Nullspace MUSIC and Improved Radio Frequency Emitter
Geolocation from a Mobile Antenna Array

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy in the Graduate School of The Ohio State
University

By

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2016

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Abstract

This work advances state-of-the-art Radio Frequency (RF) emitter geolocation from an airborne or spaceborne antenna array. With an antenna array, geolocation is based on Direction of Arrival (DOA) estimation algorithms such as MUSIC. The MUSIC algorithm is popular for DOA estimation because it has high resolution and applies to arbitrary arrays of polarization sensitive antennas. However, MUSIC fails to obtain its theoretical resolution when signals at the same carrier frequency arrive from nearby angles. Nearby signals cause a merged peak in MUSIC’s spectrum, whereas MUSIC requires a separate peak for each signal. MUSIC’s failure with closely separated signals has been addressed by many authors who have proposed modifications to MUSIC. We propose a new modification, Nullspace MUSIC, which outperforms MUSIC and its existing modifications. Nullspace MUSIC applies a divide-and-conquer approach, estimating emitters individually while other signals are nulled from the antenna pattern. Critically, Nullspace MUSIC maintains MUSIC’s orthogonality test, which results in superior performance compared to existing MUSIC modifications that apply similar logic. In Monte Carlo simulations with an example setup, Nullspace MUSIC improves MUSIC’s resolution of two diversely polarized signals from 2.0° to 0.75° without imposing any additional assumptions.

Additionally, an antenna array on an aircraft or spacecraft cannot be perfectly calibrated. RF waves are blocked, reflected, and scattered by the platform around...
the antenna array. Moreover, the platform scattering on an aircraft changes over time due to flexible wings, movable control surfaces, retractable landing gear, and variable loadouts. Consequently, a demanding full-wave electromagnetics simulation of the entire platform or a difficult measurement of the entire platform in an anechoic chamber cannot eliminate the mismatch between the true, \textit{in-situ} antenna patterns and the antenna patterns that are available for DOA estimation (the \textit{antenna array manifold}). Since MUSIC fundamentally compares an assumed set of antenna patterns with the received signals, platform-induced manifold mismatch severely degrades MUSIC’s resolution and accuracy. Therefore, we compare the performance of various modified MUSIC algorithms in the presence of array manifold mismatch. We show that Nullspace MUSIC also provides the most accurate DOAs for arbitrary signals incident on an airborne antenna array so long as they are not too close together.

Geolocation does not stop with DOA estimation. Since manifold mismatch makes individual DOA estimates inaccurate, DOAs from throughout a flight path must be used to geolocate an emitter. Conventionally, Lines of Bearing (LOB) are drawn from the antenna array along the DOAs to find the locations where the DOAs intersect with the ground. However, averaging the LOBs in the global coordinates yields large errors as a result of geometric dilution of precision for DOAs near horizon. Since averaging positions fails, a single emitter is typically located by finding the position on the ground that yields the Minimum Apparent Angular Error (MAAE) for the DOA estimates made over a flight. We extend the conventional MAAE approach to cluster lines of bearing from multiple emitters. Consequently, multiple simultaneous and co-frequency emitters can be geolocated following DOA estimation with MUSIC or Nullspace MUSIC. Note that MAAE clustering can be applied to DOA estimates
from any algorithm. Geolocation of multiple emitters using DOA estimates has received little attention in literature. Therefore, the novel MAAE clustering algorithm advances a long neglected aspect of RF emitter geolocation.

We also combat antenna array mismatch by applying the Direct Mapping Method (DMM). DMM combines DOA spectra directly by mapping them to a fixed surface such as the earth’s surface. The spectra are averaged together, and the emitter locations are estimated directly from the composite spectrum. Although DMM has been applied for low signal-to-noise-ratios (SNR) and sparse antenna arrays, we apply it to flights with antenna array mismatch. DMM and MAAE clustering yield similar position errors, although DMM is less sensitive to the particular manifold mismatch setup. Our examples include four diversely polarized emitters, which must be geolocated using a seven-element antenna array. This is too challenging for MAAE and DMM, which may not be able to locate all of the emitters. Therefore, we fuse Nullspace MUSIC and DMM in the Nullspace DMM algorithm. Nullspace DMM applies the divide-and-conquer approach by nulling an estimated emitter’s position throughout a flight. This allows another emitter to be estimated with minimal interference from the nulled emitter. Nullspace DMM is the only approach that locates all four emitters in all the scenarios that we consider. In summary, Nullspace DMM confronts the practical limitations on DOA-based geolocation and pushes the boundaries of what is possible for a small array on an independent platform.

Finally, we apply the proposed geolocation algorithm to real-world experimental data. A six-element antenna array and Data Collection System (DCS) were installed on a small aircraft. The DCS recorded signals from four live transmitters during a three-hour flight over Columbus, Ohio. The samples were post processed, and
combinations of one to four emitters were geolocated from various segments of the flight. As expected, individual DOA estimates were erratic and widespread due to the airplane’s perturbations of the measured array manifold. MAAE and DMM locate at most three of the four emitters during several flight segments. However, Nullspace DMM separates the emitters from each other and yields unambiguous estimates for every emitter in every flight segment. The successful experimental trials show that Nullspace DMM could significantly enhance airborne emitter geolocation in missions such as RF spectrum enforcement, locating unknown transmitters for defense, and search and rescue operations.
Dedicated to Linda
Acknowledgments

I would like to sincerely thank my adviser, Professor Inder “Jiti” Gupta, for his mentoring, guidance, and commitment during my education. Dr. Andrew O’Brien also served as an invaluable resource throughout my graduate student career for discussing concepts and collaborating on projects. In addition, the professors, researchers, staff, and students of The Ohio State University ElectroScience Laboratory created a unique environment for learning, research, and collaboration.

Much of this work was sponsored by a NASA Space Technology Research Fellowship under grant NNX12AM37H. I sincerely appreciate the support of the NSTRF program and staff during my Fellowship. I am particularly grateful to Dr. Obed “Scott” Sands at the NASA Glenn Research Center for his support and collaboration over the years. I also want to thank the engineers at the NASA Glenn and NASA Goddard research centers for the opportunity to work with NASA on cutting edge projects and for many invigorating conversations.

Finally, I thank my wife, Linda, my parents, my friends, and the Lord for encouragement and support throughout this effort.
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Chapter 1: Introduction

1.1 Objective and Background: RF Emitter Geolocation from a Mobile Platform

This research advances the state-of-the-art in Radio Frequency (RF) emitter geolocation for an airborne or spaceborne platform that is equipped with an antenna array. Pioneering applications for single-platform geolocation include passively locating transmitters for defense [1], civilian RF spectrum enforcement [2], and localizing the receiver relative to unknown transmitters (radio SLAM) [3]. In addition, geolocation by a mobile platform applies to traditional geolocation domains such as navigating between radio beacons [4], search and rescue, [5, 6], radar [7], sonar [8], and asset monitoring [9]. Emitter localization has been a critical topic since the advent of radio [10], and its importance will only increase as wireless communications continue their explosive growth [11].

Geolocation of RF emitters can be carried out using a single platform or multiple platforms. In the case of multiple platforms, three or more stationary receivers must be spread out over a surveillance area. All of the receivers should have line of sight reception from the emitters of interest [12, 13]. If one or more receivers moves rapidly, then two receivers may be sufficient to locate stationary transmitters [14]. In either case, however, the receivers must be precisely synchronized with a common clock,
and the receivers must share large amounts of information to compare their received signals. Although multi-platform geolocation is ideal for some setups, it cannot be applied to cases for which multiple receivers are not available. It is also inappropriate if high bandwidth communications are difficult or dangerous.

Single-platform geolocation allows a mobile receiver to operate independently, without transmitting to cooperating receivers. Single-platform geolocation relies on a single array of co-located antennas. The number of elements in the antenna array must exceed the number of transmitters, but antenna arrays with six or more elements are increasingly common [15, 16]. These antenna elements must be sampled synchronously, but a common clock is easily distributed to the antenna array electronics on a single platform. Additionally, the platform’s position and attitude must be known precisely. Fortunately, Global Navigation Satellite Systems (GNSS) such as GPS and Galileo, along with commercial-off-the-shelf Inertial Measurement Units (IMU), supply precise position and orientation information for outdoor receivers. Moreover, systems have begun replacing single antenna elements with antenna arrays to enable beamforming and interference mitigation [17]. These developments herald an upcoming golden age of antenna array signal processing, and the new antenna arrays create the potential for opportunistic geolocation on platforms that never previously possessed independent geolocation capabilities. Hence, this dissertation starts with the basics, then builds towards improved geolocation with a single receiver that locates emitters independently using a passive antenna array and data collection system, GNSS, and an IMU.
1.2 Limitations of Current Single Platform Geolocation

Single platform geolocation relies on Direction of Arrival (DOA) estimation. While DOA estimation is theoretically very accurate [18], and it allows a system to operate independently, it also has some practical drawbacks. First, DOA estimation is well known to fall short of its theoretical resolution for multiple emitters [19]. Indeed, resolving multiple signals has been a principle research topic in DOA estimation for quite some time [20]. The Bartlett beamformer, (also known as conventional beamforming), was among the earliest DOA algorithms for an antenna array that allowed multiple signals to be observed [21]. However, its resolution, (the minimum separation at which nearby signals may be distinguished), strongly depends on the aperture of the antenna array, and the beamformer’s DOA estimates are biased when multiple emitters are present [22]. The Minimum Variance Distortionless Response (MVDR) beamformer [23] improves the resolution of the classical beamformer, but it still falls short of the best possible resolution. MUltiple SIgnal Classification (MUSIC) [24] yields better resolution, which is theoretically unlimited with sufficient Signal to Noise Ratio (SNR) and many samples of the received signals [18]. However, MUSIC’s practical resolution falls short of theoretical predictions, even when all of MUSIC’s assumptions are met [19]. Finally, Maximum Likelihood (ML) [25] offers the best possible resolution, and it is not affected by the particular problems that impose resolution limits on MUSIC. Unfortunately, ML quickly becomes computationally intractable with multiple incident signals, since every possible combination of emitter parameters should be tested. Therefore, DOA estimation itself imposes some limits on single platform geolocation due to suboptimal resolution for multiple emitters.
Direction of Arrival estimation also suffers from poor accuracy if the exact gain and phase patterns of the array’s antenna elements (the *antenna array manifold*) are only approximately known [26]. Approximate antenna patterns yield approximate DOAs because DOA estimation fundamentally compares the received signals with the assumed antenna patterns. Imperfectly known antenna patterns afflict mobile systems such as aircraft mounted arrays because the host platform blocks, reflects, and scatters the received radio signals. Therefore, the *in-situ* (on-platform) antenna pattern differs from the nominal antenna pattern [27]. *In-situ* antenna array calibration determines the on-platform antenna response, but it is non-trivial for large platforms like aircraft because direct measurements are difficult and expensive. Moreover, many platform configurations change over time due to control surface movements, variable loadouts, and additions or upgrades of other antenna systems on the platform.

To overcome inaccurate DOA estimates, DOA-based geolocation systems typically estimate DOAs from diverse observation points, then combine the measurements to estimate the emitter position [28, 29]. In advanced approaches, the system essentially finds the position that would minimize squared errors in the estimated DOAs [30]. That position is assumed to be the emitter location. The estimated emitter location can be very accurate if the DOA estimates are not too erratic, and if the observations provide good geometric diversity for the emitter [31, 32].

Unfortunately, multiple simultaneous emitters pose significant challenges when geolocating emitters based on multiple sets of DOA measurements. DOA estimation does not inherently identify DOA estimates from different observations as coming from the same transmitter. Therefore, DOA estimates from different observations must be grouped together before they can be used to geolocate an emitter [33]. Lines
of Bearing (LOB) may be drawn from the receiver along the estimated DOAs to find where the LOBs intersect with each other or the earth’s surface. However, inaccurate DOA estimates may cause the LOBs from different emitters to intermix, and the DOA estimates must be carefully separated before the emitters may be located. The untangling of intermixed DOA estimates has received surprisingly little attention in academic literature, and it remains as a fundamental obstacle to locating multiple simultaneous transmitters using estimated DOAs.

DOA estimation’s limited resolution, its poor performance with an imperfectly calibrated manifold, and the difficulty of associating DOA estimates from different observations, are restraining new and existing applications for geolocating multiple emitters with a mobile-receiver. Solving the challenges of suboptimal resolution and DOA estimate association would enable multiple emitter geolocation tasks that currently cannot be considered using a single mobile receiver. Moreover, robust techniques would be a powerful enabler for opportunistic geolocation using arrays that were not originally designed or sited for direction of arrival estimation.

1.3 Current Approaches

Current approaches to geolocating multiple emitters from a mobile emitter are often based on the Multiple Signal Classification (MUSIC) algorithm [24]. MUSIC’s popularity may be attributed to its excellent performance, conceptual simplicity, and unusual versatility [34]. MUSIC is applicable to general array geometries, two-dimensional DOA estimation, dominantly-polarized or polarization-sensitive antenna arrays, and multiple emitters. Moreover, MUSIC runs at a reasonable computational
cost. However, MUSIC estimates multiple DOAs from the peaks in a single function that is calculated versus direction. This function, which known as the *MUSIC spectrum*, does not obtain MUSIC’s theoretical resolution when peaks from multiple emitters merge together. Consequently, several methods [35, 36, 37, 38] have been proposed to enhance MUSIC’s resolution. Thus, limitations on MUSIC’s practical resolution are well recognized, and they have been partially addressed.

Several authors have also studied how to improve MUSIC’s performance by correcting mismatched antenna patterns. MUSIC essentially exploits an array’s ability to weigh and sum the antenna outputs such that a signal from a particular direction is attenuated by tens of dB. Such destructive interference is referred to as *nulling* [39]. However, nulls are very sensitive to small errors in the antenna patterns [40]. Thus, MUSIC’s superb performance under ideal conditions degrades quickly in the presence of antenna array calibration errors [26]. Note that although alternative high resolution DOA algorithms such as ESPRIT [41] or Root MUSIC [35] are sometimes regarded as avoiding the antenna array calibration requirement entirely, they still require the array to be calibrated in the sense that it should meet a stipulated model. Specifically, Root MUSIC requires elements of the array to be identical, while ESPRIT requires an array to be composed of pairs of identical antennas. Therefore, Root MUSIC and ESPRIT are also affected by perturbations in the antenna patterns because mutual coupling and platform scattering make every antenna’s pattern unique. Root MUSIC and ESPRIT also require extensions in order to estimate azimuth and elevation with the same array [42, 43, 44].

Unfortunately, most approaches to improving an array’s calibration focus on mutual coupling between the antenna elements in the array [45, 46, 47, 48]. Mutual
coupling is generally modeled using a Mutual Coupling Matrix (MCM), which models each antenna’s radiation as a weighted sum of all the antenna elements’ radiation patterns in isolation from each other. Including mutual coupling effects on the antenna patterns is necessary, but not sufficient, for accurate DOA estimation. In particular, geolocation with a mobile platform assumes that the array has been mounted on some vehicle. Vehicles block, reflect, and scatter radio waves in a very direction dependent manner [27], and the direction independent MCM cannot model this behavior. Current approaches considering the environment around an array have not been demonstrated with complicated platforms [49]. Moreover, the platform around an aircraft mounted array may vary over time as a result of deployable landing gear, moving control surfaces, and a variable loadout. Therefore, the antenna patterns will also vary with time, and a static antenna array patterns are almost certain to differ from the true antenna array patterns. Robust beamforming approaches have been published for maintaining SNR in spite of errors in the assumed antenna patterns [50, 51]. These approaches are very general and model the uncertainty in the array manifold as an ellipsoid. However, similar approaches for MUSIC have not been developed.

As pointed out in the previous section, if multiple emitters are present, then DOAs from multiple observations must be grouped together before an emitter can be located. Association may use Lines of Bearing (LOB) that are drawn from the receiver along the estimated DOAs. The system finds where the LOBs intersect with each other or the earth’s surface, and estimates may be grouped with batch processing or multiple target tracking [52, 53]. However, untangling intermixed estimates is not addressed in current literature. Instead, current approaches for locating multiple
emitters using multiple observations of a scene avoid traditional DOA estimation altogether. Direct Mapping Methods (DMM) such as [53, 54, 55, 56] combine DOA spectra from various observations, and they map a composite DOA function directly to the earth’s surface. The extrema of this function are used as the emitter location estimates. DMM successfully locates multiple emitters without grouping DOAs, and it improves theoretical performance [57]. However, DMM has not been applied in cases with imperfectly calibrated antenna patterns, nor have other authors undertaken experimental demonstrations. Authors advocating DMM also do not address the question of how well multiple emitters can be located using DOA estimates from individual observations.

1.4 Contributions and Organization of This Document

Geolocation with a mobile platform would benefit from several advances. First, further improvements to resolution for multiple emitters should be pursued. In particular, algorithms for improved resolution should maintain MUSIC’s ability to work with arbitrary polarizations, since the polarization of a signal of interest may not be known ahead of time. Second, improved DOA algorithms should decrease MUSIC’s sensitivity to imperfectly calibrated antenna patterns. Imperfect calibration must be anticipated in real-world applications, especially when taking advantage of beam-forming arrays that were not originally intended for DOA estimation. Third, since geolocation is the ultimate aim of DOA estimation, a robust algorithm should be developed to locate multiple emitters using intermixed DOA estimates from various observations of a scene. Without such an approach, the DOA algorithms themselves cannot be fully evaluated. Fourth, DOA estimation and DMM from a mobile platform
should be compared for multiple emitters following a realistic flight path. If possible, DMM’s sensitivity to imperfect calibration should also be reduced. Finally, a flight experiment should be conducted, and live transmitters should be located. Such an experiment would put to rest questions regarding the oft encountered discrepancies between simulation performance and real world performance. For example, a simulation may assume perfect knowledge of the platform position and orientation, and it may also assume that the antenna electronics have been calibrated perfectly. A flight test with experimental hardware would help discern whether these assumptions are reasonable.

The research presented in this dissertation addresses all of these imperatives. First, Chapter 2 discusses the narrowband signal model with polarization and conventional DOA estimation. This provides the necessary background for the dissertation. Then, Chapter 3 examines MUSIC’s performance with two signals. Although MUSIC’s theoretical performance is excellent, simulations show that peaks in the MUSIC spectrum may merge at small signal separations, which leads to large DOA errors. The merged peaks cause MUSIC to fall far short of its theoretical performance. Next, current modified MUSIC algorithms are examined, and a new modified MUSIC algorithm, called Nullspace MUSIC, is introduced. MUSIC and its modifications are tested for various signal separations in Monte Carlo trials. The Monte Carlo simulations show that Nullspace MUSIC yields the best resolution for mixed polarization signals. It is also more robust than the best existing method (Pole-Zero MUSIC [35]), which may introduce spurious peaks into the MUSIC spectrum.

The next chapters focus on expected real world geolocation performance. Chapter 4 introduces mismatch between the receive array’s true and assumed antenna patterns.
Monte Carlo simulations test MUSIC, Pole-Zero MUSIC, and Nullspace MUSIC for various incident signal directions. Nullspace MUSIC yields the smallest DOA errors with mixed polarization signals, and it is the most robust to perturbations in the antenna patterns. That is, Nullspace MUSIC performs about the same in a ‘small’ mismatch setup as in a ‘large’ mismatch setup. However, Nullspace MUSIC only improves the DOA estimates for moderate to large signal separations (approximately 50% of the array’s beamwidth or greater). In contrast, Pole-Zero does not improve MUSIC’s DOA estimates for mixed polarization signals at any signal separation. Thus, Nullspace MUSIC advances state of the art DOA estimation with multiple emitters in practical scenarios affected by antenna pattern mismatch.

Chapter 5 applies MUSIC and Nullspace MUSIC during simulated flights with multiple emitters and antenna pattern mismatch. Due to mismatch and the presence of multiple emitters, the DOA estimates for various emitters intermix with each other. Therefore, we must extend current, single emitter geolocation algorithms to handle multiple emitters. In this approach, we iteratively group the estimated DOAs to minimize the apparent angular error for each emitter. The Minimum Apparent Angular Error (MAAE) estimates are surprisingly robust to inaccurate and intermixed DOAs. In simulations that follow a recorded flight path from a geolocation mission, MAAE yields accurate geolocation estimates for four emitters with various polarizations for some flight paths. However, for other flight paths, all four emitters may not be found. Notably, the flight paths cause emitters to appear nearby in angular space, and Nullspace MUSIC usually does not improve the final geolocation estimates as compared to using MUSIC.
Since mismatch prevented MAAE from locating all four emitters for some flights in Chapter 5, Chapter 6 applies the Direct Mapping Method to the same flight paths. While DMM has been proposed for airborne geolocation of multiple emitters, it has not been studied in the context of imperfectly calibrated antenna arrays. Like MAAE, DMM easily locates one or two emitters at a time. DMM also uses fewer computations because it only searches for peaks in the final spectrum. However, DMM also struggles to locate all four emitters for some flight paths. Therefore, we combine Nullspace MUSIC and DMM in the Nullspace DMM algorithm. Nullspace DMM is the only algorithm that locates all four emitters for the two flight paths and the two mismatch setups that were considered. Since Nullspace DMM yields qualitatively similar spectra for both mismatch setups, Nullspace DMM is an excellent candidate for robust, multi-emitter, single-platform geolocation with an imperfectly calibrated antenna array.

Finally, Chapter 7 tests real-world, single-platform, geolocation performance with an experimental flight test. A six-element antenna array was built and mounted to a Piper Saratoga (six-seater) aircraft. The in-situ antenna pattern was simulated to minimize the mismatch between the true and assumed antenna patterns. A corresponding data collection system was designed and built so that the effects of the front-end electronics on the received signals could be calibrated out. Then, a three hour flight test was conducted over Columbus, Ohio, and signals from four live transmitters were collected. After the flight, the recorded samples were post processed, and the emitter locations were estimated using MAAE, DMM, and Nullspace DMM. As in Chapters 5 and 6, mismatch between the true and assumed antenna platforms causes very erratic DOA estimates. Nevertheless, all methods combine observations throughout the flight path, and they can locate one or two emitters without too much
trouble. However, MAAE and DMM fail to locate all four emitters for some flight paths. On the other hand, Nullspace DMM successfully geolocates all four emitters for both flight paths. Additional flight segments are tested with the same result - Nullspace DMM is the only approach that can separate and localize all four emitters in all flight paths.

Ultimately, this dissertation advances state-of-the-art emitter geolocation with a single platform. The novel Nullspace MUSIC algorithm improves on state-of-the-art DOA resolution with mixed polarization signals. It is also more robust than current enhanced resolution methods when confronted with antenna array pattern mismatch. Single emitter geolocation with DOA estimates was extended to multiple emitter geolocation using a novel clustering algorithm. In addition, DMM’s robustness to mismatch was enhanced by combining it with Nullspace processing in the novel Nullspace DMM algorithm. As a result, emitters can be located unambiguously in very challenging scenarios where DMM and conventional geolocation fail.

Finally, a flight test showed that the DOA uncertainty induced by array manifold mismatch is a very real effect, and that Nullspace DMM is the most robust approach for geolocating multiple emitters in spite of platform induced antenna array pattern mismatch. Similar geolocation campaigns for multiple simultaneous emitters have not been reported in academic literature.
Chapter 2: Direction of Arrival Estimation

This chapter overviews Radio Frequency (RF) Direction of Arrival (DOA) estimation. Recall that DOA estimation provides the foundation for emitter geolocation from a single mobile platform. Thus, Section 2.1 begins by describing the narrowband signal model [58] that is commonly used for DOA estimation. The narrowband signal model represents the electromagnetic interactions between an incident signal and an antenna, and it simplifies analysis of the received signal samples. The narrowband signal model will be assumed throughout this dissertation. Next, Section 2.2 describes the antenna array that will be used for examples in this chapter and throughout most of the dissertation. The commercial off-the-shelf (COTS) array contains seven elements laid out in a circle, and it is well suited for localizing multiple signals that arrive from the antenna’s upper hemisphere with various polarizations. Then, Section 2.3 overviews beamforming [21] with an antenna array and how it is applied to DOA estimation. Beamforming is quick and straightforward, but signals that are close together may not be resolved. Hence, Section 2.4 describes Multiple Signal Classification (MUSIC) [24], a high resolution, subspace-based algorithm for estimating multiple DOAs. In this chapter, the various DOA algorithms will be introduced with straightforward examples, while theoretical and in-depth performance studies for MUSIC will be covered in Chapter 3.
Note that this chapter studies beamforming and MUSIC because they apply to arbitrary antenna arrays; estimate DOAs for multiple uncorrelated signals; estimate azimuth as well as elevation; and run quickly. Alternative methods for estimating the DOAs of multiple emitters tend to lack one or more of these qualities. For instance, ESPRIT [41] requires the antenna array to be divisible into pairs of identical antennas. This requirement is poorly suited for arrays experiencing mutual coupling and scattering that make every antenna element unique. ESPRIT also requires special considerations to estimate azimuth and elevation [44]. On the other hand, Maximum Likelihood (ML) is fully general and offers the best possible resolution. However, ML quickly becomes computationally intractable with multiple incident signals, since every possible combination of emitter parameters should be tested. Therefore, beamforming and MUSIC are the most suitable algorithm for geolocating multiple emitters from an airborne platform, and they will be the focus of this chapter.

The received signal samples also depend on the orientation of the signals’ electric fields, which defines the signal’s polarization. Therefore, arbitrary polarization is added to the signal model in Section 2.5. This section also discusses how MUSIC can locate signals with arbitrary polarization without estimating the polarization. Overall, this chapter provides the necessary background for modified MUSIC algorithms in Chapter 3. Modified MUSIC algorithms are needed because MUSIC cannot meet its theoretical resolution threshold in many scenarios with uncorrelated signals.

### 2.1 Narrowband Signal Model

Fig. 2.1 shows the basic setup for Direction of Arrival (DOA) estimation. Electromagnetic fields, (shown in orange), from $M$ transmitters are incident on a passive
array of $N$ antenna elements, (shown in black). The transmitters are in the far-field of the array’s host platform, and the array’s phase center is at the origin of the coordinate system, $O$. The $z$-axis extends along the array’s broadside angle. The $\theta$ angle is measured from the $z$-axis, while the $\phi$ angle is measured from the $x$-axis, and increases moving towards the $y$-axis.

DOA estimation commonly assumes that the incident signals are narrowband. For a signal to be narrowband, the array aperture, (measured in wavelengths), must be much smaller than the inverse of the fractional signal bandwidth [58]. The fractional bandwidth is defined as $B/f_0$, where $B$ is the signal bandwidth, and $f_0$ is the carrier frequency. If a signal is narrowband, then the various phase shifts across the aperture of the array are about the same for the lowest and highest frequencies in the signal. Critically, this implies that the time delay between when the antenna elements in the array receive the same signal can be represented as a phase shift. Therefore, the
antennas’ direction dependent gain and delay differences can be modeled simply as complex functions of angle.

If the transmitted signal is narrowband, and the signal polarization matches the polarization of the receive antenna, the then the output from the $i$th antenna element can be modeled as

$$x_i(t) = a_i(\psi)s(t)e^{j\omega_0 t}. \quad (2.1)$$

In (2.1), $s(t)$ is the baseband, transmitted signal that has propagated to the array’s phase center at time $t$; $e^{j\omega_0 t}$ shifts the baseband signal to the carrier frequency; $\omega_0 = 2\pi f_0$ is the carrier frequency in radians per second; $a_i(\psi)$ is a complex number incorporating the $i$th antenna element’s gain and phase, (relative to the array’s phase center), at the carrier frequency; $\psi$ is the direction from the array to the transmitter; and $j$ is the imaginary unit, $+\sqrt{-1}$. The direction $\psi$ may represent one angle, $\theta$, or two angles, $(\theta, \phi)$, depending upon the scenario under consideration.

Note that $a_i(\psi)$ is the in-situ antenna response, which incorporates the $i$th antenna’s offset from the array’s phase center; mutual coupling with the other elements in the array; and scattering from the environment around the antenna array. Consequently, local multipath is included in $a_i(\psi)$. In the narrowband signal model, each incident signal is in the far-field of the platform and is affected by a single gain and phase shift. Since the direct path and scattered signals have the same frequency, the signal amplitudes and phases combine linearly to yield one received signal with one amplitude and phase. Also note that although the antenna elements in the array may be identical to each other, they have different responses versus direction due to their unique positions as well as the unique impacts of mutual coupling and local scattering.
Each antenna is connected to a dedicated set of front-end electronics, as shown in Fig. 2.2. The front-end electronics include components such as mixers, filters, and amplifiers that condition the analog signals. Generally, the antenna electronics downconvert (2.1) to baseband, which removes the $e^{j\omega t}$ dependence in (2.1). Then, the baseband outputs of the antennas are sampled at discrete times, $t_k = k\Delta t$, where $\Delta t$ is the sampling interval, in seconds. The $i$th antenna’s $k$th sampled signal can be modeled as

$$x_i[k] = a_i(\psi)s[k] + n_i[k].$$

(2.2)

Here, $n_i[k]$ is noise from the $i$th antenna’s front-end electronics. The electronics noise is commonly assumed to be a zero mean, additive white Gaussian random variable [58]. The noise is independent of the transmitted signals, and it is uncorrelated between the antenna elements in the array. This equation also assumes that the front-end electronics are ideal, and that they do not change the magnitude or phase of the received signal. In practice, the front-end electronics must be calibrated, and the $N$ receive channels must be equalized. After ideal equalization, the magnitude and phase induced on the received signal is solely due to the antenna effects, $a_i(\psi)$, and additive noise, as assumed in (2.2). Note also that (2.2) assumes that the Analog to Digital Converter (ADC) samples the signals at a uniform rate; has sufficient dynamic range to avoid clipping the signals and to make the finite resolution effects negligible; and has sufficient bandwidth to avoid signal aliasing.

The sampled signal for all antennas in the array can be expressed as an $N \times 1$ snapshot vector,

$$\mathbf{x}[k] = \mathbf{a}(\psi)s[k] + \mathbf{n}[k],$$

(2.3)
where \( a(\psi) = [a_1(\psi), \ldots, a_N(\psi)]^T \) is a \( N \times 1 \) complex vector incorporating all elements’ in-situ gain and phase towards \( \psi \), and superscript \( T \) is the transpose operation. Antenna engineers often refer to \( a(\psi) \) as the \textit{steering vector} or the \textit{antenna array manifold} in the direction \( \psi \). Similarly, \( n[k] \) is the \( N \times 1 \) noise vector.

With \( M \) incident signals, the snapshot vector becomes

\[
\mathbf{x}[k] = M \sum_{m=1}^{M} a(\psi_m) s_m[k] + n[k]. \tag{2.4}
\]

While (2.4) applies for correlated and uncorrelated signals, we will always assume that the incident signals are uncorrelated in this dissertation. Uncorrelated signals appropriately model geographically separated emitters that operate independently or transmit independent information. Independent, complex Gaussian random variables [59] will be used to generate the snapshot vectors for simulated results.

The sum in (2.4) can be written as a matrix multiplication so that the snapshot vector is

\[
\mathbf{x}[k] = \mathbf{A}s[k] + \mathbf{n}[k], \tag{2.5}
\]
where the $N \times M$ array matrix $A$ equals $[a(\psi_1), \ldots, a(\psi_M)]$, and the $M \times 1$ signal vector $s[k]$ equals $[s_1[k], \ldots, s_M[k]]^T$. As we noted before, the narrowband signal model assumes that local multipath contributes to the gain and phase shifts $a(\psi_m)$ that are induced on the $m$th signal. According to the narrowband model, local multipath does not create additional, time-delayed copies of the received signals at the receive array.

Although DOA estimation can be attempted with a single snapshot vector [60, 61], beamforming and MUSIC typically use many more samples to improve the accuracy of the estimated sample covariance. For $K$ samples, (2.5) becomes

$$X = AS + N,$$  

with $X = [x[1], \ldots, x[K]]$, $S = [s[1], \ldots, s[K]]$, and $N = [n[1], \ldots, n[K]]$. The snapshot matrix, $X$, is used by the array signal processor for direction of arrival estimation.

### 2.2 Antenna Array Setup

This section discusses the antenna array that is used for examples in the beamforming and MUSIC sections. The commercial, off-the-shelf, array consists of seven patch antenna elements laid out as shown in Fig. 2.3. Each antenna element is nominally Right Hand Circularly Polarized (RHCP), and the inter-element spacing is 0.47 wavelengths at the carrier frequency.

The far-field patterns of the antenna array were obtained through a combination of measurement and simulation. First, the physical array was mounted on a four-foot circular ground plane. Then, the far-field patterns were measured in The Ohio State University’s anechoic chamber at the ElectroScience laboratory (Fig. 2.4). These
measurements had a three degree resolution in $\theta$ and $\phi$. The measured fields were then imported into ANSYS HFSS [62] and simulated on a circular ground plane with the same diameter$^1$. HFSS simulated the fields with a resolution of one degree in $\theta$ and $\phi$ for $\theta \leq 90^\circ$. In addition, the antenna patterns were simulated with 0.01$^\circ$ degree resolution in $\theta$ along the pitch cut of the array, the elevation cut for which $\phi = 0^\circ$ and $\phi = 180^\circ$, (see Fig. 2.1). Simulated antenna patterns will be used for DOA estimation throughout the dissertation so that the DOAs are minimally influenced by the discretization of the array manifold.

Fig. 2.5. shows the Right Hand Circularly Polarized (RHCP) antenna pattern for the center element in the antenna array. The left plot shows the RHCP gain, and the right plot shows the RHCP phase. The center of each plot corresponds to

$^1$The simulation of the measured fields on a circular ground plane is discussed in detail in Section 4.3.
antenna zenith. The outer edge of the plot corresponds to the antenna horizon. The azimuth angle is zero on the right, and it increases counter-clockwise. Similarly, Fig. 2.6. shows the Left Hand Circularly Polarized (LHCP) antenna pattern for the center element in the antenna array. Finally, Fig. 2.7 shows the center element’s gain for the RHCP, LHCP, $E_\theta$, and $E_\phi$ polarizations along the pitch cut of the array. From Figs. 2.5-2.7, we see that the antenna is well designed for receiving RHCP signals. However, the center element’s gain for the $E_\theta$ and $E_\phi$ polarizations also exceeds -7dB out to $\theta \approx 70^\circ$, and the $E_\phi$ gain is almost as great as the RHCP gain at some angles. In addition, diffraction from the ground plane introduces some variation into the antenna patterns and increases the cross polarization sensitivity of the antenna elements.
Figure 2.5: Center antenna element, right hand circularly polarized radiation pattern. Left: Gain (dB). Right: Phase (Deg).

Figure 2.6: Center antenna element, left hand circularly polarized radiation pattern. Left: Gain (dB). Right: Phase (Deg).

Figure 2.7: Center antenna element gain in the pitch cut along $\phi = 0^\circ$ and $\phi = 180^\circ$ for the RHCP, LHCP, $E_\theta$ and $E_\phi$ polarizations.
The seven-element antenna array will be used throughout most of the dissertation for single-polarization as well as mixed-polarization scenarios. Initially, we study single polarization DOA estimation, and we will assume that the array only senses RHCP signals. Then, we will study mixed polarization DOA estimation, for which the cross polarization of the array is non-negligible and must be considered for accurate DOA estimates. Note that, in this chapter and Chapter 3, we will assume the exact antenna patterns to be known and available for DOA estimation. In addition, the antenna patterns are assumed to be fixed and deterministic.

2.3 Beamforming

A beamformer is a processor used in conjunction with an antenna array to provide spatial filtering [63]. Conceptually, an array of $N$ antenna elements can phase-shift and sum the $N$ sampled signals such that the narrowband received signal from an angle $\psi_0$ will add constructively. If the gains of the antenna elements are the same, then the coherently summed signal voltage will increase by a factor of $N$ at the beamformer output. However, if the beamformer’s phase shifts do not match the phase shifts induced by the array in the direction $\psi_0$, then some of the received signal power will be canceled. Thus, phase shifts corresponding to $\psi_0$ maximize the signal’s received power. This spatial discrimination makes beamforming a natural starting point for DOA estimation. In this section, we will discuss conventional beamforming and an adaptive beamforming algorithm, (Capon’s method, or Minimum Variance Distortionless Response), as applied to DOA estimation.
2.3.1 Conventional Beamforming

In beamforming, the outputs of the $N$ antenna elements are weighed and summed with a set of $N$ complex weights. Using the snapshot vector $\mathbf{x}$ from (2.3), the $k$th output of the beamformer is written as

$$y[k] = \mathbf{w}^H \mathbf{x}[k].$$

(2.7)

Here, $\mathbf{w}$ is the $N \times 1$ weight vector, and superscript $H$ denotes the conjugate transpose operation.

The average output power of the beamforming system is given by the expected value of the output, $y[k]$, multiplied by its conjugate,

$$P = E\{y[k]y^*[k]\}$$

$$= E\{\mathbf{w}^H \mathbf{x}[k]\mathbf{x}^H[k]\mathbf{w}\}$$

$$= \mathbf{w}^H E\{\mathbf{x}[k]\mathbf{x}^H[k]\}\mathbf{w}$$

$$= \mathbf{w}^H \mathbf{R}\mathbf{w}. \quad (2.8)$$

In (2.8), superscript $*$ represents the conjugate operation and $E\{\cdot\}$ is the expected value operator. The matrix $\mathbf{R} = E\{\mathbf{x}\mathbf{x}^H\}$ is referred to as the covariance matrix. Many DOA algorithms, such as Beamforming and MUSIC, use the covariance matrix to estimate the signal DOAs.

In practice, a finite number of samples are available, and the expected value is replaced by an average. The finite sample average for $\mathbf{R}$ is

$$\tilde{\mathbf{R}} = \frac{1}{K} \mathbf{X}\mathbf{X}^H. \quad (2.9)$$

Throughout this dissertation, a tilde will indicate an approximate or simulation value. Also, recall that $K$ is the number of samples.
Conventional beamforming maximizes the output power for a given direction under the constraint that \( \| \mathbf{w} \| = 1 \), where \( \| \cdot \| \) is the Euclidean norm. With a single incident signal at \( \psi_0 \), selecting

\[
\mathbf{w} = \frac{\mathbf{a}(\psi_0)}{\| \mathbf{a}(\psi_0) \|}
\]

acts as a matched filter and maximizes the power of a signal from \( \psi \) [58].

If we let \( P_{BF} = E\{ \mathbf{y}[k] \mathbf{y}^*[k] \} \), and we insert (2.10) into (2.8), then we obtain a function of power versus angle,

\[
P_{BF}(\psi) = \frac{\mathbf{a}^H(\psi) \mathbf{R} \mathbf{a}(\psi)}{\| \mathbf{a}(\psi) \|^2}.
\]

(2.11)

\( P_{BF} \) is referred to as the beamforming spectrum. By calculating the beamforming spectrum for all angles of interest and finding the \( M \) highest peaks, we can estimate the DOAs for the \( M \) incident signals.

Fig. 2.8 shows the normalized beamforming spectrum for a RHCP signal incident on the seven-element antenna in Fig. 2.3. The normalized spectrum is simply the spectrum divided by its maximum value for any angle,

\[
\hat{P}_{BF}(\psi) = \frac{P_{BF}(\psi)}{\max_{\psi} P_{BF}(\psi)},
\]

(2.12)

where the hat indicates a normalized quantity. The signal arrives from \((\theta, \phi) = (10^\circ, 0^\circ)\) and has an SNR at an isotropic element of 10dB\(^2\). As expected, the spectrum peaks at the signal DOA. However, the half power beamwidth - the angular distance between the points where the beamformer power is one-half of the peak beamformer power - is about 49.5\(^\circ\), which is quite large. Note that this approximates the half

\(^2\)Since the antenna elements have gain near zenith, the actual SNR is greater than 10dB. However, the SNR at each antenna depends on the antenna gain, which varies from antenna to antenna. Therefore, SNRs will be specified relative to an isotropic antenna
power beamwidth of a linear array with the same aperture. For a linear array of isotropic antennas, the Half Power Beam Width (HPBW) is given by [64],

$$\text{HPBW} = \frac{0.886}{(\cos(\theta)D/\lambda)}.$$  \hspace{1cm} (2.13)

Here, $D$ is the aperture of the antenna array, and $\theta$ is the angle (measured from broadside) that the beam is pointed towards. Also, the HPBW has units of radians. Using the aperture of our array, $D = 0.94$ wavelengths, and the DOA of the incident signal, $\theta = 10^\circ$, the HPBW would be $54.8^\circ$.

If we assume that the conventional beamformer can separate a second signal that is added to the spectrum when the second signal is at least one HPBW away, then the classical beamformer’s resolution is limited by the aperture of the array to a little less than $\lambda/(D \cos(\theta))$. Then, if the signals are equal power, the signals should be resolved by the presence of a dip (approximately 3dB) between the signal peaks in the beamformer spectrum.
Figure 2.9: Normalized conventional beamforming spectrum for uncorrelated RHCP signals from ($\theta$, $\phi$) ($10^\circ$, $0^\circ$) and ($50^\circ$, $0^\circ$). Left: Polar plot. Right: Pitch cut.

Fig. 2.9 shows the normalized beamforming spectrum, (2.12), for two uncorrelated RHCP signals that arrive from ($\theta$, $\phi$) ($10^\circ$, $0^\circ$) and ($50^\circ$, $0^\circ$), each with 10dB SNR. Since these signals are separated by less than the half power beamwidth for the array, we do not expect conventional beamforming to resolve them. Indeed, the beams for the two signals merge, and the overall peak in the beamforming spectrum is at ($21^\circ$, $0^\circ$). Moreover, the spectrum does not contain peaks at the true signal DOAs. Although conventional beamforming maximizes the signal power for a particular direction, it does not consider any other signals that may be incident on the array when selecting weights for a given direction. As a result, a beam for one direction may also be sensitive to signals from other directions, and the peaks in the beamforming spectrum may be biased. In addition, the conventional beamformer power scales with the incident signal power. Thus, a high SNR signal may obscure a low SNR signal in the beamforming spectrum. Therefore, conventional beamforming is insufficient for multiple signal direction finding in many scenarios.
2.3.2 Minimum Variance Distortionless Response

Capon’s method is another beamforming technique that uses a different set of weights [23]. The weights enhance the Signal to Interference plus Noise (SINR) ratio by minimizing the received power with the constraint that a signal coming from a particular direction is undistorted. When used for DOA estimation, Capon’s method improves DOA resolution, and it sharpens the peak in the DOA spectrum if only one signal is present. Since a signal’s power is proportional to its variance, Capon’s method may also be referred to as the Minimum Variance Distortionless Response (MVDR) beamformer.

MVDR adapts the beampattern to the sampled signals to minimize the received power. However, to avoid the trivial solution of giving zero weight to each channel, MVDR enforces the constraint that the signal from a chosen direction is unaffected by the beamformer. Thus, MVDR method solves the optimization problem [21]

$$\min_w P(w)$$

subject to \( w^H a(\psi) = 1 \),

and \( P(w) \) is the average beamformer power, \( w^H \hat{R} w \), (2.8). The weights for angle \( \psi \), \( w(\psi) \) are given by

$$w(\psi) = \frac{\hat{R}^{-1} a(\psi)}{a^H(\psi) \hat{R}^{-1} a(\psi)}. \quad (2.15)$$

Note that these weights depend on the incident signals through \( \hat{R} \), whereas the conventional beamforming weights were independent of the incident signals. The weights’ dependence on the signal scenario makes MVDR an adaptive beamformer.

The MVDR spectrum, \( P_{MVDR} \) is the power versus angle yielded by the weights in (2.15). That is,

$$P_{MVDR}(\psi) = \frac{1}{a^H(\psi) \hat{R}^{-1} a(\psi)} \quad (2.16)$$
Figure 2.10: Normalized MVDR spectrum with one RHCP signal from $(\theta, \phi) (10^\circ, 0^\circ)$. Left: Polar plot. Right: Pitch cut.

Figure 2.11: Normalized MVDR spectrum with uncorrelated RHCP signals from $(\theta, \phi) (10^\circ, 0^\circ)$ and $(50^\circ, 0^\circ)$. Left: Polar plot. Right: Pitch cut.

If a signal is at least one beamwidth away from the direction of interest, $\psi$, then MVDR’s output SNR at $\psi$ is almost unaffected by other signal [21]. This improves MVDR’s performance relative to beamforming for incident signals with equal or unequal powers. At closer separations, MVDR still provides some suppression of the signal not of interest.
Fig. 2.10 shows the normalized MVDR spectrum for the same signal scenario that was shown in Fig. 2.8. With one RHCP signal, MVDR also peaks at the true signal location. However, MVDR’s half power beamwidth is only 5.9°, which is about an eighth of the conventional beamformer’s half power beamwidth for the same scenario. Although MVDR’s beamwidth depends on the signal to noise ratio of the incident signal, MVDR has much sharper peaks, and it will have higher resolution when multiple co-channel signals are incident on the array.

The same two signal scenario from Fig. 2.9 is revisited with MVDR in Fig. 2.11. MVDR resolves the uncorrelated RHCP signals and peaks at the true DOAs. However, the 3dB beamwidth around the signals increases significantly as compared to the one signal scenario. The beams widen because power from one signal leaks into the beam for the other signal in spite of MVDR’s minimum power optimization. Nevertheless, MVDR resolves multiple signals in many scenarios.

2.4 MUSIC

While MVDR is an optimal beamformer, it does not obtain optimal DOA resolution. If signals are too close, then the beams will merge just as in the case with the conventional beamformer. However, two nearby signals may be resolved by exploiting the eigenstructure [65] of the covariance matrix. This approach is followed by MUSIC [24], an algorithm that obtains asymptotically unlimited DOA resolution in scenarios with high SNR, many snapshots, uncorrelated signals, and uncorrelated Additive White Gaussian Noise (AWGN) in the receivers [66].
The true covariance matrix in (2.8) may also be written as

\[
R = E\{xx^H\} \\
= E\{(As + n)(As + n)^H\} \\
= E\{Ass^H A^H + Asn^H + ns^H A^H + nn^H\}.
\] (2.17)

If the signals and the noise are uncorrelated, then the cross terms involving \(s\) and \(n\) have expected values of zero. Thus,

\[
R = AE\{ss^H\} A^H + E\{nn^H\} \\
= A\Lambda A^H + \sigma^2 I,
\] (2.18)

where \(\Lambda\) is an \(M \times M\) diagonal matrix with the received signal powers on the diagonal; \(\sigma^2\) is the noise power from the antenna electronics, and \(I\) is the \(N \times N\) identity matrix. As we have already assumed, the incident signals and the noise in the antenna electronics must be uncorrelated for \(\Lambda\) and the noise covariance, \(\sigma^2 I\), to be diagonal.

Note also that the noise power is assumed to be the same in every receive channel.

If \(M < N\), then the covariance matrix in (2.18) is the sum of a rank-deficient matrix, with rank \(M\) or less, and a scaled identity matrix. Therefore, any vector that is orthogonal to \(A\) is an eigenvector of \(R\) with an associated eigenvalue of \(\sigma^2\). In other words, if \(v\) is orthogonal to every column of \(A\), then \(Rv = \sigma^2 v\). Eigendecomposition finds \(N - M\) such vectors that are also mutually orthogonal. Thus, the \(N - M\) eigenvectors of \(R\) with eigenvalues of \(\sigma^2\) are orthogonal to \(a(\psi_1) \ldots a(\psi_M)\).

The orthogonality between the \(N - M\) eigenvectors and the antenna array manifold is exploited by MUSIC [24]. Assume that the number of received snapshots (\(K\)) is greater than the number of antenna elements. Then, MUSIC eigendecomposes \(\hat{R}\) into eigenvectors \(v_1, \ldots, v_N\) and associated eigenvalues \(\lambda_1 > \lambda_2 > \cdots > \lambda_N\). The \(M\) dimensional vector space spanned by \(v_1 \ldots v_M\) is referred to as the signal-subspace.
Ideally, this vector space has the same span as $A$. MUSIC uses the orthogonal complement of the signal-subspace, the $N \times (N - M)$-dimensional noise-subspace $Q$. The noise-subspace is defined according to

$$Q = [v_{M+1}, \ldots, v_N],$$

(2.19)

where $v_{M+1}, \ldots, v_N$ are the eigenvectors corresponding to the smallest $N - M$ eigenvalues of $\hat{R}$. Under ideal conditions, any vector in the noise-subspace is orthogonal to $[a_1 \ldots a_M]$. MUSIC takes advantage of this key relationship. When the array is only sensitive to one polarization, MUSIC estimates the signal DOAs from the $M$ maxima of the MUSIC spectrum, $f_{MU}(\psi)$

$$f_{MU}(\psi) = \frac{a^H(\psi)a(\psi)}{a^H(\psi)QQ^H a(\psi)},$$

(2.20)

In (2.20), MUSIC essentially measures the length of the projection of the assumed manifold vector into the noise-subspace for all directions of interest. Note that MUSIC assumes that the antenna array manifold is known for all directions of interest so that (2.20) can be calculated.

Equation (2.20) can be simplified by noting that $QQ^H$ forms a projection matrix. A projection matrix $P$ has the special property that $P^2 = P$. Thus, (2.20) may also be written as

$$f_{MU}(\psi) = \|P_N \hat{a}(\psi)\|^{-2},$$

(2.21)

where $P_N = QQ^H$, and $\hat{a}(\psi) = a(\psi)/\|a(\psi)\|$ is the normalized antenna manifold vector.

In the direction of a signal, the norm of the manifold vector’s projection into the noise-subspace, $\|P_N \hat{a}(\psi)\|$, would be zero under ideal conditions. Away from a signal, the length of the projection would be non-zero. The MUSIC spectrum is the inverse
of the square of this length. Thus, in the direction of a signal, the MUSIC spectrum should be infinite. Therefore, MUSIC has very high dynamic range and very fine resolution under ideal conditions. MUSIC’s theoretical performance will be covered in detail in Chapter 3.

Comparing the denominators of the MVDR spectrum (2.16) and the MUSIC spectrum (2.21) also shows that MUSIC suppresses angles between signals more effectively than MVDR. Note that the MVDR spectrum may similarly be simplified as

$$P_{MVDR}(\psi) = \| R^{-1/2} a(\psi) \|^2.$$  \hfill (2.22)

Considering the eigendecomposition of the covariance matrix, it can be shown that

$$R^{-1/2} = V \Lambda^{-1/2} V^H.$$ \hfill (2.23)

Taken together, (2.22) and (2.23) show that MVDR scales down signal-subspace components of $a(\psi)$ by $\lambda^{-1/2}$, with $\sigma^2 \leq \lambda_M \leq \lambda \leq \lambda_1$. On the other hand, MUSIC maps signal-subspace components of $a(\psi)$ to zero. Thus, the MUSIC spectrum shows greater contrast between different test vectors around a signal’s DOA. This gives MUSIC a narrower beamwidth and higher peaks than MVDR. Therefore, MUSIC can distinguish between more closely separated signals than MVDR.

Figs. 2.12 and 2.13 show the MUSIC spectrum for the one and two-signal scenarios considered in Section 2.3. Note that these plots are on a 50 dB scale, whereas the beamforming plots were on a 25 dB scale. In the one signal case, we see that MUSIC has a much narrower peak in the direction of the signal. Indeed, the 3dB beamwidth is less than one degree! As with conventional beamforming and MVDR, MUSIC correctly estimates the signal’s DOA. In the case with two uncorrelated RHCP
signals, MUSIC’s peaks are much sharper than MVDR’s, and the signals are clearly resolved. As in the one signal case, MUSIC’s 3dB beamwidth is less than one degree. As with MVDR, the MUSIC spectrum peaks at the true DOAs.

Fig. 2.14 compares the normalized beamforming, MVDR, and MUSIC spectra for the one and two signal scenarios. These plots show elevation cuts along $\phi = 0^\circ$. Positive $\theta$ values correspond to $\phi = 0^\circ$, while negative $\theta$ values correspond to $\phi = 180^\circ$.  

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Figure 2.14: Comparison of normalized spectra for conventional beamforming, MVDR, and MUSIC. Left: One RHCP signal from $(\theta, \phi) (10^\circ, 0^\circ)$. Right: Uncorrelated RHCP signals from $(\theta, \phi) (10^\circ, 0^\circ)$ and $(50^\circ, 0^\circ)$.

These plots clearly show the differences in dynamic range as well as beamwidth for the various methods. It is easy to see why MUSIC is popular for high resolution DOA estimation. Like beamforming, MUSIC applies to general array geometries, has a simple formula, and does not require estimating extra parameters (such as the signal amplitude or waveform). However, MUSIC has much finer resolution capability than beamforming for scenarios involving multiple uncorrelated signals. Moreover, MUSIC is easily extended to polarization sensitive arrays, as discussed in the next section.

2.5 Mixed Polarization Signals and MUSIC

Since transverse electromagnetic (TEM) waves in the far-field do not have a radial component, they can be written as the sum of a $\theta$-oriented electric field and a $\phi$-orientated electric field [67]. An electric field’s orientation is referred to as its polarization. The previous sections assumed a particular combination of $\theta$ and $\phi$-polarized fields by considering RHCP signals. However, arbitrary signal polarizations may be encountered when geolocating unknown emitters. Moreover, each emitter may have
a unique polarization. Fortunately, the narrowband signal model from Section 2.1 can be modified to incorporate arbitrary polarization [68]. This section discusses the extension of the model, as well as the Mixed Polarization MUSIC (MP-MUSIC) algorithm, which estimates DOAs for signals with arbitrary and different polarizations.

Let $\eta$ parameterize the relative phase of a signal’s $\theta$ and $\phi$ components. Let $\zeta$ parameterize the relative magnitude of the $\theta$ and $\phi$ components. Then, a TEM wave with any polarization can be modeled as

$$e(\eta, \zeta) = E(\sin(\zeta)e^{j\eta\hat{\theta}} + \cos(\zeta)e^{j\phi}),$$

(2.24)

where $\hat{\theta}$ is the unit vector in the $\theta$ direction; $\hat{\phi}$ is the unit vector in the $\phi$ direction; and $E$ is the electric field amplitude.

The response of an antenna to a TEM wave is the sum of the antenna’s response to the wave’s $\theta$ and $\phi$ components. Let the antenna array manifold vector for a $\theta$-polarized wave be $b(\psi)$. Let the antenna array manifold vector for a $\phi$-polarized wave be $c(\psi)$. The antenna array manifold vector for an arbitrary polarization is

$$a(\psi, \eta, \zeta) = \sin(\zeta)e^{j\eta}b(\psi) + \cos(\zeta)c(\psi).$$

(2.25)

For a mixed polarization scenario, the antenna manifold matrix from (2.6) becomes

$$A = [a(\psi_1, \eta_1, \zeta_1), \ldots, a(\psi_M, \eta_M, \zeta_M)],$$

(2.26)

and $x[k] = As[k] + n[k]$ as before. Thus, the antenna manifold matrix can be used to model the received samples from multiple emitters with various polarizations.

### 2.5.1 MUSIC for Mixed Polarization Signals

Conventional beamforming and MVDR must search over the polarization parameters $\zeta$ and $\eta$ when the signal polarization is unknown. However, Schmidt formulated
MUSIC such that it may search for signals with arbitrary polarization [24]. Interestingly, MUSIC does not need to estimate a signal’s polarization to estimate its DOA. This form of MUSIC will be referred to as Mixed Polarization MUSIC (MP-MUSIC) to distinguish it from the better known single-polarization formula in (2.20).

MP-MUSIC [24] forms the matrix

\[ M(\psi) = \begin{bmatrix} \hat{b}^H(\psi) P_N \hat{b}(\psi) & \hat{b}^H(\psi) P_N \hat{d}(\psi) \\ \hat{d}^H(\psi) P_N \hat{b}(\psi) & \hat{d}^H(\psi) P_N \hat{d}(\psi) \end{bmatrix}, \]  

(2.27)

where \( P_N = QQ^H \) is the projection matrix for the noise space of \( \tilde{R} \). Note that the array manifold vectors in \( M(\psi) \) have been normalized. Additionally, \( \hat{d}(\psi) \) is a unit vector constructed such that

\[ \text{Span}\{\hat{b}(\psi), \hat{d}(\psi)\} = \text{Span}\{\hat{b}(\psi), \hat{c}(\psi)\}, \]

(2.28)

and

\[ \langle \hat{b}(\psi), \hat{d}(\psi) \rangle = 0, \]

(2.29)

where \( \langle \cdot, \cdot \rangle \) indicates the inner product operation. Thus, \( \hat{d}(\psi) \) is the orthogonal complement of \( \hat{b}(\psi) \) within the vector subspace defined by \( \hat{b}(\psi) \) and \( \hat{c}(\psi) \). The \( \hat{b}(\psi) \) and \( \hat{d}(\psi) \) vectors can be found following the Gram-Schmidt process [65].

The MP-MUSIC spectrum is

\[ f_{MP}(\psi) = \frac{1}{\lambda_{\text{min}}(M(\psi))}. \]

(2.30)

Here \( \lambda_{\text{min}}(M(\psi)) \) indicates the minimum eigenvalue of \( M(\psi) \). Since \( M \) is a \( 2 \times 2 \) matrix, its eigenvalues can be calculated quickly and analytically.

The \( M \) angles at the maxima of (2.30) are the estimated DOAs of the emitters. In Appendix A, we show that the columns of \( M(\psi) \) are linearly dependent when \( \psi \) is the direction of an incident signal and thus that (2.30) becomes unbounded. We
also discuss why the $\hat{d}$ vector should be orthonormal to $\hat{b}$, namely, to avoid spurious peaks in the MP-MUSIC spectrum that do not correspond to a signal.

Looking again at (2.27), we see that Mixed Polarization MUSIC essentially calculates three single-polarization MUSIC spectra. The $\theta$-polarized MUSIC spectrum is $M_{11}$. The ‘complement’ spectrum is $M_{22}$. The third spectrum is $M_{12}$, which uses the $\theta$-polarized manifold vectors and the complement vectors. Its conjugate is $M_{21}$. Since $M(\psi)$ essentially contains three MUSIC spectra, MP-MUSIC requires about three times as many computations as single polarization MUSIC. Naturally, a calibrated array manifold must also be available for two orthogonal polarizations. It is also shown in Appendix A that MP-MUSIC may only estimate up to $N - 2$ DOAs, whereas single polarization MUSIC may estimate up to $N - 1$ DOAs. However, the ability to handle arbitrary polarization is critical when the transmitted polarization is unknown due to an uncooperative emitter or a time varying receiver orientation.

2.5.2 Simulation Examples for Mixed Polarization MUSIC

Fig. 2.15 shows the MUSIC and MP-MUSIC spectra for a case with a single incident signal. The spectra have not been normalized, so the MUSIC spectra measure orthogonality directly. The signal is $\phi$-polarized and has SNR of 10dB relative to a isotropic element. One thousand snapshots are used to estimate the covariance matrix. First, the RHCP MUSIC spectrum is calculated with the RCHP array manifold, as in Fig. 2.12. Even though the antenna is dominantly RHCP polarized near zenith, the peak in the MUSIC spectrum is less than 15dB, as compared to about 40dB when the signal was RHCP in Fig. 2.12. Thus, the antenna’s slight sensitivity
to LHCP waves affects the received samples for the $\phi$-polarized signal. Nevertheless, RHCP MUSIC still peaks at the true signal DOA in spite of the polarization mismatch between the incident signal and MUSIC’s test vectors.

Next, the MP-MUSIC spectrum was calculated. It is shown on the right in Fig. 2.15. Because MP-MUSIC handles arbitrary polarization, it does not suffer from polarization mismatch, and the spectrum peaks sharply. For easy comparison, zoomed in views of the spectra are shown in the middle plots, and the bottom plot shows both spectra along the elevation cut where $\phi = 0^\circ$ and $\phi = 180^\circ$. Fig. 2.15 shows that even with a well-designed antenna, polarization mismatch suppresses MUSIC’s peak. However, MP-MUSIC accounts for the array’s sensitivity to other polarizations and readily identifies the signal’s DOA.

Fig. 2.16 shows an example with two uncorrelated signals. A $\phi$-polarized signal arrives from $(\theta, \phi) (10^\circ, 0^\circ)$, and a $\theta$-polarized signal arrives from $(50^\circ, 0^\circ)$. As before, the signals have SNRs of 10dB at an isotropic element, and one thousand snapshots are used to estimate the covariance matrix. First, single polarization MUSIC was calculated using the RHCP array manifold. This is shown on the left. However, the maximum of the spectrum is at $(25^\circ, 354^\circ)$, about $15^\circ$ away from the nearest signal. The second highest peak is at $(50^\circ, 0^\circ)$, which is the DOA of the $\theta$-polarized signal. Thus, polarization mismatch can also bias the DOA estimates. MP-MUSIC, however, behaves as expected with sharp peaks at the true signal DOAs. This is shown on the right of Fig. 2.16. Finally, the bottom plot shows the spectra along $\phi = 0^\circ$ and $\phi = 180^\circ$. Clearly, neglecting polarization mismatches the true and assumed array manifolds, dampens the MUSIC spectrum, and may yield biased estimates.
Figure 2.15: MUSIC with one $\phi$-polarized signal from $(\theta, \phi)$ $(10^\circ, 0^\circ)$. Top left: Single polarization MUSIC with the RHCP array manifold, in dB. Top right: Mixed Polarization MUSIC (MP-MUSIC) spectrum, in dB. Middle: Zoomed in views around the DOAs. Bottom: Line plot along $\phi = 0^\circ$ and $\phi = 180^\circ$. 
Figure 2.16: MUSIC with a $\phi$-polarized signal at $(\theta, \phi) (10^\circ, 0^\circ)$ and a $\theta$-polarized signal at $(50^\circ, 0^\circ)$. Top left: Single polarization MUSIC with the RHCP array manifold, in dB. Top right: Mixed Polarization MUSIC (MP-MUSIC), in dB. Middle: Zoomed in views around the DOAs. Bottom: Line plot along $\phi = 0^\circ$ and $\phi = 180^\circ$. 
of emitter DOAs. However, the mixed polarization form of MUSIC accounts for arbitrary polarization and restores MUSIC’s expected performance for arbitrarily polarized signals.

2.6 Summary

This chapter overviewed the narrowband signal model, beamforming, and MUSIC. A seven-element antenna array was also introduced, which will be used for much of this dissertation. Simulations with the antenna array showed the conventional beamforming, Capon’s method (MVDR), and MUSIC can all locate a single signal. However, conventional beamforming’s resolution is similar to the classical array 3dB beamwidth, a little less than $\lambda/(D \cos \theta)$. This resolution is insufficient for many multiple signal scenarios. Adaptive beamforming (MVDR) and MUSIC yield higher resolution. However, MUSIC had a smaller beamwidth, and only MUSIC can handle signals with unknown polarization. Simulation examples showed that neglecting polarization effects suppresses the peaks of the MUSIC function, even at angles where the array is nominally only sensitive to one polarization. Moreover, with multiple incident signals, we found that MUSIC’s peaks may be significantly biased by polarization mismatch. Therefore, this dissertation will discuss both single polarization MUSIC as well as Mixed Polarization MUSIC, and these algorithms will be used as the basis for multiple emitter geolocation from a mobile platform. Next, we will study MUSIC’s theoretical resolution, its practical resolution, and modified MUSIC algorithms that enhance MUSIC’s resolution for an arbitrary antenna array.
Chapter 3: Modified MUSIC Algorithms for DOA Estimation of Multiple Signals

3.1 Introduction

The previous chapter introduced Direction of Arrival (DOA) estimation, which provides the foundation for radio-frequency emitter geolocation from a single platform. While single-platform geolocation could employ many DOA algorithms, Chapter 2 focused on beamforming and MUSIC [24]. These algorithms apply to arbitrary antenna arrays and can be calculated quickly. However, MUSIC is more versatile than beamforming since it easily handles arbitrary polarization. MUSIC also yielded the highest resolution DOA spectra in examples with the seven-element antenna array. Therefore, this dissertation will focus primarily on MUSIC-based DOA estimation and geolocation. Alternative DOA algorithms will only be discussed in comparison to the MUSIC-based algorithms.

Although MUSIC offers better DOA resolution than beamforming methods, MUSIC still may not resolve signals that are very close together if the Signal to Noise Ratio (SNR) is low or if the number of samples is small [19]. Since improved DOA resolution under these conditions should enhance emitter geolocation from a moving platform, this chapter studies modified MUSIC algorithms that outperform MUSIC
at small signal separations. The study includes several such algorithms that have already been reported in open literature. The chapter also introduces a new modification to the MUSIC algorithm. In Monte Carlo simulations, the new approach performs about as well as the best existing method with a dominantly polarized array. Moreover, it significantly outperforms the best existing method for diversely polarized signals that are incident on a polarization sensitive array. The new algorithm also retains MUSIC’s advantages as a fast and general algorithm. We refer to the new algorithm as Nullspace MUSIC.

Section 3.2 discusses the limit on DOA estimation accuracy, the Cramer Rao Lower Bound (CRLB), as well as MUSIC’s theoretical estimate variance. Theoretically, MUSIC’s variance should match the CRLB for large separations, and MUSIC should still be able to resolve signals that are close together. Monte Carlo simulations in Section 3.3 show that single-polarization MUSIC performs as expected with two well-separated signals, but it does not meet its theoretical performance if the signals are very close together or arrive with low SNR. Other authors have noted this behavior, and current algorithms that improve MUSIC’s resolution are discussed in Section 3.4. Two of these algorithms, Sequential MUSIC and Pole-Zero (PZ)-MUSIC, are based on MUSIC and can be applied to an arbitrary array. Moreover, Monte Carlo simulations show that they significantly improve MUSIC’s resolution.

Section 3.5 introduces the new MUSIC algorithm, Nullspace MUSIC. Like the current modified MUSIC algorithms, Nullspace MUSIC also improves MUSIC’s resolution in Monte Carlo simulations with closely spaced signals and low SNR. Moreover, Nullspace MUSIC improves significantly on Sequential MUSIC by properly exploiting MUSIC’s key orthogonality relationship, which Sequential MUSIC neglects (Appendix
Next, Section 3.6 extends Pole-Zero and Nullspace MUSIC to mixed polarization scenarios so that they retain the versatility of Mixed Polarization (MP)-MUSIC. In Monte Carlo trials with the seven element-array, Nullspace MUSIC outperforms MP-MUSIC and PZ-MUSIC. Using one thousand snapshots, Nullspace MUSIC consistently resolves signals with different polarizations that have 10dB SNR and are only 0.75° apart. PZ-MUSIC cannot consistently resolve the same signals until they are 1.5° apart, and MP-MUSIC cannot consistently resolve the signals until they are 2.0° apart. Thus Nullspace MUSIC significantly improves MUSIC-based DOA resolution relative to state-of-the-art.

Section 3.7 compares and contrasts Nullspace MUSIC to existing approaches for enhanced resolution that were not covered earlier. Due to their limitations or poor performance when tested using the seven-element array, these additional methods are not considered further. Finally, Section 3.8 summarizes the chapter. While this chapter shows that modified MUSIC algorithms enhance MUSIC’s resolution with a perfectly known antenna array manifold, the next chapter will examine their performance with an imperfectly known antenna array manifold.

3.2 Cramer Rao Lower Bound and MUSIC Variance

The minimum variance for an unbiased estimator is given by the Cramer Rao Lower Bound (CRLB) [69]. While discussing the CRLB, we assume the narrowband signal model with uncorrelated signals and equal-power, uncorrelated, additive white Gaussian noise, which was discussed in Chapter 2. In addition, we assume that the number of incident signals is less than the number of antenna elements in the receive
array, and we assume that we are searching for DOAs along the $\theta$ angle only (the signals are incident along a known elevation cut).

Under these assumptions, the CRLB for the estimated DOAs depends on the array’s antenna elements and layout, the signal directions and strengths, the receiver noise, and the number of samples that are used to estimate the covariance matrix. The CRLB for the $m$th signal has been derived as [66, 18]

$$
\sigma_{CR}^2(\theta_m) = \frac{1}{2K\text{SNR}_m} \frac{1}{h(\Theta, \theta_m)}. \tag{3.1}
$$

Here, $\theta_m$ is the direction of the $m$th signal, $K$ is the number of snapshots collected by the receiver, and $\text{SNR}_m$ is the ratio of the $m$th signal’s power to the noise variance. Also, $\Theta$ is a vector of the $M$ incident signal’s DOAs, and $h(\Theta, \theta_m)$ is the square of the norm of the manifold derivative’s projection into the noise subspace at $\theta_m$. It is given by

$$
h(\Theta, \theta_m) = \delta^H(\theta_m)[I - A(\Theta)(A^H(\Theta)A(\Theta))^{-1}A^H(\Theta)]\delta(\theta_m)
= \delta^H(\theta_m)P_N\delta(\theta_m)
= \|P_N\delta(\theta_m)\|^2, \tag{3.2}
$$

where $\delta(\theta_m)$ is the derivative of the manifold with respect to $\theta$ at $\theta_m$; $I$ is the identity matrix; $A(\Theta) = [a(\theta_1) \ldots a(\theta_M)]$ is the antenna manifold matrix defined in (2.5); and $\|\cdot\|$ is the Euclidean norm. Note that the matrix $[I - A(\Theta)(A^H(\Theta)A(\Theta))^{-1}A^H(\Theta)]$ is better recognized as the projection matrix for the noise space, $P_N$, from Chapter 2. Recall that the signal subspace is the vector space with the same span as $A(\Theta)$, and the noise subspace is its orthogonal complement.

From (3.1), we see that the CRLB is inversely proportional to $h(\Theta, \theta_m)$. As the derivative of the array manifold becomes more orthogonal to the signal subspace, $h(\Theta, \theta_m)$ grows and the CRLB shrinks. If the derivative of the array manifold becomes
more parallel to the signal subspace, then \( h(\Theta, \theta_m) \) shrinks and the CRLB increases. Therefore, we can intuitively understand the CRLB as being inversely related to the similarity between the manifold derivative and the signal subspace. For now, let us assume that the antenna array is dominantly polarized and that the incident signals have the same polarization. If \( \theta_1 \approx \theta_2 \), then \( \delta(\theta_1) \) is approximately proportional to \( a(\theta_2) - a(\theta_1) \). In this case, \( \delta(\theta_1) \) is almost a linear combination of \( a(\theta_1) \) and \( a(\theta_2) \), and it is nearly in the signal subspace. Thus, its projection into the noise subspace must be relatively small, and the CRLB must be relatively large. Therefore, the CRLB’s inverse dependence on \( h(\Theta, \theta_m) \) gives the intuitive result that closely spaced signals are more difficult to estimate than physically well separated signals.

Another simple result is that increased antenna gain improves DOA estimation accuracy. Note that the manifold and its derivative scale with the square root of the gain of the antenna elements [67]. Therefore, the CRLB is inversely proportional to the gains of the antenna elements in a particular direction. Of course, (3.1) additionally shows that the CRLB is inversely related to SNR and the number of samples that are used to estimate the covariance matrix.

Theoretically, MUSIC is unbiased whether estimating a single signal or multiple signals [66]. In addition, the estimates should have a jointly Gaussian error distribution. The theoretical variance of single polarization MUSIC for one or many signals is given in [66] as

\[
\sigma_{\text{MU}}^2(\theta_m) = \frac{1}{2K\text{SNR}_m} \left[ 1 + \left[ (A^H(\Theta)A(\Theta))^{-1}\right]_{ii} \right] \frac{1}{h(\Theta, \theta_m)} \\
= \sigma_{\text{CR}}^2(\theta_m) \left[ 1 + \frac{[(A^H(\Theta)A(\Theta))^{-1}]_{ii}}{\text{SNR}_m} \right]. \tag{3.3}
\]
Note that subscript $\hat{i}$ refers to the $i$th element of the diagonal of the matrix. Also, (3.3) applies only if the vectors $a(\theta_1) \ldots a(\theta_M)$ are linearly independent and the number of snapshots exceeds the number of antenna elements in the receive array. These conditions are needed to ensure that the covariance matrix is not singular, which was one of the requirements for deriving (3.3) [66].

Equation (3.3) has two interesting qualities. First, since $A^H A$ is a positive-definite matrix, its inverse is also positive-definite. The diagonal elements of a positive-definite matrix are real and greater than zero. It follows that $[(A^H A)^{-1}]_{mm} > 0$ and $\sigma_{MU}^2(\theta_m) > \sigma_{CR}^2(\theta_m)$, as expected. Second, (3.3) also shows that MUSIC’s variance approaches the CRLB as the signal’s SNR approaches infinity. Recall that the CRLB itself approaches zero as SNR approaches infinity. Taken together, the last two statements imply that MUSIC should resolve arbitrarily close signals if they are uncorrelated and have sufficient SNR. On the other hand, (3.3) predicts that MUSIC will fall farther short of the CRLB with low SNR, and the CRLB itself increases with low SNR. Therefore, we should assess MUSIC and any modified MUSIC algorithms with high SNR and low SNR when characterizing their performance.

3.3 CRLB and MUSIC Variance for the Seven-Element Antenna Array

We now evaluate how well MUSIC performs with the antenna introduced in Chapter 2. The evaluation compares the CRLB and the theoretical MUSIC variance versus angular separation. Two uncorrelated RHCP signals arrive along the pitch cut of the antenna, ($\phi = 0^\circ$ and $\phi = 180^\circ$ in Fig. 2.1), and the receiver estimates DOAs along the same cut. We assume that the receiver collects one thousand snapshots to estimate the DOAs of two signals. One signal always arrives from $\theta_1 = 0^\circ$. The DOA
of the other signal ($\theta_2$) is varied from $1^\circ$ to $70^\circ$. The CRLB and MUSIC variance are calculated at $\theta_1$ and $\theta_2$ for each $\theta_2$. Note from (3.2) that $h(\Theta, \theta_m)$ depends on $\theta_2$ through $A(\Theta)$. Therefore, the CRLB and MUSIC variance at $\theta_1 = 0^\circ$ depend on $\theta_2$, and they are shown as functions of $\theta_2$. Naturally, $\sigma_{CR}^2(\theta_2)$ is also a function of $\theta_2$.

To evaluate (3.1) and (3.3), the antenna’s Right Hand Circularly Polarized (RHCP) manifold was output from HFSS with a resolution of 0.01°. The fine discretization enables accurate numerical calculation of the derivative in (3.2). The gains of the seven antenna elements are shown along the pitch cut in Fig. 3.1. Due to mutual coupling and unique scattering effects from the edge of the 4’ ground plane, each antenna element has a unique gain pattern.

First, we assume that both signals arrive with 10dB SNR. Fig. 3.2 shows the CRLB and MUSIC variance as functions of the second signal’s DOA. The left plot shows $\sigma_{CR}^2(\theta_1)$ and $\sigma_{MU}^2(\theta_1)$; the right plot shows $\sigma_{CR}^2(\theta_2)$ and $\sigma_{MU}^2(\theta_2)$; and both

---

Figure 3.1: RHCP antenna gains along the pitch cut of the seven-element antenna array. The various curves are for the different antenna elements in the array.
plots use a logarithmic $y$-axis. Note that the lines for the MUSIC variance and CRLB almost overlap. Additionally, the left plot shows that the first signal’s CRLB is lowest when the second signal arrives from far away. As expected, the CRLB generally increases as the second signal is brought closer. The right plot shows that the second signal’s CRLB starts high when the signal is near horizon, (the array has less effective aperture and less gain). The CRLB generally decreases for smaller $\theta_2$, but it increases around $\theta_2 = 35^\circ$ before decreasing again. This non-monotonic behavior depends on the signal subspace and the manifold derivative at $\theta_2$, as discussed in the previous section. As expected, however, the CRLB rises sharply when the separation between $\theta_2$ and $\theta_1$ is very small. Recall that in this case the manifold derivatives become linear combinations of the manifold vectors at $\theta_2$ and $\theta_1$. Nevertheless, for this example with good SNR and many samples, we see that MUSIC should be very close to optimum for uncorrelated signals until the signal separation is very small (less than $5^\circ$ or $1/10$th of the array’s beamwidth).

Fig. 3.3 compares the Cramer Rao Lower Bound and MUSIC variance for two RHCP signals with 0dB SNR. The other parameters are the same as before. The plots show $\sigma^2_{CR}$ and $\sigma^2_{MU}$ for $\theta_1$ and $\theta_2$. As before, MUSIC tracks the CRLB when the signals are well separated. As compared to the 10dB case, the low SNR raises the minimum CRLB and MUSIC variance at any separation from about 0.003 degrees squared to about 0.03 degrees squared. MUSIC’s expected variance also increases to over 100 degrees squared for separations of $1^\circ$! MUSIC’s variance noticeably exceeds the CRLB for signal separations less than $20^\circ$ (about $0.4$ beamwidths). As expected from (3.3), MUSIC has higher variance for lower signal to noise ratios, especially at small separations.
Even though Figs. 3.2 and 3.3 show that MUSIC departs from the CRLB at small separations and low SNR, MUSIC should still be very effective at resolving nearby signals. For instance, if two 0dB signals arrive from 0° and 10°, then the variances in the DOA estimates are only 0.33 degrees squared for $\theta_1 = 0^\circ$ and 0.90 degrees squared for $\theta_2 = 10^\circ$. If the DOA errors are zero-mean and normally distributed, then 99.7% of the estimates for $\theta_1$ should fall between $-1.72^\circ \leq \theta \leq 1.72^\circ$, and 99.7% of the estimates for $\theta_2$ should fall between $7.15^\circ \leq \theta \leq 12.85^\circ$. Note that 99.7% of estimates corresponds to all estimates within three standard deviations of the the mean. The non-overlapping ranges of the theoretical DOA estimates implies that MUSIC, using one thousand snapshots, should consistently resolve 0dB RHCP signals that arrive from 0° and 10°. Next, we will study whether this is the case in practice.
Figure 3.3: CRLB and theoretical MUSIC variance along the pitch cut of the seven-element antenna with one signal at $\theta = 0^\circ$ and a second signal at $\theta = \theta_2$, (0dB SNR 1000 snapshots). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$.

3.3.1 Monte Carlo Simulations

MUSIC’s practical performance was assessed with Monte Carlo trials. These trials used the same parameters as the theoretical studies (seven-element antenna, one thousand samples, RHCP signals). As in the theoretical calculations, one signal arrived from $0^\circ$, and the other signal arrived from $\theta_2$. The trials were repeated 20,000 times for each $\theta_2$ so that MUSIC’s error variance could be calculated. Two hundred eighty signal separations spanning $0.25^\circ$ to $70^\circ$ in $0.25^\circ$ increments were considered. Thus, 5.6 million trials were run for each Monte Carlo simulation.

Each trial was executed as follows. First, independent signal and noise samples were generated from the complex Gaussian distribution [59]. The signal samples were scaled according to a set SNR. Then, the snapshot matrix, $X$, and the sample covariance matrix were calculated according to (2.6) and (2.9). Next, the single polarization MUSIC spectrum was calculated along the pitch cut for $-90^\circ \leq \theta \leq 90^\circ$.
according to (2.20). The number of incident signals was assumed to be known when calculating the MUSIC spectrum. The spectrum was searched to find all peaks, where a peak is defined as a point higher than its two neighbors. The highest two peaks were selected as MUSIC’s estimates of $\theta_1$ and $\theta_2$. Then, the peaks were associated with the true DOAs such that the mean squared DOA errors were minimized. Minimizing the mean squared error keeps the error variance as small as possible, and it is the best method to use for performance comparisons with the Cramer Rao Lower Bound. Throughout this chapter, peaks will always be associated with the true DOAs such that the mean squared error is minimized.

After all trials for a given separation completed, MUSIC’s error variance for a fixed $\theta_2$ was calculated using the errors for the corresponding 20,000 trials. Let the number of trials be $T$, and let MUSIC’s estimate of $\theta_m$ for trial $t$ be denoted as $\theta_{m,t}$, (a tilde denotes an estimated value). Then, MUSIC’s error variance in the Monte Carlo trials is given by

$$\tilde{\sigma}^2_{\text{MU}}(\theta_m) = \frac{1}{T} \sum_{t=1}^{T} (\theta_{m,t} - \bar{\theta}_m)^2.$$  (3.4)

Here, $\bar{\theta}_m$ is the mean estimate,

$$\bar{\theta}_m = \frac{1}{T} \sum_{t=1}^{T} \theta_{m,t}.  \quad (3.5)$$

Then, the mean error, (bias), is simply

$$\bar{\varepsilon}_m = \bar{\theta}_m - \theta_m. \quad (3.6)$$

The first set of Monte Carlo trials assumes that the signals arrive with 10dB SNR. The top plots in Fig. 3.4 overlay the Monte Carlo variances, $\tilde{\sigma}^2_{\text{MU}}(\theta_m)$, onto $\sigma^2_{\text{CR}}(\theta_m)$ and $\sigma^2_{\text{MU}}(\theta_m)$, which were shown in Fig. 3.2. The simulated MUSIC results closely
track $\sigma_{MU}^2(\theta_m)$ when $\theta_2$ is greater than about $5^\circ$. This suggests that the simulations faithfully match the narrowband signal model for which (3.1) and (3.3) were derived. At smaller separations, however, MUSIC’s error variance suddenly rises in the Monte Carlo trials. The middle plots show this phenomenon in greater detail - they plot the same variances on over the range $0^\circ \leq \theta_2 \leq 10^\circ$. The bottom plots show the mean error over the same $x$-scale. While MUSIC is theoretically unbiased, the Monte Carlo results exhibit large biases for both signals with $\theta_2 \leq 3.75^\circ$. As expected, however, MUSIC is unbiased over the range $10^\circ \leq \theta_2 \leq 70^\circ$, (not shown).

Fig. 3.5 overlays the Monte Carlo results for 0dB SNR and one thousand samples onto the theoretical results from Fig. 3.3. As in the 10dB case, the Monte Carlo results closely track $\sigma_{CR}^2(\theta_m)$ and $\sigma_{MU}^2(\theta_m)$ at large separations. However, MUSIC’s performance drops sharply at about seventeen degrees. This occurred before MUSIC’s theoretical variance starts increasing sharply at about five degrees separation. Since MUSIC’s Monte Carlo performance falls well short of MUSIC’s theoretical performance, we must determine if the implementation is flawed or if MUSIC inherently diverges from its theoretical variance.

MUSIC’s departure from its expected behavior can be explained by looking at individual MUSIC spectra. Fig. 3.6 shows a MUSIC spectrum for two signals incident from $0^\circ$ and $4^\circ$ with 10dB SNR and one thousand snapshots. The other parameters are the same as in Fig. 3.4. For these snapshots, the MUSIC spectrum contains two peaks, and the DOA estimates have small errors. This is consistent with the performance predicted by (3.3). However, Fig. 3.7 shows a MUSIC spectrum for the same signal scenario, but with different snapshots generated from the same probability distribution. In this case, the spectrum contains a high peak in between the true
Figure 3.4: Cramer Rao Lower Bound and RHCP MUSIC variance in Monte Carlo trials (10dB SNR, 1000 samples). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$. Top: Variance for full $x$-scale. Middle: Variance for zoomed $x$-scale. Bottom: Bias for zoomed $x$-scale.
Figure 3.5: Cramer Rao Lower Bound and RHCP MUSIC variance in Monte Carlo trials (0dB SNR, 1000 samples). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$. Top: Variance for full $x$-scale. Middle: Variance for zoomed $x$-scale. Bottom: Bias for zoomed $x$-scale.
Figure 3.6: RHCP MUSIC spectrum versus $\theta$ with 1000 samples and 10dB SNR. The true DOAs are shown with dashed vertical lines.

Figure 3.7: RHCP MUSIC spectrum versus $\theta$ with a different set of 1000 samples and 10dB SNR. The true DOAs are shown with dashed vertical lines.
DOAs. The next highest peak in the spectrum is about nineteen degrees away from the nearest signal DOA. Since the expected variance for $\theta_1 = 0^\circ$ is only 0.064 degrees squared, errors approaching twenty degrees in some trials dramatically increase the observed variance above the level predicted by (3.3).

Figs. 3.8 and 3.9 show MUSIC spectra for two signals incident from 0$^\circ$ and 10$^\circ$ with 0dB SNR and one thousand snapshots. The other parameters are the same as in Fig. 3.5. As in the 10dB case, the signals are resolved for the first set of snapshots, but the signals are not resolved for the second set of snapshots. With the second set of snapshots, the second highest peak is also about nineteen degrees from $\theta_1$. Errors such as this one are much greater than predicted by (3.3), which yields an expected variance of 0.33 degrees squared.

This section showed that MUSIC should theoretically resolve signals that are very close together. However, Monte Carlo trials of MUSIC with the seven-element antenna array did not yield the predicted performance. Individual examples showed that MUSIC fails because the two peaks merge, and MUSIC must select a very poor peak as the second signal’s location. Note that the CRLB and theoretical MUSIC variance only consider the manifold and its derivative at the true signal DOAs. However, the behavior of the array manifold in-between the true signal DOAs significantly affects whether the MUSIC spectrum will contain a ‘dip’ in-between the DOAs, which would allow the signals to be resolved. The behavior of the manifold in-between the signals is the root cause for the discrepancy between the predicted and observed MUSIC performance. Therefore, the next section will study possible modifications to MUSIC that enhance MUSIC’s resolution and reduce the error variance in practice.
Figure 3.8: RHCP MUSIC spectrum versus $\theta$ with 1000 samples and 0dB SNR. The true DOAs are shown with dashed vertical lines.

Figure 3.9: RHCP MUSIC spectrum versus $\theta$ with a different set of 1000 samples and 0dB SNR. The true DOAs are shown with dashed vertical lines.
3.4 Modified MUSIC Algorithms for Closely Spaced Signals

We have seen that the peaks of the MUSIC spectrum may merge if closely spaced signals arrive with low SNR. Following the usual procedure and selecting the two highest peaks as the estimated DOAs yielded much higher DOA variance than predicted by theory. This problem was recognized almost as soon as researchers began studying MUSIC [35]. Therefore, this section discusses several modifications to MUSIC that should improve MUSIC’s resolution and accuracy.

3.4.1 Modified Peak Selection

Figs. 3.6-3.9 suggest that MUSIC’s performance shortfall may have an obvious solution. While the canonical MUSIC algorithm [24] states that we should “pick $M$ peaks of $P_{MU}(\theta)$,” where $M$ is the number of signals and $P_{MU}$ is the MUSIC spectrum, perhaps we should nevertheless pick the DOAs more carefully.

The simplest approach may be to decide that if the peaks merge, then the merged peak’s $\theta$ is the DOA estimate for both signals. Fig. 3.10 overlays the Monte Carlo results using this rule onto the theoretical variances for 0dB signals with small separations. If either DOA error is greater than the separation between the signals, then the highest peak in the MUSIC spectrum is used for both DOAs. As before, 20,000 trials were run for each $\theta_2$ with independent samples and MUSIC’s variance was calculated according to (3.4). The top plots show the variances of the estimates at $\theta_1$ and $\theta_2$. In this case, MUSIC’s variance still increases quickly at $\theta_2 = 17^\circ$, and it exceeds the level predicted by (3.3). However, the variance peaks between $7^\circ$ and $11^\circ$, and it decreases as the separation approaches zero.
Figure 3.10: RHCP MUSIC error in Monte Carlo trials (0dB SNR, 1000 samples) with the peak combination rule. Top Left: Variance for the signal at $\theta = 0^\circ$. Top Right: Variance for the signal at $\theta = \theta_2$. Bottom: Bias for both signals.
The estimator’s variance can be explained as follows. If the signals are separated by 16°, a merged peak will yield an average error of 8° for each signal. If some trials yield merged peaks and others do not, then some trials have errors that are only a fraction of a degree, while others have errors of several degrees. This yields a variance that exceeds the predicted variances of about 0.4 and 0.1 degrees squared. As the signals are brought even closer together, their peaks merge in almost 100% of the trials. When the error is always about the same, its variance drops. Indeed, the top plots show that MUSIC, (with this rule), appears to yield variances less than the Cramer Rao Lower Bound. However, the bottom plot shows the biases for the two signals in the Monte Carlo trials. As the signals are brought closer together, the ensemble means become more biased towards each other. At approximately 5.5° separation or less, the biases are about ±θ^2/2, and the variance is near zero. Therefore, this estimator’s variance should not be compared to the CRLB, which only applies for unbiased estimators. Since this estimator yields large biases and variances, it is an inadequate solution to the problem of merged peaks. Therefore, we will study algorithms that attempt to resolve signals when MUSIC’s peaks have merged into a single peak.

3.4.2 CLEAN Algorithm for MUSIC

One approach to resolving signals is based on CLEAN, a Radio Astronomy algorithm that removes sidelobes from interferometric images [70]. CLEAN has been applied to MUSIC with the goal of resolving closely spaced sources that yield a single peak in the MUSIC spectrum [71]. The overall idea behind CLEAN-MUSIC is to form linear combinations of a high resolution MUSIC spectrum and the array’s beam
pattern. CLEAN-MUSIC starts with a MUSIC spectrum given by
\[
f_{CL}(\theta) = 1 - \|P_N \hat{a}(\theta)\|^2
\]
(3.7)
\[
f_{CL}(\theta) = 1 - f_{MU}^{-1}(\theta).
\]
where \(f_{MU}(\theta)\) was given in (2.21). Recall that \(P_N\) is the projection matrix for the noise space estimated from the covariance matrix; \(\hat{a}(\theta)\) is the normalized steering vector at angle \(\theta\); and a hat indicates that the vector is unit norm. This spectrum has the same resolution as \(f_{MU}(\theta)\) in (2.21), but much shallower peaks. The peaks are at the same DOAs as in (2.21).

CLEAN-MUSIC estimates the first DOA from the peak of (3.7). Let the estimated DOA be \(\tilde{\theta}_1\). Then, CLEAN-MUSIC subtracts a scaled and shifted copy of the beampattern of the array at zenith. The shift centers the beampattern around the \(\tilde{\theta}_1\). The shifted beampattern is termed the ‘dirty beam,’ and it is given by
\[
f_{db}(\theta, \tilde{\theta}_1) = \delta |\hat{a}^H(0^\circ)\hat{a}(\theta - \tilde{\theta}_1)|^2,
\]
(3.8)
where \(\delta\) is a ‘gain’ that the authors recommend setting in the range of 0.05 and 0.5. Note that the beampattern is precalculated and does not depend on the incident signals. With \(\delta = 1\), (3.8) is also equivalent to (3.7) with a signal at zenith and no added noise.

The subtraction may be repeated \(p\) times until a peak appears. Then, the second signal’s DOA is estimated from the peak of
\[
g_{CL}(\theta, \tilde{\theta}_1) = 1 - \|P_N \hat{a}(\theta)\|^2 - pf_{db}(\theta, \tilde{\theta}_1).
\]
(3.9)
where \(p\) is the number of iterations applied in CLEAN-MUSIC.

Equation (3.9) shows that CLEAN-MUSIC relies on a dubious principle. CLEAN-MUSIC claims to resolve signals with a linear combination of two spectra - MUSIC
for two signals and MUSIC for one signal at zenith (i.e. the beampattern) - that cannot individually resolve the signals. Recall from Chapter 2 that our antenna has a half power beamwidth at zenith of about 50°, while Fig. 3.4 shows that MUSIC has trouble resolving signals that are separated by 5° when the SNR is 10dB. It may seem that adding or subtracting a 50° wide beam would not help narrow the peaks around two signals down to less than 5°. Nevertheless, if CLEAN’s MUSIC spectrum (3.7) around two signals is ‘lopsided,’ (meaning that it decreases more rapidly on one side than on the other), then CLEAN can force a second peak into the MUSIC spectrum by subtracting the array’s beam pattern in small steps. Assuming that the antenna has a symmetrical beam pattern at zenith, the peak of (3.9) should be on the side of the spectrum that decreases more slowly. Thus, CLEAN-MUSIC is the numerical implementation of the intuition that the second signal is most likely on the side of spectrum that decreases more slowly. Although CLEAN forms clear images for astronomy, it will not be considered further in this dissertation for direction finding.

3.4.3 Root MUSIC

Root MUSIC provides a very different approach to improving MUSIC’s resolution [35, 72]. Root MUSIC analytically finds the poles of the MUSIC function without searching the MUSIC spectrum, (2.20). Root MUSIC often yields $M$ DOA estimates even when spectral MUSIC (2.20) does not have $M$ peaks. Root MUSIC also ignores some of the noise that affects the MUSIC spectrum [72]. However, Root MUSIC’s analytical and search free approach depends on the ideal manifold of a uniform linear antenna array. While Root MUSIC has been adapted to Uniform Circular Arrays [73], this still does not incorporate an arbitrary array manifold with mutual coupling.
and scattering effects. Although the elements in the antenna array shown earlier are spaced 0.47\(\lambda\) apart and are mounted on a circular ground plane, Fig. 3.1 showed that the antenna array has unique element patterns. We will see in later chapters that mounting the array on a platform leads to very strong perturbations of the antenna patterns. Therefore, Root MUSIC improves MUSIC’s resolution, but it is not applicable in many cases of interest for which an array is mounted on a host vehicle.

### 3.4.4 Pole-Zero MUSIC

Barabell, Root MUSIC’s author, also set out an effective approach that can be applied to arbitrary array manifolds [35]. Suppose that two signals are incident on the array. MUSIC presumes that the norm squared of the projection of the array manifold vectors into the noise space, \(\|P_N\hat{a}(\theta)\|^2\), is approximately zero at \(\theta_1\) and \(\theta_2\). Thus, the MUSIC spectrum

\[
f_{MU}(\theta) = \frac{1}{\|P_N\hat{a}(\theta)\|^2}
\]  

(3.10)

has (approximate) poles at \(\theta_1\) and \(\theta_2\). But, Barabell noted that underestimating the number of signals by one often causes MUSIC to have an (approximate) pole in-between two signals. In other words, if \(P_N^+\) is the projection matrix for the noise space plus the eigenvector corresponding to the smallest signal space eigenvalue, then \(\|P_N^+a(\theta)\|^2\) is often minimized in-between the two signal DOAs. Therefore, Barabell inserts a ‘dip’ into the MUSIC spectrum by multiplying (3.10) and \(\|P_N^+a(\theta)\|^2\). For two signals, he proposes the cost function,

\[
f_{PZ}(\theta) = \frac{\|P_N^+\hat{a}(\theta)\|^2}{\|P_N\hat{a}(\theta)\|^2}\cdot
\]  

(3.11)
Ideally, this function has poles at $\theta_1$ and $\theta_2$ and an (approximate) zero in-between. The exponent $r$ can be adjusted to set the multiplicity of the poles of the function (and the prominence of the peaks). Moreover, this approach can be applied for more than two signals [35]. Since Barabell’s modified MUSIC function deliberately interlaces poles and zeros in a rational function to reveal closely spaced emitters, we will refer to it as *Pole-Zero MUSIC* or *PZ-MUSIC*. Recall that MUSIC fails to resolve two signals because of the height of the MUSIC spectrum in-between two signals. Therefore, PZ-MUSIC attempts to force a zero into the spectrum in-between the signals while still maintaining poles at the true DOAs.

### 3.4.5 Sequential MUSIC

Another approach called *Sequential MUSIC* is also reported in open literature [36]. Sequential MUSIC addresses MUSIC’s failure to resolve nearby signals by estimating DOAs from multiple spectra. In other words, since finding two peaks in one function may fail because the peaks have merged, why not generate a separate spectrum for the second signal? Removing the first signal and generating a second spectrum for the second signal would eliminate the problematic ‘in-between’ region that is ignored by the Cramer Rao Lower Bound and the theoretical MUSIC variance derivations.

Sequential MUSIC follows this line of attack and starts by generating the usual MUSIC spectrum, (3.10). The highest peak in the spectrum is taken as the estimate for one of the signal DOAs. Then, Sequential MUSIC places a null at this DOA and looks for the next signal. This should allow the second signal’s DOA to be estimated with minimal interference from the first signal. Naturally, this approach can be applied to $M$ signals by progressively placing $M - 1$ nulls. Since a single
DOA is estimated from each MUSIC spectrum, Sequential MUSIC always yields $M$ DOA estimates. Sequential MUSIC also eliminates the question of which peaks are the ‘best’ peaks in a spectrum containing more than $M$ peaks.

Sequential MUSIC follows these steps. Let the absolute maximum of the MUSIC spectrum be at $\tilde{\theta}_1$. Then, calculate a set of test vectors that are orthogonal to the steering vector at $\tilde{\theta}_1$. These residual steering vectors are given by

\[
r(\theta, \tilde{\theta}_1) = [I - \hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)]a(\theta),
\]

and the normalized residual steering vectors are

\[
\hat{r}(\theta, \tilde{\theta}_1) = \frac{r(\theta, \tilde{\theta}_1)}{\|r(\theta, \tilde{\theta}_1)\|}. \tag{3.13}
\]

With two incident signals, Sequential MUSIC estimates one angle from the standard MUSIC spectrum. Then, the second signal’s DOA is estimated from the absolute maximum of the second spectrum,

\[
f_{\text{SEQ,2}}(\theta) = \frac{1}{\|\mathbf{P}_N\hat{r}(\theta, \tilde{\theta}_1)\|^2}. \tag{3.14}
\]

If additional signals are present, the manifold vectors for their estimated DOAs may be added to the antenna matrix in (3.14). In other words, $\hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)$ is replaced by $[\mathbf{A} \mathbf{A}^H]$, where $\mathbf{A} = [\hat{a}(\tilde{\theta}_1), \ldots, \hat{a}(\tilde{\theta}_m)]$. Then, another spectrum is generated and the next DOA is estimated. Sequential MUSIC stops when all $M$ DOAs have been estimated.

**3.4.6 Monte Carlo Trials with Sequential MUSIC and PZ-MUSIC**

We now study PZ-MUSIC and Sequential MUSIC in Monte Carlo simulations with 20,000 trials per separation. Fig. 3.11 considers uncorrelated signals with SNRs
of 10dB, and the setup is the same as in Fig. 3.4. The left plots are for the signal at $0^\circ$, and the right plots are for the signal at $\theta_2$. The top plots show variances over the entire range of $\theta_2$, while the middle plots zoom in to $0^\circ \leq \theta_2 \leq 10^\circ$. The CRLB has been dropped from these plots to make room for the new algorithms. Finally, the bottom plots show the biases on the zoomed scale. We see that PZ-MUSIC and sequential MUSIC perform nearly the same as MUSIC for separations larger than $5^\circ$. For separations less than $5^\circ$, Sequential and PZ-MUSIC have significantly lower variance than MUSIC. In addition, both methods are less biased than MUSIC. Note that MUSIC’s large bias results from selecting merged peaks and low peaks around $\theta = -20^\circ$ when the peaks had merged. PZ-MUSIC yields the lowest variance for both signals. For separations greater than about $3.5^\circ$ (such that signals are usually resolved by both methods), PZ-MUSIC also yields the lowest bias.

Another figure of merit is the minimum separation at which the algorithms can resolve two signals. Let us assume that DOA errors have Gaussian distributions around their mean until the peaks start merging. Note that MUSIC’s estimation errors for multiple signals have been analyzed and found to be jointly Gaussian distributed with zero mean [66]. Then, let us then say that the signals are consistently resolved if

$$\bar{\theta}_1 + 3\sigma(\theta_1) < \bar{\theta}_2 - 3\sigma(\theta_2)$$

Here, $\bar{\theta}_m$ is the mean estimate for $\theta_m$, and $\sigma(\theta_m)$ is the estimate’s standard deviation for signal $m$. By comparing the mean DOA estimates plus/minus three standard deviations, we assert that the signals are resolved if 99.7% of estimates for $\theta_1 = 0^\circ$ are less than 99.7% of estimates for $\theta_2$.

Table 3.1 shows the minimum separation that each algorithm resolved according to this definition. Pole-Zero MUSIC improved MUSIC’s resolution by 41% in the
Figure 3.11: Error variances and means in Monte Carlo trials with RHCP signals (10dB SNR, 1000 samples). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$. Top: Variance, full $x$-scale. Middle: Variance, zoomed $x$-scale. Bottom: Bias, zoomed $x$-scale.
Table 3.1: Minimum separation for signals to be resolved according to (3.15)

<table>
<thead>
<tr>
<th>Algorithm/SNR</th>
<th>10dB</th>
<th>0dB</th>
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<tbody>
<tr>
<td>Monte Carlo MUSIC</td>
<td>4.25°</td>
<td>12.0°</td>
</tr>
<tr>
<td>Monte Carlo Sequential MUSIC</td>
<td>3.5°</td>
<td>9.5°</td>
</tr>
<tr>
<td>Monte Carlo Pole-Zero MUSIC</td>
<td>2.5°</td>
<td>7.25°</td>
</tr>
</tbody>
</table>

Monte Carlo trials with 10dB RHCP signals, and it improved MUSIC’s resolution by 40% with 0dB RHCP signals. Sequential MUSIC improved MUSIC’s resolution by 18% and 21% in the 10dB and 0dB cases, respectively.

Fig. 3.12 shows Monte Carlo results with signal SNRs of 0dB. PZ-MUSIC and Sequential MUSIC perform the same as MUSIC for separations larger than about 17°. For separations less than 13°, Sequential and PZ-MUSIC yield significantly lower variance than MUSIC. In addition, both methods are unbiased for large signal separations, and they are less biased than MUSIC for small signal separations. Both methods reduce the minimum resolvable separation between the signals, though PZ-MUSIC yields the least variance and best resolution, (Table 3.1). For separations greater than 9.5° (such that both signals are resolved by both methods), PZ-MUSIC also yields the lowest bias for both signals.

From Figs. 3.11 and 3.12, we have seen that PZ-MUSIC and Sequential MUSIC significantly improve MUSIC’s bias, error variance, and resolution for RHCP signals using the seven-element antenna array. These algorithms are particularly interesting due to their general applicability, sound principles, and excellent performance. In the next section, we introduce a new modification to the MUSIC algorithm, Nullspace MUSIC, which similarly improves MUSIC’s performance for arbitrary antenna arrays.
Figure 3.12: Error variances and means in Monte Carlo trials with RHCP signals (0dB SNR, 1000 samples). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$. Top: Variance, full $x$-scale. Middle: Variance, zoomed $x$-scale. Bottom: Bias, zoomed $x$-scale.
3.5 Nullspace MUSIC

In the previous section, we saw that several authors have addressed MUSIC’s inability to resolve closely spaced sources. While some approaches may not apply to emitter geolocation from a single mobile platform, two modifications to the MUSIC algorithm, Pole-Zero MUSIC [35] and Sequential MUSIC [36], were shown to separate merged peaks corresponding to closely spaced sources. These algorithms are general, and they may be applied whenever MUSIC may be applied. In this section, we introduce a new modification to the MUSIC algorithm that also separates peaks from nearby sources. The new approach is motivated by an adaptive array’s ability to null undesired signals [39]. When we are localizing a particular signal, all other signals are interference. Therefore, signals should be isolated from each other in the received data prior to the direction finding step.

In adaptive array processing, a system often sums all signals together using a weight vector, \( w \). The array output is

\[
y[m] = w^H x[m].
\]  

(3.16)

Setting \( w \) properly causes contributions to \( y[m] \) from a given direction to add constructively. This is called beamforming. Alternatively, setting \( w \) so that the contributions to \( y[m] \) from a given direction cancel out is called nullsteering.

Conceptually, nullsteering is a straightforward process. Suppose the manifold vector in the direction of a signal is \( a(\theta_0) \). Then, using any \( N \) dimensional vector orthogonal to \( a(\theta_0) \) as the weight vector will create a beam with a null at \( \theta_0 \). A total of \( N - 1 \) orthogonal sets of weights share this property. These vectors define the nullspace of \( a(\theta) \).
The essence of the new approach to resolve emitter locations in the presence of manifold mismatch is as follows. If one signal’s DOA, $\theta_1$, and the true manifold are known, a system can form $N-1$ orthogonal beams that have a null at $\theta_1$. Multiplying the received signals by the orthogonal weight vectors will yield $N-1$ weighted signals that are unaffected by the signal at $\theta_1$. Since the weight vectors are orthogonal, signals and noise that were uncorrelated in the original samples will also be uncorrelated after nullsteering. The system may then use MUSIC to estimate the DOA of another signal from the $N-1$ weighted signals with minimal interference from the signal at $\theta_1$. Naturally, MUSIC will use $(N-1) \times 1$ dimensional test vectors corresponding to the $N-1$ nullpatterns in place of the $N \times 1$ array manifold vectors.

As an example of this principle, Fig. 3.13 shows the gains of six orthogonal beams formed by the seven-element antenna array. Every beam places a null at $(\theta, \phi) = (35^\circ, 0^\circ)$. Naturally, if a signal arrives from $(35^\circ, 0^\circ)$, then the nulls will severely attenuate it. If a second signal is incident on the array, then its DOA can be estimated with the six outputs from nullsteering, and with minimal interference from the signal at $(35^\circ, 0^\circ)$. At the same time, the weight vectors for the beams in Fig. 3.13 are orthonormal. Consequently, uncorrelated noise from the receiver electronics will also be uncorrelated in the six null-steered signals. Once the second signal’s DOA has been estimated, a new set of six nullpatterns can be formed so that the second signal is attenuated, and the first signal can be re-estimated. Therefore, the new approach applies an intuitive, divide-and-conquer strategy that does not impose any additional assumptions relative to MUSIC.

In practice, the DOAs are unknown when direction finding begins. Therefore, an initial estimate for $\theta_1$ must be found before nulls can be placed. The initial estimate
Figure 3.13: Gains, in dB, for six orthogonal beams formed by the seven-element array. Each nullpattern places a null at $(35^\circ, 0^\circ)$, which is shown by a white star.
can come from the usual MUSIC algorithm. Then, the second DOA can be found from the beams that place a null at the first signal’s estimated DOA. Since we want to estimate all signals individually, we will alternate between nulling one signal and estimating the other signal until the DOA estimates are constant from one iteration to the next.

For two signals, the above approach is implemented with the following steps:

1. Calculate MUSIC, (3.10), and find the maximum. Assume the maximum is at the DOA of an emitter. Call this angle $\tilde{\theta}_1$.

2. Find $N - 1$ mutually orthonormal weight vectors orthogonal to $a(\tilde{\theta}_1)$. These may be found from Graham-Schmidt orthonormalization [65] or the dominant eigenvectors of $I - \hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)$. Let the weight vectors be $w_1, \ldots, w_{N-1}$.

3. Form an $N \times (N - 1)$ mapping matrix, $W = [w_1, \ldots, w_{N-1}]$. Note that $W$ is the nullspace of $a(\tilde{\theta}_1)$, and that it maps vectors from $N$ dimensional vector space to $N - 1$ dimensional vector space [65].

4. Calculate the weighted snapshot vectors, $x'[k] = WHx[k]$.

5. Calculate the nullspace covariance matrix, $\tilde{R}' = \frac{1}{K} \sum_{k=1}^{K} x'[k]x'^H[k]$. Its dimensions are $(N - 1) \times (N - 1)$.

6. Eigendecompose the nullspace covariance matrix into eigenvectors $[\hat{v}'_1, \ldots, \hat{v}'_{N-1}]$ and corresponding eigenvalues $\lambda'_1 > \lambda'_2 > \cdots > \lambda'_{N-1}$. Assuming that we have nulled one signal out of $x'[k]$, the noise space still contains $N - M$ eigenvectors, while the signal subspace contains $M - 1$ eigenvectors.

7. Calculate $P'_N$, the projection matrix for the noise space of $\tilde{R}'$. Note that $P'_N = [\hat{v}'_2, \ldots, \hat{v}'_{N-1}][\hat{v}'_2, \ldots, \hat{v}'_{N-1}]^H$. 

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8. Calculate the nullspace test vectors,

\[ \hat{a}'(\theta) = \frac{W^H a(\theta)}{\|W^H a(\theta)\|}. \]

9. Calculate the Nullspace MUSIC Spectrum,

\[ f_{NS}(\theta) = \|P_N \hat{a}'(\theta)\|^{-2}. \] (3.17)

10. Assume that the maximum of the Nullspace MUSIC spectrum is at the DOA of the second emitter.

11. Repeat steps 2-10, nulling emitter 2 and estimating the DOA of emitter 1.

   If the estimated DOA of emitter 1 does not change, then the new algorithm has converged on the DOA estimates. Otherwise, continue nulling one emitter and estimating the DOA of the other emitter until the estimated DOAs do not change between iterations.

   We call this approach Nullspace MUSIC because it is based on mapping signals into the nullspace of the antenna manifold vectors at the signal locations. Using the nullspace as the mapping matrix minimizes the power of the ‘interferer’ while we estimate the ‘signal of interest’ (SOI). Although all signals may be equally important, we regard all signals except for the SOI as interference while we estimate the SOI’s DOA.

   In practice, Step 4 of Nullspace MUSIC is conceptual. Instead of multiplying the snapshot vectors by the mapping matrix, we can left and right multiply the covariance
matrix by $W$ and obtain the same nullspace covariance matrix. That is,

$$
\tilde{R}^\prime = \frac{1}{K} \sum_{k=1}^{K} x^\prime[k] x^{H}[k] \\
\tilde{R}^\prime = \frac{1}{K} \sum_{k=1}^{K} W^H x[k] x^H[k] W \\
\tilde{R}^\prime = W^H \left( \frac{1}{K} \sum_{k=1}^{K} x[k] x^H[k] \right) W \\
\tilde{R}^\prime = W^H \tilde{R} W.
$$

Calculating the covariance matrix from (3.18) significantly reduces the computational cost of the nullsteering step.

When more than two signals are present, Nullspace MUSIC would proceed as follows. Steps 1-10 would be executed to find one signal’s DOA and null out its contribution to the weighted signals. The maximum of the second spectrum would be taken as the DOA of a second signal. Then, steps 2-10 would be executed with weight vectors that are the dominant eigenvectors of $I - \tilde{A} \tilde{A}^H$, where $\tilde{A} = [\hat{a}(\tilde{\theta}_1) \hat{a}(\tilde{\theta}_2)]$. If $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are close to their true values, then the next spectrum will isolate the signal from $\theta_3$. This DOA can be estimated with minimal interference from the first signals. Then, Nullspace MUSIC would run again and estimate $\theta_1$ by using $\tilde{A} = [\hat{a}(\tilde{\theta}_2) \hat{a}(\tilde{\theta}_3)]$. This repeats, where $\tilde{A}$ is formed from the steering vectors of the previous two DOA estimates until $\theta_1, \theta_2,$ and $\theta_3$ are constant from one iteration to the next. The same approach can be applied to $M$ signals, so long as $M \leq N - 1$.

Although Nullspace MUSIC is motivated by an array’s ability to null out particular directions, it follows a similar idea as Sequential MUSIC; Sequential and Nullspace MUSIC attempt to minimize the power from one signal while estimating the other signal’s DOA. Thus, the next subsection examines the differences between the two algorithms.
3.5.1 Differences between Sequential MUSIC and Nullspace MUSIC

Unlike Sequential MUSIC, Nullspace MUSIC maintains the orthogonality test that is fundamental to MUSIC. In contrast, Sequential MUSIC only approximates it. Sequential MUSIC only uses a projection matrix in its denominator if the covariance matrix has been estimated perfectly, (as in the case of noiseless data), and if the first DOA is estimated without error. Of course, Sequential MUSIC is most useful when MUSIC cannot resolve two signals by itself. This implies that the first DOA estimate may not be exact. By not using a projection matrix, Sequential MUSIC’s denominator, \((P_N[I - \hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)])\), cannot map test vectors to zero as is the case in MUSIC. This is shown explicitly in Appendix B. As a result, Sequential MUSIC’s peaks are less sharp and less accurate. On the other hand, Nullspace MUSIC calculates a new projection matrix for the denominator via eigendecomposition. This preserves the orthogonality test that is fundamental to MUSIC. Consequently, Nullspace and Sequential MUSIC calculate different spectra and estimate different DOAs. Therefore, orthogonality makes Nullspace MUSIC a novel approach with significantly better resolution and accuracy.

Fig. 3.14 compares Sequential and Nullspace MUSIC for a two-signal scenario. The uncorrelated RHCP signals are incident from 0° and 10° with 0dB SNR. The left plot shows that the MUSIC spectrum peaks at 1.67°, has a second small peak in the middle of the signals, and then falls off without peaking near 10°. Sequential and Nullspace MUSIC place a null at 1.67°, and each generates a second spectrum. Sequential MUSIC then peaks at 10.83°, whereas Nullspace MUSIC’s second spectrum peaks at 10.12°. Nullspace MUSIC’s peak is also about 2.3dB higher than Sequential...
Figure 3.14: Sequential and Nullspace MUSIC example for two 0dB signals from $\theta = 0^\circ$ and $\theta = 10^\circ$. Left: MUSIC, second Sequential MUSIC spectrum, and second Nullspace MUSIC spectrum. Right: Sequential MUSIC spectra and the final Nullspace MUSIC spectra.

MUSIC’s peak. Thus, even without iteration, Nullspace MUSIC yielded a higher peak and a more accurate DOA than Sequential MUSIC. The right plot adds the final spectra for Nullspace MUSIC. Nullspace MUSIC converged after calculating five spectra, (not including the initial MUSIC spectrum). During the iterations, Nullspace MUSIC refines its DOA estimates to $0.75^\circ$ and $9.99^\circ$. The peak at $9.99^\circ$ is also 1.2 dB higher than Nullspace MUSIC’s peak in the left plot at $10.12^\circ$, (and 3.5 dB higher than Sequential MUSIC’s peak). Thus, Nullspace MUSIC significantly outperformed Sequential MUSIC thanks to orthogonality and extra iteration.

Considering the last example, we may ask how much of Nullspace MUSIC’s performance improvement relative to Sequential MUSIC is due to orthogonality and how much is due to iteration? Surprisingly, iteration generally makes Sequential MUSIC worse for uncorrelated signals, as explained below. Therefore, we may attribute almost all of Nullspace MUSIC’s performance improvement to orthogonality, without which iteration is usually counterproductive.
Stoica’s analysis of Sequential MUSIC [37] provides important context. Stoica found that Sequential MUSIC only yields more accurate estimates for the second DOA in certain cases involving correlated signals. For uncorrelated signals, Stoica states plainly that Sequential MUSIC is asymptotically worse than MUSIC. In addition, MUSIC outperforms Sequential MUSIC in every case that Stoica reported with simulated signals. The apparent discrepancy between the positive results in [36] and the negative results in [37] arises from the former principally considering DOA resolution (not accuracy), while the latter only considers DOA accuracy (not resolution). If Stoica is correct, then estimating the first signal’s DOA from MUSIC should be better than estimating it after placing a null at Sequential MUSIC’s estimate of $\theta_2$. Logically, Sequential MUSIC’s error for $\theta_2$ should compound into its subsequent estimate of $\theta_1$. Applying the same logic again implies that Sequential MUSIC’s performance will degrade with additional iteration. Interestingly, [36] seems to agree, stating that “additional iterations may not necessarily improve the estimation performance” of Sequential MUSIC.

It is worth noting that in [37], Stoica proposed a modification “Improved Sequential MUSIC” that further improves DOA variances with correlated signals. However, Improved Sequential MUSIC also operates in $N$ dimensional space, does not guarantee projection matrices, and it provides very little performance improvement over MUSIC when the incident signals are uncorrelated. Therefore, Sequential MUSIC’s performance degradation with uncorrelated signals remains unaddressed.

We assessed iteration’s impact on Sequential MUSIC in simulations. For Iterated Sequential MUSIC, we place a null at Sequential MUSIC’s second estimated DOA and re-estimate the first DOA. Fig. 3.15 shows the ratios of the variance of Iterated
Figure 3.15: Error variances for Sequential MUSIC with three spectra relative to the error variance for Sequential MUSIC in Monte Carlo trials. Left: 10dB trials. Right: 0dB trials. Top: Relative variance, full x-scale. Bottom: Relative variance, zoomed x-scale.
Sequential MUSIC divided by the variance of Sequential MUSIC without iteration. The left plots consider the 10dB SNR case on $x$-scales of $0^\circ$ to $70^\circ$ and $0^\circ$ to $10^\circ$. The right plots consider the 0dB cases on $x$-scales of $0^\circ$ to $70^\circ$ and $0^\circ$ to $30^\circ$. For the 10dB signals, iteration always increases Sequential MUSIC’s variance! In the 0dB case, the third iteration improves both Sequential MUSIC estimates for $10^\circ \leq \theta_2 \leq 14^\circ$. Otherwise, however, at least one signal’s variance increases. Usually, both signals’ variances increase. Therefore, Fig. 3.15 corroborates [36], which stated that iteration was unnecessary in Sequential MUSIC. It also agrees with Stoica, who found that Sequential MUSIC’s method of nulling a signal and estimating another signal generally increases the estimate variance. Ultimately, Sequential MUSIC is hamstrung by its neglect of MUSIC’s fundamental orthogonality test. The orthogonality test is preserved by Nullspace MUSIC, which significantly outperforms Sequential MUSIC as a result.

### 3.5.2 Nullspace MUSIC Performance

We ran Nullspace MUSIC for the same Monte Carlo trials as in Figs. 3.11 and 3.12. Nullspace MUSIC’s minimum resolution according to (3.15) was $3^\circ$ in the 10dB case and $7.5^\circ$ in the 0dB case. Thus, Nullspace MUSIC had better resolution than Sequential MUSIC and about the same resolution as PZ-MUSIC for RHCP signals.

Fig. 3.16 adds the Monte Carlo results for Nullspace MUSIC to the results for PZ-MUSIC, and Sequential MUSIC (without iteration) with uncorrelated signals that arrive with 10dB SNR. As before, the left plots are for the signal at $0^\circ$, and the right plots are for the second signal. The top and middle plots show the error variances, and the bottom plots show the DOA biases. For the 10dB trials, Nullspace MUSIC
yields slightly less variance than Sequential MUSIC. Nullspace MUSIC is the least biased method when \( \theta_2 \geq 3^\circ \) (such that Nullspace MUSIC resolves the signals).

Fig. 3.17 considers the same setup with 0dB uncorrelated signals. Nullspace MUSIC clearly outperforms Sequential MUSIC. The variances for both signals were much lower whenever Nullspace MUSIC resolves the signals, and Nullspace MUSIC was less biased over the same range of angular separations. In addition, Nullspace MUSIC’s variance tied or even beat PZ-MUSIC’s variance for separations over 8°. Nullspace MUSIC also yields the same or less bias as PZ-MUSIC when both methods resolve the signals. Thus, Nullspace MUSIC and Pole-Zero MUSIC yielded very similar performance with low SNR signals.

Finally, Fig. 3.18 shows the average number of iterations that Nullspace MUSIC applied in order to converge. One iteration is defined as estimating each signal with the other signals nulled. The minimum number of iterations is one, which is sufficient to check convergence by comparing the last peak to the peak of the initial MUSIC spectrum. The left and right plots show the average number of iterations in the 10dB and 0dB trials, respectively. On average, Nullspace MUSIC applies two iterations or less when the signals are separated by more than 4°. With 0dB SNR, Nullspace MUSIC applies two iterations or less when the signals are separated by more than 11°. Nullspace MUSIC uses more iterations at smaller separations, but, in many of these trials, Nullspace MUSIC could not resolve the signals. Note that the number of iterations was capped at 4.5 during the trials, and Nullspace MUSIC halted whether the DOAs had converged or not. Although Nullspace MUSIC requires iteration and performs about the same as PZ-MUSIC with a dominantly polarized
Figure 3.16: Error variances and means in Monte Carlo trials with RHCP signals (10dB SNR, 1000 samples). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$. Top: Variance, full $x$-scale. Middle: Variance, zoomed $x$-scale. Bottom: Bias, zoomed $x$-scale.
Figure 3.17: Error variances and means in Monte Carlo trials with RHCP signals (0dB SNR, 1000 samples). Left: Signal at $\theta = 0^\circ$. Right: Signal at $\theta = \theta_2$. Top: Variance, full $x$-scale. Middle: Variance, zoomed $x$-scale. Bottom: Bias, zoomed $x$-scale.
array and a perfectly known array manifold, it outperforms Sequential MUSIC. Additionally, the relative performance with mixed polarization and an imperfectly known array manifold still needs to be addressed.

This section introduced a new algorithm, Nullspace MUSIC, for resolving signals that conventional MUSIC cannot resolve. While Nullspace MUSIC and Sequential MUSIC apply the same principle of estimating separate signals from separate spectra, Nullspace MUSIC consistently outperforms Sequential MUSIC. Sequential MUSIC neglects the orthogonality test between the noise space and the antenna array manifold that is fundamental to MUSIC. In contrast, Nullspace MUSIC meets MUSIC’s assumptions using beams that steer nulls towards the signal-not-of-interest. While MUSIC and Nullspace MUSIC compare test vectors to a projection matrix, Sequential MUSIC’s matrix is only a projection matrix when the covariance matrix and the first DOA are estimated perfectly. As a result, Sequential MUSIC’s performance
falls short of MUSIC’s performance when signals are well separated [37], and iterating Sequential MUSIC generally degrades its DOA estimates without much effect on resolution (Fig. 3.15). Since Nullspace MUSIC correctly applies the same ideas as MUSIC and Sequential MUSIC while also minimizing interference between signals, its resolution and estimate variance improve significantly relative to Sequential MUSIC. Although Pole-Zero MUSIC generally yields the best resolution and the lowest error variance, Nullspace MUSIC was very competitive, and it tied Pole-Zero MUSIC for most signal separations. Therefore, we will continue to study PZ-MUSIC and Nullspace MUSIC in mixed polarization scenarios. Thus far, the modified MUSIC algorithms have assumed that the antenna array is dominantly polarized and that the received signals match this polarization.

3.6 Mixed Polarization

The previous section introduced several modifications to the single polarization MUSIC algorithm that improve MUSIC’s resolution. In this section, we consider Mixed Polarization (MP)-MUSIC’s performance with diversely polarized signals. We also extend Pole-Zero MUSIC and Nullspace MUSIC to Mixed Polarization scenarios. The ability to handle mixed polarization is critical if the array is sensitive to more than one polarization and if signals arrive with unknown polarization. Such cases arise when an array is mounted on a platform, on which reflections and diffractions increase an antenna’s sensitivity to other polarizations. In addition, the apparent polarization of a signal can vary based on the orientation of the platform. Extending Pole-Zero MUSIC and Nullspace MUSIC to handle any polarization retains the versatility of
the original MUSIC algorithm, which can be applied for polarization diverse antennas and signals.

Recall from Chapter 2 that MP-MUSIC forms the matrix

\[
M(\theta) = \begin{bmatrix}
\hat{b}^H(\theta) P_N \hat{b}(\theta) & \hat{b}^H(\theta) P_N \hat{d}(\theta) \\
\hat{d}^H(\theta) P_N \hat{b}(\theta) & \hat{d}^H(\theta) P_N \hat{d}(\theta)
\end{bmatrix},
\]

(3.19)

where \( P_N \) is the projection matrix for the noise space of \( \tilde{R} \), and \( \hat{d}(\theta) \) is a unit vector constructed such that \( \text{Span}\{\hat{b}(\theta), \hat{d}(\theta)\} = \text{Span}\{\hat{b}(\theta), \hat{c}(\theta)\} \), and \( \langle \hat{b}(\theta), \hat{d}(\theta) \rangle = 0 \). Recall also that the manifold vectors for the \( \theta \) and \( \phi \) polarizations are \( b \) and \( c \), respectively. The MP-MUSIC spectrum is then

\[
f_{\text{MP}}(\theta) = \frac{1}{\lambda_{\text{min}}(M(\theta))},
\]

(3.20)

where \( \lambda_{\text{min}}(M(\theta)) \) is the minimum eigenvalue of the \( 2 \times 2 \) matrix \( M \). As usual, the \( M \) highest peaks of the MP-MUSIC spectrum are the DOA estimates for the \( M \) incident signals. MP-MUSIC can estimate up to \( N - 2 \) DOAs, (Appendix A).

MP-MUSIC’s estimate variance versus separation will behave very differently with signals that have different polarizations as compared to signals that have the same polarization. Recall that in single polarization studies, the manifold derivative was nearly a linear combination of the manifold vectors at small signal separations. As a result, the CRLB in (3.1) becomes inversely related to signal separation when the signals are close together. MUSIC’s variance correspondingly increases, and MP-MUSIC’s variance must similarly increase if the incident signals have the same polarization. However, if incident signals have different polarizations, then the manifold derivative for one polarization generally differs from the manifold for another polarization. As a result, small signal separations do not imply a pole in the CRLB.
function, and MP-MUSIC may be able to resolve much closer signals due to their polarization diversity.

The CRLB still sets a lower bound for an unbiased DOA estimator in the Mixed Polarization case. Note that the CRLB requires linearly independent manifold vectors and zero-mean, Gaussian white noise that is uncorrelated between antenna elements [66]. These conditions are met even if the incident signals polarizations are unknown. To calculate the CRLB, we simply need to use the manifold vectors and derivatives corresponding to the signals’ polarizations. However, we expect an estimator’s performance to fall short of the CRLB for finite SNR since the unknown polarization introduces uncertainty that is not present for a dominantly polarized array. For example, Mixed Polarization MUSIC implicitly estimates the polarization of the signal in the eigenvector of the $M$ matrix, (3.19). Errors in this estimate will compound the error in the DOA estimate. Therefore, MP-MUSIC’s variance for well separated signals should increase as compared to single polarization MUSIC estimating signals that match the array’s dominant polarization. Of course, the seven-element array’s gain decreases for non-RHCP signals, and this also increases the CRLB and MP-MUSIC’s expected variance for non-RHCP signals.

### 3.6.1 Pole-Zero MUSIC

Pole Zero MUSIC is easily extended to mixed polarization. Recall that for two signals, Pole-Zero MUSIC is a product of the usual MUSIC spectrum (raised to the $r$th power) and the inverse of the MUSIC spectrum calculated with the assumption that only one signal is incident on the array. This forces a zero in between poles (corresponding to DOAs) that are not easily resolved. The exact same logic can be
applied to mixed polarization scenarios using Mixed Polarization MUSIC, (3.20). For Mixed Polarization scenarios, the Pole-Zero Spectrum with two signals is simply

\[ f_{PZ,MP}(\theta) = \frac{\lambda_{\text{min}}(M^+(\theta))^{r\min(M(\theta))}}{\lambda_{\text{min}}(M(\theta))}, \]  (3.21)

where \( M^+ \) is the \( M \) matrix calculated assuming that only one signal is present. Recall that \( r \) is a factor controlling the multiplicity of the poles in the Pole-Zero spectrum. When calculating Mixed Polarization PZ-MUSIC in simulations, we will use \( r = 2 \).

### 3.6.2 Nullspace MUSIC

Nullspace MUSIC is also readily extended to mixed polarization scenarios. The principles of estimating one signal while nulling other signals simply requires us to be able to place nulls for signals with unknown polarization. Fortunately, we can attempt to null a signal because MP-MUSIC also estimates the signal’s normalized steering vector, \( \hat{a}(\theta, \eta, \zeta) \), where \( \eta \) and \( \zeta \) are the signal polarization parameters. In [24], Schmidt notes that the eigenvector corresponding to the minimum eigenvalue of \( M \) in (3.19) has the form \( [1 \quad q]^T \). The complex scalar \( q \) gives the relative magnitude and phase of \( \hat{d} \) to \( \hat{b} \). Let the first estimated DOA be \( \tilde{\theta}_1 \), and let the associated signal’s polarization parameters be \( \zeta_1 \) and \( \eta_1 \). It follows that \( \hat{a}(\theta_1, \eta_1, \zeta_1) \) can be estimated according to

\[ \hat{a}(\theta_1, \eta_1, \zeta_1) \approx \frac{\hat{b}(\theta_1) + q\hat{d}(\theta_1)}{\|\hat{b}(\theta_1) + q\hat{d}(\theta_1)\|} \]  (3.22)

where \( \| \cdot \| \) indicates the Euclidean norm. Although \( \eta_1 \) and \( \zeta_1 \) appear on the left of (3.22), they do not appear on the right because they are embedded in \( q \).
For two uncorrelated signals and a polarization sensitive antenna array, Nullspace MUSIC is implemented with the following steps:

1. Calculate MP-MUSIC, (3.20), and find the maximum. Assume the maximum is at the DOA of an emitter. Call this angle $\tilde{\theta}_1$.

2. Estimate the manifold vector $\tilde{a}(\tilde{\theta}_1, \eta_1, \zeta_1)$ from (3.22). Recall that $q$ comes from the eigenvector of $M(\tilde{\theta}_1)$ corresponding to the minimum eigenvalue.

3. Find $N - 1$ mutually orthonormal weight vectors orthogonal to $\tilde{a}(\tilde{\theta}_1, \eta_1, \zeta_1)$. These may be found from Graham-Schmidt orthonormalization [65], or the eigenvectors of $I - \tilde{a}\tilde{a}^H$. Let the weight vectors be $w_1, \ldots, w_{N-1}$.

4. Form an $N \times (N - 1)$ mapping matrix, $W = [w_1, \ldots, w_{N-1}]$.

5. Calculate the weighted snapshot vectors, $x'[k] = W^Hx[k]$.

6. Calculate the “nullspace covariance matrix,” $\tilde{R}' = \frac{1}{K} \sum_{k=1}^{K} x'[k]x'^H[k]$.

7. Eigendecompose the nullspace covariance matrix into eigenvectors $[\hat{v}'_1, \ldots, \hat{v}'_{N-1}]$ and corresponding eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_{N-1}$.

8. Calculate $P'_N$, the projection matrix for the noise space of $\tilde{R}'$. Note that $P'_N = [\hat{v}'_2, \ldots, \hat{v}'_{N-1}][\hat{v}'_2, \ldots, \hat{v}'_{N-1}]^H$.

9. Calculate the nullspace test vectors,

$$\hat{b}'(\theta) = \frac{W^Hb(\theta)}{\|W^Hb(\theta)\|}.$$

$$\hat{c}'(\theta) = \frac{W^Hc(\theta)}{\|W^Hc(\theta)\|}.$$

Note that $\hat{d}'(\theta)$ is a unit vector constructed such that $\text{Span}\{\hat{b}'(\theta), \hat{d}'(\theta)\} = \text{Span}\{\hat{b}'(\theta), \hat{c}'(\theta)\} = \text{Span}\{\hat{b}'(\theta), \hat{d}'(\theta)\}$, and $\langle \hat{b}'(\theta), \hat{d}'(\theta) \rangle = 0.$
10. Calculate the modified $M$ matrix,

$$
M'(\theta) = \begin{bmatrix}
\hat{b}'(\theta) & \hat{b}'(\theta) \\
\hat{d}'(\theta) & \hat{d}'(\theta)
\end{bmatrix} 
\begin{bmatrix}
P_N' \hat{b}'(\theta) \\
P_N' \hat{d}'(\theta)
\end{bmatrix} 
\begin{bmatrix}
P_N' \hat{b}'(\theta) \\
P_N' \hat{d}'(\theta)
\end{bmatrix}.
$$

(3.23)

11. Calculate the Nullspace MUSIC Spectrum,

$$
f_{\text{MP}}'(\theta) = \frac{1}{\lambda_{\min}(M'(\theta))}.
$$

(3.24)

12. Assume that the maximum of the Nullspace MUSIC spectrum is at the DOA of the second emitter.

13. Repeat steps 2-12, nulling emitter 2 and estimating the DOA of emitter 1. If the estimated DOA of emitter 1 does not change, then the algorithm has converged on the DOA estimates. Otherwise, continue nulling one emitter and estimating the DOA of the other emitter until the estimated DOAs do not change between iterations.

As in the single polarization algorithm, the above approach may be extended to three or more signals by placing nulls at $M - 1$ angles while estimating the DOA of the signal of interest. Note that Nullspace MUSIC has the same polarization estimation capabilities as MUSIC, and polarization could be used for emitter classification. However, we will focus exclusively on DOA estimation and emitter geolocation without making use of the estimated polarization.

3.6.3 Simulations with Mixed Polarization Signals

We applied Mixed Polarization forms of MUSIC, PZ-MUSIC, and Nullspace MUSIC in Monte Carlo trials with two uncorrelated signals. A $\theta$-polarized signal is always
present from $0^\circ$, and a $\phi$-polarized signal arrives from $\theta_2$. As before, 20,000 trials were run for $0.25^\circ \leq \theta_2 \leq 70^\circ$ in $0.25^\circ$ increments. Thus, a total of 5.6 million trials were completed. Each trial estimated the covariance matrix using one thousand samples.

Fig. 3.19 shows the Monte Carlo results when the signals arrive with 10dB SNR. In addition to the Monte Carlo variances, Fig. 3.19 shows the CRLB (3.1) and theoretical MUSIC variance (3.3), which are calculated using the appropriate steering vectors for the $\theta$ and $\phi$ polarizations. At large separations, all methods perform similarly, and they come close to the CRLB. At small separations, the Cramer Rao Lower Bound no longer has an inverse relationship with signal separation, as expected.

In the Monte Carlo trials, MP-MUSIC, PZ-MUSIC, and Nullspace MUSIC maintain small variances and biases for small signal separations. However, MP-MUSIC encounters merged peaks in a few trials starting at $1.75^\circ$ separation. Without two peaks near the signals, a low peak far away from any signal is chosen as a DOA, and MUSIC’s variance increases. PZ-MUSIC always yields two peaks until the signals are separated by $1.25^\circ$. On the other hand, Nullspace MUSIC’s error variance and bias stay small for all signal separations. According to the definition of resolution in (3.15), an estimator meeting the CRLB would have a resolution of $0.5^\circ$. In comparison, Nullspace MUSIC’s resolution in the trials is $0.75^\circ$, PZ-MUSIC’s resolution is $1.5^\circ$, and MP-MUSIC’s resolution is $2.0^\circ$. Thus, Nullspace MUSIC performs best in the mixed polarization, Monte Carlo trials with 10dB SNR.

Fig. 3.20 shows the Monte Carlo results when the incident signals arrive with 0dB SNR. In this case, MP-MUSIC yields increased variances starting at about $11^\circ$ separation. Pole-Zero MUSIC encounters the same problem at about $5.25^\circ$ separation. Nullspace MUSIC, however, remains unbiased and highly accurate for all signal
Figure 3.19: Error variances and means in Monte Carlo trials (10dB SNR, 1000 samples). Left: $\theta$-polarized signal at $\theta = 0^\circ$. Right: $\phi$-polarized signal at $\theta = \theta_2$. Top: Variance, full $x$-scale. Middle: Variance, zoomed $x$-scale. Bottom: Bias, zoomed $x$-scale.
Table 3.2: Minimum separation for $\theta$-polarized and $\phi$-polarized signals to be resolved according to (3.15)

<table>
<thead>
<tr>
<th>Algorithm/SNR</th>
<th>10dB</th>
<th>0dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo MP-MUSIC</td>
<td>2.0°</td>
<td>9.25°</td>
</tr>
<tr>
<td>Monte Carlo Pole-Zero MUSIC</td>
<td>1.5°</td>
<td>4.75°</td>
</tr>
<tr>
<td>Monte Carlo Nullspace MUSIC</td>
<td>0.75°</td>
<td>2.75°</td>
</tr>
<tr>
<td>Theoretical MUSIC Variance</td>
<td>0.5°</td>
<td>1.75°</td>
</tr>
<tr>
<td>Theoretical CRLB</td>
<td>0.5°</td>
<td>1.25°</td>
</tr>
</tbody>
</table>

separations. According to the definition of resolution in (3.15), MP-MUSIC should have a theoretical resolution of 1.75°. Nullspace MUSIC’s resolution in the trials is 2.75°; PZ-MUSIC’s resolution is 4.75°, and MP-MUSIC’s resolution is 9.25°. The resolutions of the various methods are summarized in Table 3.2.

In addition, Pole-Zero MUSIC yielded spurious peaks in some trials with separations greater than 65°. Although a peak was also present near 0°, the fact that a higher peak was found implies that Pole-Zero MUSIC should always use a sophisticated peak finding algorithm. On the other hand, Nullspace MUSIC had no problems with well separated signals. Therefore, Nullspace MUSIC again had the best performance in the mixed polarization, Monte Carlo trials with 0dB SNR.

Fig. 3.21 shows the average number of iterations that Nullspace MUSIC requires for convergence. With mixed polarization signals, fewer iterations were required to converge than in the RHCP cases. In the 10dB case and the 0dB case, an average of two iterations was sufficient for all signal separations. In the 0dB case, two iterations was sufficient for separations greater than 3.5°. Thus, Nullspace MUSIC’s
Figure 3.20: Error variances and means in Monte Carlo trials (0dB SNR, 1000 samples). Left: \( \theta \)-polarized signal at \( \theta = 0^\circ \). Right: \( \phi \)-polarized signal at \( \theta = \theta_2 \). Top: Variance, full x-scale. Middle: Variance, zoomed x-scale. Bottom: Bias, zoomed x-scale.
computational requirements are very reasonable even if the signal polarizations are not known.

In summary, this section extended Pole-Zero MUSIC and Nullspace MUSIC to mixed polarization scenarios. These methods and MP-MUSIC were applied to Monte Carlo simulations with a $\theta$-polarized signal and a $\phi$-polarized signal. We observed that the Cramer Rao Lower Bound stayed bounded as the separation between signals approached zero. Therefore, MUSIC based approaches should be able to resolve very closely spaced signals that have different polarizations. However, MP-MUSIC and Pole-Zero MUSIC sometimes yielded a merged peak in between the signals. The next highest peak may be far away from either signal. Consequently, the merged peak increased the estimator variances and biases well above theoretical predictions. On the other hand, Nullspace MUSIC estimates different signals from different spectra. Because the signals have different polarizations, one signal can be nulled out without also nulling the signal of interest. Therefore, Nullspace MUSIC always found a peak
for each signal. As a result, Nullspace MUSIC yielded superior robustness and resolution with the \( \theta \) and \( \phi \)-polarized signals. Nullspace MUSIC maintained very small variances and biases at all separations. Indeed, Nullspace MUSIC’s DOA errors were small even at separations less than the separations in Table 3.2, which used a conservative definition of resolution. Therefore, Nullspace MUSIC significantly advances state-of-the-art DOA estimation for mixed polarization signals that are incident on a polarization-sensitive antenna array.

### 3.7 Comparison with Other Methods

Additional methods have also been proposed to address MUSIC’s poor resolution with closely separated signals. This section discusses a few of them. However, because of their limitations or poor performance that we observed in the Monte Carlo trials, these algorithms will not be considered outside of this section.

#### 3.7.1 Alternating Projection for Maximum Likelihood Estimation

The Alternating Projection to Maximum Likelihood Estimation (AP) [74] is another approach that can separate closely resolved signals. AP fixes an emitter direction, \((\theta_1)\) and searches for the emitter direction \((\theta_2)\) that maximizes the likelihood function \( L \),

\[
L(\Theta) = \text{tr}[P_{A(\Theta)} \tilde{R}].
\]  

In (3.25), \( P_{A(\Theta)} \) is the projection matrix onto the space spanned by \( a(\theta_1) \) and \( a(\theta_2) \), and \( \text{tr}[:] \) indicates the trace operation. The vector, \( \Theta \), refers to the set of parameters for \( A \), e.g. \( \theta_1 \) and \( \theta_2 \). Like Nullspace MUSIC, AP alternates between estimating the DOA of emitter 1 and emitter 2 until the DOAs converge.
MUSIC based approaches like Pole-Zero MUSIC and Nullspace MUSIC have an advantage over AP when the array is sensitive to more than one polarization. AP can only estimate one parameter at a time. Hence, adding unknown polarization would add many iterations to AP. For two signals, six parameters must be estimated instead of two. With so many parameters to estimate, AP would also run the risk of converging to local solutions instead of global solutions.

When the same authors extended AP to diversely polarized arrays, they employed “Simulated Annealing,” (SA) [75]. SA avoids local minima by guessing and checking solutions over the entire solution space. New candidate solutions are generated based on the current best solution. Solutions are always accepted if they increase the likelihood function (3.25). Solutions lower likelihood functions can also be accepted based on random number generation and the relative $L(\Theta)$ compared to the current best solution. As SA keeps iterating, the likelihood of accepting a solution with a lower $L(\Theta)$ decreases. This allows SA to escape local minima early on, and it causes the algorithm to converge as the probability of accepting a worse solution drops. SA therefore requires a good generator of new candidate solutions based on the current solution, a good schedule regarding how many iterations will be run and what the criteria are for accepting a new candidate solution, and SA may not always yield the same answer. SA also lacks the visual appeal of spectra methods. Therefore, SA will not be considered further in this dissertation.

3.7.2 Recursively Applied and Projected MUSIC

Recursively Applied and Projected MUSIC (RAP-MUSIC) [38] is a modified form of Sequential MUSIC. RAP-MUSIC applies Sequential MUSIC’s projection matrix
to the estimated steering vectors as well as the estimated signal subspace. Then, RAP-MUSIC estimates DOAs from the peaks of the signal-subspace form of the MUSIC spectrum. The authors show that RAP-MUSIC lowers DOA variances with correlated signals. However, they also note that RAP-MUSIC does not yield more accurate estimates with uncorrelated signals.

RAP-MUSIC was also included in the single polarization Monte Carlo trials with uncorrelated signals (not shown). Like Sequential MUSIC, RAP-MUSIC was found to yield higher variances than the usual MUSIC algorithm. In addition, iterating by calculating a third RAP-MUSIC spectrum and re-estimating the DOA of the first signal increased RAP-MUSIC’s error variances. Recall that this also occurred for Sequential MUSIC, (Fig. 3.15). Like Sequential MUSIC, RAP-MUSIC still does not use the orthogonality principle that is fundamental to MUSIC if the covariance matrix is imperfectly estimated. Consequently, Nullspace MUSIC is a superior approach for uncorrelated signals, and RAP-MUSIC will not be considered further.

### 3.7.3 Min Norm

The Min-Norm algorithm [76] is another approach for improving MUSIC’s resolution. Min-Norm is a variation of MUSIC that uses a single eigenvector in the noise space to form the projection matrix. This eigenvector is carefully chosen so that it does not add spurious peaks to the MUSIC spectrum. Note that it is well known that only using one eigenvector for the noise space will generally yield spurious peaks, as shown in [76] Fig. 3. However, the analysis of why the Min-Norm vector does not yield spurious peaks relies on the special manifold of a uniform linear array.
Min-Norm may not have the same behavior when it is applied to an array of elements that couple with each other and are affected by their host platform.

3.7.4 Beamspace MUSIC

Finally, Nullspace MUSIC is also similar to Beamspace MUSIC [19]. Beamspace MUSIC forms a mapping matrix similar to $W$; calculates a covariance matrix similar to $\tilde{R}'$; and calculates a new MUSIC spectrum similar to the Nullspace MUSIC spectrum. However, Beamspace MUSIC has different motivations and different criteria for the mapping matrix. Beamspace MUSIC chooses an orthonormal basis of the entire signal subspace – plus an orthonormal vector – as the columns of the mapping matrix. Beamspace MUSIC uses as few columns in the mapping matrix as possible. This reduces the computational burden as compared to MUSIC. It also has the potential to improve resolution for very closely separated signals. However, Beamspace MUSIC did not perform well in simulations with the seven-element antenna.

Beamspace MUSIC’s authors considered two incident signals on a dominantly polarized array. The authors recommend using mapping matrix spans the signal subspace as well as the vector at the midpoint in between the signal DOAs. For two signals, the Beamspace mapping matrix should contain orthonormal vectors spanning the same space as

$$B = [a(\theta_1) a(\theta_2) a(\theta_{\text{mid}})]$$

(3.26)

where $\theta_{\text{mid}} = (\theta_1 + \theta_2)/2$. The Beamspace mapping matrix is then

$$W = A(B^H B)^{-\frac{1}{2}}$$

(3.27)

The Beamspace covariance matrix is

$$R' = W^H R W,$$

(3.28)
and it is eigendecomposed into eigenvectors \( \hat{v}_1', \hat{v}_2', \hat{v}_3' \) and corresponding eigenvalues \( \lambda'_1 > \lambda'_2 > \lambda'_3 \). The eigenvector \( \hat{v}_3' \) is used to form a new projection matrix for the new noise space, \( P'_N = \hat{v}_3' \hat{v}_3'H \).

The Beamspace test vectors are
\[
a'(\theta) = W^H \hat{\mathbf{a}}(\theta),
\]
and the normalized test vector is \( \hat{\mathbf{a}}' = a'(\theta)/\|a'(\theta)\| \). The Beamspace MUSIC spectrum is
\[
f_{BS}(\theta) = \frac{1}{\|P'_N \hat{\mathbf{a}}'(\theta)\|^2}.
\]

Like Nullspace MUSIC, Beamspace MUSIC uses a projection matrix for the mapped space in its denominator. Thus, it also preserves the orthogonality principle that is fundamental to MUSIC. However, Min-Norm [76] showed that using a single dimensional noise space, as recommended for Beamspace MUSIC, introduces spurious peaks into the MUSIC spectrum ([76], Fig. 3). Since our seven element array consists of coupled elements uniquely affected by scattering from a platform, it seems unlikely that \( \hat{v}_3' \) will avoid spurious peaks. On the other hand, Nullspace MUSIC uses the same dimensional noise subspace as MUSIC. By using more eigenvectors, Nullspace MUSIC increases the test vector’s projection into the noise subspace at angles away from the signal DOAs, which suppresses spurious peaks.

Beamspace MUSIC was implemented as follows. The mapping matrix was formed as \( B = [\hat{v}_1, \hat{v}_2, \hat{\mathbf{a}}(\tilde{\theta}_{\text{mid}})] \). Note that \([\hat{v}_1, \hat{v}_2]\) span approximately the same subspace as \([a_1, a_2]\). The midpoint angle, \( \tilde{\theta}_{\text{mid}} \) was estimated as the angle at which the MUSIC spectrum peaked when MUSIC assumed that a single signal is present. That is,
\[
\tilde{\theta}_{\text{mid}} = \arg \max_{\theta} \frac{1}{\|P'_N \hat{\mathbf{a}}'(\theta)\|}.
\]
where $P_N^+$ will be recognized from its use in Pole-Zero MUSIC, (3.11). Although [19] does not use this for estimating $\tilde{\theta}_{\text{mid}}$, we have seen that Pole-Zero MUSIC consistently inserts a zero into the MUSIC spectrum in between the two signals by using $P_N^+$ to estimate the signals’ midpoint.

With this implementation, Beamspace MUSIC consistently yielded the poorest performance. Its minimum resolution according to (3.15) was 8.25° for 10dB RHCP signals and 18.25° for 0dB RHCP signals. For comparison, MUSIC’s minimum resolutions are 4.5° and 12.25°, while Nullspace MUSIC’s minimum resolutions are 3.25° and 7.75°. Beamspace MUSIC was also notable as the only algorithm that was prone to have a spurious peak in between the true DOAs in addition to peaks near the DOAs. Thus, while Beamspace MUSIC uses much of the same math as Nullspace MUSIC and maintains MUSIC’s orthogonality principle, Beamspace MUSIC did not fare well when applied to the seven-element antenna array. In addition, Beamspace MUSIC estimates multiple signals from a single spectrum. Therefore, it is vulnerable to spurious peaks, which are especially likely when using a small noise subspace. Consequently, Beamspace MUSIC may not be applicable for geolocation from a mobile platform, and it will not be considered further.

3.8 Summary and Conclusions

In this chapter, we studied MUSIC’s performance with a perfectly known array manifold. The array manifold corresponded to the seven-element array on a ground plane with 0.47λ spacing between elements. Thus, the array should be well suited to estimating DOAs of signals arriving from near antenna broadside. Indeed, MUSIC’s
theoretical variance predicted that MUSIC would consistently resolve closely separated signals having the same polarization. If the signal polarizations were different, then MUSIC’s resolution should have been even better. However, MUSIC’s reliance on peakfinding from a spectrum meant that DOAs could not be estimated if the peaks from two signals merged. Alternatively, spurious peaks in other directions could be taken as signal DOAs when no signal was present nearby. MUSIC falls short of its theoretical performance because the theoretical derivation ignores the behavior of the antenna array manifold in between the true DOAs. However, the region in-between DOAs significantly impacts whether MUSIC can resolve two signals.

MUSIC’s inability to resolve closely separated signals has prompted many authors to propose new approaches that overcome MUSIC’s peakfinding problem. Root MUSIC [35] avoids peakfinding by finding analytical poles in the MUSIC function. However, it relies on ideal array manifolds of uncoupled antennas that are arranged in particular layouts. Similarly, the Minimum Norm algorithm [76] generates a high resolution MUSIC spectrum, but it relies on the special structure of a uniform linear array to avoid generating spurious peaks. Consequently, these methods may not perform well with general antenna arrays that are mounted on a host platform.

Sequential MUSIC [36], Improved Sequential MUSIC [37], and Recursively Applied Projection MUSIC [38], proposed estimating each signal’s DOA from a different spectrum. This innovative approach avoids the problem of merged peaks and spurious peaks. However, it was shown in Appendix B that these methods do not use projection matrices to calculate their spectra if the covariance matrix and signal DOAs must be estimated. Therefore, these methods neglect the orthogonality principle that is fundamental to MUSIC. In addition, the authors of [36, 37, 38] found that the
methods did not improve MUSIC’s error variance when the incident signals were uncorrelated. Even so, Monte Carlo simulations with Sequential MUSIC showed that estimating different signals from different spectra lowered the minimum resolvable signal separation with the seven-element antenna that was introduced in Chapter 2.

Into this crowded field, we have proposed a new modification of the MUSIC algorithm for separating closely spaced signals. We named the algorithm Nullspace MUSIC. Like the sequential methods, Nullspace MUSIC generates multiple DOA spectra and estimates one DOA from each spectrum. However, Nullspace MUSIC creates beams that physically null signals incident from a particular direction. The beams are orthogonal, and they map the received signals to the nullspace of the estimated array manifold vector. Uncorrelated signals and antenna noise in the element space are also uncorrelated in the nullspace. Nullspace MUSIC thus matches MUSIC’s assumptions. Nullspace MUSIC also employs a projection matrix to test the nullspace steering vectors for orthogonality. As a result, Nullspace MUSIC may be iterated to further refine the estimates. Iterating Nullspace MUSIC reduces the DOA variance, whereas iterating Sequential or RAP-MUSIC usually increases the error variance. Nullspace MUSIC also further reduces the minimum separation between signals that can be reliably resolved. While Beamspace MUSIC also employs orthogonal beams and a projection matrix, Beamspace MUSIC’s use of as few beams as possible led to poor performance with a physical antenna array.

Pole-Zero (PZ) MUSIC is another approach, which may be applied whenever MUSIC is applied. Pole-Zero MUSIC is based on the sound principle of separating merged peaks by inserting zeros in between the poles of the MUSIC spectrum. Monte Carlo simulations verified that Pole-Zero MUSIC significantly enhances MUSIC’s resolution...
and estimate variance. In Monte Carlo trials, PZ-MUSIC is the best existing MUSIC modification because it provides the greatest performance improvement while maintaining MUSICs generality.

Nullspace MUSIC and PZ MUSIC performed the best in Monte Carlo trials with RHCP signals, and they also have straightforward extensions to Mixed Polarization scenarios. The CRLB also shows that signals with different polarizations be resolved when they are much closer together. This is because the array manifold depends on the signal polarization, and the array manifolds for different signals do not become identical as the signals move towards each other. In mixed polarization scenarios with a $\theta$-polarized signal and a $\phi$-polarized signal, MP-MUSIC, Nullspace MUSIC, and PZ-MUSIC performed the same at most separations. However, at close separations, MUSIC often yields spurious peaks that bias the results and increase the variance. PZ-MUSIC improved MP-MUSIC’s resolution by avoiding spurious peaks until the signals were very close together, (or very far apart). On the other hand, Nullspace MUSIC avoided spurious peaks at all separations and resolved signals that were just $0.75^\circ$ apart with 10dB SNR, according to (3.15). This was half of PZ-MUSIC’s resolution of $1.5^\circ$, and very close to the resolution corresponding to the CRLB, $0.5^\circ$. Similar results were obtained with 0dB signals.

Therefore, we see that Nullspace MUSIC is a novel and worthwhile approach to the well-studied problem of how MUSIC’s resolution can be restored when peaks for nearby signals merge together. Nullspace MUSIC performs similarly to Pole-Zero MUSIC when the array is dominantly polarized. Pole-Zero MUSIC is easier to compute since it does not iterate, and PZ-MUSIC provided the best resolution for strong RHCP signals. With differently polarized signals, however, Nullspace MUSIC
was more robust, resulting in superior bias and error variance for closely separated signals.

Recall that the purpose of this study is to improve airborne geolocation from a single platform. While this chapter assumed a perfectly known manifold, mounting an array on a platform perturbs the antenna manifold, and perfect calibration should not be assumed. Therefore, the next chapter will study DOA estimation when the array manifold is only approximately known. The study will examine whether PZ-MUSIC and Nullspace MUSIC also improve performance with a mismatched array manifold.
Chapter 4: Modified MUSIC Algorithms in the Presence of Antenna Array Manifold Mismatch

4.1 Introduction

Direction finding algorithms commonly assume that the exact antenna array manifold is known and can be used to estimate the Directions of Arrival (DOAs) of incident signals. However, geolocation from a mobile platform requires mounting direction finding arrays on vehicles such as aircraft or spacecraft. On a platform, blockages, reflections, and diffractions of the radio waves significantly perturb the array manifold. For example, diffraction from the edge of the ground plane in Chapters 2 and 3 caused ripples in the seven-element antenna array’s radiation patterns, (see Figs. 2.5 and 3.1). If the array is installed on a platform, the ripples from the ground plane will be replaced by scattering effects from the host vehicle. The scattering effects will cause manifold mismatch unless the array manifold for direction finding is calibrated to match the true, in-situ, antenna array manifold.

In this chapter, we study how manifold mismatch affects MUSIC [24]. In particular, we study whether the algorithms that enhanced MUSIC’s resolution in Chapter 3 also improve MUSIC’s performance when the array manifold is only approximately
known. Recall that MUSIC applies to arbitrary antenna patterns, handles polarization sensitive arrays, and runs quickly. Therefore, MUSIC provides a better foundation for single-platform geolocation than other DOA algorithms, such as ESPRIT [41] or Maximum Likelihood [25].

Studying MUSIC in the presence of manifold mismatch provides critical insight for real-world geolocation applications. Mismatch is difficult to avoid because exact, in-situ array manifolds are difficult to obtain. To measure the in-situ manifold directly, the entire platform must be assembled, brought to a large antenna range, and mounted such that the array response can be measured for all angles of interest. Alternatively, full-wave electromagnetic simulations may require enormous computing resources to complete a high-fidelity simulation of a platform that spans hundreds of wavelengths. Simulation and measurement may therefore be prohibitively expensive. Moreover, platform configurations may change over time due to moving control surfaces, flexing wings, retracting landing gear, or changing the aircraft’s loadout. Even a perfectly measured or simulated array manifold does not consider time varying effects of the environment around the array. Therefore, DOA systems mounted on aircraft should anticipate some level of mismatch between the true, in-situ array manifold and the assumed array manifold that is available for direction finding.

First, Section 4.2 extends the simulations with the ground plane manifold to estimate elevation and azimuth (2D-DOA estimation). While 1D-DOA estimation facilitated comparisons of MUSIC and its theoretical performance in Chapter 3, airborne geolocation generally requires estimating both elevation and azimuth. As expected, Nullspace MUSIC and Pole-Zero (PZ)-MUSIC [35] significantly improve conventional
MUSIC’s DOA estimates when the signals are close together and the array manifold is perfectly known. Although Nullspace MUSIC and PZ-MUSIC essentially tied when estimating RHCP signals in Chapter 3, Nullspace MUSIC significantly outperforms PZ-MUSIC in simulated 2D-DOA estimation for RHCP signals using the seven-element array. Nullspace MUSIC also outperforms PZ-MUSIC for diversely polarized signals and mixed-polarization DOA estimation.

Next, Section 4.3 introduces a generic aircraft platform. Three variations of the platform are simulated with the seven-element antenna array. The simulations confirm that the gains and phases of the antennas are significantly perturbed by scattering from the various models. The in-situ manifolds are used for DOA simulations in Section 4.4, which considers signals that are incident on the full aircraft model. However, the manifold for a simpler version of the platform is available for DOA estimation. The mismatch between the generating manifold and the available manifold mimics real-world manifold mismatch effects. As a result of the mismatch, MUSIC’s DOA estimates contain significant biases. In Monte Carlo trials, the modified MUSIC algorithms reduce MUSIC’s DOA errors and improve MUSIC’s resolution. Overall, Nullspace MUSIC yields the best performance improvement relative to MUSIC.

Finally, Section 4.5 summarizes the chapter and makes some final observations. Nullspace MUSIC significantly enhances resolution when the array manifold is perfectly known, and it reduces errors with mismatch so long as the signals are well separated. On the other hand, mismatch causes Pole-Zero MUSIC to always underperform Mixed-Polarization (MP)-MUSIC for diversely polarized signals, and PZ-MUSIC may also underperform conventional MUSIC when the array is assumed to be dominantly polarized. The performance characterization in this chapter motivates
us to apply conventional MUSIC, MP-MUSIC, and Nullspace MUSIC to simulated geolocation flights in Chapter 5.

4.2 Monte Carlo Simulations for 2D-DOA Estimation with a Perfectly Known Antenna Array Manifold

We ran additional Monte Carlo Simulations with the seven-element antenna array from Chapters 2 and 3. These simulations also studied DOA accuracy for various separations between uncorrelated signals. However, the sweep versus angular separation was implemented differently than in the previous chapter. In this chapter, both signals arrived from random directions that spanned most of the array’s upper hemisphere. Since the signals are separated in elevation and azimuth, the distance between two angles \( (\theta_1, \phi_1) \) and \( (\theta_2, \phi_2) \) is measured as arc-length or great-circle distance [77],

\[
\Delta \sigma(\theta_1, \phi_1, \theta_2, \phi_2) = 2 \sin^{-1} \left[ \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + \cos \left( \frac{\pi}{2} - \theta_1 \right) \cos \left( \frac{\pi}{2} - \theta_2 \right) \sin^2 \left( \frac{\phi_1 - \phi_2}{2} \right) \right]. \tag{4.1}
\]

Note that 2D-DOA estimation allows more diverse setups and is more appropriate for studying geolocation from a mobile receiver. In this context, the arc-length metric allows us to concisely discuss angular separations and DOA errors for arbitrary signal directions.

4.2.1 DOA Selection

To test a specific angular separation between signals, \( \Delta \sigma_0 \), we determined the DOAs as follows. The first DOA was chosen from uniform random distributions for \( 0^\circ \leq \theta \leq 70^\circ \) and \( 0^\circ \leq \phi < 360^\circ \). Then, we found all directions with \( \theta \leq 70^\circ \) that were \( \Delta \sigma_0 \pm 0.25^\circ \) away from the first DOA. From this set of angles, one was chosen
at random to be the second DOA. If the signals had different polarizations, then the polarizations were randomly assigned to the signals. Thus, both signals share the same probability density function. Note, however, that large $\theta$, (low-elevation), angles are more likely with large angular separations so that the second DOA meets the specified separation from the first DOA. In addition, the array manifold for the seven-element array on the ground plane was tabulated in $1^\circ$ increments in $\theta$ and $\phi$. Thus, the elevation and azimuth angles of the incident signals were always integers.

4.2.2 Peak Selection and Association

After the DOAs were chosen, the snapshot vectors were generated according to (2.5). As before, one thousand random signal and noise samples from the complex Gaussian distribution were used to calculate the covariance matrix. Then, the MUSIC, Pole-Zero MUSIC, and Nullspace MUSIC spectra were calculated for $0^\circ \leq \theta \leq 90^\circ$ and $0^\circ \leq \phi < 360^\circ$. All DOA algorithms assumed that the number of incident signals was known a-priori and did not need to be estimated.

Generally, the MUSIC and Pole-Zero spectra contained several peaks. Thus, peaks were chosen as follows. First, we found every point in the spectrum that was higher than its immediate neighbors. In other words, we found all of the peaks in the spectrum. The highest peak was used as one DOA estimate. The other DOA estimate was the highest peak that was at least $\Delta\sigma_0/4$ degrees away from the first DOA estimate. This uses knowledge about the true signal scenario, but it enables automated peak finding to avoid selecting two nearby peaks when an obvious - but slightly lower - peak is located some distance away. Monte Carlo simulations with and without this rule confirmed that it improves MUSIC and PZ-MUSIC’s error statistics. Additionally,
peaks were chosen from the middle of the spectrum, \((\theta < 90^\circ)\), if possible. However, if too few peaks were available for \(\theta < 90^\circ\), then peaks with \(\theta = 90^\circ\) were added as needed. Nullspace MUSIC also chose peaks for \(\theta < 90^\circ\) before using peaks with \(\theta = 90^\circ\). However, since Nullspace MUSIC generates a separate spectrum for each signal, the minimum separation rule was not needed.

After the peaks were selected, they were compared with the true signal DOAs. The true and estimated DOAs were paired so that the Mean Squared Error was minimized. Minimum Mean Square Error (MMSE) DOA association is the same rule that was applied in Chapter 3. Note that DOA error is measured according to the arclength between the peaks and the true DOAs, (4.1), which incorporates the error in \(\theta\) and \(\phi\). We will refer to error measured in arc length as \textit{arc-error}, and we will denote it as \(\sigma\). Note also that arc-error is always positive. Thus, positive mean error does not imply that the DOA estimators are biased in a particular direction.

4.2.3 Monte Carlo Results

First, we considered two uncorrelated 0dB RHCP signals and single polarization DOA estimation. DOA spectra were calculated with conventional MUSIC (2.21), PZ-MUSIC (3.11), and Nullspace MUSIC (Section 3.5), and peaks were selected from the spectra according to Section 4.2.2. We ran 20,000 trials at separations between 2° and 70° in 1° increments. Thus, 1.38 million trials were run with two RHCP signals. Each trial used random signal locations that were chosen according to Section 4.2.1. The trials yielded the mean arc-error and standard deviation that are shown in the top plots of Fig. 4.1. Since both signals arrive from random directions, their statistics are combined into a single series for each plot. Note that since the signal DOAs are
chosen randomly, both signals arrive from near the horizon in many cases. Large \( \theta \) angles lower the effective aperture (\( D \cos \theta \), where \( D \) is the array aperture), as well as the antenna gain. Therefore, these simulations are more challenging than the simulations in Chapter 3, which always included a signal arriving from zenith.

As expected, conventional MUSIC often fails to resolve signals at larger separations than in Chapter 3. Nevertheless, Pole-Zero MUSIC and Nullspace MUSIC continue to significantly reduce the DOA errors as compared to conventional MUSIC. In a change from the 1D-DOA estimation results, Nullspace MUSIC outperforms PZ-MUSIC with 0dB RHCP signals. For example, when the signals are separated by 15°, MUSIC’s average error is 4.0°, PZ-MUSIC’s mean error is 1.4°, and Nullspace MUSIC’s mean error is 1.1°. At the same separation, the three methods’ standard deviations are 12.7°, 4.8°, and 1.7° respectively. These results suggest that Nullspace MUSIC is more robust than PZ-MUSIC when the signals are allowed to arrive from any angle. As expected from studying the CRLB in Section 3.2, all methods’ DOA errors grow significantly as the signals are brought together because the array manifold vectors become increasingly similar.

Next, we considered a \( \theta \)-polarized (\( E_\theta \)) and a \( \phi \)-polarized (\( E_\phi \)) signal, each with 0dB SNR. The results are shown in the bottom plots of Fig. 4.1. The relative performances of the three algorithms are generally the same as in the RHCP case. For example, at 15° separation, MUSIC, PZ-MUSIC, and Nullspace MUSIC have mean errors of 4.1°, 2.5°, and 1.8°, respectively. Their standard deviations are 14.2°, 9.7°, and 6.1°. Thus, PZ-MUSIC and Nullspace MUSIC reduce MUSIC’s DOA errors in 2D-DOA estimation with diversely polarized signals, but Nullspace MUSIC yields the best performance.
Figure 4.1: Mean error and error variance for 0 dB signals incident on the seven-element array, which is mounted on a 4’ ground plane. Top: RHCP signals and RHCP DOA estimation. Bottom: An $E_\theta$ signal, an $E_{\phi}$ signal, and mixed polarization DOA estimation. Left: Mean arc-error. Right: Standard deviation of arc-errors.
Fig. 4.1 also shows that PZ-MUSIC’s standard deviation actually increases after the signal separation increases beyond 45°. Similar effects were seen in the top left plot of Fig. 3.20, which also considered mixed polarization DOA estimation. If diversely polarized signals are far apart, then PZ-MUSIC’s assumption that underestimating the number of signals will yield a spectrum with a zero in between the true DOAs may not hold. Instead, a spurious peak can be introduced into the spectrum and be mistaken for a signal DOA. Therefore, PZ-MUSIC occasionally causes unnecessary errors with large signal separation, diverse polarization, and low SNR. Ultimately, Nullspace MUSIC provides the best 2D direction finding performance in both polarization scenarios with 0dB signals and the ground plane array manifold regardless of the signal separation.

In the final study with a perfectly known manifold, we increased the SNR of the signals from 0dB to 30dB. All other parameters were kept the same as in Fig. 4.1. The error means and standard deviations for the RHCP scenario are shown in the top plots of Fig. 4.2. The statistics for the mixed polarization scenario are shown in the bottom plots. The plots show that raising the SNR to 30dB essentially eliminates DOA error in the mixed polarization scenarios for all methods. In the RHCP scenarios, MUSIC is essentially error free until the signals are separated by five degrees or less. PZ-MUSIC avoids DOA errors until the signals are separated by only three degrees. At two degrees separation, Nullspace MUSIC’s mean error is only 0.03° with a standard deviation of 0.2°. Thus, Nullspace MUSIC has the best resolution in a high SNR scenario as well as the low SNR scenario discussed previously. Moreover, the high SNR enabled all three methods to correctly estimate
Figure 4.2: Mean error and error variance for 30 dB signals incident on the seven-element array, which is mounted on a 4’ ground plane. Top: RHCP signals and RHCP DOA estimation. Bottom: An $E_\theta$ signal, an $E_\phi$ signal, and mixed polarization DOA estimation. Left: mean arc-error. Right: Standard deviation of arc-errors.
the DOAs if the signal separation was greater than five degrees and the array manifold was perfectly known.

In summary, this section extended MUSIC-based DOA estimation to estimating elevation and azimuth. Both angles will need to be estimated for geolocation from a mobile platform. As expected, PZ-MUSIC and Nullspace MUSIC enhance MUSIC’s DOA resolution. However, Nullspace MUSIC yields the best performance for RHCP signals and diversely polarized signals. Nullspace MUSIC also outperforms MUSIC and PZ-MUSIC in low and high SNR scenarios. While this section assumed a perfectly known array manifold for the seven-element array on a ground plane, the chapter will move on to in-situ array manifolds and DOA estimation in the presence of array manifold mismatch. For the rest of this chapter, we will assume that the incident signal SNR is 30dB, and we will continue to generate sample covariance matrices with one thousand samples. The high SNR and large number of samples allows DOA errors to be attributed solely to manifold mismatch.

4.3 In-Situ Antenna Manifold Simulation

In practice, mismatch between the true, in-situ antenna array manifold and the assumed antenna array manifold biases DOA estimates. Therefore, Fig. 4.2 may not represent real-world performance with strong signals. In this section, we consider the same seven-element antenna array as before, but we study its antenna patterns on a generic aircraft platform.

Recall that we have been using a commercial, off-the-shelf (COTS) array that contains seven RHCP patch antennas. Six patches are distributed around a circle with a radius of 0.47 wavelengths, and the seventh patch is located at the array
center. While we know the layout of the antenna elements, we do not know the exact details of the antenna element design or the radome that covers the array. Therefore, we cannot simulate the array on a platform directly. Instead, we must measure the far-field pattern and find a set of electric currents over the array aperture that approximates the measured pattern. Note that antenna engineers refer to the problem of reconstructing sources from a radiation pattern as the Inverse Source Problem (ISP), [78, 79].

After finding sources that approximate the measured radiation pattern, we simulate the same sources on a platform of interest. Since the sources only approximate the antenna’s radiation, we do not expect the *in-situ* source simulation to yield the exact, *in-situ*, antenna array manifold. Instead, we expect that the scattering-induced changes in the array manifold to resemble the effects of the physical platform. Thus, the simulated manifolds for different platform will allow us to investigate the effects of platform-induced antenna array manifold mismatch on DOA estimation with MUSIC.

We obtained the antenna array manifolds in a four-step process combining measurement and simulation. First, we measured the antenna array in the Ohio State University ElectroScience Laboratory’s anechoic chamber. The chamber measurement of the array is shown in Fig. 4.3. During the measurement, the array was mounted on a circular ground plane with a diameter of 4’. Second, we decomposed the measured radiation patterns into a set of basis functions and excitations. Third, we simulated the same basis functions on the generic aircraft platform. Finally, we applied the excitations from Step 2 to the basis functions in Step 3. This yields an approximate the *in-situ* radiation pattern for the commercial antenna array on the generic aircraft platform. The rest of this section explains the process in greater detail
4.3.1 In-Situ Antenna Patterns

Fig. 4.4 shows four different platforms in HFSS. Counter-clockwise from the top left, these are the 4’ ground plane that the array was measured on; a generic aircraft platform with a body, tail structure, and tilted wings with pylons; the same model with flat-wings and without pylons; and the flat plate that forms the body of the generic aircraft model. The model with flat wings represents a base configuration for an aircraft; the full model represents the same aircraft carrying some loadout; and the body plate represents a basic calibration for the aircraft. The body plate, which is identical in all three aircraft models, measures 12.5 wavelengths (λ) long and 5λ wide. The attached wings are each 3.75λ long, the tail structure is 2.5λ tall, and the pylons are 1.25λ tall. Thus, the entire model would fit into a box that is 12.5λ long, 12.5λ wide, and 2.5λ tall. This aircraft model is well suited to studying manifold mismatch because it is generic and simple to simulate. The antenna coordinate system is also
shown in Fig. 4.4. Note that the $\theta$ angle is measured off of array zenith, and the $\phi$ angle is measured counter-clockwise from the nose of the generic aircraft model.

The platforms models were used to simulate the in-situ radiation of seven pairs of perpendicular slots. Each pair of slots was located at the center of a patch antenna in the physical array. Thus, the simulated patterns are basis functions that can be used to represent the radiation of an antenna in the array. The measured radiation patterns of the array on the 4' ground plane can be decomposed with the basis functions for the simulated ground plane. The decomposition finds the weights that, when applied to the basis functions, approximately synthesize the measured radiation pattern. Similarly, the in-situ radiation pattern of the same array on the platform is synthesized by applying the weights to the basis functions for the platform.

Decomposition and synthesis follow a simple procedure. Let the measured $E_\theta$ and $E_\phi$ pattern for an antenna element be a $2P \times 1$ vector $z$, where $P$ is the number of angles that the antenna element was measured at\(^3\). Both gain and phase are accounted for by the electric field measurements in $z$, whose entries are complex numbers. Let the simulated $E_\theta$ and $E_\phi$ pattern for the 14 slots be a $2P \times 14$ matrix $Y$. Let the rows (corresponding to angles) in $Y$ be arranged in the same order that they are arranged in $z$. Then, the measured antenna pattern can be approximated as

$$z \approx Yw,$$  \hspace{1cm} (4.2)

where $w$ is a $14 \times 1$ weight vector. The matrix equation is overdetermined, and an optimal set of basis function weights can be found from the least squares solution,

$$w = (Y^H Y)^{-1} Y^H z.$$  \hspace{1cm} (4.3)

\(^3\)Since two polarizations were measured, the total number of measurements at a given frequency is $2P$. 

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Figure 4.4: HFSS models. Top Left: 4’ circular ground plane matching the ground plane that the array was measured on. Top Right: Body plate for the generic aircraft model. Bottom Left: Flat-wing model with a tail structure. Bottom Right: Full model with 10° wing tilt, pylons, and the same tail structure.
Note that the weights correspond to currents exciting the slot antenna elements.

When we measured the array, the gain and phase were tabulated in three-degree increments in $\theta$ and $\phi$. However, this implies a dense sampling near zenith, since adjacent angles with the same $\theta$ angle are only $(\phi_1 - \phi_2) \sin \theta$ degrees apart. Therefore, we found the least squares solution to the equation

$$z \odot s \approx Y w \odot S,$$

where $\odot$ is the Hadamard (entrywise) product, $s = \sin \theta$, and $\theta$ is a $2P \times 1$ vector of the $\theta$ angle for each measurement angle. The $2P \times 14$ matrix $S$ is simply $s$ repeated 14 times, once for each slot basis function. Thus, we found the current decomposition minimizing the error between the measured and synthesized patterns without overemphasizing the measurements near zenith. Note that, although the slots have very little coupling with each other, the least squares solution uses slots at every element’s location to decompose a single antenna element’s radiation pattern. Thus, the least squares solution models the mutual coupling between the physical elements by assuming radiation from slots that are distributed over the array aperture. Consequently, the synthesized patterns implicitly include mutual coupling.

The measured gain and phase for the center element are shown in the top of Fig. 4.5. The simulated gain and phase are shown in the middle row of plots. Finally, the bottom row shows the difference in gain and phase between the measured and simulated radiation pattern. The centers of these plots correspond to antenna zenith ($\theta = 0^\circ$), the outer edge corresponds to horizon ($\theta = 90^\circ$), and zero degrees azimuth ($\phi$) is on the right. Comparing the top and middle rows shows that the simulated gain and phase match the measured patterns very well over the upper hemisphere. The bottom row shows that gain differences are usually within 1.5dB, while phase
differences are usually within 10°. This confirms that the basis functions and excitations model the radiation from the physical antenna element reasonably well. Note that our mismatch study does not rely on an exact match between the measured and simulated antenna patterns. Instead, we require a realistic manifold mismatch setup, which we will obtain by completing full-wave electromagnetics simulations of the antenna array on the different variations of the same platform.

Fig. 4.6 shows the in-situ gains and phases for the center element on the plate model, the flat model, and the full model. For these patterns, the excitations found from least squares were applied to the slot patterns for the respective platform. Note that a zero-degree azimuth angle corresponds to the nose of the platform, 90° and 270° correspond to the wings (on the flat and full models), and 180° corresponds to the tail. Comparing Figs. 4.5 and 4.6 shows that swapping the 4’ circular ground plane for the 10’ by 4’ body plate significantly perturbs the antenna pattern. Although radio waves diffract from the edge of the platform at the antenna horizon, the antenna pattern near zenith is affected as well. Furthermore, the middle and bottom plots show that the wings and tail structure add rapid variations along the elevation cut between the wings. Finally, the bottom plot shows that the pylons block radiation from near the horizon and add strong reflections just above the nulls. Thus, Fig. 4.6 shows that the same antenna radiates very differently depending on its host platform. Moreover, the next section will show that platform effects also strongly influence DOA estimation.
Figure 4.5: Ground plane antenna patterns. Top Row: Measured RHCP pattern for the center element mounted on the 4' Ground Plane. Middle Row: Synthesized HFSS pattern for the center element mounted on the 4' ground plane. Bottom Row: Difference between measurement and simulation. Left Column: RHCP gain, in dB. Right Column: RHCP phase, in Degrees.
Figure 4.6: *In-situ* RHCP antenna patterns. Top Row: HFSS pattern for the center element mounted on the platform ground plane (body-plate). Middle Row: Synthesized HFSS pattern for the center element mounted on flat-wing model. Bottom Row: Synthesized HFSS pattern for the center element mounted on the full model. Left Column: RHCP gain, in dB. Right Column: RHCP phase, in Degrees.
4.4 Monte Carlo Simulations with Manifold Mismatch

Fundamentally, DOA estimation compares the received signals to an assumed array manifold. In the case of MUSIC, this comparison is made through the estimated noise subspace. If the array is mounted on a platform and radiates similarly to the bottom row of Fig. 4.6, but only a simple manifold for a ground plane is available, then MUSIC generally will not find any manifold vectors that are perpendicular to the observed noise subspace. Consequently, MUSIC’s DOA estimates will be biased. Biases will occur for high SNR as well as low SNR.

In this section, we quantify mismatch’s effects on MUSIC in Monte Carlo simulations. The simulations consider examples for which the signals are incident on the full platform, but the MUSIC spectra are calculated using a simpler manifold. First, we considered a single incident signal. Cases with a single incident signal approximate the best possible DOA performance for a particular mismatch setup. Then, we consider cases with two uncorrelated signals. Aside from simulating manifold mismatch by changing the array manifolds, we apply the same DOA selection, peak selection, SNR, and other parameters that were used for Fig. 4.2.

Studying specific examples of manifold mismatch has several advantages as compared to stochastic modeling. First, using a full-wave electromagnetics solver to find the antenna patterns ensures that the platform effects are realistic. The example array manifolds approximately solve Maxwell’s equations, and they represent realistic radiation from an antenna array on a generic aircraft platform. Consequently, the simulated DOA estimates approximate real-world DOA estimates for signals that satisfy the narrowband signal model. Stochastic models for antenna array manifold mismatch must be careful to similarly maintain fidelity to Maxwell’s equations, or
the antenna patterns and the subsequent conclusions may not apply to real-world DOA estimation. Secondly, the generic aircraft platform is simple to analyze, and features in the antenna patterns can be associated with particular features on the aircraft model. Stochastic models may not have a similarly intuitive description, which would make interpreting the results more difficult. Thus, relating stochastic model parameters to a particular real-world DOA system may be challenging. Finally, a realistic stochastic model may contain too many variables for a comprehensive analysis. A stochastic model must make a tradeoff between fidelity and simplicity, whereas the array manifolds on the generic aircraft platform are both simple and high fidelity. Therefore, we prefer to study particular examples of manifold mismatch and the effects on DOA estimation. Note that Chapter 7 will use a completely different antenna array and platform for experimental direction finding and emitter geolocation, yet array manifold mismatch will cause similar effects as are observed in this chapter.

4.4.1 Smaller Mismatch Setup

In the first mismatch setup, we estimated signal DOAs using the antenna manifold for the array on the flat-wing platform. Since the flat-wing and full models only differ due to wing tilt and pylons, we will refer to this as the ‘smaller’ mismatch setup. First, we ran 20,000 trials with a 30dB RHCP signal incident from a randomly chosen angle for $0^\circ \leq \theta \leq 70^\circ$. Conventional MUSIC then sought the DOA over the range $0^\circ \leq \theta \leq 90^\circ$. The mean error and standard deviation are shown in the left column of Table 4.1. Note that these statistics exclude 0.2% of trials for which the error exceeded 45°. Excluding outliers gives us a robust indication of the best possible
Table 4.1: MUSIC error statistics with a single incident signal in the smaller mismatch setup.

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<th>RHCP</th>
<th>Mixed Polarization</th>
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<tr>
<td>Mean</td>
<td>2.1°</td>
<td>3.3°</td>
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<tr>
<td>Std. Dev</td>
<td>2.8°</td>
<td>4.2°</td>
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<tr>
<td>Outliers</td>
<td>0.2%</td>
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DOA performance for a specific mismatch setup. Next, we ran 20,000 trials with a 30dB $E_\theta$ signal and mixed polarization MUSIC. The other parameters were the same as for the RHCP signal. Finally, we ran 20,000 trials with a 30dB $E_\phi$ signal and mixed polarization MUSIC. The combined mean error and standard deviation for the trials with the $E_\theta$ and $E_\phi$ signals are shown in the right column of Table 4.1. For the right column, 0.5% of trials were excluded from the statistics because the DOA error exceeded 45°.

We can make three important observations from the single-signal results. First, the presence of outliers with a single signal demonstrates that manifold mismatch makes some large DOA errors unavoidable. In these cases, peakfinding was very simple - the DOA estimate was the highest peak with $\theta \leq 90^\circ$. Nevertheless, dozens of trials yielded very large DOA errors. Second, we should not expect the MUSIC based algorithms to yield better error statistics with two incident signals than with a single incident signal. Naturally, estimating two signals poses a greater challenge. Even with a perfectly known array manifold, the Cramer Rao Lower Bound increases as the number of signals increases; see (3.2). Therefore, we should evaluate DOA methods by how close they come to MUSIC’s single-signal performance rather than by
the absolute DOA error statistics. Third, mismatch has less effect on single polarization DOA estimation than mixed polarization DOA estimation. Mixed polarization MUSIC is affected by mismatch in the RHCP and LHCP manifolds, and much of the antenna’s LHCP radiation is due to reflections from the platform. Therefore, the LHCP manifold is particularly sensitive to changes in the host platform. Consequently, mixed polarization MUSIC yielded outliers more than twice as often as RHCP MUSIC. In addition, the mean error and standard deviation for the successful (non-outlier) trials were about 50% greater than in the RHCP case. We expect a similar increase in error and outliers with two incident signals.

Fig. 4.7 shows the mean arc-error and standard deviation versus separation when two uncorrelated signals are incident on the array. The array is mounted on the full platform, and the DOAs are estimated with the manifold for the array on the flat-wing platform. The top plots show the RHCP results, and the bottom plots show the results for the mixed polarization scenarios. Each plot has the corresponding error for a single incident signal from Table 4.1 drawn as a horizontal dashed line. Note however, that every trial was counted for the two-signal error statistics; outliers with large errors were not excluded. The plots consider angular separations between the two incident signals ranging from 10° to 70°. Due to mismatch, closer separations cannot be resolved. For example, with RHCP signals separated by 10°, Conventional MUSIC’s mean error is about 7°, PZ-MUSIC’s mean error is 5.9°, and Nullspace MUSIC mean error is 8.0°. Since the average errors exceed half the separation between the signals, all methods fail to resolve the signals in many trials when the signal separation is 10° or less. However, when the signals are separated by 40°, conventional MUSIC, PZ-MUSIC, and Nullspace MUSIC have mean errors of 3.7°, 3.5°, and 2.9°.
Figure 4.7: Mean error and error variance for two 30dB signals incident on the seven-element array, which is mounted on the full platform with pylons and wing tilt. The flat-wing manifold is available for DOA estimation. Top: RHCP signals and RHCP DOA estimation. Bottom: An $E_\theta$ signal, an $E_\phi$ signal, and mixed polarization DOA estimation. Left: Mean arc-error. Right: Standard deviation of arc-errors.

The respective standard deviations of 5.9°, 5.9°, and 3.9° follow a similar pattern. With large signal separations, PZ-MUSIC performs slightly worse than conventional MUSIC, while Nullspace MUSIC almost attains the empirical lower bound set by conventional MUSIC’s single-signal performance.

The mixed polarization scenarios yield somewhat different results. MP-MUSIC’s mean error is between 6° and 8° for all separations. PZ-MUSIC’s mean error is higher, ranging between 7° and 10° for all separations, but it initially decreases as the separation between the $E_\theta$ and $E_\phi$ signals increases. Meanwhile, Nullspace MUSIC’s mean
error is greater than 12° at the smallest separations, but the mean error decreases to less than 5° at 50° separation. When the signals are well separated, Nullspace MUSIC is effective at placing nulls and estimating each DOA individually. Nullspace MUSIC yields smaller errors than MP-MUSIC and PZ-MUSIC, which must estimate both DOAs simultaneously.

Interestingly, the mixed-polarization methods’ mean error and standard deviation increase at signal separations greater than 60°. At these separations, the DOA selection described in Section 4.2.2 is biased to select lower elevations in order to satisfy the large separation. Since mismatch is more severe near horizon (especially around the pylons), the corresponding DOA estimates are more erratic, and the mean error and standard deviation increase. Mixed Polarization MUSIC is more sensitive to mismatch, and the combination of lower signal elevations and greater sensitivity causes the mixed polarization statistics to degrade with increasing separation. This contrasts with conventional MUSIC when assuming a dominantly polarized array. The top plots showed that the MUSIC error statistics approached the empirical lower bound, (the single-signal performance), in spite of the lower elevation angles at large signal separations. At large separations, Nullspace MUSIC yields the lowest error because it removes the ambiguity regarding which peak is the ‘best’ peak to use as a second DOA estimate.

However, the most surprising result was that both modified MUSIC algorithms yield larger errors than MP-MUSIC for nearby signals in mixed polarization scenarios. Nullspace MUSIC also underperforms conventional MUSIC with nearby RHCP signals. This is the opposite of the behavior in Fig. 4.1, where modified MUSIC algorithms outperformed conventional MUSIC at small separations and performed the
same at large separations. To understand why mismatch reverses the trends in Fig. 4.1, we will look at an example.

Fig. 4.8 shows an example for which PZ-MUSIC and Nullspace MUSIC yield larger errors than MUSIC. Two RHCP signals are incident on the full platform from \((\theta, \phi) = (55^\circ, 105^\circ)\) and \((55^\circ, 120^\circ)\). These angles, which are separated by \(\Delta \sigma = 12.3^\circ\), are shown with crosses on top of the DOA spectra. With the flat-wing model available, the MUSIC spectrum (top left) peaks in between the signals at \((54^\circ, 114^\circ)\). MUSIC’s second peak is very shallow, but relatively nearby at \((56^\circ, 127^\circ)\). The peaks are marked in the MUSIC spectrum by white stars. PZ-MUSIC (top right) and Nullspace MUSIC (bottom row) also peak at \((54^\circ, 114^\circ)\). However, PZ-MUSIC’s second peak is slightly farther away at \((55^\circ, 128^\circ)\). Thus, PZ-MUSIC increases the DOA error for the signal at \((55^\circ, 120^\circ)\) from 5.85\(^\circ\) to 6.55\(^\circ\). Note that PZ-MUSIC doubles the multiplicity of the poles in the MUSIC spectrum, and the color scale for PZ-MUSIC spans 50 dB compared to 25 dB for MUSIC and Nullspace MUSIC. For the same setup, Nullspace MUSIC places a null at \((54^\circ, 114^\circ)\) and largely nulls out both signals. The subsequent Nullspace MUSIC spectrum is low everywhere, but its highest peak is at \((83^\circ, 176^\circ)\). Consequently, Nullspace MUSIC’s error for the second signal is 58.4\(^\circ\)!

Nullspace MUSIC’s large mean error and standard deviation for small signal separations in Fig. 4.7 shows that the unfortunate outcome in Fig. 4.8 is not uncommon for Nullspace MUSIC when estimating nearby signals with a mismatched array manifold. However, Fig. 4.7 also showed that Nullspace MUSIC significantly reduces the mean error and standard deviation if the signals are sufficiently separated. If the signals are closely spaced, then placing a null in between the signals, as in the example above, severely degrades the signal to noise ratio for the signal-of-interest (SOI)
Figure 4.8: Example setup with 30dB SNR and RHCP signals incident from $(\theta, \phi) = (55, 105)$ and $(55, 120)$. Top Left: MUSIC Spectrum. Top Right: PZ-MUSIC Spectrum. Bottom: Nullspace MUSIC spectra for the individual signals. The true DOAs are shown with purple crosses. The estimated DOAs are shown with white stars.
without effectively nulling the other signal (signal-not-of-interest, or SNOI). Consequently, it is reasonable for Nullspace MUSIC to have worse performance at small separations - for which nulling is counterproductive - and better performance at large separations, for which nulling is effective and the signals can be estimated individually. Even if the nulling-DOA or nulling-weights for the SNOI contain some errors, Nullspace MUSIC still attenuates the SNOI by 20-30dB. Meanwhile, the SOI may be unaffected due to the large separation between the signals. Therefore, it should not be surprising that Nullspace MUSIC is more sensitive than MUSIC to manifold mismatch when signals are close together, but Nullspace MUSIC is less sensitive to manifold mismatch when the signals are moderately separated or far apart.

In summary, the smaller mismatch setup demonstrated that mismatch introduces significant DOA errors, even with a single incident signal. Mismatch has more effect with mixed polarization DOA estimation than single polarization DOA estimation. With two signals, PZ-MUSIC only reduced errors with RHCP signals that were close together. Otherwise, PZ-MUSIC actually increased the DOA errors relative to MUSIC. On the other hand, Nullspace MUSIC increases DOA errors if the signals are close together, as mismatch makes nulling less effective or even counter-productive. However, with separations greater than about 25°, (50% of the array’s beamwidth at zenith), Nullspace MUSIC significantly reduced the mean mismatch-induced errors for both polarization setups. Nullspace MUSIC reduces the error standard deviation for both setups when the signals are separated by more than about 75% of a beamwidth.
4.4.2 Larger Mismatch Setup

We also tested MUSIC, PZ-MUSIC, and Nullspace MUSIC when the manifold for the body plate was available for DOA estimation. As before, 30dB signals were incident on the full platform. Since the wings and tail structure have been removed from the available manifold in the previous examples, the manifold mismatch is greater than in the previous example. We will refer to this example as the ‘larger’ mismatch setup.

As before, we ran trials with a single incident signal to find an empirical lower bound for the DOA error with the mismatch setup. The outlier removed mean error and standard deviation are shown in Table 4.2. As before, the RHCP results are in the left column, and the mixed polarization results are in the right column. The rate of outliers (with errors greater than 45°) approximately doubles in comparison to the smaller mismatch setup. In addition, the error statistics for the remaining trials generally increase. As expected, the single-signal trials demonstrate that larger changes to the environment around the antenna induce larger DOA errors.

The Monte Carlo results for the larger mismatch setup with two incident signals are shown in Fig. 4.9. The top row is for RHCP trials, and the bottom row is for the mixed polarization trials. The single-signal DOA errors from Table 4.2 are shown with dashed horizontal lines. As expected, the increased mismatch increases all methods’ error statistics across the range of signal separations. As before, errors are largest for small separations and generally decrease with increasing angular separation. In comparison to the smaller mismatch setup, MUSIC and PZ-MUSIC’s performance
Table 4.2: MUSIC error statistics with a single incident signal in the larger mismatch setup.

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<td>Outliers</td>
<td>0.4%</td>
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degraded more than Nullspace MUSIC’s performance. As a result, Nullspace MUSIC performs the best regardless of angular separation with RHCP signals and conventional MUSIC. Therefore, Nullspace MUSIC may still improve the resolution of conventional, single polarization MUSIC, but the relative performance depends on the particular mismatch setup. In the same cases, PZ-MUSIC improves errors relative to MUSIC for small and moderate separations. At large separations, however, PZ-MUSIC performs slightly worse than MUSIC.

In the mixed polarization case, Nullspace MUSIC yields the smallest error means for separations greater than about 26°, and it yielded the smallest standard deviations for separations greater than about 37°. Meanwhile, mixed polarization PZ-MUSIC cannot reduce MUSIC’s error statistics at any signal separation. As in Section 4.2, PZ-MUSIC’s poor performance may be explained as follows. If diversely polarized signals are far apart, then PZ-MUSIC’s assumption that underestimating the number of signals will yield a spectrum with a zero in between the true DOAs may not hold. Instead, a spurious peak can be introduced into the spectrum and be mistaken for a signal DOA. Therefore, PZ-MUSIC may cause unnecessary errors, and it underperforms MUSIC in Monte Carlo simulations as a result. Also recall that we attempted to avoid peakfinding errors by stipulating that conventional MUSIC and PZ-MUSIC
would select peaks that were at least $\Delta \sigma / 4$ degrees apart. This used knowledge about the true signal scenario, but Monte Carlo simulations with and without this rule showed that it benefited MUSIC and PZ-MUSIC’s statistics. Consequently, PZ-MUSIC’s poor performance may not be mitigated by more advanced peakfinding.

In summary, the results from the larger mismatch setup show that if the signals are somewhat well separated, (by about 74% of a beamwidth for this array at zenith)
Nullspace MUSIC tends to be the most robust to manifold mismatch. Nullspace MUSIC performed better than other methods for all separations with RHCP signals, and it was the only method that outperformed MUSIC with mixed polarization signals.

4.5 Summary and Conclusions

This chapter studied the best performing MUSIC-based algorithms from Chapter 3 in the presence of array manifold mismatch. We began by extending the Monte Carlo simulations to estimating both elevation and azimuth angles. Initially, the array manifold was assumed to be perfectly known. With low and high SNR, we found that Pole-Zero (PZ) MUSIC and Nullspace MUSIC reduced DOA errors for signals incident on the ground plane array manifold. Nullspace MUSIC yielded the smallest errors when we assumed that the array was dominantly polarized, as well as when an $E_\theta$ and an $E_\phi$ signal were incident on the array.

However, with manifold mismatch, the array manifold that is available for direction finding does not match the true, in-situ array manifold. Manifold mismatch is likely when an array is mounted on an aircraft, which has moving control surfaces and may fly in different configurations. Therefore, manifold mismatch must be considered for single platform geolocation by a mobile platform.

We simulated the seven-element array from Chapters 2 and 3 on three variations of a generic aircraft platform. The simulations demonstrated that the same antenna element has markedly different radiation patterns based on the environment surrounding the antenna. We then set up simulations for which the incident signal samples were generated using the full version of the platform, but the DOA algorithms used the array manifold corresponding to a simpler version of the platform.
Using particular *in-situ* antenna array manifolds simulated in HFSS ensured that the manifold mismatch effects were realistic, and it also made the perturbations in the antenna patterns easy to associate with features on the platform. On the other hand, a stochastic model of array manifold mismatch effects may not represent realistic radiation, and it may be difficult to relate the results to a real world geolocation platform. In the example setups with a single incident signal at 30dB SNR, the manifold mismatch induces a couple degrees of error and several degrees of standard deviation (on average) in MUSIC’s DOA estimates. In addition, 0.2% to 1.0% of trials yielded errors greater than 45°, indicating that mismatch makes some outliers unavoidable. Naturally, estimating DOAs for two signals will be more challenging.

As compared to low SNR, manifold mismatch significantly changes the relative performance of the modified MUSIC algorithms versus separation. With low SNR and a known antenna array manifold, the modified MUSIC methods outperform conventional MUSIC at small signal separations, and they perform the same with large signal separations. Manifold mismatch makes estimating nearby signals much more challenging, and modified MUSIC algorithms often underperform MUSIC in the case of high SNR and an imperfectly calibrated antenna array manifold. However, with large signal separations, modified MUSIC algorithms may significantly improve on MUSIC’s accuracy in the presence of array manifold mismatch. This reversal of trends as compared to a perfectly known array manifold emphasizes the importance of studying DOA accuracy with a mismatched manifold, which operational systems should anticipate.

Overall, Nullspace MUSIC was the most effective at reducing DOA errors for two incident signals in the presence of manifold mismatch. We tested ‘smaller’ and ‘larger’
manifold mismatch setups by changing how closely the platform for the available array manifold matched the true, *in-situ* array manifold. When we assumed that the array was dominantly polarized, Nullspace MUSIC yielded the smallest DOA errors at moderate to large signal separations. Nullspace MUSIC also yielded the smallest DOA errors at all separations with the larger mismatch setup and RHCP signals. In comparison, Pole-Zero MUSIC reduced MUSIC’s DOA errors for small signal separations, but at moderate or large signal separations, PZ-MUSIC increased DOA errors relative to MUSIC. This behavior was also observed in Chapter 3, (see Fig. 3.20). Notably, Nullspace MUSIC saw the smallest increase in its error statistics when we moved from the smaller mismatch setup to the larger mismatch setup. Thus, Nullspace MUSIC was the most robust to the particular details of the manifold mismatch in the examples with RHCP signals. However, nearby signals combined with array manifold mismatch can cause Nullspace MUSIC’s attempt to null the signal-not-of interest to fail, leading to large DOA errors in such cases.

We observed somewhat different results when we assumed that an $E_\theta$ and an $E_\phi$ signal were incident on the array and we used mixed polarization DOA estimation. In this case, Pole-Zero MUSIC was unable to improve the DOA errors relative to MUSIC for any signal separation. However, Nullspace MUSIC yielded smaller average errors for separations greater than $26^\circ$ (52% of a beamwidth) and smaller error standard deviations for separations greater than $37^\circ$ (74% of a beamwidth) in both mismatch setups. The combination of mismatch and small signal separations may make the nulling in Nullspace MUSIC unreliable or counterproductive, leading to spurious peaks far away from ether signal. However, if the signals are well separated, Nullspace MUSIC can reliably null the signals and reduce the mean error and
standard deviation relative to MUSIC, which estimates both signals from the same spectrum. Therefore, Nullspace MUSIC improves DOA performance for signals that are well separated.

Overall, Nullspace MUSIC improves DOA accuracy with a mismatched array manifold. Nullspace MUSIC yields smaller mean errors and standard deviations for signals that arrive from angles that are at least moderately far apart (about 74% of a beamwidth). However, Nullspace does not consistently improve DOA resolution because mismatch may render nulling counterproductive. Whether or not Nullspace MUSIC improves airborne geolocation of multiple emitters cannot be determined from the individual DOA simulations that have been considered thus far. Instead, a full flight should be considered, and Nullspace MUSIC’s performance should be compared to conventional MUSIC. Also, since mixed polarization PZ-MUSIC underperformed MP-MUSIC for both mismatch setups at all separations, we will not study it further. In the next chapter, we will apply MP-MUSIC and Nullspace MUSIC to geolocating multiple emitters with various polarizations during a simulated flight.
Chapter 5: Geolocation with MUSIC and Nullspace MUSIC

5.1 Introduction

In this chapter, we investigate geolocation accuracy for stationary emitters on the ground that are observed from an aircraft. Airborne geolocation from a single platform relies Direction of Arrival (DOA) estimation. DOA estimates are made at various points along the flight path, and Lines of Bearing (LOB) are drawn from the aircraft along the estimated DOAs to find the points where they intersect the ground. The points on the ground are processed to yield the final geolocation estimate. Estimating DOAs and drawing Lines of Bearing forms the basis for conventional geolocation from a mobile platform, and this approach is also known as triangulation.

In a favorable geolocation flight path, the emitters are observed from diverse positions and orientations. The diverse observations of the scene provide the opportunity to accurately locate the emitters in spite of mismatch-induced biases in the DOA estimates (Chapter 4). While every estimate for an emitter may be biased, biases for different observations should not always be the same, and the biases for many diverse observations of the same emitters may cancel out. Therefore, conventional geolocation using the MUSIC DOAs [24] or the Nullspace MUSIC DOAs (Chapter
3) has the potential to accurately geolocate multiple emitters in spite of manifold mismatch.

In this chapter, we apply Mixed Polarization (MP)-MUSIC and Nullspace MUSIC to RF emitter geolocation during a recorded flight path from an experimental aircraft campaign. The flight path includes the aircraft’s recorded orientation as well as its position. However, the transmitted and received signals are simulated. By simulating the received signals over a recorded flight, we capture an aircraft’s true flight dynamics while still allowing control over parameters such as transmitter location, power, and polarization. We also control manifold mismatch by generating the received signals with the array manifold for the full platform in Chapter 4, but estimating the DOAs using the array manifold for the array on one of the three variations of the platform. Simulating the received signals also avoids signals-not-of-interest, which may be encountered during a live data collection.

The simulation results reveal that manifold mismatch poses a severe challenge for geolocating multiple emitters, even during a flight path that encloses all emitters of interest. The simulations include scenarios with up to four diversely polarized emitters. The estimated Lines of Bearing for multiple emitters intermix, and at first they appear to be inseparable. If we cannot assign estimates to a particular emitter, then we cannot use the estimates to locate the emitter. However, after introducing a new clustering technique for the DOA estimates, we are able to accurately locate three to four of the four emitters. In general, MP-MUSIC’s DOA estimates lead to more accurate location estimates than Nullspace MUSIC’s estimates because of the small angular separations between the emitters at some observations during the flights.
The chapter proceeds as follows. First, Section 5.2 discusses the parameters for the simulated flights. Then, Section 5.3 discusses how DOA estimates are processed in the angular domain to estimate the emitter location. Since the aircraft position and orientation change throughout the flight, the true DOA to an emitter also changes. Therefore, a few steps are required to find the point on the ground that minimizes the apparent DOA error throughout the flight. Next, Section 5.4 shows that diverse observations allow a single emitter to be located very accurately in spite of array manifold mismatch. In addition, processing the estimated DOAs in the angular domain yields much more accurate location estimates than simply processing the points where the Lines of Bearing intersect with the ground.

Section 5.5 applies MP-MUSIC and Nullspace MUSIC to scenarios with two uncorrelated emitters, and it discusses how estimates from different observations may be grouped together when multiple emitters are present in the scene. Indeed, associating DOA estimates with a particular emitter poses the primary challenge for simultaneous emitters that overlap in the frequency domain. Therefore, we extend the current approach of Minimizing the Apparent Angular Error (MAAE) for a single emitter to clustering DOA estimates for multiple emitters. Although the DOA estimates are widespread, contain large errors, and intermix, MAAE clustering determines both emitter locations very accurately.

Section 5.6 applies the same approach with four uncorrelated and diversely polarized emitters. However, this pushes conventional geolocation to the limit. MP-MUSIC and Nullspace MUSIC may estimate a maximum of five emitters with unknown polarizations, and the flight paths bring emitters close together in the angular domain.
Furthermore, the emitters are usually seen from near the horizon such that the array’s effective aperture is reduced. Consequently, a particularly challenging emitter cannot be found over one of the flight paths due to the DOA errors that are induced by manifold mismatch.

The examples with multiple emitters also show that Nullspace MUSIC improves the final geolocation estimates only in a few cases. In other cases for which the emitters are not well separated in angle space, Nullspace MUSIC yields less accurate location estimates than MP-MUSIC. Consequently, Chapter 6 will study a different approach to combining observations from the same examples considered in this chapter. While Nullspace MUSIC generally does not improve geolocation in this chapter, it plays a critical role in Chapter 6.

5.2 Simulated Flight Setup

This chapter considers two segments of a flight path that was flown by a Piper Saratoga aircraft in October, 2015. During the flight, the airplane’s position and orientation were recorded by an aviation-grade Inertial Measurement Unit (IMU). The IMU post-processed the GPS pseudo ranges and inertial measurements, and it output the aircraft’s precise position and attitude during the flight.

DOA estimation during the flight employed simulated sample covariance matrices. We generated the covariance matrices with one thousand samples from each antenna. The sampled noise assumed 1MHz receiver bandwidth, a 6dB receiver noise figure, and a 300 Kelvin receiver electronics temperature. The receiver electronics were assumed to be the only source of noise, and the noise was assumed to be Additive White Gaussian Noise (AWGN). The sampled signals were simulated using the array
manifold for the generic aircraft platform with wing tilt and pylons. The platform was orientated with the array facing downward so that the array generally had line of sight to the emitters. The emitters transmitted signals with power of 100 Watts, and the received power assumed isotropic radiation and free-space path loss. The received power and noise parameters yielded signal to noise ratios (SNR) that would have been 30dB to 60dB throughout the flight for an isotropic receiver. However, the actual SNR for a particular antenna element depends on the emitter polarization and the antenna’s gain towards the transmitter at a given observation. Nevertheless, excellent SNR, uncorrelated Gaussian noise, and using many samples to estimate the covariance matrix ensure that any DOA errors will be due to mismatches between the true and available antenna array manifolds. High SNR and narrowband signals generally makes it easy to estimate the number of incident signals using a method such as the Minimum Description Length (MDL) [80] or the Akaike Information Criteria (AIC) [81]. Thus, this chapter and the remainder of the dissertation will assume that the number of signals is known a-priori and does not need to be estimated.

Fig. 5.1 shows a forty-five minute long segment of the flight path, for which the aircraft flew back and forth over Columbus, Ohio. The segment began in the south and worked its way north following the flight path shown in the top left plot. The flight path recorded by the IMU is shown in purple. The top right plot shows the aircraft’s X, Y, and Z coordinates versus time (relative to the center of the search area), and the bottom plots show the roll, pitch, and yaw throughout the flight. During the flight, we simulated Direction of Arrival estimation at the ‘x’ marks in the flight path (top left plot), which occur at thirty-second intervals. The received signals arrived from one to four transmitters, whose locations are shown as black stars.
Figure 5.1: Long flight path overview. Top Left: Flight path and emitter locations. Top Right: Receiver X, Y, and Z vs time during the flight. Bottom Left: Roll and pitch during the flight. Bottom Right: Yaw.

All of the emitter locations were encompassed by the flight path, which should have provided favorable geometry for locating any of the emitters. However, the aircraft altitude was only about 2km, while the northwest and southeast emitters were about 22km apart. Thus, emitters that were far away from the aircraft were seen from near the horizon. Moreover, the airplane flew with the nose pitched upwards by about eight degrees, and the airplane rolled more than ten degrees during turns. Therefore, emitters were actually seen from above horizon (no line of sight) at several points during the flight. Since the array manifold near the horizon is strongly affected by changes to the aircraft body, we can expect large DOA errors for many observations.
Figure 5.2: Angles to emitters during the long flight path. Top Left: $\theta$ angles. Top Right: Azimuth angles. Bottom left: Angular separation between emitters. Bottom Right: SNR for an isotropic antenna element during the flight.

during the flight if the true, *in-situ* array manifold is perturbed from the manifold that is available for DOA estimation.

Fig. 5.2 shows the $\theta$ and $\phi$ angles from the receiving array on the aircraft towards the four emitters during the flight (top plots). The bottom plots show the angular separations between pairs of emitters as well as each emitter’s SNR throughout the flight path assuming an isotropic receiver. Recall that the $\theta$ and $\phi$ angles were shown in the aircraft coordinate system in Fig. 4.4, and that elevation is simply $90^\circ - \theta$. Any $\theta$ angles greater than $90^\circ$ indicate that the array did not have line of sight to the emitter. Since the flight encompasses all of the emitters, every emitter is seen from diverse azimuth angles. Since the flight also passes near each emitter, every emitter
is briefly seen for \( \theta \) angles less than 45°. However, emitters are typically observed at \( \theta \) angles greater than 80° throughout the flight. Finally, even though the emitters are very well spread out on the ground, they are often separated by 20° or less in angle space. Note that angular separation is calculated according to (4.1), and it incorporates separation in \( \theta \) and \( \phi \). Recall in Chapter 4 that MP-MUSIC yielded an average error of over 8° for signals arriving from \( \theta \leq 70° \) when two diversely polarized signals were separated by 30°. Since this flight setup considers larger \( \theta \) angles and smaller separations between emitters, DOA estimation will be very challenging in the presence of array manifold mismatch.

Fig. 5.3 shows the same details for the second segment of the flight path. This segment lasts just over ten minutes. Since the segment was shorter, DOA estimates were made every six seconds, such that the total number of DOA estimates was similar for both flight segments. The top plots show that the aircraft started in the northeast flying towards the center, then proceeded clockwise around a large circle. The aircraft continued flying at an altitude of about 2km. The second row of plots show the aircraft’s roll pitch and yaw. Although it flew in a circle, the aircraft’s roll varied considerably during the flight. The third row of plots show the \( \theta \) and \( \phi \) angles to the emitters. The emitter locations are the same as in the previous flight path. Even though the circular flight path is in the middle of the emitters, \( \theta \) angles less than 60° were uncommon, and \( \theta \) angles greater than 80° were typical for the eastern emitters. In addition, the circular flight path did not encompass any of the emitters, so the Geometric Dilution of Precision (GDOP) [1] was larger for the circular flight path than for the long flight path. Finally, the bottom plots show the angular separations between the emitters as well as the SNR for an isotropic receiver.
Figure 5.3: Circle flight path overview. Top left: Flight path and emitter locations. Top Right: Receiver X, Y, and Z vs time during the flight. Second Row: Roll and pitch (left) and yaw (right). Third Row: $\theta$ (left) and azimuth (right) angles to the emitters during the flight. Bottom Left: Angular separations between emitters. Bottom Right: SNR for an isotropic receiver during the flight.
Angular separations were almost always greater than 40°, with typical separations greater than 60°. The eastern and western pairs of emitters tended to be the closest together in angular space. Thus, DOA estimation should generally be easier than in the long flight path, but separating the eastern and western pairs of emitters may still be challenging. Finally, each emitter’s SNR varied from about 35dB to 60dB throughout the flight, (assuming an isotropic antenna at the receiver location).

5.3 Emitter Geolocation

We estimated the emitter positions using DOA estimates made throughout the flight segments. For both of the flight segments, we limited DOA estimation to cover a 40km × 40km area on the ground around the emitters. This area was divided into a grid with 200m resolution in x and y. At the observation points along the flight (every 30 seconds for the long flight path and every 6 seconds for the circular flight path), we calculated the angle from the aircraft to every grid point. DOAs were then estimated over the corresponding angular space in the antenna coordinate system. This transform from the global coordinates to the aircraft coordinate system took the aircraft position and orientation into account, and it yielded a subset of the far-field sphere over which the array manifolds had been simulated. By limiting the angular search space to correspond to a fixed area on the ground, we avoided extreme outliers, and we avoided DOA estimates that would not intersect with the Earth’s surface. This approach also ensured that we sought DOAs over the angles that the signals arrived from. Since emitters were occasionally seen from θ > 90° during the flight, estimating DOAs below the aircraft horizon would have been insufficient.
Two methods were applied to geolocate an emitter from the many DOA estimates that were made throughout the flight. First, we simply averaged the points on the ground that corresponded to the estimated DOAs. This is equivalent to drawing Lines of Bearing (LOBs) from the aircraft along the estimated DOAs and finding averaging the locations that the LOBs intersect with the ground. Averaging locations is sufficient when the array manifold is perfectly known and the individual location estimates are very close to the true locations. However, the low elevation angles imply that small errors in the DOA estimates can yield large errors in the location estimate. Large location errors from small DOA errors disproportionately bias the average location. Therefore, we also estimated emitter positions by searching the same 40km × 40km grid for the grid point that yielded the smallest average apparent angular error over the flight. In other words, we treated every point in the grid as a candidate emitter location. Then, we found the grid point that best explained the observed DOAs by minimizing the sum of the apparent DOA errors when we assumed that the grid point was the true emitter location.

Mathematically, minimizing the average apparent DOA error proceeds as follows. For the tth observation, let the angles from the aircraft to the grid point \((x_0, y_0)\) be \(\theta_t(x_0, y_0)\) and \(\phi_t(x_0, y_0)\). These angles are in the antenna coordinate system, which incorporates the attitude of the host aircraft. Let the estimated DOA for the tth observation be \((\tilde{\theta}_t, \tilde{\phi}_t)\). Then, the average apparent angular error at the grid point is given by

\[
\bar{e}(x_0, y_0) = \frac{1}{T} \sum_{t=1}^{T} \Delta \sigma(\tilde{\theta}_t, \tilde{\phi}_t, \theta_t(x_0, y_0), \phi_t(x_0, y_0)),
\]

where the arclength or great-circle distance, \(\Delta \sigma\), was given in (4.1). Recall also that arclength is always positive.
The average apparent angular error in (5.1) essentially shows how well a position on the ground would explain the estimated Directions of Arrival. Suppose that the emitter is located at \((x_0, y_0)\). If the estimated DOAs are exact, then \(\bar{e}(x_0, y_0) = 0\). All other grid points will have positive values that depend on far away the point appeared in angle space during the flight. Greater average angular errors indicate a lower likelihood that \((x_0, y_0)\) was the source of the signal whose DOA was estimated.

Equation (5.1) exploits a well-known concept in geolocation that the best emitter location estimate corresponds to the location that minimizes the error in the DOA estimates [30]. While [30] used a gradient descent approach to find the location that would minimize the error in the DOA estimates, it is even more effective to calculate the apparent error function (5.1) over a large area. By evaluating the function over the region of interest rather than following gradient descent from an initial point, we can avoid local minima. We directly observe the average angular error, as well as the steepness of the gradient of the apparent angular error around the estimated location. A small average angular error and steep gradient around the minimum of (5.1) would indicate a high confidence location estimate. On the other hand, a large average angular error with a shallow or elongated minimum would indicate some uncertainty in the emitter’s location.

Note that minimization problems generally prefer to minimize the sum of the squared error [82]. However, squared-error places a large weight on outliers [82], and minimizing absolute error is a common strategy to reduce the influence of a few large outliers [83]. In addition, we have found through Monte Carlo simulations that minimizing the sum of the angular errors reduces the sensitivity to outliers and yielded more accurate location estimates. Therefore, we will evaluate (5.1) and find
the minimum of the average apparent angular error. This minimum will be used as the estimated emitter location, and examples in this chapter will demonstrate that the approach works very well. Naturally, it would be simple to modify (5.1) to consider squared errors for setups that yield fewer outliers.

5.4 Single Emitter Results

First, we show examples with a single active emitter. Fig. 5.4 considers the south-west emitter, which broadcasts a simulated RHCP signal. Although the emitter is RHCP, its polarization is assumed to be unknown, and MP-MUSIC is used to estimate its location. For simplicity, the dissertation will assume that the received signal polarization is always the same as the transmitted signal polarization, regardless of the aircraft’s orientation. In addition, the signals are received by the array on the full platform with pylons and tilted wings. Throughout this chapter, the same model will always be used as the true array manifold.

Initially, the simulations assume that the true, in-situ, array manifold is available for signal processing. Since the array manifold is matched and the SNR is very high, every DOA estimate is at the exact emitter location. The left plot shows the observation points along the flight, the true emitter location, and the individual location estimates. However, the emitter location estimates are hidden beneath the marker for the true emitter location. Since all of the emitter location estimates are correct, averaging the estimated x-y points from throughout the flight yields the emitter’s true location.

The right plot shows the average apparent angular error - $\bar{e}(x, y)$ in (5.1) - as a function of position on the ground. As expected, the true emitter location minimized
the average apparent error. The average apparent error at other points in the grid is non-zero, and it increases moving away from the emitter. Due to the relative positions of the emitter and the aircraft, as well as the aircraft attitude, the average apparent error does not increase symmetrically moving radially away from the emitter. Instead, it increases rapidly moving northeast, and more slowly moving southwest. The shape of the \( \bar{e}(x, y) \) around the emitter location arises from geometric dilution of precision [1].

Fig. 5.5 considers the same flight path and emitter setup when the assumed array manifold does not match the true array manifold. The top row assumes that the available manifold is for the array on the flat-wing platform. This is the same as in the “smaller” mismatch setup from Chapter 4. The bottom row assumes that the available manifold is for the array on the body-plate platform (the “larger” mismatch setup). The estimated DOAs are shown in the left plots as blue crosses.

Figure 5.4: RHCP emitter location estimation with a known array manifold. Left: MP-MUSIC estimates on the ground. Right: Mean angular error as a function of position.
Since the true and available array manifolds are different, many DOA estimates have significant errors. For the smaller mismatch setup, the average of all the location estimates was 0.45km away from the true emitter location. For the larger mismatch setup, the average location was 0.86km away from the true emitter location. Although outlier removal may improve averaging in the x-y domain, we have already limited outliers by limiting our search area. In addition, it would be difficult to extend further outlier removal to cases with multiple emitters. Therefore, additional outlier removal will not be considered in this dissertation. On the other hand, the point with the smallest average angular error was 200 meters away from the true location for the smaller mismatch setup (top right plot), while for the larger mismatch setup, the point with the lowest average angular error corresponded to the exact emitter location (bottom right plot). Comparing the left and right plots implies that averaging in the x-y domain is disproportionately influenced by outliers, but minimizing average angular error can recover the true emitter location. This is particularly impressive because the average apparent angular error for the smaller mismatch setup with the flat-wing array manifold was 17.1°. For the larger mismatch setup with the body-plate manifold, the average angular error at the true emitter location was 25.2°! Obtaining the true emitter location within 200m in spite of these errors demonstrates that minimizing the average angular error is robust to very large errors in the DOA estimates.

Next, we considered the long flight path when only the northwest emitter was active. The northwest emitter transmitted a simulated \( \phi \)-polarized (\( E_\phi \)) signal. The \( E_\phi \) emitter was more difficult to locate than the RHCP signal because \( E_\phi \) polarized signals are tangential to the aircraft body. Reflections from the metal aircraft shift
Figure 5.5: RHCP emitter location estimation with a mismatched array manifold for the long flight path. Left: MP-MUSIC estimates on the ground. Right: Mean angular error as a function of position. Top: The available manifold is for the array on the flat-wing platform. Bottom: The available manifold is for the array on the body-plate platform.
the phase of the tangential electric fields by $180^\circ$ \[84\]. Therefore, changes to the aircraft strongly perturb the $E_\phi$ array manifold. On the other hand, RHCP signals are a combination of an $E_\phi$ component and an $E_\theta$ component. The $E_\theta$ component does not undergo an $180^\circ$ phase shift when it is reflected, and it is less affected by changes to the platform around the antenna. Consequently, the $E_\phi$ array manifold is more mismatched than the RHCP manifold, and DOA errors for the $E_\phi$ signal tend to be larger than for the RHCP signal.

Fig. 5.6 shows the results for the long flight path with the $E_\phi$ emitter. The left plots show the estimated DOAs on the ground, and the right plots show the average apparent DOA error (5.1) over the grid. We see that the DOA estimates were even more spread out than in the case with the RHCP emitter. The mean locations were 1.7km and 4.1km away from the true emitter locations for the smaller and larger mismatch setups, respectively. However, the points with the minimum average error were at the true emitter location for both of the mismatched manifolds. The average angular errors at the emitter location were $27.1^\circ$ and $32.3^\circ$. Thus, minimizing the average apparent angular error found the true emitter locations in spite of large and persistent DOA errors.

Fig. 5.7 considers the circular flight path with the northwest ($E_\phi$) emitter active. Averaging locations on the ground yielded errors of 0.6km and 1.8km for the smaller and larger mismatch setups, respectively. Using the same DOA estimates, but finding the location that would minimize the average angular error, yielded errors of 0.0km and 0.2km for the smaller and larger mismatch setups. Again, the average angular errors were large (over 30 degrees for both mismatch setups). Therefore, minimizing
Figure 5.6: $E_\phi$ emitter location estimation with a mismatched array manifold for the long flight path. Left: MP-MUSIC estimates on the ground. Right: Mean apparent angular error (5.1) as a function of position. Top: The available manifold is for the array on the flat-wing platform. Bottom: The available manifold is for the array on the body-plate platform.
angular error accurately estimates the emitter locations for the long and short flight paths, for either mismatch setup, and for either of the western emitters.

Finally, Table 5.1 summarizes the geolocation errors for the examples shown in Figs. 5.4 to 5.7. In every case with a mismatched array manifold, minimizing the apparent angular error yielded more accurate location estimates than averaging the points where the lines of bearing intersect the ground. This trend continues when
estimating more than one emitter location. Therefore, the rest of this chapter will focus on the location estimates yielded by minimizing the average apparent DOA error in (5.1). Moreover, the location errors in this section were 0.2km or less in spite of average DOA errors as large as 32° and DOA estimates that intersected the ground all over the 40km×40km search region.

Table 5.1: Position errors for single emitter scenarios.

<table>
<thead>
<tr>
<th>Emitter (RHCP)</th>
<th>Flight Path</th>
<th>Available Manifold</th>
<th>Error: Mean XY</th>
<th>Error: Minimum Apparent Angular Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Southwest</td>
<td>Long</td>
<td>Full Platform</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td>Northwest (E)</td>
<td>Long</td>
<td>Flat-Wing</td>
<td>0.4 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td>Northwest (E)</td>
<td>Long</td>
<td>Body-Plate</td>
<td>0.9 km</td>
<td>0.0 km</td>
</tr>
</tbody>
</table>

5.5 Two Emitter Results

Next, we considered cases with the RHCP and $E_\phi$ emitters transmitting simultaneously on the same frequency. The flight paths and emitter parameters were the same as in the single signal cases. Note that since the received signal power depends
on the aircraft’s location, the signals arrive with unequal signal to noise ratios. MP-MUSIC and Nullspace MUSIC estimated the signal DOAs, and the number of signals was assumed to be known for both algorithms.

Since two emitters were present, we must separate the DOA estimates into two groups in order to use the averaging and angle error minimization techniques that were used with a single emitter. The grouping requirement applies for MP-MUSIC and Nullspace MUSIC alike, as both techniques consider observations individually without associating DOA estimates from different observations. Initially, we separated the DOA estimates into groups using the k-means algorithm [85]. K-means iteratively finds clusters that minimize the sum of the distances from the cluster center to the estimates. In k-means, different numbers of estimates may be associated with a single emitter. Unequal cluster sizes are appropriate for DOA estimation with manifold mismatch, as mismatch may cause the two best peaks to be located near one of the emitters. For MP-MUSIC with multiple emitters, we required the distance between peaks that are selected as DOA estimates to be at least 1/4 of the true distance between the closest pair of emitters. While this used information about the true signal scenario, it allowed a computer to choose reasonable DOA estimates without human intervention after every observation. On the other hand, Nullspace MUSIC estimates each signal’s DOA from a separate spectrum, so a minimum separation rule was not applied.

Fig. 5.8 shows MP-MUSIC results in the left column and Nullspace MUSIC results in the right column. Every plot in Fig. 5.8 assumes that the body-plate manifold

\(^4\text{k-means may also minimize the sum of the squares of the distances from the cluster centers to the estimates. However, we prefer to use the sum of the distances instead of the sum of the squares because this metric is less influenced by large outliers in the x-y domain.}\)
is available for DOA estimation (larger mismatch setup). The top row shows the flight path, emitter positions, and the individual DOA estimates. The individual DOA estimates are marked with blue ‘x’\'s and orange crosses to indicate the k-means cluster that they were associated with. Unfortunately, k-means separated the MP-MUSIC and Nullspace MUSIC estimates into an eastern group and a western group. These groups did not correspond to the emitters, which were separated along the northing direction. Consequently, the cluster centers were several kilometers from the true emitter locations. The next two rows show the apparent angular errors for the k-means clusters. Surprisingly, the average apparent error for both MP-MUSIC clusters and both Nullspace MUSIC clusters was minimized near the RHCP emitter. Since k-means failed to associate DOA estimates with a particular emitter, every k-means cluster contained DOA estimates corresponding to both emitters. However, manifold mismatch was not as severe for the RHCP emitter, and its DOA estimates were more accurate. Consequently, the point minimizing the apparent DOA error appears to be near the RHCP emitter regardless of which cluster we consider. Note angle error minimization located the RHCP emitter to within 0.20km using the MP-MUSIC DOA estimates. The same process yielded an error of 0.72km using the Nullspace MUSIC estimates. These errors were much better than the corresponding k-means centroid locations, all of which had errors greater than 5km. However, our best guess regarding the second emitter is that it is at the same location as the first emitter, which is not the case. Therefore, we must improve the grouping/association step in order to locate the second emitter.

Section 5.4 showed that the average position estimates were much less accurate than finding the point corresponding to the smallest apparent DOA error over the
Figure 5.8: $E_{\phi}$ and RHCP emitter location estimation with the larger mismatch setup for the long flight path and k-means clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped by k-means. Middle and Bottom Rows: Mean angle error for each cluster of estimates as a function of location.
flight. Therefore, we would expect clustering in the angular domain to also be more accurate than clustering in the x-y domain. However, the angle from the antenna to a fixed point on the ground is different at every observation. Consequently, k-means and other standard clustering algorithms are not directly applicable to DOA estimates. Nevertheless, angle domain clustering can be implemented iteratively without too much difficulty.

Assume that \( M \) emitters transmitted throughout the flight, and that \( M \) is known. Then, the following steps cluster DOA estimates into groups that minimize the apparent angular error throughout the flight:

1. First, initialize \( M \) clusters using the k-means algorithm in the x-y domain.

2. Find the locations that would minimize the apparent angular error for the DOAs in the clusters, (for example, the bottom two rows of Fig. 5.8). Let the cluster centers be \((x_1, y_1), \ldots, (x_M, y_M)\).

3. If two cluster centers are identical, then shift one of the centers to an adjacent grid point.

4. For all DOAs in the first cluster, calculate \( \Delta \sigma(\tilde{\theta}_t, \tilde{\phi}_t, \theta_t(x_1, y_1), \phi_t(x_1, y_1)) \). This gives the apparent DOA error from the DOA estimate \( \tilde{\theta}_t, \tilde{\phi}_t \) to the first emitter’s estimated location for observation \( t \).

5. Next, calculate the apparent DOA errors if the estimates were associated with the other cluster centers, \( \Delta \sigma(\tilde{\theta}_t, \tilde{\phi}_t, \theta_t(x_m, y_m), \phi_t(x_m, y_m)) \) for \( m = 2, \ldots, M \).

6. Then, assign the DOA estimate to the cluster for which it has the minimum apparent error.
7. Repeat the reassignment for every DOA estimate in every cluster.

Finally, steps 2-7 are repeated until DOA estimates no longer swap between clusters. The location estimates are the positions minimizing (5.1) for the DOAs assigned to each cluster. We will refer to the angle domain clustering and location estimation as Minimum Apparent Angular Error (MAAE) clustering.

Fig. 5.9 shows the MAAE clusters and DOA error functions for the same setup as in Fig. 5.8. Recall that this setup uses the larger mismatch setup. As before, the results with MP-MUSIC are shown on the left, while the results with Nullspace MUSIC are shown on the right. First, we note that the clusters, which are indicated by different colors and markers, mixed together in the x-y domain. Nevertheless, these groupings minimize the apparent angular error for the DOA estimates. Finding the final groupings required four iterations with the MP-MUSIC estimates and four iterations with the Nullspace MUSIC estimates. Nevertheless, MAAE found both emitter locations using both the MP-MUSIC and Nullspace MUSIC estimates. With MP-MUSIC DOAs, the estimated locations of the RHCP and $E_\phi$ emitters were correct (0km error). For the Nullspace MUSIC results on the right, the RHCP emitter was correct, but the $E_\phi$ location error was 0.63km. Thus, MAAE separated the DOA estimates into clusters corresponding to the two emitters, but large outliers can still corrupt the association step and bias the final location estimates.

Fig. 5.10 considers the same flight path and emitter setup, but with the “smaller” manifold mismatch setup. Using the MP-MUSIC DOAs, MAAE found the RHCP and $E_\phi$ emitters with location errors of 0.0km and 0.2km. MAAE’s location estimates were 0.4km and 0.2km when using the Nullspace MUSIC results. However, MAAE required six iterations to finalize the clusters of MP-MUSIC estimates, and it required
Figure 5.9: $E_\phi$ and RHCP emitter location estimation with the larger mismatch setup for the long flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Middle and Bottom Rows: Mean angle error for each cluster of estimates as a function of location.
eleven iterations to finalize the clusters of Nullspace MUSIC estimates. Thus, MAAE clustering worked very well, but grouping estimates together was computationally expensive.

Fig. 5.11 shows the short circular flight path with the same emitter setups. The body-plate array manifold is available for DOA estimation. As noted before, this flight path maintained a larger angular separation between the emitters from the perspective of the array. The top row shows the flight path, emitter locations, and MAAE clusters for the MP-MUSIC DOAs (left) and the Nullspace MUSIC DOAs (right). These clusters required three iterations and two iterations for convergence, respectively. MAAE required fewer iterations than for the long flight path because the estimates were already well separated in angle space. In addition, the estimates in the clusters are generally well separated in the x-y domain, indicating that DOA estimates tended to land near the emitters. Therefore, the initial k-means clusters were a better match for the final MAAE clusters, which helped to reduce the number of iterations. The plots in the bottom rows show the angular error functions. We immediately see that MAAE located both emitters using the MP-MUSIC and Nullspace MUSIC DOAs. In each case, the location error is 0.2km. The results with the smaller mismatch setup, (not shown), are similar, with errors of 0.2km or less. Therefore, MAAE can locate both the RHCP and the $E_{\phi}$ emitter with very small error when the aircraft follows a favorable flight path. Table 5.2 summarizes the results for this section. Note that plots for the circular flight path with the flat-wing manifold were similar to the results with the body-plate manifold, and they are not shown in a figure.

In summary, locating multiple simultaneous emitters requires the estimates to be grouped before they can be processed to locate an individual emitter. With high
Figure 5.10: $E_{\phi}$ and RHCP emitter location estimation with the smaller mismatch setup for the long flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Middle and Bottom Rows: Mean angle error for each cluster of estimates as a function of location.
Figure 5.11: $E_\phi$ and RHCP emitter location estimation with the larger mismatch setup for the circular flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Middle and Bottom Rows: Mean angle error for each cluster of estimates as a function of location.
Table 5.2: Position errors for two-emitter scenarios.

<table>
<thead>
<tr>
<th>Flight Path</th>
<th>Available Manifold</th>
<th>Emitter</th>
<th>Error: MP-MUSIC &amp; MAAE</th>
<th>Error: Nullspace MUSIC &amp; MAAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long</td>
<td>Body-Plate</td>
<td>Northwest (Eϕ)</td>
<td>0.0 km</td>
<td>0.6 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest (RHCP)</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td>Long</td>
<td>Flat-Wing</td>
<td>Northwest (Eϕ)</td>
<td>0.2 km</td>
<td>0.4 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest (RHCP)</td>
<td>0.0 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td>Circle</td>
<td>Body-Plate</td>
<td>Northwest (Eϕ)</td>
<td>0.2 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest (RHCP)</td>
<td>0.2 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td>Circle</td>
<td>Flat-Wing</td>
<td>Northwest (Eϕ)</td>
<td>0.2 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest (RHCP)</td>
<td>0.2 km</td>
<td>0.2 km</td>
</tr>
</tbody>
</table>

SNR and a perfectly known array manifold, grouping is trivial because the DOA estimates are clearly separated on the ground. However, manifold mismatch biases the DOA estimates from the individual observations, causing estimates throughout the flight to intermix on the ground. Then, DOA estimates cannot be separated by looking at their position on the ground; estimates corresponding to both emitters end up in both clusters. However, estimates can be separated by grouping them so that the average apparent error for the DOAs and the estimated emitter locations is minimized. Clustering and estimating locations by Minimizing the Apparent Angular Error (MAAE) yielded errors of 0.2km or less in every case with the circular flight path. For the long flight path, the RHCP emitter was always located with errors of 0.2km or less. However, the error for the Eϕ emitter was as much as 0.6km using the
Nullspace MUSIC estimates. The $E_\phi$ emitter is relatively close to the RHCP emitter, and its signal is more strongly affected by scattering. Therefore, while MAAE is much more robust than k-means clustering on the ground, it may still be biased because of erratic DOA estimates that preferentially lie around another nearby emitter. In addition, MAAE may be non trivially expensive, as eleven iterations were required to finalize the DOA estimate groups for Nullspace MUSIC on the long flight path with the smaller mismatch setup.

### 5.6 Four Emitter Results

Next, we considered cases with all four co-frequency emitters transmitting simultaneously. The southeast emitter transmitted an $E_\theta$ signal, and the northeast emitter transmitted an RHCP signal. As before, the northwest and southwest emitters transmit $E_\phi$ and RHCP signals, respectively. All other parameters are the same as in previous results. Since Mixed Polarization MUSIC and Nullspace MUSIC can only estimate up to $N - 2 = 5$ emitters (where $N$ is the number of antenna elements in the array), this is a very challenging geolocation setup.

#### 5.6.1 Long Flight Path

Fig. 5.12 shows results for the long flight path with the larger mismatch setup (the body-plate array manifold is available for DOA estimation). As before, the left plots use MP-MUSIC’s DOA estimates and MAAE location estimation. The right plots use Nullspace MUSIC’s DOA estimates. The top row shows the MAAE clusters. Here, we see that the DOA estimates were very spread out, and the clusters intermixed considerably in the x-y domain. MAAE required twelve iterations to form clusters from the MP-MUSIC estimates and seventeen iterations to form clusters from
the Nullspace MUSIC estimates. Nevertheless, the MP-MUSIC DOAs gave relatively good location estimates for three emitters. Considering the emitters counter-clockwise from the northwest, the errors were 19.8km, 2.4km, 0.6km, and 0.8km. However, the Nullspace MUSIC estimates yielded a very poor cluster that does not correspond to the $E_{\phi}$ emitter, which could not be located. The errors for the other three emitters were 10.8km, 0.4km, and 0.6km. While the $E_{\phi}$ emitter could not be located from the long flight path using MP-MUSIC or Nullspace MUSIC, it is nevertheless impressive that the DOA estimates located any emitters with errors less than 1km considering the unknown polarization, low elevation angles, small separations, and the spread of estimates shown in the top row.

Fig. 5.13 considers the same setup with the exception that the manifold for the array on the flat plate model was available for DOA estimation (smaller mismatch setup). As before, the top row shows that the DOA estimates are very spread out, and the clusters intermixed on the ground. The next four rows show the average apparent angular error for the MP-MUSIC estimates (left) and the Nullspace MUSIC estimates (right). In spite of the fact that this setup had less mismatch than the previous case, MAAE iterated fourteen times using the MP-MUSIC DOAs and fifteen times using the Nullspace MUSIC estimates. As before, MAAE could not locate the $E_{\phi}$ emitter using the MP-MUSIC estimates. The large number of iterations and the inability to find the $E_{\phi}$ emitter suggest that MAAE is not sufficiently robust to poor DOA estimates. Otherwise, the location errors for the emitters (starting in the southwest and proceeding counter-clockwise) were 0.2km, 1.4km, 0.2km. Using the Nullspace MUSIC estimates, the southwest and southeast emitters were located with errors of 0km and 5.1km. As in the larger mismatch case, Nullspace MUSIC and MAAE do
Figure 5.12: Four emitter location estimation with the larger mismatch setup for the long flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Bottom Rows: Mean angle error for each cluster as a function of location.
not yield a good position estimate for the $E_\phi$ emitter. However, the northwest emitter also cannot be found.

Table 5.3 summarizes the errors for the long flight path. The $E_\phi$ emitter was never located, and Nullspace MUSIC only yielded accurate location estimates for two emitters at a time. Clearly, Nullspace MUSIC is not yielding improved performance for the long flight path as compared to MP-MUSIC. Based on the previous chapter, we expect MP-MUSIC to be more accurate with manifold mismatch for small signal separations, for which Nullspace may suppress both signals and select a spurious peak that is far away from any signal. Spurious peaks also challenge clustering, and we have seen that correctly associating DOA estimates together is critical for obtaining an accurate geolocation estimate. However, these examples show that, with manifold mismatch, Nullspace MUSIC may not be sufficient to separate emitters during a flight for which the emitters appear close together in angle space.

<table>
<thead>
<tr>
<th>Available Manifold</th>
<th>Emitter</th>
<th>Error: MP-MUSIC &amp; MAAE</th>
<th>Error: Nullspace MUSIC &amp; MAAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body-Plate</td>
<td>Northwest</td>
<td>19.8 km</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>2.6 km</td>
<td>10.8 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>0.6 km</td>
<td>0.4 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>0.8 km</td>
<td>0.6 km</td>
</tr>
<tr>
<td>Flat-Wing</td>
<td>Northwest</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.2 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>1.4 km</td>
<td>5.1 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>0.2 km</td>
<td>19.8 km</td>
</tr>
</tbody>
</table>

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Figure 5.13: Four emitter location estimation with the smaller mismatch setup for the long flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Bottom Rows: Mean angle error for each cluster as a function of location.
5.6.2 Circular Flight Path

Figs. 5.14 and 5.15 show the results for the circular flight path with the larger and smaller mismatch setups, respectively. As before, the left column considers the MP-MUSIC DOAs, while the right column considers the Nullspace MUSIC estimates. Although DOA estimates continue to spread out, the MAAE clusters intermix less with the circular flight path, which keeps the emitters well separated in angle space. Five to nine iterations were required for the MAAE clusters to converge. However, all four emitters can be located with both mismatch setups and with both algorithms’ DOA estimates. The northwest ($E_{\phi}$) emitter yields a 6.0km location error for MP-MUSIC in the smaller mismatch setup due to its difficult polarization and its proximity to the southwest/RHCP emitter. The same emitter was located with much less error using MP-MUSIC’s DOAs for the larger mismatch setup, indicating that MP-MUSIC and MAAE are sensitive to the particular manifold mismatch setup. In addition the southeast ($E_{\theta}$) emitter challenges Nullspace MUSIC and MAAE due to geometric dilution of precision.

Table 5.4: Position errors for the circular flight path with four emitters.

<table>
<thead>
<tr>
<th>Available Manifold</th>
<th>Emitter</th>
<th>Error: MP-MUSIC &amp; MAAE</th>
<th>Error: Nullspace MUSIC &amp; MAAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body-Plate</td>
<td>Northwest</td>
<td>0.2 km</td>
<td>0.3 km</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.6 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>1.0 km</td>
<td>5.5 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>1.0 km</td>
<td>1.1 km</td>
</tr>
<tr>
<td>Flat-Wing</td>
<td>Northwest</td>
<td>6.1 km</td>
<td>1.5 km</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.3 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>2.1 km</td>
<td>5.2 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>1.1 km</td>
<td>0.8 km</td>
</tr>
</tbody>
</table>
Figure 5.14: Four emitter location estimation with the larger mismatch setup for the circular flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Bottom Rows: Mean angle error for each cluster as a function of location.
Figure 5.15: Four emitter location estimation with the smaller mismatch setup for the circular flight path and iterative clustering. Left Column: MP-MUSIC results. Right Column: Nullspace MUSIC results. Top Row: Estimates on the ground, grouped to minimize the sum of angular errors. Bottom Rows: Mean angle error for each cluster as a function of location.
The location errors for the circular flight path with the larger and smaller mismatch setups are summarized in Table 5.4. We see that Nullspace MUSIC improves some of the geolocation estimates when the emitters are well separated in angle space throughout the flight. However, when geolocating unknown emitters, qualities such as the angular separation between emitters are not known a-priori. Therefore, the next chapter will apply another approach to geolocating emitters in spite of array manifold mismatch. This new method avoids the challenge of grouping highly erratic estimates. However, it is still unable to locate all four emitters using the long flight path. All four emitters can only be found when the new method is combined with Nullspace processing.

5.7 Summary and Conclusions

This chapter applied Mixed Polarization (MP)-MUSIC [24] and Nullspace MUSIC (Chapter 3) to geolocate ground-based RF emitters during simulated flights. Conventional geolocation employs DOA estimates from many points along the flight path to draw Lines of Bearing (LOBs) from the airplane along the estimated DOAs. Ideally, the DOAs would intersect with the ground at the emitter’s position. In practice, inconsistent LOBs must be processed to determine the emitter’s position. The emitters’ signals were uncorrelated, and they arrived with SNRs greater than 30dB throughout the flights. Nevertheless, antenna array manifold mismatch caused MP-MUSIC’s DOA estimates to be very erratic, even with a single incident emitter. In addition, geometric dilution of precision at low elevation angles spread DOA estimates out over a large area. Consequently, processing DOA estimates in the x-y domain makes the final location estimates inaccurate and susceptible to outliers.
On the other hand, the point on the ground that minimizes the apparent angular error throughout the flight reveals the true emitter location very accurately in spite of array manifold mismatch. Indeed, Minimizing the Average Apparent Error (MMAE) in the DOA estimates yielded a single emitter’s location to within 0.2km in spite of average DOA errors of more than 25°. The MAAE method is a well-known method for combining noisy DOA estimates, and the minimum error point has previously been found using gradient descent so that geolocation does not become computationally burdensome [30].

However, manifold mismatch may severely challenge conventional geolocation if multiple co-channel emitters transmit simultaneously. In this case, the DOA estimates must be separated into groups before they can be used to localize an emitter. This applies to MP-MUSIC and Nullspace MUSIC, which independently process different observations that were made during a flight. Clustering in the x-y domain yielded groups containing DOA estimates for multiple emitters. In our example with two emitters, only one emitter could be found after the estimates were clustered in the x-y domain. In fact, both clusters yielded the same approximate emitter location.

Consequently, DOA estimates should be clustered to Minimize the Apparent Angular Error (MAAE) for the estimates in a group. The novel MAAE clustering algorithm successfully located two simultaneous emitters over the long and circular flight paths with both mismatch setups. However, with four emitters, the northwest ($E_\phi$) emitter could not be located using the long flight path. The $E_\phi$ manifold is the most susceptible to changes in the aircraft structure because tangential electric fields that are reflected from the platform are phase shifted by 180°. Remarkably, however, the other emitters in the scene were often located with small error in spite of manifold
mismatch, poor angular separation between emitters, unknown polarization, and the fact that the array can estimate at most five DOAs using Mixed Polarization MUSIC. MAAE approximately located all four emitters with the circular flight path. We also observed that MAAE’s location estimates were sensitive to the particular mismatch setup. The same flight path and DOA algorithm may yield geolocation estimates that are several kilometers apart for the same emitter depending on which antenna array manifold is available.

Throughout the chapter, we did not see a significant improvement in the geolocation estimates when using Nullspace MUSIC for DOA estimation. This is not surprising because the long flight path brought the emitters close together in the angle domain. Chapter 4 showed that Nullspace MUSIC outperforms MP-MUSIC’s mean error and error variance for signals with $\theta \leq 70^\circ$ when the signals are separated by more than about $37^\circ$, or 74% a beamwidth at zenith. In addition, the antenna array beamwidth is very large near horizon because the planar array of patch elements does not have much vertical aperture. Even for the circular flight path, the northwest and southwest emitters were often seen from $\theta \geq 80^\circ$, and they were often separated by $40^\circ$. During the long flight path, signals were sometimes separated by less than $20^\circ$. Consequently, Nullspace MUSIC generally did not yield more accurate DOA estimates over the flight paths that we considered.

In addition to not always finding all emitters and being sensitive to the particular mismatch setup, MMAE may require many iterations to converge on the best groupings of DOA estimates. In these examples, five to twelve iterations were common, with as many as seventeen iterations being required for convergence. Therefore, the next chapter will introduce another method to locate emitters during a flight. This
method is less sensitive to changes in the mismatch setup, and it requires significantly fewer computations for peakfinding, and the DOA association step is removed entirely. Although Nullspace MUSIC did not improve geolocation in this chapter, Nullspace processing plays a critical role in Chapter 6.
Chapter 6: Direct Mapping Method for RF Emitter Geolocati

6.1 Introduction

Traditionally, Radio-Frequency (RF) emitter geolocation with an airborne receiver relies on Direction of Arrival (DOA) estimation followed by triangulation [1]. In triangulation, Lines of Bearing (LOBs) are drawn from the receiver along the estimated DOAs. Lines of Bearing from different locations towards the same emitter should intersect with the ground at the emitter’s location. However, DOA errors cause the LOBs to point towards different locations on the ground. Consequently, the DOA estimates are inconsistent with each other, and the inconsistent measurements must be carefully processed in order to find the most likely position of the emitter [28, 29]. For the long and winding flight path in Chapter 5, differences between the true and assumed antenna array manifolds caused DOA estimates for a single emitter to land throughout a 40km×40km area. Nevertheless, the location minimizing the sum of the apparent DOA errors throughout a flight yielded a very good estimate of a single emitter’s location. For the western emitters, these errors were 0.2km or less, which is very small as compared to the size of the search area and the length of the flight path.
DOA errors may cause geolocation to fail for multiple emitters. The DOAs for simultaneous emitters that overlap in the frequency domain must be separated into groups. Each group of DOAs must correspond to a single emitter for the emitter’s location to be estimated. However, grouping estimates is non-trivial because inaccurate DOA estimates from multiple emitters often intermix, and DOA estimation does not associate DOAs with specific emitter. Consequently, inconsistent estimates turn association into a significant challenge when geolocating multiple emitters.

In general, the association problem must be solved by studying the DOAs themselves. In Chapter 5, we iteratively clustered DOAs to find the groups and emitter location estimates that yielded the Minimum the Apparent Angular Error (MAAE) throughout the flight. MAAE clustering enabled us to consistently locate two emitters in spite of large DOA errors that were induced by array manifold mismatch. In some cases, however, manifold mismatch prevented us from locating four simultaneous and overlapping emitters. In addition, we observed that different manifold mismatch setups with the same flight path and emitters could shift an emitter’s estimated location by several kilometers. Since relatively small changes in the estimated DOAs can lead to large changes in the final clusters and estimated emitter positions, we concluded that MAAE clustering was not sufficiently robust for the cases considering multiple emitters with different polarizations. While Chapters 3 and 4 showed that Nullspace MUSIC can improve DOA estimates in single-look DOA estimation, the flight paths in Chapter 5 often brought emitters close together in angular space, and applying Nullspace MUSIC before MAAE clustering usually did not improve the final location estimates.
In this chapter, we apply another method [53, 54, 55, 86] for locating emitters on the ground to the same mismatch scenarios that we considered in Chapter 5. In this approach, the direction of arrival spectra are mapped directly to the ground. The mapped spectra from different observations along the flight path are averaged, and the maxima of the composite spectrum are taken as the estimated emitter locations. Since this method maps DOA spectra from the angular domain directly to the ground, we will refer to it as the Direct Mapping Method (DMM). In comparison with traditional geolocation (triangulation), DMM reduces the variance of the location estimate in cases with low Signal to Noise Ratio (SNR), [57]. DMM also removes DOA ambiguities associated with sparse antenna arrays [86]. However, we will continue to assume a filled array and a good signal to noise ratio. Instead, we present a novel study on DMM’s performance in cases with manifold mismatch.

We find that DMM is robust to mismatch in that it yields almost the same results for both manifold mismatch setups. However, DMM may not uniquely identify the emitter locations in challenging scenarios. Like MUSIC, DMM may yield more peaks than the number of emitters. In such cases, the emitter location estimates are ambiguous, and the $M$ emitter locations may not correspond to the $M$ ‘best’ peaks in the DMM spectrum. Therefore, we combine DMM with Nullspace MUSIC in a novel approach that we call \textit{Nullspace DMM}. In Nullspace DMM, the emitter location estimates are the $M$ absolute maxima of $M$ different spectra. As a result, Nullspace DMM’s emitter location estimates are unambiguous. In addition, we show that this approach definitively locates all four emitters during both flight paths and with both mismatch setups that were considered in Chapter 5.
The rest of this chapter proceeds as follows. Section 6.2 explains DMM, and Section 6.3 shows results with a single emitter. DMM easily locates a single emitter using the true or mismatched array manifolds. Section 6.4 combines DMM with Nullspace MUSIC so that multiple emitter locations may be estimated individually. We refer to the combined approach as Nullspace DMM. The flight paths with two simultaneous emitters are considered in Section 6.5. Both emitters are easily found by Nullspace DMM. Finally, Section 6.6 returns to the four-emitter cases from Chapter 5. This scenario is too challenging for DMM, and the spectra do not uniquely identify all four emitter locations. However, Nullspace DMM yields an accurate location estimate for every emitter in both mismatch scenarios. Therefore, Nullspace DMM is the most robust approach to locating multiple emitters in spite of antenna array manifold mismatch.

6.2 The Direct Mapping Method

The Direct Mapping Method (DMM) maps DOA spectra for several observations along a flight path directly to a fixed area of interest on the ground\(^5\). The fixed area is divided into a grid, and the DOA spectra from all observations are mapped to the same grid. Note that in Chapter 5, we mapped a 40km × 40km grid to angle space and estimated DOAs over the limited domain. Mapping from the ground to the angular domain provides a simple way to ensure that the true DOAs are included in the angular search area. Recall that the receiver sometimes did not have line of sight to a transmitter. Therefore, estimating DOAs underneath the airplane would

\(^5\)DMM can also be applied in other situations, such as several stationary towers that can observe emitters from a common area. However, this dissertation will only consider the case with an airborne receiver.
have been insufficient. Mapping also avoids extreme outliers, and it avoids LOBs that never intersect with the ground. After estimating DOAs over the grid on the ground in Chapter 5, we treated the peaks on the grid as conventional line of bearing estimates.

However, since the DOA spectra from different observations along the flight path cover the same points on the ground, the spectra themselves can be averaged together to form a composite spectrum. The \( M \) extrema of the averaged spectrum are the estimated locations for the \( M \) emitters. If spectra from a diverse set of observations are averaged together, then spurious extrema should be de-emphasized, and the true emitter locations should be revealed by consistently minimizing or maximizing the DOA spectrum that was mapped to the ground. Naturally, mapping spectra to the ground requires precise knowledge of the aircraft position and attitude. However, the conventional geolocation approach requires the same information to draw lines of bearing from the aircraft to the ground along the estimated DOAs. In addition, triangulation and DMM both assume that the emitters are stationary, and that they transmit for the duration of the receiver’s flight. Therefore, DMM has the same fundamental requirements as traditional airborne geolocation.

Let \( g(\theta, \phi) \) be a generic function versus angle that is applied for Direction of Arrival estimation. For instance, Chapter 2 discussed classical beamforming, the Minimum Variance Distortionless Response (MVDR) beamformer, and MUSIC. Depending on the DOA function, \( g(\theta, \phi) \) may be applied to dominantly polarized arrays or polarization sensitive arrays. Whichever function is used, \( g(\theta, \phi) \) can be rewritten as a function of \( x \) and \( y \) location on the earth by mapping from the \((x, y)\) domain to the \((\theta, \phi)\) domain. As in Chapter 5, let the angles from the aircraft to the grid point
\((x_0, y_0)\) at observation \(t\) be \(\theta_t(x_0, y_0)\) and \(\phi_t(x_0, y_0)\). These angles are in the antenna coordinate system, which incorporates the position and attitude of the host aircraft. Then, the DMM spectrum is written as the average of the \(g(x, y)\) spectra that were observed throughout the flight. In this dissertation, we will write the DMM spectrum as

\[
G(x, y) = \frac{1}{T} \sum_{t=1}^{T} g^r(\theta_t(x, y), \phi_t(x, y))
\] (6.1)

where \(r\) is an exponent controlling how the various observations are combined.

The exponent \(r\) facilitates linear averaging of an algorithm’s underlying metric. For example, single polarization MUSIC (2.20) measures the square of the projection of the manifold vectors into the noise space, \(\|P_N \hat{a}(\theta)\|^2\), and the minimum value corresponds to the estimated direction of arrival. However, MUSIC is usually plotted as the inverse, \(\|P_N \hat{a}(\theta)\|^{-2}\), and the DOA is estimated from the peak. Naturally, adding the inverses of the squared projections would be non-linear, and it may not have the same extrema as the sum of the projections. Regardless of how the DOA function \(g(x, y)\) is written, the exponent \(r\) may be adjusted so that observations are combined linearly.

Note that \(G(x, y)\) is more robust to spurious DOA estimates if the spectra calculated throughout the flight are averaged linearly. According to the Central Limit Theorem [59], averaging independent random variables tends to yield a Gaussian distribution of averages, even if the summed variables themselves are non-Gaussian. At a grid point, the averaged spectra from diverse observations should tend to have a more normal distribution than individual spectra, or non-linear sums of the spectra. In other words, the mean spectrum at each grid point should reflect how well the antenna array manifold tends to match the observed signal subspace with less influence.
from outliers caused by manifold mismatch at particular angles. Therefore, we expect a linearly averaged spectrum to contain fewer outliers than a non-linearly averaged spectrum.

While any spectral DOA algorithm may be used for DMM, we employ Mixed Polarization MUSIC (2.30) for DMM in this chapter. Thus, the DMM spectrum is given by

$$G_{MP}(x, y) = \frac{1}{T} \sum_{t=1}^{T} \lambda_{\min}^{\frac{1}{2}}(M(\theta_t(x, y), \phi_t(x, y))),$$

(6.2)

where the $2 \times 2$ matrix $M$ is given by (2.27) in Chapter 2. Thus, (6.2) averages the square root of the minimum eigenvalue of $M$ at a point over the flight. Note that the eigenvalues are proportional to the squared projections in $M$, and the square roots of the eigenvalues are proportional to the projections of the test vectors into the noise subspace.

Emitter locations are estimated by calculating (6.2) on a grid of $(x, y)$ points spanning an area of interest. If $(x_0, y_0)$ corresponds to the true location of an emitter, then $\lambda_{\min}^{\frac{1}{2}}(M(\theta_t(x_0, y_0), \phi_t(x_0, y_0)))$ is ideally zero for all $t$ throughout the flight. For consistency with previous MUSIC results, we will plot the inverse of (6.2), $G_{MP}^{-1}(x, y)$. Thus, the peaks of the plotted spectrum identify the estimated emitter locations. Just as we chose to sum the DOA errors (and not their squares) in Chapter 5 for MAAE, we choose to sum the square roots of the minimum eigenvalues because linear combinations of estimates are less susceptible to outliers [83]. Consequently, (6.2) yields more accurate geolocation estimates in the presence of antenna array manifold mismatch.
Note that for $T$ observations, DMM requires peaks to be found only once. In contrast, conventional DOA estimation requires a peak search following every observation. DMM’s deferral of the location estimate until all information is available also avoids discarding information from the MUSIC spectra. If an individual DOA spectrum contains more peaks than the number of signals and several peaks that have similar heights, then some peaks will be ignored by the conventional approach. The fact that these points were possible emitter locations is lost and does not contribute to the final location estimate. On the other hand, DMM retains that information from the observations and uses it to estimate the emitter location from the final DMM spectrum. Conventional DOA estimation should also be careful to avoid DOA estimates that do not intersect with the ground. On the other hand, DMM naturally incorporates the assumption that the emitter is ground based. Finally, the MAAE clustering and location estimation steps discussed in Chapter 5 can also be expensive to compute. Therefore, DMM is actually simpler to implement, uses all of the available information from MUSIC, and it requires fewer computations than conventional, two-step, geolocation.

### 6.2.1 Optional Weighting

If desired, per-point weighting can also be applied to the DMM function. For example, the patch antennas in the seven-element array have a smaller 3dB beamwidth at their broadside angles as compared to the horizon. Therefore, we may want $G(x_0, y_0)$ to emphasize the observations of $(x_0, y_0)$ for which $\theta$ was near broadside and deemphasize the observations of $(x_0, y_0)$ for which $\theta$ was near horizon. In that case, (6.1) can be easily modified to provide a different set of weights for each point
on the ground. Let the weighted version of $G(x, y)$ be

$$G_W(x, y) = \frac{1}{W(x, y)} \sum_{t=1}^{T} w(\theta_t(x, y)) g^r(\theta_t(x, y), \phi_t(x, y)),$$  \hspace{1cm} (6.3)

where

$$W(x, y) = \sum_{t=1}^{T} w(\theta_t(x, y)),$$  \hspace{1cm} (6.4)

and $w(\theta_t(x, y))$ is a weight function emphasizing observations with small $\theta$ angles. For example, $w(\theta_t)$ could simply be $\cos(\theta_t)$ for $\theta \leq 90^\circ$ and zero otherwise. Since each point on the ground is normalized by $W(x, y)$, each point in the grid receives the same total weight throughout the flight.

Note also that a weight function could be derived such that zero weight was given to observations for which a point on the ground was obstructed by the aircraft structure and the array did not have line of sight to the point on the ground. Weights could also favor observations for which a system expects high SNR, (based on the distance to a grid point), or for which the array manifold is known to be well calibrated. While the weighting functions (6.3) and (6.4) further increase the versatility and robustness of DMM, the results shown in this dissertation will use (6.2), which employs uniform weighting. Using uniform weighting demonstrates that DMM can be applied simply and yield excellent results.

### 6.3 Single Emitter DMM

We applied DMM using the same recorded flight paths, emitter locations, and emitter polarizations that were discussed in Section 5.2. The first flight path wound back and forth over a $40\text{km} \times 40\text{km}$ area of interest, while the second flight path made a short circle in the middle of the same search area. Both flight paths employed the
aircraft’s recorded attitude to determine the angle towards the emitter and the angles towards the DMM grid points on the ground. The signal to noise ratio at the receiver is a function of the receiver’s location. However, it is always greater than 30dB for all emitters, and one thousand simulated snapshots are used to estimate the antenna array covariance matrix. In addition, DMM assumed that the number of emitters was known. Thus, any observed geolocation errors will be due to mismatch between the true and assumed antenna array manifold. As before, we assumed that the true, *in-situ*, array manifold corresponded to the array on the full platform with pylons and wing tilt.

First, we considered a case with a perfectly known array manifold, and the northwest \((E_\phi)\) emitter transmitted throughout the flight. Fig. 6.1 shows the DMM spectrum for each flight path, in dB. Recall that the plotted DMM spectrum is the inverse of (6.2), which was calculated over the \(40\text{km} \times 40\text{km}\) area of interest using the \(T\) observations along the flight path. The observation points are shown with magenta ‘x’ markers in Fig. 6.1. Since the array manifold was perfectly known and the SNR was very high, DMM peaked very sharply at the true emitter location. Although the peak heights are greater than 50dB, we clipped both color scales to 15dB so that they match later plots. Thus, DMM worked as expected with a known antenna array manifold.

Next, we considered the same flights with array manifold mismatch. Manifold mismatch was introduced by simulating the received signal samples with the *in-situ* array manifold, while DMM estimated the emitter locations using the array manifold corresponding to the array on a simpler version of the platform. In other words, \(A\) in (2.6), \(X = AS + N\), was taken from *in-situ* array manifold, while the test vectors
Figure 6.1: Single emitter DMM spectra for the northwest ($E_\phi$) emitter. The true, \textit{in-situ} manifold is available for DMM.

\textbf{b} and \textbf{d} in (2.27) were taken from the array manifold for the seven-element array on the body-plate or flat-wing models. Since the flat-wing model is more similar to the true platform, we refer to cases using the flat-wing manifold for DMM as the \textit{smaller mismatch setup}. We will refer to cases for which DMM uses the body-plate manifold as the \textit{larger mismatch setup}.

Fig. 6.2 shows the DMM spectra for the larger mismatch setup, (top row), and the smaller mismatch setup, (bottom row). Mismatch causes the peaks of the DMM spectrum to be much lower than in Fig. 6.1. Nevertheless, DMM’s position error was 0.28km or less in all four cases. Moreover, the peak in each DMM spectrum was clear and unambiguous. Recall from Figs. 5.6 and 5.7 that the conventional approach’s DOA estimates for the same scenarios were spread out all over the area of interest, and accurate geolocation estimation required finding the point on the ground that would minimize the apparent angular error over the flight. On the other hand, DMM did not require DOA estimates for each observation, and DMM did not require a
Figure 6.2: Single emitter DMM spectra for the northwest ($E_\phi$) emitter. Top Row: The body-plate manifold is available for DMM (larger mismatch setup). Bottom Row: The flat-wing model is available for DMM (smaller mismatch setup). Left: Long flight path. Right: Circular flight path.

search for the position minimizing the angular error. Consequently, DMM was also less computationally expensive than traditional geolocation.

Fig. 6.3 considers the same flight paths and mismatch setups with the southwest (RHCP) emitter active. This time, the position errors were 0.2km or less in all four cases. Since manifold mismatch is less severe for the RHCP array manifolds, the peaks are also about 5dB higher than in Fig. 6.2. As before, the DMM spectra clearly indicated the emitter location without the outliers that were present when DOAs were
estimated for every observation in Chapter 5. In addition, we note that the top row (for which DMM used the body-plate array manifold) and the bottom row (for which DMM used the flat-plate array manifold) are visually very similar. Thus, DMM is robust to mismatch in the sense that a slightly different manifold mismatch setup did not change the final DMM spectrum by much. This is an advantage of DMM, since the user operating the geolocation system would prefer the DMM spectrum and the emitter position estimates to be insensitive to the configuration of the host aircraft during the flight.
6.4 Nullspace DMM

The Direct Mapping Method can also be applied to cases with multiple simultaneous emitters. In this case, $M$ peaks corresponding to the $M$ emitters must be selected from the DMM spectrum. However, selecting multiple peaks from a single spectrum was problematic in Chapter 3, where MUSIC spectra contained a merged peaks between two emitters (Figs. 3.6-3.9). Similarly, manifold mismatch caused merged peaks and spurious peaks in Chapter 4. In this case, MUSIC did not indicate which peaks corresponded to the emitter DOAs. Consequently, the DOA estimates were ambiguous. Chapters 4 removed the ambiguity by applying Nullspace MUSIC to mismatch setups and estimating each DOA from a separate MUSIC spectrum. The resulting DOA estimates were more accurate, so long as the signals were not too close together.

The same logic may also be used to extend Nullspace MUSIC to DMM. In this approach, the first emitter’s location is initially estimated from the usual DMM spectrum. Let the estimated location be $(\tilde{x}_1, \tilde{y}_1)$. Then, a new DMM spectrum is generated from beams that place nulls along the direction of the estimated emitter location at every observation. Recall that $N$ antenna elements can be used to generate $N - 1$ orthogonal beams, each of which destructively cancels a signal arriving from a specified direction. As in DMM, $T$ individual spectra from the $T$ observations are averaged together, and the second emitter’s location is estimated from the composite spectrum. However, the composite spectrum has an approximate null at the first emitter’s location. Thus, the second emitter’s location is estimated after direction-constrained nulling has been applied throughout the flight to minimize the received power from the first emitter’s estimated position. If two emitters are present, the next spectrum
nulls the second emitter’s position throughout the flight and re-estimates the first emitter’s position. This process repeats until the emitter location estimates do not change between iterations. After the algorithm converges, we obtain two emitter location estimates that were made with minimal interference from each other.

If more than two emitters are present, then emitters are nulled progressively until a single emitter is left. Then, emitter locations are estimated in round-robin fashion such that each emitter’s location is estimated with minimal interference from the other emitters. This continues until every emitter’s position is unchanged from one iteration to the next, or until a maximum number of iterations has been reached. We refer to this algorithm as Nullspace DMM.

Note that if we apply Nullspace MUSIC to every observation individually, then, nulls will generally be placed towards different locations at each observation. This was the case in Chapter 5, which applied Nullspace MUSIC to estimate emitter positions over a flight. However, nulling different locations is clearly suboptimal if we assume that the emitter locations are fixed throughout the flight. If the location that is nulled changes while the emitter is stationary, then many nulls will not be directed at the emitter. We would expect nulling to be more effective if the best estimate of the emitter’s location were used as the nulling direction for all observations. This is the approach that is used by Nullspace DMM, which incorporates the assumption that the emitters are stationary by always placing nulls at the same locations on the ground.

The steps to implement Nullspace DMM for a mixed polarization receive array are given below. These steps assume that two uncorrelated emitters transmit during the flight.
1. Calculate $G_{MP}^{-1}(x, y)$ from (6.2), and find the maximum. Assume the maximum is at the location of an emitter. Call this location $(\tilde{x}_1, \tilde{y}_1)$.

2. For the $t$th observation along the flight path, estimate the manifold vector $\tilde{a}_t$ in the direction of $(\tilde{x}_1, \tilde{y}_1)$ from (3.22). Recall that this vector is unit norm.

3. Find $N - 1$ mutually orthonormal weight vectors that are also orthogonal to $\tilde{a}_t$, where $N$ is the number of antenna elements in the receive antenna array\(^6\). These may be found from Graham-Schmidt orthonormalization [65], or the eigenvectors of $I - \tilde{a}_t\tilde{a}_t^H$, where $I$ is the identity matrix. Let the weight vectors be $w_{t,1}, \ldots, w_{t,N-1}$.

4. Form a $N \times (N - 1)$ mapping matrix, $W_t = [w_{t,1}, \ldots, w_{t,N-1}]$.

5. Calculate the “nullspace covariance matrix,” $\tilde{R}'_t = W_t^H \tilde{R}_t W_t$, where $\tilde{R}_t$ is the covariance matrix for the $t$th observation, and it is calculated from (2.9). The dimensions of $\tilde{R}'_t$ are $(N - 1) \times (N - 1)$.

6. Eigendecompose the nullspace covariance matrix into eigenvectors $[\hat{v}'_{t,1}, \ldots, \hat{v}'_{t,N-1}]$ and corresponding eigenvalues $\lambda_{t,1} > \lambda_{t,2} > \cdots > \lambda_{t,N-1}$.

7. Calculate $P'_t$, the projection matrix for the noise space of $\tilde{R}'_t$. Note that $P'_t = [\hat{v}'_{t,2}, \ldots, \hat{v}'_{t,N-1}] [\hat{v}'_{t,2}, \ldots, \hat{v}'_{t,N-1}]^H$.

8. Calculate the nullspace test vectors. Recall that $b$ and $c$ are the array manifolds for the $E_\theta$ and $E_\phi$ polarizations, $\psi$ is a generic direction, and a hat indicates that a vector is unit norm.

$$\hat{b}'_t(\psi) = \frac{W_t^H b(\psi)}{\|W_t^H b(\psi)\|}.$$\(^6\) These weights form the beams, each of which has a null for the emitter’s assumed position and polarization.
\[
\hat{c}_t'(\psi) = \frac{W_t^H c(\psi)}{\|W_t^H c(\psi)\|}.
\]
\[
\hat{d}_t'(\psi) = \frac{c_t' - b_t'b_t'^H c_t'}{\|c_t' - b_t'b_t'^H c_t'\|}.
\]

9. Calculate the modified \( M \) matrices for the \( t \)th observation,

\[
M'_t(\psi) = \begin{bmatrix}
\hat{b}_t'^H(\psi) P_t' \hat{b}_t'(\psi) & \hat{b}_t'^H(\psi) P_t' \hat{d}_t'(\psi) \\
\hat{d}_t'^H(\psi) P_t' \hat{b}_t'(\psi) & \hat{d}_t'^H(\psi) P_t' \hat{d}_t'(\psi)
\end{bmatrix}.
\] (6.5)

10. Calculate the Nullspace DMM Spectrum,

\[
G'_M(x, y) = \frac{1}{T} \sum_{t=1}^{T} \lambda_{\text{min}}^{\frac{1}{2}}(M'_t(\theta_t(x, y), \phi_t(x, y))).
\] (6.6)

11. Assume that the maximum of the \( G'_M(x, y) \) is at the location of the second emitter.

12. Repeat steps 2-11, nulling emitter 2 and estimating the location of emitter 1.

If the estimated location of emitter 1 does not change, then Nullspace DMM has converged. Otherwise, continue nulling one emitter and estimating the location of the other emitter until the estimated locations do not change between iterations or until a maximum number of iterations has been reached.

As with Nullspace MUSIC (for DOA estimation), the above approach may be extended to three or more signals by placing nulls at \( M - 1 \) locations while estimating the location of the emitter of interest. In this case, Nullspace DMM converges when the \( M \) current location estimates match the \( M \) previous location estimates. A minimum of \( 2M - 1 \) spectra must be calculated to check whether Nullspace DMM has converged.

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Note that Nullspace DMM is more expensive to compute than DMM because Step 8 requires new sets of test vectors to be calculated at every observation throughout the flight. However, Nullspace MUSIC is still less expensive than applying Nullspace MUSIC at every observation, in which case new test vectors must be calculated and peaks must be found in the individual spectra. Nevertheless, Nullspace DMM’s extra expense is worthwhile if it is the only method that is able to locate all emitters in the scene.

6.5 Two-Emitter DMM and Nullspace DMM

First, DMM was applied to simulated flights with two uncorrelated emitters. Fig. 6.4 shows the DMM spectra for the western emitters over the long flight path and the circular flight path. The top row and bottom rows assume the larger and smaller mismatch setups, respectively. As before, the northwest emitter transmitted an $E_\phi$ signal, while the southwest emitter transmitted a RHCP signal. As in the single emitter cases, DMM yields a higher peak around the RHCP emitter than around the $E_\phi$ emitter. Nevertheless, all four spectra in Fig. 6.4 have peaks near the true emitter locations. The maximum error for the RHCP emitter is 0.2km, and the maximum error for the $E_\phi$ emitter is 0.4km. Note that the DMM spectra are similar to each other for both mismatch setups, indicating that DMM is robust to small perturbations in the array manifold. In addition, the DMM spectra are unambiguous in the sense that they do not contain additional peaks that could be mistaken for an emitter. Finally, DMM never had to select between ambiguous peaks for an individual observations. Therefore, for the two-emitter setup, DMM was simpler to apply than conventional DOA estimation and clustering, and the accuracy was about the same.
Fig. 6.5 shows the Nullspace DMM spectra for the long flight path. Recall that Nullspace DMM generates a separate spectrum for each emitter. The top row shows the Nullspace DMM spectra when the body-plate manifold is available, and the bottom row shows the Nullspace DMM spectra when the flat-wing manifold is available. The left and right columns show the spectra for the northwest and southwest emitters, respectively. Note that the color scale spans 10dB, while the DMM spectra spanned 15dB. The plots show that Nullspace DMM effectively isolates the emitters from each other, and it generates a clear peak for each emitter. When using the body-plate manifold, Nullspace DMM’s estimate of the $E_{\phi}$ emitter is 0.3km away from the true emitter location. However, each of the other emitter location estimates is correct (zero error). In addition, Nullspace DMM converged after three iterations, which is the minimum number of iterations for two emitters.

Fig. 6.6 shows the Nullspace DMM spectra for the circular flight path. As before, Nullspace MUSIC generated the spectra in the top row using the body-plate manifold, and the flat-wing manifold was used for the bottom row. The plots show that Nullspace DMM isolates the emitters from each other and generates a single clear peak for each emitter. The maximum error for any emitter position is 0.3km, which is very good. As with the long flight path, Nullspace DMM converged after the minimum number of iterations.

The location errors for both flight paths and mismatch setups are summarized in Table 6.1. These examples show that Nullspace MUSIC successfully nulls out an emitter over the flight paths and can yield the same or better results as compared to DMM. In contrast, applying Nullspace MUSIC in Chapter 5 generally did not improve the emitter location estimates. Nullspace DMM’s performance improvement can be
Figure 6.4: Two-emitter DMM spectra for both flight paths. Top Row: Larger mismatch setup. Bottom Row: Smaller mismatch setup. Left: Long flight path. Right: Circular flight path.
Figure 6.5: Two-emitter Nullspace DMM spectra for the long flight path. Top Row: Larger mismatch setup. Bottom Row: Smaller mismatch setup.
Figure 6.6: Two-emitter Nullspace DMM spectra for the circle flight path. Top Row: Larger mismatch setup. Bottom Row: Smaller mismatch setup.
attributed to correctly employing the assumption that the emitters are stationary by always placing a null at the emitter location, which was estimated from diverse observations. Nullspace DMM likewise generates visually similar spectra for the two mismatch setups, indicating that Nullspace DMM is also robust to changes in the mismatch between the true and assumed antenna array manifolds. However, since conventional DOA estimation was also able to locate these emitters for the same flight setups, Nullspace DMM’s real test will come from four-emitter cases, for which the conventional approach could not accurately locate all four emitters.

<table>
<thead>
<tr>
<th>Flight Path</th>
<th>Available Manifold</th>
<th>Emitter</th>
<th>Error: DMM</th>
<th>Error: Nullspace DMM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long</td>
<td>Body-Plate</td>
<td>Northwest ((E_\phi)) Southwest (RHCP)</td>
<td>0.4 km</td>
<td>0.3 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest ((E_\phi))(RHCP)</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td>Long</td>
<td>Flat-Wing</td>
<td>Northwest ((E_\phi)) Southwest (RHCP)</td>
<td>0.2 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest ((E_\phi))(RHCP)</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td>Circle</td>
<td>Body-Plate</td>
<td>Northwest ((E_\phi)) Southwest (RHCP)</td>
<td>0.2 km</td>
<td>0.3 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest ((E_\phi))(RHCP)</td>
<td>0.2 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td>Circle</td>
<td>Flat-Wing</td>
<td>Northwest ((E_\phi)) Southwest (RHCP)</td>
<td>0.0 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Southwest ((E_\phi))(RHCP)</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
</tbody>
</table>
6.6 Four-Emitter DMM and Nullspace DMM

Next, we consider the case for which all four emitters transmit simultaneously. Starting from the northwest emitter and proceeding clockwise, the transmitter polarizations are $E_\phi$, RHCP, $E_\theta$, and RHCP. This is the same setup that was used in Chapter 5.

Fig. 6.7 shows the DMM spectra for the long flight path (top) and the circular flight path (bottom). The left and right columns assume that the body-plate or flat-wing manifold is available for DMM, respectively. All but the northwest emitter are visible in the four spectra. For the long flight path, the other three emitters are located with errors of at most 0.2km. However, it is difficult to judge whether the fourth emitter is at its true location or somewhere in the middle.

The circular flight path yields similar results. The peak near the northwest ($E_\phi$) emitter is not very high, and with many other peaks in the spectrum, it is hard to tell whether we should select that peak or a peak in the middle of the flight path. In addition, geometric dilution of precision causes some error for the southeastern emitter. Thus, while DMM was simple to calculate, insensitive to changes in the mismatch scenario, and yielded good estimates for three of the four emitters, manifold mismatch prevented DMM from locating the $E_\phi$ emitter when the three other emitters were also transmitting. While DMM performs as well as the conventional approach with a lower computational cost, the inability to definitively locate the $E_\phi$ emitter motivates us to apply Nullspace DMM to the same scenarios.

Fig. 6.8 shows the final Nullspace DMM results for the long flight path with four emitters. The rows show the spectra for individual emitters. The left and right columns correspond to the larger and smaller mismatch setups, respectively.
Figure 6.7: Four-emitter DMM spectra for both flight paths. Left: Larger mismatch setup. Right: Smaller mismatch setup. Top: The long flight path. Bottom: The circle flight path.
Figure 6.8: Four-emitter Nullspace DMM spectra for the long flight path. Left: Larger mismatch setup. Right: Smaller mismatch setup.
Figure 6.9: Four-emitter Nullspace DMM spectra for the circle flight path. Left: Larger mismatch setup. Right: Smaller mismatch setup.
Nullspace DMM required ten iterations to converge for the larger mismatch setup, while a minimum of seven iterations are required to check whether Nullspace DMM has converged. Nullspace DMM completed twenty iterations with the smaller mismatch setup without converging. However, the Nullspace DMM estimates did not change by much between iteration 8 and iteration 20.

Although Nullspace MUSIC required several iterations, the location estimates are accurate enough to justify the computational effort. Moreover, the location estimates are unambiguous since each estimate is simply the maximum of its corresponding spectrum, and the spectra do not require the ‘best’ peaks to be selected. With both mismatch setups, Nullspace DMM’s error for the $E_\phi$ emitter was 0.44km. In all other cases, the error was 0.3km or less. The position errors for the long flight path are summarized in Table 6.2. Thus, Nullspace DMM effectively separates the emitters from each other and accurately estimates their locations. Note that although Nullspace DMM’s peak for the $E_\phi$ emitter is only about 5.7dB for either mismatch setup, the DMM peak height for the same emitter by itself was only 1-2dB higher in Fig. 6.2. Thus, the small dynamic range is mostly due to manifold mismatch. Finally, the Nullspace DMM spectra are visually similar for the larger and smaller mismatch setups. Thus, Nullspace DMM yields excellent and robust performance in this challenging scenario with four emitters, mixed polarizations, manifold mismatch, and near horizon angles of arrival.

Fig. 6.9 shows the final Nullspace DMM spectra for the circular flight path with four emitters. As before, the left and right columns correspond to the larger and smaller manifold mismatch setups. These setups required ten iterations or less to converge, which is only three iterations more than the minimum. As compared to
Table 6.2: DMM errors for the long flight path with four emitters.

<table>
<thead>
<tr>
<th>Available Manifold</th>
<th>Emitter</th>
<th>Error: DMM</th>
<th>Error: Nullspace DMM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body-Plate</td>
<td>Northwest</td>
<td>N/A</td>
<td>0.4 km</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>0.2 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>0.0 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td>Flat-Wing</td>
<td>Northwest</td>
<td>N/A</td>
<td>0.4 km</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.0 km</td>
<td>0.0 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>0.2 km</td>
<td>0.3 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>0.2 km</td>
<td>0.3 km</td>
</tr>
</tbody>
</table>

Fig. 6.7, all emitters are clearly separated from each other. As usual, the eastern emitters are affected by geometric dilution of precision [1]. Nevertheless, each emitter is clearly identified. Moreover, the error for the $E_\phi$ emitter is 0.2 km or less with both mismatch setups. Recall that the conventional approach could not locate this emitter with the smaller mismatch setup (see Fig. 5.15). The other errors for this flight path are summarized in Table 6.3. We also observe that the Nullspace DMM spectra are not very sensitive to the particular mismatch configuration. Therefore, Nullspace DMM provides several advantages for the mismatch, emitter, and flight setups that have been considered so far.
Table 6.3: DMM errors for the circle flight path with four emitters.

<table>
<thead>
<tr>
<th>Available Manifold</th>
<th>Emitter</th>
<th>Error: DMM</th>
<th>Error: Nullspace DMM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body-Plate</td>
<td>Northwest</td>
<td>N/A</td>
<td>0.0 km</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.3 km</td>
<td>0.6 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>4.8 km</td>
<td>4.2 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>0.3 km</td>
<td>2.1 km</td>
</tr>
<tr>
<td>Flat-Wing</td>
<td>Northwest</td>
<td>N/A</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td>Southwest</td>
<td>0.2 km</td>
<td>0.2 km</td>
</tr>
<tr>
<td></td>
<td>Southeast</td>
<td>1.1 km</td>
<td>0.6 km</td>
</tr>
<tr>
<td></td>
<td>Northeast</td>
<td>0.8 km</td>
<td>0.4 km</td>
</tr>
</tbody>
</table>

6.7 Summary and Conclusions

This chapter applied the Direct Mapping Method (DMM) to emitter geolocation from an airborne platform. DMM maps Direction of Arrival (DOA) spectra directly to the ground so that they can be averaged. By averaging DOA spectra from diverse observations, DMM suppresses spurious peaks that challenge conventional geolocation. In addition, DMM only needs to search once for peaks, and it does not need to group DOA estimates that have mixed together due to estimation errors. Since DMM can be applied to any generic DOA function, DMM can be applied for a dominantly polarized array or a polarization sensitive array by selecting an appropriate DOA function to map to the ground.

We applied DMM to the same flight setups that were considered in Chapter 5. These flights assumed that the receive array was mounted on the full platform with pylons and wing-tilt. As expected, DMM easily located an emitter with a perfectly known array manifold. However, DMM also performed very well with manifold mismatch. DMM located all emitters in the single and two-emitter scenarios, and DMM
yielded very similar results whether the body-plate or flat-wing manifolds were available for geolocation. Consequently, DMM is insensitive to the particular details of the manifold mismatch. Recall in Chapter 5 that an emitter’s estimated position could change by several kilometers depending on the mismatch setup. When all four emitters transmitted simultaneously, DMM located all emitters except for the $E_\phi$ emitter. Thus DMM performed similarly to conventional triangulation, but DMM required fewer computations since peakfinding was only run once. Conventional DOA estimation must find peaks in $T$ spectra, where $T$ is the number of observations of the scene.

This chapter also introduced Nullspace DMM, which combines the advantages of Nullspace MUSIC and the Direct Mapping Method. The first emitter’s location is estimated from the usual DMM spectrum. Next, this location is nulled by the antenna array throughout the flight path, and another emitter’s location is estimated. When all emitters except one have been nulled by the array, each emitter’s location can be estimated individually with minimal interference from the other emitters. Nullspace DMM converges when the estimated emitter locations do not change between iterations, or when a maximum number of iterations has been reached. In spite of manifold mismatch, Nullspace DMM located all emitters in the two and four-emitter scenarios. Thus, Nullspace DMM was the only approach that provided an accurate estimate of the $E_\phi$ emitter’s location when all emitters transmitted simultaneously. Moreover, the Nullspace DMM spectra were visually similar for the larger and smaller mismatch setups, which shows that Nullspace MUSIC is also insensitive to the exact platform
configuration. Therefore, Nullspace DMM is the most promising approach for locating emitters in real-life scenarios with an antenna array mounted on an aircraft or spacecraft that induces unknown scattering on the received radio waves.

While results up to this point in the dissertation have been simulated, the next chapter will investigate whether the conclusions in this dissertation hold true with experimental flight data.
Chapter 7: Flight Experiment for RF Emitter Geolocation

7.1 Introduction

Thus far, this dissertation has discussed Direction of Arrival (DOA) estimation, Nullspace MUSIC, antenna array manifold mismatch, triangulation, and the Direct Mapping Method (DMM). The ultimate goal of the study has been to improve geolocation accuracy for an airborne or spaceborne antenna array that observes multiple simultaneous emitters, which also transmit in the same frequency band. While DOA estimation has been studied for more than a century, resolving nearby emitters still challenges modern DOA methods such as MUSIC. Therefore, we introduced Nullspace MUSIC, which improves resolution for dominantly polarized arrays, as well as polarization sensitive arrays. Nevertheless, antenna array manifold mismatch prevents MUSIC or Nullspace MUSIC from resolving nearby emitters. In the examples that we considered, simple changes to the array’s host platform were sufficient to degrade an array’s resolution, (the minimum separation at which signals can reliably be separated), from less than a degree to twenty or even thirty degrees.

Poor resolution and large biases induced by array manifold mismatch pose a formidable challenge when locating multiple emitters. Even with a single emitter in Chapter 5, the average DOA error over a simulated flight path could be more than
thirty degrees. Although Lines of Bearing (LOBs) drawn from the aircraft along the estimated DOAs intersected the ground throughout a 40km × 40km area of interest, a single emitter could be located by finding the location that minimized the apparent angular error observed throughout the flight. Two emitters, however, were significantly more challenging. The estimated Lines of Bearing intermixed, and they had to be separated before they could be used to geolocate the emitter. Indeed, clustering had to be based on Minimizing the Apparent Angular Error (MAAE) over the flight. While MAAE clustering was impressively robust to inaccurate DOA estimates, it required many iterations, and it sometimes could not locate every emitter in challenging scenarios with four emitters.

Chapter 6 attempted to address the conventional method’s shortcomings with the Direct Mapping Method (DMM). DMM directly combines multiple observations of a scene by mapping DOA spectra to a fixed grid on the ground. DMM only searches for peaks in final, averaged spectrum. Averaging the spectra throughout the flight is both more robust, (since spurious peaks from individual observations are suppressed), and more computationally efficient, (since peakfinding is only run once). Although DMM was less sensitive to the particular details of the mismatch scenario, DMM still could not locate all emitters in all cases with four simultaneous transmitters. Therefore, we combined the principles behind Nullspace MUSIC and DMM to form Nullspace DMM. In Chapter 6, Nullspace DMM was the only approach that successfully located all emitters in all of the scenarios. Although Nullspace DMM is computationally expensive, it is still tractable, and its expense was justified by its superior performance.
While we have found Nullspace DMM to be a promising approach for airborne emitter geolocation, our conclusions thus far have been based on simulations. Thus, this chapter does not introduce new techniques. Instead, it verifies the results and the conclusions that have been discussed thus far through an actual geolocation flight with live signals. For this flight, we built a six element array of vertical monopole antennas. We mounted the array to a small aircraft and flew around Columbus, Ohio. During the flight, we recorded signals from four known transmitters. We subsequently post-processed the recorded samples, and we tested conventional DOA estimation, DMM, and Nullspace DMM.

The experimental results were similar to the simulation results in Chapters 5 and 6. In the experimental results, individual DOA estimates were also spread out, but MAAE successfully located each of the four emitters when it transmitted by itself. MAAE also performed well with two simultaneous emitters. However, MAAE was unable to locate all four emitters for some flight paths. Similarly, DMM easily located one or two emitters. However, DMM could not provide clear estimates with four incident signals. On the other hand, Nullspace DMM successfully located all four emitters in the same scenarios. Therefore, Nullspace DMM has been experimentally verified as a particularly robust means of geolocating multiple simultaneous emitters that share a common frequency band.

This chapter proceeds as follows. First, Section 7.2 describes the flight experiment, including the antenna array, the aircraft, the Data Collection System (DCS), and the flight path. Next, Section 7.3 describes the signal preprocessing routines that separated the signals of interest from co-channel interference. Co-channel interference
had to be mitigated because the transmitters of interest occupy the 900MHz Industrial, Scientific, and Medical (ISM) band, which allows unlicensed transmission by a wide variety of radio services. We suppressed the undesired signals from other radio services by cross correlating the received samples with the known spreading codes of the desired transmitters. The post-processing routines also allow us to locate individual emitters in Section 7.4 as if they were the only active transmitter, even though multiple emitters transmit simultaneously. While the DOA estimates generally have errors of around ten degrees, MAAE and DMM easily geolocate the emitters. Section 7.5 considers two emitters at a time. As in Chapter 5, two emitters can also be geolocated without too much trouble. Finally, Section 7.6 attempts to geolocate all four emitters at the same time. Conventional DOA estimation and DMM cannot locate all of the emitters. However, Nullspace DMM succeeds at locating every emitter. Additional flight paths are also tested with the same result. Finally, Section 7.7 reviews the chapter and makes some final observations.

7.2 Flight Setup

The airborne geolocation campaign collected signals from four vertically polarized transmitters in the 902-928 MHz ISM band. The transmitters were not under our control, but instead belonged to an existing radio-location system. Therefore, we built an array of six monopole antennas that received signals in the same frequency band. The monopoles were mounted vertically to a 20” mounting disk, and they were arranged in a circle with a six-inch radius. Note that the six-inch radius corresponds to a little less than half a wavelength at the carrier frequency. The array is shown in Fig. 7.1. After the array was built, it was measured in the ElectroScience Laboratory’s
Figure 7.1: Experimental receive array and measurement in The Ohio State University’s anechoic chamber at the ElectroScience Laboratory.

...and measurement in the anechoic chamber. During the chamber measurement, the 20" disk was mounted on a 4-foot diameter ground plane with 3" standoff spacers (right).

After the measurement, the 20" disk and monopoles were mounted on the bottom of a six-seater Piper Saratoga aircraft, which is owned by Ohio University in Athens, Ohio. Fig. 7.2 shows the aircraft, as well as the installed antenna array. The array was mounted to the bottom of the aircraft using the same 3" spacers that were used for the antenna measurement. The spacers allowed the antenna cables to enter the aircraft cabin without excessively bending the cables.

Since the aircraft was too large and heavy to measure in an anechoic chamber, we obtained in-situ antenna array patterns from electromagnetics simulations. In-situ simulation of each element’s antenna pattern followed a simple, four-step approach. First, the 20" mounting disk and the 4’ ground plane were modeled in HFSS. Then, HFSS simulated the radiation patterns for a set of basis current elements distributed...
over the mounting disk. Second, a chosen antenna’s measured radiation pattern was decomposed into a weighted sum of the radiation patterns from the basis current elements. The decomposition employed least-squares fitting to find the excitations for the basis elements that minimized the integral of the squared error in the measured and simulated antenna patterns. Third, the radiation patterns of the basis current elements were simulated on the 20” mounting disk when the body of the Piper Saratoga was also modeled in HFSS. The HFSS model offset the plate from the Saratoga by 3” to match the installed configuration of the array. The HFSS model of the mounting disk and the Saratoga are shown in Fig. 7.3. In the fourth step, the currents found in step two were applied to the in-situ basis-element patterns, which were simulated in step three. The sum of the radiated patterns from the currents yielded the simulated in-situ radiation pattern of the chosen monopole antenna. The procedure was followed for all elements of the array to synthesize the in-situ antenna array manifold. Note that this is the same procedure that we used to model the in-situ radiation pattern on the generic aircraft platform in Chapter 4.
Fig. 7.4 shows the measured, equivalent, and *in-situ* array $E_\theta$ radiation patterns for a monopole in the antenna array. The left column shows the antenna gain, and the right plot shows the antenna phase. The top row shows the measured pattern; the middle row shows the equivalent radiation pattern for the simulated basis current elements on the 20” mounting disk and the 4’ ground plane; and the bottom row shows the final, simulated, *in-situ* radiation pattern. Note that the plot centers correspond to antenna zenith (pointing down from the aircraft), and the outer edges correspond to horizon. The aircraft nose is on the right side of the plots at $\phi = 0^\circ$, the wings are at $\phi = 90^\circ, 270^\circ$, and $\phi = 180^\circ$ corresponds to the tail. Since the top and middle row differ only slightly, we see that the equivalent currents recreate the measured radiation pattern with high fidelity. When the equivalent currents are simulated on the Saratoga, the body and the wings introduce quick variations in the gain and phase patterns. These variations are particularly noticeable around the wings of the aircraft. Naturally, the gain and phase patterns for the other five elements of the array are similarly perturbed (not shown). These *in-situ* patterns form the antenna array

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Figure 7.4: $E_\theta$ polarized gain (left) and phase (right) for one of the vertical monopoles in the six-element antenna array. Top: Measured pattern from the anechoic chamber. Middle: Equivalent current pattern, simulated in HFSS. Bottom: Simulated *in-situ* pattern.
manifold, which was used to estimate the emitter locations. Of course, the received signals were affected by the true, physical array manifold, which is only approximated by the simulated, in-situ array manifold. Nevertheless, the in-situ array manifold is representative of the array manifolds that would be used by an operational system, since operational aircraft cannot easily be measured in an anechoic chamber.

7.2.1 Data Collection System

A Data Collection System (DCS) was also required to record the signals that were received by the antenna array elements. We employed a custom built, rack mountable, and software defined system for our DCS. A single channel of the DCS is diagrammed in the left plot of Fig. 7.5. First, the signals received by the antenna passed through a bandpass filter, which was tuned to the 902-928MHz ISM band. The bandpass filter rejected any out-of-band signals that were received by the antenna element. Next, the received signal passed through a switch, and it was amplified and filtered by an Ettus Research Universal Software Radio Peripheral (USRP) software defined radio [87]. The USRP downconverted and digitized the signal at a rate of twenty megasamples per second (MSPS), and it sent the samples to a server. The server was tasked with time-tagging, organizing, and storing the sampled signals that were received from six dedicated USRPs (one for each monopole in the antenna array). The dedicated USRPs were time synchronized by an Ettus Research Octoclock [88], which distributed a stable 10MHz clock as well as a Pulse Per Second (PPS) signal that was time-synchronized with GPS. The common clocks ensured that the USRPs sampled the antenna elements' outputs at the same times, as required for array signal processing.
The DCS was carefully designed and built so that gain and phase differences between channels could be observed and calibrated out. Note from Fig. 7.5 that the switch after the bandpass filter could also pass a calibration signal from a transmitter USRP. The transmitter USRP output a known signal with 20MHz of bandwidth, which was split by an eight-way power divider into every receiver chain\(^7\). The calibration signal then received the same amplification and filtering that was applied to the signals from the antenna array. Comparing the frequency domain representations of the recorded calibration signals to each other enabled the frequency-dependent gain and phase differences in the receive USRPs to be observed and equalized. In addition to network analyzer measurements of the bandpass filters, cables, and switches, and the eight-way power divider, periodic recordings of the calibration signal enabled differences in the front-end electronics chain to be calibrated out of the received signals.

\(^7\)The power divider’s extra output ports were terminated with 50 Ohm loads.
As a result, the residual differences between channels were mostly due to the differences in the direction-dependent gain and phase of the various antenna elements. While calibration cannot be perfect, antenna induced effects should dominate the observable differences between the calibrated signals. The DCS was also physically compact, and the right plot of Fig. 7.5 shows the DCS installed in the Saratoga.

### 7.2.2 Flight Path

After the array and DCS were installed in the aircraft, we collected data during a three-hour flight over Columbus, Ohio. We took off from Ohio University’s airport in Athens, Ohio, then proceeded to Columbus, where the transmitters were located. Over Columbus, we executed a long, meandering flight path, as well as three circles at various locations. In addition to recording the samples from the antenna elements, we also recorded the aircraft’s precise position and orientation using an Inertial Measurement Unit (IMU) and an aviation grade GPS receiver. The GPS receiver recorded the pseudoranges to individual GPS satellites [89], and it later post processed the pseudoranges to determine the best-fit position and orientation of the aircraft during the flight.

The recorded flight path, as well as the emitter locations, are shown in Fig. 7.6. The left plot shows the entire flight path, while the right plot shows a zoomed in view over Columbus. Note that the transmitters operated continuously throughout the flight. Thus, we may pick any flight segment and attempt to locate the transmitters with the corresponding received signals. For instance, Chapters 5 and 6 simulated received signals for a forty-five minute segment and a ten minute segment of the flight path shown in Fig. 7.6. Note that the aircraft altitude was about 2km, and the aircraft
flew slightly pitched up. As a result, the array did not always have line of sight to the emitters. The aircraft altitude, orientation, and angles to the emitters during the two segments were shown in Fig. 5.1 and 5.3. Note also that the simulated emitter locations in Chapter 5 matched the true emitter locations during the experimental flight. The experimental geolocation sections in this chapter will continue studying the same segments of the flight path.

7.3 Signal Preprocessing

Since the transmitters use an ISM frequency band that is shared by many radio services, they take several steps to ensure that users can receive the signal in spite of interference. First, the emitters broadcast Binary Phase Shift Keyed (BPSK) signals with a 2MHz bandwidth and a known spreading sequence. Thus, receivers can apply cross correlation to separate the desired signal from co-channel interference. Moreover, the four transmitters share the same frequency band, and BPSK spreading codes also enable receivers to separate the simultaneous emitters from each other.
Despreading the BPSK signals provides at least 21dB of isolation between the emitters in the system [90]. Second, the system broadcasts the same signal on two slightly offset center frequencies. If one channel is in use by a competing service, the other may be free. Third, each emitter broadcasts the signal with 30 Watts of Equivalent Isotropic Radiated Power (EIRP), which is more power than unlicensed transmitters are allowed to use.

The transmitters’ system design ensured that the signals were often clearly visible in the frequency domain. The left plot of Fig. 7.7 shows the magnitude of a 1024-point FFT that has been averaged over 10mS of recorded data. The system’s center frequencies are shown with dashed red lines, and the band edges of the transmitted signals are shown with dashed gray lines. The sinc-squared shape of the desired signals shows up clearly, and most of the narrowband interference signals are outside of the frequency bands of interest. Note that all six receive channels are plotted. However, since the receiver channels are very nearly uniform, the frequency spectra overlap such that only one line is apparent.

Unfortunately, co-channel interference is very apparent at other observations. The right plot of Fig. 7.7 shows the frequency spectra from an observation taken sixty seconds earlier while still in the middle of the long flight path. Although the emitters are still close to the receiver, narrowband interference overpowers the signals of interest within their bandwidth. In addition, a very strong interferer transmits a few MHz above the upper frequency of interest. Consequently, it would be unwise to simply form covariance matrices directly from the samples and attempt to locate the emitters. Following a naive approach may yield a covariance matrix that is strongly affected by an intermittent transmitter, which we are not interested in, and whose
Frequency domain processing proceeded as follows. First, the sampled signals were downshifted by 3MHz such that the upper frequency band was centered around zero Hertz. Then, the signals were low-pass filtered by a factor of ten such that two MHz of bandwidth remained. Next, the signals were downsampled by a factor of ten. By low pass filtering before downsampling, we prevented energy from outside the two MHz band of interest from aliasing into the band.
After decimating the samples, we cross correlated three milliseconds of the received signal from a reference element with the known spreading code. The three milliseconds of data were split into three, one-millisecond (mS) segments, and the magnitudes of the 1mS cross correlations were averaged. We applied a range of delay and Doppler frequencies to the spreading code prior to cross correlation, and we found the delay and Doppler that maximized the absolute value of the cross correlation with the received signal. The delay and Doppler estimation follows the same process that is used by a GPS receiver for GPS signal acquisition, [89]. Next, we generated a clean replica of the incident signal using the known spreading code and the estimated delay and Doppler. Finally, we estimated the array manifold vector in the (unknown) direction of the $m$th transmitter by cross correlating the replica with each element’s sampled signals. That is,

$$\tilde{a}_m = \frac{1}{K} \sum_{k=1}^{K} \begin{bmatrix} r_m^*[k]x_1[k] \\ r_m^*[k]x_2[k] \\ \vdots \\ r_m^*[k]x_N[k] \end{bmatrix},$$

(7.1)

where $x_n[k]$ is the $k$th sample at receive antenna $n$; $r_m[k]$ is the $k$th sample of the replica for emitter $m$; a '*' denotes a conjugate; $K$ is the total number of samples; and $N$ is the number of antennas in the receive array. Note that $r_m$ incorporates the estimated delay and Doppler for the $m$th emitter, and a tilde denotes an estimated quantity. However, the estimated manifold vector does not measure the antenna’s absolute gain or phase, since (7.1) is affected by the magnitude and phase of the received signal. Instead, the estimated manifold vector approximates the magnitudes and phases of the antenna elements \textit{relative to each other} in the direction of the BPSK signal. The relative manifold vectors are parallel to the actual antenna manifold.
vectors; ideally, they only differ by a complex scale factor. Therefore, the estimated manifold vectors are sufficient for direction finding with MUSIC.

When we selected a flight segment, we generated covariance matrices as follows. For any arbitrary combination of $M$ emitters, we can synthesize the Signal Subspace due to the transmitters of interest as

$$\tilde{S} = [\tilde{a}_1, \ldots, \tilde{a}_M].$$

(7.2)

Note that one to four of the emitters may be included in the synthesized signal subspace. Then, a corresponding covariance matrix may be synthesized as

$$\tilde{R} = \tilde{S}\tilde{S}^H$$

(7.3)

The synthesized covariance matrix is rank deficient - although its dimensions are $N \times N$, it is only rank $M$. Nevertheless, MUSIC's usual procedure of eigendecomposing the covariance matrix and forming the noise space from the eigenvectors corresponding smallest $N - M$ eigenvalues still applies. Although we do not add noise to the synthesized covariance matrix, it is still affected by noise and interference in the received samples, which biases the cross correlations in (7.1). Indeed, we were unable to estimate manifold vectors at some observations because of interference, and because the receive antennas do not always have line of sight to the emitters. If we could not estimate one or more manifold vectors at a point along the flight path, then that observation was not used for emitter geolocation in Sections 7.4 to 7.6. Note that MUSIC and DMM will always assume that the number of signals is known a-priori and does not need to be estimated.

In summary, the preprocessing was well suited to experimental data collection in an unlicensed frequency band. Preprocessing ensured that the signals of interest are
present in the recorded data; it mitigated co-channel interference; and it allowed us to consider various combinations of emitters. Although we separated the emitters from each other via despreading, we geolocated \( M \) simultaneous emitters by synthesizing a covariance matrix from \( M \) estimated antenna manifold vectors. The synthesized covariance matrices were rank deficient, but the noise and interference in the sampled signals still biased the estimated manifold vectors. Therefore, the synthesized covariance matrices were not noise free. Ultimately, the estimated noise spaces used by MUSIC were similar to what would be obtained under an ideal scenario for which the signals of interest were the only signals present in the scene.

7.4 Single Emitter Geolocation

After the flight and preprocessing, we were ready to geolocate the transmitters. Since the transmitters were \( E_\theta \) polarized and the receive array was dominantly \( E_\theta \) polarized, DOA estimation and geolocation relied on single polarization MUSIC (2.21) instead of Mixed Polarization MUSIC, which was used in Chapter 5. In addition, DMM and MUSIC’s Line of Bearing intersections with the ground were calculated using Digital Elevation Maps (DEMs) for the terrain around Columbus [91]. MUSIC and DMM sought the emitters on a 40km by 40km grid with 200 meter resolution. The height of the grid at a given \((x, y)\) point was determined by querying the DEM to obtain the z-coordinate and adding 150 feet to account for the average height of the transmitter towers.

We calculated the MUSIC DOA estimates as follows. First, we used the estimated manifold vectors to generate covariance matrices along the long flight path. The covariance matrices were synthesized at regular intervals, although some gaps occurred
corresponding to points where the preprocessing could not estimate the manifold vectors for all four emitters. Then, we calculated the MUSIC spectra on the grid for every observation. At each observation, the aircraft position and orientation were used to calculate the angle towards every grid point in the antenna coordinate system. Then, the \textit{in-situ} array manifold was interpolated at the corresponding angles. Next, the interpolated manifold vectors were compared to the estimated noise space to generate the MUSIC spectrum (2.21). We used the peaks of the individual, mapped DOA spectra as the estimated DOAs over the flight. Note that by estimating DOAs over the grid instead of over angle space, we avoided DOA estimates that never intersected with the ground. We also ensured that we sought the emitters over the correct region of the antenna array manifold, even if the receive array did not have line of sight to the emitter at a particular observation. Moreover, this is the same process that was followed for the conventional DOA estimates in Chapter 5.

The results for conventional DOA estimation with a single emitter over the long flight path are shown in Fig. 7.8. DOA estimates were made from covariance matrices that were synthesized every thirty seconds throughout the flight segment. Each row of Fig. 7.8 corresponds to a particular emitter, (northwest, southwest, southeast, and northeast); the left column shows the individual DOA estimates on the ground; and the right column shows the apparent angular error for the DOA estimates in the left column as a function of position. Each emitter’s DOA estimates approximately spanned a 20km by 20km region around the emitter, which is actually better than the simulation examples in Chapter 5. The fact that the location errors are smaller in the experimental data suggests that the experimental setup does an outstanding job of synchronizing the samples with the flight path, calibrating the front end, and
suppressing interference. Recall that Chapter 5 assumed that manifold mismatch was the only source of error, implying that channel synchronization was ideal, the channel calibration was ideal, the aircraft position was perfectly known, and the emitter of interest was the only signal received by the array. Nevertheless, the simulated DOA estimates for a single emitter were spread over the entire search area in Chapter 5. In the right column of Fig. 7.8, each emitter’s location is clearly shown by a minimum in the apparent angular error near the true emitter location. Starting with the northwest emitter and proceeding counter-clockwise, the location errors are 0.3km, 0.9km, 0.3km, and 0.5km. These errors are remarkably small for experimental data, a large search area, and a relatively low flight path.

Fig. 7.9 shows the DMM results for the same setups. As in Chapter 6, these spectra summed the inverse of the square root of the usual MUSIC spectrum. Recall, however, that we used single polarization MUSIC because the transmitters and the receive antennas were dominantly vertically polarized. Let $\tilde{a}_{\theta,t}(x, y, z)$ be the manifold vector the $\theta$ polarization in the direction of the grid point at the coordinates $(x, y, z)$ for the $t$th observation, and let $T$ be the total number of observations. Let $P_t$ be the projection matrix for the noise space of the $t$th covariance matrix, $\tilde{R}_t$. Then, the DMM spectrum is

$$G(x, y) = \frac{1}{T} \sum_{t=1}^{T} \| P_t \tilde{a}_{\theta,t}(x, y, z) \|,$$  \hspace{1cm} (7.4)

where $\| \cdot \|$ is the Euclidean norm. Note that the DMM spectrum was inverted for plotting for consistency with how MUSIC is usually plotted.

As expected, each DMM spectrum in Fig. 7.9 unambiguously identifies the emitter location, and the DOA biases of individual estimates in Fig. 7.8 have been suppressed. Starting with the northwest emitter and proceeding counter-clockwise, the location
Figure 7.8: Single emitter geolocation for the long flight path using conventional MUSIC and minimizing the apparent angular error over the grid.
Figure 7.9: Single emitter geolocation for the long flight path using DMM.

errors are 1.1km, 0.6km, 0.3km, and 0.5km, which is also remarkably small for the experimental setup. In addition, Fig. 7.9 experimentally verifies DMM’s ability to locate an emitter in spite of array manifold mismatch.

Fig. 7.10 shows the conventional DOA estimates for a single emitter at a time following the circular flight segment, which is in between the four emitters. Since the circular flight segment is shorter than the previous segment, we synthesized co-variance matrices every six seconds. Thus, conventional DOA estimation and DMM worked with a similar number of observations throughout the circular flight segment as in the long flight segment. As in the long flight segment, the estimated emitter
Table 7.1: Summary of position errors for single emitter scenarios.

<table>
<thead>
<tr>
<th>Geolocation</th>
<th>Northwest</th>
<th>Southwest</th>
<th>Southeast</th>
<th>Northeast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional (Long)</td>
<td>0.3km</td>
<td>0.9km</td>
<td>0.3km</td>
<td>0.5km</td>
</tr>
<tr>
<td>DMM (Long)</td>
<td>1.1km</td>
<td>0.6km</td>
<td>0.3km</td>
<td>0.5km</td>
</tr>
<tr>
<td>Conventional (Circle)</td>
<td>0.6km</td>
<td>1.3km</td>
<td>7.0km</td>
<td>1.2km</td>
</tr>
<tr>
<td>DMM (Circle)</td>
<td>1.0km</td>
<td>1.7km</td>
<td>10.3km</td>
<td>1.6 km</td>
</tr>
</tbody>
</table>

locations span a relatively small area, suggesting that the sources of error from the experimental setup are small. Starting with the northwest emitter and proceeding counter-clockwise, the MAAE location errors are 0.6km, 1.3km, 7.0km, and 1.2km. As in Chapter 5, the southeast emitter is affected by geometric dilution of precision, and the minimum of the MAAE spectrum is very shallow along the direction from the circular flight segment towards the emitter. Thus, the 7.0km error is not a cause for concern.

Fig. 7.11 shows the DMM estimates for the individual emitters and the circular flight segment. As with the conventional estimates, the emitters show up clearly in the DMM spectra. These results again verify DMM’s utility in overcoming antenna manifold mismatch. Recall also that DMM only needs to search for peaks once, and DMM actually requires fewer computations than the conventional DOA estimation. Table 7.1 summarizes the DOA errors for the single emitter scenarios. Table 7.1 shows that both techniques work quite well, although the conventional approach yields slightly better overall location errors with a single emitter.
Figure 7.10: Single emitter geolocation for the circle flight path using conventional MUSIC and minimizing the apparent angular error over the grid.
Figure 7.11: Single emitter geolocation for the circle flight path using DMM.
7.5 Two Emitter Geolocation

Next, we geolocated the western emitters following the same flight paths. In this case, the covariance matrices (7.3) were generated using two estimated manifold vectors in the signal subspace. Since conventional MUSIC must estimate two DOAs from a single spectrum, we required the selected DOA estimates to be at least 5km apart on the ground, which is about half of the separation between the emitters.

Fig. 7.12 shows the results for the long flight path. The left column shows the conventional DOA and geolocation estimates, while the right column shows the DMM and Nullspace DMM geolocation estimates. In the left column, we see that the DOAs are spread out over the western half of the grid. Nevertheless, MAAE clusters the estimates after four iterations, and the final location estimates are 1.1km and 0.3km away from the northwest and southwest emitters, respectively. In the same scenario, DMM yields a clean spectrum with two peaks. The peaks are 0.7km and 0.2km away from the northwest and southwest emitters, respectively. The DMM spectrum is also the starting point for Nullspace DMM, which separates the peaks so that each emitter’s location is estimated from an individual spectrum. Nullspace DMM required seven iterations to converge on the final spectra. The peaks of these spectra are 1.0km and 0.7km away from the true emitter locations. Note that DMM spectrum is on a 15dB scale, while the Nullspace DMM spectra are on 10dB scales.

Fig. 7.13 considers the same emitters and the circular flight segment. MAAE clusters MUSIC’s DOA estimates after four iterations, and it locates both emitters. However, the location errors are a little more than 1km for both emitters. DMM yields a clean spectrum with a peak for each emitter. The location errors are 0.8km and 1.2km for the northwest and southwest emitters, respectively. Nullspace DMM refines
Figure 7.12: Geolocation estimates for the western emitters and the long flight path. Top Left: DOA estimates and clusters. Middle and Bottom Left: Positions minimizing the apparent angular error over the grid for the two clusters. Top Right: DMM spectrum. Middle and Bottom Right: Nullspace DMM spectra.
DMM’s location estimates. After five iterations, the Nullspace DMM spectra converge with estimates that are 0.3km and 1.0km away from the true emitter locations.

The errors for the two emitter setup are summarized in Table 7.2. The location errors with two simultaneous emitters were similar to the errors with the individual errors. In summary, conventional DOA estimation and DMM both succeeded at estimating the emitter locations, and Nullspace DMM successfully separated the emitters. Thus, the results in this section serve as an experimental verification of MAAE, DMM, and Nullspace DMM with multiple emitters.

<table>
<thead>
<tr>
<th>Geolocation (Flight path)</th>
<th>Northwest</th>
<th>Southwest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional (Long)</td>
<td>1.1km</td>
<td>0.3km</td>
</tr>
<tr>
<td>DMM (Long)</td>
<td>0.7km</td>
<td>0.2km</td>
</tr>
<tr>
<td>Nullspace DMM (Long)</td>
<td>1.0km</td>
<td>0.7km</td>
</tr>
<tr>
<td>Conventional (Circle)</td>
<td>1.5km</td>
<td>1.3km</td>
</tr>
<tr>
<td>DMM (Circle)</td>
<td>0.8km</td>
<td>1.2km</td>
</tr>
<tr>
<td>Nullspace DMM (Circle)</td>
<td>0.3km</td>
<td>1.0km</td>
</tr>
</tbody>
</table>
Figure 7.13: Geolocation estimates for the western emitters and the circular flight path. Top Left: DOA estimates and clusters. Middle and Bottom Left: Positions minimizing the apparent angular error over the grid. Top Right: DMM spectrum. Middle and Bottom Right: Nullspace DMM spectra.
7.6 Four Emitter Results

In this section, we push the geolocation methods to their limit by estimating the locations of all emitters at the same time. Thus, the estimated manifold vectors for all emitters were included in the synthesized covariance matrices, (7.2) and (7.3).

First, we estimated the emitter locations over the long flight path with conventional MUSIC and MAAE clustering. The top plot of Fig. 7.14 shows the individual DOA estimates. The colors indicate the groupings, which MAAE found after seven iterations. The location estimates cover the 40km by 40km area of interest, showing the increased difficulty of estimating four simultaneous emitters. Nevertheless, MAAE yields a clear minimum apparent angular error near each of the emitters. Starting with the northwest emitter and working counter-clockwise the errors are 3.2km, 0.7km, 0.0km, and 0.7km.

The DMM and Nullspace DMM spectra for the same flight are shown in Fig. 7.15. The DMM spectrum shows the general area corresponding to the four emitters. However, the DMM spectrum contains more than four peaks, which are shallow and are not well separated. The highest peaks that were at least 5km away from each other were selected as the emitter location estimates. The corresponding location errors (counter-clockwise from the northwest) are 3.0km, 2.0km, 0.3km, and 0.8km. On the other hand, Nullspace DMM yields a separate spectrum for each emitter. Consequently, the location estimates are unambiguous. The location errors for each emitter (counter clockwise from the northwest) are 1.0km, 0.0km, 0.3km, and 0.3km. Nullspace DMM generated 11 spectra before converging; recall that a minimum of seven spectra are required for convergence with four emitters. Thus, Nullspace DMM converged relatively quickly. Overall, all three approaches yielded impressive results.
Figure 7.14: Conventional estimates for four emitters over the long flight path. Top: DOA estimates and clusters. Bottom Rows: Apparent angular error functions for the four clusters over the grid.
However, Nullspace DMM yielded the clearest location estimates and the least total error.

Fig. 7.16 shows the conventional DOA estimates for the circular flight path with all four emitters. As before, the individual DOA estimates are spread over the area of interest. However, MAAE does not yield a cluster for the southwest emitter. Instead, two clusters minimize the apparent angular error near the northeastern emitter. Recall from Chapter 5 that the apparent error for multiple clusters may be minimized at nearly the same location when the DOA estimates for different emitters are grouped into the same cluster (Fig. 5.8). Since the southwest emitter could not be located, MAAE is not sufficiently robust for this DOA errors caused by the challenging flight setup. The errors for the other emitters are given in Table 7.3.

In contrast with the conventional approach, Fig. 7.17 shows that DMM localized all four emitters. As is usually the case, the flight path’s low altitude and distance from the southeast emitter’s location caused geometric dilution of precision. Nevertheless, DMM clearly separated the emitters and revealed their location relative to the aircraft. Moreover, Nullspace DMM converged after twelve iterations, and it yielded a single strong peak for every emitter. Nullspace DMM also yielded the smallest total errors for the circular flight path. The errors for the flight paths discussed thus far are given in Table 7.3. This table shows that Nullspace DMM outperformed the other methods in the experimental flights with four emitters present at the same time.

We compared conventional geolocation, DMM, and Nullspace DMM for two additional flight segments. Fig. 7.18 shows the conventional geolocation results for a $7\frac{1}{2}$ minute long segment of the long flight path that has been previously considered. During the short segment, the airplane flew straight through the middle of the emitters,
Figure 7.15: DMM and Nullspace DMM estimates for four emitters over the long flight path. Top: DMM spectrum. Bottom Rows: Nullspace MUSIC spectra for the four emitters.
Figure 7.16: Conventional estimates for four emitters over the inner-circle flight path. Top: DOA estimates and clusters. Bottom Rows: Apparent angular error functions for the four clusters over the grid.
Figure 7.17: DMM and Nullspace DMM estimates for four emitters over the inner-circle flight path. Top: DMM spectrum. Bottom Rows: Nullspace MUSIC spectra for the four emitters.
Table 7.3: Summary of position errors for four simultaneous emitters.

<table>
<thead>
<tr>
<th>Geolocation (Flight path)</th>
<th>Northwest</th>
<th>Southwest</th>
<th>Southeast</th>
<th>Northeast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional (Long)</td>
<td>3.2 km</td>
<td>0.7 km</td>
<td>0.0 km</td>
<td>0.7 km</td>
</tr>
<tr>
<td>DMM (Long)</td>
<td>3.0 km</td>
<td>2.0 km</td>
<td>0.3 km</td>
<td>0.8 km</td>
</tr>
<tr>
<td>Nullspace DMM (Long)</td>
<td>1.0 km</td>
<td>0.1 km</td>
<td>0.3 km</td>
<td>0.3 km</td>
</tr>
<tr>
<td>Conventional (Circle)</td>
<td>2.7 km</td>
<td>N/A</td>
<td>5.9 km</td>
<td>1.1 km</td>
</tr>
<tr>
<td>DMM (Circle)</td>
<td>0.6 km</td>
<td>1.0 km</td>
<td>12.0 km</td>
<td>1.0 km</td>
</tr>
<tr>
<td>Nullspace DMM (Circle)</td>
<td>0.4 km</td>
<td>0.7 km</td>
<td>7.1 km</td>
<td>1.1 km</td>
</tr>
</tbody>
</table>

and DOA estimates were made every six seconds. As a result, MAAE had approximately the same number of observations to work with as in the long flight path. As usual, the DOA estimates from the straight flight segment intermixed on the ground. However, MAAE required eighteen iterations to converge, and it did not produce a cluster corresponding to the southeast emitter. Only the northwest and northeast emitter locations were well estimated. The corresponding location errors were 2.7km and 1.6km. MAAE’s inability to estimate all four emitter locations during the flight segment through the middle of the emitters again implies that conventional DOA estimation is not robust enough for our experimental flight setup.

DMM also performed suboptimally. The top plot of Fig. 7.19 shows that DMM produced long plateaus on either side of the flight path as well as at the actual emitter locations. Selecting the highest peaks that are at least 5km apart yielded errors of
1.9km for the northwest emitter, 2.9km for the southwest emitter, and 2.0km for the northeast emitter. The southeast emitter’s location was missed because of a higher peak near the center of the spectrum. On the other hand, Nullspace DMM suppressed the plateaus and yielded good location estimates. The errors for the emitters ranged from 0.8km to 2.0km. The errors for the individual emitters are shown in Table 7.4. Thus, Nullspace DMM again yielded the best performance with experimental flight data. Moreover, it was the only approach that found the southeast emitter during this flight segment.

Finally, we considered the circular flight segment to the south of all the emitters. This flight segment was especially challenging because it was the only segment for which the aircraft did not fly in between the emitters. Thus, the emitters were always to one side of the aircraft, and the true DOAs were close together. Fig. 7.20 shows the conventional DOA estimates, and Fig. 7.21 shows the DMM/Nullspace DMM estimates. Note that preprocessing could not estimate manifold vectors for all emitters during the southwestern portion of the circle. Hence, the circle contained a gap in observations. Thus, this circular segment likely was the lowest SNR portion of the flight that we considered. Even so, the conventional MAAE approach yields acceptable estimates for the four emitters. The errors for the emitters (counter-clockwise from northwest), were 2.6km, 3.3km, 7.4km, and 9.0km. The last cluster is in the direction of the northeast emitter, but its minimum error point is in the middle of the four emitters. Using the same data, DMM produces several elongated peaks. Picking the highest peaks that are at least 5km apart yields errors (starting from the northwest emitter) of 3.9km, 4.7km, 5.8km, and 4.9km. Finally, Nullspace DMM converges after thirteen iterations, and produces a peak for each emitter. Starting
Figure 7.18: Conventional estimates for four emitters over the straight flight path. Top: DOA estimates and clusters. Bottom Rows: Apparent angular error functions for the four clusters over the grid.
Figure 7.19: DMM and Nullspace DMM estimates for four emitters over the straight flight path. Top: DMM spectrum. Bottom Rows: Nullspace MUSIC spectra for the four emitters.
Table 7.4: Summary of position errors for four simultaneous emitters with the straight and south circle flight segments.

<table>
<thead>
<tr>
<th>Geolocation (Flight path)</th>
<th>Northwest</th>
<th>Southwest</th>
<th>Southeast</th>
<th>Northeast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional (Straight)</td>
<td>2.7km</td>
<td>13.1km</td>
<td>N/A</td>
<td>1.6km</td>
</tr>
<tr>
<td>DMM (Straight)</td>
<td>1.9km</td>
<td>2.9km</td>
<td>N/A</td>
<td>2.0km</td>
</tr>
<tr>
<td>Nullspace DMM (Straight)</td>
<td>0.9km</td>
<td>2.0km</td>
<td>1.8km</td>
<td>0.8km</td>
</tr>
<tr>
<td>Conventional (South Circle)</td>
<td>2.6km</td>
<td>3.3km</td>
<td>7.4km</td>
<td>9.0km</td>
</tr>
<tr>
<td>DMM (South Circle)</td>
<td>3.9km</td>
<td>4.7km</td>
<td>5.8km</td>
<td>4.9km</td>
</tr>
<tr>
<td>Nullspace DMM (South Circle)</td>
<td>3.9km</td>
<td>1.1km</td>
<td>6.6km</td>
<td>6.8km</td>
</tr>
</tbody>
</table>

from the northwest emitter, the errors are 3.9km, 1.1km, 6.6km, and 6.8km. While the errors for the eastern emitters are large, Nullspace DMM separates the emitters in spite of poor SNR and very poor geometry, and it indicates the general area of the emitters relative to the flight path. Although the three methods performed similarly, Nullspace DMM again yielded the smallest total error using experimental flight data with four simultaneous emitters. The location errors for the straight and south-circle flight paths are summarized in Table 7.4.
Figure 7.20: Conventional estimates for four emitters over the south circle flight path. Top: DOA estimates and clusters. Bottom Rows: Apparent angular error functions for the four clusters over the grid.
Figure 7.21: DMM and Nullspace DMM estimates for four emitters over the south circle flight path. Top: DMM spectrum. Bottom Rows: Nullspace MUSIC spectra for the four emitters.
7.7 Summary and Conclusions

This chapter discussed a flight data campaign, which we used to geolocate four emitters around Columbus, Ohio that broadcast live signals. We recorded the signals during a three-hour flight using a six-element antenna array and a six-channel data collection system (DCS). The data collection system was carefully designed so that the signals from the antenna elements were sampled simultaneously. The DCS was also designed so that the effects of the front-end electronics could be calibrated out. Thus, the residual gain and phase differences between the channels were mostly due to the differences between the antenna elements in the array. In addition, the antenna patterns were simulated in HFSS on a model of the experimental aircraft. Simulating the in-situ array manifold minimized the mismatch between the true, in-situ antenna array manifold and the manifold available for geolocation.

Geolocation followed several steps. First, we minimized the effects of interference by decimating the sampled signals and cross correlating with the emitters’ known spreading codes. Interference mitigation was necessary because the emitters operate in the 900 MHz ISM band, and co-channel interference from other transmitters was apparent in the frequency domain. Cross correlation suppressed interference and yielded estimated manifold vectors in the (unknown) directions of the emitters throughout the flight path. Then, covariance matrices were synthesized for desired combinations of emitters over chosen segments of the flight path. This allowed us to test geolocation of one, two, and four simultaneous emitters over various flight segments.

Conventional DOA estimates and DMM easily located a single emitter at a time. This showed that our experimental procedure was very good, as the results required
accurate time synchronization between the samples and the IMU. The results also could not have been obtained without effective calibration of the front-end electronics in each receive channel. As expected, however, DOA errors were also persistent throughout the flight path, similar to those observed in Chapter 5. These errors implied that the true, in-situ antenna array manifold did not match the approximate array manifold that we simulated in HFSS. These experimental results confirm that DOA systems should anticipate array manifold mismatch and the need to combine inconsistent DOA estimates when geolocating a transmitter.

Next, we geolocated two simultaneous emitters by synthesizing covariance matrices with two estimated manifold vectors at a time. Conventional MUSIC estimation followed by MAAE clustering located both emitters. DMM, and Nullspace DMM likewise located both emitters. Thus, we experimentally demonstrated the novel MAAE clustering algorithm, as well as DMM and Nullspace DMM’s utility for overcoming antenna manifold mismatch.

Finally, we geolocated all four emitters at the same time. Conventional DOA estimation and MAAE clustering were able to locate all four emitters during with the long flight segment and the southern circle flight segment. However, one or two emitters were not located during the straight flight path and the circular flight path in the middle of the emitters. As in Chapter 5, we observed that MAAE clustering is not robust enough in challenging scenarios with very intermixed estimates. Ultimately, manifold mismatch may cause geolocation based on conventional DOA estimation to fail.

We applied DMM and Nullspace DMM to combine DOA spectra and avoid the spurious peaks that foiled the MUSIC and MAAE. DMM always yielded ‘hot spots’
around the true emitter locations. However, the hot spots for the emitters were only well separated during the circular flight path in between the emitters. Although DMM provided good geolocation estimates for at least three emitters in the other flight segments, the merged plateaus created ambiguity regarding the ‘best’ points to choose for the emitter location estimates. In contrast, Nullspace DMM yielded unambiguous estimates, as each location estimate is the absolute maximum of a separate spectrum. Nullspace DMM also yielded the smallest total error for all flight segments with four emitters. Nullspace DMM’s results were particularly good for the straight flight path, for which it was the only algorithm that located every emitter. Since Nullspace DMM was most robust approach in experimental flight data, we recommend it for real world systems that must similarly locate multiple simultaneous emitters that overlap in the frequency domain in spite of an imperfectly calibrated antenna array.
Radio Frequency (RF) emitter geolocation from an airborne or spaceborne platform is critical for diverse applications such as defense, spectrum enforcement, and search and rescue. Single platform geolocation is preferred to Time Difference of Arrival (TDOA) geolocation if multiple receivers are unavailable or if high bandwidth communication between nodes is unacceptable. While emitter geolocation has been essential since the advent of radio, the current explosive growth of wireless communications demands new and improved emitter geolocation capabilities for myriad systems.

Conventional geolocation from a mobile platform relies on Direction of Arrival (DOA) estimation algorithms such as MUSIC. MUSIC estimates DOAs for multiple signals; has asymptotically unlimited resolution; applies to arbitrary and polarization-sensitive antenna arrays; and runs quickly. To geolocate an emitter, Lines of Bearing (LOB) are drawn from the array along the estimated DOAs until they intersect the ground. If the DOAs are estimated without error, then all LOBs intersect the ground at the emitter’s true location.

Unfortunately, MUSIC’s practical resolution and accuracy fall short of theoretical predictions. MUSIC estimates multiple DOAs from the peaks of a single spectrum. Peaks for nearby signals may merge together and prevent MUSIC from distinguishing...
the signals. In addition, MUSIC assumes that the antenna array’s gain and phase patterns – the antenna array manifold – is perfectly known. However, when the array is mounted on a platform, the platform perturbs the manifold by blocking, reflecting, and diffracting the RF signals. Mismatches between the on-platform (\textit{in-situ}) and assumed antenna array manifolds bias the estimated DOAs and further degrade DOA resolution. Moreover, LOBs for multiple emitters may intermix on the ground. If the intermixed position estimates cannot be grouped together and associated with a particular emitter, then conventional geolocation fails completely.

We introduced the novel Nullspace MUSIC algorithm to combat MUSIC’s limited resolution. Nullspace MUSIC applies a divide-and-conquer approach and estimates DOAs by nulling all signals except for the current signal of interest. Nullspace MUSIC proceeds in a round-robin fashion and estimates the DOAs of all signals from individual DOA spectra. Moreover, Nullspace MUSIC maintains MUSIC’s crucial orthogonality test. Monte Carlo trials with single polarization signals showed that Nullspace MUSIC improves MUSIC’s resolution, and it is about as accurate as the best existing modifications to MUSIC. Moreover, Monte Carlo trials with diversely polarized signal show that Nullspace MUSIC can cut the minimum resolvable separation between signals in half. Note that alternative DOA algorithms (not based on MUSIC) rely on the same antenna manifold as MUSIC, or they use models of the antenna array that will not hold when the antenna elements couple to each other and are mounted on a platform. For instance, alternatives require an array containing identical antennas or identical subarrays. Consequently, Nullspace MUSIC provides the most accurate DOAs for arbitrary signals incident on polarization sensitive array, and it outperforms state-of-the-art DOA estimation.
Nullspace MUSIC also improves DOA accuracy in setups with array manifold mismatch. For mismatch studies, the seven-element array was simulated on three versions of a generic aircraft platform. Then, incident signals were simulated assuming the full version of the platform, while DOA estimation used the manifold for a simpler version of the platform. This setup used high fidelity antenna array manifolds and realistically modeled mismatch between the true and assumed antenna array manifolds. The antenna array manifold mismatch setup caused MUSIC’s mean error and error standard deviation to increase dramatically. On average, Nullspace MUSIC improved MUSIC’s accuracy for single polarization and mixed polarization signals for both mismatch setups, so long as the signals were separated by more than about one-half of the array’s beamwidth. On the other hand, the best existing modification to MUSIC could not improve MUSIC’s accuracy for any separation between diversely polarized signals. Therefore, Nullspace MUSIC is the most robust approach for improving MUSIC’s accuracy in realistic scenarios with an imperfectly calibrated antenna array manifold.

Array manifold mismatch led us to introduce a new clustering and geolocation algorithm for DOA estimates that are made throughout a flight. We simulated the received signals over two flight segments, and we applied MUSIC and Nullspace MUSIC to estimate DOAs throughout the flights. The flights assumed antenna array manifold mismatch for the seven-element array on the generic aircraft. The mismatch caused very erratic DOAs, even for a single incident signal. Averaging the points where the LOBs intersect with the ground yielded poor location estimates. On the other hand, Minimizing the Apparent Average Error (MAAE) recovered the emitter positions very accurately in spite of large errors in the individual DOAs. While
MAAE has been known in literature for some time as the best way to locate a single emitter, multiple emitters had not been addressed. With multiple emitters, averaging positions on the ground failed completely. Therefore, we introduced a novel extension of MAAE that successfully grouped the DOA estimates for two emitters, then accurately geolocated the emitters. Although MAAE performs impressively considering the widespread DOA estimates that invariably mix together, it was not sufficiently robust for the most challenging scenarios.

To improve robustness to manifold mismatch, we applied the Direct Mapping Method (DMM) to the same flights. DMM maps DOA spectra to the ground and averages the spectra themselves. Then, emitter locations are estimated directly from the averaged spectrum. DMM does not impose any additional constraints on emitter geolocation, and it actually requires fewer computations than MAAE because peak-finding only needs to be run once for the entire flight. Moreover, inconsistent DOA estimates do not need to be untangled, and the point minimizing the apparent angular does not need to be sought. Any spectral DOA algorithm that calculates a function versus angle can be used as the basis for DMM. While DMM has been applied for low signal-to-noise ratios and sparse antenna arrays, our application of DMM to overcome manifold mismatch was novel. In this work, we used mixed polarization MUSIC as the basis for DMM. We found that DMM was more robust than MAAE in that the DMM spectra were very similar for both mismatch setups. Insensitivity to the exact mismatch setup affirms DMM’s fitness for real-world geolocation, for which the location estimates should not depend on the particular wing tilt or loadout during a mission. However, DMM performs similarly to MAAE in that it could not locate all emitters in the most challenging scenarios with four diversely polarized emitters.
Geolocating all emitters required combining Nullspace MUSIC and DMM in the Nullspace DMM algorithm. Nullspace DMM estimates the first emitter’s position from the usual DMM spectrum. Next, the first emitter’s location is nulled throughout the flight path. A new DMM spectrum is formed for which the first emitter was nulled at every observation. The second emitter’s location is estimated with minimal interference from the first emitter. Emitter estimation continues in a round-robin fashion until every emitter has been estimated individually with minimal interference from the other emitters. Nullspace DMM is the only method that can locate all four diversely polarized emitters in the flight simulations.

Finally, we applied MAAE, DMM, and Nullspace DMM to recorded data from a real world geolocation experiment. A six-element array and Data Collection System (DCS) were built and installed in a small aircraft. The aircraft flew around Columbus, Ohio and recorded signals from four live transmitters. The signals were post processed, and geolocation was attempted for one to four emitters over various segments of the flight path. The experimental flight confirmed the findings earlier in this work. Specifically, individual DOA estimates were erratic, but MAAE accurately located a single emitter. MAAE also separated DOA estimates for two emitters, which were accurately located. The one and two emitter scenarios were also easily handled by DMM. However, DMM and MAAE struggled with four emitter cases. On the other hand, Nullspace DMM located every emitter in every flight segment. Nullspace MUSIC’s robust geolocation - handling four real-world emitters with a six-element array while following low flight paths - demonstrates that Nullspace DMM significantly advances state-of-the-art RF emitter geolocation from an airborne antenna array.
Future Work

RF emitter geolocation will continue to be the subject of intense research for the foreseeable future. While this work focused on simulation and experimental results, theoretical analyses should also be conducted on Nullspace MUSIC and Nullspace DMM. In the presence of antenna array manifold mismatch, Nullspace MUSIC currently underperforms MUSIC for closely separated signals. Future analysis of Nullspace MUSIC should address this poor performance region. For instance, nearby signals with the same polarization generate an increasingly large dynamic range in the eigenvalues of the covariance matrix. The dynamic range of the eigenvalues may indicate that an imperfectly calibrated array should only estimate a single incident signal. While estimators such as the Akaike Information Criteria (AIC) [81] and Minimum Description Length (MDL) [80] may be used to estimate the number of signals that are incident on the array, this work always assumed that the number of signals was known apriori. MDL and AIC may provide a basis for determining when Nullspace MUSIC should not be applied. Alternatively, Maximum Likelihood (ML) could be used to check whether Nullspace MUSIC or conventional MUSIC yields the estimates with the highest ML criteria. Either approach may prevent Nullspace MUSIC’s performance degradation with small signal separations in the presence of array manifold mismatch.

In addition, a system’s limited resolution and accuracy due to manifold mismatch may be addressed through in-situ antenna array calibration with signals of opportunity. In-situ calibration updates the assumed array manifold using estimates of the array manifold vectors and associated directions of arrival. Note that the dominant eigenvector of the covariance matrix is a scaled version of the array manifold vector.
for a single incident signal. Therefore, the manifold vector in the direction of a single incident signal may be easily estimated. In addition, approaches using higher order statistics, such as Independent Component Analysis [92], may estimate manifold vectors for multiple signals. However, as shown in this work, a poorly calibrated array manifold yields biased DOA estimates. We also showed that diverse observations of the scene combined with MAAE, DMM, or Nullspace DMM can accurately recover the emitter location. Knowing the emitter position in addition to the aircraft position and orientation allows the DOAs towards the emitter to be inferred throughout the flight. Therefore, the DOA biases caused by platform-induced array manifold mismatch may be mitigated over time through in-situ calibration using the techniques in this work to geolocate the emitters.

Finally, we note that Nullspace processing may be applied to various other algorithms. For instance, Nullspace processing could be fused with Maximum Likelihood estimation to estimate one signal’s parameters while other signals of interest are nulled. This is in contrast to alternating projection [74], which holds parameters fixed while estimating a single parameter, but still uses the sample covariance matrix. Nulling other signals that have already been estimated, then generating a new covariance matrix and test vectors (as in Nullspace MUSIC), may improve the Alternating Projection approach to Maximum Likelihood estimation. Other parameter estimators may benefit similarly.
Appendix A: Motivation for Orthonormalizing the Test Vectors in MP-MUSIC

We have found that the best practice for calculating Mixed Polarization MUSIC, [24], is to use orthonormal test vectors for $M$ in (2.27). We first explain why the Mixed Polarization (MP)-MUSIC spectrum peaks at a signal DOA. Then, we explain why the test vectors should be normalized prior to calculating MP-MUSIC. Lastly, we explain why the manifold vectors for the different polarizations should be orthogonalized as well. In short, non-orthonormal test vectors lead to false peaks in the MP-MUSIC spectrum.

A.1 MP-MUSIC at a Signal DOA

In [24], the $2 \times 2$ matrix $M$ of (2.27) is written as

$$M(\psi) = \begin{bmatrix} b^H(\psi) & b^H(\psi) c(\psi) \\ c^H(\psi) & c^H(\psi) c(\psi) \end{bmatrix}.$$  \hspace{1cm} (A.1)

Note that because $P$ is a projection matrix, $P^H = P$ and $P^H P = P$. Thus, we can also write (A.1) as

$$M(\psi) = \begin{bmatrix} (Pb(\psi))^H Pb(\psi) & (Pb(\psi))^H Pc(\psi) \\ (Pc(\psi))^H Pb(\psi) & (Pc(\psi))^H Pc(\psi) \end{bmatrix}. \hspace{1cm} (A.2)$$

Let $Pb = \beta$. Let $Pc = \gamma$. Then, (A.2) is

$$M(\psi) = \begin{bmatrix} \beta^H \beta & \beta^H \gamma \\ \gamma^H \beta & \gamma^H \gamma \end{bmatrix}. \hspace{1cm} (A.3)$$
Note that \( b \) and \( c \) span a 2D subspace. If \( \psi_1 \) is the DOA of a signal, then \( P \) will project \( b \) and \( c \) onto a space excluding some linear combination of \( b \) and \( c \). This makes \( \beta \) and \( \gamma \) linearly dependent at \( \psi_1 \). In this case, we can write

\[
\gamma = \rho \beta \tag{A.4}
\]

and

\[
M(\psi_1) = \begin{bmatrix}
\beta^H \beta & \rho \beta^H \beta \\
\rho^* \beta^H \beta & |\rho|^2 \beta^H \beta
\end{bmatrix} . \tag{A.5}
\]

where \( \cdot^* \) denotes the conjugate and \( | \cdot | \) is the absolute value.

Let \( \mathbf{v} = [\beta^H \beta \quad \rho^* \beta^H \beta]^T \). Then,

\[
M(\psi_1) = [\mathbf{v} \quad \rho \mathbf{v}] . \tag{A.6}
\]

The columns of \( M(\psi_1) \) are linearly dependent, and \( M(\psi_1) \) has an eigenvalue equal to zero. This produces a singularity in \( \lambda_{\min}^{-1}(M(\psi)) \) at \( \psi_1 \). The singularity indicates that a signal was incident from \( \psi_1 \), as desired.

### A.2 Test Vector Normalization

Now suppose that the all antennas in the array are purely \( E_\theta \) polarized at \( \psi_1 \). Then, \( c = 0, \gamma = 0 \) and (A.3) becomes

\[
M(\psi_1) = \begin{bmatrix}
\beta^H \beta & 0 \\
0 & 0
\end{bmatrix} . \tag{A.7}
\]

This matrix is also singular and produces a singularity in \( \lambda_{\min}^{-1}(M(\psi_1)) \) regardless of whether a signal is present. In practice, all antennas have some cross-pol for all angles. However, if the array is dominantly \( E_\theta \) polarized at \( \psi_1 \), then \( \lambda_{\min}^{-1}(M(\psi_1)) \) will be large regardless of whether a signal is present. Therefore, the test vectors in (2.27) should be normalized in order to avoid distorting the MP-MUSIC spectrum.
A.3 Test Vector Orthogonalization

Now suppose that the array consists of identical elements in free space with no mutual coupling between them. In this case, the Array Factor of the array is also the antenna array manifold. The $E_\theta$ and $E_\phi$ manifold vectors are scaled versions of the Array Factor. Their scale factors are the magnitudes and phases of the antenna elements’ $E_\theta$ and $E_\phi$ responses in a given direction. We can write, $c = \rho b$, $Pb = \beta$, and $Pc = \rho \beta$. Once again, (A.2) becomes

\begin{equation}
M(\psi) = \begin{bmatrix}
\beta^H \beta & \rho \beta^H \beta \\
\rho^* \beta^H \beta & |\rho|^2 \beta^H \beta
\end{bmatrix}.
\end{equation}

(A.8)

Hence, when the $E_\theta$ and $E_\phi$ test vectors are parallel, $M(\psi)$ has linearly dependent columns and $\lambda_{\text{min}}^{-1}(M(\psi))$ is singular. When the $E_\theta$ and $E_\phi$ test vectors are almost parallel, $\lambda_{\text{min}}^{-1}(M(\psi))$ is large and may have a spurious peak caused by the test vectors and not by the incident signals. Therefore, the test vectors should be orthogonalized for MP-MUSIC to avoid distorting its spectrum. This is why Schmidt notes that orthogonal test vectors provide the best results in [93].

Here, we also see that the projection matrix’s rank must be at least two in order to avoid making (A.8) singular for all angles. Thus, MP-MUSIC and Nullspace MUSIC may only search for up to $N-2$ signals.

This appendix shows that a simple preprocessing rule for MUSIC is to always orthonormalize the test vectors prior to calculating $M(\psi)$. This is the approach we followed in (2.27) and for all MP-MUSIC simulations examples in the dissertation.
Appendix B: Sequential MUSIC Projection Matrices

The Sequential MUSIC spectrum that is used to estimate the second signal’s DOA can be written as

\[
f_{\text{SEQ,2}}(\theta) = \frac{1}{\|P_N \hat{r}(\theta, \tilde{\theta}_1)\|^2} \]

\[
f_{\text{SEQ,2}}(\theta) = \frac{[I - \hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)]a(\theta)}{\|P_N [I - \hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)]a(\theta)\|^2} \tag{B.1} \]

\[
f_{\text{SEQ,2}}(\theta) = \frac{P_{\perp}a(\theta)}{\|P_NP_{\perp}a(\theta)\|^2}. \]

Recall that \(P_N\) is the projection matrix for the noise space of the sample covariance matrix; \(a(\theta)\) is the array manifold vector towards angle \(\theta\); \(\tilde{\theta}_1\) is the estimated angle for a signal in the first step of Sequential MUSIC; \(P_{\perp}\) is the projection matrix for the space orthogonal to \(\hat{a}(\tilde{\theta}_1)\); a hat (\(\hat{\cdot}\)) indicates that a vector is unit norm; and \(r(\theta)\) is the residual vector in Sequential MUSIC.

This appendix shows that the matrix in the denominator, \(P_NP_{\perp}\) is only a projection matrix if \(\hat{a}(\tilde{\theta}_1)\) is in the noise subspace or the signal subspace. If \(\hat{a}(\tilde{\theta}_1)\) has components in both the signal subspace and the noise subspace, then \(P_NP_{\perp}\) is not a projection matrix. Recall that MUSIC is fundamentally an orthogonality test [24]. The orthogonality test is carried out by left-multiplying the array manifold vectors with a projection matrix for the noise subspace. Sequential MUSIC’s failure to always
use a projection matrix is the likely cause for its higher DOA variance in Monte Carlo trials (See Chapter 3), as well as in theoretical analyses [37].

Projection matrices have two properties. First, $PP = P$. Second, $P^H = P$. We now show that $P_N P_\perp$ is not a projection matrix if $\hat{a}(\tilde{\theta}_1)$ has a noise space component and a signal space component. The proof is a proof by contradiction.

Let us assume that $P_N P_\perp$ is a projection matrix. Recall that $P_\perp = I - \hat{a}(\tilde{\theta}_1)\hat{a}^H(\tilde{\theta}_1)$. For now, let us write $\hat{a}(\tilde{\theta}_1)$ as $\hat{a}_1$. Then,

$$P_N P_\perp = P_N P_\perp P_N P_\perp$$

$$P_N(I - \hat{a}_1\hat{a}_1^H) = P_N(I - \hat{a}_1\hat{a}_1^H)P_N(I - \hat{a}_1\hat{a}_1^H)$$

(\text{B.2})

$$P_N - P_N\hat{a}_1\hat{a}_1^H = P_N P_N - P_N P_\perp\hat{a}_1\hat{a}_1^H - P_N\hat{a}_1\hat{a}_1^H P_N$$

$$+ P_N\hat{a}_1\hat{a}_1^H P_N\hat{a}_1\hat{a}_1^H.$$  

(\text{B.3})

$P_N$ is explicitly constructed as a projection matrix. Therefore, $P_N P_N = P_N$ and

$$P_N - P_N\hat{a}_1\hat{a}_1^H = P_N - P_N\hat{a}_1\hat{a}_1^H - P_N\hat{a}_1\hat{a}_1^H P_N$$

$$+ P_N\hat{a}_1\hat{a}_1^H P_N\hat{a}_1\hat{a}_1^H.$$  

(\text{B.3})

Subtracting the left hand side from both sides,

$$0 = -P_N\hat{a}_1\hat{a}_1^H P_N + P_N\hat{a}_1\hat{a}_1^H P_N\hat{a}_1\hat{a}_1^H$$

(\text{B.4})

$$P_N\hat{a}_1\hat{a}_1^H P_N = P_N\hat{a}_1\hat{a}_1^H P_N\hat{a}_1\hat{a}_1^H.$$  

This must be true for all $\hat{a}_1$. Obviously, it is satisfied if $P_N\hat{a}_1 = 0$. Therefore, our starting assumption - that $P_N P_\perp$ is a projection matrix - holds if $\hat{a}_1$ is in the signal subspace that is perpendicular to the noise subspace. However, if $\hat{a}_1 \neq 0$, then,

$$P_N\hat{a}_1(P_N\hat{a}_1)^H = P_N\hat{a}_1(P_N\hat{a}_1)^H \hat{a}_1\hat{a}_1^H.$$  

(\text{B.5})
Recall that $P_N^H = P_N$. In addition, if $M_1$ and $M_2$ are two matrices, then $(M_1M_2)^H = M_2^HM_1^H$. Taking the Hermitian of both sides,

$P_N\hat{a}_1(P_N\hat{a}_1)^H = \hat{a}_1\hat{a}_1^HP_N\hat{a}_1(P_N\hat{a}_1)^H$.  \hspace{1cm} (B.6)

The above equation holds if

$P_N\hat{a}_1 = \hat{a}_1\hat{a}_1^HP_N\hat{a}_1$. \hspace{1cm} (B.7)

Recall that $x$ is an eigenvector of $M$ if $Mx = \lambda x$. Therefore, $P_N\hat{a}_1$ is an eigenvector of $\hat{a}_1\hat{a}_1^H$ with eigenvalue 1. Since $\hat{a}_1\hat{a}_1^H$ is rank 1, its eigenvector is $\hat{a}_1$, and

$P_N\hat{a}_1 = \hat{a}_1$. \hspace{1cm} (B.8)

In other words, $\hat{a}_1$ is in the noise subspace. Therefore, our original assumption that $P_NP_\perp$ is a projection matrix holds with $P_N\hat{a}_1 \neq 0$ if $\hat{a}_1$ is in the estimated noise subspace. Next, we will look at whether $P_NP_\perp$ is a projection matrix if $\hat{a}_1$ has a component in the estimated noise space as well as the signal subspace.

We found that if $P_NP_\perp$ is a projection matrix, then

$\hat{a}_1\hat{a}_1^HP_N\hat{a}_1(P_N\hat{a}_1)^H = P_N\hat{a}_1(P_N\hat{a}_1)^H$. \hspace{1cm} (B.9)

Suppose that $\hat{a}_1 = a_N + a_S$ with $a_N \perp a_S$, $a_N \neq 0$, and $a_S \neq 0$. Note that $P_N\hat{a}_1 = a_N$. Then,

$\hat{a}_1\hat{a}_1^Ha_Na_N^H = a_Na_N^H$

$(a_N + a_S)(a_N + a_S)^Ha_Na_N^H = a_Na_N^H$ \hspace{1cm} (B.10)

$(a_Na_N^H + a_Na_S^H + a_Sa_N^H + a_Sa_S^H)a_Na_N^H = a_Na_N^H.$

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Since $a_N \perp a_S$, the terms inside of the parenthesis ending in $a_S^H$ will go to zero when right-multiplied by $a_N$, 

\[(a_Na_N^H + a_Sa_N^H)a_Na_N^H = a_Na_N^H \]

\[a_Na_N^Ha_Na_N^H = a_Na_N^H. \tag{B.11} \]

Note that $a_N^Ha_N = ||a_N||^2$. We find this term inside both terms on the left side, 

\[||a_N||^2(a_Na_N^H + a_Sa_N^H) = a_Na_N^H \]

\[||a_N||^2a_Na_N^H = a_Na_N^H - ||a_N||^2a_Na_N^H \]

\[||a_N||^2a_Sa_N^H = (1 - ||a_N||^2)a_Na_N^H \]

\[a_Sa_N^H = \frac{(1 - ||a_N||^2)}{||a_N||^2}a_Na_N^H. \tag{B.12} \]

Note that the right side of the above equation is Hermitian. For equality to hold, the left side must also be Hermitian. This implies that 

\[a_Sa_N^H = a_Na_S^H. \tag{B.13} \]

If we multiply both sides by $a_Sa_N^H$, then 

\[a_Sa_N^Ha_Sa_N^H = a_Na_N^Ha_Sa_N^H \]

\[0 = ||a_S||^2a_Na_N^H. \tag{B.14} \]

Since $a_S \neq 0$ and $a_N \neq 0$, the last equation is false. Therefore, if $\hat{a}_1$ has a signal subspace component and a noise subspace component ($\hat{a}_1 = a_N + a_S$), then $P_NP_\perp$ is not a projection matrix. This completes the proof.
Bibliography


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