A Study of Black Hole Formation and Evaporation via the D1D5 CFT Dual

Dissertation

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Abstract

Several advancements in our understanding of fuzzball black have been made via the study of a dual Conformal Field Theory. In this approach, the full picture of black hole formation and eventual evaporation is expected to correspond to thermalization in the CFT dual. In other words, an initial set of high-energy wavepackets that 'break up' into a large quantity of low-energy excitations in the CFT should be the dual to an initial set of high-energy infalling particles forming a black hole that then evaporates via a large quantity of low-energy Hawking radiation in the gravity description.

The nature of a fuzzball’s CFT dual depends on particular parameters of the underlying gravity theory, its moduli space, and it is conjectured that some choice of parameters is dual to a particularly simple ‘orbifold’ CFT. Calculations at this orbifold point match some semi-classical results, but it is also clear that the orbifold point is does not exactly match the behavior of gravity as we know it. It is thus necessary to move away from this orbifold point via the perturbative application of a deformation operator.

A framework for studying deformations away from the orbifold point was developed some time ago, but its application was previously limited to the simplest of tree-level processes. My work extends this framework to all tree-level processes. At this stage, there is no clear indication of thermalization. In addition, analysis is muddied by a change in the Hilbert space of the CFT. This makes comparisons between
initial and final states difficult to perform. To address this, the framework was further extended to include a simple one-loop process. This process maintains a consistent Hilbert space between in and out states, allowing a more straightforward search for thermalization effects. Furthermore, certain patterns occurred when comparing the one-loop results to the tree-level process that constitutes the loop’s first half. It is thus natural to conjecture that some of these patterns may remain at even higher orders.

In addition to this work, I also present detailed analysis of a simplifying regime known as the ‘continuum limit.’ In this limit, the energy of the system is taken to be much larger than the size of the individual quanta, so that excitation levels appear continuous rather than discrete. This is exactly the regime of black hole formation. It also greatly simplifies the results of our calculations, which is of particular importance when dealing with the otherwise intractable expressions found in even the simplest one-loop case.

While my work has not by any means succeeded in completing the ideal picture of black hole formation and subsequent evaporation, it provides a solid foundation upon which one may continue the search. By maintaining a consistent Hilbert space, the new one-loop result is readily extendable via a method of repeated application, which is not the case for the tree-level calculations. For the same reason it provides a more accessible place to search for wave packet splitting. And while the analytic results are particularly messy, they are accessible enough to facilitate future numerical work. Furthermore, a large number of useful relations have been proven to all orders of the twist portion of the deformation.
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# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>iv</td>
</tr>
<tr>
<td>List of Figures</td>
<td>ix</td>
</tr>
<tr>
<td>List of Tables</td>
<td>x</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 The Semi-Classical Black Hole</td>
<td>2</td>
</tr>
<tr>
<td>1.1.1 Entropy and Black Hole Microstates</td>
<td>2</td>
</tr>
<tr>
<td>1.2 The Information Paradox</td>
<td>4</td>
</tr>
<tr>
<td>1.3 Fuzzballs</td>
<td>5</td>
</tr>
<tr>
<td>2. The CFT Dual</td>
<td>9</td>
</tr>
<tr>
<td>2.1 The Orbifold Point</td>
<td>9</td>
</tr>
<tr>
<td>2.1.1 Copy Notation</td>
<td>10</td>
</tr>
<tr>
<td>2.1.2 Symmetries</td>
<td>11</td>
</tr>
<tr>
<td>2.1.3 Field Content</td>
<td>12</td>
</tr>
<tr>
<td>2.2 Vacuum States</td>
<td>14</td>
</tr>
<tr>
<td>2.3 Deforming the CFT</td>
<td>15</td>
</tr>
<tr>
<td>2.4 General Results</td>
<td>17</td>
</tr>
<tr>
<td>2.4.1 Transpose Relation for Bosons</td>
<td>20</td>
</tr>
<tr>
<td>2.4.2 Supersymmetry Relations: Bogoliubov Coefficients</td>
<td>21</td>
</tr>
<tr>
<td>2.4.3 Supersymmetry Relations: Transition Amplitudes</td>
<td>23</td>
</tr>
</tbody>
</table>
3. First Order Deformations ........................................... 26
   3.1 The ACM Method ............................................... 26
      3.1.1 Emptying t Plane ......................................... 28
      3.1.2 Relations for the Coefficients of Interest .............. 31
      3.1.3 Mapping the Modes to the Empty t Plane ................ 34
      3.1.4 Computing the Bogoliubov Coefficients ................. 38
      3.1.5 Computing the Transition Amplitudes ................... 40
   3.2 Applying the Supercharge ..................................... 43

4. One-Loop Calculation ............................................. 45
   4.1 The Covering Map ............................................. 45
   4.2 Deforming the Modes .......................................... 49
      4.2.1 Handy Symmetries ........................................ 50
   4.3 The Bogoliubov Coefficients .................................. 52
      4.3.1 Fermion Zero Modes ....................................... 54
   4.4 The Transition Amplitudes .................................... 54
      4.4.1 Fermion Zero Modes ....................................... 57
   4.5 Handling the Supercharge ..................................... 58
      4.5.1 Supercharge Mapping ...................................... 59
      4.5.2 The First Surpercharge Integral ......................... 60
      4.5.3 The Second Supercharge Integral ......................... 71
      4.5.4 Deforming and Inverting .................................. 72
      4.5.5 Writing the Full Supercharge Result ..................... 76

5. The Continuum Limit .............................................. 81
   5.1 First-Order Continuum Limit ................................. 82
      5.1.1 The Bogoliubov Coefficient ............................... 83
      5.1.2 The Transition Amplitude ................................ 84
   5.2 The One-Loop Case ........................................... 85
      5.2.1 Bogoliubov Coefficients ................................ 86
      5.2.2 The Transition Amplitudes ................................ 87
   5.3 The Supercharge Coefficient .................................. 90

6. Analysis and Conclusions ....................................... 91
   6.1 First Order Results .......................................... 92
   6.2 Second Order Results ........................................ 93
   6.3 Conclusions .................................................. 94
Appendices

A. R Vacuum Notation ............................................. 98

B. Proof of the the $|\chi\rangle$ Form. ............................... 100
   B.1 Middle expression ........................................... 100
   B.2 Right expression ............................................ 101

C. Proof that $G_{A,0}^+$ Commutes with $\sigma_2^+(w_0) = 0$ ............. 104
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Smooth Capping Process</td>
<td>6</td>
</tr>
<tr>
<td>1.2 Fuzzball Geometry</td>
<td>7</td>
</tr>
<tr>
<td>2.1 Removal of the G Contour</td>
<td>18</td>
</tr>
<tr>
<td>3.1 Single Deformation Schematic</td>
<td>28</td>
</tr>
<tr>
<td>3.2 First Order $t$ Plane</td>
<td>29</td>
</tr>
<tr>
<td>4.1 Two-Twist Schematic</td>
<td>46</td>
</tr>
<tr>
<td>4.2 Second Order Complex Planes</td>
<td>47</td>
</tr>
<tr>
<td>5.1 Amplitude Scaling of One-Loop Bogoliubov Coefficient</td>
<td>87</td>
</tr>
<tr>
<td>5.2 Oscillatory Behavior of One-Loop Bogoliubov Coefficient</td>
<td>88</td>
</tr>
<tr>
<td>5.3 One-Loop Transition Amplitude Fit</td>
<td>89</td>
</tr>
<tr>
<td>5.4 One-Loop Supercharge Coefficient Fit</td>
<td>90</td>
</tr>
</tbody>
</table>
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Tracking Supercurrent Contour Deformations</td>
<td>72</td>
</tr>
</tbody>
</table>
Chapter 1: Introduction

The Standard Model is one of the greatest accomplishments in physics over the past century. Containing a quantum-mechanical description of the electromagnetic, weak, and strong nuclear interactions along with a classification of all directly-detected fundamental particles, the theory boasts some of the most precise experimental verification in all of science. At the same time, it is known to be incomplete. There is currently much speculation and study of many potential gaps in the Standard Model, most prominently the nature of dark matter and the possibility of supersymmetry, but my study focuses on the most model’s prominent gap — the complete absence of nature’s fourth fundamental force, gravity.

Currently, our best understanding of gravitational interactions comes from the theory of General Relativity. Here, the gravitational force is described as curvature in the spacetime geometry. While successful at large scales, this treatment of gravity is fundamentally incompatible with our current understanding of quantum mechanics at small distances, and by extension, large energies. Thinking in terms of small distances, incompatibility arises from the presence of quantum virtual particles that break the the typical “smoothness” of the spacetime geometry. However, my work approaches the problem from the perspective of large energies, where the incompatibility manifests in the process of black hole formation and evaporation.
1.1 The Semi-Classical Black Hole

Einstein’s equations of General Relativity permit solutions that contain singularities and event horizons, marking causally disconnected regions. A simple example is the Schwarzschild black hole described by the measure:

\[ ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{r}{2M}\right)^{-1} dr^2 - r^2 d\Omega, \quad (1.1) \]

where \( d\Omega \) is the Euclidean spherical measure. This metric describes the spacetime structure surrounding a spherically-symmetric mass \( M \). The event horizon is the surface \( r = 2M \), where the radial and temporal coordinates trade roles. Any path stretching from the interior of this region to the exterior is necessarily spacelike. Even light cannot escape the event horizon.

While more complicated black hole metrics exist, a particularly troublesome pattern soon emerged. The parameters describing the metric external to a classical black hole are its mass, angular momentum, and electric charge. These are the macrostate parameters accessible to outside observers, but what about black hole microstates? For other objects like stars and planets the question can in principal be addressed through direct interactions with the object’s surface and interior, but black holes hide such interactions are behind an event horizon. It thus seems that a black hole’s microstate information is completely hidden. This naturally leads us to two important questions. What kinds of microstates are hidden inside a black hole have, and does the particular microstate configuration really have no effect on the outside world?

1.1.1 Entropy and Black Hole Microstates

Partial progress on our first question was made some time ago by the combined work of Bekenstein and Hawking [1] [2]. Together, they found that the entropy of a
black hole is one fourth the area of its event horizon as measured in Planck units.

\[ S_{BH} = \frac{A}{4}. \] (1.2)

This formula allows us to rest easy knowing that an entropic system falling into a black hole does not in fact violate the second law of thermodynamics, but it also presents some more questions. First off, it turns out that the entropy in a black hole is enormous. For a Schwarzschild black hole, the radius is proportion to the mass and thus the energy. We then have the highly unusual relation:

\[ S_{BH} \sim E^2. \] (1.3)

Compare this to an ideal gas, where the entropy grows as the log of the energy. At astronomical scales the difference quickly gets out of hand. Yet at the same time, all of the matter in a classical black hole is crushed into a singularity. This is quite the conundrum. How can there by so many more ways to configure a singularity as compared to more familiar systems like an ideal gas?

The second oddity of the Bekenstein-Hawking formula is as follows. Any time one tries to cram too much energy into too small a volume gravity takes over and a black hole forms, so in a manner of speaking black holes are the densest objects nature can hold. It then stands to reason that they would also contain the maximally-allowed information density. Indeed, we just saw that the entropy of a black hole is very large. But then it would be natural to expect a relationship between entropy and volume, rather than area. Why is it that the maximum amount of information nature can hide within a region of space depends on the surface area of that space instead of its volume?
On top of all this, we still have our original second question. There are a large number of different microstates that any particular black hole can inhabit, but does that ever effect the universe outside the horizon?

1.2 The Information Paradox

If black holes have varying entropy, then thermodynamically they ought to have a temperature. This would in turn imply black body radiation even though such radiation could never escape the event horizon in any classical theory. Indeed, Hawking identified a quantum-mechanical mechanism by which energy can escape a black hole \[3\]. Hawking radiation can occur when a quantum virtual pair is created in the vicinity of an event horizon, wherein one of the two particles escapes to infinity while its negative-energy counterpart is absorbed by the black hole. Hawking also showed that this radiation is thermal, with a temperature that precisely fits with the Bekenstein entropy in the expected thermodynamic fashion.

\[
\frac{1}{T_{BH}} \sim \frac{\partial S}{\partial E} \sim E, \tag{1.4}
\]

where the last relation is for a simple Schwarzschild black hole. More generally the temperature of a black hole is proportional to the surface gravity at the event horizon.

Unfortunately, the radiation mechanism found by Hawking poses a serious problem for semi-classical physics. The pair-production that causes radiation takes place in the near-horizon region, but in the semi-classical black hole this region is in a vacuum state since the initial matter has all collapsed into a singularity. Thus the Hawking radiation can have no relation to any potential microscopic description of its source. This means that two black holes which have the same macroscopic parameters — the same mass, charge, and angular momentum — will both radiate in the same
manner, even if the black holes were formed from wildly different collapses. Indeed, Hawking demonstrated that if a black hole evaporates completely,\(^1\) the final collection of Hawking radiation is related to the initial collapsing matter that formed the black hole through a density matrix, \textit{not} a unitary S-matrix \[^2\]. Furthermore, any 'small corrections' coming from quantum gravity will be insufficient to alter this conclusion \[^4\]. This means that semi-classical black hole evaporation is markedly different from the evolution of an ordinary quantum state, to the point that an observer who collects all the radiation will be \textit{unable} to identify the initial collapsing state. This loss of information resulting from the non-unitary evolution of black holes is known as the information paradox. It is a striking illustration of the incompatibility of General Relativity with the Standard Model.

1.3 Fuzzballs

In attempting to resolve the above quandaries, it is natural to look towards a quantum theory of gravity. Indeed, string-theoretic models describing black holes as bound states of various p-branes, termed fuzzballs, have found some success in this area. For my research, I have focused on states consisting of \(n_1\) D1-branes and \(n_5\) D5-branes in type IIB string theory, which is termed the D1D5 black hole. Leading order microstate counts in the macroscopic limit for this model have been shown to agree with the Bekenstein entropy calculated from their classical horizon area \[^5\]. However, the microstates themselves generate geometries which lack both horizons and singularities \[^6\]. Instead, the geometries end in a smooth cap in the classical

\(^1\)A proposed alternative, which will not be discussed here, is that the black hole leaves a plank-sized remnant which nonetheless possesses enormous information content
horizon region where one of the compact dimensions shrinks to zero size. This process is depicted in Figure 1.1.

![Diagram of fuzzball geometry](image)

Figure 1.1: The process by which a fuzzball geometry caps smoothly. One of the compact dimensions shrinks to zero size in the classical horizon region. There is no spacetime interior to this region.

While fuzzball geometries tend to approach the traditional, singularity-containing black hole geometry outside the classical horizon region, they look very different within that region. In the D1D5 model of interest, the theory contains a total of 9 + 1 dimensions. One spatial dimension is compactified into a circle \( S^1 \), while four others are compactified onto a smaller torus \( T^4 \) (other 4-manifolds are possible, but we consider only the torus here). The D1-branes wrap around \( S^1 \), while the D5-branes have one direction wrapped around \( S^1 \) and the other four wrapped around the much smaller \( T^4 \). The expected geometry at infinity is then \( \mathcal{R}^{1,1} \times S^1 \times T^4 \), where the noncompact dimensions are flat. Near the classical horizon region we obtain a geometry close to that of a naive black hole, which can be approximated by \( AdS_3 \times \) Compactifications. This AdS region is called the 'throat,' and these compactifications turn out to be \( S^3 \times T^4 \), giving a throat geometry of \( AdS_3 \times S^3 \times T^4 \). The geometry ends in a smooth 'cap' as one of the compact dimensions shrinks to zero (see Figure 1.1). There is also a 'neck,' which connects the throat to flat infinity. A schematic of
this geometry is depicted in Figure 1.2, which comes from [7]. The dashed line is the boundary of the AdS throat, allowing for a CFT dual description of that region.

Figure 1.2: The geometry of a generic fuzzball microstate. Spacetime is flat at infinity and AdS in the throat. There is a 'neck' which connects infinity with the throat. The geometry ends not in a singularity, but in a 'fuzzball cap' whose structure depends on the choice of microstate. A conformal dual to the AdS region lies on the boundary $r_b$ and describes the region interior to that boundary. This figure is reproduced from [7].

Because of the complete lack of any interior spacetime, the fuzzball structure remains near the classical horizon rather than collapsing inwards. Such structure would entail that Hawking radiation does not occur in a vacuum, but at a surface containing microstate information. This offers a mechanism whereby Hawking radiation could carry that information with it, thereby avoiding the transition into a mixed state that Hawking identified. Furthermore, the fact that all the structure is located near the surface of the classical-horizon directly resolves the question of why black hole entropy scales with the area of that surface rather than the volume it surrounds.
Despite these successes, the account of fuzzball formation and evaporation is still incomplete. A detailed microscopic description of these processes would be ideal. Fortunately, the gravity description of string theory is dual to a CFT dual at the boundary of the fuzzball’s AdS throat \[8\]. It is in this dual CFT that I have conducted my research.

The remainder of this thesis will proceed as follows. In Chapter 2, I describe the dual CFT and the perturbative calculations that must be performed. In Chapter 3, I present a full account of all first-order configurations. In Chapter 4, I present a simple second-order case. In Chapter 5, I take a look at a simplifying large-energy limit. In Chapter 6, I assess the pertinent outcomes of my work and describe some promising directions for future research.
Chapter 2: The CFT Dual

There are several different parameters that can impact a fuzzball’s precise throat geometry. Vacuum expectation values, the string coupling $g$, and the size of the compact dimensions $S^1$ and $T^4$ are a few examples these parameters, which in total define the moduli space of the theory. Naturally, moving through points in this moduli space is dual to moving through the moduli space of the CFT at the boundary. But since the exact geometry is complicated, early work instead began at a simple point in the CFT moduli space termed the orbifold point\(^2\). While there is no good reason to expect this point to be dual to the geometry we are interested in, its simplicity still makes it a good place to begin our investigations. It also provides many fruitful results \cite{5,9–14}.

For the most part, this work will follow the notational conventions laid out in \cite{15}.

\subsection{The Orbifold Point}

At the orbifold point, the CFT dual is a 1+1 dimensional sigma model with central charge $c = 6$. The base space is spanned by the $S^1$ coordinate $y$ and the time coordinate $t$. Naturally, the coordinate $y$ is compactified with $0 \leq y < 2\pi R$, where $R$ is the radius of $S^1$. We take this to be much larger than the string scale. It is also

\(^2\)Technically the existence of the orbifold point within the CFT moduli space is only a conjecture, but we will not concern ourselves with that technicality here.
convenient to work with the rescaled, Euclideanized coordinates:

\[ \tau = i \frac{t}{R}, \quad \sigma = \frac{y}{R}, \]  

\[ (2.1) \]

which of course restricts \( 0 \leq \sigma < 2\pi \).

The target space of the orbifold sigma model is the symmetrized product of \( n_1 n_5 \) copies of the torus \( T^4 \). Each copy of \( T^4 \) gives four bosonic excitations \( X^i \) and four fermionic excitations \( \psi^i \), and each of these fields can be left- or right-moving around \( S^1 \). Parenthetical superscripts will denote which copy of \( T^4 \) the operators belong to, while a bar over an operator indicates that it is part of the right-moving sector. Since the left- and right-moving sectors are almost completely factorized (they are equivalently the holomorphic and antiholomorphic sectors of a 1+1 dimensional CFT), analysis is typically restricted to only the left-moving sector. Where the right-moving sector is included, redundant bars are placed on operator indices to avoid confusion.

Since the target space is a symmetrized product of \( n_1 n_5 \) identical copies of the relevant CFT, any states we construct at the orbifold point must be symmetric under permutations of these copies. For example, if we wish to construct a state with a left-moving bosonic excitation in a single copy of \( T^4 \), we must use something of the form:

\[ |\psi\rangle = (X^{(1)} + X^{(2)} + X^{(3)} + \ldots + X^{(n_1 n_5)})|0\rangle \]  

\[ (2.2) \]

At this point, operators from one copy of \( T^4 \) do not interact at all with operators from other copies.

### 2.1.1 Copy Notation

In later chapters, we will deform away from the orbifold point in a manner that joins and un-joins various copies of the \( c = 6 \) CFT. The notation of [15] was tailored
to the simple case of two singly-wound CFTs joining into one doubly-wound CFT. We must therefore make a few modifications in order to handle arbitrary windings and arbitrary numbers of copies.

We henceforth label an arbitrary CFT copy with the index \((k)\). If this CFT is specifically before all deformations of interest, the index \((i)\) is used instead. The index \((j)\) is used if the CFT is specifically after the deformations. In short:

\[
(i) \implies \text{initial copy}, \quad (j) \implies \text{final copy}, \quad (k) \implies \text{any copy}.
\] (2.3)

When the copy number must be identified explicitly, a prime is used for copies located after the deformations. So \((1)\) means Copy 1 before the deformations while \((1')\) means Copy 1 after the deformations. We will never need to talk about copies in-between deformations of interest. Each copy’s winding number will be denoted as \(N_{(k)}\).

2.1.2 Symmetries

Each CFT copy has an \(SO(4)\) symmetry arising from the rotational symmetry in the 4 non-compact directions of \(R^{4,1}\) at flat infinity. Following the notation of [15], this symmetry is described in terms of \(SU(2)\) symmetries for the left- and right-moving fermions:

\[
SO(4)_E \sim SU(2)_L \times SU(2)_R,
\] (2.4)

where the \(E\) stands for ‘external’ and is used to indicate that the symmetry stems from rotations in the non-compact directions. The corresponding quantum numbers thus give the angular momentum of quanta in the gravity description.

One can also perform rotations in the four directions of the compactified \(T^4\). The symmetry is broken by the compactification, but it still serves as a useful organizing
principle. We thus write:

\[ SO(4)_I \sim SU(2)_1 \times SU(2)_2, \]  

(2.5)

where the \( I \) stands for ‘internal’ and is used to indicate that the symmetry stems from rotations in the compact \( T^4 \) directions. In grouping our fields, we use indices \( \alpha, \dot{\alpha} \) for \( SU(2)_L \) and \( SU(2)_R \), respectively, and indices \( A, \dot{A} \) for \( SU(2)_1 \) and \( SU(2)_2 \), respectively. Any bars used to indicate the right-moving sector will be placed beneath any relevant dots. All of these \( SU(2) \) charges are raised and lowered via the antisymmetric \( \epsilon \) tensor:

\[ \epsilon_{++} = \epsilon^{+-} = -\epsilon_{-+} = -\epsilon^{-+} = 1. \]  

(2.6)

2.1.3 Field Content

The four left-moving fermions can be grouped into complex fermions that transform under \( SU(2)_L \) and \( SU(2)_1 \), while the right-moving fermions are similarly grouped into complex fermions that transform under \( SU(2)_R \) and \( SU(2)_1 \). The bosons are vectors in the torus, and thus transform under \( SO(4)_I \sim SU(2)_1 \times SU(2)_2 \). We therefore work with:

\[ \psi^{\alpha A}, \quad \bar{\psi}^{\dot{\alpha} \dot{A}}, \quad X_{A\dot{A}}, \quad \bar{X}_{A\dot{A}} \]  

(2.7)

where the various \( SU(2) \) indices take on the values + or − and are raised and lowered by an antisymmetric tensor \( \epsilon \). When describing behavior that generalizes over all values for some of the indices, I will suppress those indices for ease of reading.

As with any field theory, it is useful to identify the mode expansions of the various fields, but there are two subtleties which must be discussed before proceeding. First, I must briefly discuss the deformation. Part of the deformation, which will be discussed
in more detail in the next section, involves a ‘twist field’ that joins together two of the independent copies of our CFT. The two joined copies (1) and (2) can be described by a single CFT that is wound twice around the $S^1$ direction (so $0 \leq \sigma < 4\pi$). Operators from copy (1) and their mirrors from copy (2) become the same operator, which behaves as the (1)$^{st}$ version for $0 \leq \sigma < 2\pi$ and the (2)$^{nd}$ version for $2\pi \leq \sigma < 4\pi$. Since the deformation is perturbative, we can additionally twist in a third winding, a fourth winding, and so on. It is thus necessary to construct mode expansions that also consider the CFT’s winding number $N_{(k)}$. Note that all deformations conserve total winding number.\footnote{This is important. The orbifold point has $n_1n_5$ CFT copies, which brings information on the number of string theory components in the gravity dual. This information had better not disappear from our theory.}

That is:

$$\sum_{(i)} N_{(i)} = \sum_{(j)} N_{(j)} \quad (2.8)$$

The second subtlety is that the bosonic excitation $X^{(k)}$ has the wrong conformal weight for a chiral primary. This is not uncommon in CFTs, and the solution is to work instead with the chiral primary $\partial_w X^{(k)}$ (and the right-moving $\overline{\partial_w X^{(k)}}$), where $w = \tau + i\sigma$, and $\overline{w} = \tau - i\sigma$ coordinatize the cylinder formed by $y$ and $t$. The fermion fields do not have this problem, and can be expanded directly.

Taking these subtleties into account, the left-moving mode expansions for our sector of interest\footnote{The Ramond sector. The Neveu-Schwarz sector is not used on the cylinder} are as follows:

$$\alpha^{(k)}_{AA,n} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi N_{(k)}} \partial_w X^{(i)}_{AA}(w)e^{\frac{n}{N_{(k)}}w} dw$$

$$d^{(k),AA}_{n} = \frac{1}{2\pi i \sqrt{N_{(k)}}} \int_{\sigma=0}^{2\pi N_{(k)}} \psi^{(i),AA}(w)e^{\frac{n}{N_{(k)}}w} dw. \quad (2.9)$$

The right-moving modes are analogous. The (anti)commutators are:
\[
\begin{align*}
\alpha^{(k)}_{A,A,m}, \alpha^{(k')}_{B,B,n} &= -m \epsilon_{AB} \epsilon_{AB} \delta^{(k)} \delta^{(k')} \delta_{m+n,0} \\
\{ d^{(k),\alpha A}_{m}, d^{(k'),\beta B}_{n} \} &= -\epsilon^{\alpha\beta} \epsilon_{AB} \delta^{(k)} \delta^{(k')} \delta_{m+n,0}. \quad (2.10)
\end{align*}
\]

Note that the bosonic modes are not canonically normalized.

### 2.2 Vacuum States

For each copy of the CFT, the lowest energy state of the left-moving sector is the NS vacuum,

\[
|0_{NS}\rangle, \quad h = 0, \quad m = 0, \quad (2.11)
\]

where \( h \) is the \( L_0 \) eigenvalue. However, our interest lies mostly in the R sector of the CFT. The vacua of this sector are:

\[
\begin{align*}
|0_{R}^{\pm}\rangle, & \quad h = \frac{1}{4}, \quad m = \pm \frac{1}{2} \\
|0_{R}\rangle, |\bar{0}_{R}\rangle, & \quad h = \frac{1}{4}, \quad m = 0. \quad (2.12)
\end{align*}
\]

These vacua are related through fermion zero modes. In addition, we will often have two copies of a CFT, which means that the theory’s available vacuum states are any tensor product of each copy’s vacuum states. This creates ordering (sign) ambiguities for the fermion zero modes. Appendix A provides a full accounting of the ordering conventions used in this work.

One can further relate the R and NS sectors via spectral flow \([16]\). Spectral flow by a single unit in the left-moving sector produces the transformations:

\[
\begin{align*}
\alpha = 1 : & \quad |0_{R}\rangle \rightarrow |0_{NS}\rangle, \quad |0_{NS}\rangle \rightarrow |0_{R}^{+}\rangle \\
\alpha = -1 : & \quad |0_{R}^{+}\rangle \rightarrow |0_{NS}\rangle, \quad |0_{NS}\rangle \rightarrow |0_{R}^{-}\rangle. \quad (2.13)
\end{align*}
\]
The other R vacua can flow to the NS sector by first relating them to $|0^R_\pm\rangle$ via fermion zero modes. Alternatively, one can think of this transformation as adding or removing spin fields $S^\mu$, which themselves are used to transform the NS vacuum into $|0^R_\pm\rangle$. We will use this spin field language freely in later chapters.

2.3 Deforming the CFT

A particle that falls into the fuzzball throat is described in the dual CFT by the creation of excitation pairs. At the orbifold point, these excitations propagate undisturbed until they re-collide, at which point they can annihilate and the black hole emits a quanta. Emission rates have been shown to match semi-classical calculations [7,17], but the undisturbed propagation of the excitations indicates that the energy of an absorbed quanta would eventually be radiated in precisely the same form it took when it entered. This is very much not an expected feature of Hawking radiation, and indicates a breakdown in the usefulness of the orbifold CFT. As anticipated, the orbifold point is not dual to the point in the gravity moduli space that actually matches our fuzzball geometry. An accurate picture of the evaporation process thus requires us to deform the CFT away from its orbifold point.

The deformation of the CFT dual can be described perturbatively, which allows us to work with repeated applications of a basic ‘deformation operator.’ This operator carries a supercharge that swaps fermions with bosons along with the aforementioned twist field that connects two copies of the $T^4$ in the target space [18]. Consider, for example, the simplest nontrivial case $n_1n_5 = 2$. The orbifold point is the symmetrized product of two singly-wound CFTs, so the only available action for the twist operator is to join the two copies together. At second order, the twist operator must re-split
these copies. They are then joined again at third order, split again at fourth order, etc. Each time a supercharge is also applied.

It has been shown that the deformation operator can be taken as a particular combination of a zero mode of the supercurrent and a twist field, with the supercurrent applied all around the insertion point of the twist field \[15\]. Thus we will need to know the mode expantions of the supercurrents in CFTs of general winding number.

For the left-moving sector, the supercurrents and their zero modes are given by:

\[
G_A^{(k),\alpha}(w) = \psi^{(k)\alpha A}(w)\partial_w x_A^{(k)}(w)
\]

\[
G_{A,n}^{(k),\alpha} = \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi N(k)} G^{(i),\alpha}_A(w)e^{\frac{w}{N(k)}w}dw.
\] (2.14)

The \(G\) modes can also be expressed in terms of boson and fermion modes:

\[
G_{A,n}^{(k),\alpha} = -\frac{i}{\sqrt{N(k)}} \sum_m d_m^{(k),\alpha A} G_{A,A,n-m}^{(k)}.
\] (2.15)

Similar results hold for the right-moving sector. We also define the ‘full \(G\)’ operator and modes:

\[
G_A^{\alpha}(w) = \sum_{(k)} G_A^{(k),\alpha}(w)
\]

\[
G_{A,n}^{\alpha} = \sum_{(k)} G_{A,n}^{(k),\alpha}.
\] (2.16)

The \(G\) modes have straightforward (anti)commutation with our bosonic and fermionic excitations:

\[
\left[ G_A^{\alpha}, G_B^{(k)}_{\beta B,m} \right] = -\frac{im}{\sqrt{N(j)}} \epsilon_{AB}\epsilon_{AB} d_{m+n}^{(k)\alpha A}
\]

\[
\{ G_A^{\alpha}, d_m^{(k)\beta B} \} = \frac{i}{\sqrt{N(j)}} \epsilon_{A\beta}^{\alpha} \epsilon_{AB}^{(i)\beta} \delta_{A, A,m+n}^{(j)\alpha A}.
\] (2.17)

In general, we want to apply supercurrents of the form \(G_A^{\pm}, \overline{G}_A^{\pm}\), along with the twist field \(\sigma_{2}^{\pm\pm}\) that twists two copies of the orbifold CFT together. However, there is
a proportionality between $G^{-}\sigma^{+}$ and $G^{+}\sigma^{-}$ [15]. We can thus write the deformation operator as:

$$\hat{O}_{A\bar{A}} = \oint_{w_0} G_{\bar{A}}^{-}(w)\sigma_2^{+}(w_0) \oint_{\bar{w}_0} \bar{G}_{\bar{A}}^{-}(\bar{w})\bar{\sigma}_2^{+}(\bar{w}_0)$$  \hspace{1cm} (2.18)$$

where the twist field has been split into its $SU(2)_L$ and $SU(2)_R$ portions. We will work entirely with the left-moving portion of this operator, which we will term $\hat{O}_{\bar{A}}$. The contour around $w_0$ can be split into two contours enclosing the compact coordinate $\sigma$, one above the twist in the positive direction and the other below the twist in the negative direction. This process is described in Figure 2.1. With no exponential weight, these contours become zero modes. We then have:

$$\hat{O}_{\bar{A}} = G_{\bar{A},0}^{-}\sigma_2^{+}(w_0) - \sigma_2^{+}(w_0)G_{\bar{A},0}^{-}.$$  \hspace{1cm} (2.19)$$

With this form we can typically work with the twist and the supercharge separately.

### 2.4 General Results

At this stage it is possible to demonstrate a few general results that hold through the application of any number of twist insertions. In doing so, we will use the symbol $\hat{\sigma}$ to denote an arbitrary combination of positive-charge twists $\sigma_2^{+}$. The vacuum states before and the after these twists will be tensor products of R vacua across all copies, but these products need not have the same dimension. We thus introduce the notation:

$$|\emptyset\rangle = \prod_{(i)} |0_{R}^{(i)}\rangle$$

$$|\emptyset'\rangle = \prod_{(j)} |0_{R}^{(j)}\rangle,$$  \hspace{1cm} (2.20)$$

where $|0_{R}^{(i)}\rangle$ is used to indicate an unspecified type of R vacuum.
Figure 2.1: The deformation operator involves the application of a supercurrent integrated around the twist operator. The contour may be stretched as shown, so that the deformation operator can be written in terms of a combination of applying the supercharge before and after the twist. The resulting contours that circle the cylinder form zero modes.

The most important general results cannot be proven without the use of techniques that we will not cover until Chapter 3.

The first key relation is:

$$\hat{\sigma} \vert \emptyset \rangle \equiv \vert \chi \rangle \propto \exp \left[ \gamma_{B,mn}^{(j)(j')} \left( -\alpha_{++,-m}^{(j)} \alpha_{--,-n}^{(j')} + \alpha_{+-,-m}^{(j)} \alpha_{-+,--}^{(j')} \right) \right] \times \exp \left[ \gamma_{F,mn}^{(j)(j')} \left( d_{-m}^{(j),++} \delta_{-n}^{(j'),--} - d_{-m}^{(j),--} d_{-n}^{(j'),++} \right) \right] \vert \emptyset' \rangle,$$

where the mode indices are summed over all values that do not annihilate $\vert \emptyset' \rangle$ and the copy indices are summed over all available values. The $\gamma$ coefficients will in general depend on the insertion points of the various twist operators in $\hat{\sigma}$. We have termed them Bogoliubov coefficients. The constant of proportionality can in principal be calculated, but will ignore it in favor of working with amplitude ratios. Proof of
this relation for the bosonic content was presented in [15]. We extended this proof for the fermionic content in [22]. The fermionic proof is reproduced here with more convenient notation in Appendix B.

The last two key relations deal with an initial excitation, so there are bosonic and fermionic cases.

\[
\hat{\sigma} \alpha_{A,-m}^{(i)} |0\rangle = f_{mp}^{B}(i)(j) \alpha_{A,-p}^{(j)} |\chi\rangle,
\]

\[
\hat{\sigma} d_{-m}^{(i),\alpha A} |0\rangle = f_{mp}^{F}(i)(j) d_{-p}^{(j),\alpha A} |\chi\rangle,
\]

where again the mode indices are summed over non-annihilation values and the copy indices are summed over all available values. The fermion coefficient depends on the fermion’s SU(2) R charge because the twist fields also carry that charge. We call these \( f \) coefficients transition amplitudes. Proofs for these relations follow naturally from the proofs for form of \( |\chi\rangle \) in Equation.

We will also make use of a lemma that requires the methods of Chapter 3 to demonstrate. The full proof can be found in Appendix C. Here we will simply present the relation:

\[
\oint_{w_0} G_{A}^{+}(w) \sigma_2^{+}(w_0) \, dw = 0.
\]

As presented in Figure 2.1, we can now deform the contour to produce two contours circling the compact \( y \) coordinate in opposite directions (this process is also detailed more thoroughly in Chapter 3. This then gives the relation:

\[
\oint_{w_0} G_{A}^{+}(w) \sigma_2^{+}(w_0) \, dw = G_{A,0}^{+} \sigma_2^{+}(w_0) - \sigma_2^{+}(w_0) G_{A,0}^{+}
\]

\[
= \left[ G_{A,0}^{+}, \sigma_2^{+}(w_0) \right] = 0.
\]

The full \( G_{A,0}^{+} \) mode is then seen to commute with the basic twist operator \( \sigma_2^{+} \). Since all twist conjunctions of interest are constructed from a combination of \( \sigma_2^{+} \) operators,
we find:

\[ G_{A,0}^+ \sigma = \sigma G_{A,0}^+ \]  

(2.25)

We now proceed with a series of general relations that can be proven from the results presented above, without further use of the techniques detailed in Chapter 3.

### 2.4.1 Transpose Relation for Bosons

Consider the situation in which the winding configuration of the final state is identical to that of the initial state:

\[(i) = (j) \implies N(i) = N(j),\]  

(2.26)

for all \((i), (j)\). Now consider the amplitude:

\[ \mathcal{A} \equiv \langle \emptyset' | \alpha_+^{(j)} | \alpha_-^{(i)} | \emptyset \rangle. \]  

(2.27)

Passing the initial boson through the twists yields:

\[ \mathcal{A} = \sum_{(j'), p} f_{mp}^{B,(i)(j')} \langle \emptyset' | \alpha_+^{(j)} | \alpha_-^{(j')} | \emptyset \rangle. \]  

(2.28)

Applying the commutation relations (2.10) now gives:

\[ \mathcal{A} = -nf_{mn}^{B,(i)(j)} \langle \emptyset' | \emptyset \rangle. \]  

(2.29)

The bosonic operators have no SU(2) R charge, so their behavior is in general independent of both the choice of particular R vacua as well as the charge of the twists. The first independence renders the amplitude \( \mathcal{A} \) independent of the combination of R vacua chosen for both \( |\emptyset\rangle \) and \( \langle \emptyset'| \). Since both cover the same winding configurations, it is possible to swap the two vacuum choices. We thus find:

\[ \mathcal{A} = \sum_{(j'), p} f_{mp}^{B,(i)(j')} \langle \emptyset | \alpha_+^{(j)} | \alpha_-^{(j')} | \emptyset' \rangle. \]  

(2.30)
We now make use of the independence from the twist operator’s SU(2) R charge. Ignoring this charge, the fact that the initial and final states have the same copy configurations makes the conjunction of twist operators \( \hat{\sigma} \) its own inverse.\(^5\) Furthermore, each twist operator’s inverse is its hermitian conjugate. We thus take the conjugate of both sides in Equation (2.30) to obtain:

\[
\mathcal{A}^* = \langle \emptyset' | \alpha^{(i)}_{+++,+m} \hat{\sigma}^{-1} \alpha^{(j)}_{+++,n} | \emptyset \rangle \\
= -m \left( \tilde{f}_{nm}^{B,(j)(i)} \right)^* \langle \emptyset' | \hat{\sigma} | \emptyset' \rangle,
\]

(2.31)

where here the tilde denotes the transition amplitude for the reversed process. But for bosons the reversed twist conjunction is the same as the initial set of twists, so \( \tilde{f} = f \). Combining this with Equation (2.29) gives:

\[
nf_{nm}^{B,(i)(j)} = m \left( f_{nm}^{B,(j)(i)} \right)^*.
\]

(2.32)

### 2.4.2 Supersymmetry Relations: Bogoliubov Coefficients

We will now prove a relationship between the bosonic and fermionic Bogoliubov coefficients. We start by applying \( G_{0,+}^+ \) to \( |\chi\rangle \):

\[
G_{0,+}^+ |\chi\rangle = G_{0,+}^+ \hat{\sigma} |\emptyset\rangle \\
= \hat{\sigma} G_{0,+}^+ |\emptyset\rangle \\
= 0.
\]

(2.33)

where we have used the fact that \( G_{0,A}^+ \) commutes with all \( \sigma_2^+ \) operators as well as the fact that it annihilates all R vacua.

We now make use of the general form of \( |\chi\rangle \) from Equation 2.4. This gives:

\(^5\)In general one would also need a global interchange of SU(2) R charges to invert the twists.
0 = G_{0,+}^+ |\chi\rangle \\
\propto G_{0,+}^+ \exp \left[ \gamma_{mn}^{B,(j)(j')} \left( -\alpha_{++,-m}^{(j)} \alpha_{--,n}^{(j')} + \alpha_{+,-m}^{(j)} \alpha_{--,-n}^{(j')} \right) \right] \\
\times \exp \left[ \gamma_{mn}^{F,(j)(j')} \left( d_{-m}^{(j),++} d_{-n}^{(j'),--} - d_{-m}^{(j),--} d_{-n}^{(j'),++} \right) \right] |\theta'\rangle \\
\propto \sum_{j,j',m,n \geq 0} \sum \left[ \gamma_{mn}^{B,(j)(j')} \left[ G_{0,+}^+ \alpha_{++,-m}^{(j)} \alpha_{--,-n}^{(j')} + \alpha_{+,-m}^{(j)} \alpha_{--,-n}^{(j')} \right] \\
+ \gamma_{mn}^{F,(j)(j')} \left\{ G_{0,+}^+ d_{-m}^{(j),++} d_{-n}^{(j'),--} - d_{-m}^{(j),--} d_{-n}^{(j'),++} \right\} \right] |\chi\rangle \\
\propto \sum_{(j),(j'),k,l \geq 0} \sum \left[ \gamma_{mn}^{Bn,(j)(j')} \left[ -\alpha_{++,-m}^{(j)} \left[ G_{0,+}^+ \alpha_{--,-n}^{(j')} + \left[ G_{0,+}^+ \alpha_{+,-k}^{(j')} \alpha_{--,-l}^{(j')} \right] \right] \right] \\
+ \gamma_{mn}^{F,(j)(j')} \left[ -d_{-m}^{(j),++} \left\{ G_{0,+}^+ d_{-n}^{(j'),--} + \left\{ G_{0,+}^+ d_{-m}^{(j'),++} d_{-n}^{(j'),--} \right\} \right\} \right] |\chi\rangle. \\
(2.34)

Note that in all cases \( \gamma^B \) is treated as zero when either mode index are zero. This is because boson zero modes annihilate all vacua and are thus not included in \(|\chi\rangle\).

We now apply the (anti)commutation relations from Equation 2.17. This gives:

\[ 0 \propto \sum_{(j),(j'),m,n} \left[ \gamma_{mn}^{B,(j)(j')} \left( -\alpha_{++,-m}^{(j)} \left( \frac{i n}{\sqrt{N(j')}} \right) d_{-n}^{(j),++} + \left( \frac{-i m}{\sqrt{N(j)}} \right) d_{-m}^{(j),--} \alpha_{--,-n}^{(j')} \right) \\
+ \gamma_{mn}^{F,(j)(j')} \left( -d_{-m}^{(j),++} \left( \frac{i}{\sqrt{N(j')}} \right) \alpha_{++,-m}^{(j)} + \left( \frac{-i}{\sqrt{N(j)}} \right) \alpha_{+,-m}^{(j')} \alpha_{--,-n}^{(j'),--} \right) \right] |\chi\rangle \\
\propto \sum_{(j),(j'),m,n} \left[ \left( \frac{n}{\sqrt{N(j')}} \right) \gamma_{mn}^{B,(j)(j')} \alpha_{++,-m}^{(j)} d_{-n}^{(j),++} + \left( \frac{m}{\sqrt{N(j)}} \right) \gamma_{mn}^{B,(j)(j')} \alpha_{+,-m}^{(j)} \alpha_{--,-n}^{(j'),--} \\
+ \left( \frac{1}{\sqrt{N(j')}} \right) \gamma_{mn}^{F,(j)(j')} \alpha_{+,-m}^{(j')} d_{-m}^{(j),++} + \left( \frac{1}{\sqrt{N(j)}} \right) \gamma_{mn}^{F,(j)(j')} \alpha_{--,-n}^{(j)} d_{-m}^{(j),++} \right] |\chi\rangle. \tag{2.35} \]

where we have absorbed an overall factor of \(-i\) into the proportionality.

We now make the following convenient modifications to only the first term of (2.35):

\[ m \leftrightarrow n, \quad \gamma_{mn}^B \rightarrow \gamma_{mn}^B, (j) \leftrightarrow (j'). \tag{2.36} \]

22
Here we use the fact that the bosonic coefficient is symmetric since bosonic creation operators commute. We now have

\[ 0 \propto \sum_{(j),(j') \, m,n \geq 0} \left[ \alpha_{j,j'}^{(j')} - n d_{j,m}^{(j')} + \left( \frac{m \gamma_{mn}^{B,(j)(j')}}{\sqrt{N(j)}} + \frac{\gamma_{mn}^{F,(j)(j')}}{\sqrt{N(j')}} \right) \right] |\chi\rangle. \]  

(2.37)

We have now expressed the relation as a sum over orthogonal creation operators. This means that each term in the sum must separately vanish. For \( m,n > 0 \), this gives the relations:

\[ \gamma_{mn}^{B,(j)(j')} = \gamma_{mn}^{B,(j')(j)} \]
\[ \gamma_{mn}^{F,(j)(j')} = -\frac{m \gamma_{mn}^{B,(j')(i)}}{\sqrt{N(j')}}. \]  

(2.38)

When \( n = 0 \), the bosonic modes annihilate \(|\chi\rangle\) so the relationship is trivial. When \( m = 0, n \neq 0 \) we obtain:

\[ \gamma_{0,n}^{F,(j')(j')} = 0, \quad n > 0. \]  

(2.39)

### 2.4.3 Supersymmetry Relations: Transition Amplitudes

Here we find relations between the bosonic and fermionic transition amplitudes for arbitrary twisting. First we note that again that \( G_{A,0}^+ \) annihilates all R vacua:

\[ G_{A,0}^+ |\chi\rangle = G_{A,0}^+ \hat{\sigma} |\theta\rangle = \hat{\sigma} G_{A,0}^+ |\theta\rangle = 0. \]  

(2.40)

We will now apply these relations to the two possible SU(2) R charges for fermion modes.
Fermions with negative SU(2) R charge

Let us here consider the state with an initial negative-charge fermion mode and act on it with a $G^+$ zero mode. Since this zero mode commutes with all twists, we have for instance:

$$G^+_{+,0} \hat{\sigma} d^{(i),--}_{-m} |\emptyset\rangle = \hat{\sigma} G^+_{-,0} d^{(i),--}_{-m} |\emptyset\rangle. \quad (2.41)$$

For $m = 0$ both sides vanish. Setting aside this trivial case, we proceed with $(2.41)$ in two ways. Beginning with the left-hand side:

$$G^+_{+,0} \hat{\sigma} d^{(i),--}_{-m} |\emptyset\rangle = \sum_{p,(k)} G^+_{+,0} f^F_{mp} d^{(j),--}_{-p} |\chi\rangle$$

$$= \sum_{p,(j)} f^F_{mp} \left\{ G^+_{+,0}, d^{(j),--}_{-p} \right\} |\chi\rangle$$

$$= \sum_{p,(j)} \frac{-i}{\sqrt{N(j)}} f^F_{mp} \left(-1\right) \epsilon^{+,-},\epsilon^{+,-}, \alpha^{(i)}_{++,\ldots, p} |\chi\rangle$$

$$= \sum_{p,(j)} \frac{-i}{\sqrt{N(j)}} f^F_{mp} \alpha^{(j)}_{++,\ldots, p} |\chi\rangle. \quad (2.42)$$

We now turn to the right-hand side of $(2.41)$:

$$\sigma^+ G^+_{-,0} d^{(i),++}_{-m} |\emptyset\rangle = \hat{\sigma} \left\{ G^+_{-,0}, d^{(i),++}_{-m} \right\} |\emptyset\rangle$$

$$= -\frac{i}{\sqrt{N(i)}} \sigma^+ \left(-1\right) \epsilon^{+,-},\epsilon^{+,-}, \alpha^{(i)}_{++,\ldots, m} |\emptyset\rangle$$

$$= -\frac{i}{\sqrt{N(i)}} \hat{\sigma} \alpha^{(i)}_{++,\ldots, m} |\emptyset\rangle$$

$$= -\frac{i}{\sqrt{N(i)}} \sum_{p,(j)} f^B_{mp} \alpha^{(j)}_{++,\ldots, p} |\chi\rangle. \quad (2.43)$$

Combining this with $(2.42)$, we find that for any arbitrary twisting:

$$\sqrt{N(i)} f^B_{mp}^{(i)(j)} = \sqrt{N(j)} f^F_{mp}^{-(i)(j)}, \quad n, p > 0. \quad (2.44)$$
Fermion with positive SU(2) R charge

We now consider a state with an initial boson mode and act on it with a $G^+_0$ mode. We have for instance:

$$G^+_{+,0} \hat{\sigma} \alpha^{(i)}_{-,m} |\emptyset\rangle = \hat{\sigma} G^+_{-,0} \alpha^{(i)}_{-,m} |\emptyset\rangle. \quad (2.45)$$

Again we set aside the trivial $m = 0$ case and proceed with (2.45) in the same two ways. For the left-hand side:

$$G^+_{+,0} \hat{\sigma} \alpha^{(i)}_{-,m} |\emptyset\rangle = \sum_{p,j} G^+_{+,0} f_{mp}^{B,(i)(j)} \alpha^{(j)}_{-,p} |\chi\rangle$$

$$= \sum_{p,j} f_{mp}^{B,(i)(j)} \left[ G^+_{+,0}, \alpha^{(j)}_{-,p} \right] |\chi\rangle$$

$$= \sum_{p,j} \frac{-i}{\sqrt{N(j)}} f_{mp}^{B,(i)(j)} (-p) \epsilon_{-,p}^{--} \alpha^{(j)}_{-,p} |\chi\rangle$$

$$= -i \sum_{p,j} \frac{p}{\sqrt{N(j)}} f_{mp}^{B,(i)(j)} d^{(j),++}_{-p} |\chi\rangle. \quad (2.46)$$

For the right-hand side of (2.45):

$$\hat{\sigma} G^+_{+,0} \alpha^{(i)}_{-,m} |\emptyset\rangle = \hat{\sigma} \left[ G^+_{+,0}, \alpha^{(i)}_{-,m} \right] |\emptyset\rangle$$

$$= \frac{-i}{\sqrt{N(i)}} \hat{\sigma} (-n) \epsilon_{-,n}^{--} \alpha^{(i)}_{-,m} |\emptyset\rangle$$

$$= -i \frac{m}{\sqrt{N(i)}} \hat{\sigma} d^{(i),++}_{-m} |\emptyset\rangle$$

$$= -i \frac{m}{\sqrt{N(i)}} \sum_{p,j} f_{mp}^{F,(i)(j)} d^{(j),++}_{-m} |\chi\rangle. \quad (2.47)$$

Combining this with (2.46), we find that for any arbitrary twisting:

$$p \sqrt{N(i)} f_{mp}^{F,(i)(j)} = m \sqrt{N(j)} f_{mp}^{B,(i)(j)}, \quad n, p > 0. \quad (2.48)$$
Chapter 3: First Order Deformations

Earlier work in [15, 19] examined the simplest case of first-order deformations, where the initial CFT contains two singly-wound copies. The deformation joins these two copies into a single doubly-wound CFT. The methods developed for this simplest case were extended by this author and others in [20, 21]. The extended results, presented here, allows the two initial CFTs to have arbitrary winding numbers. This is akin to applying a deformation operator after many such operators have already twisted together several copies of the orbifold CFT.

3.1 The ACM Method

At first order, the treatment of the supercurrent is trivial — it simply applies a superposition of a zero mode before and after the twist. It is thus the twist insertion that deserves our attention. The methods used to handle such insertions were first developed by Avery, Chowdhury, and Mathur in [15, 19] for the simplest case of two singly-wound CFT copies joining into one doubley-wound copy. Fortunately this method generalizes to some of the more complicated cases.\(^6\)

\(^6\)Caveat: With larger numbers of twist insertions, the method requires inverting polynomials of larger order. This quickly becomes analytically impossible, though numerical methods can still be used. Furthermore, the separation of the supercurrent from the twists is nontrivial at higher orders.
The deformation is applied at the coordinate \( w_0 = \tau_0 + i\sigma_0 \). With the initial CFT copies having winding \( N(1) \) and \( N(2) \), the twist portion of the deformation will result in a final CFT will have winding \( N(1) + N(2) \). Since there is only a single out state CFT, we will not use any copy index when working with it.

Unfortunately, in the course of the calculation we quickly run into expressions where winding numbers are raised to the power of other winding numbers. It is thus handy to introduce notation that does not make use of subscripts.

\[
M \equiv N(1) \\
N \equiv N(2)
\]  

(3.1)

The out state CFT then has winding \( M + N \). The deformation is depicted using this notation in Figure 3.1. Note that these multi-wound in and out states imply the use of higher order twist operators in their construction.\(^7\) This means we have several twists to deal with: An order \( M \) twist from initial copy 1, and order \( N \) twist from initial copy 2, an order \( M + N \) twist from the final copy, and the twist of interest at \( w_0 \). Each of these regions also has an overall SU(2) R charge, and thus a spin field. On top of this, there is the supercurrent. The first step is to separate the supercurrent from the twist. This can be expressed in terms of zero modes before and after the twist. Thus we will devote most of our attention to handling the twist insertions and then apply the supercharge zero modes at the end of the computation.

To deal with the twists, we will map to a covering space where all fields are single valued. We will then use spectral flow to remove all spin insertions in this plane. The result is an NS covering plane with no punctures or spin insertions. Any modes from

\(^7\) Higher-order twist operators can be built from repeated application of the \( \sigma_2 \) operators, so they are nothing special.
Figure 3.1: The cylinder parameterized by $w = \tau + i\sigma$. The state before the twist has two CFT copies with windings $M, N$. The twist operator $\sigma_2$ links these into a single component string of winding $M + N$.

Our in or out states will map to contours in this plane, which we can then deform freely. This allows us to contract in and out modes together in useful ways.

3.1.1 Emptying $t$ Plane

We begin the process of finding a good covering space by first mapping the cylinder $w$ to the complex plane via:

$$z = e^w. \quad (3.2)$$

In the present case, The twist operators $\sigma_M, \sigma_N$ from the in state both map to the origin of the $z$ plane ($\tau \to -\infty$). The interaction twist operator $\sigma_2(w_0)$, maps to the point $z_0 = e^{(w_0)}$. The out state twist $\sigma_{M+N}$ maps to $t = \infty$. Each of these regions also bring a spin field with the appropriate SU(2) R charge: $S^+$ for the twist of interest and for the negative R vacuum from the out state, and $S^-$ for each of the negative R vacuua from the in state. See Figure 3.2 for a visual representation.
Figure 3.2: The covering plane with coordinate $t$. The bifurcation points of the map are shown. There are spin fields $S^-$ at $t = 0, a$ and $S^+$ at $t = \frac{aM}{M+N}, \infty$.

We now map the $z$ plane to a covering space $t$ where our fields are single valued everywhere. A simple map that achieves this single-valued nature is:

$$z = t^M(t-a)^N.$$  \hspace{1cm} (3.3)

Here the initial copy 1 CFT maps to the neighborhood of $t = 0$, the initial copy 2 CFT maps to the neighborhood of $t = a$, and the final CFT maps to $t \to \infty$. The powers of the leading-order coordinate in these neighborhoods removes the multivalued nature of the corresponding CFTs. However, the spin fields still remain (see Figure 3.2)

Boson modes ignore the problematic spin fields since they have no SU(2) R charge, but in order to freely deform our fermion modes we will need to remove them. This is done using spectral flow at the locations of the spin fields, which of course requires us to identify those locations. This requires us to pin down the coordinate $a$ in addition to locating the image of the twist interaction at $w_0$. In general twists map to bifurcation points in the $t$ plane, so we need to identify the last bifurcation point of the map 3.3.
In addition to $t = 0, a, \infty$, the derivative of the 3.3 vanishes at:

$$t = \frac{aM}{M + N}. \quad (3.4)$$

This corresponds to:

$$z_0 = a^{M+N} \frac{M^M N^N}{(M + N)^{M+N}} (-1)^N \quad (3.5)$$

Solving for $a$ now requires us to specify how to deal with the fractional exponent. We do this by choosing

$$(-1)^N = e^{i\pi N} \quad (3.6)$$

which determines the quantity $a$ in terms of $z_0 = e^{w_0}$ as

$$a = e^{-i\pi \frac{N}{M+N}} \left( \frac{z_0}{M^M N^N} \right)^{\frac{1}{M+N}} (M + N). \quad (3.7)$$

This gives the image of the in state’s copy 2 CFT. Plugging the expression into Equation 3.4 also gives the image of $w_0$.

Now that we know the location of all the spin fields, we can remove them via spectral flow. This results in an empty $t$ plane in the NS vacuum. Any in or out state modes will map to contours in this plane, and we can now freely deform those contours however we wish. In particular, we will find it useful to expand these contours in terms of modes natural to the NS sector at various points in the $t$ plane. These natural modes are:

$$\tilde{a}_{A\dot{\alpha},n}^{t\to t_0} = \frac{1}{2\pi} \int_{t=t_0}^{\infty} \partial_t X_{A\dot{\alpha}}(t) t^n \, dt$$

$$\tilde{d}^{A\dot{\alpha},t\to t_0} = \frac{1}{2\pi i} \int_{t=t_0}^{\infty} \psi^{A\dot{\alpha}}(t) t^{r-\frac{3}{2}} \, dt. \quad (3.8)$$
The point \( t_0 \) is the image point of the region of interest. Note that the fermion index is a half-integer, since we are in the NS sector. The (anti)commutation relations for these natural modes are only straightforward when the modes are natural to the same neighborhoods. In this case one has:

\[
\begin{align*}
[\tilde{\alpha}_{A \dot{A},m}^{t \to t_0}, \tilde{\alpha}_{A \dot{A},n}^{t \to t_0}] &= -\epsilon_{AB}\epsilon_{\dot{A}B}m\delta_{n+m,0} \\
\{ \tilde{d}_{q}^{A,t \to t_0}, \tilde{d}_{r}^{A,t \to t_0} \} &= -\epsilon^{\alpha\beta}\epsilon^{AB}\delta_{q+r,0}.
\end{align*}
\] (3.9)

We also have the natural behavior:

\[
\tilde{\alpha}_{A \dot{A},n}^{t \to t_0}|0_{NS}\rangle_t = 0, \quad n \geq 0
\]

\[
\tilde{d}_{q}^{A,t \to t_0}|0_{NS}\rangle_t = 0, \quad q > 0.
\] (3.10)

### 3.1.2 Relations for the Coefficients of Interest

We will now identify expressions for the Bologjubov coefficients and transition amplitudes in terms of amplitudes in the \( t \) plane. Since contours can be deformed freely in this pane, all of our modes can be directly contracted, yielding analytic expressions for the coefficients of interest. Since these relations hold for any general twist conjunction, we will use the generalized notation from Chapter 2.

**The Bogoliubov Coefficients**

Consider first the amplitude:

\[
A_1 = \langle \emptyset' | \alpha_{++,-}^{(j)} \alpha_{--,-}^{(j')} | \tilde{\sigma} | \emptyset \rangle \\
= \langle \emptyset' | \alpha_{++,-}^{(j)} \alpha_{--,-}^{(j')} | \chi \rangle,
\] (3.11)

for \( m, n > 0 \). Now we now the form of \( |\chi\rangle \). In particular, it carries pairs of bosonic excitations that do not annihilate \( |\emptyset'\rangle \). But this means its excitations do annihilate
\[ \langle 0'|. \] Similarly, the modes explicitly inserted into \( A_1 \) annihilate \( |0'\rangle \). It is hence clear that only the first-order term from \( |\chi\rangle \) can provide a nonzero contribution to \( A_1 \). This gives:

\[
A_1 = \langle 0'|a^{(j)}_{++n}a^{(j')}_{--m}\gamma_{jk}^{B,\hat{\sigma}}a^{(\hat{\sigma})}_{++j}a^{(\hat{\sigma})}_{--k}|0 \rangle \\
= -mn\gamma_{mn}^{B,(j)(j')}\langle 0'|0'\rangle \\
= -mn\gamma_{mn}^{B,(j)(j')}\langle 0'|\hat{\sigma}|0'\rangle, \tag{3.12}
\]

where the last line uses the fact that only the zeroth order term contributes to \( \langle 0'|\chi \rangle \).

Denoting the remaining vacuum amplitude as \( A_0 \), we have:

\[
\gamma_{mn}^{B,(j)(j')} = -\frac{1}{mn}A_1/A_2. \tag{3.13}
\]

We now map both the relevant amplitudes to the \( t \) plane (using a \( t \) plane appropriate to the twist conjunction \( \hat{\sigma} \)). Here the twist fields are removed, while the states \( |\chi\rangle \) and \( \langle 0'| \) map to the NS vacuum in and out states, respectively. This gives:

\[
A_1 = t\langle 0_{NS}|a^{(j)}_{++n}a^{(j')}_{--m}|0_{NS}\rangle_t \\
A_0 = t\langle 0_{NS}|0_t\rangle, \tag{3.14}
\]

where the primes on the boson modes are used to indicate that they have undergone the effects of the coordinate maps and spectral flows used to empty the covering plane.\(^8\) The colons are used to indicate that the mode whose contour is on the interior should be written to the right (it acts on the in state).

We now re-express Equation 3.13 as:

\[
\gamma_{mn}^{B,(j)(j')} = -\frac{1}{mn}t\langle 0_{NS}|a^{(j)}_{++n}a^{(j')}_{--m}|0_{NS}\rangle_t/t\langle 0_{NS}|0_t\rangle. \tag{3.15}
\]

\(^8\) The spectral flows have no effect on the boson modes, but they will be important when we look at fermions.
We can now express the transformed boson modes as a linear combination of the natural modes from Equation 3.8. We then use the known commutation relations of those natural modes to obtain an analytic expression for the Bogoliubov coefficient.

The fermion coefficient follows the same procedure. The resulting expression is:

$$\gamma_{mn}^{F,(j)(j')} = \frac{t\langle 0_{NS}| :d_n^{(j),++}d_m^{(j'),--}:|0_{NS}\rangle_t}{t\langle 0_{NS}|0\rangle_t}. \quad (3.16)$$

We now return to the present case of interest:

$$\hat{\sigma} \rightarrow \sigma_2^+(w_0)$$

$$|\chi\rangle \rightarrow |\chi(w_0)\rangle$$

$$|\emptyset\rangle \rightarrow |0_{-R}^{(1)}\rangle|0_{-R}^{(2)}\rangle$$

$$\langle \emptyset'| \rightarrow \langle 0_{-R-}\rangle. \quad (3.17)$$

This gives:

$$\gamma_{mn}^B = -\frac{1}{mn} \frac{t\langle 0_{NS}|\alpha'_{++,-n}\alpha'_{--,-m}|0_{NS}\rangle_t}{t\langle 0_{NS}|0\rangle_t}$$

$$\gamma_{mn}^F = \frac{t\langle 0_{NS}|d_n^{++}d_m^{--}|0_{NS}\rangle_t}{t\langle 0_{NS}|0\rangle_t}. \quad (3.18)$$

Here the mode ordering is the same as that of the cylinder as there is only one out state CFT.

The Transition Amplitudes

We now apply an analogous setup to obtain expression for the transition amplitudes \(f\). Begin with the amplitude:

$$\mathcal{A}_2 = \langle \emptyset'|\alpha_{--,-n}^{(j)}\hat{\sigma}\alpha_{++,-m}^{(i)}|\emptyset\rangle$$

$$= f_{mp}^{B,(i)(j')}(\emptyset'|\alpha_{--,-p}^{(j)}\alpha_{++,-p'}^{(i')}\chi)$$

$$= -pf_{mp}^{B,(i)(j)}\mathcal{A}_0. \quad (3.19)$$
Here only the zeroeth order of $|\chi\rangle$ can contribute to the amplitude. We again map into the emptied $t$ to obtain the expression:

$$f_{mp}^{B,(i)(j)} = -\frac{1}{p} \frac{t\langle 0_{NS}\mid \alpha^{(j)}_{-p}\alpha^{(i)}_{++},m \rangle \mid 0_{NS}\rangle_t}{t\langle 0_{NS}\mid 0\rangle_t}.$$  \hspace{1cm} (3.20)

The process for fermions is identical, except that we cannot expect different initial fermion SU(2) R charges to yield the same coefficients. We thus have:

$$f_{mp}^{F+, (i)(j)} = -\frac{t\langle 0_{NS}\mid d_{p}^{(j),-},d_{m}^{(i),++} \rangle \mid 0_{NS}\rangle_t}{t\langle 0_{NS}\mid 0\rangle_t}$$

$$f_{mp}^{F-, (i)(j)} = -\frac{t\langle 0_{NS}\mid d_{p}^{(j),++},d_{m}^{(i),-} \rangle \mid 0_{NS}\rangle_t}{t\langle 0_{NS}\mid 0\rangle_t}.$$  \hspace{1cm} (3.21)

In terms of the present case, these are:

$$f_{mp}^{B,(i)} = -\frac{1}{p} \frac{t\langle 0_{NS}\mid \alpha^{(i)}_{-p}\alpha^{(i)}_{++},m \rangle \mid 0_{NS}\rangle_t}{t\langle 0_{NS}\mid 0\rangle_t}$$

$$f_{mp}^{F+, (i)} = -\frac{t\langle 0_{NS}\mid d_{p}^{(i),-},d_{m}^{(i),++} \rangle \mid 0_{NS}\rangle_t}{t\langle 0_{NS}\mid 0\rangle_t}$$

$$f_{mp}^{F-, (i)} = -\frac{t\langle 0_{NS}\mid d_{p}^{(i),++},d_{m}^{(i),-} \rangle \mid 0_{NS}\rangle_t}{t\langle 0_{NS}\mid 0\rangle_t},$$  \hspace{1cm} (3.22)

where the fact that the out modes always map to large $t$ has fixed our ordering.

When performing contractions in the $t$ plane, we will need to first deform the contours such that they are living in the same neighborhood. This allows us to express them in terms of modes natural to the same point in the $t$ plane, which lets us use the simple (anti)commutators from Equation 3.9. When doing this, we will add a superscript $t \rightarrow t_0$ to our deformed modes, to indicate which neighborhoods we are working in.

### 3.1.3 Mapping the Modes to the Empty $t$ Plane

There are two steps involved in obtaining our desired empty cover: Mapping to the cover and removing the spin fields via spectral flow. Bosons are affected only by...
the first step, while fermions are affected by both steps. We will handle each step in turn.

The Coordinate Maps

Under a general coordinate transformation, the fields $\psi$ and $\partial X$ transform according to their conformal weight:

$$\partial_{w}X^{(k)}_{AA}(w) \rightarrow \frac{dw'}{dw}\partial_{w'}X^{(k)}_{AA}(w')$$

$$\psi^{(k),\alpha A}(w) \rightarrow \left(\frac{dw'}{dw}\right)^{\frac{1}{2}}\psi^{(k),\alpha A}(w').$$ \hspace{1cm} (3.23)

Since the modes are integrals, there is also a jacobian factor:

$$dw \rightarrow \left(\frac{dw'}{dw}\right)^{-1} dw.$$ \hspace{1cm} (3.24)

The jacobian conveniently cancels the transformation of the bosonic field. For the fermionic field, the net result is:

$$\psi^{(i),\alpha A}(w) dw \rightarrow \left(\frac{dw'}{dw}\right)^{-\frac{1}{2}} \psi^{(i),\alpha A}(w') dw'.$$ \hspace{1cm} (3.25)

For the map $z = e^{w}$, we have:

$$\frac{dz}{dw} = z.$$ \hspace{1cm} (3.26)

Thus the fermion field and jacobian combination pick up a factor of $z^{-\frac{1}{2}}$. Our cylinder modes from Equation 2.9 then become:

$$\alpha^{(k)}_{AA,n} \rightarrow \frac{1}{2\pi i} \int_{\text{arg}(z)=0}^{2\pi N(k)} \partial_{w}X^{(i)}_{AA}(w) z^{\frac{n}{N(k)}} dw$$

$$d^{(k),\alpha A}_{n} \rightarrow \frac{1}{2\pi i} \sqrt{N(k)} \int_{\text{arg}(z)=0}^{2\pi N(k)} \psi^{(i),\alpha A}(w) z^{\frac{n}{N(k)}} z^{-\frac{1}{2}} dw.$$ \hspace{1cm} (3.27)

Here the contour circles the relevant image point: $z = 0$ for either in state CFT copy and $z = \infty$ for the out state CFT.
We must now map these modes into the $t$ plane. Here the removal of the multi-valued nature of the fields also serves to reduce all of our integrals to a single contour around the relevant image point. Looking at Equation 3.3, we find the derivative of the map:

$$
\frac{dt}{dz} = \left( \frac{dz}{dt} \right)^{-1} = \left[ (M + N) \frac{z}{t(t-a)} \left( t - \frac{Ma}{M+N} \right) \right]^{-1}. \quad (3.28)
$$

The single power of $z$ is intentionally left in. Since the fermion field and jacobian combination gain a factor of the above derivative raised to the negative one-half power, the left-in $z$ factor well bring a power of $z^{1/2}$. This exactly cancels the analogous factor from the first coordinate map.

We can now write our cylinder modes in terms of a $t$ plane that still has spin fields.

$$
\alpha^{(k)}_{\alpha A,m} \rightarrow \frac{1}{2\pi} \oint_{t=t_0} \partial_t X_{\alpha A}(t) \left( t^N (t-a)^M \right)^{\frac{m}{N(k)}} dt
$$

$$
d^{(k),\alpha A}_m \rightarrow \frac{1}{2\pi i \sqrt{N(k)}} \oint_{t=t_0} \psi^{\alpha A}(t) \left( t^M (t-a)^N \right)^{\frac{m}{N(k)}}
$$

$$
\times \left[ (M + N) \frac{1}{t(t-a)} \left( t - \frac{aM}{M+N} \right) \right]^{1/2} dt. \quad (3.29)
$$

The Spectral Flows

Looking back at Figure 3.2, we see four spin fields. We will eliminate these spin fields via a series of spectral flows at finite points. Such a spectral flow applies/removes a spin field at that point in addition to affecting the point at infinity in the opposite direction. We thus apply the following three spectral flows:

$$
\alpha = +1 \text{ at } t = 0, t = a, \quad \alpha = -1 \text{ at } t = \frac{aM}{M+N}. \quad (3.30)
$$
The first spectral flow removes both the $S^-$ at $t = 0$ and the $S^+$ at infinity. The second spectral flow removes the $S^-$ at $t = a$ while inserting an $S^-$ at infinity. The final spectral flow then removes both the $S^+$ at the image of $w_0$ and the new $S^-$ at infinity. The $t$ plane is then free of spin fields.

Spectral flow also has an affect on any operators with an SU(2) R charge. This effect is straightforward for operators whose fermion content can be expressed as a simple exponential in the language in which the fermions are bosonized. For such operators with charge $j$, spectral flow by $\alpha$ units produces the transformation:

$$\hat{O}_j(t) \rightarrow t^{-\alpha j} \hat{O}_j(t).$$

(3.31)

The fermion fields and spin fields are of this form with $j = \pm \frac{1}{2}$. However, since we work only with ratios of amplitudes we will not need to take account of the transformation of the spin fields themselves.

From our three spectral flows, we have the following affects on the fields $\psi^\pm$.

$$\psi^{+A}(t) \rightarrow \psi^{+A}(t) \left[ t^{-\frac{1}{2}}(t-a)^{-\frac{1}{2}} \left( t - \frac{a M}{M + N} \right)^{\frac{1}{2}} \right]$$

$$\psi^{-A}(t) \rightarrow \psi^{-A}(t) \left[ t^{\frac{1}{2}}(t-a)^{\frac{1}{2}} \left( t - \frac{a M}{M + N} \right)^{-\frac{1}{2}} \right].$$

(3.32)

Applying these transformations to Equation 3.29, we find:

$$d_{m}^{(k),+A} = \frac{\sqrt{M + N}}{2\pi i \sqrt{N(k)}} \oint_{t=t_0} \psi^{+A}(t) \left( t^M (t - a)^N \right)^{\frac{m}{N(k)}}$$

$$\times t^{-1}(t-a)^{-1} \left( t - \frac{a M}{M + N} \right) \, dt$$

$$d_{m}^{(k),-A} = \frac{\sqrt{M + N}}{2\pi i \sqrt{N(k)}} \oint_{t=t_0} \psi^{-A}(t) \left( t^M (t - a)^N \right)^{\frac{m}{N(k)}} \, dt.$$

(3.33)

The bosons are unaffected by spectral flow.

$$\alpha_{AA,m}^{(k)} = \frac{1}{2\pi} \oint_{t=t_0} \partial_t X_{AA}(t) \left( t^M (t - a)^N \right)^{\frac{m}{N(k)}} \, dt.$$
3.1.4 Computing the Bogoliubov Coefficients

The Bogoliubov coefficients are given by the relations of Equation 3.18. These relations use only the deformed out state modes. We thus work with \( N_{(k)} = M + N \).

In this case, our modes in the \( t \) plane are:

\[
\alpha'_{A,\overline{A},m} = \frac{1}{2\pi} \oint_{t=\infty} \partial_t X_{A\overline{A}}(t) \left( t^M (t-a)^N \right)^{\frac{m}{M+N}} dt.
\]

\[
d'^A_m = \frac{1}{2\pi i} \oint_{t=\infty} \psi^+(t) \left( t^M (t-a)^N \right)^{\frac{m}{M+N}} \times t^{-1} (t-a)^{-1} \left( t - \frac{aM}{M+N} \right) dt
\]

\[
d'^{-A}_m = \frac{1}{2\pi i} \oint_{t=\infty} \psi^{-A}(t) \left( t^M (t-a)^N \right)^{\frac{m}{M+N}} dt.
\]

We must now express these modes in terms of modes natural to the \( t \) plane at \( t_0 = \infty \).

These modes are integrands weighted by powers of \( t \), so we expand the integrands in Equation 3.35 in powers of \( t \) at large \( t \). These powers of \( t \) then directly translate into mode indices, as per Equation 3.8 The result is:

\[
\alpha'_{++,n} = \sum_{p \geq 0} \frac{nN}{M+N} C_p (-a)^p \tilde{\alpha}_{A,\overline{A},n-p}
\]

\[
\alpha'_{--,m} = \sum_{q \geq 0} \frac{mN}{M+N} C_q (-a)^q \tilde{\alpha}_{A,\overline{A},m-q}
\]

\[
d'^{++}_n = \sum_{p \geq 0} \frac{nM}{M+N} \frac{1}{\frac{nM}{M+N}} C_p \left( \frac{N}{M+N} a^{p+1} \tilde{d}_{n-p}^{++} + a^p \tilde{d}_{n-p}^{-+} \right)
\]

\[
d'^{--}_m = \sum_{q \geq 0} \frac{mM}{M+N} \frac{1}{\frac{mM}{M+N}} C_q a^q \tilde{d}_{m-q}^{--}.
\]

Plugging this result into Equation 3.18, we find:

\[
\gamma^B_{mn} = -\frac{1}{mn} \sum_{p \geq 0} \sum_{q \geq 0} \frac{nN}{M+N} C_p \frac{nN}{M+N} C_q (-a)^{p+q} \frac{t\langle 0|\tilde{\alpha}_{++,l-p} \tilde{\alpha}_{--,k-q}|0 \rangle t}{t\langle 0|0 \rangle t}
\]

\[
\gamma^F_{mn} = \sum_{p,q \geq 0} \left[ \frac{N}{M+N} a^{p+q} \frac{mN}{M+N} C_q \frac{nM}{M+N} \frac{1}{\frac{nM}{M+N}} C_p \langle 0_N S|\tilde{d}_{n-p}^{++} \tilde{d}_{m-q}^{--} |0_N S \rangle t \right] / t \langle 0_N S|0 \rangle t
\]

38
Let us start with the bosons. Using the commutation relations from Equation 3.9, we have

\[ k - q = -(l - p) \Rightarrow q = (k + l) - p. \]  

(3.38)

Furthermore, \( \tilde{\alpha}_{++} \) needs to be an annihilation operator, and thus \( p \leq n \). This gives

\[ \gamma_{mn}^B = \frac{(-a)^{m+n}}{mn} \sum_{p=0}^{n} \frac{nN}{M+N} C_p \frac{mN}{M+N} C_{(m+n)-p} (n - p). \]  

(3.39)

Evaluating this sum gives:

\[ \gamma_{mn}^B = -\frac{(-a)^{m+n}}{\pi^2} \sin \left( \frac{mM \pi}{M + N} \right) \sin \left( \frac{nM \pi}{M + N} \right) \frac{MN}{(M + N)^2} \times \frac{1}{(m+n)} \frac{\Gamma \left( \frac{mM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right)}{\Gamma \left( \frac{nM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right)} \frac{\Gamma \left( \frac{m}{M+N} \right) \Gamma \left( \frac{n}{M+N} \right)}{\Gamma \left( \frac{nM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right)}. \]  

(3.40)

Turning to the fermions, the anticommutation relations from Equation 3.9 give for the first term:

\[ n - p - \frac{1}{2} = - \left( m - q + \frac{1}{2} \right) \Rightarrow q = m + n - p. \]  

(3.41)

while for the second term we similarly find:

\[ q = m + n - p - 1. \]  

(3.42)

We then have:

\[ \gamma_{mn}^F = -a^{m+n} \left[ \sum_{p=0}^{n-1} \frac{nM}{M+N}^{-1} C_p \frac{nM}{M+N} C_{m+n-p} + \frac{N}{M+N} \sum_{p=0}^{n-2} \frac{nM}{M+N}^{-1} C_p \frac{nM}{M+N} C_{m+n-p-1} \right]. \]  

(3.43)

Evaluating this sum gives:

\[ \gamma_{mn}^F = \frac{a^{m+n}}{\pi^2} \sin \left( \frac{mM \pi}{M + N} \right) \sin \left( \frac{nM \pi}{M + N} \right) \frac{MN}{(M + N)^2} \times \frac{m}{m+n} \frac{\Gamma \left( \frac{mM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right) \Gamma \left( \frac{nM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right) \Gamma \left( \frac{m}{M+N} \right) \Gamma \left( \frac{n}{M+N} \right)}{\Gamma \left( \frac{nM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right) \Gamma \left( \frac{nM}{M+N} \right) \Gamma \left( \frac{nN}{M+N} \right) \Gamma \left( \frac{m}{M+N} \right) \Gamma \left( \frac{n}{M+N} \right)}. \]  

(3.44)
3.1.5 Computing the Transition Amplitudes

The transition amplitudes are given by the relations of Equation 3.22. These relations use both the in and out state modes. In this case, our in state modes map to the \( t \) plane as:

\[
\alpha'_{A,m}(1) = \frac{1}{2\pi} \oint_{t=0} dt \psi(t) X_A(t) \left( t^M (t-a)^N \right)^{\frac{m}{M}}
\]

\[
\alpha'_{A,n}(2) = \frac{1}{2\pi} \oint_{t=a} dt \psi(t) X_A(t) \left( t^M (t-a)^N \right)^{\frac{n}{N}}
\]

\[
d_{m}^{(1),+A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{M}} \oint_{t=0} dt \psi^{(1),+A}(t) (t-a)^{\frac{mN}{M}-1} \left( t - \frac{aM}{M+N} \right) dt
\]

\[
d_{m}^{(1),-A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{M}} \oint_{t=0} dt \psi^{(1),-A}(t) (t-a)^{\frac{mN}{N}-1} \left( t - \frac{aN}{M+N} \right) dt
\]

\[
d_{n}^{(2),+A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{t=a} dt \psi^{(2),+A}(t+a) (t-a)^{\frac{nM}{M}-1} \left( t + \frac{aN}{M+N} \right) dt
\]

\[
d_{n}^{(2),-A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{t=a} dt \psi^{(1),-A}(t+a) (t-a)^{\frac{nN}{M}} dt.
\]  

(3.45)

We now apply a simple coordinate shift to the the copy 2 modes:

\[
t \leftrightarrow t + a.
\]  

(3.46)

Under this transformation, we have:

\[
\alpha'_{A,n}(2) = \frac{1}{2\pi} \oint_{t=0} dt \psi(t) X_A(t) \left( (t+a)^M t^N \right)^{\frac{n}{N}}
\]

\[
d_{n}^{(2),+A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{t=0} dt \psi^{(2),+A}(t+a) (t-a)^{\frac{nM}{M}-1} \left( t + \frac{aN}{M+N} \right) dt
\]

\[
d_{n}^{(2),-A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{t=a} dt \psi^{(1),+A}(t+a) (t+a)^{\frac{nN}{M}} dt.
\]  

(3.47)

Now these modes differ from the copy 1 modes by the transformations:

\[
M \leftrightarrow N, \quad a \rightarrow -a.
\]  

(3.48)
However, looking at Equation 3.36 we find that the mode expansions of the out state excitations are symmetric under the conjunction of these two transformations. This means the above transformations map the copy 1 transition amplitudes into the copy 2 transition amplitudes. We with thus calculate only the copy 1 transition amplitudes explicitly.

Now the out state operators map to contours at large $t$, and we have already expanded these operators in terms of modes natural to the $t$ plane in this region. The operators on the initial copy 1 CFT instead map to contours around $t = 0$. We must now expand these operators in terms of modes natural to this region, which amounts to expanding in powers of $t$ for small $t$. This directly gives the copy 1 in modes in terms of modes natural to the $t$ plane at $t = 0$.

\[
\begin{align*}
\alpha_{++,m}^{(1)} &= \sum_{q \geq 0} -\frac{mN}{M} C_q(-a) -\frac{mN}{M} q \tilde{\alpha}_{++,q-m} \\
\tilde{d}_{-m}^{(1),++} &= \sum_{q \geq 0} -\frac{mN}{M} C_q(-a) -\frac{mN}{M} q \tilde{d}_{q-m+\frac{1}{2}}^{++} \\
\tilde{d}_{-m}^{(1),--} &= \sum_{q \geq 0} -\frac{mN}{M} C_q(-a) -\frac{mN}{M} q \tilde{d}_{q-m+\frac{1}{2}}^{--}.
\end{align*}
\] (3.49)

We now use these expansions along with the out state expansions from Equation 3.36 (with mode index labeled $p$) to compute the amplitude ratios found in Equation 3.22. For the bosons, we find:
\[
\begin{align*}
\mathbf{f}_{m_0}^{(1)} & = -\frac{1}{p} \sum_{q,q' \geq 0} \frac{-m N^2}{M} \frac{C_q}{\frac{p^N}{M^p} + N^2} C_q' \left(-a\right)^{\frac{-m N}{p^p} - q + q'} \left\langle 0 | \tau \alpha_{\mathbf{-p-q'}, \mathbf{p+q-m}} | 0 \right\rangle_t \\
& = -\left(-a\right)^p \frac{M}{M+N} \frac{1}{p} \sum_{q = \max(m-p,0)}^{m-1} \frac{-m N}{M} \frac{C_q}{\frac{p^N}{M^p} + N^2} C_{p+q-m}(q - m) \\
& = \begin{cases} \\
\frac{M}{M+N} \frac{1}{\pi(M+N)} \frac{(-\left(-a\right)^p \frac{m(M+N)}{M} \frac{C_q}{p^N} C_{p+q-m}(q - m)}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} = \frac{m}{M} \\
\frac{M}{M+N} \frac{1}{\pi(M+N)} \frac{(-\left(-a\right)^p \frac{m(M+N)}{N} \frac{C_q}{p^N} C_{p+q-m}(q - m)}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} \neq \frac{m}{N}.
\end{cases}
\end{align*}
\]

Applying the transformation from Equation 3.48, we also have:

\[
\mathbf{f}_{m_0}^{(2)} = \begin{cases} \\
\frac{N}{M+N} \frac{1}{\pi(M+N)} \frac{(-\left(-a\right)^p \frac{m(M+N)}{M} \frac{C_q}{p^N} C_{p+q-m}(q - m)}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} = \frac{m}{N} \\
\frac{N}{M+N} \frac{1}{\pi(M+N)} \frac{(-\left(-a\right)^p \frac{m(M+N)}{N} \frac{C_q}{p^N} C_{p+q-m}(q - m)}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} \neq \frac{m}{N}.
\end{cases}
\]

Note the interesting relation:

\[
f_{m_0}^{(1)} + f_{m_0}^{(2)} = \delta_{mp}.
\]

An analogous relationship at second order will be noted in Chapter 5.

Similarly, one finds for the fermions:

\[
f_{m_0}^{+,(1)} = \sqrt{\frac{M+N}{M}} \frac{(-\left(-a\right)^p \frac{m(M+N)}{M} \frac{C_q}{p^N} C_{p+q+1}}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} = \frac{m}{M} \\
+ \sqrt{\frac{M}{M+N}} \frac{(-\left(-a\right)^p \frac{m(M+N)}{N} \frac{C_q}{p^N} C_{p+q+1}}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} \neq \frac{m}{N}.
\]

\[
\frac{\sqrt{M+N}}{\sqrt{M}} \left(-a\right)^p \frac{m(N)}{M} \frac{C_q}{p^N} \sum_{q = \max(m-p-1,0)}^{m-1} \frac{C_{p-m+q+1}}{p^p} \frac{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]}
\]

\[
= \begin{cases} \\
\frac{\sqrt{M+N}}{\sqrt{M}} \left(-a\right)^p \frac{m(N)}{M} \frac{C_q}{p^N} \sum_{q = \max(m-p,0)}^{m-1} \frac{C_{p-m+q+1}}{p^p} \frac{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} = \frac{m}{M} \\
\frac{\sqrt{M+N}}{\sqrt{M}} \left(-a\right)^p \frac{m(N)}{N} \frac{C_q}{p^N} \sum_{q = \max(m-p,0)}^{m-1} \frac{C_{p-m+q+1}}{p^p} \frac{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]} & \frac{p}{M+N} \neq \frac{m}{N}.n
\end{cases}
\]

\[
\frac{\sqrt{M+N}}{\sqrt{M}} \left(-a\right)^p \frac{m(N)}{M} \frac{C_q}{p^N} \sum_{q = \max(m-p,0)}^{m-1} \frac{C_{p-m+q+1}}{p^p} \frac{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]}{\Gamma[\frac{M}{M+N}] \Gamma[\frac{N}{M+N}] \Gamma[p]}
\]
\[
\begin{align*}
    f_{mp}^{-(1)} &= \frac{\sqrt{M+N}}{\sqrt{M}} (-a)^{p - \frac{m(M+N)}{M}} \Bigg\{ \sum_{q = \max(m-p,0)}^{m-1} pN_{\frac{m+N}{M} - q}^{-1} C_p - m + q \cdot mN_{\frac{M+q}{M}} C_q \Bigg\} \\
    &= \frac{\sqrt{M+N}}{\sqrt{M}} \frac{\sqrt{M+N}}{\sqrt{M}} (-a)^{p - \frac{m(M+N)}{M}} \Bigg\{ \sum_{q = \max(m-p+1,0)}^{m} pN_{\frac{m+N}{M} - q}^{-1} C_p - m + q \cdot mN_{\frac{M+q}{M}} C_q \Bigg\} \\
\end{align*}
\]

and by extension:

\[
\begin{align*}
    f_{mp}^{+(2)} &= \frac{\sqrt{N}}{\sqrt{M+N}} \frac{\sqrt{M+N}}{\sqrt{M}} (-a)^{p - \frac{m(M+N)}{N}} \Bigg\{ \sum_{q = \max(m-p,0)}^{m-1} pN_{\frac{m+N}{N} - q}^{-1} C_p - m + q \cdot mN_{\frac{N+q}{N}} C_q \Bigg\} \\
    &= \frac{\sqrt{N}}{\sqrt{M+N}} \frac{\sqrt{M+N}}{\sqrt{M}} (-a)^{p - \frac{m(M+N)}{N}} \Bigg\{ \sum_{q = \max(m-p+1,0)}^{m} pN_{\frac{m+N}{N} - q}^{-1} C_p - m + q \cdot mN_{\frac{N+q}{N}} C_q \Bigg\} \\
\end{align*}
\]

(3.53)

3.2 Applying the Supercharge

In addition to the effects of the twist operator as outlined above, one also has a superposition of \(G^{-}_{A,0}\) modes before and after the twist. The interactions of this mode with an initial excitation, whether before or after the excitation is brought through the twist, is trivial to obtain by expanding the supercharge in terms of boson and fermion modes and and applying the appropriate \((\text{anti})\)commutation relations:

\[
\begin{align*}
    \left[ G^{-}_{A,0}, \alpha^{(k)}_{B,B',-m} \right] &= \frac{i m}{\sqrt{N_{(k)}}} d^{(k),-A}_{-m} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \\
    \left\{ G^{-}_{A,0}, d^{(k),+B}_{-m} \right\} &= \frac{i}{\sqrt{N_{(k)}}} \alpha^{(k)}_{A\dot{A},-m} \epsilon^{AB} \\
    \left\{ G^{-}_{A,0}, d^{(k),-B}_{-m} \right\} &= 0. \\
\end{align*}
\]

(3.55)
The nontrivial behavior of the supercharge is when it hits the twisted vacuum $|\chi(w_0)\rangle$. Working only with the left-moving sector, we write:

$$|\psi\rangle = G_{A,0}^-|\chi(w_0)\rangle.$$ (3.56)

We wish to write (3.56) with only negative index modes acting on $|0_-\rangle$. We thus find it convenient to write

$$G_{A,0}^- = \frac{i}{\sqrt{M+N}} \left( \sum_{l>0} d_{-l}^- \alpha_{A,\bar{A},l} + \sum_{l>0} d_{l}^- \alpha_{A,\bar{A},-l} + d_0^- \alpha_{A,\bar{A},0} \right)$$ (3.57)

Acting with the annihilators on $|\chi\rangle$ then brings in factors of $\gamma^B$ and $\gamma^F$:

$$\frac{i}{\sqrt{M+N}} \sum_{l \geq 1} d_{-l}^- \alpha_{A,\bar{A},l} |\chi\rangle = \frac{i}{\sqrt{M+N}} \sum_{l \geq 1} \sum_{k \geq 1} l \gamma_{kl}^B d_{-l}^- \alpha_{A,\bar{A},-k} |\chi\rangle$$

$$\frac{i}{\sqrt{M+N}} \sum_{k \geq 1} d_{k}^- \alpha_{A,\bar{A},-k} |\chi\rangle = -\frac{i}{\sqrt{M+N}} \sum_{k \geq 1} \sum_{l \geq 1} \gamma_{kl}^F d_{-l}^- \alpha_{A,\bar{A},-k} |\chi\rangle$$

$$\frac{i}{\sqrt{M+N}} d_0^- \alpha_{A,\bar{A},0} |\chi\rangle = 0,$$ (3.58)

and thus:

$$G_{A,0}^- |\chi\rangle = \frac{i}{\sqrt{M+N}} \sum_{k \geq 1, l \geq 1} \left( l \gamma_{kl}^B - \gamma_{kl}^F \right) d_{-l}^- \alpha_{A,\bar{A},-k} |\chi\rangle.$$ (3.59)

Plugging in our earlier results for the Bogoliubov coefficients (Equations 3.40, 3.44), we find that the $l$ and $k$ sums factorize. This gives:

$$|\psi\rangle = G_{A,0}^- |\chi\rangle = -\frac{i}{\pi^2 (M+N)^2} \left( \sum_{l \geq 1} a^l \sin \left[ \frac{N \pi l}{M+N} \right] \frac{\Gamma \left[ \frac{Ml}{M+N} \right]}{\Gamma [l]} \right) d_{-l}^- \alpha_{A,\bar{A},l} |\chi\rangle \times \left( \sum_{k \geq 1} a^k \sin \left[ \frac{N \pi k}{M+N} \right] \frac{\Gamma \left[ \frac{Mk}{M+N} \right]}{\Gamma [k]} \right) |\chi\rangle.$$ (3.60)

This is the nontrivial effect of the supercharge.
Chapter 4: One-Loop Calculation

Having performed an exhaustive assessment of all first-order deformations, we now turn to a simple second-order calculation. We begin with two singly-wound CFT copies in the negative R sector. The first deformation operator joins them into one doubly-wound copy, and then the second deformation splits them back into two singly-wound copies. We use the same methods outlined in Chapter 3, except that we have a different $t$ plane map and the stripping of the supercurrent contours is much more complicated. As before, the bulk of this chapter deals with calculating the Bogoliubov coefficients and the transition amplitudes. The effect of the supercurrent contours is left for the end.

4.1 The Covering Map

Ignoring the supercurrent contours for the moment, we have two twist operators inserted at coordinates $w_1, w_2$. For concreteness, we choose $\tau_2 > \tau_1$. The twist configuration, and its affects on an initial negative R-R vacuum state, is depicted in Figure 4.1.

From the various SU(2) charges, we anticipate a final state that is SU(2) R neutral. There are many such vacuum state to build on. We choose to build upon the vacuum $|\emptyset\rangle = |0_R^+(1)\rangle |0_R^-(2)\rangle$ for convenience. We thus have six spin fields on the cylinder: $S^-$
Figure 4.1: The cylinder with twist insertions at $w_1$ and $w_2$. Below the first twist we have the negative Ramond vacuum on each component string. Above both twists we have the state $|\chi(w_1, w_2)\rangle$, which we will compute. In the intermediate regions we have a single doubly-wound component string in the state $|\chi(w_1)\rangle$. This state was first computed in [15]. It is a simple case of the calculation outlined in Chapter 3. This intermediate state is not used here.

on both initial copies and on the out state copy 1’, and $S^+$ at both twist insertions and on the out state copy 2’ (recall that primes are used when specifying out state CFT copies).

The full details of the the coordinate maps performed in this section are depicted visually in Figure 4.2. We begin by mapping to the $z$ plane via $z = e^w$. Both in state copies map to the $z = 0$, both out state copies map to $z = \infty$, and the twist insertions map to $z_1, z_2$. Since $\tau_2 > \tau_1$, we have $|z_2| > |z_1|$.

We now map the $z$ plane into a double cover of itself. This map must separate out the two distinct copies for both the in and out states, so we choose the simple map:

$$z = \frac{(t + a)(t + b)}{t}. \quad (4.1)$$
Here, the point $z = \infty$ maps to both $t = \infty$ and $t = 0$. Near $t = \infty$ we have the leading order behavior $z \sim t$, while near $t = 0$ we have the behavior $z \sim t^{-1}$. The point $z = 0$ also has two images, at $t = -a$ and $t = -b$. We now have single single-valued fields in the $t$ plane.

![Figure 4.2: The $z$ plane (a) and $t$ plane (b) with all relevant image points labeled. The intermediate state $|\chi(w_1)\rangle$ is not depicted. In the $z$ plane, $\tau$ maps to the radial coordinate $\sigma$ coordinate maps to the phase. In the $t$ plane there are no simple directions corresponding to $\tau$ and $\sigma$.](image)

We now have some freedom to choose which CFT copies map to which regions in the $t$ plane. The physics is independent of which component string we call copy 1 and which we call copy 2, and indeed the map (4.1) is symmetric under the interchange $a \leftrightarrow b$. Similarly, we may choose which component string of the out state maps to $t = \infty$ and which component string maps to $t = 0$. The conventions we’ve chosen are:
We also need to know the images of the twist insertions at $w_1$ and $w_2$. These are the bifurcation points of the double cover:

$$\frac{dz}{dt} = 1 - \frac{ab}{t^2} = 0.$$  \hspace{1cm} (4.3)

This relation has two solutions, corresponding to our two twist insertions.

$$t_1 = -\sqrt{ab}$$

$$t_2 = \sqrt{ab}.$$  \hspace{1cm} (4.4)

This in turn gives us a relationship between the map parameters $a, b$ and our insertion points $w_1, w_2$:

$$z_1 = a + b - 2\sqrt{ab} = e^{w_1}$$

$$z_2 = a + b + 2\sqrt{ab} = e^{w_2},$$  \hspace{1cm} (4.5)

where we choose the branch of $\sqrt{ab}$ so as to maintain $|z_2| > |z_1|$. As expected, this relationship is unaffected by the interchange $a \leftrightarrow b$.

We will now note the locations of our spin fields:

$$S^+ \text{ at } t = -\sqrt{ab}, +\sqrt{ab}, 0$$

$$S^- \text{ at } t = -a, -b, \infty.$$  \hspace{1cm} (4.6)

We remove these spin fields by performing the following spectral flows:
\[ \alpha = -1 \quad \text{at} \quad t = -\sqrt{ab}, +\sqrt{ab}, 0 \]
\[ \alpha = +1 \quad \text{at} \quad t = -a, -b. \]  \hfill (4.7)

As usual this will only affect our fermion modes.

### 4.2 Deforming the Modes

For both our in and out states we have \( N_{(k)} = 1 \). This simplifies our mode expansions:

\[ \alpha_{AA, n}^{(k)} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi} X_{AA}^{(k)}(w)e^{nw}dw \]
\[ d_{n}^{(i)f, AA} = \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi} \psi^{(i)AA}(w)e^{nw}dw. \]  \hfill (4.8)

The contours wrap the cylinder above both twists for out state modes and below both twists for in state modes.

Upon mapping to the \( z \) plane, our fermion integrands once again pick up a factor of \( z^{-\frac{1}{2}} \). Mapping to the \( t \) plane produces a further factor:

\[
\left( \frac{dt}{dz} \right)^{-\frac{1}{2}} = \left( \frac{dz}{dt} \right)^{\frac{1}{2}} = \left( 1 - \frac{ab}{t^2} \right)^{\frac{1}{2}} = (t - t_1)^{\frac{1}{2}} (t - t_2)^{\frac{1}{2}}. \]  \hfill (4.9)

We then have the following modes in the \( t \) plane:

\[ \alpha_{AA, n}^{(k)} \rightarrow \frac{1}{2\pi} \int_{t_0} X_{AA}(t) \left( \frac{(t + a)(t + b)}{t} \right)^n dt \]
\[ d_{n}^{(k), AA} \rightarrow \frac{1}{2\pi i} \int_{t_0} \psi^{AA}(t)t^{-n - \frac{1}{2}}(t + \sqrt{ab})^{\frac{1}{2}}(t - \sqrt{ab})^{\frac{1}{2}}(t + a)^{-\frac{1}{2}}(t + b)^{n - \frac{1}{2}} dt. \]  \hfill (4.10)
where \( t_0 \) is the appropriate image point for the mode’s CFT copy. However, there is a subtlety to be noted here. When the copy 2’ modes at large \( z \) map to small \( t \), the behavior \( z \sim 1/t \) reverses the direction of the contour. We thus gain an overall minus sign:

\[
\alpha^{(2')}_{AA,n} \to -\frac{1}{2\pi} \oint_{t=0} X_{AA}(t) \left( \frac{(t+a)(t+b)}{t} \right)^n \, dt
\]

\[
d_{n}^{(2'),+A} \to -\frac{1}{2\pi i} \int_{t_0} \psi^{+A}(t)t^{-n} (t + \sqrt{ab})^{\frac{1}{2}} (t - \sqrt{ab})^{\frac{1}{2}} (t + a)^{n-\frac{1}{2}} (t + b)^{n-\frac{1}{2}} \, dt.
\]

We now apply the spectral flows listed in Equation 4.7. This has no effect on the bosons. For the fermions, we have:

\[
\psi^{\pm A}(t) \to t^{\pm \frac{1}{2}} (t + \sqrt{ab})^{\pm \frac{1}{2}} (t - \sqrt{ab})^{\pm \frac{1}{2}} (t + a)^{\pm \frac{1}{2}} (t + b)^{\pm \frac{1}{2}} \psi^{\pm A}(t).
\]

This gives:

\[
\alpha^{(k)}_{AA,n} = \frac{1}{2\pi} \oint_{t_0} X_{AA}(t) \left( \frac{(t+a)(t+b)}{t} \right)^n \, dt
\]

\[
d_{n}^{(k),+A} = \frac{1}{2\pi i} \int_{t_0} \psi^{+A}(t)t^{-n} (t + \sqrt{ab})(t - \sqrt{ab})(t + a)^{n-1}(t + b)^{n-1} \, dt
\]

\[
d_{n}^{(k),-A} = \frac{1}{2\pi i} \int_{t_0} \psi^{+A}(t)t^{-n-1}(t + a)^n(t + b)^n \, dt,
\]

though again the modes from copy 2’ have an overall minus sign from their contour reversal:

\[
\alpha^{(2')}_{AA,n} = -\frac{1}{2\pi} \oint_{t=0} X_{AA}(t) \left( \frac{(t+a)(t+b)}{t} \right)^n \, dt
\]

\[
d_{n}^{(2'),+A} = -\frac{1}{2\pi i} \int_{t_0} \psi^{+A}(t)t^{-n} (t + \sqrt{ab})(t - \sqrt{ab})(t + a)^{n-1}(t + b)^{n-1} \, dt
\]

\[
d_{n}^{(2'),-A} = -\frac{1}{2\pi i} \int_{t_0} \psi^{+A}(t)t^{-n-1}(t + a)^n(t + b)^n \, dt.
\]

### 4.2.1 Handy Symmetries

At this point, we will take advantage of a few symmetries of the current configuration. First we note that outside of the fermion zero modes, a global exchange of
copy labels has no physical effect. That is, the non-zero mode physics is symmetric under the transformation:

\[ (1) \leftrightarrow (2) \text{ and } (1') \leftrightarrow (2'). \] (4.15)

This alone halves the number of quantities of interest for non-zero modes. We can now look solely at, say, the copy combinations \((1)(1')\) and \((2)(1')\) and deduce the remaining two combinations by symmetry.

A close examination of Equation 4.13 also reveals a useful symmetry. The integrands are invariant under the interchange \(a \leftrightarrow b\). Now this interchange can affect \(t_0\), but whenever it doesn’t, the interchange is a symmetry. This holds for both out state CFT copies, where \(t_0 = 0, \infty\). However, for the in state copies \(t_0 = -a, -b\). Thus the interchange \(a \leftrightarrow b\) swaps our in state copies while leaving the out state copies unaffected. It is in fact the formalization of the first transformation in Equation 4.15.

On top of this, we have the transition amplitude relation from Equations 2.44 and 2.48, as well as the Bogoliubov coefficient relations from Equations 2.38. This allows us to convert bosonic results into fermionic results for non-zero modes.

We have now cut our task down significantly. We will proceed by calculating the Bogoliubov coefficient explicitly for only the bosonic \((1')(1')\) and \((1')(2')\) cases. By the global copy exchange symmetry, this also gives us the \((2')(2')\) and \((2')(1')\) cases, respectively. We will then use Equation 2.38 to obtain the corresponding fermionic quantities for nonzero modes. The fermon zero modes will not be computed explicitly. Instead I will merely present the results from [22], which not only computed the zero mode coefficients but put much effort into simplifying their form.

For the transition amplitudes, we will first calculate only the bosonic \((1)(1')\) case explicitly. Global copy symmetry and the \(a \leftrightarrow b\) will then give us the rest of the
bosonic cases, while Equations 2.44, and 2.48 will give us the fermionic amplitudes for nonzero modes. Again the fermion zero modes will simply be reproduced from [23], where their form was simplified.

4.3 The Bogoliubov Coefficients

We now compute \( \gamma_{mn}^{B,(1)(1)} \) directly. From Equation 4.13, we have:

\[
\alpha_{AA,n}^{(1')}(1') = \frac{1}{2\pi} \oint_{t \to \infty} X_{AA}(t) \left( \frac{(t + a)(t + b)}{t} \right)^n dt
\]

\[
\alpha_{AA,n}^{(2')}(2') = -\frac{1}{2\pi} \oint_{t = 0} X_{AA}(t) \left( \frac{(t + a)(t + b)}{t} \right)^n dt
\]

(4.16)

We now expand the copy 1’ integrand at large \( t \), thereby expressing the cylinder mode in terms of modes natural to the \( t \) plane in this region.

\[
\alpha_{AA,n}^{(1')} = \sum_{j,j' \geq 0} C_j^n C_{j'} a_j b_j' \tilde{\alpha}_{AA,n-j-j'}.
\]

(4.17)

Plugging this into Equation 3.15, we have:

\[
\gamma_{mn}^{B,(1')(1')} = -\frac{1}{mn} \sum_{j,j',k,k' \geq 0} C_j^n C_{j'} m C_{k'} n C_{k+m+j+k'} \tilde{\alpha}_{AA,n-j-j'}
\]

\[
\times t(0_{NS}|\tilde{\alpha}_{AA,n-j-j'}|^0_{NS}) t(0_{NS}|0)^t.
\]

(4.18)

Applying creation/annihilation constraints to the amplitude ratio in Equation 4.18, as well as the commutation relation from Equation 3.9, we find:

\[
\gamma_{mn}^{B,(1')(1')} = \frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m+n-j-j'} (n - j - j') C_j^n C_{j'} m C_{k'} n C_{m+n-j-j'-k} a^{j+k} b^{m+n-j-k}.
\]

(4.19)

where in the last line we have used the global copy interchange symmetry.
We now turn to the copy 2’ modes. Expanding them at small $t$ allows us to express them in terms of modes natural to the $t$ plane in this region:

$$\bar{\alpha}_{AA,n}^{(2')} = - \sum_{j,j' \geq 0} n C_j^n C_{j'}^{n-j} b^{n-j'} \bar{\alpha}_{AA,j+j'-n}.$$

(4.20)

We will need to expand the contours to large $t$ in order to contract with the copy 1’ modes. This is trivial, except that we should take care to realize that the copy 2’ modes form contours that remain inside the copy 1’ contours. Thus the copy 2’ modes must be the modes to form creation operators. We then have:

$$\gamma_{mn}^{B,(2')(1')} = - \frac{1}{mn} \sum_{j,j',k,k' \geq 0} n C_j^n C_{j'}^{m} C_k^m C_{k'}^{m-j+k-j'} a^{m-j+k-j'}$$

$$\times \frac{t(0|NS|0++_{n-j-j'}-k+k'-m|0NS)}{t(0|NS|0)}.$$ 

(4.21)

Applying creation/annihilation constraints to the amplitude ratio in Equation 4.21, as well as the commutation relation from Equation 3.9, we find:

$$\gamma_{mn}^{B,(2')(1')} = \frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{j+1} \sum_{k=0}^{m-n+j+j'} (n-j-j') n C_j^n C_{j'}^{m} C_k^m C_m^{m-n+j+j'-k} a^{m-j+k-j'} b^{n-j+k}$$

$$= \gamma_{mn}^{B(1')(2')}.$$ 

(4.22)

While it is not immediately obvious, there is a handy relation:

$$\gamma_{mn}^{B(1')(1')} = - \gamma_{mn}^{B(2')(1')}.$$ 

(4.23)

This was shown in [22]

Because our winding numbers are always 1, Equation 2.38 tells us we can obtain the fermion coefficient (for non-zero modes) via multiplication by $-m$. Using Equation 4.23 as well, we find:

$$\gamma_{mn}^{F,(1')(1')} = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{j'=0}^{j+1} \sum_{k=0}^{m-n+j+j'} (n-j-j') n C_j^n C_{j'}^{m} C_k^m C_m^{m-n+j+j'-k} a^{m-j+k-j'} b^{n-j+k}.$$ 

$$= \gamma_{mn}^{F,(2')(2')} = - \gamma_{mn}^{F,(2')(1')} = - \gamma_{mn}^{F,(1')(2')}.$$ 

(4.24)
4.3.1 Fermion Zero Modes

When looking at the fermion zero modes, we can immediately rule out any terms with modes that annihilate $|0^+_{R^{(1)}}0^-_{R^{(2)}}\rangle$. Each one rules out particular bogoliubov coefficients:

\[
d_{0}^{(1'),+A} \implies \gamma_{0,n}^{F,(1')(j)} = 0
\]
\[
d_{0}^{(2'),-A} \implies \gamma_{m,0}^{F,(j)(2')} = 0.
\] (4.25)

Furthermore, Equation 2.39 tells us that all coefficients vanish for $m = 0, n > 0$. This leaves us with the following coefficients to calculate:

\[
\gamma_{m,0}^{F,(2')(1')}, \text{ all } m
\]
\[
\gamma_{m,0}^{F,(1')(1')}, \text{ m > 0}
\] (4.26)

These coefficients were found in [22]. There a general relationship was noted:

\[
\gamma_{m,0}^{F,(2')(1')} = -\gamma_{m,0}^{F,(1')(1')}, \text{ m > 0.}
\] (4.27)

It is thus sufficient to present only $\gamma_{m,0}^{F,(2')(1')}$ for general $m$.

\[
\gamma_{R+,-:m,0}^{F,(1)} = -\sum_{j=0}^{m} a_{j}^{m-j} m C_{j}^{m} C_{m-j} = -b^{m} 2F_{1}\left(-m, -m, 1, \frac{a}{b}\right).
\] (4.28)

4.4 The Transition Amplitudes

Here we have the luxury of computing only a single case, $f_{mp}^{B,(1')(1')}$. The out state copy 1' bosonic modes were expanded in Equation 4.17. We now examine the in state copy 1 modes.
\[ \alpha_{A\dot{A},m}^{(1)} = \frac{1}{2\pi} \oint_{t=-a} X_{A\dot{A}}(t) \left( \frac{(t+a)(t+b)}{t} \right)^m \, dt \]
\[ = \frac{1}{2\pi} \oint_{t'=0} X_{A\dot{A}}(t) \left( \frac{t'(t'-a+b)}{t'-a} \right)^m \, dt', \quad (4.29) \]

where we have performed the coordinate shift \( t' = t + a \). Now the local expansion is one of small \( t' \). This gives:

\[ \alpha_{A\dot{A},n}^{(1)} = \sum_{k,k'=0}^{\infty} -^m C_k^m C_{k'}^{b-a} (-a)^{m-k} (-1)^{m-k'} \hat{\alpha}_{A\dot{A},k+k'-m} \to a, \quad (4.30) \]

where the hat indicates that the modes are natural to the shifted \( t' \) plane. If we proceed naively from here we will run into a mess of infinite sums. We will avoid this by dropping all modes that annihilate the local NS vacuum. This caps the sum at \( k + k' < m \).

Unfortunately the local modes natural for our copy 1 in states are not currently compatible with our local modes for our copy 1’ out state. This is because the former are natural to the shifted \( t' \) plane, while the latter are natural to the original \( t \) plane. We thus need to expand the \( t' \) modes in powers of \( t \). Since the out state copy 1’ maps to large \( t \), we perform the expansion in that region. This gives:

\[ \hat{\alpha}_{A\dot{A},k+k'-m} \to a = \sum_{k'\neq 0}^{\infty} -^m C_{k'\neq 0}^{k'+m} C_{k'}^{b-a} (-1)^{m-k'} \hat{\alpha}_{A\dot{A},k+k'-m-k'} \to a, \quad (4.31) \]

Plugging this into Equation 4.30, we have:

\[ \alpha_{A\dot{A},m}^{(1)} = \sum_{k,k',k''=0}^{\infty} -^m C_k^m C_{k'}^{k+k'-m} C_{k''} (-1)^{m-k'} (b-a)^{m-k} a^{m+k''-k'} \hat{\alpha}_{A\dot{A},k+k'-m-k''} \to a, \quad (4.32) \]

where we still have the constraint \( k + k' < m \).

We now combine Equation 4.32 with Equations 4.17 and 3.20 to obtain:
\[ f_{m_p}^{B,(1)(1')} = \sum_{j,j',k,k',k''=0}^{\infty} p C_j p C_{j'} m C_{k'\,k''} C_{k'' m} (-1)^{m-k'} a^{m+k'-k'+j} b^{j'} (b - a)^{-m-k}
\times \left( -\frac{1}{p} t^{\langle 0|\tilde{\alpha}_{k'\rightarrow \infty} + p-j-j'\tilde{\alpha}_{-k'\rightarrow m-k'|0}\rangle_t} \right), \quad (4.33) \]

with the constraint:

\[ k + k' < m. \quad (4.34) \]

Computing the amplitude via the commutation relations of Equation 3.9, eliminates the \( k'' \) sum, while creation/annihilation considerations bound the \( j, j' \) sums. We then have only finite sums:

\[ f_{m_p}^{B,(1)(1')} = \sum_{k=\max(m-p,0)}^{m-1} \sum_{k'=\max(m-p-k,0)}^{m-k-1} \sum_{j=0}^{p-m+k+k'} \sum_{j'=0}^{p-m+k+k'-j} p C_j p C_{j'} m C_k m C_{k'} \]

\[ \times k + k'-m C_{p+k+k'-m-j-j'} (b - a)^{-m-k}. \]

\[ = f_{m_p}^{B,(2)(2')}, \quad (4.35) \]

where in the last line we have made use of the global copy interchange symmetry. We now apply \( a \leftrightarrow b \) to obtain:

\[ f_{m_p}^{B,(2)(1')} = \sum_{k=\max(m-p,0)}^{m-1} \sum_{k'=\max(m-p-k,0)}^{m-k-1} \sum_{j=0}^{p-m+k+k'} \sum_{j'=0}^{p-m+k+k'-j} p C_j p C_{j'} m C_k m C_{k'} \]

\[ \times k + k'-m C_{p+k+k'-m-j-j'} (b - a)^{-m-k}. \]

\[ = f_{m_p}^{B,(1)(2')}, \quad (4.36) \]

Equations 2.44 and 2.48 with \( N_{(j)} = 1 \) then give us the non-zero mode fermion amplitudes:

\[ f_{m_p}^{F-,(i)(j)} = f_{m_p}^{B,(i)(j)} \]

\[ f_{m_p}^{F+,(i)(j)} = \frac{n}{p} f_{m_p}^{B,(i)(j)} \quad (4.37) \]
4.4.1 Fermion Zero Modes

A detailed computation of the fermion zero modes can be found in [23]. Since we can not make use of zero modes that annihilate $|0^{-}_R \rangle (1)|0^{-}_R \rangle (2)$ (the in state) or $|0^+_R \rangle (1)|0^{-}_R \rangle (2)$ (the vacuum upon which the out state is built), we have several constraints.

\[
\begin{align*}
    f^{F-, (i)(j)}_{0,p} &= 0 \\
    f^{F-, (i)(2')}_{m,0} &= 0 \\
    f^{F+, (i)(1')}_{m,0} &= 0.
\end{align*}
\]  

(4.38)

The remaining nonzero results are:

\[
\begin{align*}
    f^{F+, (1')(1')}_{0,p} &= - \sum_{j=0}^{p-1} \sum_{j'=0}^{p-j-1} p C_j^p C_{j'} (-1)^{p-j-j'-1} a^{p-j'} b^{j'} \\
    f^{F-, (1')(1')}_{m,0} &= \sum_{k=0}^{m-1} m C_k^m C_{m-k-1} (-1)^{m-k-1} a^{m-k-1} (b - a)^{-2m+k+1} \\
    f^{F+, (2')(1')}_{0,p} &= - \sum_{j=0}^{p-1} \sum_{j'=0}^{p-j-1} p C_j^p C_{j'} (-1)^{p-j-j'-1} b^{p-j'} a^{j'} \\
    f^{F-, (2')(1')}_{m,0} &= \sum_{k=0}^{m-1} m C_k^m C_{m-k-1} (-1)^{m-k-1} b^{m-k-1} (a - b)^{-2m+k+1} \\
    f^{F^+, (2')(2')}_{m,0} &= - \sum_{k=0}^{m-k} \sum_{k'=0}^{m-k} (-1)^{m-k} C_{k'}^{m+2} C_k b^k - m C_k a^k (-1)^{m-k} b^{k'} (a - b)^{-m+k-1} \\
    f^{F^+, (2')(2')}_{0,p} &= - \sum_{j=0}^{p-1} \sum_{j'=0}^{p-j-1} p C_j^p C_{j'} (-1)^{p-j-j'} a^{p-j} b^{j} \\
    f^{F^+, (1')(2')}_{m,0} &= - \sum_{k=0}^{m-k} \sum_{k'=0}^{m-k} (-1)^{m-k} C_{k'}^{m+2} C_k a^k - m C_k b^k (-1)^{m-k} a^{k'} (b - a)^{-m+k-1} \\
    f^{F^+, (1')(2')}_{0,p} &= - \sum_{j=0}^{p-1} \sum_{j'=0}^{p-j-1} p C_j^p C_{j'} (-1)^{p-j-j'} b^{p-j} a^{j}.
\end{align*}
\]  

(4.39)
4.5 Handling the Supercharge

We must now tackle the arduous task of stripping the supercharge contours off the twists. This is not as simple at this order, as simply attempting the same approach form the first order calculation would leave $G_{A,0}^-$ modes positioned between the two twists. Instead we will map the contours into the $t$ plane and move them past the twists in the same manner as the various excitations we worked with earlier.

Recall the left-moving sector of the two deformations:

\[
\hat{O}_B \hat{O}_A = \oint_{w_2} G_A^- (w) \sigma_2^+(w_2) \, dw \oint_{w_1} G_A^- (w') \sigma_2^+(w_1) \, dw'.
\]  

(4.40)

We now take the supercurrent around $\tau_2$ and stretch it, yielding three parts:

- A positive-direction contour around the cylinder at $\tau > \tau_2$, for both copies.
- A negative-direction contour around $t_1$, outside the $A$ supercurrent.
- A negative-direction contour around the cylinder at $\tau < \tau_1$, for both copies.

This gives three terms:

\[
\hat{O}_B \hat{O}_A = \int_{\sigma=0}^{\sigma=2\pi} G_B^-(w) \, dw \sigma_2^+(w_2) \int_{w_1} G_A^- (w') \sigma_2^+(w_1) \, dw' \\
- \sigma_2^+(w_2) \oint_{w_1} \oint_{|w' - w_1| < |w - w_1|} G_B^-(w)G_A^-(w') \sigma_2^+(w_1) \, dw' \, dw \\
+ \sigma_2^+(w_2) \oint_{w_1} G_B^-(w) \sigma_2^+(w_1) \, dw \int_{\sigma=0}^{\sigma=2\pi} G_B^-(w') \, dw',
\]  

(4.41)

where the extra change in sign for the last line of (4.41) comes from reversing the written order of the supercurrents. Integrals that wrap completely around the cylinder are simply supercharge modes, so we will write them as such.

58
\[ \hat{O}_B \hat{O}_\bar{A} = G^-_{B,0} \sigma_2^+(w_2) \oint_{w_1} G^-_\bar{A}(w) \sigma_2^+(w_1) \, dw \]
\[ - \sigma_2^+(w_2) \oint_{w_1 \mid w' - w_1 < |w - w_1|} G^-_B(w) G^-_\bar{A}(w') \sigma_2^+(w_1) \, dw' \, dw \]
\[ + \sigma_2^+(w_2) \oint_{w_1} G^-_\bar{A}(w) \sigma_2^+(w_1) \, dw G^-_{B,0}. \]  

(4.42)

We have now done all that we can on the cylinder. To progress, we must employ our covering space. We will use this to deal with the two remaining integrals:

\[ I_1 \equiv \sigma_2^+(w_2) \oint_{w_1} G^-_\bar{A}(w) \sigma_2^+(w_1) \, dw \]
\[ I_2 \equiv \sigma_2^+(w_2) \oint_{w_1 \mid w' - w_1 < |w - w_1|} G^-_B(w) G^-_\bar{A}(w') \sigma_2^+(w_1) \, dw' \, dw. \]  

(4.43)

4.5.1 Supercharge Mapping

The supercurrent \( G \) has conformal weight 3/2, so combined with the jacobian we have:

\[ G_-^\alpha \, dw = \left( \frac{dw'}{dw} \right)^{\frac{1}{2}} G_-^\alpha(w') \, dw' = \left( \frac{dw}{dw'} \right)^{-\frac{1}{2}} G_-^\alpha(w') \, dw'. \]  

(4.44)

This is the inverse of the factors involved in the mapping of fermion modes. It is thus easy to deduce the \( t \) plane expressions from the fermion behavior.

\[ G_-^\alpha(w) \, dw = (t + a)^{\frac{1}{2}}(t + b)^{\frac{1}{2}} t^{-\frac{1}{2}} t(t - t_1)^{-\frac{1}{2}} t(t - t_2)^{-\frac{1}{2}} G_-^\alpha(t) \, dt \]
\[ = G_-^\alpha(t)(t + a)^{\frac{1}{2}}(t + b)^{\frac{1}{2}} (t - t_1)^{-\frac{1}{2}} (t - t_2)^{-\frac{1}{2}} t^2 \, dt. \]  

(4.45)

By contrast, the supercurrent has the same type of SU(2) R charge as the fermion field, and by extension the same effects under spectral flow. We thus find:

\[ G_-^\alpha(w) \, dw \to G_-^\alpha_B(t)(t + a)(t + b)(t - t_1)^{-1}(t - t_2)^{-1} \, dt. \]  

(4.46)

\[ \text{There is also an overall constant from the effects of each spectral flow on non-local spin fields. This constant will be undone when we invert our mapping, so we do not bother to calculate it.} \]
4.5.2 The First Surpercharge Integral

Using Eqution 4.46, our first integral becomes:

\[ I_1 \rightarrow C \oint_{t_1} G^{-}_{A}(t)(t + a)(t + b)(t - t_1)^{-1}(t - t_2)^{-1} \, dt. \]  

(4.47)

There are now three steps left. First, we must stretch the contour away from \( t_1 \), until we are left with contours around the punctures at the images of the initial and final states \( t = -a, -b, 0, \infty \). Once there, we must undo our spectral flows and coordinate shifts in order to return to the cylinder. This will give us an expression in terms of contours that wrap around the cylinder above and below the twists, but with integrands still expressed in terms of the fiducial coordinate \( t \). The final step is to re-express these integrands in terms of the physical coordinate \( w \) and expand the result as modes of the supercharge. We will perform each of these three steps in turn.

Deforming the Contour

In deforming the contour, we note that there is a pole not just at \( t_1 \) but also at \( t_2 \). Since we do not want to leave a contour around \( t_2 \), we must remove this pole. We do so by making use of the fact that we have locally the \( NS \) vacuum within the contour, and can thus keep or drop terms which annihilate this vacuum as we please. We thus expand the problematic factor:

\[ (t - t_2)^{-1} = (t_1 - t_2)^{-1} + \sum_{n=1}^{\infty} C_n(t - t_1)^n. \]  

(4.48)

For any \( n \geq -1 \), the contour:

\[ \oint_{t_0} G^{-}_{A}(t)(t - t_0)^n \, dt \]  

(4.49)

annihilates the vacuum \( |0_{NS}\rangle_t \). Since we have only a power of \( -1 \) in the integrand already, any extra positive powers of \( t - t_1 \) will result in a contour which annihilates
the vacuum. We can thus write:

\[ I_1 \rightarrow -C(t_1 - t_2)^{-1} \oint_{t_1} G_{\tilde{A}}(t)(t + a)(t + b)(t - t_1)^{-1} \, dt \]

\[ = -\frac{1}{2t_1}C \oint_{t_1} G_{\tilde{A}}(t)(t + a)(t + b)(t - t_1)^{-1} \, dt, \quad (4.50) \]

where in the second line we have used the fact that \( t_2 = -t_1 \).

we are free to deform the contour through the point \( t_2 \). We will get caught at the images of the initial and final states, which results in negative-direction contours around \( t = -a, -b, 0 \). The remaining portion will be stretched into a positive-direction contour at large \( t \). Thus:

\[ I_1 \rightarrow \frac{1}{2t_1}C \left( \oint_{t=-a} + \oint_{t=-b} + \oint_{t=0} - \oint_{t \to \infty} \right) G_{\tilde{A}}(t)(t + a)(t + b)(t - t_1)^{-1} \, dt. \quad (4.51) \]

**Returning to the Cylinder**

We must now invert our spectral flows and map back to the cylinder. This removes the factor \( C \), reintroduces the twists, and applies the inverse of (4.46):

\[ G_{\tilde{B}}(t) \, dt \rightarrow G_{\tilde{B}}^{\alpha}(w)(t + a)^{-1}(t + b)^{-1}(t - t_1)(t - t_2) \, dw. \quad (4.52) \]

Recalling the various image points (See Figure 4.2), we have:

\[ \oint_{t=-a} \rightarrow \oint_{\sigma=0, \tau < \tau_1}^{\sigma=2\pi}, \quad \text{Copy 1} \]

\[ \oint_{t=-b} \rightarrow \oint_{\sigma=0, \tau < \tau_1}^{\sigma=2\pi}, \quad \text{Copy 2} \]

\[ \oint_{t \to \infty} \rightarrow \oint_{\sigma=0, \tau > \tau_2}^{\sigma=2\pi}, \quad \text{Copy 1} \]

\[ \oint_{t=0} \rightarrow -\oint_{\sigma=0, \tau > \tau_2}^{\sigma=2\pi}, \quad \text{Copy 2}, \quad (4.53) \]

where the minus sign in the last line comes from the fact that the direction of the contour is reversed when mapping from \( t \sim 0 \) to \( z \sim \infty \). With this, the signs for the out state Copy 1' and Copy 2' contours are found to match once more.
Using (4.51), (4.52), and (4.53) we obtain a new expression for $\mathcal{I}_1$ on the cylinder:

$$
\mathcal{I}_1 = \frac{1}{2t_1} \left( \oint_{\sigma=0,\tau>\tau_2} G^{-\alpha}_{\tilde{A}}(w)(t-t_2) \, dw \, \sigma_2^+(w_2) \sigma_2^+(w_1) 
- \sigma_2^+(w_2) \sigma_2^+(w_1) \oint_{\sigma=0,\tau<\tau_1} G^{-\alpha}_{\tilde{A}}(w)(t-t_2) \, dw \right)
= -\frac{1}{2t_2} \left( \oint_{\sigma=0,\tau>\tau_2} G^{-\alpha}_{\tilde{A}}(w)(t-t_2) \, dw \, \sigma_2^+(w_2) \sigma_2^+(w_1) 
- \sigma_2^+(w_2) \sigma_2^+(w_1) \oint_{\sigma=0,\tau<\tau_1} G^{-\alpha}_{\tilde{A}}(w)(t-t_2) \, dw \right),
$$

(4.54)

where we recall that in general:

$$
G^{-\alpha}_{\tilde{C}} = G^{-\alpha(1)}_{\tilde{C}} + G^{-\alpha(2)}_{\tilde{C}}
$$

(4.55)

**Simple Minkowski Coordinates**

We must now take a small detour in order to simplify our calculation. If we rotate back to Minkowski signature and choose the midpoint between the two deformations to be $w = 0$, we obtain a simple relationship between $a$, $b$, and $\Delta w = w_2 - w_1$.

$$
e^{w_1} = e^{-\frac{\Delta w}{2}}
= z_1
= \frac{(t_1 + a)(t_1 + b)}{t_1}
= \frac{(-\sqrt{ab} + a)(-\sqrt{ab} + b)}{-\sqrt{ab}}
= -\frac{ab - (a + b)\sqrt{ab} + ab}{\sqrt{ab}}
= (a + b) - 2\sqrt{ab},
$$

(4.56)

and similarly:
\[ e^{w_2} = e^{\frac{\Delta w}{2}} \]

\[ = z_2 \]

\[ = \frac{(t_2 + a)(t_2 + b)}{t_1} \]

\[ = \frac{(\sqrt{ab} + a)(\sqrt{ab} + b)}{\sqrt{ab}} \]

\[ = ab + (a + b)\sqrt{ab} + ab \]

\[ = (a + b) + 2\sqrt{ab} . \quad (4.57) \]

Combining these relationships, we have:

\[ \cos \left( \frac{\Delta w}{2i} \right) = a + b, \quad \text{sum} \]

\[ i \sin \left( \frac{\Delta w}{2i} \right) = 2\sqrt{ab}, \quad \text{diff} \]

\[ 1 = (a - b)^2, \quad \text{prod}. \quad (4.58) \]

One can now check to see that this is consistent with:

\[ a = \cos^2 \left( \frac{\Delta w}{4i} \right) \]

\[ b = -\sin^2 \left( \frac{\Delta w}{4i} \right) \quad (4.59) \]

These relationships will be used when mapping back to the cylinder.

**Mode Expansions on the Cylinder**

We must now expand Equation 4.54 in terms of modes on the cylinder, which amounts to expanding the integrand \((t - t_2)\) in powers of \(e^w\). To do so, we will note that \(e^w = z\) and then expand the integrand in powers of \(z\). We thus require an expression for \(z\) in terms of \(t\). Recalling the map from Equation 4.1, we solve the relation to obtain: We must now expand Equation 4.54 in terms of modes on the
cylinder, which amounts to expanding the integrand \((t - t_2)\) in powers of \(e^w\). To do so, we will note that \(e^w = z\) and then expand the integrand in powers of \(z\). We thus require an expression for \(z\) in terms of \(t\). Recalling the map from Equation 4.1, we solve the relation to obtain:

\[
z = \frac{(t + a)(t + b)}{t} \implies t = \frac{1}{2} \left( z - (a + b) \pm \sqrt{(z - (a + b))^2 - 4ab} \right).
\]

(4.60)

Of course there are two options, as each point in \(z\) maps to two different values of \(t\). We use our image point reference for the in and out copies to determine which choice is appropriate for which copy. This amounts to fixing the principal branch of the square root in (4.60).

Let us first look at the contours above the twists, which are at large \(z\). Here our relation has the leading order behavior:

\[
t \sim \frac{1}{2} \left( z \pm \sqrt{z^2} \right)
\]

\[
\sim \frac{1}{2} (z \pm z),
\]

(4.61)

where we have chosen the branch where:

\[
\sqrt{z^2} = z.
\]

(4.62)

Since we have labeled as Copy 1 the copy which here maps to large \(t\), we notice that this corresponds to the choice \(+\) in the inverse map. Similarly, the choice \(-\) gives small \(t\), as appropriate for Copy 2. Keeping these same choices for the same copies, we now look at the contours below the twists. Here \(z\) is small, so we have:

\[
t \sim \frac{1}{2} \left( -(a + b) \pm \sqrt{(a + b)^2 - 4ab} \right)
\]

\[
\sim \frac{1}{2} \left( -(a + b) \pm \sqrt{(b - a)^2} \right).
\]

(4.63)
We want the choice $+$ to correspond to Copy 1, which maps to $t = -a$. This means we are on the branch defined by:

$$\sqrt{(b-a)^2} = b - a. \quad (4.64)$$

One can then check that the choice $-$ now leads to an image near $t = -b$, as expected.

With these technicalities out of the way, we now expand the contours in terms of modes on the cylinder. This expansion will involve large $z$ for post-twist contours and small $z$ for pre-twist contours, as well as a choice of $+$ for Copy 1 and $-$ for Copy 2. We thus have four possibilities. However, we will soon find that it is more convenient to write (4.60) in a different form:

$$t = \frac{1}{2} \left( z - (a + b) \pm \sqrt{(z - (a + b))^2 - 4ab} \right)$$

$$= \frac{1}{2} \left( z - (a + b) \pm \sqrt{(z - (a + b))^2 - (2t_2)^2} \right)$$

$$= \frac{1}{2} \left( z - (a + b) \pm \left[ (z - (a + b) - 2t_2)^{\frac{1}{2}} (z - (a + b) + 2t_2)^{\frac{1}{2}} \right] \right). \quad (4.65)$$

From here, we can make a significant simplification by rotating to Minkowski signature. In such signature, we have the handy relations:

$$a = \cos^2(\theta)$$

$$b = -\sin^2(\theta), \quad (4.66)$$

where the angle $\theta$ is defined by:

$$\theta \equiv \frac{\Delta w}{4i}. \quad (4.67)$$

With these relations and the fact that $t_2 = \sqrt{ab}$, we find:

$$a + b = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

$$t_2 = i \cos(\theta) \sin(\theta) = \frac{i}{2} \sin(2\theta). \quad (4.68)$$
We then have:

$$-(a + b) \pm 2t_2 = -\cos(2\theta) \pm i\sin(2\theta) = -e^{\mp 2i\theta} = -e^{\mp \frac{\Delta w}{2}} \quad (4.69)$$

From here, (4.65) becomes:

$$t = \frac{1}{2} \left( z - (a + b) \pm \left[ \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z - e^{-\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \right] \right). \quad (4.70)$$

**Copy 1 Above the Twists**

For the contour on Copy 1 above the twists, we make the choice +. We then have:

$$t = \frac{1}{2} \left( z - (a + b) + \left[ \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z - e^{-\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \right] \right) = \frac{1}{2} \left[ z - (a + b) + \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z - e^{-\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \right] \equiv \frac{1}{2} [z - (a + b) + B_f], \quad (4.71)$$

Where $B_f$ contains the two binomials we wish to expand. Each of these binomials takes the form:

$$\left( z + x \right)^{\frac{1}{2}}. \quad (4.72)$$

For large $z$, the expansion is:

$$\left( z + x \right)^{\frac{1}{2}} = z^{\frac{1}{2}} \left( 1 + \frac{x}{z} \right)^{\frac{1}{2}} = z^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{1}{2} C_p x^p z^{-p} = \sum_{p=0}^{\infty} \frac{1}{2} C_p x^p z^{\frac{1}{2}-p}. \quad (4.73)$$

Substituting in each value of $x$ and combining the two expansions, we find:

$$B_f = \sum_{p,q=0}^{\infty} \frac{1}{2} C_p \frac{1}{2} C_q \left( -e^{\frac{\Delta w}{2}} \right)^p \left( -e^{-\frac{\Delta w}{2}} \right)^q z^{1-p-q} = \sum_{p,q=0}^{\infty} \frac{1}{2} C_p \frac{1}{2} C_q (-1)^{p+q} e^{\frac{\Delta w}{2} (p-q)} z^{1-p-q}. \quad (4.74)$$
At this point, the symmetry of the $p$ and $q$ sums can be used to make a nice simplification. For $p = q$, we have:

$$e^{\Delta w (p-q)} = 1 = \cos \left( \frac{\Delta w}{2i}(p-q) \right), \quad (4.75)$$

while for $p \neq q$ we have:

$$e^{\Delta w (p-q)} + e^{\Delta w (q-p)} = 2 \cos \left( \frac{\Delta w}{2i}(p-q) \right)$$

$$= \cos \left( \frac{\Delta w}{2i}(p-q) \right) + \cos \left( \frac{\Delta w}{2i}(q-p) \right). \quad (4.76)$$

We can thus replace the exponential in (4.74) with a simple cosine. We then have:

$$B_f = \sum_{p,q=0}^{\infty} (-1)^n z^{1-n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos(ik\Delta w), \quad (4.77)$$

We will now relabel the summation indices in (4.77). A natural choice is to have one index that directly gives the power of $z$. Let us thus choose indices:

$$n \equiv p + q, \quad 2k \equiv p - q, \quad (4.78)$$

where we have chosen the value $2k$ because the parameter $p - q$ has a step size of 2 for any fixed value of $p + q$. Now our index $k$ will have a step size of 1, running over integers or half-integers depending on whether $n$ is odd or even. We now rewrite (4.77) as:

$$B_f = \sum_{n=0}^{\infty} (-1)^n z^{1-n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos(ik\Delta w), \quad (4.79)$$

where we have used the fact that $\cos(x) = \cos(-x)$ to move the factor of $i$ in the cosine’s argument into the numerator. One can in principal perform the $k$ sum, but the result is messy and involves expressions that give ratios of infinities for odd $n$. 

67
Plugging (4.79) into (4.87), we find:

\[
t = \frac{1}{2} \left[ z - (a + b) + \sum_{n=0}^{\infty} (-1)^n z^{1-n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos(ik\Delta w) \right].
\] (4.80)

Now our integrand is actually \( t - t_2 \). Furthermore, each power \( z^n \) is also \( e^{nw} \), which integrated around the cylinder gives a mode of order \( n \). We thus find:

\[
\oint_{\sigma=0, r>\tau_2} G_{A,1}^{-(1)}(w)(t - t_2) \, dw = G_{A,1}^{-(1)} - (t_2 + a + b)G_{A,0}^{-(1)}
\]

\[
+ \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \right) G_{A,1-n}^{-(1)}.
\] (4.81)

We can simplify this expression via:

\[
t_2 + a + b = i \cos(\theta) \sin(\theta) + \cos^2(\theta) - \sin^2(\theta)
\]

\[
= i \sin(2\theta) + \cos(2\theta)
\]

\[
= e^{2i\theta}
\]

\[
= e^{\frac{\Delta w}{2}}.
\] (4.82)

We then have:

\[
\oint_{\sigma=0, r>\tau_2} G_{A,1}^{-(1)}(w)(t - t_2) \, dw = G_{A,1}^{-(1)} - e^{\frac{\Delta w}{2}}G_{A,0}^{-(1)}
\]

\[
+ \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \right) G_{A,1-n}^{-(1)}.
\] (4.83)

**Copy 2 Above the Twists**

For the contour on Copy 2 above the twists, we make the choice \( - \). We then have:

\[
t = \frac{1}{2} \left[ z - (a + b) - \left[ \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z - e^{-\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \right] \right]
\]

\[
= \frac{1}{2} \left[ z - (a + b) - \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z - e^{-\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \right]
\]

\[
\equiv \frac{1}{2} \left[ z - (a + b) - B_f \right],
\] (4.84)
where $B_f$ is expanded exactly as before. The result of this expansion was presented in (4.79). Plugging this result into (4.84) yields:

$$
t = \frac{1}{2} \left[ z - (a + b) - \sum_{n=0}^{\infty} (-1)^n z^{1-n} \sum_{k=\frac{-n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos(ik\Delta w) \right]. \quad (4.85)
$$

Now our integrand is still $t - t_2$. We thus find:

$$
\oint_{\sigma=0, r > r_2} G_A^{(2)}(w)(t - t_2) \, dw = G_A^{(1)} - e^{\frac{\Delta w}{2}} G_A^{(1)}
$$

$$
- \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n \sum_{k=\frac{-n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos(ik\Delta w) G_A^{(1)}
$$

Copy 1 Below the Twists

For the contour on Copy 1 below the twists, we again make the choice +. We then have:

$$
t = \frac{1}{2} \left[ z - (a + b) + \left( z - e^{\Delta w} \right)^{\frac{1}{2}} \left( z - e^{-\Delta w} \right)^{\frac{1}{2}} \right]
$$

$$
= \frac{1}{2} \left[ z - (a + b) + \left( z - e^{\Delta w} \right)^{\frac{1}{2}} \left( z - e^{-\Delta w} \right)^{\frac{1}{2}} \right]
$$

$$
\equiv \frac{1}{2} \left[ z - \cos \left( \frac{\Delta w}{2t} \right) + B_i \right]. \quad (4.87)
$$

Where $B_i$ contains the two binomials we wish to expand. Each of these binomials takes the form:

$$
(z + x)^{\frac{1}{2}}. \quad (4.88)
$$

For small $z$, the expansion is:

$$
(z + x)^{\frac{1}{2}} = x^{\frac{1}{2}} \left( 1 + \frac{z}{x} \right)^{\frac{1}{2}}
$$

$$
= x^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{1}{2} C_p x^{-p} z^p
$$

$$
= \sum_{p=0}^{\infty} \frac{1}{2} C_p x^{\frac{1}{2}-p} z^p. \quad (4.89)
$$
Substituting in each value of \( x \) and combining the two expansions, we find:

\[
B_i = \sum_{p,q=0}^{\infty} \frac{1}{2} C_p \frac{1}{2} C_q \left( -e^{-\frac{\Delta w}{2i}} \right)^{\frac{1}{2}-p} \left( -e^{\frac{\Delta w}{2i}} \right)^{\frac{1}{2}-q} z^{p+q}
\]

\[
= \sum_{p,q=0}^{\infty} \frac{1}{2} C_p \frac{1}{2} C_q \left( -e^{-\frac{\Delta w}{2i}} \right)^{\frac{1}{2}-p} (1-1^{-p-q} e^{\frac{\Delta w}{2i}(p-q)} z^{p+q})
\]

\[
= \sum_{p,q=0}^{\infty} \frac{1}{2} C_p \frac{1}{2} C_q \left( -e^{\frac{\Delta w}{2i}} \right)^{\frac{1}{2}-p} (1-1^{-p-q} \cos \left( \frac{\Delta w}{2i} (p-q) \right) z^{p+q}). \tag{4.90}
\]

We now perform a change of indices:

\[
n \equiv p + q, \quad 2k \equiv p - q,
\]

which gives:

\[
B_i = \sum_{n=0}^{\infty} (1-1^{-n} z^n) \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos \left( ik \Delta w \right). \tag{4.91}
\]

The \( k \) sum is the same one we saw for the post-twist contours, though the corresponding powers of \(-1\) and \( z \) have been exchanged. We thus find:

\[
B_i = -1 + z \cos \left( \frac{\Delta w}{2i} \right) + \sum_{n=2}^{\infty} (1-1^{-n} z^n) \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos \left( ik \Delta w \right), \tag{4.92}
\]

and then:

\[
t = \frac{1}{2} \left[ -\left( 1 + \cos \left( \frac{\Delta w}{2i} \right) \right) + \left( 1 + \cos \left( \frac{\Delta w}{2i} \right) \right) z \right.
\]

\[
+ \sum_{n=2}^{\infty} (1-1^{-n} z^n) \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos \left( ik \Delta w \right) \right]. \tag{4.93}
\]

Subtracting \( t_{2} \) and noting that the contour turns powers of \( z \) into supercharge modes, we find:

\[
\oint_{\sigma=2\pi}^{\sigma=0, \tau=r_{1}} G_{A_{0}}^{-1} (w) (t - t_{2}) \, dw = \frac{1}{2} \left[ -\left( 1 + t_{2} + \cos \left( \frac{\Delta w}{2i} \right) \right) G_{A_{0}}^{-1}
\right.
\]

\[
+ \left( 1 + \cos \left( \frac{\Delta w}{2i} \right) \right) G_{A_{0}}^{(1)}
\]

\[
+ \sum_{n=2}^{\infty} (1-1^{-n} z^n) \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos \left( ik \Delta w \right) G_{A_{n}}^{-1} \right]. \tag{4.94}
\]
Copy 2 Below the Twists

For the contour on Copy 1 below the twists, we again make the choice \(+\). We then have:

\[
t = \frac{1}{2} \left( z - (a + b) - \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z + e^{\frac{-\Delta w}{2}} \right)^{\frac{1}{2}} \right)
\]

\[
= \frac{1}{2} \left[ z - (a + b) - \left( z - e^{\frac{\Delta w}{2}} \right)^{\frac{1}{2}} \left( z - e^{\frac{-\Delta w}{2}} \right)^{\frac{1}{2}} \right]
\]

\[
\equiv \frac{1}{2} \left[ z - \cos \left( \frac{\Delta w}{2i} \right) - B_i \right], \quad (4.96)
\]

Where \(B_i\) is expanded in the same way as before (small \(z\)). We thus have:

\[
t = \frac{1}{2} \left[ \left( 1 - \cos \left( \frac{\Delta w}{2i} \right) \right) + \left( 1 - \cos \left( \frac{\Delta w}{2i} \right) \right) \right] z
\]

\[
- \sum_{n=2}^{\infty} (-1)^{1-n} \sum_{k=-\left\lceil \frac{n}{2} \right\rceil}^{\left\lceil \frac{n}{2} \right\rceil} \frac{1}{2} C_n^{\frac{n}{2}+k} \frac{1}{2} C_n^{\frac{n}{2}-k} \cos \left( ik \Delta w \right) \right]. \quad (4.97)
\]

Subtracting \(t_2\) and noting that the contour turns powers of \(z\) into supercharge modes, we find:

\[
\oint_{\sigma=0, \tau<\tau_1} G_{A}^{-2}(w)(t - t_2) \, dw = \frac{1}{2} \left[ \left( 1 - t_2 - \cos \left( \frac{\Delta w}{2i} \right) \right) G_{A,0}^{-2}
\right.
\]

\[
+ \left( 1 - \cos \left( \frac{\Delta w}{2i} \right) \right) G_{A,1}^{-2}
\]

\[
- \sum_{n=2}^{\infty} (-1)^{1-n} \sum_{k=-\left\lceil \frac{n}{2} \right\rceil}^{\left\lceil \frac{n}{2} \right\rceil} \frac{1}{2} C_n^{\frac{n}{2}+k} \frac{1}{2} C_n^{\frac{n}{2}-k} \cos \left( ik \Delta w \right) G_{A,n}^{-2} \right]. \quad (4.98)
\]

4.5.3 The Second Supercharge Integral

Using the results (4.46) and (4.43), we find that after all coordinate shifts and spectral flows the second integral becomes:

\[
\mathcal{I}_2 \rightarrow C \oint_{t_1} \oint_{t_1} G_B^{-}(t)(t + a)(t + b)(t - t_1)^{-1}(t - t_2)^{-1}
\]

\[
\times G_A^{-}(t')(t' + a)(t' + b)(t' - t_1)^{-1}(t' - t_2)^{-1} \, dt' \, dt. \quad (4.99)
\]
Table 4.1: A table of sources of overall sign changes for each of the sixteen combinations of contour destinations (left). An $\dot{A}$ indicates that the $\dot{A}$ contour has changed direction, a $\dot{B}$ indicates that the $\dot{B}$ contour has changed direction, an $S$ indicates that the contours must be written with their orders swapped because the $\dot{B}$ contour is before the twists while the $\dot{A}$ contour is after the twists, and an $S^*$ indicates that the contours must be written with their orders swapped because they are both wrapping the same finite point. Each of these possibilities brings its own minus sign. Thus an odd number of symbols indicates an overall sign change while an even number of symbols indicates no overall sign change (right).

We now perform the same method as before. As expected much of it will be similar to the calculations performed for $\mathcal{I}_1$.

### 4.5.4 Deforming and Inverting

Looking at (4.99), we basically have two copies of $\mathcal{I}_1$, one outside the other. Since we already know how each copy behaves when we deform it and invert the maps and spectral flows, we can easily combine the two behaviors together. The trickiness comes in the fact that there are sixteen terms, with four possible destinations for each of the two contours. Any contours around finite points change direction, and any time the $\dot{B}$ contour wraps an initial state image while the $\dot{A}$ contour wraps a final state image or both contours wrap around the same finite point, we must swap their order. We analyze all of these possibilities in Table 4.1.

Noting the effects of Table 4.1, we have:
where each integral symbol is used to represent two integrals and the function in the integrand is always:

$$f(x) \equiv (x + a)(x + b)(x - t_1)^{-1}. \quad (4.101)$$

We must now take (4.100) and invert the spectral flows and coordinate shifts. We know the factors this process brings into the integrands — they amount to changing the function $f$:

$$f(x) \to x - t_2. \quad (4.102)$$

Now let us look closely at the origin. The map $t \to z$ at small $t$ reverses the direction of any contours there, but when two contours wrap $t = 0$ it also swaps their order once more. Thus whenever one or both contours wrap $t = 0$ we get either one or three sign changes, for one overall sign change. Thus, on the cylinder we have:

$$I_2 = \frac{1}{4t_2^2} \left[ \oint_{t > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \oint_{t > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma^+_2(w_2)\sigma^+_2(w_1) \right.
\left. + \oint_{t > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \oint_{t > \tau_2} G_A^{-2}(w)(t - t_2) \, dw \sigma^+_2(w_2)\sigma^+_2(w_1) \right.
\left. - \oint_{t < \tau_1} G_B^{-1}(w)(t - t_2) \, dw \sigma^+_2(w_2)\sigma^+_2(w_1) \oint_{t < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \right.
\left. - \oint_{t < \tau_1} G_B^{-1}(w)(t - t_2) \, dw \sigma^+_2(w_2)\sigma^+_2(w_1) \oint_{t < \tau_1} G_A^{-2}(w)(t - t_2) \, dw \right.
\left. + \oint_{t > \tau_2} G_B^{-2}(w)(t - t_2) \, dw \oint_{t > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma^+_2(w_2)\sigma^+_2(w_1) \right. \right.$$
This result is now written entirely in terms of contours we have already solved, though it is tediously long. Let’s simplify by grouping terms based on where the twists are.

For terms where the twists come last, we have:

\[
T_l = \oint_{\tau > \tau_2} G_B^{-} \frac{1}{(w-t_2)} \, dw \oint_{\tau < \tau_1} G_A^{-} \frac{1}{(w-t_2)} \, dw \sigma_2^+(w_2) \sigma_2^+(w_1)
+ \oint_{\tau > \tau_2} G_B^{-} \frac{1}{(w-t_2)} \, dw \oint_{\tau < \tau_1} G_A^{-} \frac{1}{(w-t_2)} \, dw \sigma_2^+(w_2) \sigma_2^+(w_1)
+ \oint_{\tau > \tau_2} G_B^{-} \frac{1}{(w-t_2)} \, dw \oint_{\tau > \tau_2} G_A^{-} \frac{1}{(w-t_2)} \, dw \sigma_2^+(w_2) \sigma_2^+(w_1)
+ \oint_{\tau > \tau_2} G_B^{-} \frac{1}{(w-t_2)} \, dw \oint_{\tau > \tau_2} G_A^{-} \frac{1}{(w-t_2)} \, dw \sigma_2^+(w_2) \sigma_2^+(w_1)
+ \oint_{\tau > \tau_2} G_B^{-} \frac{1}{(w-t_2)} \, dw \oint_{\tau > \tau_2} G_A^{-} \frac{1}{(w-t_2)} \, dw \sigma_2^+(w_2) \sigma_2^+(w_1)
= \oint_{\tau > \tau_2} G_B^{-} \frac{1}{(w-t_2)} \, dw \oint_{\tau > \tau_2} G_A^{-} \frac{1}{(w-t_2)} \, dw \sigma_2^+(w_2) \sigma_2^+(w_1). \quad (4.104)
\]
For terms where the twists come first, we have:

\[
T_i = \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_A^{-2}(w)(t - t_2) \, dw
+ \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_B^{-2}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw
- \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw
- \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-2}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-2}(w)(t - t_2) \, dw.
\]

Here we anticommute the operators in the first two lines. This gives:

\[
T_i = -\sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-2}(w)(t - t_2) \, dw
- \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-2}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw
- \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw
- \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-2}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-2}(w)(t - t_2) \, dw
= -\sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw. \tag{4.106}
\]

There are eight terms where the twists are in the middle. These can be simplified as:

\[
T_m = -\int_{\tau > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw
- \int_{\tau > \tau_2} G_B^{-2}(w)(t - t_2) \, dw \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw
- \int_{\tau > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-2}(w)(t - t_2) \, dw
- \int_{\tau > \tau_2} G_B^{-2}(w)(t - t_2) \, dw \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_A^{-2}(w)(t - t_2) \, dw
+ \int_{\tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw
+ \int_{\tau > \tau_2} G_A^{-2}(w)(t - t_2) \, dw \sigma_2^+(w_2)\sigma_2^+(w_1) \int_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw
\]

75
\[ + \oint_{\tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_B^{-2}(w)(t - t_2) \, dw \]
\[ + \oint_{\tau > \tau_2} G_A^{-2}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_B^{-2}(w)(t - t_2) \, dw \]
\[ = - \oint_{\tau > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \]
\[ + \oint_{\tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw. \tag{4.107} \]

Pulling all of this together, we find that (4.103) reduces to:

\[
\mathcal{I}_2 = \frac{1}{4t_2^2} \left[ \oint_{\tau > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \oint_{\tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \right. \\
- \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \oint_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw \\
- \oint_{\tau > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_A^{-1}(w)(t - t_2) \, dw \\
+ \oint_{\tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\tau < \tau_1} G_B^{-1}(w)(t - t_2) \, dw \right] \tag{4.108} \]

### 4.5.5 Writing the Full Supercharge Result

From (4.42) and (4.43), we find that the full result can be written as:

\[
\hat{O}_B \hat{O}_A = G_{B,0} \mathcal{I}_1 + \mathcal{I}_1 G_{B,0}^{-1} - \mathcal{I}_2. \tag{4.109} \]

Plugging in the results (4.54) and (4.108), we have:

\[
\hat{O}_{B,A} = -\frac{1}{2t_2} G_{B,0}^{-1} \left( \oint_{\sigma = 0, \tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \right. \\
- \sigma_2^+(w_2) \sigma_2^+(w) \oint_{\sigma = 0, \tau < \tau_2} G_A^{-1}(w)(t - t_2) \, dw \\
- \frac{1}{2t_2} \left( \oint_{\sigma = 0, \tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \right) G_{B,0}^{-1} \\
- \frac{1}{4t_2^2} \left[ \oint_{\tau > \tau_2} G_B^{-1}(w)(t - t_2) \, dw \oint_{\tau > \tau_2} G_A^{-1}(w)(t - t_2) \, dw \sigma_2^+(w_2) \sigma_2^+(w) \right. \\
\]

76
\[- \sigma^+_2(w_2) \sigma^+_2(w_1) \oint_{\tau < \tau_1} G^-_A(w)(t - t_2) \, dw \oint_{\tau < \tau_1} G^-_B(w)(t - t_2) \, dw \]

\[- \oint_{\tau > \tau_2} G^-_B(w)(t - t_2) \, dw \sigma^+_2(w_2) \sigma^+_2(w_1) \oint_{\tau < \tau_1} G^-_A(w)(t - t_2) \, dw \]

\[+ \oint_{\tau > \tau_2} G^-_A(w)(t - t_2) \, dw \sigma^+_2(w_2) \sigma^+_2(w_1) \oint_{\tau < \tau_1} G^-_B(w)(t - t_2) \, dw \]

\[= - \frac{1}{4t^2} \left( 2t_2 G^-_{B,0} + I_B^> \right) \left( I_A^> \dot{\sigma} - \dot{\sigma} I_A^< \right) + \left( I_A^> \dot{\sigma} - \dot{\sigma} I_A^< \right) \left( 2t_2 G^-_{B,0} + I_B^< \right), \]

(4.110)

where we have introduced the shorthand notation:

\[\dot{O}_{B,A} \equiv \dot{O}_B(w_2) \dot{O}_A(w_1)\]

\[\dot{\sigma} \equiv \sigma^+_2(w_2) \sigma^+_2(w_1)\]

\[I_C^> \equiv \oint_{\tau > \tau_1} G^-_C(w)(t - t_2) \, dw\]

\[I_C^< \equiv \oint_{\tau < \tau_1} G^-_C(w)(t - t_2) \, dw.\]

(4.111)

We now recall the value of these cylinder integrals for each copy both above and below the twists, which are found in equations (4.83), (4.86), (4.95), and (4.98). Unfortunately, it turns out that these “simplified” expressions are not terribly useful. It is better to not remove the \(n = 0, 1\) terms in the sum. Leaving them in, we have:

\[\oint_{\sigma=0, \tau > \tau_2} G^{- (1)}_C(w)(t - t_2) \, dw = \frac{1}{2} \left[ G^{- (1)}_{C,1} - \left( 2t_2 + \cos \left( \frac{\Delta w}{2i} \right) \right) G^{- (1)}_{C,0} \right.\]

\[\left. + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{i}{2} C_{\frac{n}{2} + k} \frac{i}{2} C_{\frac{n}{2} - k} \cos (ik \Delta w) G^{- (1)}_{C,1-n} \right] \]

\[\oint_{\sigma=0, \tau > \tau_2} G^{- (2)}_C(w)(t - t_2) \, dw = \frac{1}{2} \left[ G^{- (2)}_{C,1} - \left( 2t_2 + \cos \left( \frac{\Delta w}{2i} \right) \right) G^{- (2)}_{C,0} \right.\]

\[\left. - \sum_{n=0}^{\infty} (-1)^n \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{i}{2} C_{\frac{n}{2} + k} \frac{i}{2} C_{\frac{n}{2} - k} \cos (ik \Delta w) G^{- (2)}_{C,1-n} \right] \]
\[
\int_{\sigma=0}^{\sigma=2\pi} G_{C}^{-1}(w)(t - t_2) \, dw = \frac{1}{2} \left[ G_{C,1}^{-1} - \left( \frac{2t_2 + \cos \left( \frac{\Delta w}{2i} \right) }{2} \right) G_{C,0}^{-1} \right]
+ \sum_{n=0}^{\infty} \left( -1 \right)^{1-n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2} + k} \bar{C}_{\frac{n}{2} - k} \cos (ik\Delta w) \bar{G}_{C,n}^{-1}
\]

\[
\int_{\sigma=0}^{\sigma=2\pi} G_{C}^{-1}(w)(t - t_2) \, dw = \frac{1}{2} \left[ G_{C,1}^{-2} - \left( \frac{2t_2 + \cos \left( \frac{\Delta w}{2i} \right)}{2} \right) G_{C,0}^{-2} \right]
- \sum_{n=0}^{\infty} \left( -1 \right)^{1-n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2} + k} \bar{C}_{\frac{n}{2} - k} \cos (ik\Delta w) \bar{G}_{C,n}^{-2}
\]

This allows us to make the following simplifications:

\[
2t_2 + \cos \left( \frac{\Delta w}{2i} \right) = e^{\Delta w/2i}
\]

\[
G_{C}^{\alpha(1)} + G_{C}^{\alpha(2)} = G_{C}^{\alpha}
\]

\[
G_{C}^{\alpha(1)} - G_{C}^{\alpha(2)} \equiv \bar{G}_{C}^{\alpha}.
\]

With this notation, we can now write our integrals as:

\[
I_{C}^{>} = \frac{1}{2} \left( G_{C,1}^{-1} - e^{\Delta w/2i} G_{C,0}^{-1} + \sum_{n=0}^{\infty} \left( -1 \right)^{n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2} + k} \bar{C}_{\frac{n}{2} - k} \cos (ik\Delta w) \bar{G}_{C,1-n}^{-1} \right)
\]

\[
I_{C}^{<} = \frac{1}{2} \left( G_{C,1}^{-1} - e^{\Delta w/2i} G_{C,0}^{-1} + \sum_{n=0}^{\infty} \left( -1 \right)^{1-n} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{2} C_{\frac{n}{2} + k} \bar{C}_{\frac{n}{2} - k} \cos (ik\Delta w) \bar{G}_{C,n}^{-1} \right)
\]

We will also note here that:

\[
2t_2 G_{B,0}^{-1} = -i \sin \left( \frac{\Delta w}{2i} \right) G_{B,0}^{-1},
\]

as well as:

\[
- i \sin(x) + \frac{1}{2} e^{ix} = \frac{1}{2} e^{-ix}.
\]
We thus find that (4.110) becomes:

\[
\hat{O}_{BA} = -\frac{1}{16t^2} \left[ \left( G_{-B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{-B,1-n}^- \right) \hat{\sigma} \right. \\
\times \left( G_{-A,1}^+ - e^{\frac{\Delta w}{2}} G_{A,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{-A,1-n}^- \right) \hat{\sigma} \\
- \frac{1}{2} \left( G_{-B,1}^+ - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{-B,1-n}^- \right) \hat{\sigma} \\
\times \left( G_{-A,1}^+ - e^{\frac{\Delta w}{2}} G_{A,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{-A,1-n}^- \right) \hat{\sigma} \\
+ \left( G_{A,1}^- - e^{\frac{\Delta w}{2}} G_{A,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{A,1-n}^- \right) \hat{\sigma} \\
\times \left( G_{B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{B,1-n}^- \right) \hat{\sigma} \\
- \hat{\sigma} \left( G_{A,1}^- - e^{\frac{\Delta w}{2}} G_{A,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{A,1-n}^- \right) \\
\times \left( G_{B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w) \tilde{G}_{B,1-n}^- \right) \right]
\]

(4.116)

For future reference, we will define the coefficient:

\[
C_n \equiv (-1)^n \sum_{k=-n}^{n} \frac{1}{2} C_{\frac{n}{2}+k} \frac{1}{2} C_{\frac{n}{2}-k} \cos (ik\Delta w). \tag{4.117}
\]

(4.116) then becomes:
\[ \hat{O}_{B\dot{A}} = -\frac{1}{16t_2^2} \left[ \left( G_{B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} C_n \tilde{G}_{B,1-n}^- \right) \left( G_{\dot{A},1}^- - e^{-\frac{\Delta w}{2}} G_{\dot{A},0}^- + \sum_{n=0}^{\infty} C_n \tilde{G}_{\dot{A},1-n}^- \right) \hat{\sigma} - \left( G_{B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} C_n \tilde{G}_{B,1-n}^- \right) \hat{\sigma} \left( G_{\dot{A},1}^- - e^{-\frac{\Delta w}{2}} G_{\dot{A},0}^- - \sum_{n=0}^{\infty} C_n \tilde{G}_{\dot{A},n}^- \right) \right] \]

\[ + \left( G_{\dot{A},1}^- - e^{-\frac{\Delta w}{2}} G_{\dot{A},0}^- - \sum_{n=0}^{\infty} C_n \tilde{G}_{\dot{A},1-n}^- \right) \hat{\sigma} \left( G_{B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- + \sum_{n=0}^{\infty} C_n \tilde{G}_{B,n}^- \right) \]

\[ - \hat{\sigma} \left( G_{\dot{A},1}^- - e^{-\frac{\Delta w}{2}} G_{\dot{A},0}^- - \sum_{n=0}^{\infty} C_n \tilde{G}_{\dot{A},n}^- \right) \left( G_{B,1}^- - e^{-\frac{\Delta w}{2}} G_{B,0}^- - \sum_{n=0}^{\infty} C_n \tilde{G}_{B,n}^- \right) \] (4.118)

We now have the supercharges expressed entirely in terms of modes acting before and after the twist conjunction.
Looking back at the results from Chapters 3 and 4, the analytic expressions can be difficult to parse. This is particularly true of the results from our one-loop calculation, which are only expressed in terms of nested finite sums. To alleviate this mess, we will work with an approximation termed the 'continuum limit,’ where the excitation level is taken to be large:

\[ \alpha_{-m}, d_{-n} \quad m, n \gg 1. \] (5.1)

The continuum limit gives us several simplifications. First, we can completely ignore fermion zero mode behavior in this limit. This removes all of the dependency on the charges of the particular R vacuum we chose to build upon. It also reduces our parameters to a single linearly-independent Bogoliubov coefficient and a single linearly-independent transition amplitude (we will also look at the coeffitient \( C_n \) from our treatment of the supercharge contours in the one-loop calculation). Second, a variety of methods allowed us to obtain simpler analytic expressions that approximate the exact results in this limit. These simpler expressions are much easier to analyze, especially in the one-loop case.

Now before we proceed with out continuum limit analysis, it is prudent to first ask whether this limit has any bearing on the physics of interest – the regime of black
hole formation and evaporation. This is an important question, and fortunately the answer is straightforward. Since black hole temperature scales inversely with its energy, astronomical black holes are extremely cold.\textsuperscript{10} Any infalling quanta will have energies much larger than this scale, and so in the CFT will appear as wave packets with central energy much larger than the characteristic energy scale of the CFT itself. This tells us that the continuum limit should approximate astronomical black hole behavior quite well.

5.1 First-Order Continuum Limit

We will now turn to our results from Chapter 3 and see what simplifications we can obtain. Since we have only one linearly-independent Bogoliubov coefficient and transition amplitude, we will perform our analysis solely with the bosons.

To refresh our memories, the complicated bosonic results from Chapter 3 are:

\[
\gamma^B_{mn} = -\frac{(-a)^{m+n}}{\pi^2} \sin \left[ \frac{m M \pi}{M + N} \right] \sin \left[ \frac{n M \pi}{M + N} \right] \frac{M N}{(M + N)^2} \times \frac{1}{(m+n)} \frac{\Gamma[m + N]}{\Gamma[m]} \frac{\Gamma[n + M]}{\Gamma[n]} (5.2)
\]

\[
\left[ f^B_{mp} \right]_{\frac{pM}{M+N} + \frac{mN}{N}} = \frac{(-1)^m \sin(\pi \frac{pM}{M+N})}{\pi(M + N)} \frac{(-a)^{p - \frac{m(M+N)}{M}}}{\frac{m(M+N)}{M}} \times \frac{\Gamma[m + N]}{\Gamma[m]} \frac{\Gamma[p + N]}{\Gamma[p]} (5.3)
\]

where we have ignored the already-simple case for the transition amplitude as well as the analogous copy 2 amplitude. We now ask how these expressions can be simplified for the case where the energies involved are much larger than the characteristic energy

\textsuperscript{10}As a point of reference, a solar mass Schwarzschild black hole has a temperature on the order of $10^{-8} K$. 

82
of the cylinder. To do this, we will first translate into notation where the limit of interest is easier to express. We will then simplify the above expressions primarily through \( \Gamma \) function approximations.

### 5.1.1 The Bogoliubov Coefficient

Rather than work with mode numbers, let us express the above results directly in terms of energy. For the Bogoliubov coefficient, this amounts to:

\[
s \equiv \frac{m}{M + N}, \quad s' = \frac{n}{M + N}.
\]

(5.4)

We will also substitute the \( a \) dependence for \( z_0 \) dependence via Equation 3.7. This gives:

\[
\gamma^{B}_{ss'} = \frac{\mu_s \mu_{s'}}{s + s'} \frac{MN}{(M + N)^3} \left( \frac{(M + N)^{M+N}}{M^MN^N} \right)^{s+s'} \frac{\Gamma[Ms]\Gamma[Ns] \Gamma[Ms']\Gamma[Ns']}{\Gamma[(M + N)s] \Gamma[(M + N)s']},
\]

(5.5)

where we have introduced the shorthand:

\[
\mu_s \equiv 1 - e^{2\pi i Ms} = 1 - e^{-2\pi i Ns}.
\]

(5.6)

In this notation, the continuum limit is simply \( s, s' \gg 1 \). We will now use a few simple properties of the \( \Gamma \) function to condense this expression.

The first \( \Gamma \) function property that we use is the relation, for positive integer \( K \):

\[
\Gamma[Ks] = (2\pi)^{\frac{1}{2}K}K^{Ks-\frac{1}{2}}\prod_{k=0}^{K-1} \Gamma[s + k/K].
\]

(5.7)

Aside from \( k = 0 \), we will now make an approximation for each of the \( \Gamma \) functions in the product on the right-hand side. Since \( k/K < 1 \) and \( s \gg 1 \), we have:

\[
\Gamma[s + k/K] \approx s^k \Gamma[s].
\]

(5.8)
From here, Equation 5.7 gives:

\[
\Gamma[Ks] \approx (2\pi)^{1-K} K^{Ks - \frac{1}{2}} (\Gamma[s])^{K \sum_{k=1}^{K} \frac{1}{s}} \\
\approx (2\pi)^{\frac{1}{2} - K} K^{Ks - \frac{1}{2}} (\Gamma[s])^{K \frac{K-1}{s^2}}.
\]  

(5.9)

We can then approximate each \( \Gamma \) function block as:

\[
\frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M + N)s]} \approx \frac{(2\pi)^{1-M-N} (M^M N^N)^s M^{-\frac{1}{2}} N^{-\frac{1}{2}} (\Gamma[s])^{M+N} s^{\frac{M+N-2}{2}}}{(2\pi)^{\frac{1}{2} - M-N} ((M + N)^{M+N})^s (M + N)^{-\frac{1}{2}}} (\Gamma[s])^{M+N} s^{\frac{M+N-1}{2}} \\
\approx \sqrt{2\pi} \left( \frac{M^M N^N}{(M + N)^{M+N}} \right)^s \sqrt{\frac{M + N}{MN}} \frac{1}{\sqrt{s}} \\
\frac{\Gamma[Ms']\Gamma[Ns']}{\Gamma[(M + N)s']} \approx \sqrt{2\pi} \left( \frac{M^M N^N}{(M + N)^{M+N}} \right)^{s'} \sqrt{\frac{M + N}{MN}} \frac{1}{\sqrt{s'}}.
\]  

(5.10)

Substituting these expressions back into Equation 5.5 now produces significant simplifications:

\[
\gamma_{ss'}^B \approx \frac{1}{\sqrt{s + s'}} \frac{1}{s} \mu_{s}s' \mu_{s'} \frac{1}{\sqrt{ss'}} \frac{1}{s + s'}.
\]  

(5.11)

We have thus eliminated all of the \( \Gamma \) functions. In this form, it is now clear that the Bogoliubov coefficient (and thus the deformed vacuum behavior) is dominated by low-energy excitations.

5.1.2 The Transition Amplitude

We again begin by shifting to energy indices:

\[
q \equiv \frac{m}{M}, \quad s \equiv \frac{p}{M + N}.
\]  

(5.12)

Re-introducing the \( z_0 \) dependence directly, we have:

\[
f_{qs,qs'}^{B,(1)} = \frac{i z_0^{s-q} \mu_s}{2\pi} \frac{1}{s - q M + N} \left( \frac{(M + N)^{M+N}}{M^M N^N} \right)^{s-q} \frac{\Gamma[Ms]\Gamma[Ns] \Gamma[(M + N)q]}{\Gamma[(M + N)s] \Gamma[Mq] \Gamma[Nq]}.
\]  

(5.13)
The Γ function blocks are the same ones we encountered before.

\[
\frac{\Gamma[M_s] \Gamma[N_s]}{\Gamma[(M + N)s]} \approx \sqrt{2\pi} \left( \frac{M^MN^N}{(M + N)^{M+N}} \right)^s \sqrt{\frac{M + N}{MN}} \frac{1}{\sqrt{s}}
\]

\[
\frac{\Gamma[(M + N)q]}{\Gamma[Mq] \Gamma[Nq]} \approx \frac{1}{\sqrt{2\pi}} \left( \frac{M^MN^N}{(M + N)^{M+N}} \right)^{-q} \sqrt{\frac{MN}{M + N}} \sqrt{s}.
\]

Equation 5.11 then gives:

\[
f_{qs}^{B,(1)} \approx \frac{i}{2\pi} \frac{1}{M + N} z_0^{s-q} \mu_s \frac{1}{s} \sqrt{\frac{q}{s - q}}.
\]  

(5.15)

The copy 2 transition amplitude can be obtained via \( M \leftrightarrow N \). Note that under this transformation, \( \mu_s \rightarrow -\mu_s \). We thus have:

\[
f_{rs}^{B,(2)} \approx -\frac{i}{2\pi} \frac{1}{M + N} z_0^{s-r} \mu_s \frac{1}{s} \sqrt{\frac{r}{s - r}},
\]

(5.16)

where the energy is now \( r = m/N \).

The important thing to note from this analysis are the inverse factors of \( s - q \) and \( s - r \). These are the changes in energy \( \Delta E \) between the final and initial modes. This tells us that the twist interaction strongly favors leaving an initial excitation near its starting energy even as the excitation level corresponding to that energy changes.

5.2 The One-Loop Case

When turning to our one-loop calculation. Since all winding numbers are 1, there is no special translation between excitation level and excitation energy. While this is convenient, it is inconvenient that we do not have any well-studied functions to exploit. Instead, we applied to numerical analysis to obtain good continuum limit approximations. The process was mostly one of plotting exact values in our good Minkowski coordinates and attempting to find good approximate fits.
For all sums analyzed, the continuum limit is well-approximated by an oscillating function in $\Delta w$. The scaling of the amplitude of this function is easily found for all cases and is in fact identical to the first-order scaling for the functions which have such analogs. However, the behavior of the oscillations themselves is only fully identified for the transition amplitudes and the supercharge coefficients. As before, we will work only with the bosonic content for our Bogoliubov coefficients and transition amplitudes.

5.2.1 Bogoliubov Coefficients

Figure 5.1 shows the magnitude of an exact one-loop bosonic Bogoliubov coefficient at $\Delta w = i\pi/2$ for varying mode indices. We overlay these points with the approximation:

$$|\gamma_{mn}^{B,(1)(1)}| \sim \frac{1}{m + n} \frac{\Gamma[m + 1/2]\Gamma[n + 1/2]}{\Gamma[m + 1]\Gamma[n + 1]} \approx \frac{1}{\sqrt{mn(m + n)}}.$$

Both of these approximations break down for small mode indices, with the latter approximation breaking faster than the former. Remarkably, this continuum limit behavior presents the same scaling as our first-order result.

We now turn to the oscillatory behavior, shown in Figure 5.2. Here made two different choices for $m, n$ while maintaining constant $m + n$. The precise oscillatory behavior is not understood, but it is clear that the number of peaks remains the same between the two plotted cases. This is not in true in general. Rather, it is a feature of the oscillations that the number of peaks is determined solely by the combination $m + n$.

We can now summarize our best understanding of the Bogoliubov coefficient in the continuum limit. When $m, n \gg 1$, we find:
Figure 5.1: Numerical values of $|\gamma_{mn}^{B,(1)(1)}|$ for $\Delta w = i\pi/2$. For $n = m$ (left), a fit of the form $\Gamma^2(m + \frac{1}{2})/[m\Gamma^2(m + 1)]$ is shown. For $n = 12$ (right), a fit of the form $\Gamma(m + \frac{1}{2})/[\Gamma(m + 1)(m + 12)]$ is shown. The coefficient clearly behaves as $\Gamma(m + \frac{1}{2})\Gamma(n + \frac{1}{2})/[\Gamma(m + 1)\Gamma(n + 1)(m + n)]$. Both plots show only $m > 4$, as these simple fits do not work well for very low values of $m$ and $n$. The plot on the right shows only even $m$, as the coefficient vanishes at $\Delta w = i\pi/2$ when $m + n$ is odd.

$$\gamma_{mn}^{B,(1)(1)} \approx \frac{C}{\sqrt{mn(m+n)}} h(m,n\Delta w), \quad (5.18)$$

where $C$ is constant and $h$ is an oscillating function in $\Delta w$ with main frequency determined by $m + n$ and total period $4\pi i$. This oscillatory function is also mirrored about $\Delta w = 2\pi i$. Notably, the energy scaling of the amplitude is identical to the first-order case.

5.2.2 The Transition Amplitudes

Here we are able to do a little better. Figure 5.3 shows the bosonic $(1) \rightarrow (1')$ transition amplitude for various cases. We overlay these points with the approximation:

$$f_{mp}^{B,(1)(1')} \approx \begin{cases} \frac{1}{(m-p)\pi} \sqrt{\frac{m}{p}} \sin \left( (m - p) \frac{\Delta w}{2i} \right) \text{sgn} \left( \frac{\Delta w}{2\pi i} - 1 \right) & m \neq p \\ \frac{\Delta w}{2\pi i} & m = p. \end{cases} \quad (5.19)$$
This time we have precisely pinned down the oscillatory behavior as a simple sine function with frequency set by $\Delta E = p - m$. We again find the same energy scaling as the first-order case as well as the mirroring about $\Delta w = 2\pi i$. We also note that exact energy conservation is once more a special case.\footnote{Note that while the continuum limit approximation for exact energy conservation has a discontinuous derivative at $\Delta w = 0, 2\pi$, these points are always differentiable in the exact expressions.}

Repeating this analysis for the $(2) \rightarrow (1')$ amplitude, we found an interesting relationship:

$$f_{mp}^{B,(1)(1')} + f_{mp}^{B,(2)(1')} = \delta_{mp}. \quad (5.20)$$

This is analogous to the first-order relationship expressed in Equation 3.52. While we have not proven this relationship from the exact analytic expressions found in Chapter 4, it has held exactly for all $m, p \leq 20$. Global copy symmetry then gives
the similar relation:

\[ f_{m}\delta^{(1)}_{mp} + f_{m}\delta^{(2)}_{mp} = \delta_{mp}. \]  

(5.21)

This in turn tells us that an initial symmetric excitation \( \alpha^{(1)}_{-m} + \alpha^{(2)}_{-m} \) commutes with our pair of twist operators (and the same is true for fermions). This may motivate the use of (1) + (2) and (1) - (2) as a new basis choice for the CFT copies, though it is not yet clear how well such an idea generalizes to other orders and configurations.
Figure 5.4: Numerical plot of $C_n$ in blue with our continuum limit approximation in red. The index values are $n = 20$ (left) and $n = 31$ (right).

5.3 The Supercharge Coefficient

Lastly, we will turn to the coefficient $C_n$ that appears in our analysis of the supercharge contours at second order.

$$C_n \equiv (-1)^n \sum_{k=-n/2}^{n/2} \frac{1}{2} C_{n+k} \frac{1}{2} C_{n-k} \cos (ik\Delta w). \quad (5.22)$$

While this sum is difficult to handle exactly, Figure 5.4 shows that for large $n$, the coefficient behaves as:

$$C_n \approx (-1)^n \frac{\Gamma[n/2]}{3.4\Gamma[n/2 + 3/2]} \left| \sin \left( \frac{\Delta w}{2i} \right) \right|^{1/2} \sin \left( n - 1 \frac{\Delta w}{2i} \right) \text{sgn} \left( \frac{\Delta w}{2i} - 1 \right) \quad (5.23)$$

where the factor of 3.4 is a numerical fit. The main oscillation is a sine function with frequency determined by $n$, while the envelope is the square root of a sine function of constant frequency. Note that there is no amplitude falloff with $n$. Since each $C_n$ factor is paired with a supercharge mode, we must be careful not to think that truncating the $n$ sum is a good approximation unless we can show that the supercharge modes begin annihilating our state at some cutoff value.
Chapter 6: Analysis and Conclusions

In analyzing the results presented in this work, it is important to keep in mind what we are looking for. When a particle falls into a fuzzball’s AdS throat, the process is dual to the creation of two wave packets in the boundary CFT — one left-moving and one-right-moving. These wave packets travel unperturbed in the orbifold CFT. They loop around the cylinder and can recombine, emitting the same particle back out of the throat. This is not the behavior expected for a black hole.

Since a black hole’s temperature scales inversely with its mass, an astronomical black hole is extremely cold. We thus expect only low-energy radiation. In the CFT this should look like thermalization, which the initial wave packets break up into a large number of low-energy excitations. One plausible perturbative mechanism for this breakup is for a wave packet of central energy $E_0$ to split into multiple packets of central energy some fraction of $E_0$. Let us call this the ‘splitting’ mechanism. An alternative mechanism is to have each order of the deformation reduce the packet’s central energy by some fixed amount. This is possible because each CFT copy brings its own box size, thereby giving us a characteristic energy scale to work with. This alternative mechanism would naturally generate large numbers of order 1 excitations. We’ll call this the ‘siphoning’ mechanism.
6.1 First Order Results

And exhaustive treatment of first-order deformations was presented in Chapter 3. Perhaps the most important result is the fact that each transition amplitude $f_{mp}$ carries a factor:

$$
\frac{1}{\frac{p}{M+N} - \frac{m}{N_{(i)}}}
$$

where $N_{(1)} = M$ and $N_{(2)} = N$. The two terms in the denominator are then mode numbers divided by the relevant CFT’s box size. These are then direct representations of the energy carried by the initial and final modes. As seen in our continuum limit expressions from Chapter 5, the above factor is simply:

$$
\frac{1}{\Delta E^i}.
$$

Since this factor appears directly in the amplitude, the probability of transitioning into modes of radically different energies is sharply suppressed. Furthermore, the supercurrent does not alter this conclusion. Only $G_{A,0}^-$ modes are present, and while such modes relate bosons to fermions, as zero modes they do not change an excitation’s energy level. We can thus rule out the splitting mechanism.

When looking for signs of the siphoning mechanism, we run headlong into a signal-to-noise dilemma. Low-energy excitations also dominate the $|\chi(w)\rangle$ state. We must thus differentiate any low-energy siphoning effects from these vacuum effects. Even worse, low-energy excitations dominate the $|\chi\rangle$ states, so the problem persists at all orders. On top of this, the first-order case also presents yet another difficulty: The Hilbert spaces for the out and in states do not match. We would thus need to disentangle the siphoning effects from both the ordinary vacuum effects and any odd
effects we might encounter from the change in Hilbert space. We have not found any success in performing such an analysis.

6.2 Second Order Results

We now turn to the simple one-loop case detailed in Chapter 4. Unfortunately, any straightforward analytic approach appears intractable. The exact expressions are so convoluted that many terms can only be expressed in the form of nested sums. Even the behavior of the supercharge, simple at first order, now brings additional complicated sums into the picture.

Despite the above difficulties, we have made some progress through numerical analysis is Chapter 5. After rotating into convenient Minkowski coordinates, plots of the exact results for specific large mode-number choices allowed us to pin down clean analytic approximations for most quantities of interest in the continuum limit. These functions all oscillate in \( \Delta w \) with a period of \( 4\pi i \). Unfortunately, we were unable to pin down the precise oscillatory behavior of the Bogoliubov coefficients.

Once these simplified approximations were obtained, we noticed that the amplitudes of the oscillations for the Bogoliubov coefficients and transition amplitudes all had the same functional forms as their first-order counterparts. This leads one to suspect that the pattern may hold to higher orders. We have yet to make any serious attempts at verifying such a conjecture, but if the pattern did hold it would mean that the dreaded \( \Delta E \) suppression plagues the transition amplitude at all orders. This likely rules out the splitting mechanism at second order and possibly beyond, though the story is complicated somewhat by the fact that the supercurrents now produce nonzero \( G^A_{\tilde{A}} \) modes. These modes alter excitation levels in addition to swapping
bosons and fermions. However, the coefficients involved have nothing to do with the energy carried by the initial excitation. Any thermalization effects of these modes are therefore more likely to work via the siphoning mechanism.

Another approach to analyzing our one-loop results is to continue pursuing purely numerical methods. While there are two infinite, non-dimining sums that bring a linear combination of single $G^-_A$ modes, the unbounded mode index runs negative for the out state and positive for the in state. This means that for any amplitude expression, each unbounded $G^-_A$ mode will annihilate the vacuum applied on its end of the twist unless it contracts with another excitation on that same end other twist. Thus the infinite tower of excitations from the $|\chi(w_1, w_2)\rangle$ state is not an option. Furthermore, two $G^-_A$ modes cannot contract with each other. It is therefore necessary for each $G^-_A$ mode to contract with an initial-state excitation at some point. This allows one to construct simple amplitude quantities that completely circumvent these infinite sums, or quantities that ensure that only finitely many terms can contribute. Amplitudes of this type, while messy, are exactly calculable. As a proof-of-concept, I have produced a Mathematica program that calculates one such amplitude explicitly for a wide range of initial excitation levels.

6.3 Conclusions

While we have not identified any clear evidence of thermalization in the CFT dual, we have managed to rule out several potential avenues. The splitting mechanism has been completely ruled out as a first-order effect, and its most direct avenue has also been ruled out in the specific second configuration we chose to study. Furthermore, we noticed a pattern that, if general, would produce great difficulties for this mechanism.
at all orders. In addition, our one-loop results presents a scenario in which the in
and out Hilbert spaces are identical. This could simplify the analysis of the proposed
siphoning method.

On top of the above specific conclusions, we have also identified a large number of
relationships that hold rigorously at all orders. These relationships drastically reduce
the number of linearly-independent Bogoliubov coefficients and transition amplitudes
for all twist configurations. In addition, we developed a more generalizable method
for stripping the supercurrent contours off of the twist insertions even when multiple
twists are present. These results should prove useful to any future researchers working
with new deformation configurations.
Bibliography


Appendix A: R Vacuum Notation

Here we lay out our notation for the various R vacua in the left-moving, two-copy sector. We start with the vacuum

\[ |0^-_R\rangle^{(1)}|0^-_R\rangle^{(2)} \equiv |v\rangle \]  \hspace{1cm} (A.1)

and act on it with various fermion zero modes to construct the other R vacua. In order to be consistent with earlier work, we also require something along the lines of

\[ |0^+_R\rangle^{(i)} = d_0^{(i)++}d_0^{(i)+-}|0^-_R\rangle^{(i)} \]  \hspace{1cm} (A.2)

though the theory does not actually have states containing only one of the two copies.

Below is an exhaustive detailing of our notation for the various vacua.

\[ |v\rangle = |0^-_R\rangle^{(1)}|0^-_R\rangle^{(2)} \]
\[ d_0^{(1)+-}|v\rangle = |0_R\rangle^{(1)} \otimes |0^-_R\rangle^{(2)} \]
\[ d_0^{(1)++}|v\rangle = |0_R\rangle^{(1)} \otimes |0^-_R\rangle^{(2)} \]
\[ d_0^{(1)++}d_0^{(1)+-}|v\rangle = |0^+_R\rangle^{(1)} \otimes |0^-_R\rangle^{(2)} \]  \hspace{1cm} (A.3)
\begin{align}
\langle v \mid d_0^{(2)++} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \\
\langle v \mid d_0^{(1)+-} d_0^{(2)++} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(1)+-} d_0^{(2)+-} \rangle &= |0_{R}^{+}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \quad (A.4) \\
\langle v \mid d_0^{(2)++} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \\
\langle v \mid d_0^{(1)+-} d_0^{(2)++} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(2)++} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(1)+-} d_0^{(2)++} \rangle &= |0_{R}^{+}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \quad (A.5) \\
\langle v \mid d_0^{(2)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{+}\rangle^{(2)} \\
\langle v \mid d_0^{(1)+-} d_0^{(2)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{+}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(2)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{+}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(1)+-} d_0^{(2)++} \rangle &= |0_{R}^{+}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \quad (A.6) \\
\langle v \mid d_0^{(2)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{+}\rangle^{(2)} \\
\langle v \mid d_0^{(1)+-} d_0^{(2)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{+}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(2)++} d_0^{(2)+-} \rangle &= |0_{R}^{-}\rangle^{(1)} \otimes |0_{R}^{+}\rangle^{(2)} \\
\langle v \mid d_0^{(1)++} d_0^{(1)+-} d_0^{(2)++} \rangle &= |0_{R}^{+}\rangle^{(1)} \otimes |0_{R}^{-}\rangle^{(2)} \quad (A.6)
\end{align}
Appendix B: Proof of the the $|\chi\rangle$ Form.

Here we prove the exponential form of $|\chi\rangle$ presented in Equation 2.4 for the fermionic components. The bosons follow in an analogous fashion. This proof was first presented with different notation in [22].

The method of the proof is straightforward. We look at capping the state $|\chi\rangle$ with a general femionic state $\langle \Theta | \hat{Q}^\dagger$. This provides the relationship

$$ A(\hat{Q}) \equiv \frac{\langle \Theta' | \hat{Q} | \chi \rangle}{\langle \Theta | \chi \rangle} = \frac{i\langle 0_{NS} | \hat{Q}' | 0_{NS} \rangle_t}{i\langle 0_{NS} | 0 \rangle_t}. $$

(B.1)

The rightmost expression can be calculated from the behavior of the fermion modes under coordinate transformations and spectral flows, while the middle expression can be calculated from our guess for the form of $|\chi\rangle$. Showing that these two methods are consistent demonstrates that we have the correct form for $|\chi\rangle$.

B.1 Middle expression

Let us begin by taking a close look at the form presented in Equation 2.4. Because we are working with pairs of fermion creation operators in the exponent, all of the terms in the exponent commute. We can thus rewrite the fermion portion as:

$$ |\chi\rangle^F = e^{\sum_{(j),(j') \sum m,n \geq 0} \gamma_{mn}^{(j)(j')} (d_{m}^{(j)+} d_{-m}^{(j')-} - d_{m}^{(j)+} d_{-m}^{(j')-})} |\emptyset\rangle $$

$$ = \prod_{A,B} \prod_{(j),(j')} \prod_{m,n \geq 0} \left[ 1 + \epsilon_{AB} \gamma_{mn}^{(j)(j')} d_{m}^{(j)+} d_{n}^{(j')-} \right] |\emptyset\rangle. $$

(B.2)
From this form, we can see clearly that any state with an odd number of fermion excitations will have no overlap with $|\chi\rangle$. We thus begin with a two-excitation state.

$$A\left(d_{m}^{(j)},+A_{m}^{(j)},-B\right) = \frac{\langle\emptyset|d_{m}^{(j)},+A_{m}^{(j)}f,-B|\chi\rangle}{\langle\emptyset|\chi(w_{1},w_{2})\rangle} = \epsilon^{AC}\epsilon^{BD}\epsilon^{CD}\gamma^{F_{mn}}(j)(j').$$

The proportionality to $\epsilon^{AB}$ simply means that the combinations $d^{++}d^{--}$ and $d^{+-}d^{-+}$ yield coefficients which differ only by an overall sign. Note that the amplitude is directly an anticommutator of the relevant fermion mode pairs.

What if we cap with a state that contains more fermion excitations? We know that we must have an even number of excitations, so they come in pairs. We can then calculate $A$ in two steps. First, we write out all possible ways to group the fermion excitations into pairs, accounting for the overall sign required to anticommutate the operators into the appropriate pairings. Then each pairing combination provides a contribution equal to the product of the amplitude for each individual pair within that combination. Adding the contributions for each combination then gives us the total amplitude. We can write this schematically as:

$$A\left(d_{1}d_{2}d_{3}\ldots\right) = \sum_{\text{pairing combinations}}(-1)^{p}\left(\prod_{\{d_{i}d_{j}\}}A(d_{i}d_{j})\right),$$

where $p$ is the number of anticommutations we need to perform to achieve the pairing configuration.

### B.2 Right expression

Let us compare the previous result to the calculation in the $t$ plane. The transformations to map into an empty covering plane involve a combination of coordinate
maps and spectral flows. Under each such transformation, the modes \( d_{n}^{(j),\alpha A} \) behave in general as:

\[
d_{n}^{(j),\alpha A} \rightarrow \frac{1}{2\pi i \sqrt{N(j)}} \oint_{t_{0}} \psi_{\alpha A}^{\ast}(t) h_{n}^{(j),\alpha}(t) \, dt.
\]  

(B.5)

Since each relevant transformation separately applies a factor \( h_{n}^{(j),\alpha}(t) \), any any combination of such transformations will simply apply the product of each component’s factor. We’ll denote this overall factor as \( H_{n}^{(j),\alpha}(t) \).

We can now expand the function \( H_{n}^{(j),\alpha}(t) \) as a polynomial in \( t \), which can in general include negative powers of \( t \). This gives:

\[
d_{m}^{(j),\alpha A} = \sum_{p=-\infty}^{\infty} C_{mpq}^{(j)\alpha} \frac{1}{2\pi i \sqrt{N(j)}} \oint_{t_{0}} \psi_{\alpha A}^{\ast}(t) t^{q} \, dt
\]

\[
= \sum_{q=-\infty}^{\infty} C_{mpq}^{(j)\alpha} \tilde{d}_{q}^{\alpha A}.
\]  

(B.6)

From this expression we see that when capping with a two-excitation state, the right side of (B.1) becomes:

\[
\mathcal{A} \left( d_{m}^{(j),+A} d_{n}^{(j'),+B} \right) = \sum_{p,q=-\infty}^{\infty} C_{mpq}^{(j)\alpha} C_{mpq}^{(j')\beta} t^{0_{NS}} \left< 0_{NS} \right| : \tilde{d}_{p}^{+A} \tilde{d}_{q}^{-B} : |0_{NS} \rangle \right> t^{0_{NS}} |0_{NS} \rangle.
\]  

(B.7)

The amplitude in the numerator is only nonzero when the two fermion modes can contract together, with the outer mode annihilating the inner mode. The anticommutator is:

\[
\left\{ \tilde{d}_{p}^{+A}, \tilde{d}_{q}^{-B} \right\} = -\epsilon^{+-} \epsilon^{AB} \delta_{p+q,0},
\]  

(B.8)

which is proportional to \( \epsilon^{AB} \). We thus find that for any fermion pair,

\[
\mathcal{A} \left( d_{m}^{(j),+A} d_{n}^{(j),-B} \right) = \epsilon^{AB} D_{mn}^{(j)(j')},
\]  

(B.9)

for some coefficient \( D \). This form matches the result from (B.3).
Now what happens when we cap with a state that contains more fermion excitations? Schematically, we have:

\[
\mathcal{A}(d_1 d_2 d_3 \ldots) = \sum_{p_1, p_2, p_3, \ldots} (C_{p_1} C_{p_2} C_{p_3} \ldots) \frac{\langle 0_{NS} : \tilde{d}_{p_1} \tilde{d}_{p_2} \tilde{d}_{p_3} \ldots : 0 \rangle_t}{\langle 0_{NS} | 0 \rangle_t}. \tag{B.10}
\]

It is clear that the right side vanishes when there are an odd number of fermion modes, as such a case will always leave at least one uncontracted mode to annihitate the vacuum bra or ket. With an even number of modes, the amplitude can be calculated by performing all possible combinations of contractions in the expression on the right side and adding each contribution together. However, we also have the relation

\[
\frac{\langle 0_{NS} | \tilde{d}_{p_1} \tilde{d}_{p_2} \tilde{d}_{p_3} \ldots | 0_{NS} \rangle_t}{\langle 0_{NS} | 0 \rangle_t} = \frac{\langle 0_{NS} : \tilde{d}_{p_3} \tilde{d}_{p_4} : 0 \rangle_t}{\langle 0_{NS} | 0 \rangle_t}. \tag{B.11}
\]

We thus find:

\[
\mathcal{A}(d_1 d_2 d_3 \ldots) = \sum_{\text{pairing combinations}} (-1)^p \left[ \prod_{\{p_i, p_j\}} \left( \sum_{p_i, p_j} C_{p_i} C_{p_j} \frac{\langle 0_{NS} : \tilde{d}_{p_i} \tilde{d}_{p_j} : 0 \rangle_t}{\langle 0_{NS} | 0 \rangle_t} \right) \right]
= \sum_{\text{pairing combinations}} (-1)^p \left( \prod_{\{d_i, d_j\}} \mathcal{A}(d_i d_j) \right), \tag{B.12}
\]

where \(p\) is again the number of anticommutations we must perform to reach the particular pair combination. Since this overall sign depends only on how we group the operators in \(\hat{Q}\) and not on any of the specifics of the coordinate maps or spectral flows, each \(p\) that appears here takes the same value as in in (B.4). Thus the two relations are identical, and we see that the claimed form for \(|\chi(w_1, w_2)\rangle\) is correct.

The proof for the bosonic content in [15] is analogous, though there is no need to track minus signs. A natural extension of this proof is to note that an in state mode also maps into a linear combination of modes natural to the \(t\) plane. This results in nonzero contractions only with capping states that contain odd numbers of excitations, thus demonstrating the relations found in Equation 2.22.
Appendix C: Proof that $G^+_A$ Commutes with $\sigma^+_2(w_0) = 0$

Here we that that a $G^+_A$ contour wrapped around a single $\sigma^+_2$ with zero weight vanishes. This in turn indicates that $G^+_A$ commutes with $\sigma^+_2(w_0) = 0$.

We start by considering the following operator on the cylinder:

$$\hat{Q} = \frac{1}{2\pi i} \int_{w_0} G^+_A(w)\sigma^+_2(w_0) \, dw.$$  \hfill (C.1)

Mapping to the $z$ plane with $z = e^w$, we find:

$$\hat{Q} = z_0^{1/2} \frac{1}{2\pi i} \int_{z_0} z^{1/2}G^+_A(z)\sigma^+_2(z_0) \, dz,$$  \hfill (C.2)

where the factor $z_0^{1/2}$ comes from the transformation of $\sigma^+_2(z_0)$.

We must now map into a multi-cover of our $z$ plane. While the form of this map depends on the number of deformations we’re working with, we know that it has bifurcation points at all twist insertions. Furthermore, we can always shift our $t$ coordinate so that this particular insertion maps to the origin. With $n$ additional insertions, the map obeys:

$$\frac{dz}{dt} \sim t \prod_{k=1}^n(t - t_k).$$  \hfill (C.3)

Equation C.2 is then:

$$\hat{Q} \propto \frac{1}{2\pi i} \int_{t=0} G^+_A(t)z^{1/2} \left( \frac{dz}{dt} \right)^{-1/2} \, dt \, S^+(0)|0_{NS}\rangle_{t=0}$$

$$\propto \frac{1}{2\pi i} \int_{t_0} dt \, z^{1/2}t^{-\frac{1}{2}} \prod_{k=1}^n(t - t_k)^{-\frac{1}{2}}G^+_A(t) \, dt \, S^+(0)|0_{NS}\rangle_{t=0}.$$  \hfill (C.4)
We now perform a single spectral flow by $\alpha = -1$ to remove $S^+(0)$. This gives:

$$\hat{Q} \propto \frac{1}{2\pi i} \int_{t=0} \frac{1}{2} n \prod_{k=1}^{n} (t - t_k)^{-\frac{1}{2}} G_A^+(t) \, dt \, |0_{NS}\rangle_{t=0} \quad (C.5)$$

At this point our integrand contains the factor:

$$z^{1/2} \prod_{k=1}^{n} (t - t_k)^{-\frac{1}{2}} \quad (C.6)$$

Since we integrate around the origin, we expand this factor in powers of $t$ for small $t$. We do not know the exact form of $z$, but Equation C.3 tells us that it is well behaved. We thus find:

$$z^{1/2} \prod_{k=1}^{n} (t - t_k)^{-\frac{1}{2}} = \sum_{p=0}^{\infty} f_p (\{t_k\}) t^p. \quad (C.7)$$

This allows us to express (C.5) as:

$$\hat{Q} \propto \sum_{p=0}^{\infty} f_p (\{t_k\}) \frac{1}{2\pi i} \int_{t=0} G_A^+(t) t^p \, dt \, |0_{NS}\rangle_{t=0}. \quad (C.8)$$

We will now define $G_A^+$ natural to the $t$ plane at $t = 0$. Since the behavior of $G_A^+ \, dw$ has the opposite coordinate transformation behavior as $\psi^{\alpha A}(w) \, dw$, the natural modes are:

$$\tilde{G}_A^+, t \to 0 = \frac{1}{2\pi i} \int_{0} G_A^+(t) t^{r+1/2} \, dt, \quad r \in \mathbb{Z} + 1/2. \quad (C.9)$$

We now express (C.8) in terms of these modes:

$$\hat{Q} \propto \sum_{p=0}^{\infty} f_p (\{t_k\}) \tilde{G}_A^+, t \to 0 |0_{NS}\rangle_{t=0}. \quad (C.10)$$

But it is a well-known property of the NS vacuum that it is annihilated by each of these modes. This can be seen by expressing the $G_A^+$ modes in terms of natural fermion and boson modes:

$$G_{A,k-\frac{1}{2}} = -i \sum_{r \in \mathbb{Z} + \frac{1}{2}} \tilde{\alpha}_{A,k,r} \tilde{d}_r^A \tilde{\alpha}_{A,k-r-\frac{1}{2}}. \quad (C.11)$$
The fermion annihilates the vacuum whenever \( r > 0 \), but the boson annihilates the vacuum unless \( r + \frac{1}{2} > k \). We thus need \( 0 > r > k - \frac{1}{2} \implies k < 0 \) to avoid annihilating the vacuum. As such, Equation gives simply:

\[
\dot{Q} = \frac{1}{2\pi i} \int_{w_0} G_\Lambda^+(w)\sigma_2^+(w_0)dw = 0. \tag{C.12}
\]