Nonlinear compliance elements are used, which interacts with the nonlinear resistance element to improve the amplitude sensitivity predictions. The MHBM procedure of Chai, et al. [4.17] is also refined with improved generating solution approximations for the new system equations which contain both nonlinear compliance and resistance elements.

In summary, this chapter extends the literature [4.15-4.18], and the specific contributions include (1) new and experimentally validated quasi-linear and nonlinear models which improve amplitude sensitivity in dynamic stiffness predictions, and (2) a
refined semi-analytical method which captures and offers physical insight into multiple interacting nonlinearities in a system of three equations. The results suggest that nonlinear models of resistance and compliance are necessary. While these two nonlinear elements yield reasonable dynamic stiffness predictions, further work developing a nonlinear pumping area expression (as given by equation (4.13)) should improve characterization of dynamic stiffness amplitude sensitivity.
Chapter 4 References


### Chapter 4 Nomenclature

- **A**: effective pumping area (m²)
- **C**: fluid compliance (m⁵/N)
- **Д**: Fourier domain derivative
- **D**: diameter (m)
- **F**: force (N)
- **F**: discrete Fourier transform
- **I**: fluid inertance (N·s²/m³)
- **K**: stiffness (N/m)
- **p**: pressure (Pa)
- **l**: length of flow passage (m)
- **M**: ensemble size
- **m**: mean component (dc value)
- **N**: number of harmonics considered
- **q**: volume flow rate (m³/s)
- **R**: fluid resistance (N·s/m⁵)
- **S**: scaling
- **t**: time (s)
- **V**: volume (m³)
- **x**: displacement (m)
- **z**: state variable
- **β**: nonlinear model parameter
\( \theta \) normalized time

\( \phi \) loss angle (rad)

\( \varepsilon \) approximation error

\( \Gamma \) residual in Fourier domain

\( \gamma \) residual in time domain

\( \Lambda \) eigenvalue matrix

\( \lambda \) eigenvalue

\( \mu \) dynamic viscosity (N\cdot s/ m\(^2\))

\( \rho \) fluid density (kg/m\(^3\))

\( \Omega \) excitation frequency (rad/s)

**Subscripts**

\( a \) alternating component

\( b \) baseline, or reference value

\( c \) compliance chamber

\( d \) measured data

\( f \) fluid system

\( i \) inertia track

\( j \) index

\( k \) index

\( L \) left chamber

\( R \) right chamber

\( r \) rubber path
$T$ transmitted

$\alpha$ transfer function coefficient

$\_\_\_\_\_\_$ vector

$=\_\_\_\_\_\_$ matrix

**Superscripts**

$\sim$ complex valued

$\wedge$ normalized

$\ast$ complex conjugate

**Abbreviations**

DFT discrete Fourier transform

HBM single-term harmonic balance method

LTI linear-time-invariant

MHBM multi-term harmonic balance method
Chapter 5. Nonlinear Analysis of Hydraulic Bushings with Fractionally Damped Pumping Chambers

5.1. Introduction

In a recent paper, Fredette, et al. [5.1] developed nonlinear hydraulic bushing models with multiple nonlinear elements. That article offered new insight into the physics of these devices, capturing additional physical effects and developing nonlinear a compliance model for elastomeric chambers. However, the results of [5.1] left room for further improvement since hydraulic bushings often have complex designs with interacting features. In particular, no damping formulation was given for the elastomeric chambers (containing the hydraulic fluid), although their material exhibits significant viscoelastic behavior. Several other authors have studied hydraulic bushings, but the majority of the literature uses linear system principles [5.2-5.5], despite significant amplitude sensitivity observed in such devices [5.1,5.6-5.8]. A few researchers have proposed amplitude sensitive models, using either a quasi-linear or nonlinear approach. For instance, Svensson and Håkansson [5.6] developed a hydraulic bushing model which included a nonlinear elastic element for the rubber path, but used the linear system principles for the fluid path. Chai, et al. [5.7] proposed a model with a nonlinear fluid resistance term, which introduces significant amplitude sensitivity. An experimental study by Fredette, et al. [5.8] identified amplitude sensitive compliance behavior in the
fluid system, which led to models with both nonlinear fluid resistance and chamber compliance elements [5.1,5.8].

The major goal of this chapter is to extend the prior formulation [5.1] by introducing fractional calculus based viscoelasticity and additional nonlinear elements to enhance modeling capabilities and to better understand the underlying physics of hydraulic bushings. This approach is based, in part, on the work of Sjoberg and Kari [5.11], who combined nonlinear elasticity, fractional viscoelasticity, and smoothened dry friction damping to mimic the dynamic behavior of a carbon black filled rubber. In this chapter, the focus will be placed on the damping effects in the pumping chambers. The literature on this topic is limited even though the fractional calculus based constitutive laws have been shown to represent the viscoelasticity of many elastomeric materials in a compact, accurate, and physically meaningful way [5.9-5.12]. Such viscoelastic behavior would be present in many types of elastomeric isolators and hydraulic bushings.

5.2. Problem Formulation

The scope of this chapter is on a class of production grade hydraulic bushings, schematically described via a baseline model in Figure 5.1 (equivalent to [5.1]). Steady-state sinusoidal excitation and transmitted force response are used to estimate the dynamic stiffness $\tilde{K}$ in the example case of the production bushing used in [5.1]. The design features of the example case are modeled with a lumped parameter framework. Figure 5.1 shows a schematic of the component split into two parallel load paths, where $x(t) = (x_a/2)\sin(\Omega t)$ is the inner sleeve’s displacement excitation, with peak-to-peak amplitude $x_a$ and frequency $\Omega$ with units of rad/s. The forces transmitted to the outer
sleeve through the rubber and fluid paths are denoted $F_r$ and $F_f$, respectively. Here, $p_1$ and $p_2$ represent the dynamic pressures in each pumping chamber, while $q_i$ denotes the volume flow rate in the inertial track. The fluid resistance and inertance of the inertia track are given by $R_i$ and $I_i$, and $C_f$ is the fluid compliance of the pumping fluid. The effective pumping area of the inner sleeve is given by $A_s$. Finally, the rubber path stiffness is denoted $k_r$, with rubber path damping force defined in a function form as $g_r(x, \dot{x})$. Typically, viscous damping would be employed, implying that $g_r(x, \dot{x}) = \eta \dot{x}$, where $\eta$ is the viscous damping coefficient.

The governing equations of the fluid system are given by the continuity equations for each pumping chamber,

\begin{align*}
C_f \dot{p}_1 &= A_s \dot{x} - A_s \dot{y}_1 - q_i, \quad (5.1-a) \\
C_f \dot{p}_2 &= -A_s \dot{x} - A_s \dot{y}_2 + q_i, \quad (5.1-b) \\
I_i \dot{q}_i &= p_1 - p_2 - R_i q_i. \quad (5.1-c)
\end{align*}

The transmitted force through each path are calculated as,

\begin{align*}
F_f &= A_s (p_1 - p_2), \quad (5.2) \\
F_r &= k_r x + g_r(x, \dot{x}), \quad (5.3)
\end{align*}

and the total transmitted force is, $F_T = F_f + F_r$.

The focus of this chapter is on the damping effects resulting from the deformation of the pumping chamber walls. The particular objectives of this chapter are to (1) develop
quasi-linear models which capture pumping chamber damping effects leading to
amplitude-dependent dynamic stiffness behavior of a production bushing, (2) Improve
the nonlinear models of prior paper [5.1] by including fractional calculus and dry-friction
type damping in the pumping chambers, and (3) validate the predictions of the new
models against dynamic stiffness measurements. Table 5.1 describes each model used in
this chapter. The scope of this chapter is limited to component-level, uniaxial, steady-
state analysis of a single example case up to a 50 Hz bandwidth with peak-to-peak
displacement amplitudes of 0.1, 0.5, 1.0, and 2.0 mm.

5.3. Fractional Damping Formulation

Fractional dampers are governed by fractional calculus constitutive laws. In
particular, the damping force in each fractional damper is given by
\[ F(t) = \eta D_t^\alpha z(t), \]
where \( z(t) \) is the displacement of the element, \( \alpha \) is the fractional order, and \( D_t^\alpha \) is a
differential operator defined,
\[ D_t^\alpha (\cdot) = \frac{d^\alpha}{dt^\alpha} (\cdot). \] (5.4)

It should be noted that viscous damping is a special case of fractional damping where
\( \alpha = 1 \), and structural damping is the case when \( \alpha = 0 \) and the coefficient \( \eta \) is purely imaginary. In general, the Caputo definition of a fractional derivative is the Cauchy
integral [5.9,5.10],
\[ D_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dz}{dt} d\tau, \] (5.5)
Figure 5.1. Schematic of the prior model of hydraulic bushing where the (a) fluid path and (b) rubber path subsystems correspond to the analysis of prior work [5.1]. Here, \( x(t) \) is the inner sleeve’s displacement excitation while \( F_r \) and \( F_f \) are the forces transmitted to the outer sleeve through the rubber and fluid paths, respectively. The pressures in each pumping chamber are denoted by \( p_1 \) and \( p_2 \), and \( q_i \) is the volume flow rate in the inertial track. The fluid resistance and inertance of the inertia track are given by \( R_i \) and \( I_i \), and \( C_f \) is the fluid compliance of the pumping fluid (independent of its container). \( A_s \) is the effective pumping area of the inner sleeve. The rubber path stiffness is denoted \( k_r \), with damping force \( g_r(x, \dot{x}) \).
Table 5.1. Description of quasi-linear (QL) and nonlinear (NL) models used in this chapter and comparison with prior paper [5.1]. See Figure 5.2 for symbols.

<table>
<thead>
<tr>
<th>Model</th>
<th>Type</th>
<th>From [5.1] (Fig. 1)</th>
<th>New (Fig. 2)</th>
<th>Description</th>
<th>$g_r(x,\dot{x})$</th>
<th>$g_c(y,\dot{y})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>QL</td>
<td>×</td>
<td></td>
<td>Undamped pumping chambers</td>
<td>$\eta_1\dot{x}$</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>QL</td>
<td>×</td>
<td></td>
<td>Fractionally damped pumping chamber</td>
<td>$\eta_1 D_t^\alpha(x)$</td>
<td>$\eta_1 D_t^\alpha(y)$</td>
</tr>
<tr>
<td>III</td>
<td>NL</td>
<td>×</td>
<td></td>
<td>Nonlinear $R_i(q)$ and $C(p)$ elements and undamped pumping chambers</td>
<td>$\eta_1 D_t^\alpha(x)$</td>
<td>0</td>
</tr>
<tr>
<td>IV-0</td>
<td>NL</td>
<td>×</td>
<td></td>
<td>Undamped, equivalent mechanical spring in place of $C(p)$</td>
<td>$\eta_1 D_t^\alpha(x)$</td>
<td>0</td>
</tr>
<tr>
<td>IV-1</td>
<td>NL</td>
<td>×</td>
<td></td>
<td>Fractional and friction-type damping elements for pumping chamber with constant $A_x$</td>
<td>$\eta_1 D_t^\alpha(x)$</td>
<td>$\eta_1 D_t^\alpha(y)$ + $\mu_c \tanh(\sigma_c \dot{y})$</td>
</tr>
<tr>
<td>IV-2</td>
<td>NL</td>
<td>×</td>
<td></td>
<td>Fractional and friction chamber damping elements with QL $A_x$ parameter</td>
<td>$\eta_1 D_t^\alpha(x)$</td>
<td>$\eta_1 D_t^\alpha(y)$ + $\mu_c \tanh(\sigma_c \dot{y})$</td>
</tr>
</tbody>
</table>

which has the convenient Laplace transform, $\mathcal{L}\{D_t^\alpha z(t)\} = s^\alpha Z(s)$. This facilitates the frequency-domain quasi-linear models developed in section 5.4. For numerical integration, however, the Caputo definition is prohibitively cumbersome. Therefore the numerical, time-domain analysis of section 5.5 uses the Grünwald-Letnikov (GL) definition [5.9,5.10],
\[ D_t^\alpha z(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\infty} \left( \frac{\alpha}{j} \right) (-1)^j z(t-jh), \]  

(5.6)

where \( h \) is the length of one time-step. This formulation is easily converted to a difference equation,

\[ D_t^\alpha z(t_0) \approx h^{-\alpha} \sum_{j=0}^{N_j} \left( \frac{\alpha}{j} \right) (-1)^j \bar{z}[N_j - j], \]  

(5.7)

where square brackets denote indexing a discrete vector, the overbar denotes a history variable such that \( \bar{z} \) is a vector of the history of \( z(t) \), sampled at a period of \( h \), such that \( \bar{z}[0] = z(t_0 - N_j h) \) and \( \bar{z}[N_j] = z(t_0) \). Equation (5.7) accurately approximates the fractional derivative of the signal at time \( t_0 \), given a sufficiently large number of history points, \( N_j \), and a sufficiently small \( h \).

5.4. Development of the Quasi-linear Model (II)

A more detailed system description is required in order to capture the chamber damping effects and overcome the deficiencies of prior work [5.1]. The new configuration is proposed in Figure 5.2, where all previously defined symbols retain their meaning. Additionally, \( A_c \) is the effective piston area of the pumping chamber, and the spring coefficient \( k_c \) and effective mass \( m_c \) represent the elastic and inertial properties of the compliance chamber walls. The fractional damping coefficient and order of the pumping chamber are given by \( \eta_c, \alpha_c \), respectively. The friction element is characterized by a coefficient \( (\mu_c) \) and smoothing parameter \( (\sigma_c) \); it will be discussed in more detail.
later since it is a nonlinear element and cannot therefore be captured in the quasi-linear model.

Figure 5.2. Schematic of new quasi-linear and nonlinear models of hydraulic bushing with parallel fluid path (a) and rubber path (b) subsystems. Here, $A_c$ is the effective pumping area of the pumping chambers. The spring coefficient $k_c$ and effective mass $m_c$ represent the elastic and inertial properties of the pumping chamber walls. The fractional damping coefficient and order of the pumping chamber are given by $\eta_c, \alpha_c$, respectively. The friction coefficient is given by $\mu_c$ with smoothing parameter $\sigma_c$. The rubber path stiffness is denoted $k_r$, with fractional rubber damping $g_r(x, \dot{x}) = \eta_r D_t^{\alpha_r}(x)$. 
The general governing equations of the new, more general fluid system of Figure 5.2 are given by

\[ C_f \dot{p}_1 = A_c \dot{x} - A_c \dot{y}_1 - q_i, \]  
(5.8-a)

\[ C_f \dot{p}_2 = -A_c \dot{x} - A_c \dot{y}_2 + q_i, \]  
(5.8-b)

\[ m_c \ddot{y}_1 = A_c p_1 - k_c y_1 - g_c (y_1, \dot{y}_1), \]  
(5.8-c)

\[ m_c \ddot{y}_2 = A_c p_2 - k_c y_2 - g_c (y_2, \dot{y}_2), \]  
(5.8-e)

\[ I \dot{q}_i = p_1 - p_2 - R_i q_i. \]  
(5.8-f)

The chamber and rubber path damping forces in model II (corresponding to Figure 5.2) are defined using the fractional derivative,

\[ g_c (y, \dot{y}) = \eta_c D^\alpha_c y, \]  
(5.9)

\[ g_r (x, \dot{x}) = \eta_r D^\alpha_r (x). \]  
(5.10)

Certain simplifications enhance the tractability of quasi-linear and nonlinear analyses of these equations. For instance, has been previously demonstrated that the compliance of the fluid is much smaller than the compliance associated with the pumping chamber [5.1,5.8], suggesting that \( C_f \approx 0 \) would be a reasonable assumption. Finally, since the frequency range of interest (up to 50 Hz) is relatively low, the effects of the effective chamber wall mass are negligible, \( m_c \approx 0 \). The numerical implications of this are discussed in section 5.5.

The quasi-linear analysis is based on the linear-time-invariant system theory. Using linearized (constant) model parameters, the dynamic stiffness of model I is defined in the Laplace domain,
\[
\tilde{K}_I(s) = \frac{\tilde{F}_c(s)}{X(s)} = \frac{\tilde{F}_r(s)}{X(s)} + \frac{\tilde{F}_r(s)}{X(s)},
\]
(5.11)

where \(X(s) = \mathcal{L}\{x(t)\}\). The \(\tilde{K}(s)\) transfer function is derived,

\[
\tilde{K}_I(s) = \frac{B_1 s^{2+\alpha_c} + B_3 s^{2+\alpha_c} + B_5 s^2 + B_7 s^{1+\alpha_c} + B_9 s^{1+\alpha_c}}{s^2 + B_{13} s + B_{12} s^{\alpha_c} + B_{11}},
\]
(5.12-a)

where

\[
B_1 = \frac{2k_r k_c}{I_i A_c^2}, \quad B_2 = \frac{2\eta_c k_r}{I_c A_c^2}, \quad B_3 = \frac{2k_r \eta_r}{I_i A_c^2},
\]
(5.12-b,c,d)

\[
B_4 = \frac{2\eta_c \eta_r}{I_c A_c^2}, \quad B_5 = \frac{R_i}{I_i} \left( k_r + \frac{2A_r^2 k_c}{A_c^2} \right), \quad B_6 = \frac{2R_i A_r^2 \eta_c}{I_i A_c^2},
\]
(5.12-e,f,g)

\[
B_7 = \frac{R_i \eta_r}{I_i}, \quad B_8 = \left( k_r + \frac{2A_r^2 k_c}{A_c^2} \right), \quad B_9 = \frac{2\eta_c A_r^2}{A_c^2},
\]
(5.12-h,i,j)

\[
B_{10} = \eta_r, \quad B_{11} = \frac{2k_r}{I_i A_c^2}, \quad B_{12} = \frac{2\eta_c}{I_i A_c^2}, \quad B_{13} = \frac{R_i}{I_i}.
\]
(5.12-k,l,m,n)

This leaves ten physical parameters: \(A_c, k_c, \eta_c, \alpha_c, k_r, \eta_r, \alpha_r, I_i, R_i\), and \(A_c\), compared with the six physical parameters in model I. Also, the transfer function of model I was third-order over second-order with 6 coefficients, whereas Model II is has a fractional-order lead term and 13 coefficients.

With a reasonable initial guess, the parameters are optimized using a Levenberg-Marquardt least-squares curve fitting algorithm [5.13]. Model II’s parameters are tabulated and normalized by their 0.1 mm amplitude values in Table 5.2 to evaluate each one’s amplitude sensitivity. It may be observed that the \(A_c, k_c,\) and \(\eta_c\) parameters are
only found in the combinations $k_c/A_c^2$ or $\eta_c/A_c^2$. Specifying all three parameters is therefore redundant, so $k_c$ and $\eta_c$ are scaled such that $A_c$ remains constant for all amplitudes. Figures 5.3 and 5.4 compare the measured dynamic stiffness spectra of model II at each amplitude with the measurement and model I from [5.1] in terms of magnitude and phase, respectively. Table 5.3 compares the root mean squared (RMS) error of the two quasi-linear models (I and II) defined as,

$$\varepsilon = \frac{1}{K_0} \sqrt{ \frac{1}{N_\Omega} \sum_{\Omega} (\tilde{K}_I(\Omega) - \tilde{K}_I(\Omega))^2 },$$

(5.13)

where $N_\Omega$ is the number of frequencies considered and $K_0$ is the static stiffness. Model II achieves a dramatic improvement in accuracy.

Several insights emerge from the quasi-linear models when the amplitude dependence of each parameter is studied. First, it is apparent that certain parameters are much more amplitude sensitive than others: $R$, $\eta$, and $\alpha$, all change by a factor of two between the minimum and maximum amplitudes while $k$, $\eta$, and $A_x$ vary by less than 20%. Second, certain parameters interact in ways which obfuscates their true nature from a curve-fitting analysis. For example, the denominator natural frequency is related to $k_c/I_i$. While the curve-fitting procedure is effective at locating the value of the $k_c/I_i$ ratio, it may tend to target either the $k_c$ or $I_i$ parameters to do so. Though the inertance should be relatively insensitive to changes in amplitude [5.1], the quasi-linear model shows a decrease in $I_i$ by 35%. The resulting parameters may not reflect the true amplitude sensitive physics in the device, but this would be difficult to verify in the quasi-linear model framework. A time-domain approach which directly captures the
nonlinearities in the system is needed to better understand the governing physics of this device.

Table 5.2. Amplitude sensitivity in the parameters of quasi-linear model II. Each parameter is normalized by the baseline value at 0.1 mm peak-to-peak excitation. See Figures 5.1 and 5.2 for symbols.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Peak-to-peak amplitude ($x_a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1 mm*</td>
</tr>
<tr>
<td>$A_x$ (m²)</td>
<td>1.00</td>
</tr>
<tr>
<td>$k_c$ (N/m)</td>
<td>1.00</td>
</tr>
<tr>
<td>$\eta_c$ (Ns$^{oc}$/m)</td>
<td>1.00</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>1.00</td>
</tr>
<tr>
<td>$k_r$ (N/m)</td>
<td>1.00</td>
</tr>
<tr>
<td>$\eta_r$ (Ns$^{or}$/m)</td>
<td>1.00</td>
</tr>
<tr>
<td>$\alpha_r$</td>
<td>1.00</td>
</tr>
<tr>
<td>$I_i$ (kg/m$^4$)</td>
<td>1.00</td>
</tr>
<tr>
<td>$R_i$ (kg/m$^4$/s)</td>
<td>1.00</td>
</tr>
<tr>
<td>$A_c$ (m²)</td>
<td>1.00</td>
</tr>
</tbody>
</table>

* baseline
Table 5.3. RMS errors in the quasi-linear models with respect to measured dynamic stiffness spectra, revealing a substantial improvement in accuracy.

<table>
<thead>
<tr>
<th>Model</th>
<th>Error at peak-to-peak amplitude ($x_a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1 mm</td>
</tr>
<tr>
<td>Model I, from [5.1]</td>
<td>20.28%</td>
</tr>
<tr>
<td>Model II</td>
<td>3.77%</td>
</tr>
</tbody>
</table>

Figure 5.3. Dynamic stiffness magnitude of quasi-linear models compared with measurement at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm peak-to-peak excitation amplitudes. Key: ■ – measurement; — model I [5.1]; – model II.
Figure 5.4. Loss angle of quasi-linear models compared with measurement at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm peak-to-peak excitation amplitudes. Key: ■ – measurement; – model I [5.1]; – model II.

5.5. Development of the Nonlinear Model (IV)

5.5.1 Nonlinear elements

Two nonlinear elements employed in model IV are inherited from the nonlinear model III developed in [5.1]. The fluid resistance element, \( R_i(q_i) \), assumes turbulent flow in the inertial track where \( \beta_i \) captures the geometric and fluid properties of the inertial track, [5.1,5.7,5.14]

\[
R_i(q_i) = \frac{\varphi}{\sigma R_i} |q_i|^{0.75}.
\]  

(5.14)
Here, $\xi > 1$ is added as a correction factor to account for entrance/exit effects, bends in the channel, and other losses. A nonlinear fluid compliance element has been used [5.1,5.8] to capture the elastic behavior of the pumping chambers where the $\beta_c$ parameters are experimentally derived, $C(p) = 3\beta_{c3}p^3 + 2\beta_{c2}p + \beta_{c1}$. While this formulation proved useful in prior analysis, an equivalent, nonlinear, mechanical spring allows a more flexible model framework.

To establish equivalence with the compliance model, the nonlinear spring must satisfy the volume relation, from which fluid compliance is derived:

$$V_c(p) = \beta_{c3}p_0^3 + \beta_{c2}p_0^2 + \beta_{c3}p + \beta_{c0} = V_{c0} + A_c y_c,$$  

(5.15)

where $V_{c0}$ is the volume of a single pumping chamber at the resting static pressure, $p_0$. Ignoring damping forces (as in the quasi-static compliance experiment [5.8]), the elastic force is related to the chamber pressure as, $F_c = A_c p_c$, leading to the equation,

$$\frac{\beta_{c3}}{A_c^3} F_c^3 + \frac{\beta_{c2}}{A_c^2} F_c^2 + \frac{\beta_{c1}}{A_c} F_c + \beta_{c0} = V_{c0} + A_c y_c.$$  

(5.16)

This equation must be solved for $F_c(y_c)$ to characterize the spring, which yields three solutions. Two of these solutions are complex-valued, while one is real. The physically meaningful, real-valued solution is given by the following,

$$F_c(y_c) = G_5(y_c) - \frac{G_1 - G_2^2}{G_5(y_c)} - G_2,$$  

(5.17-a)

substituting the $G$ terms for brevity,

$$G_0 = \frac{(\beta_{c0} - V_{c0}) A_c^3}{2\beta_{c3}}, \quad G_1 = \frac{\beta_{c1} A_c^2}{3\beta_{c3}}, \quad G_2 = \frac{\beta_{c2} A_c}{3\beta_{c3}},$$  

(5.17-b,c,d)

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\[ G_3 = \frac{\beta_c c_2 A_c^3}{6\beta_{c_3}^2}, \quad G_4(y_c) = G_0 + G_2^2 - G_3 - \frac{A^4}{2\beta_{c_3}^3} y_c, \quad \text{(5.17-e,f)} \]

\[ G_5(y_c) = \left( \sqrt{G_4(y_c)^2 + (G_1 - G_2^2)^3} - G_4(y_c) \right)^{1/3}. \quad \text{(5.17-g)} \]

The nonlinear stiffness of the pumping chamber is given by,

\[ k_c(y_c) = \frac{F_c(y_c)}{y_c}. \quad \text{(5.18)} \]

Practically, the above expression would not be used since it contains a singularity at \( y_c = 0 \), but the \( k_c(y_c) y_c \) product is always used together, such that

\[ k_c(y_c) y_c \bigg|_{y_c=0} = F_c(0) = A_c p_0, \quad \text{where} \quad p_0 \quad \text{is the static pressure in the chambers when the device is at rest.} \]

For this example case, \( p_0 \) equals one atmospheric plus a small positive pressure which results from the assembly process. The equivalence may be mathematically verified at the element level by simply evaluating,

\[ V_c \left( \frac{F_c(y)}{A_c} \right) - V_{c_0} - A_c y = 0. \quad \text{(5.19)} \]

In addition, a nonlinear dry friction element is added to capture the amplitude dependence in chamber damping. Although fractional calculus based damping elements capture viscoelastic effects, \( D^\alpha_t \) is a linear operator and therefore will not exhibit amplitude dependence except by interacting with other, nonlinear elements. Many definitions exist for “smoothened” or “soft” friction laws, and would result in varying dynamic properties [5.11]. For simplicity, a smoothened kinetic friction element is employed in parallel with the fractional damping element,
where the friction coefficient is \( \mu_c \) and the smoothing parameter is \( \sigma_c \). The friction element is nonlinear and directly exhibits amplitude sensitive behavior as well as capturing hysteresis effects not described by the viscoelastic terms. The unique effects of frictional and fractional damping elements are discussed in section 5.6.

5.5.2 Numerical simulation

Numerical solution of the nonlinear model IV is accomplished using a 4\textsuperscript{th} order Runge-Kutta (RK4) numerical integration algorithm [5.15]. This method solves the initial value problem,

\[
\dot{z} = g(t_n, z_n), \quad z(t_0) = z_0, \quad (5.21-a,b)
\]

using the following increments at each time-step \( (n) \) where \( z \) is the state and \( h \) is the step time:

\[
\kappa_{1n} = g(t_n, z_n), \quad (5.22-a)
\]

\[
\kappa_{2n} = g\left(t_n + \frac{h}{2}, z_n + \frac{h}{2} \kappa_{1n}\right), \quad (5.22-b)
\]

\[
\kappa_{3n} = g\left(t_n + \frac{h}{2}, z_n + \frac{h}{2} \kappa_{2n}\right), \quad (5.22-c)
\]

\[
\kappa_{4n} = g(t_n + h, z_n + h \kappa_{3n}), \quad (5.22-d)
\]

\[
t_{n+1} = t_n + h, \quad (5.22-e)
\]

\[
z_{n+1} = z_n + \frac{h}{6} \left( \kappa_{1n} + 2 \kappa_{2n} + 2 \kappa_{3n} + \kappa_{4n} \right). \quad (5.22-f)
\]
The time resolution is defined in terms of the desired number of points per period, $N_h$, and the excitation frequency, $\Omega$ in rad/s. Step time is calculated as,

$$h = \frac{2\pi}{\Omega N_h}. \quad (5.23)$$

Given the simplifying assumptions stated in section 5.2, the governing equations may be rewritten as,

\begin{align*}
\dot{y}_1 &= \frac{A_c \dot{x} - q_c}{A_c}, \quad (5.24-a) \\
\dot{y}_2 &= -\frac{A_c \dot{x} - q_c}{A_c}, \quad (5.24-b) \\
p_1 &= \frac{1}{A_c} \left(k_c (y_1) y_1 + \eta_i D_i^{\mu_c} [y_1] + \mu_c \tanh (\sigma_c \dot{y}_1) \right), \quad (5.24-c) \\
p_2 &= \frac{1}{A_c} \left(k_c (y_2) y_2 + \eta_i D_i^{\mu_c} [y_2] + \mu_c \tanh (\sigma_c \dot{y}_2) \right), \quad (5.24-d) \\
\dot{q}_i &= \frac{1}{I_i} \left(p_1 - p_2 - R_i (q_i, q_i) \right). \quad (5.24-e)
\end{align*}

The state is \( \{y_1, y_2, q_i\}^T \) since the pressures are not calculated from differential equations and therefore don’t require integration.

The significance of the simplifying assumptions made in section 5.4 \( C_f = 0, m_c = 0 \) is highlighted by consideration of the state vector, which would be \( \{y_1, \dot{y}_1, y_2, \dot{y}_2, p_1, p_2, q_i\}^T \) without these assumptions. The reduction from dimension 7 to 3 speeds the algorithm to some extent, but the principal benefit lies in the reduction from second-order behavior to first order. Numerical stability is problematic with the second
order systems, in part since the linearized resonant frequency of the mechanical (pumping
chamber) systems are much higher than the fluid system resonance. A very small step
size would be required, which would interfere with the efficient computation of fractional
derivatives as well as the integration time. Finally, numerical damping has been found to
be necessary to ensure numerical stability for the second-order systems across varying
amplitudes and frequency. Such artificial damping parameters may match or exceed
magnitude of the physical damping, thus masking the physical damping effects. Since the
assumptions are reasonable, the substantial numerical convenience they offer justifies
their use.

The fractional derivatives in equations (5.24-c,d) are calculated using the GL
difference equation (5.7), which has inherent history dependence. Efficiently
incorporating this term into the RK4 algorithm is non-trivial, and many canned numerical
integration solvers would be unable to handle this feature. If a single step time is used in
both the integration and fractional derivative algorithms, then the history of the variable
can align with previous time-steps of the state vector. This obviates the need for
interpolation or estimation of the history, which could substantially slow the algorithm.
However, the RK4 algorithm evaluates slopes at the midpoints between time-steps for the
\( \kappa_{2n}, \kappa_{3n} \) terms and the endpoint for the \( \kappa_{4n} \) term, requiring a unique specification of the
state history for each chamber and for each \( \kappa \) term,

\[
\kappa_{in} : \bar{y} = \{ y(t-N_h h), \ldots, y(t-jh h), \ldots, y(t) \},
\]

\[ (5.25-a) \]

\[
\kappa_{2n} : \bar{y} = \left\{ \frac{y(t-N_h h) + y(t-h-N_h h)}{2}, \ldots, \frac{y(t-h) + y(t-h-jh h)}{2}, \ldots, \frac{y(t-h) + y(t) + h/2 \kappa_{in}}{2} \right\},
\]

\[ (5.25-b) \]
to produce a uniformly sampled history ending at $t_n$, $t_{n+1}/2$, or $t_{n+1}$, respectively.

Early in the simulation when $t < N/n$, this required history extends before the initial condition. In this case, the state is assumed to have been at the initial condition for all $t < t_0$. This would create a contradiction simulating a second order system in state space with nonzero initial conditions since both $y$ and $\dot{y}$ would be included in the state; however, the governing differential equations (5.24-a,b,e) are first order (ignoring the fractional derivatives in equations (5.24-c,d)), so the only notable result of this assumption is an artificial transient effect near $t_0$ which is indistinguishable from the other transients generated by the initial condition.

Since a steady-state harmonic solution is desired, the system is integrated until consecutive periods are identical to within some threshold. Depending on the excitation frequency and initial condition, the amount of time required varies greatly. For this example case, when two consecutive periods of $F_t(t)$ compare with an RMS error of less than 0.1% of the peak-to-peak amplitude, a discrete Fourier transform (DFT) of the final period is taken to find the amplitude and phase of the response at the excitation frequency,

$$\hat{F}_f(\Omega) = \frac{1}{N_h} \sum_{n=1}^{N_h} F_f(nh) e^{i\Omega nh},$$ (5.26)
where the overhead tilde represents a complex-valued number. Figure 5.5 shows an example of the simulated response, comparing the DFT sinusoidal fit with the total response and illustrating the fluid and rubber constituents. The force transmitted through the fluid path exhibits a relatively non-sinusoidal waveform, indicating relatively large harmonic distortion. However, the sinusoidal approximation of the combined rubber and fluid constituents achieves a high degree of accuracy at all amplitudes and frequencies (suggesting a much lower harmonic distortion) and is used in the calculation of the dynamic stiffness,

\[ \tilde{K}_H(\Omega) = \frac{\tilde{F}(\Omega)}{x_a}. \]  

(5.27)

It would also be possible to do analytical work on the transmitted force superharmonics, but since the amplitudes are relatively small compared to the first harmonic, this is left to future work.

5.6. Sensitivity of Fractional and Friction Damping Parameters

5.6.1 Fractional damping parameters

Two fractional damping parameters have been introduced to model IV \((\eta_\epsilon, \alpha_\epsilon)\), and their effects on the dynamic properties of the system must be understood. To this end, a sensitivity study is carried out, showing the effect of each damping parameter on
Figure 5.5. Predicted time domain response of forces from the nonlinear model IV-1.

Here, $F_f$ exhibits a significantly non-harmonic response, demonstrating the nonlinearity in the system. Key: $-$ $F_r$; $-$ $F_f$; $-$ $F_T(t)$ response; $-$ sinusoidal fit.

the dynamic stiffness at each amplitude. Figure 5.6 shows the effect of parameter $\eta_c$ on the dynamic stiffness magnitude and loss angle for both small and large excitation amplitude. Several insights emerge. First, the effect is insensitive to excitation amplitude. For both cases, the loss angle is raised slightly and the dynamic stiffness magnitude increases at frequencies higher than the peak value. Additionally, a stiffening effect is observed in both the magnitude and phase spectra due to the combined stiffness and damping effects present in a fractional derivative based element. Figure 5.7 shows the effect of $\alpha_c$, which is also similar for each amplitude. A factor of 2 decrease in $\alpha_c$...
produces a minimal effect at both amplitudes, while doubling the parameter significantly flattens the stiffness magnitude peak and increases the loss angle at higher frequencies. This effect is predictable given the nature of the fractional element and limiting cases, where the baseline value of $\alpha_c = 0.3$ represents light damping. As $\alpha_c \to 0$, the fractional damper becomes more like a spring element, but the coefficient $\eta_c$ is much smaller than the elastic spring coefficient $k_e$, so any resulting effect is minimal. On the other hand, the limiting case as $\alpha_c \to 1$ is a purely dissipative viscous damper which is physically distinct from the elastic element and produces a noticeable, qualitatively different effect.

Figure 5.6. Effect of fractional damping coefficient ($\eta_c$) on the dynamic stiffness magnitude at (a) 0.1 mm and (b) 2.0 mm peak-to-peak displacement excitation and loss angle at (c) 0.1 mm and (d) 2.0 mm excitation. Key: $\cdots$ – baseline $\eta_c$; $\text{--} \eta_c / 2$; $\text{---} 2\eta_c$. 
Figure 5.7. Effect of fractional damping order ($\alpha_c$) on the dynamic stiffness magnitude at (a) 0.1 mm and (b) 2.0 mm peak-to-peak displacement excitation and loss angle at (c) 0.1 mm and (d) 2.0 mm excitation. Key: \ldots -- baseline $\alpha_c$; -- $\alpha_c / 2$; -- $2\alpha_c$.

5.6.2 Friction damping parameters

The sensitivity of the two friction parameters is also of interest, particularly since the behavior of a nonlinear element may be less intuitive than with the linear fractional dampers. Figure 5.8 displays the effect of increasing or decreasing $\mu_c$ by a factor of 2. The plots demonstrate substantial amplitude dependence since the 2.0 mm simulation shows virtually no sensitivity to the $\mu_c$ parameter, while the 0.1 mm dynamic stiffness magnitude is strongly affected near its peak value. Similarly, altering the $\sigma_c$ parameter by a factor of 10 only affects the low amplitude tests to any significant degree, as seen in
Figure 5.9. Here, the dynamic stiffness and loss angle peaks decrease with higher values of $\sigma_c$, but the loss angle increases substantially at frequencies higher than the peak. This behavior results directly from the local amplitude sensitivity of the dry friction element. In other words, the particular behavior which exists at amplitudes near some operating point is not representative of the global behavior at all amplitudes. This is illustrated in Figure 5.10, which presents a force displacement curve across the chamber damping elements at three frequencies and two amplitudes. It is clear that the friction element induces substantially different behavior at different excitation amplitudes.

Figure 5.8. Effect of dry friction coefficient ($\mu_c$) on the dynamic stiffness magnitude at
(a) 0.1 mm and (b) 2.0 mm peak-to-peak displacement excitation and loss angle at (c) 0.1 mm and (d) 2.0 mm excitation. Key: $\cdots \cdots$ – baseline $\mu_c$; – $\mu_c / 2$; – $2\mu_c$. 

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5.7. Verification and Validation of New Nonlinear Models

Model IV is verified (as model IV-0) by using the same parameters as model III [5.1], so the only differences would be in the integration algorithms used. Figures 5.11 and 5.12 compare the model IV-0 dynamic stiffness spectra with the spectra of model III given in [5.1] and the measurement, showing only minimal differences between these two models. Two differences justify some discussion. First, the loss angle spectra near the peaks show a small discrepancy between the two nonlinear models. This is due to the fact that a different phase estimation algorithm was employed in model IV (DFT) vs. model III (least squares regression). The new algorithm is more precise, whereas the resolution...
of the regression method was directly dependent on $N_b$. Second, small peaks exist in the loss angle spectra at high frequencies, particularly at the 0.1 mm amplitude. These are numerical artifacts of the integration algorithm and may be reduced by enhanced resolution at the cost of increased computational time, although for most practical purposes they may simply be ignored. The new model has more of these erroneous points than the former since the RK4 algorithm uses a fixed step whereas the previous method allowed for an adaptive step size, and thus would require lower resolution.

Having reproduced the results of [5.1], a new parameter set can be employed with model IV-1 to improve its predictive accuracy compared with measured dynamic stiffness spectra. Figures 5.13 and 5.14 show the resulting dynamic stiffness spectra with

\[ g_r(y) \] at (a) 0.1 mm and (b) 1.0 mm peak-to-peak displacement excitation. Key: 

- $\Omega = 8$ Hz;
- $\Omega = 16$ Hz;
- $\Omega = 32$ Hz.

![Figure 5.10](attachment:figure5_10.png)
nonzero fractional and friction damping in the pumping chambers. Improvement is achieved over model III at all amplitudes in terms of both magnitude and phase, but the magnitude spectra offers some additional insight. In particular, model IV tends to under-predict $|\tilde{K}_m|$ at the lower amplitudes, and over-predict it at the higher ones, balancing with an excellent prediction near 1.0 mm. Since the over/under-prediction vs. amplitude relationship appears nearly linear, it would suggest that at least one amplitude-dependent parameter is being linearized in model IV.

Figure 5.11. Dynamic stiffness magnitude of nonlinear models compared with measurement at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm. Key: ■ – measurement; – model III [5.1]; – model IV-0.
Figure 5.12. Loss angle of nonlinear models compared with measurement at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm. Key: ■ – measurement; — model III [5.1]; — model IV-0.

Table 5.4. RMS error of the new nonlinear models with respect to measured dynamic stiffness spectra. Each new model (IV-1 or IV-2) achieves a substantial improvement.

<table>
<thead>
<tr>
<th>Model</th>
<th>Error at peak-to-peak amplitude ($x_a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1 mm</td>
</tr>
<tr>
<td>Model III, from [5.1]</td>
<td>49.28%</td>
</tr>
<tr>
<td>Model IV-1</td>
<td>28.94%</td>
</tr>
<tr>
<td>Model IV-2</td>
<td>21.84%</td>
</tr>
</tbody>
</table>
It has been previously suggested [5.1, 5.8] that a nonlinear characterization of pumping area may improve model accuracy, so a quasi-linear $A_x(x_o)$ parameter is introduced to demonstrate the potential effectiveness of model IV with limited further refinement. Figures 5.15 and 5.16 depict the resulting dynamic stiffness spectra in terms of magnitude and phase, showing excellent agreement at all amplitudes. Table 5.4 compares the RMS error of each nonlinear model, showing general improvement over model III and exceptional accuracy with model IV-2, particularly at the higher amplitudes where the error is reduced by over half. This suggests a need for further study to characterize the physical nature of $A_x(x)$.

![Graph 1](image1)

![Graph 2](image2)

Figure 5.13. Dynamic stiffness magnitude of nonlinear models at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm. Key: ■ = measurement; – model III [5.1]; – model IV-1.
Figure 5.14. Loss angle of nonlinear models at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm. Key: ■ – measurement; – model III [5.1]; – model IV-1.

Figure 5.15. Dynamic stiffness magnitude of nonlinear models with quasi-linear \( A_s(x_o) \) at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm. Key: ■ – measurement; – model III [5.1]; – model IV-1.
Figure 5.16. Loss angle of nonlinear models with quasi-linear $A_x(x_a)$ at (a) 0.1 mm, (b) 0.5 mm, (c) 1.0 mm, and (d) 2.0 mm. Key: ■ — measurement; — model III [5.1]; — model IV-1.

8. Conclusion

This chapter significantly extends the prior article [5.1] by offering improved quantification of damping for amplitude sensitive reduced order hydraulic bushing models under steady state harmonic excitation up to 50 Hz between 0.1-2.0 mm peak-to-peak excitation amplitudes. In particular, fractional calculus viscoelastic elements as well as smoothened dry friction dampers have been introduced to the fluid pumping chambers, capturing unique dynamic behavior which is not related to the nonlinear dissipative elements used in prior models [5.1,5.7,5.8]. Thus this chapter offers three principle
contributions. First, new quasi-linear models which include fractionally damped pumping chambers substantially reduce the RMS error at all amplitudes compared with the prior literature [5.1]. Second, new nonlinear models exhibit new amplitude sensitive behavior which may better describe dynamic stiffness measurements at multiple excitation amplitudes; additionally, interactions between the new damping elements and other nonlinear elements in the system are observed. Third, measured dynamic stiffness spectra validate both quasi-linear and nonlinear models at several amplitudes, demonstrating an enhanced description of the underlying physics in a hydraulic bushing’s interacting design features.

The scope of this chapter has been limited to uniaxial, steady-state harmonic excitation up to 50 Hz with 0.1, 0.5, 1.0, and 2.0 mm peak-to-peak excitation amplitudes. Physical effects not captured by the new models (II, IV) may become dominant in other loading directions, or beyond the 2.0 mm amplitude, or at a higher frequency range. For example, at very high amplitudes, a nonlinear stiffness element (stopper) would engage, altering the dynamic properties substantially. Or, at very high frequencies (say around 1 kHz) internal resonances of both the rubber material and fluid system would be observed, and the lumped parameter modeling framework [5.1-5.8] would be inadequate. Finally, this chapter suggests the need for improved characterization of pumping area, $A_x$. The previous paper [5.1] indicated that $A_x$ was of critical importance, and this chapter demonstrates that it indeed exhibits amplitude sensitivity, suggesting that an experimental study (perhaps similar to the fluid compliance experiment of [5.8]) would be needed to better characterize the nonlinear behavior of $A_x(x,p)$. 

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Chapter 5 References


Chapter 5 Nomenclature

A effective pumping area (m$^2$)
B transfer function coefficient
C fluid compliance (m$^5$/N)
D derivative operator
F force (N)
G elastic model parameter
g damping forcing function (N·s/m)
h time step (s)
I fluid inertance (N·s$^2$/m$^5$)
j fractional derivative history term index
K dynamic stiffness (N/m)
k elastic spring coefficient (N/m)
L Laplace transform
m effective mass (kg)
N number of terms
n time step index
p pressure (Pa)
q volume flow rate (m$^3$/s)
R fluid resistance (N·s/m$^5$)
s Laplace domain variable (1/s)
t time (s)
V volume (m$^3$)
\( X \) displacement of inner sleeve in the Laplace domain (m)
\( x \) displacement of inner sleeve in the time domain (m)
\( y \) effective displacement of pumping chamber wall (m)
\( Z \) arbitrary displacement in Laplace domain (m)
\( z \) arbitrary displacement in time domain (m)
\( \alpha \) fractional derivative order
\( \beta \) model parameter
\( \Gamma \) the gamma function
\( \varepsilon \) root mean squared error
\( \eta \) fractional/viscous damping coefficient
\( \kappa \) Runge-Kutta increment
\( \mu \) friction coefficient (N)
\( \xi \) empirical correction factor
\( \sigma \) smoothing factor (1/s)
\( \tau \) variable of integration (s)
\( \phi \) loss angle (rad)
\( \Omega \) excitation frequency (rad/s)
\([ \ ]\) discrete variable indexing

Subscripts

0 initial or static value
1, 2 pumping chambers 1 and 2
\( a \) amplitude (peak-to-peak)
\( c \) pumping chamber
\( f \) fluid path
\( i \) inertia track
\( r \) rubber path
\( T \) total transmitted force

**Superscripts**

\(-\) history

\(\sim\) complex-valued variable

**Abbreviations**

DFT discrete Fourier transform
GL Grünwald-Letnikov fractional derivative
NL nonlinear
RK4 4\textsuperscript{th} order Runge-Kutta algorithm
RMS root mean squared
QL quasi-linear
Chapter 6. Conclusion

6.1 Summary

This dissertation proposes new analytical and semi-analytical models for a class of elastomeric and hydraulic vibration isolators. This work extends the prior literature [6.1-6.5] by proposing new models and characterization methods of the viscoelastic properties of elastomeric isolators (in the time and frequency domains) and then developing nonlinear hydraulic bushing models which capture the amplitude-sensitive dynamic behavior. The proposed analytical and computational models are validated with dynamic stiffness measurements of production and laboratory isolation devices.

First, the multi-axis dynamic stiffness terms of elastomeric isolators are determined over a broad range of frequencies using a spectral element approach with fractional damping. The dynamic properties are modeled by using the continuous system theory in terms of homogeneous rods or Timoshenko beams. The transfer matrix type dynamic stiffness expressions are developed from exact harmonic solutions given translational or rotational displacement excitations. Broadband dynamic stiffness magnitudes (say up to 5 kHz) are computationally verified for axial, torsional, shear, flexural, and coupled stiffness terms using a finite element model. Some discrepancies are found between finite element and spectral element models for the axial and flexural motions, illustrating certain limitations of each method. Experimental validation is provided for an isolator with two cylindrical elements (that work primarily in the shear
mode) using dynamic measurements, as reported in the prior literature, up to 600 Hz. Superiority of the fractional damping formulation over structural or viscous damping models is illustrated via experimental validation.

Second, the frequency dependent properties of elastomeric vibration isolators are addressed the context of lumped parameter models with fractional damping elements. A mass is placed between two fractional calculus Kelvin-Voigt elements to develop a minimal order system for the example case of a conventional elastomeric bushing typical of automotive suspension systems. The minimal order system model accurately predicts dynamic stiffness in both broadband resonant behavior as well as the lower-frequency regime that is controlled by damping. For transient response analysis, an inverse Laplace transform of the dynamic stiffness spectrum is taken via the Residue Theorem. Since the fractional calculus based solution is given in terms of problematic integrals, a new time-frequency domain estimation technique is proposed which approximates time-domain responses for a class of transient excitation functions. The approximation error is quantified and found to be small, and tractable closed-form transient response functions are provided along with a discussion of numerical issues.

Next, quasi-linear and nonlinear models are proposed with a focus on the amplitude sensitivity in magnitude and loss angle spectra (up to 50 Hz). Since the model parameters of a production bushing are unknown, dynamic stiffness tests and laboratory experiments are utilized to extract model parameters. Nonlinear compliance and resistance elements are incorporated, including their interactions in order to improve amplitude sensitive predictions. New solution approximations for the new system equations (containing both nonlinear compliance and resistance elements) refine the
multi-term harmonic balance term method. Quasi-linear models yield excellent accuracy but lack the ability to predict trends in amplitude sensitivity since they rely on available dynamic stiffness measurements. Nonlinear models containing both nonlinear resistance and compliance elements yield superior predictions to those of prior models (with a single resistance or compliance nonlinearity), while also providing more physical insight.

Finally, the research of Chapter 4 is extended to include damping effects within the fluid pumping (elastomeric) chambers. The conventional fluid compliance element is replaced by an equivalent mechanical spring representing the nonlinear elasticity of the pumping chambers. Fractional calculus based and friction-type damping elements are added in parallel to the spring elements of pumping chambers. Improved quasi-linear models are proposed at four amplitudes, demonstrating amplitude sensitivity in model parameters. New nonlinear models are proposed and numerically simulated, predicting dynamic properties at multiple amplitudes. The sensitivity of dynamic stiffness properties to the fractional and frictional damping parameters is qualitatively evaluated. Finally, experimental validation is provided for both quasi-linear and nonlinear models.

6.2 Contributions

This dissertation contributes to the state-of-the-art of elastomeric vibration isolator modeling and characterization in two distinct but related areas. Improved characterization of elastomeric vibration isolators is achieved in time and frequency domains through the introduction of fractional calculus based damping elements in the first half of this dissertation. In the second half, new nonlinear models are proposed and experimentally validated for the interacting nonlinear features of a hydraulic isolator,
capturing dynamic properties over a range of excitation amplitude and frequency. Specific contributions are discussed below.

First, the spectral element method provides a compact and elegant modeling framework which can accurately capture the dynamic properties of certain elastomeric isolators at very high frequencies using advanced, fractional calculus based viscoelasticity. This allows for the direct extension of prior work by Noll, et al. [6.1] and addresses the limitations of the previous work by Östberg and Kari [6.2]. In particular, the frequency range is expanded up to 5 kHz, (and could be extended to larger bandwidths without difficulty) all six degrees of freedom are captured (including coupling terms), dynamic behavior in the damping-controlled region (say below 100 Hz) is modeled, and the isolator geometry may involve short aspect ratios where the length of a cylindrical isolator is less than its diameter.

Second, a procedure for estimating the transient response of a fractionally damped isolator with resonant behavior is proposed, overcoming the limitations of prior research by Rossikhin and Shitikova [6.3] and others. The proposed method estimates the time-domain implications of frequency-domain properties, which is non-trivial, and done using system-specific solutions in the previous work. The solutions in this dissertation are more general, and suitable for analyzing elastomeric isolators over a frequency range which includes internal resonance. Other solution methods are possible, but this research provides physical insight into the relationship between dynamic properties in the time and frequency domains.

Next, the focus moves to hydraulic bushings, where previous literature has offered only sparse nonlinear characterization. This dissertation extends previous models
develop by Chai, et al. [6.4] which had a single nonlinear fluid resistance element to include an experimentally obtained nonlinear fluid compliance parameter, and to focus on production bushings rather than a more controlled laboratory device. Furthermore, the multi-term harmonic balance method reveals superharmonic terms in the dynamic response which may be used to gain insight into or identify measured nonlinear behavior in hydraulic bushings. New nonlinear models show increased accuracy compared with dynamic stiffness measurements compared with previous studies.

Finally, the new nonlinear hydraulic bushing models (of Chapter 4) are further extended to include advanced viscoelastic damping mechanisms in the pumping chambers. An equivalent nonlinear stiffness of the chamber is developed to replace the fluid compliance element and facilitate the introduction of additional model features, including fractional calculus and friction dampers. This new configuration further improves the amplitude sensitive dynamic stiffness predictions and offers physical insight into the damping mechanisms which influence the dynamic properties of a hydraulic bushing.

6.3 Future Work

Several avenues of future research are suggested.

1. Extend the time-frequency transformation procedure of the fractionally damped, lumped model (of Chapter 3) to the continuous system, spectral element models (as developed in Chapter 2). The transient response estimation procedure relies on a finite number of singular points existing in the transfer function, but the
relatively compact, closed-form solutions generated by the spectral element method may have interesting solutions despite having infinite singular points.

2. Develop an inverse-identification procedure for elastomeric joints based on spectral element models; this would extend the work of Noll et al. [6.5] (which assumed spectrally invariant dynamic properties). Such an assumption is reasonable only over a relatively limited frequency range, while the dynamic properties of the larger structure supported by several isolators may easily be tuned to have rich dynamic behavior at very high frequencies. The modified procedure should identify the isolator’s damping parameters as well as providing more robust experimental validation to the spectral element model.

3. Perform an experimental study into the nonlinear nature of the pumping area of hydraulic bushings. This dissertation has identified the effective pumping area to be both a significant factor and source of amplitude sensitivity in the example bushing, but only empirical identification has been offered. The pumping area is closely related to the fluid compliance, so an experiment very similar to the setup suggested by Fredette, et al. [6.6,6.7] could be used to measured the nonlinear compliance. A physically meaningful nonlinear pumping area parameter would improve the fidelity and predictive capabilities of hydraulic bushing models.

Further extensions to the scope may include nonlinear models for elastomeric bushings, multi-axis loading of hydraulic bushings, and transient characterization of hydraulic bushings. These extensions could help to overcome some of the assumptions and limitations of the models in this dissertation. Finally, the methods proposed in this
research may be applicable to a wide range of vibration isolators, absorbers and vibro-acoustic materials. Further, the proposed mathematical techniques may be applied to a broad array of other fields including viscoelasticity, rheology, biology or biophysics, control theory, signal and image processing, and mechatronics [6.8].
Chapter 6 References


Bibliography


