Barycentric Straightening, Splitting Rank and Bounded Cohomology

Dissertation

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By

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Abstract

This work is devoted to the study of bounded cohomology from a geometric point of view. In the context of non-compact, connected, semisimple Lie groups with finite center, Dupont raised the question of whether the comparison map is always surjective. The main goal of this work is to show Dupont’s conjecture in high degrees. We introduce the barycentric straightening and show that the straightened simplices in higher rank symmetric spaces of non-compact type has uniformly bounded volume, when the dimension of the simplices are close to the dimension of the symmetric spaces. We also give counterexamples where the barycentrically straightened simplices have unbounded volume when the dimension is equal to the splitting rank. This provides an obstruction to our straightening method. In addition, we compute explicitly this splitting rank for all symmetric spaces.
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Chapter 1: Introduction

The notion of bounded cohomology was first introduced by Gromov in the 1980s, and has played an important role in geometry, topology and dynamical system since then. Unlike the ordinary cohomology, the bounded cohomology is often hard to compute and little is known about these groups in general. When studying the bounded cohomology, one important theme is the comparison map from the bounded cohomology to the ordinary cohomology. It is natural to ask whether this comparison map is surjective, or equivalently speaking, does any cohomology class have a bounded representative. In the context of semisimple Lie groups, Dupont first raised the following question (see also Monod’s ICM address [19, Problem A’], and [5, Conjecture 18.1]):

**Conjecture 1.** Let $G$ be a connected semisimple Lie group with finite center, then the comparison map $\eta : H^*_c(G, \mathbb{R}) \rightarrow H^*_c(G, \mathbb{R})$ is always surjective.

This problem is still open in most generality. However, people have pursued various results via different methods. The rank one case is solved by Gromov and Thurston, which is due to the fact that the geodesic simplices in rank one symmetric spaces have uniformly bounded volume. In degree two, Guichardet and Wigner showed that the continuous cohomology $H^2_c(G)$ is either one dimensional or zero,
depending on whether the associated symmetric space is of Hermitian type or not [13]. In the former case, Domic and Toledo [9] showed the generator is bounded (See also Clerc and Orsted [6]), thus showing Dupont’s conjecture in degree two. In top degree, Lafont and Schmidt [17] showed surjectivity in all cases excluding SL(3, \mathbb{R}), followed by Bucher-Karlsson’s complementary result [4], thus completing an equivalent conjecture of Gromov: that the simplicial volume of any closed locally symmetric space of noncompact type is positive. One of the key step in the approach of [17] is to show boundedness of a certain Jacobian, which relies heavily on previous work of Connell and Farb [7], [8]. Recently, Inkang Kim and Sungwoon Kim [15] extended the Jacobian estimate to codimension one (but the codimension one surjectivity of the comparison map is automatic), and they also gave detailed investigation on rank two cases. Besides, Hartnick and Ott [14] showed that the even generators are always bounded, and in particular the conjecture holds for Lie groups of Hermitian type.

Our main purpose of the work is to show the following two results:

**Main Theorem 1.** Let $X = G/K$ be an $n$-dimensional irreducible symmetric space of non-compact type of rank $r \geq 2$, excluding $SL(3, \mathbb{R})/SO(3)$ and $SL(4, \mathbb{R})/SO(4)$, and \( \Gamma \) a cocompact torsion-free lattice in $G$. Then the comparison maps $\eta : H^*_{c,b}(G, \mathbb{R}) \to H^*_c(G, \mathbb{R})$ and $\eta' : H^*_b(\Gamma, \mathbb{R}) \to H^*(\Gamma, \mathbb{R})$ are both surjective in all degrees $* \geq n - r + 2$.

**Main Theorem 2.** Let $X = G/K$ be an $n$-dimensional symmetric space of non-compact type of rank $r \geq 2$, and $\Gamma$ a cocompact torsion-free lattice in $G$. Assume $X$ has no direct factors of $\mathbb{H}^2$, $SL(3, \mathbb{R})/SO(3)$, $Sp(2, \mathbb{R})/U(2)$, $G_2^2/SO(4)$ and $SL(4, \mathbb{R})/SO(4)$, then the comparison maps $\eta : H^*_{c,b}(G, \mathbb{R}) \to H^*_c(G, \mathbb{R})$ and $\eta' : H^*_b(\Gamma, \mathbb{R}) \to H^*(\Gamma, \mathbb{R})$ are both surjective in all degrees $* \geq \text{srk}(X) + 2$.
Remark. Notice $\text{srk}(X)$ in Main Theorem 2 denotes the splitting rank of $X$, which is defined later in Definition 4.3.5. For irreducible symmetric spaces, we always have $\text{srk}(X) \leq n - r$, hence Main Theorem 2 generalizes Main Theorem 1. But in order to show Main Theorem 2, we need a better estimate on the Jacobian that requires a more detailed investigation on the symmetric spaces.

Remark. We note that this work is based on two papers [18] [21] by the author, and the former paper is joint work with Jean Lafont.

We structure our thesis as follows: Chapter 2 includes basic knowledge about symmetric spaces and Patterson-Sullivan measures in higher rank. We define our barycentric straightening in Chapter 3, and also estimate the Jacobian of straightened simplices in detail. In Chapter 4, we prove Main Theorem 1, and define the splitting rank to give obstruction to our straightening method. In Chapter 5, we give the splitting rank of all symmetric spaces, and show Main Theorem 2 after a better estimate of the Jacobian. Finally, in the Appendix, we work case by case on all symmetric spaces in order to compute and estimate the splitting rank.
Chapter 2: Preliminary

2.1 Symmetric Spaces of Non-compact Type

In this section, we give a quick review of some results on symmetric spaces of non-compact type; for more details, we refer the reader to Eberlein’s book [11]. Let \( X = G/K \) be a symmetric space of non-compact type, where \( G \) is semisimple and \( K \) is a maximal compact subgroup of \( G \). Geometrically \( G \) can be identified with Isom\(_0\)(\( X \)), the connected component of the isometry group of \( X \) that contains the identity, and \( K = \text{Stab}_p(G) \) for some \( p \in X \). Fixing a basepoint \( p \in X \), we have a Cartan decomposition \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) of the Lie algebra \( \mathfrak{g} \) of \( G \), where \( \mathfrak{k} \) is the Lie algebra of \( K \), and \( \mathfrak{p} \) can be isometrically identified with \( T_pX \) using the Killing form. Let \( \mathfrak{a} \subseteq \mathfrak{p} \) be a maximal abelian subalgebra of \( \mathfrak{p} \). We can identify \( \mathfrak{a} \) with the tangent space of a flat \( \mathcal{F} \) at \( p \) — that is to say, an isometrically embedded Euclidean space \( \mathbb{R}^r \subseteq X \), where \( r \) is the rank of \( X \). Given any vector \( v \in T_pX \), there exists a flat \( \mathcal{F} \) that is tangent to \( v \). We say \( v \) is regular if such a flat is unique, and singular otherwise.

Now let \( v \in \mathfrak{p} \) be a regular vector. This direction defines a point \( v(\infty) \) on the visual boundary \( \partial X \) of \( X \). \( G \) acts on the visual boundary \( \partial X \). The orbit set \( Gv(\infty) = \partial_FX \subseteq \partial X \) is called a Furstenberg boundary of \( X \). Since both \( G \) and \( K \) act transitively on \( \partial_FX \), \( \partial_FX \) is compact. In fact, a point stabilizer for the \( G \)-action on \( \partial_FX \) is a minimal parabolic subgroup \( P \), so we can also identify \( \partial_FX \) with
the quotient \( G/P \). In the rest of this paper, we will use a specific realization of the Furstenberg boundary – the one given by choosing the regular vector \( v \) to point towards a barycenter of a Weyl chamber in the flat.

For each element \( \alpha \) in the dual space \( a^* \) of \( a \), we define \( g_\alpha = \{ Y \in g \mid [A,Y] = \alpha(A)Y \text{ for all } A \in a \} \). We call \( \alpha \) a root if \( g_\alpha \) is nontrivial, and in such case we call \( g_\alpha \) the root space of \( \alpha \). We denote the finite set of roots \( \Lambda \), and we have the following root space decomposition

\[
g = g_0 \bigoplus_{\alpha \in \Lambda} g_\alpha
\]

where \( g_0 = \{ Y \in g \mid [A,Y] = 0 \text{ for all } A \in a \} \), and the direct sum is orthogonal with respect to the canonical inner product on \( g \).

Let \( \theta \) be the Cartan involution at the point \( p \). Then \( \theta \) is an involution on \( g \), which acts by \( I \) on \( k \) and \(-I\) on \( p \), hence it preserves Lie bracket. We can define \( k_\alpha = (I + \theta)g_\alpha \subseteq k \), and \( p_\alpha = (I - \theta)g_\alpha \subseteq p \), with the following properties:

**Proposition 2.1.1.** [11, Proposition 2.14.2] (1) \( I + \theta : g_\alpha \to k_\alpha \) and \( I - \theta : g_\alpha \to p_\alpha \) are linear isomorphisms. Hence \( \dim(k_\alpha) = \dim(g_\alpha) = \dim(p_\alpha) \).

(2) \( k_\alpha = k_{-\alpha} \) and \( p_\alpha = p_{-\alpha} \) for all \( \alpha \in \Lambda \), and \( k_\alpha \oplus p_\alpha = g_\alpha \oplus g_{-\alpha} \).

(3) \( k = k_0 \oplus \bigoplus_{\alpha \in \Lambda^+} k_\alpha \) and \( p = a \oplus \bigoplus_{\alpha \in \Lambda^+} p_\alpha \), where \( k_0 = g_0 \cap k \), and \( \Lambda^+ \) is the set of positive roots.

**Remark.** Since \( p_\alpha = (g_\alpha + g_{-\alpha}) \cap p \), the direct sum of \( p \) in (3) of Proposition 2.1.1 is also orthogonal with respect to the canonical inner product on \( p \).
We now analyze the adjoint action of $k$ on $a$. Let $u \in k_\alpha$ and $v \in a$, we can write $u$ as $(I + \theta)w$ where $w \in g_\alpha$, hence we have

$$[u, v] = [(I + \theta)w, v] = [w, v] + [\theta w, v] = -\alpha(v)w + \theta[w, -v]$$

$$= -\alpha(v)w + \theta(\alpha(v)w) = -\alpha(v)(I - \theta)(w)$$

$$= -\alpha(v)(I - \theta)(I + \theta)^{-1} u$$

This gives the following proposition.

**Proposition 2.1.2.** Let $\alpha$ be a root. The adjoint action of $k_\alpha$ on $a$ is given by

$$[u, v] = -\alpha(v)(I - \theta)(I + \theta)^{-1} u$$

for any $u \in k_\alpha$ and $v \in a$. In particular, $k_\alpha$ maps $v$ into $p_\alpha$.

Assume $v \in a \subseteq T_x X$ is inside a fixed flat through $x$, and let $K_v$ be the stabilizer of $v$ in $K$. Then the space $K_v a$ is the tangent space of the union of all flats that goes through $v$. Equivalently, it is the union of all vectors that are parallel to $v$, hence it can be identified with $a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(v) = 0} p_\alpha$. In particular, if $v$ is regular, then the space is just $a$. Moreover, if we denote by $k_v$ the Lie algebra of $K_v$, then $k_v = \{u \in k \mid [u, v] = 0\} = k_0 \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(v) = 0} k_\alpha$.

### 2.2 Patterson-Sullivan Measures

Let $X = G/K$ be a symmetric space of non-compact type, and $\Gamma$ be a cocompact lattice in $G$. In [1], Albuquerque generalizes the construction of Patterson-Sullivan to higher rank symmetric spaces. He showed that for each $x \in X$, we can assign a probability measure $\mu(x)$ that is $G$-equivariant and is fully supported on the Furstenberg boundary $\partial F(X)$. Moreover, for $x, y \in X$ and $\theta \in \partial F(X)$, the Radon-Nikodym
derivative is given by
\[ \frac{d\mu(x)}{d\mu(y)}(\theta) = e^{hB(x,y,\theta)} \]
where \( h \) is the volume entropy of \( X/\Gamma \), and \( B(x, y, \theta) \) is the Busemann function on \( X \).

Recall that, in a non-positively curved space \( X \), the Busemann function \( B \) is defined by
\[ B(x, y, \theta) = \lim_{t \to \infty} (d_X(y, \gamma_{\theta}(t)) - t) \]
where \( \gamma_{\theta} \) is the unique geodesic ray from \( x \) to \( \theta \). Fixing a basepoint \( O \) in \( X \), we shorten \( B(O, y, \theta) \) to just \( B(y, \theta) \). Notice that for fixed \( \theta \in \partial F(X) \) the Busemann function is convex on \( X \), and by integrating on \( \partial F(X) \), we obtain, for any probability measure \( \nu \) that is fully supported on the Furstenberg boundary \( \partial F X \), a strictly convex function
\[ x \mapsto \int_{\partial F X} B(x, \theta)d\nu(\theta) \]
(See [7, Proposition 3.2.1] for a proof of this last statement.)

Hence we can define the barycenter \( \text{bar}(\nu) \) of \( \nu \) to be the unique point in \( X \) where the function attains its minimum. It is clear that this definition is independent of the choice of basepoint \( O \).
3.1 Barycentric Straightening

In this section, we discuss the barycentric straightening introduced by Lafont and Schmidt [17] (based on the barycenter method originally developed by Besson, Courtois, and Gallot [3]). Let $X = G/K$ be a symmetric space of non-compact type, and $\Gamma$ be a cocompact lattice in $G$. We denote by $\Delta^k_s$ the standard spherical $k$-simplex in the Euclidean space, that is

$$\Delta^k_s = \left\{ (a_1, \ldots, a_{k+1}) \mid a_i \geq 0, \sum_{i=1}^{k+1} a_i^2 = 1 \right\} \subseteq \mathbb{R}^{k+1},$$

with the induced Riemannian metric from $\mathbb{R}^{k+1}$, and with ordered vertices $(e_1, \ldots, e_{k+1})$. Given any singular $k$-simplex $f : \Delta^k_s \to X$, with ordered vertices $V = (x_1, \ldots, x_{k+1}) = (f(e_1), \ldots, f(e_{k+1}))$, we define the $k$-straightened simplex

$$\text{st}_k(f) : \Delta^k_s \to X$$

$$\text{st}_k(f)(a_1, \ldots, a_{k+1}) := \bar{\text{bar}} \left( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \right)$$

where $\mu(x_i)$ is the Patterson-Sullivan measure at $x_i$. We notice that $\text{st}_k(f)$ is determined by the (ordered) vertex set $V$, and we denote $\text{st}_k(f)(\delta)$ by $\text{st}_V(\delta)$, for $\delta \in \Delta^k_s$.

Observe that the map $\text{st}_k(f)$ is $C^1$, since one can view this map as the restriction of the $C^1$-map $\text{st}_n(f)$ to a $k$-dimensional subspace (see e.g. [17, Property (3)]). For any
\[ \delta = \sum_{i=1}^{k+1} a_i e_i \in \Delta^k_s, \text{ st}_k(f)(\delta) \text{ is defined to be the unique point where the function} \]
\[ x \mapsto \int_{\partial_F X} B(x, \theta) d \left( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \right) (\theta) \]
is minimized. Hence, by differentiating at that point, we get the 1-form equation
\[ \int_{\partial_F X} dB_{(st_V(\delta), \theta)}(\cdot) d \left( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \right) (\theta) \equiv 0 \]
which holds identically on the tangent space \( T_{st_V(\delta)}X \). Differentiating in a direction \( u \in T_\delta(\Delta^k_s) \) in the source, one obtains the 2-form equation
\[ \sum_{i=1}^{k+1} 2a_i \langle u, e_i \rangle \delta \int_{\partial_F X} dB_{(st_V(\delta), \theta)}(v) d(\mu(x_i))(\theta) \]
\[ + \int_{\partial_F X} DdB_{(st_V(\delta), \theta)}(D_\delta(st_V)(u), v) d \left( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \right) (\theta) \equiv 0 \]
which holds for every \( u \in T_\delta(\Delta^k_s) \) and \( v \in T_{st_V(\delta)}(X) \). Now we define two semi-positive definite quadratic forms \( Q_1 \) and \( Q_2 \) on \( T_{st_V(\delta)}(X) \):
\[ Q_1(v, v) = \int_{\partial_F X} dB^2_{(st_V(\delta), \theta)}(v) d \left( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \right) (\theta) \]
\[ Q_2(v, v) = \int_{\partial_F X} DdB^2_{(st_V(\delta), \theta)}(v) d \left( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \right) (\theta) \]
In fact, \( Q_2 \) is positive definite since \( \sum_{i=1}^{k+1} a_i^2 \mu(x_i) \) is fully supported on \( \partial_F X \) (see [7, Section 4]). From the 2-form equation above, we obtain, for \( u \in T_\delta(\Delta^k_s) \) a unit vector and \( v \in T_{st_V(\delta)}(X) \) arbitrary, the following
\[ |Q_2(D_\delta(st_V)(u), v)| = \left| - \sum_{i=1}^{k+1} 2a_i \langle u, e_i \rangle \delta \int_{\partial_F X} dB_{(st_V(\delta), \theta)}(v) d(\mu(x_i))(\theta) \right| \quad (3.1.1) \]
\[ \leq \left( \sum_{i=1}^{k+1} \langle u, e_i \rangle^2 \right)^{1/2} \left( \sum_{i=1}^{k+1} 4a_i^2 \left( \int_{\partial_F X} dB_{(st_V(\delta), \theta)}(v) d(\mu(x_i))(\theta) \right)^2 \right)^{1/2} \]
\[ \leq 2 \left( \sum_{i=1}^{k+1} a_i^2 \int_{\partial_F X} dB^2_{(st_V(\delta), \theta)}(v) d(\mu(x_i))(\theta) \int_{\partial_F X} 1 d(\mu(x_i)) \right)^{1/2} \]
\[ = 2Q_1(v, v)^{1/2} \]
via two applications of the Cauchy-Schwartz inequality.

We restrict these two quadratic forms to the subspace \( S = \text{Im}(D_\delta(st_V)) \subseteq T_{st_V(\delta)}(X) \), and denote the corresponding \( k \)-dimensional endomorphisms by \( H_\delta \) and \( K_\delta \), that is

\[
Q_1(v, v) = \langle H_\delta(v), v \rangle_{st_V(\delta)}
\]

\[
Q_2(v, v) = \langle K_\delta(v), v \rangle_{st_V(\delta)}
\]

for all \( v \in S \).

For points \( \delta \in \Delta_k^s \) where \( st_V \) is nondegenerate, we now pick orthonormal bases \( \{u_1, \ldots, u_k\} \) on \( T_\delta(\Delta_k^s) \), and \( \{v_1, \ldots, v_k\} \) on \( S \subseteq T_{st_V(\delta)}(X) \). We choose these so that \( \{v_i\}_{i=1}^k \) are eigenvectors of \( H_\delta \), and \( \{u_1, \ldots, u_k\} \) is the resulting basis obtained by applying the orthonormalization process to the collection of pullback vectors \( \{(K_\delta \circ D_\delta(st_V))^{-1}(v_i)\}_{i=1}^k \). So we obtain

\[
\det(Q_2|_S) \cdot |Jac_\delta(st_V)| = |\det(K_\delta) \cdot Jac_\delta(st_V)| = |\det(\langle K_\delta \circ D_\delta(st_V)(u_i), v_j \rangle)|
\]

By the choice of bases, the matrix \( \langle (K_\delta \circ D_\delta(st_V))(u_i), v_j \rangle \) is upper triangular, so we have

\[
|\det(\langle K_\delta \circ D_\delta(st_V)(u_i), v_j \rangle)| = \prod_{i=1}^k |\langle K_\delta \circ D_\delta(st_V)(u_i), v_i \rangle| \\
\leq \prod_{i=1}^k 2^{\langle H_\delta(v_i), v_i \rangle}^{1/2} \\
= 2^k \det(H_\delta)^{1/2} = 2^k \det(Q_1|_S)^{1/2}
\]

where the middle inequality is obtained via Equation (3.1.1). Hence we get the inequality

\[
|Jac_\delta(st_V)| \leq 2^k \cdot \frac{\det(Q_1|_S)^{1/2}}{\det(Q_2|_S)}
\]
We summarize the above discussion into the following proposition.

**Proposition 3.1.1.** Let $Q_1, Q_2$ be the two positive semidefinite quadratic forms defined as above (note $Q_2$ is actually positive definite). Assume there exists a constant $C$ that only depends on $X$, with the property that

$$\frac{\det(Q_1|_S)^{1/2}}{\det(Q_2|_S)} \leq C$$

for any $k$-dimensional subspace $S \subseteq T_{stV(\delta)}X$. Then the quantity $|\text{Jac}(st_V)(\delta)|$ is universally bounded – independent of the choice of $(k+1)$-tuple of points $V \subset X$, and of the point $\delta \in \Delta^k_s$.

### 3.2 Jacobian Estimate

Let $X = G/K$ be an irreducible symmetric space of non-compact type. We fix an arbitrary point $x \in X$ and identify $T_x X$ with $p$. Let $\mu$ be a probability measure that is fully supported on the Furstenberg boundary $\partial_F X$. Using the same notation as in Section 3.1, we define a semi-positive definite quadratic form $Q_1$ and a positive definite quadratic form $Q_2$ on $T_x X$

$$Q_1(v, v) = \int_{\partial_F X} dB^2(x, \theta)(v) d\mu(\theta)$$

$$Q_2(v, v) = \int_{\partial_F X} DdB(x, \theta)(v, v) d\mu(\theta)$$

for $v \in T_x(X)$. We will follow the techniques of Connell and Farb [7], [8], and show the following theorem.

**Theorem 3.2.1.** Let $X$ be an irreducible symmetric space of non-compact type excluding $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ and $\text{SL}(4, \mathbb{R})/\text{SO}(4)$, and let $r = \text{rank}(X) \geq 2$. If $n = \dim(X)$,
then there exists a constant $C$ that only depends on $X$, such that

$$\frac{\det(Q_1|_S)^{1/2}}{\det(Q_2|_S)} \leq C$$

for any subspace $S \subseteq T_xX$ with $n - r + 2 \leq \dim(S) \leq n$.

In view of Proposition 3.1.1, this implies that the barycentrically straightened simplices of dimension $\geq n - r + 2$ have uniformly controlled Jacobians. The reader whose primary interest is bounded cohomology, and who is willing to take Theorem 3.2.1 on faith, can skip ahead to Section 4 for the proof of the Main Theorem 1.

The rest of this Section will be devoted to the proof of Theorem 3.2.1. In Section 3.2.1, we explain some simplifications of the quadratic forms, allowing us to give geometric interpretations for the quantities involved in Theorem 3.2.1. In Section 3.2.2, we formulate the “weak eigenvalue matching” Theorem 3.2.3 (which will be established in Section 3.3). Finally, in Section 3.2.3, we will deduce Theorem 3.2.1 from Theorem 3.2.3.

### 3.2.1 Simplifying the Quadratic Forms

Following [7, Section 4.3], we fix a flat $\mathcal{F}$ going through $x$, and denote the tangent space by $\mathfrak{a}$, so $\dim(\mathfrak{a}) = r$ is the rank of $X$. By abuse of notation, we identify $\mathfrak{a}$ with $\mathcal{F}$. Choose an orthonormal basis $\{e_i\}$ on $T_xX$ such that $\{e_1, ..., e_r\}$ spans $\mathcal{F}$, and assume $e_1$ is regular so that $e_1(\infty) \in \partial \mathcal{F}X$. Then $Q_1, Q_2$ can be expressed in the following matrix forms.

$$Q_1 = \int_{\partial \mathcal{F}X} O_{\theta} \begin{pmatrix} 1 & 0 \\ 0 & 0^{(n-1)} \end{pmatrix} O_{\theta}^* d\mu(\theta)$$

$$Q_2 = \int_{\partial \mathcal{F}X} O_{\theta} \begin{pmatrix} 0^r \\ 0 \\ 0^{(n-r)} \end{pmatrix} O_{\theta}^* d\mu(\theta)$$
where \( D_\lambda = \text{diag}(\lambda_1, \ldots, \lambda_{n-r}) \), and \( O_\theta \) is the orthogonal matrix corresponding to the unique element in \( K \) that sends \( e_1 \) to \( v(x, \theta) \), the direction at \( x \) pointing towards \( \theta \). Moreover, there exists a constant \( c > 0 \) that only depends on \( X \), so that \( \lambda_i \geq c \) for \( 1 \leq i \leq n - r \). For more details, we refer the readers to the original [7].

Denote by \( \bar{Q}_2 \) the quadratic form given by

\[
\bar{Q}_2 = \int_{\partial F} O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & I^{(n-r)} \end{pmatrix} O_\theta^* d\mu(\theta)
\]

Then the difference \( Q_2 - c\bar{Q}_2 \) is positive semi-definite, hence \( \det(Q_2|_S) \geq \det(c\bar{Q}_2|_S) \).

So in order to show Theorem 3.2.1, it suffices to assume \( Q_2 \) has the matrix form

\[
\int_{\partial F} O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & I^{(n-r)} \end{pmatrix} O_\theta^* d\mu(\theta)
\]

Given any \( v \in T_x X \), we have the following geometric estimates on the value of the quadratic form

\[
Q_1(v, v) = \int_{\partial F} v^t O_\theta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} O_\theta^* v d\mu(\theta) \tag{3.2.1}
= \int_{\partial F} \langle O_\theta^* v, e_1 \rangle^2 d\mu(\theta)
\leq \int_{\partial F} \sum_{i=1}^{r} \langle O_\theta^* v, e_i \rangle^2 d\mu(\theta)
= \int_{\partial F} \sin^2(\angle(O_\theta^* v, F)) d\mu(\theta)
\]

Roughly speaking, \( Q_1(v, v) \) is bounded above by the weighted average of the time the \( K \)-orbit spends away from \( F^\perp \). Similarly, we can estimate

\[
Q_2(v, v) = \int_{\partial F} v^t O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & I^{(n-r)} \end{pmatrix} O_\theta^* v d\mu(\theta) \tag{3.2.2}
= \int_{\partial F} \sum_{i=r+1}^{n} \langle O_\theta^* v, e_i \rangle^2 d\mu(\theta)
= \int_{\partial F} \sin^2(\angle(O_\theta^* v, F)) d\mu(\theta)
\]
So again, $Q_2(v,v)$ roughly measures the weighted average of the time the $K$-orbit spends away from $\mathcal{F}$.

### 3.2.2 Eigenvalue Matching

In their original paper, Connell and Farb showed an eigenvalue matching theorem [7, Theorem 4.4], in order to get the Jacobian estimate in top dimension. For the small eigenvalues of $Q_2$ (there are at most $r$ of them), they want to find twice as many comparatively small eigenvalues of $Q_1$. Then by taking the product of those eigenvalues, they obtain a uniform upper bound on the ratio of determinants $\det(Q_1)^{1/2}/\det(Q_2)$, which yields an upper bound on the Jacobian. However, as was pointed out by Inkang Kim and Sungwoon Kim, there was a mistake in the proof. Connell and Farb fixed the gap by showing a weak eigenvalue matching theorem [8, Theorem 0.1], which was sufficient to imply the Jacobian inequality.

We generalize this method and show that in fact we can find $(r - 2)$ additional small eigenvalues of $Q_1$ that are bounded by a universal constant times the smallest eigenvalue of $Q_2$. This allows for the Jacobian inequality to be maintained when we pass down to a subspace of codimension at most $(r - 2)$. We now state our version of the weak eigenvalue matching theorem.

**Definition 3.2.2.** We call a set of unit vectors $\{w_1, \ldots, w_k\}$ a $\delta$-orthonormal $k$-frame if $\langle w_i, w_j \rangle < \delta$ for all $1 \leq i < j \leq k$.

**Theorem 3.2.3.** (Weak eigenvalue matching.) Let $X$ be an irreducible symmetric space of non-compact type, with $r = \text{rank}(X) \geq 2$, excluding $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ and $\text{SL}(4, \mathbb{R})/\text{SO}(4)$. There exist constants $C'$, $C$, $\delta$ that only depend on $X$ so that the following holds. Given any $\epsilon < \delta$, and any orthonormal $k$-frame $\{v_1, \ldots, v_k\}$ in $T_x X$
with \( k \leq r \), whose span \( V \) satisfies \( \angle(V, F) \leq \epsilon \), then there is a \((C')\)-orthonormal \((2k + r - 2)\)-frame given by vectors \( \{v'_1, \ldots, v'_r, v''_1, \ldots, v''_r, v'_1, v''_1, \ldots, v'_k, v''_k\} \), such that for \( i = 1, \ldots, k \), and \( j = 1, \ldots, r \), we have the following inequalities:

\[
\angle(hv'_i, F^\perp) \leq C \angle(hv_i, F) \\
\angle(hv''_i, F^\perp) \leq C \angle(hv_i, F) \\
\angle(hv''_i, F^\perp) \leq C \angle(hv_1, F)
\]

for all \( h \in K \), where \( hv \) is the linear action of \( h \in K \) on \( v \in T_xX \simeq p \).

The proof of Theorem 3.2.3 will be delayed to Section 3.3.

### 3.2.3 Proof of Theorem 3.2.1

In this section, we will prove Theorem 3.2.1 using Theorem 3.2.3. Before starting the proof, we will need the following three elementary results from linear algebra.

**Lemma 3.2.4.** Let \( Q \) be a positive definite quadratic form on some Euclidean space \( V \) of dimension \( n \), with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). Let \( W \subseteq V \) be a subspace of codimension \( l \), and let \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-l} \) be the eigenvalues of \( Q \) restricted to \( W \). Then \( \lambda_i \leq \mu_i \leq \lambda_{i+l} \) holds for \( i = 1, \ldots, n - l \).

**Proof.** We argue by contradiction. Assume \( \mu_i > \lambda_{i+l} \) for some \( i \). Take the subspace \( W_0 \subseteq W \) spanned by the eigenvectors corresponding to \( \mu_i, \mu_{i+1}, \ldots, \mu_{n-l} \); clearly \( \dim(W_0) = n - l - i + 1 \). So for any nonzero vectors \( v \in W_0 \), we have \( Q(v, v) \geq \mu_i \|v\|^2 > \lambda_{i+l} \|v\|^2 \). However, if we denote \( V_0 \subseteq V \) the \((i + l)\)-dimensional subspace spanned by the eigenvectors corresponding to \( \lambda_1, \ldots, \lambda_{i+l} \), we have \( Q(v, v) \leq \lambda_{i+l} \|v\|^2 \) for any \( v \in V_0 \). But \( \dim(W_0 \cap V_0) \geq \dim(W_0) + \dim(V_0) - \dim(V) = 1 \) implies \( W_0 \cap V_0 \)
is nontrivial, so we obtain a contradiction. This establishes \( \mu_i \leq \lambda_{i+1} \). A similar argument shows \( \lambda_i \leq \mu_i \).

\[ \text{Lemma 3.2.5.} \text{ Let } Q \text{ be a positive definite quadratic form on some Euclidean space } V \text{ of dimension } n, \text{ with eigenvalues } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \text{ If } \{v_1, \ldots, v_n\} \text{ is any orthonormal frame of } V, \text{ ordered so that } Q(v_1, v_1) \leq Q(v_2, v_2) \leq \cdots \leq Q(v_n, v_n), \text{ then } Q(v_i, v_i) \geq \lambda_i / n \text{ for } i = 1, \ldots, n. \]

**Proof.** We show this by induction on the dimension of \( V \). The statement is clear when \( n = 1 \), so let us now assume we have the statement for \( \dim(V) = n - 1 \). Now if \( \dim(V) = n \), we restrict the quadratic form \( Q \) to the \((n - 1)\)-dimensional subspace \( W \) spanned by \( v_1, \ldots, v_{n-1} \), and denote the eigenvalues of \( Q|_W \) by \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \). By the induction hypothesis and Lemma 3.2.4, we obtain

\[
Q(v_i, v_i) \geq \frac{\mu_i}{n - 1} \geq \frac{\lambda_i}{n - 1} \geq \frac{\lambda_i}{n}
\]

for \( 1 \leq i \leq n - 1 \). Finally, for the last vector, we have

\[
Q(v_n, v_n) \geq \frac{Q(v_1, v_1) + \cdots + Q(v_{n}, v_{n})}{n} = \frac{\text{tr}(Q)}{n} = \frac{\lambda_1 + \cdots + \lambda_n}{n} \geq \frac{\lambda_n}{n}
\]

This completes the proof of the lemma.

\[ \text{Lemma 3.2.6.} \text{ Let } Q \text{ be a positive definite quadratic form on some Euclidean space } V \text{ of dimension } n. \text{ If } \{v_1, \ldots, v_k\} \text{ is any } \tau\text{-orthonormal } k\text{-frame for } \tau \text{ sufficiently small (only depends on } n), \text{ ordered so that } Q(v_1, v_1) \leq \cdots \leq Q(v_k, v_k), \text{ then there is an orthonormal } k\text{-frame } \{u_1, \ldots, u_k\} \text{ such that } Q(u_i, u_i) \leq 2Q(v_i, v_i). \]

**Proof.** We do the Gram-Schmidt process on \( \{v_1, \ldots, v_k\} \) and obtain an orthonormal \( k\)-frame \( \{u_1, \ldots, u_k\} \). Notice \( \{v_1, \ldots, v_k\} \) is \( \tau\)-orthonormal, so we have \( u_i = v_i + O(\tau)v_1 + \ldots + O(\tau)v_{i-1} + O(\tau)v_{i+1} + \ldots + O(\tau)v_k \) for \( i = 1, \ldots, k \). By the induction hypothesis and Lemma 3.2.4, we obtain

\[
Q(u_i, u_i) \leq \frac{Q(v_1, v_1) + \cdots + Q(v_{k}, v_{k})}{k} = \frac{\text{tr}(Q)}{k} = \frac{\lambda_1 + \cdots + \lambda_k}{k} \leq \frac{\lambda_k}{k}
\]

for \( 1 \leq i \leq k \).
... + \(O(\tau)v_i\), where by \(O(\tau)\) we denote a number that has universal bounded (only depends on \(n\)) ratio with \(\tau\). This implies

\[
Q(u_i, u_i) = Q(v_i, v_i) + O(\tau) \sum_{1 \leq s \leq t \leq i} Q(v_s, v_t)
\]

Since \(\left|Q(v_s, v_t)\right| \leq \sqrt{Q(v_s, v_s)Q(v_t, v_t)} \leq Q(v_i, v_i)\), we obtain

\[
Q(u_i, u_i) \leq Q(v_i, v_i) + O(\tau)Q(v_i, v_i) \leq 2Q(v_i, v_i)
\]

for \(\tau\) sufficiently small. This completes the proof of the lemma.

We are now ready to establish Theorem 3.2.1.

**Proof.** As was shown in [7, Section 4.4], for any fixed \(\epsilon_0 \leq 1/(r + 1)\), there are at most \(r\) eigenvalues of \(Q_2\) that are smaller than \(\epsilon_0\) (we will choose \(\epsilon_0\) in the course of the proof). By Lemma 3.2.4 the same is true for \(Q_2|_S\). We arrange these small eigenvalues in the order \(L_1 \leq L_2 \leq \ldots \leq L_k\), where \(k \leq r\). Observe that, if no such eigenvalue exists, then by Lemma 3.2.4, \(\det(Q_2|_S)\) is uniformly bounded below, and the theorem holds (since the eigenvalues of \(Q_1|_S\) are all \(\leq 1\)). So we will henceforth assume \(k \geq 1\). We denote the corresponding unit eigenvectors by \(v_1, \ldots, v_k\) (so that \(v_i\) has eigenvalue \(L_i\)). Although \(V = \text{span}\{v_1, \ldots, v_k\}\) might not have small angle with \(F\), it is shown in [8, Section 3] that there is a \(k_0 \in K\) so that \(\angle(k_0v_i, F) \leq 2\epsilon_0^{1/4}\) for each \(i\).

Let \(\epsilon\) be a constant small enough so that \(\epsilon < \delta\), where \(\delta\) is from Theorem 3.2.3, and also \(\tau := C'\epsilon\) satisfies the condition of Lemma 3.2.6 (where \(C'\) is obtained from Theorem 3.2.3). Hence the choice of \(\epsilon\) only depends on \(X\). We now make a choice of \(\epsilon_0\) such that \(2\epsilon_0^{1/4} < \epsilon\), and hence \(\angle(k_0V, F) < \epsilon\). (Note again the choice of \(\epsilon_0\) only depends on \(X\).)
Apply Theorem 3.2.3 to the frame \( \{k_0v_1, \ldots, k_0v_k \} \), and translate the obtained \((C'\epsilon)\)-orthonormal frame by \( k_0^{-1} \). This gives us a \((C'\epsilon)\)-orthonormal \((2k+r-2)\)-frame \( \{v'_1, v''_1, \ldots, v'_{(r)}, v''_1, v''_2, \ldots, v'_k, v''_k \} \), such that for \( i = 1, \ldots, k \), and \( j = 1, \ldots, r \), we have
\[
\angle(hv'_i, F^\perp) \leq C \angle(hv_i, F)
\]
\[
\angle(hv''_i, F^\perp) \leq C \angle(hv_i, F)
\]
\[
\angle(hv_{(j)}^i, F^\perp) \leq C \angle(hv_1, F)
\]
for all \( h \in K \) (note that we have absorbed the \( k_0 \)-translation into the element \( h \)).

We notice \( \angle(hv'_i, F^\perp) \leq C \angle(hv_i, F) \) implies \( \sin^2(\angle(hv'_i, F^\perp)) \leq C_0 \sin^2(\angle(hv_i, F)) \) for some \( C_0 \) depending on \( C \). For convenience, we still use \( C \) for this new constant. Hence, we obtain
\[
Q_1(v'_i, v'_i) \leq \int_{\partial_F X} \sin^2(\angle(O_\theta^*v'_i, F^\perp))d\mu(\theta)
\]
\[
\leq C \int_{\partial_F X} \sin^2(\angle(O_\theta^*v_i, F))d\mu(\theta) = CQ_2(v_i, v_i) = CL_i
\]
An identical estimate gives us \( Q_1(v''_i, v''_i) \leq CL_i \), and \( Q_1 \left(v_{(j)}^i, v_{(j)}^i \right) \leq CL_1 \).

We rearrange the \((C'\epsilon)\)-orthonormal \((2k+r-2)\)-frame as \( \{u'_1, u''_1, \ldots, u'_{(r)}, u'_2, u''_2, \ldots, u'_k, u''_k \} \) so that it has increasing order when applying \( Q_1 \). Then the inequalities still hold for this new frame:
\[
Q_1(u'_i, u'_i) \leq CL_i
\]
\[
Q_1(u''_i, u''_i) \leq CL_i
\]
\[
Q_1 \left(u_{(j)}^i, u_{(j)}^i \right) \leq CL_1
\]
Since the choice of \( \epsilon \) makes \( C'\epsilon \) satisfy the condition of Lemma 3.2.6, we apply the lemma to this \( C'\epsilon \)-orthonormal frame. This gives us an orthonormal \((2k+r-2)\)-frame
\[ \{ \overrightarrow{u}_1', \overrightarrow{u}_1'', \ldots, \overrightarrow{u}_r', \overrightarrow{u}_r'', \overrightarrow{u}_2', \overrightarrow{u}_2'', \ldots, \overrightarrow{u}_k', \overrightarrow{u}_k'' \}, \] such that

\[
Q_1(\overrightarrow{u}_i', \overrightarrow{u}_i') \leq 2Q_1(u_i', u_i') \leq 2CL_i
\]

\[
Q_1(\overrightarrow{u}_i'', \overrightarrow{u}_i'') \leq 2Q_1(u_i'', u_i'') \leq 2CL_i
\]

\[
Q_1(\overrightarrow{u}_i^{(j)}, \overrightarrow{u}_i^{(j)}) \leq 2Q_1(u_i^{(j)}, u_i^{(j)}) \leq 2CL_1
\]

Again, we can rearrange the orthonormal basis to have increasing order when applying \(Q_1\), and it is easy to check that, for the resulting rearranged orthonormal basis, the same inequalities still hold.

We denote the first \((2k + r - 2)\) eigenvalues of \(Q_1\) by \(\lambda_1' \leq \lambda_1'' \leq \ldots \leq \lambda_r' \leq \lambda_2 \leq \lambda_r'' \leq \ldots \leq \lambda_k' \leq \lambda_k''\), and the first \(2k\) eigenvalues of \(Q_1|S\) by \(\mu_1' \leq \mu_1'' \leq \ldots \leq \mu_k' \leq \mu_k''\).

Applying Lemma 3.2.5, we have

\[
\lambda_i' \leq nQ_1(\overrightarrow{u}_i', \overrightarrow{u}_i') \leq 2nCL_i
\]

\[
\lambda_i'' \leq nQ_1(\overrightarrow{u}_i'', \overrightarrow{u}_i'') \leq 2nCL_i
\]

\[
\lambda_i^{(j)} \leq nQ_1(\overrightarrow{u}_i^{(j)}, \overrightarrow{u}_i^{(j)}) \leq 2nCL_1
\]

for \(1 \leq i \leq k\) and \(1 \leq j \leq l\).

Notice \(\dim(S) \geq n - r + 2\). We apply Lemma 3.2.4 and obtain

\[
\mu_1' \leq \lambda_1^{(r-1)} \leq 2nCL_1
\]

\[
\mu_1'' \leq \lambda_1^{(r)} \leq 2nCL_1
\]

\[
\mu_i' \leq \lambda_i' \leq 2nCL_i
\]

\[
\mu_i'' \leq \lambda_i'' \leq 2nCL_i
\]
for \(2 \leq i \leq k\). The eigenvalues of \(Q_1|_S\) are bounded above by 1, and \(L_1, ..., L_k\) are the only eigenvalues of \(Q_2|_S\) that are below \(\epsilon_0\) (and recall the choice of \(\epsilon_0\) only depends on \(X\)). Therefore,

\[
\det(Q_1|_S) \leq \prod_{i=1}^{k} \mu'_i \mu''_i \leq \prod_{i=1}^{k} (2nC L_i)^2 \leq (2nC)^{2k} \left( \frac{\det(Q_2|_S)}{\epsilon_0^{\dim(S)-k}} \right)^2 \\
\leq \overline{C} \det(Q_2|_S)^2
\]

where \(\overline{C}\) only depends on \(X\). This completes the proof of Theorem 3.2.1. \(\square\)

3.3 Reduction to the Combinatorial Problem

In this section, we will prove the “weak eigenvalue matching” Theorem 3.2.3, which was introduced in Section 3.2.2. The approach is to follow [8], and reduce the theorem to a combinatorial problem. Then we apply Hall’s Marriage theorem to solve it.

3.3.1 Hall’s Marriage Theorem

We introduce the classic Hall’s Marriage Theorem, and later on we will apply a slightly stronger version (Corollary 3.3.3 below) in the proof of Lemma 3.3.5.

**Theorem 3.3.1** (Hall’s Marriage Theorem). Suppose we have a set of \(m\) different species \(A = \{a_1, ..., a_m\}\), and a set of \(n\) different planets \(B = \{b_1, ..., b_n\}\). Let \(\phi : A \rightarrow \mathcal{P}(B)\) be a map which sends a species to the set of all suitable planets for its survival. Then we can arrange for each species a different planet to survive if and only if for any subset \(A_0 \subseteq A\), we have the cardinality inequality \(|\phi(A_0)| \geq |A_0|\).

**Corollary 3.3.2.** Under the assumption of Theorem 3.3.1, we can arrange for each species two different planets if and only if for any subset \(A_0 \subseteq A\), we have the cardinality inequality \(|\phi(A_0)| \geq 2|A_0|\).
**Proof.** Assume there exists such arrangement, the cardinality condition holds obviously. On the other hand, assume we have the cardinality condition, we want to show there is an arrangement. We make an identical copy on each species and form the set \( A' = \{ a'_1, ..., a'_m \} \). We apply the Hall’s Marriage Theorem to the set \( A \cup A' \) relative to \( B \). Then for each \( i \), both species \( a_i \) and \( a'_i \) have its own planet, and that means there are two planets for the original species \( a_i \).

To see why the cardinality condition holds, we choose an arbitrary subset \( H \cup K' \subseteq A \cup A' \) where \( H \subseteq A \) and \( K' \subseteq A' \). Let \( K \) be the corresponding identical copy of \( K' \) in \( A \). We have \( \phi(H \cup K') = \phi(H \cup K) \geq 2|H \cup K| \geq |H| + |K| = |H \cup K'| \). This completes the proof.

**Corollary 3.3.3.** Suppose we have a set of vectors \( V = \{ v_1, ..., v_r \} \), and for each \( v_i \), the selectable set is denoted by \( B_i \subseteq B \). If for any subset \( V_0 = \{ v_{i_1}, ..., v_{i_k} \} \subseteq V \), we have \( |B_{i_1} \cup ... \cup B_{i_k}| \geq 2k + r - 2 \), then we can pick \( (3r - 2) \) distinct element \( \{ b'_1, ..., b'^{(r)}_1, b'_i, b''_i (2 \leq i \leq r) \} \) in \( B \) such that \( b'_1, ..., b'^{(r)}_1 \in B_1 \) and \( b'_i, b''_i \in B_i \).

**Proof.** First we choose \( V_0 \) the singleton set that consists of only \( v_1 \). By hypothesis, we have \( |B_1| \geq r \geq (r - 2) \), hence we are able to choose \( (r - 2) \) elements \( b^{(3)}_1, ..., b^{(r)}_1 \) for \( v_1 \). Next we can easily check the cardinality condition and apply Corollary 3.3.2 to the set \( V \) with respect to \( B \setminus \{ b^{(3)}_1, ..., b^{(r)}_1 \} \) to obtain the pairs \( \{ b'_i, b'_i \} \) (for each \( 1 \leq i \leq r \)). This completes the proof of this corollary.

### 3.3.2 Angle Inequality

Throughout this section, we will work exclusively with unit vectors in \( T_x X \simeq \mathfrak{p} \). We embed the point stabilizer \( K_x \) into \( \text{Isom}(T_x X) \simeq O(n) \), and endow it with the induced metric. This gives rise to a norm on \( K \), defined by \( \| k \| = \max_{v \in T_x X} \angle(v, kv) \),
\( \forall k \in K \). We denote the Lie algebra of \( K_x \simeq K \) by \( \mathfrak{k} \), which has root space decomposition \( \mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda^+} \mathfrak{k}_\alpha \). For each small element \( k \in K \), the action on a vector \( v \) can be approximated by the Lie algebra action, that is, if \( k = \exp(u) \) is small, then 
\[ ||[u, v]|| \approx ||kv - v|| \sim \angle(v, kv), \]
where we write \( A \sim B \) if \( A/B \) and \( B/A \) are both universally bounded. By abuse of notation, we do not distinguish between \( ||k|| \) and \( ||u|| \) inside a very small neighborhood \( \mathcal{U} \) of 0 inside \( \mathfrak{k} \). Although \( ||\cdot|| \) is not linear on \( \mathcal{U} \), it is linear up to a universal constant, that is, \( ||tu|| \sim t ||u|| \), for all \( u \in \mathcal{U} \) and \( t \) such that \( tu \in \mathcal{U} \). We now show the following lemmas.

**Lemma 3.3.4.** (Compare [8, Lemma 1.1]) Let \( X = G/K \) be a rank \( r \geq 2 \) irreducible symmetric space of non-compact type, and fix a flat \( \mathcal{F} \subseteq T_xX \) at \( x \). Then for any small \( \rho > 0 \), there is a constant \( C(\rho) \) with the following property. If \( v \in \mathcal{F} \) is arbitrary, and \( v^* \in \mathcal{F} \) is a maximally singular vector in the \( \rho \)-neighborhood of \( v \) (in the sense that the dimension of \( K_{v^*} \) is as large as possible), then
\[ \angle(hu, \mathcal{F}^\perp) \leq C \angle(hv, \mathcal{F}) \]
for any \( h \in K \), and \( u \in (K_{v^*}\mathcal{F})^\perp \simeq \bigoplus_{\alpha \in \Lambda^+, \alpha(v^*) \neq 0} \mathfrak{p}_\alpha \), where \( \Lambda^+ \) is the set of all positive roots. Moreover, we have
\[ \angle(hu, \mathcal{F}^\perp) \leq C \angle(hk_0v, K_{v^*}\mathcal{F}) \]
for any \( h \in K \), \( u \in (K_{v^*}\mathcal{F})^\perp \), and \( k_0 \in K_{v^*} \).

**Proof.** We only need to verify the inequality when \( \angle(hv, \mathcal{F}) \) is small. Notice for any vector \( v \in \mathcal{F} \), and any small element \( w \in \mathfrak{k}_\alpha = (I + \theta)g_\alpha = (I + \theta)(I - \theta)^{-1}p_\alpha \), the Lie algebra action (see Proposition 2.1.2) has norm
\[ ||[w, v]|| = || - \alpha(v) \cdot (I - \theta)(I + \theta)^{-1}w|| \sim ||\alpha(v)|| \cdot ||w||. \]
This is due to the fact that \((I + \theta)(I - \theta)^{-1}\) is a linear isomorphism between \(\mathfrak{f}_\alpha\) and \(\mathfrak{p}_\alpha\) (see Proposition 2.1.1), and when restricted to \(\mathfrak{f}_\alpha \cap U\), it preserves the norms up to a uniform multiplicative constant.

Infinitesimally speaking, for \(h = \exp(w)\), we have that \(hv - v = [w, v]\), so the estimate on the Lie algebra action tells us about the infinitesimal growth of \(\|hv - v\|\). We also see that, since \([w, v] \in \mathfrak{p}_\alpha\), \(h\) moves the vector \(v\) in the direction \(\mathfrak{p}_\alpha\) (which we recall is orthogonal to the flat \(\mathcal{F}\), see Proposition 2.1.1). Now \(v^*\) is a maximally singular vector in the \(\rho\)-neighborhood of the unit vector \(v\), so once \(\rho\) is small enough, if \(\alpha\) is any root with \(\alpha(v^*) \neq 0\), then \(\alpha(v)\) will be uniformly bounded away from zero (depending only on the choice of \(\rho\)). This shows that if a root \(\alpha\) satisfies \(\alpha(v^*) \neq 0\), then \(\angle(hv, \mathcal{F}) \sim \|h\|\) for all \(h \in \exp(\mathfrak{f}_\alpha \cap U)\).

Now we move to analyzing the general case \(h = \exp(w)\), where \(w \in \mathfrak{f}\) is arbitrary. If \(\angle(hv, \mathcal{F})\) is small, then it follows that the components of \(hv\) on each \(\mathfrak{p}_\alpha\) must be small. From the discussion above, this implies that the component of \(w\) in each \(\mathfrak{f}_\alpha|_{\alpha(v^*) \neq 0}\) is small, i.e. \(w\) almost lies in \(\mathfrak{f}_{v^*} = \mathfrak{f}_0 \oplus \bigoplus_{\alpha(v^*) \neq 0} \mathfrak{f}_\alpha\). Since \(h\) almost lies in \(K_{v^*}\), there exists an element \(h_0 \in K_{v^*}\) such that \(h_0^{-1}h\) is close to the identity. We write \(h = h_0h_1\), where \(h_1 = \exp(w_1) \in \exp(\mathfrak{f}_{v^*}) = \exp \left( \bigoplus_{\alpha(v^*) \neq 0} \mathfrak{f}_\alpha \right) \), and observe that the analysis in the previous paragraph applies to the element \(h_1\). Now observe that, infinitesimally, \(h_1v - v = [u_1, v] \in \bigoplus_{\alpha(v^*) \neq 0} \mathfrak{p}_\alpha\), so \(h_1\) moves \(v\) in a direction lying in \(\bigoplus_{\alpha(v^*) \neq 0} \mathfrak{p}_\alpha\). On the other hand, infinitesimally, \(K_{v^*}\) moves the entire flat \(\mathcal{F}\) in the directions \(\bigoplus_{\alpha(v^*) = 0} \mathfrak{p}_\alpha\) (corresponding to the action of its Lie algebra \(\mathfrak{f}_{v^*}\)). But these two directions are orthogonal, which means that \(h_1v\) leaves not just \(\mathcal{F}\) orthogonally, but actually leaves orthogonally to the entire orbit \(K_{v^*}\). This allows us to estimate

\[
\angle(hv, \mathcal{F}) = \angle(h_1v, h_0^{-1}h) \geq \angle(h_1v, K_{v^*}) \sim \|h_1\|, 
\]

(3.3.2)
where at the last step, we use that $h_1$ moves $v$ orthogonally off the $K_{v^*}$ orbit of $F$.

On the other hand, we are assuming that the vector $u$ lies in $(K_{v^*} F)^\perp$, hence also in $h_0^{-1} F^\perp$. So we have the sequence of inequalities

$$
\angle(hu, F^\perp) = \angle(h_1 u, h_0^{-1} F^\perp) \leq \angle(h_1 u, u) \leq ||h_1||.
$$

Combining equations (3.3.2) and (3.3.3) gives us the first inequality.

Similarly, $\angle(hk_0 v, K_{v^*} F)$ being small also implies that the component of $h$ on each $\mathfrak{t}_{\alpha} |_{\alpha(v^*)\neq 0}$ is small. So by writing $h = h_0 h_1$ in the same manner, we get

$$
\angle(hk_0 v, K_{v^*} F) = \angle(h_1 k_0 v, K_{v^*} F) = \angle(k_0^{-1} h_1 k_0 v, K_{v^*} F).
$$

Notice that $K_{v^*}$ conjugates $\mathfrak{t}_{v^*}$ to itself, so $k_0^{-1} h_1 k_0$ is an element in $\exp(\mathfrak{t}_{v^*})$. In view of equation (3.3.1) and the fact that $k_0^{-1} h_1 k_0 v$ leaves orthogonally to $K_{v^*} F$, we obtain $\angle(k_0^{-1} h_1 k_0 v, K_{v^*} F) \sim ||k_0^{-1} h_1 k_0|| = ||h_1||$. Combining this estimate with equation (3.3.3) gives the second inequality.

\[\square\]

**Lemma 3.3.5.** Let $X = G/K$ be a rank $r \geq 2$ irreducible symmetric space of non-compact type excluding $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ and $\text{SL}(4, \mathbb{R})/\text{SO}(4)$, and fix a flat $F \subseteq T_x X$ at $x$. Then there exists a constant $C > 0$ that only depends on $X$, such that for any $\frac{1}{2}$-orthonormal $r$-frame $\{v_1, ..., v_r\}$ in $F$, there is an orthonormal $(3r - 2)$-frame $\{v'_1, v''_1, ..., v'_r, v''_i (2 \leq i \leq r)\}$ in $F^\perp$ such that

$$
\angle(hv'_i, F^\perp) \leq C \angle(hv_i, F)
$$

$$
\angle(hv''_i, F^\perp) \leq C \angle(hv_i, F)
$$

$$
\angle(hv''_i, F^\perp) \leq C \angle(hv_i, F)
$$

$$
\angle(hv''_i, F^\perp) \leq C \angle(hv_i, F)
$$

for all $h \in K$, $i = 2, ..., r$, $j = 1, ..., r$. 

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Proof. Once we have chosen a parameter $\rho$, we will denote by $v_i^*$ a maximally singular vector in $\mathcal{F}$ that is $\rho$-close to $v_i$, and we will let $Q_i = (Kv_i^*\mathcal{F})^\perp \simeq \bigoplus_{\alpha \in \Lambda^+, \alpha(v_i^*) \neq 0} p_\alpha$. We now fix an $\rho$ small enough so that, for every $\frac{1}{2}$-orthonormal $r$-frame $\{v_1, \ldots, v_r\} \subset \mathcal{F}$, the corresponding $\{v_i^*\}_{i=1}^r$ are distinct. For each $v_i$, the vectors in $Q_i$ are the possible choice of vectors that satisfy the angle inequality provided by Lemma 3.3.4. So it suffices to find $r$ vectors in $Q_1$, and two vectors in each $Q_i (i \neq 1)$, such that the chosen $(3r - 2)$ vectors form an orthonormal frame.

Now for each root $\alpha$, we pick an orthonormal frame $\{b_\alpha\}$ on $p_\alpha$, we collect them into the set $B := \{b_\alpha\}_{i=1}^{\lceil n-r \rceil}$, which forms an orthonormal frame on $\mathcal{F}^\perp$. We will pick the $(3r - 2)$-frame from the vectors in $B$. For instance, vector $v_1$ has selectable set $B_1 := Q_1 \cap B$, in which we want to choose $r$ elements, while for $i = 2, \ldots, r$, vector $v_i$ has selectable set $B_i := Q_i \cap B$, from which we want to choose two elements. Most importantly, the $(3r - 2)$ chosen vectors have to be distinct from each other. This is a purely combinatorial problem, and can be solved by using Hall’s Marriage theorem.

In view of Corollary 3.3.3, we only need to check the cardinality condition. We notice the selectable set of $v_i$ is $B_i$ which spans $Q_i$, so $|B_i| = \dim(Q_i)$. The next lemma will estimate the dimension of the $Q_i$, and hence will complete the proof of Lemma 3.3.5.

\begin{lemma}
Let $X = G/K$ be a rank $r \geq 2$ irreducible symmetric space of non-compact type, excluding $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ and $\text{SL}(4, \mathbb{R})/\text{SO}(4)$, and fix a flat $\mathcal{F}$. Assume $\{v_1^*, \ldots, v_r^*\}$ spans $\mathcal{F}$, and let $Q_i = Kv_i^*\mathcal{F}$. Then for any subcollection of vectors $\{v_i^*, \ldots, v_k^*\}$, we have $\dim(Q_{i_1} + \ldots + Q_{i_k}) \geq (2k + r - 2)$.
\end{lemma}
Proof. Since \( Q_\iota = (K v^*_\iota F)^\perp \cong \bigoplus_{\alpha \in \Lambda^+, \alpha(v^*_\iota) \neq 0} \mathfrak{p}_\alpha \), we obtain \( Q_\iota + ... + Q_k \)
= \( \bigoplus_{\alpha \in \Lambda^+, \alpha(V) \neq 0} \mathfrak{p}_\alpha \), where \( V = \text{Span}(v^*_1, ..., v^*_k) \). We can estimate
\[
\dim(Q_\iota + ... + Q_k) = \sum_{\alpha \in \Lambda^+, \alpha(V) \neq 0} \dim(\mathfrak{p}_\alpha) \geq \left| \{ \alpha \in \Lambda, H_\alpha \in V^\perp \} \right|
= \frac{1}{2} \left( |\Lambda| - \left| \{ \alpha \in \Lambda, H_\alpha \in V^\perp \} \right| \right),
\]
where \( V^\perp \) is the orthogonal complement of \( V \) in \( F \), and \( H_\alpha \) is the vector in \( F \) that represents \( \alpha \).

Now we denote \( t_\iota = \frac{1}{2} \max_{U \subseteq F, \dim(U) = i} \left| \{ \alpha \in \Lambda, H_\alpha \in U \} \right| \), the number of positive roots in the maximally rooted \( i \)-dimensional subspace. We use the following result that appears in the proof of [7, Lemma 5.2]. For completeness, we also add their proof here.

Claim 3.3.7. [7, Lemma 5.2] \( t_\iota - t_{\iota - 1} \geq i, \) for \( 1 \leq i \leq r - 1 \).

Proof. This is proved by induction on \( i \). For \( i = 1 \), the inequality holds since \( t_0 = 0 \) and \( t_1 = 1 \). Assuming \( t_{\iota - 1} - t_{\iota - 2} \geq i - 1 \) holds, we let \( V_{\iota - 1} \) be an \((i - 1)\)-dimensional maximally rooted subspace. By definition, the number of roots that lie in \( V_{\iota - 1} \) is \( 2t_{\iota - 1} \). There exists a root \( \alpha \) so that \( H_\alpha \) does not lie in \( V_{\iota - 1} \), and also does not lie on its orthogonal complement (by irreducibility of the root system). So \( H_\alpha^\perp \cap V_{\iota - 1} := Z \) is a codimension one subspace in \( V_{\iota - 1} \). By the induction hypothesis, there are at least \( i - 1 \) pairs of root vectors that lie in \( V_{\iota - 1} - Z \), call them \( \pm H_{\alpha_1}, ..., \pm H_{\alpha_{i-1}} \). Hence by properties of root system [11, Proposition 2.9.3], either \( \pm (H_\alpha + H_{\alpha_l}) \) or \( \pm (H_\alpha - H_{\alpha_l}) \) is a pair of root vectors, for each \( 1 \leq l \leq i - 1 \). Along with \( \pm H_\alpha \), these pairs of vectors lie in \((V_{\iota - 1} \oplus \langle H_\alpha \rangle) - V_{\iota - 1} \). We have now found \( 2i \) root vectors in the \( i \)-dimensional subspace \( V_{\iota - 1} \oplus \langle H_\alpha \rangle \), which do not lie on the maximally rooted subspace \( V_{\iota - 1} \). This shows \( t_\iota - t_{\iota - 1} \geq i \), proving the claim. \( \square \)
Finally, we can estimate \( \dim(Q_{i_1} + \ldots + Q_{i_k}) \geq \frac{1}{2} \left( |\Lambda| - \left| \{ \alpha \in \Lambda, H_\alpha \in V^\perp \} \right| \right) \geq t_r - t_{r-k} \). Using the Claim, a telescoping sum gives us \( t_r - t_{r-k} \geq (r-1) + \ldots + (r-k+1) = k(2r-k+1)/2 \), whence the lower bound \( \dim(Q_{i_1} + \ldots + Q_{i_k}) \geq k(2r-k+1)/2 \).

When \( r \geq 4 \), or \( k < r = 3 \), or \( k < r = 2 \), it is easy to check that \( k(2r-k+1)/2 \geq 2k + r - 2 \). This leaves the case when \( r = k = 3 \), or \( r = k = 2 \). When \( r = k = 3 \), we can instead estimate \( \dim(Q_1 + Q_2 + Q_3) = \dim(F^\perp) = n - 3 \geq 7 = 2k + r - 2 \), provided \( n \geq 10 \), which only excludes the rank three symmetric space \( \text{SL}(4, \mathbb{R})/\text{SO}(4) \).

A similar analysis when \( r = k = 2 \) only excludes the rank two symmetric space \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \). This completes the proof of Lemma 3.3.6, hence completing the proof of Lemma 3.3.5.

\[ \square \]

**Remark.** In the rank two case, both Theorem 3.2.3 and Theorem 3.2.1 only give you statements about degree = \( n \). Our Main Theorem then only gives surjectivity of comparison maps in top degree, which agrees with the result of [17], and the corresponding Jacobian estimate is consistent with [7] [8].

### 3.3.3 Proof of Theorem 3.2.3

We assume \( k = r \) without loss of generality since otherwise we can always extend the \( k \)-frame to an \( r \)-frame that has small angle to \( F \). Our first step is to move the frame so as to lie in \( F \), while controlling the angles between the resulting vectors (so that we can apply Lemma 3.3.5). This is done by first moving the vectors to the respective \( K_{v_f}F \), and then moving to \( F \).

As in the proof of Lemma 3.3.4, \( \angle(v_i, F) \) being small implies that the components of \( v_i \) on each \( p_\alpha \) is small. The \( K \)-orbit of \( v_i \) intersects \( F \) finitely many times (exactly once in each Weyl chamber), and if each of these intersections is \( \rho \)-close to a maximally
singular vector, choose \( v^*_i \) to be the one closest to \( v_i \). The element in \( K \) moving \( v_i \) to \( F \) will almost lie in \( K_{v_i^*} \) (by an argument similar to the one in Lemma 3.3.4). By decomposing this element as a product \( \hat{k}_i k_i \), we obtain a small \( k_i \) which sends \( v_i \) to \( K_{v_i^*} F \) (and \( \hat{k}_i \in K_{v_i^*} \)). If \( k_i^{-1} = \exp(u_i) \), we have \( u_i \in \bigoplus_{\alpha \in \Lambda^+, \alpha(v^*) \neq 0} t_{\alpha} \).

We now estimate the norm \( ||k_i|| \). From the identification of norms in a small neighborhood of the identity, we have \( ||k_i|| = ||u_i|| \). Since \( \hat{k}_i \) is an element in \( K_{v_i^*} \) that sends \( k_i v_i \) to \( F \), an argument similar to the proof of second inequality in Lemma 3.3.4 gives us

\[
\angle(v_i, K_{v_i^*} F) = \angle((\hat{k}_i k_i^{-1} \hat{k}_i^{-1})(\hat{k}_i k_i v_i), K_{v_i^*} F) \sim_{\rho} ||\hat{k}_i k_i^{-1} \hat{k}_i^{-1}|| = ||k_i||
\]

(where the constant will depend on the choice of \( \rho \)). On the other hand, since \( F \subset K_{v_i^*} F \), we obtain \( \angle(v_i, K_{v_i^*} F) \leq \angle(v_i, F) \). But by hypothesis, \( \angle(v_i, F) < \epsilon \). Putting all this together, we see that, for each fixed \( \rho \), there exists a constant \( C' \) that only depends on \( X \), so that each of the \( ||k_i|| \) is bounded above by \( \frac{1}{2} C' \epsilon \). In particular, any \( \{k_i\}_{i=1}^r \) perturbation of an orthonormal frame gives rise to a \( C' \epsilon \)-orthonormal frame, and hence the collection \( \{k_1 v_1, \ldots, k_r v_r\} \) forms a \( C' \epsilon \)-orthonormal frame.

Next, since \( \hat{k}_i \) is an element in \( K_{v_i^*} \), it leaves \( v^*_i \) fixed. From triangle inequality we obtain

\[
\angle(\hat{k}_i k_i v_i, k_i v_i) \leq 2 \angle(k_i v_i, v^*_i) < 2 \rho.
\]

It follows that the collection of vectors \( \{\hat{k}_1 k_1 v_1, \ldots, \hat{k}_r k_r v_r\} \subset F \) is obtained from the \( C' \epsilon \)-orthonormal frame \( \{k_1 v_1, \ldots, k_r v_r\} \) by rotating each of the various vectors by an angle of at most \( 2 \rho \) hence forms a \( (C' \epsilon + 4 \rho) \)-orthonormal basis in \( F \). In particular, once \( \rho \) and \( \delta \) are chosen small enough, it gives us a 1/2-orthonormal basis inside \( F \).
Applying Lemma 3.3.5 to the 1/2-orthonormal frame \( \{ \hat{k}_1 k_1 v_1, \ldots, \hat{k}_r k_r v_r \} \subset \mathcal{F} \) gives us an orthonormal \((3r - 2)\)-frame \( \{ v'_1, \ldots, v'^{(r)}, v''_i, v''_i \ (2 \leq i \leq r) \} \) such that the angle inequalities hold. Now by the second inequality of Lemma 3.3.4, we have the following inequalities:

\[
\angle(hv'_i, \mathcal{F}^\perp) \leq C \angle(hk_i v_i, K v_i^* \mathcal{F}) \leq C \angle(hk_i v_i, \mathcal{F})
\]

\[
\angle(hv''_i, \mathcal{F}^\perp) \leq C \angle(hk_i v_i, K v_i^* \mathcal{F}) \leq C \angle(hk_i v_i, \mathcal{F})
\]

\[
\angle(hv^{(j)}_1, \mathcal{F}^\perp) \leq C \angle(hk_1 v_1, K v_1^* \mathcal{F}) \leq C \angle(hk_1 v_1, \mathcal{F})
\]

for \(2 \leq i \leq r, 1 \leq j \leq r\) and any \( h \in K \). Finally, we translate each of the vectors \( v'_i, v''_i \) by \( k_i^{-1} \), and each \( v^{(j)}_1 \) by \( k_1^{-1} \), producing a \( C' \epsilon \)-orthonormal \((3r - 2)\)-frame that satisfies the inequalities in Theorem 3.2.3, hence completing the proof.
Chapter 4: Surjectivity of the Comparison Map in Bounded Cohomology

In this Section, we provide some background on cohomology (see Section 4.1), establish the Main Theorem 1 (Section 4.2), establish some limitations on our technique of proof (Section 4.3), and work out a detailed class of examples (Section 4.4).

4.1 Bounded Cohomology

Let $X = G/K$ be a symmetric space of non-compact type, and $\Gamma$ be a cocompact lattice in $G$. We recall the definition of group cohomology, working with $\mathbb{R}$ coefficients (so that we can relate these to the de Rham cohomology). Let $C^n(\Gamma, \mathbb{R}) = \{ f : \Gamma^n \to \mathbb{R} \}$ be the space of $n$-cochains. Then the coboundary map $d : C^n(\Gamma, \mathbb{R}) \to C^{n+1}(\Gamma, \mathbb{R})$ is defined by

$$
d(f(\gamma_1, \ldots, \gamma_{n+1})) = f(\gamma_2, \ldots, \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i \gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{n+1})$$

$$+ (-1)^{n+1} f(\gamma_1, \ldots, \gamma_n)$$

The homology of this chain complex is $H^*(\Gamma, \mathbb{R})$, the group cohomology of $\Gamma$ with $\mathbb{R}$ coefficients. Moreover, if we restrict the cochains above to bounded functions, we obtain the space of bounded $n$-cochains $C^*_b(\Gamma, \mathbb{R}) = \{ f : \Gamma^n \to \mathbb{R} \mid f \text{ is bounded} \}$.
and the corresponding bounded cohomology $H^*_b(\Gamma, \mathbb{R})$ of $\Gamma$. The inclusion of the bounded cochains into the ordinary cochains induces the comparison map $H^*_b(\Gamma, \mathbb{R}) \rightarrow H^*(\Gamma, \mathbb{R})$.

Similarly, we can define the (bounded) continuous cohomology of $G$, by taking the space of continuous $n$-cochains $C^n_c(G, \mathbb{R}) = \{ f : G^n \rightarrow \mathbb{R} \mid f \text{ is continuous} \}$ or the space of bounded continuous cochains $C^n_{c,b}(G, \mathbb{R}) = \{ f : G^n \rightarrow \mathbb{R} \mid f \text{ is continuous and bounded} \}$. With the same coboundary maps as above, this gives two new chain complexes, whose homology will be denoted by $H^*_c(G, \mathbb{R})$ and $H^*_{c,b}(G, \mathbb{R})$ respectively. Again, one has a naturally induced comparison map $H^*_{c,b}(G, \mathbb{R}) \rightarrow H^*_c(G, \mathbb{R})$.

Now let $M = X/\Gamma$ be the closed locally symmetric space covered by $X$. Note that $M$ is a $K(\Gamma, 1)$, so

$$H^*_dR(M, \mathbb{R}) \simeq H^*_{sing}(M, \mathbb{R}) \simeq H^*(\Gamma, \mathbb{R})$$

The isomorphism between the de Rham cohomology and group cohomology is explicitly given by

$$\phi : H^k_dR(M, \mathbb{R}) \rightarrow H^k(\Gamma, \mathbb{R})$$

$$\omega \mapsto f_\omega$$

where $f_\omega(\gamma_1, \ldots, \gamma_k) = \int_{\Delta(\gamma_1, \ldots, \gamma_k)} \tilde{\omega}$. Here, $\tilde{\omega}$ is a lift of $\omega$ to $X$, and $\Delta(\gamma_1, \ldots, \gamma_k)$ is any natural $C^1$ $k$-filling with ordered vertices $\{ x, \gamma_1 x, (\gamma_1 \gamma_2) x, \ldots, (\gamma_1 \gamma_2 \cdots \gamma_k) x \}$ for some fixed basepoint $x \in X$ (for instance, one can choose $\Delta(\gamma_1, \ldots, \gamma_k)$ to be the geodesic coning simplex, see Dupont [10]). Alternatively, we can use the barycentric straightened $C^1$ simplex $st(\Delta(\gamma_1, \ldots, \gamma_k))$ (which we defined in Section 3.1). That is to say, if we define $\overline{f}_\omega(\gamma_1, \ldots, \gamma_k) = \int_{st(\Delta(\gamma_1, \ldots, \gamma_k))} \tilde{\omega}$, then $\overline{f}_\omega$ represents the same cohomology class as $f_\omega$. This is due to the fact that the barycentric straightening
is $\Gamma$-equivariant (see [17, Section 3.2]). We call $\bar{f}_\omega$ the barycentrically straightened cocycle.

On the other hand, there is a theorem of van Est [20] which gives the isomorphism between the relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{t}, \mathbb{R})$ and the continuous bounded cohomology $H^c_*(G, \mathbb{R})$. A class in $H^k(\mathfrak{g}, \mathfrak{t}, \mathbb{R})$ can be expressed by an alternating $k$-form $\varphi$ on $\mathfrak{g}/\mathfrak{t} \simeq T_xX$. By left translation, it gives a closed $C^\infty$ $k$-form $\tilde{\varphi}$ on $X = G/K$. In [10], this isomorphism is explicitly given by

$$\phi : H^k(\mathfrak{g}, \mathfrak{t}, \mathbb{R}) \to H^k_*(G, \mathbb{R})$$

$$\varphi \mapsto f_\varphi$$

where $f_\varphi(g_1, \ldots, g_k) = \int_{\Delta(g_1, \ldots, g_k)} \tilde{\varphi}$, and $\Delta(g_1, \ldots, g_k)$ is the geodesic simplex with ordered vertices consisting of $\{x, g_1x, (g_1g_2)x, \ldots, (g_1g_2 \cdots g_k)x\}$ for some fixed basepoint $x \in X$. Again, we can replace $\Delta(g_1, \ldots, g_k)$ by the barycentric straightened $C^1$ simplex $st(\Delta(g_1, \ldots, g_k))$, and the resulting barycentrically straightened function $\bar{f}_\varphi(g_1, \ldots, g_k) = \int_{st(\Delta(g_1, \ldots, g_k))} \tilde{\varphi}$ is in the same cohomology class as $f_\varphi$.

### 4.2 Proof of the Main Theorem 1

In this section, we use Theorem 3.2.1 to establish the **Main Theorem 1**. We need to show both comparison maps $\eta$ and $\eta'$ are surjective. Let us start with $\eta$. We use the van Est isomorphism (see Section 4.1) to identify $H^*_c(G, \mathbb{R})$ with $H^*(\mathfrak{g}, \mathfrak{t}, \mathbb{R})$. For any class $[f_\varphi] \in H^k_*(G, \mathbb{R})$ where $f_\varphi(g_1, \ldots, g_k) = \int_{\Delta(g_1, \ldots, g_k)} \tilde{\varphi}$, we instead choose the barycentrically straightened representative $\bar{f}_\varphi$. Then for any $(g_1, \ldots, g_k) \in G^k$, we have

$$|\bar{f}_\varphi(g_1, \ldots, g_k)| = \left| \int_{st(\Delta(g_1, \ldots, g_k))} \tilde{\varphi} \right| \leq \left| \int_{\Delta^k} st^*_V \tilde{\varphi} \right| = \int_{\Delta^k} |Jac(st_V)| \cdot \|	ilde{\varphi}\| d\mu_0 \quad (4.2.1)$$

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where \( d\mu_0 \) is the standard volume form of \( \Delta^k_s \). But from Proposition 3.1.1 and Theorem 3.2.1, the expression \( |\text{Jac}(st_V)| \) is uniformly bounded above by a constant (independent of the choice of vertices \( V \) and the point \( \delta \in \Delta^k_s \)), while the form \( \tilde{\varphi} \) is invariant under the \( G \)-action, hence bounded in norm. It follows that the last expression above is less than some constant \( C \) that depends only on the choice of alternating form \( \varphi \).

We have thus produced, for each class \([f\varphi]\) in \( H^k_c(G, \mathbb{R}) \), a bounded representative \( \tilde{f}_\varphi \). So the comparison map \( \eta \) is surjective. The argument for surjectivity of \( \eta' \) is virtually identical, using the explicit isomorphism between \( H^k(\Gamma, \mathbb{R}) \) and \( H^k_{dR}(M, \mathbb{R}) \) discussed in Section 4.1. For any class \([f\omega]\) \( \in H^k(\Gamma, \mathbb{R}) \), we choose the barycentrically straightened representative \( \tilde{f}_\omega \). The differential form \( \tilde{\omega} \) has bounded norm, as it is the \( \Gamma \)-invariant lift of the smooth differential form \( \omega \) on the compact manifold \( M \). So again, the estimate in Equation (4.2.1) shows the representative \( \tilde{f}_\omega \) is bounded, completing the proof.

### 4.3 Obstruction to Straightening Methods

In this section, we give a general obstruction to the straightening method that is applied in Section 4.2. In the next section, we will use this to give some concrete examples showing that Theorem 3.2.1 is not true when \( \dim(S) \leq n - r \). Throughout this section, we let \( X = G/K \) be an \( n \)-dimensional symmetric space of non-compact type, and we give the following definitions.

**Definition 4.3.1.** Let \( C^0(\Delta^k, X) \) be the set of singular \( k \)-simplices in \( X \), where \( \Delta^k \) is assumed to be equipped with a fixed Riemannian metric. Assume that we are given a collection of maps \( \text{st}_k : C^0(\Delta^k, X) \to C^0(\Delta^k, X) \). We say this collection of maps forms a straightening if it satisfies the following properties:
(a) the maps induces a chain map, that is, it commutes with the boundary operators.

(b) \( st_n \) is \( C^1 \) smooth, that is, the image of \( st_n \) lies in \( C^1(\Delta^n, X) \).

For a subgroup \( H \leq G \), we say the straightening is \( H \)-equivariant if the maps \( st_k \) all commute with the \( H \)-action.

Since \( X \) is simply connected, property (a) of Definition 4.3.1 implies that the chain map \( st_* \) is actually chain homotopic to the identity. Also, property (b) of Definition 4.3.1 implies the image of any straightened \( k \)-simplex is \( C^1 \)-smooth, i.e. \( \text{Im}(st_k) \subset C^1(\Delta^k, X) \). The barycentric straightening introduced in Section 3.1 is a \( G \)-equivariant straightening. As we saw in Section 4.2, obtaining a uniform control on the Jacobian of the straightened \( k \)-simplices immediately implies a surjectivity result for the comparison map from bounded cohomology to ordinary cohomology. This motivates the following:

**Definition 4.3.2.** We say the straightening is \( k \)-bounded, if there exists a constant \( C > 0 \), depending only on \( X \) and the chosen Riemannian metric on \( \Delta^k \), with the following property. For any \( k \)-dimensional singular simplex \( f \in C^0(\Delta^k, X) \), and corresponding straightened simplex \( st_k(f) : \Delta^k \to X \), the Jacobian of \( st_k(f) \) satisfies:

\[
|\text{Jac}(st_k(f))(\delta)| \leq C
\]

where \( \delta \in \Delta^i \) is arbitrary (and the Jacobian is computed relative to the fixed Riemannian metric on \( \Delta^k \)).

Our Theorem 3.2.1 and Proposition 3.1.1 then tells us that, when \( r = \mathbb{R}\text{-rank}(G) \geq 2 \) (excluding the two cases \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \) and \( \text{SL}(4, \mathbb{R})/\text{SO}(4) \)), our barycentric straightening is \( k \)-bounded for all \( k \geq n - r + 2 \). One can wonder whether this range
can be improved. In order to obtain obstructions, we recall [17, Theorem 2.4]. Restricting to the case of locally symmetric spaces of non-compact type, the theorem says:

**Theorem 4.3.3.** [17, Theorem 2.4] Let $M$ be an $n$-dimensional locally symmetric space of non-compact type, with universal cover $X$, and $\Gamma$ be the fundamental group of $M$. If $X$ admits an $n$-bounded, $\Gamma$-equivariant straightening, then the simplicial volume of $M$ is positive.

**Corollary 4.3.4.** If $X$ splits off an isometric $\mathbb{R}$-factor, then $X$ does not admit an $n$-bounded, $G$-equivariant straightening.

*Proof.* Let $X \simeq X_0 \times \mathbb{R}$ for some symmetric space $X_0$. If $X$ admits an $n$-bounded, $G$-equivariant straightening, then consider a closed manifold $M \simeq M_0 \times S^1$, where $\tilde{M}_0 \simeq X_0$. According to Theorem 4.3.3, the simplicial volume $||M||$ is positive. But on the other hand $||M|| = ||M_0 \times S^1|| \leq C \cdot ||M_0|| \cdot ||S^1|| = 0$. This contradiction completes the proof. □

We will use subspaces satisfying Corollary 4.3.4 to obstruct bounded straightenings.

**Definition 4.3.5.** For $X$ a symmetric space of non-compact type, we define the splitting rank of $X$, denoted $\text{srk}(X)$, to be the maximal dimension of a totally geodesic submanifold $Y \subset X$ which splits off an isometric $\mathbb{R}$-factor.

For the irreducible symmetric spaces of non-compact type, the splitting rank is calculated in the next chapter, see Theorem 5.1.3.

**Theorem 4.3.6.** If $k = \text{srk}(X)$, then $X$ does not admit any $k$-bounded, $G$-equivariant straightening.
Proof. We show this by contradiction. Assume $X = G/K$ admits a $k$-bounded, $G$-equivariant straightening $st_i$, and let $Y \subset X$ be a $k$-dimensional totally geodesic subspace which splits isometrically as $Y' \times \mathbb{R}$. Denote by $p : X \to Y$ the orthogonal projection from $X$ to $Y$, and note that the composition $p \circ st$ is a straightening on $Y$, which we denote by $\overline{st}$. Notice $Y$ is also a symmetric space and can be identified with $G_0/K_0$, for some $G_0 < G$, and $K_0 < K$. Then the straightening $\overline{st}$ is certainly $G_0$-equivariant. We claim it is also $k$-bounded. This is because the projection map $p$ is volume-decreasing, hence

$$|Jac(\overline{st}(f))| = |Jac(p(st(f)))| \leq |Jac(st(f))| \leq C$$

when $k \leq i \leq n$. Therefore, we conclude that $Y$ admits a $G_0$-equivariant, $k$-bounded straightening. This contradicts Corollary 4.3.4.

Remark. In view of Proposition 3.1.1 and the arguments in Section 4.2, we can view Theorem 4.3.6 as obstructing the bounded ratio Theorem 3.2.1. Specifically, if $k = \text{srk}(X)$, then Theorem 4.3.6 tells us that one has a sequence $f_i : \Delta^k_s \to X$ with the property that the Jacobian of $st_k(f_i)$ is unbounded. From the definition of our straightening maps $st_k$, this means one has a sequence $V_i = \left\{v^{(i)}_0, \ldots, v^{(i)}_k\right\} \subset X$ of $(k + 1)$-tuples of points (the vertices of the singular simplices $f_i$), and a sequence of points $\delta_i = \left(a^{(i)}_0, \ldots, a^{(i)}_k\right)$ inside the spherical simplex $\Delta^k_s \subset \mathbb{R}^{k+1}$, satisfying the following property. If one looks at the corresponding sequence of points

$$p_i := (st_k(f_i))(\delta_i) = \text{Bar} \left( \sum_{j=0}^{k} a^{(i)}_j \mu(v^{(i)}_j) \right),$$

one has a sequence of $k$-dimensional subspace $S_i \subset T_{p_i}X$ (given by the tangent spaces $D(st_{V_i})(T_{\delta_i} \Delta^k_s)$ to the straightened simplex $st_k(f_i)$ at the point $p_i$), and the sequence

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of ratios $\det(Q_1|_S^i)/\det(Q_2|_S^i)$ tends to infinity. It is not too hard to see that, for each dimension $k' \leq k$, one can find a $k'$-dimensional subspace $\bar{S}_i \subset S_i$ such that the sequence of ratios of determinants, for the quadratic forms restricted to the $\bar{S}_i$, must also tend to infinity. Thus the bounded ratio Theorem 3.2.1 fails whenever $k' \leq \text{srk}(X)$.

4.4 The Case of $SL(m, \mathbb{R})$

We conclude this chapter with a detailed discussion of the special case of the Lie group $G = SL(m, \mathbb{R})$, $m \geq 5$. The continuous cohomology has been computed (see e.g. [12, pg. 299]) and can be described as follows. If $m = 2k$ is even, then $H^*_c(SL(2k, \mathbb{R}))$ is an exterior algebra in $k$ generators in degrees $5, 9, \ldots, 4k - 3, 2k$. If $m = 2k + 1$ is odd, then $H^*_c(SL(2k + 1, \mathbb{R}))$ is an exterior algebra in $k$ generators in degrees $5, 9, \ldots, 4k + 1$.

The associated symmetric space is $X = SL(m, \mathbb{R})/SO(m)$, and we have that

$$n = \dim(X) = \dim(SL(m, \mathbb{R}))-\dim(SO(m)) = (m^2 - 1) - \frac{1}{2}m(m-1) = \binom{m+1}{2} - 1,$$

while the rank of the symmetric space is clearly $r = m - 1$. Thus, our Main Theorem 1 tells us that, for these Lie groups, the comparison map

$$H^*_b,c(SL(m, \mathbb{R})) \rightarrow H^*_c(SL(m, \mathbb{R}))$$

is surjective within the range of degrees $* \geq \binom{m+1}{2} - m + 2$.

Observe that the exterior product of all the generators $H^*_c(SL(m, \mathbb{R}))$ yields the generator for the top-dimensional cohomology, which lies in degree $\binom{m+1}{2} - 1$. Dropping off the 5-dimensional generator in the exterior product yields a non-trivial class in degree $\binom{m+1}{2} - 6$. Comparing with the surjectivity range in our Main Theorem 1, we
see that the first interesting example occurs in the case of $SL(8, \mathbb{R})$, where our results imply that $H_{c,b}^{30}(SL(8, \mathbb{R})) \neq 0$ (as well as $H_{c,b}^{35}(SL(8, \mathbb{R})) \neq 0$, which was previously known). Of course, as $m$ increases, our method provides more and more non-trivial bounded cohomology classes. For example, once we reach $SL(12, \mathbb{R})$, we get new non-trivial bounded cohomology classes in $H_{c,b}^{68}(SL(12, \mathbb{R}))$ and $H_{c,b}^{72}(SL(12, \mathbb{R}))$.

Finally, let us consider Theorem 4.3.6 in the special case of $X = SL(m, \mathbb{R})/SO(m)$. Choose a maximally singular direction in the symmetric space $X$, and let $X_0$ be the set of geodesics that are parallel to that direction. Without loss of generality, we can take $X_0 = G_0/K_0$, where

$$G_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & a \end{bmatrix} \mid \det(A) \cdot a = 1, \; a > 0 \right\}$$

and $K_0 = SO(m) \cap G_0$. Moreover, $X_0$ clearly splits off an isometric $\mathbb{R}$-factor, and can be isometrically identified with $SL(m - 1, \mathbb{R})/SO(m - 1) \times \mathbb{R}$. This is the maximal dimensional subspace of $SL(m, \mathbb{R})$ that splits off an isometric $\mathbb{R}$-factor (see Table 5.1), and the splitting rank is just $\dim(X_0) = \binom{m}{2}$. So in this special case, Theorem 4.3.6 tells us that our method for obtaining bounded cohomology classes fails once we reach degrees $\leq \binom{m}{2}$. Comparing this to the range where our method works, we see that, in the special case where $G = SL(m, \mathbb{R})$, the only degree which remains unclear is $\binom{m}{2} + 1$. This example shows our **Main Theorem 1** is very close to the optimal possible.
Chapter 5: Splitting Rank

The notion of splitting rank is defined in Section 4.3, as an obstruction in degree to a Jacobian being uniformly bounded. In Section 5.1, we will list in Table 5.1-5.2 the splitting rank of all irreducible symmetric spaces of non-compact type. In Section 5.2, we will define and analyze the $k$-th splitting rank. Finally in Section 5.3, we will consider the reducible cases and give a corresponding estimate on the $k$-th splitting rank.

5.1 Totally Geodesic Submanifolds with $\mathbb{R}$-factor

Let $X$ be a symmetric space of non-compact type. We write $X = G/K$ where $G = \text{Isom}_0(X)$ is the connected component of isometry group of $X$ and $K$ is a maximal compact subgroup of $G$. Fixing a base point $p \in X$, we have the Cartan decomposition $g = k + p$ and $p$ can be identified with the tangent space of $X$ at $p$. We recall the splitting rank of a symmetric space $X$ of non-compact type, denoted by $\text{srk}(X)$, is the maximal dimension of a totally geodesic submanifold $Y \subset X$ which splits off an isometric $\mathbb{R}$-factor. For a totally geodesic submanifold, its tangent space can be identified with a Lie triple system.

Remark. We notice a similar notion of maximal totally geodesic submanifolds is discussed in [2], but our definition is slightly different. Indeed, if $X = G_2^2/\text{SO}(4)$, then
the submanifold that has dimension equal to the splitting rank is $\mathbb{H}^2 \times \mathbb{R}$. This is not maximal totally geodesic, since $\mathbb{H}^2 \times \mathbb{R} \subset \text{SL}(3, \mathbb{R})/\text{SO}(3) \subset G_2^2/\text{SO}(4)$ gives a chain of totally geodesic inclusions.

**Proposition 5.1.1.** Suppose a totally geodesic submanifold $Y \times \mathbb{R} \subset X$ has dimension equal to the splitting rank of $X$. Then the corresponding Lie triple system $[p', [p', p']] \subset p'$ has the form $p' = a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V) = 0} p_\alpha$, where $a$ is a choice of maximal abelian subalgebra in $p$ that contains the $\mathbb{R}$-factor $V$.

**Proof.** We identify the tangent space of $X$ with $p$ via the Cartan decomposition, and the tangent space of $Y \times \mathbb{R}$ with a Lie triple system $p'' \subset p$. The product structure implies that any vector $v \in p''$ commutes with the $\mathbb{R}$-factor $V$. Hence $p'' \subset p'$, where $p' = \{ Z \in p \mid [Z, V] = 0 \}$. Notice that $p'$ is itself a Lie triple system. To see this, we first extend $V$ to a maximal abelian subalgebra $a \subset p$, and form the restricted root space decomposition $p = a \oplus \bigoplus_{\alpha \in \Lambda^+} p_\alpha$. Then $p'$ decomposes as $a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V) = 0} p_\alpha$. By a standard Lie algebra computation, we see that $[p', p'] \subset \mathfrak{k}'$, where $\mathfrak{k}' = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V) = 0} \mathfrak{k}_\alpha$, and also $[\mathfrak{k}', p'] \subset p'$. Therefore $p'$ is a Lie triple system that contains $p''$. By the assumption that $p''$ has maximal dimension, we conclude $p' = p''$. This completes the proof.

**Remark.** We comment that the totally geodesic submanifold in the above proposition is the same as $F(\gamma)$ — the union of all flats that goes through the geodesic $\gamma$ corresponding to the $\mathbb{R}$-factor. In general, $F(\gamma) = F_s(\gamma) \times \mathbb{R}^t$ where $F_s(\gamma)$ is also a symmetric space of non-compact type and $t$ is some integer that measures the singularity of $\gamma$ (see [11, Proposition 2.20.10] for more details). We see in the next proposition that $F(\gamma)$ attains maximal dimension when $t = 1$. 

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**Proposition 5.1.2.** Suppose a totally geodesic submanifold $Y \times \mathbb{R} \subset X$ has dimension equal to the splitting rank of $X$. Then $Y$ is also a symmetric space of non-compact type (i.e. it does not split off an $\mathbb{R}$-factor).

**Proof.** The proposition is a direct consequence of [11, Proposition 2.20.10] and Proposition 5.2.3 below.

We continue to analyze $Y$ via the above splitting of the Lie algebra. Let $\mathfrak{a}$ be a maximal abelian subalgebra containing the $\mathbb{R}$-factor $V$, and denote by $\mathfrak{a}' \subset \mathfrak{a}$ the orthogonal complement of $V$. Then the Lie triple system of $Y \times \mathbb{R}$ can be written as $V \oplus \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V)=0} p_\alpha$, where $V$ represents the $\mathbb{R}$-factor, and $\mathfrak{a}' \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V)=0} p_\alpha$ is the Lie triple system of $Y$. As $Y$ corresponds to a maximal parabolic subalgebra in $\mathfrak{g}$, we can choose a simple system $\Omega = \{\alpha_1, \ldots, \alpha_r\} \subset \Lambda$ corresponding to $X$ such that $\Omega' = \{\alpha_1, \ldots, \alpha_{r-1}\} \subset \ker(V) \cap \Lambda$ is a simple system corresponding to $Y$ (See [16, Proposition 7.76]). In other words, $Y$ has a truncated simple system generated by throwing away one element from the simple system of $X$. We give more detailed information in the next theorem, by simply working through all the cases of irreducible symmetric spaces of non-compact type.

**Theorem 5.1.3.** Let $X$ be an irreducible symmetric space of non-compact type. Assume $\dim(X) = n$ and $\text{rank}(X) = r \geq 2$. We give in the following tables the splitting rank of $X$, as well as all totally geodesic submanifolds $Y \times \mathbb{R}$ whose dimension attains the splitting rank.

**Remark.** In the table, we write $\text{SO}_0^{i,j}/\text{SO}_i \text{SO}_j$ short for $\text{SO}_0(i,j)/\text{SO}(i) \times \text{SO}(j)$, and similarly for $\text{SU}_{i,j}/S(U_i \times U_j)$ and $\text{Sp}_{i,j}/\text{Sp}_i \times \text{Sp}_j$. 

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<table>
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<tr>
<th>$X$</th>
<th>$Y$</th>
<th>srk$(X)$</th>
<th>$n$</th>
<th>Comments</th>
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<td>SL$(r, \mathbb{C})/\text{SU}(r)$</td>
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<td>$r(r + 2)$</td>
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<td>SU$(2r)/\text{Sp}(r)$</td>
<td>$n - 4r$</td>
<td>$r(2r + 3)$</td>
<td>$r \geq 2$</td>
</tr>
<tr>
<td>$E_6^{-26}/F_4$</td>
<td>$\mathbb{H}^9$</td>
<td>10</td>
<td>26</td>
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</tr>
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<td>SO$<em>{r,r+k}^0/\text{SO},\text{SO}</em>{r+k}$</td>
<td>SO$<em>{r-1,r-1+k}^0/\text{SO}</em>{r-1}\text{SO}_{r-1+k}$</td>
<td>$n - (2r + k - 2)$</td>
<td>$r(r + k)$</td>
<td>$r \geq 2$, $k \geq 1$</td>
</tr>
<tr>
<td>SO$(2r + 1, \mathbb{C})/\text{SO}(2r + 1)$</td>
<td>SO$(2r - 1, \mathbb{C})/\text{SO}(2r - 1)$</td>
<td>$n - (4r - 2)$</td>
<td>$r(2r + 1)$</td>
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<td>Sp$(r - 1, \mathbb{R})/U(r - 1)$</td>
<td>$n - (2r - 1)$</td>
<td>$r(r + 1)$</td>
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<td>SU$<em>{r-1,r-1}/S(U</em>{r-1}U_{r-1})$</td>
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<td>$2r^2$</td>
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<td>Sp$(r, \mathbb{C})/\text{Sp}(r)$</td>
<td>Sp$(r - 1, \mathbb{C})/\text{Sp}(r - 1)$</td>
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<tr>
<td>SO$(4r)/U(2r)$</td>
<td>SO$(4r - 4)/U(2r - 2)$</td>
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<td>$2r(2r - 1)$</td>
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<td>$E_6^{-26}/F_4$</td>
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<td>$r^2$</td>
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<td>$n - (4r - 4)$</td>
<td>$r(2r - 1)$</td>
<td>$r \geq 4$</td>
</tr>
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<td>SU$<em>{r-1,r-1+k}/S(U</em>{r-1}U_{r-1+k})$</td>
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<td>$2r(r + k)$</td>
<td>$r \geq 1$, $k \geq 1$</td>
</tr>
<tr>
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<td>Sp$_{r-1,r-1+k}/\text{Sp}<em>r\text{Sp}</em>{r-1+k}$</td>
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<td>$4r(r + k)$</td>
<td>$r \geq 1$, $k \geq 1$</td>
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<tr>
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<td>SO$(4r - 2)/U(2r - 1)$</td>
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<td>$2r(2r + 1)$</td>
<td>$r \geq 2$</td>
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<td>$E_6^{-14}/\text{Spin}(10) \times U(1)$</td>
<td>SU$_{1,5}/S(U_1U_5)$</td>
<td>11</td>
<td>32</td>
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Table 5.1: Splitting rank of irreducible symmetric spaces of non-compact type ($A_r$-$BC_r$)
<table>
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<th>$Y$</th>
<th>srk($X$)</th>
<th>$n$</th>
<th>Comments</th>
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<td>$E_6(\mathbb{C})/E_6$</td>
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<td>13</td>
<td>28</td>
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<td>$E_6^2/\text{SU}(6) \times \text{Sp}(1)$</td>
<td>$\text{SU}(3,3)/\text{S}(\text{U}(3) \times \text{U}(3))$</td>
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Table 5.2: Splitting rank of irreducible symmetric spaces of non-compact type ($E_6$-$G_2$)
Remark. In the above table, the symmetric spaces are listed according to their Dynkin diagrams. The groups are listed in the order $A_r, B_r, C_r, D_r, (BC)_r, E_6, E_7, E_8, F_4$ and $G_2$. Notice that a symmetric space of non-compact type is uniquely determined by its Dynkin diagram together with the multiplicities of simple roots.

Proof. We prove the case of $\text{SL}(r+1,\mathbb{R})/\text{SO}(r+1)$ (the first line in Table 5.1). If $X = \text{SL}(r+1,\mathbb{R})/\text{SO}(r+1)$, the Dynkin diagram is shown in Figure 5.1, with multiplicities all ones. By the previous discussion, $Y$ is generated by the truncated simple system \{$\alpha_1, ... , \hat{\alpha}_i, ... , \alpha_r$\} for some $i = 1, ... , r$, preserving the same multiplicities and configurations. Hence we have $Y = \text{SL}(i,\mathbb{R})/\text{SO}(i) \times \text{SL}(r-i+1,\mathbb{R})/\text{SO}(r-i+1)$, for some $i = 1, ... , r$. The dimension of $Y$ equals $(i-1)(i+2)/2 + (r-i)(r-i+3)/2$, so the codimension of $Y \times \mathbb{R} \subset X$ is $-i^2 + (r+1)i$, which attains its minimal codimension $r$ when $i = 1, r$. In both cases we have $Y = \text{SL}(r,\mathbb{R})/\text{SO}(r)$, and hence $\text{srk}(X) = \dim(Y \times \mathbb{R}) = n-r$, where $n = \dim(X) = r(r+3)/2$. The remaining cases are analyzed similarly, and can be found in the Appendix.

By looking at the tables above, we immediately have the following corollaries.

**Corollary 5.1.4.** Under the same assumptions of Theorem 5.1.3, $Y$ is also irreducible.
Corollary 5.1.5. If $X$ is an irreducible symmetric space of non-compact type, then $\text{srk}(X) \leq n - r$, and $\text{srk}(X) = n - r$ if and only if $X = \text{SL}(r+1, \mathbb{R})/\text{SO}(r+1)$.

Before moving to the next section, we will need a little more information about the dimensions of totally geodesic submanifolds $Y \times \mathbb{R} \subset X$. Here we focus on the cases where $Y$ is of non-compact type, that is, $Y$ is generated by a truncated simple system $\{\alpha_1, \ldots, \check{\alpha}_i, \ldots, \alpha_r\}$ where $\{\alpha_1, \ldots, \alpha_r\}$ is a simple system of $X$. Besides the largest dimension case at $\text{srk}(X)$, we are also curious about the second largest dimension, and we will need to verify that there is enough gap between the two.

Proposition 5.1.6. Let $X$ be an irreducible symmetric space of non-compact type. Assume $\dim(X) = n$ and $\text{rank}(X) = r \geq 4$. If $Y \times \mathbb{R}$ and $Y' \times \mathbb{R}$ are two totally geodesic submanifolds in $X$ where $\dim(Y \times \mathbb{R}) = \text{srk}(X)$ and $\dim(Y' \times \mathbb{R})$ attains the second largest dimension among all truncated simple systems generating $Y'$, then the gaps between the two dimensions $(\dim(Y) - \dim(Y'))$ are given in the following tables.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y'$</th>
<th>Gap</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}(r+1, \mathbb{R})/\text{SO}(r+1)$</td>
<td>$\mathbb{H}^2 \times \text{SL}(r-1, \mathbb{R})/\text{SO}(r-1)$</td>
<td>$r - 2$</td>
<td>$r \geq 4$</td>
</tr>
<tr>
<td>$\text{SL}(r+1, \mathbb{C})/\text{SU}(r+1)$</td>
<td>$\mathbb{H}^3 \times \text{SL}(r-1, \mathbb{C})/\text{SU}(r-1)$</td>
<td>$2r - 4$</td>
<td>$r \geq 4$</td>
</tr>
<tr>
<td>$\text{SU}^*(2r+2)/\text{Sp}(r+1)$</td>
<td>$\mathbb{H}^5 \times \text{SU}^*(2r-2)/\text{Sp}(r-1)$</td>
<td>$4r - 8$</td>
<td>$r \geq 4$</td>
</tr>
<tr>
<td>$\text{SO}^{0}_{r,r+k}/\text{SO}<em>r\text{SO}</em>{r+k}$</td>
<td>$\mathbb{H}^2 \times \text{SO}^{0}<em>{r-2,r-2+k}/\text{SO}</em>{r-2}\text{SO}_{r-2+k}$</td>
<td>$2r + k - 5$</td>
<td>$r + 2k &gt; 7$</td>
</tr>
</tbody>
</table>

Table 5.3: Dimension gap after splitting rank $(A_r-B_r)$
\[
\begin{array}{|c|c|c|c|}
\hline
X & Y' & \text{Gap} & \text{Comments} \\
\hline
\text{SO}^0_{1,5}/\text{SO}_4\text{SO}_5 & \text{SL}(4, \mathbb{R})/\text{SO}(4) & 3 & r = 4, \\
& & & k = 1 \\
\text{SO}^0_{5,6}/\text{SO}_5\text{SO}_6 & \mathbb{H}^2 \times \text{SO}^0_{3,4}/\text{SO}_3\text{SO}_4 \\
& & \text{or } \text{SL}(5, \mathbb{R})/\text{SO}(5) & 6 & r = 5, \\
& \text{SO}(2r + 1, \mathbb{C})/\text{SO}(2r + 1) & \mathbb{H}^3 \times \text{SO}(2r - 3, \mathbb{C})/\text{SO}(2r - 3) & 4r - 8 & r > 5 \\
& \text{SO}(9, \mathbb{C})/\text{SO}(9) & \text{SL}(4, \mathbb{C})/\text{SU}(4) & 6 & r = 4 \\
& \text{SO}(11, \mathbb{C})/\text{SO}(11) & \mathbb{H}^3 \times \text{SO}(7, \mathbb{C})/\text{SO}(7) & 12 & r = 5 \\
& & \text{or } \text{SL}(5, \mathbb{C})/\text{SU}(5) & & \\
\text{Sp}(r, \mathbb{R})/U(r) & \mathbb{H}^2 \times \text{Sp}(r - 2, \mathbb{R})/U(r - 2) & 2r - 4 & r > 5 \\
\text{Sp}(4, \mathbb{R})/U(4) & \text{SL}(4, \mathbb{R})/\text{SO}(4) & 3 & r = 4 \\
\text{Sp}(5, \mathbb{R})/U(5) & \mathbb{H}^2 \times \text{Sp}(3, \mathbb{R})/U(3) & 6 & r = 5 \\
& & \text{or } \text{SL}(5, \mathbb{R})/\text{SO}(5) & & \\
\text{SU}_{r,r}/S(U_rU_r) & \mathbb{H}^3 \times \text{SU}_{r-2,r-2}/S(U_{r-2}U_{r-2}) & 4r - 9 & r > 6 \\
\text{SU}_{4,4}/S(U_4U_4) & \text{SL}(4, \mathbb{C})/\text{SU}(4) & 3 & r = 4 \\
\text{SU}_{5,5}/S(U_5U_5) & \text{SL}(5, \mathbb{C})/\text{SU}(5) & 8 & r = 5 \\
\text{SU}_{6,6}/S(U_6U_6) & \mathbb{H}^3 \times \text{SU}_{4,4}/S(U_4U_4) & 15 & r = 6 \\
& & \text{or } \text{SL}(6, \mathbb{C})/\text{SU}(6) & & \\
\text{Sp}(r, \mathbb{C})/\text{Sp}(r) & \mathbb{H}^3 \times \text{Sp}(r - 2, \mathbb{C})/\text{Sp}(r - 2) & 4r - 8 & r > 5 \\
\text{Sp}(4, \mathbb{C})/\text{Sp}(4) & \text{SL}(4, \mathbb{C})/\text{SU}(4) & 6 & r = 4 \\
\text{Sp}(5, \mathbb{C})/\text{Sp}(5) & \mathbb{H}^3 \times \text{Sp}(3, \mathbb{C})/\text{Sp}(3) & 12 & r = 5 \\
& & \text{or } \text{SL}(5, \mathbb{C})\text{SU}(5) & & \\
\text{SO}^*(4r)/U(2r) & \mathbb{H}^4 \times \text{SO}^*(4r - 8)/U(2r - 4) & 8r - 19 & r > 6 \\
\text{SO}^*(16)/U(8) & \text{SU}^*(8)/\text{Sp}(4) & 3 & r = 4 \\
\text{SO}^*(20)/U(10) & \text{SU}^*(10)/\text{Sp}(5) & 12 & r = 5 \\
\text{SO}^*(24)/U(12) & \text{SU}^*(12)/\text{Sp}(6) & 25 & r = 6 \\
\hline
\end{array}
\]

Table 5.4: Dimension gap after splitting rank \((B_r-C_r)\)
<table>
<thead>
<tr>
<th>X</th>
<th>Y'</th>
<th>Gap</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sp_{r,r}/Sp_r</td>
<td>$\mathbb{H}^5 \times Sp_{r-2,r-2}/Sp_{r-2} \times Sp_{r-2}$</td>
<td>$8r - 17$</td>
<td>$r &gt; 5$</td>
</tr>
<tr>
<td>Sp_{4,4}/Sp_4Sp_4</td>
<td>$SU^*(8)/Sp(4)$</td>
<td>9</td>
<td>$r = 4$</td>
</tr>
<tr>
<td>Sp_{5,5}/Sp_5Sp_5</td>
<td>$SU^*(10)/Sp(5)$</td>
<td>20</td>
<td>$r = 5$</td>
</tr>
<tr>
<td>SO_{0,r}/SO_rSO_r</td>
<td>$\mathbb{H}^2 \times SO_{r-2,r-2}/SO_{r-2}SO_{r-2}$</td>
<td>$2r - 5$</td>
<td>$r &gt; 7$</td>
</tr>
<tr>
<td>SO_{4,4}/SO_4SO_4</td>
<td>$\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$</td>
<td>3</td>
<td>$r = 4$</td>
</tr>
<tr>
<td>SO_{6,5}/SO_6SO_5</td>
<td>SL(5,$\mathbb{R}$)/SO(5)</td>
<td>2</td>
<td>$r = 5$</td>
</tr>
<tr>
<td>SO_{6,6}/SO_6SO_6</td>
<td>SL(6,$\mathbb{R}$)/SO(6)</td>
<td>5</td>
<td>$r = 6$</td>
</tr>
<tr>
<td>SO_{7,7}/SO_7SO_7</td>
<td>$\mathbb{H}^2 \times SO_{5,5}/SO_5SO_5$ or SL(7,$\mathbb{R}$)/SO(7)</td>
<td>9</td>
<td>$r = 7$</td>
</tr>
<tr>
<td>SO(2r,$\mathbb{C}$)/SO(2r)</td>
<td>$\mathbb{H}^3 \times SO(2r - 4,$\mathbb{C}$)/SO(2r - 4)</td>
<td>$4r - 10$</td>
<td>$r &gt; 7$</td>
</tr>
<tr>
<td>SO(8,$\mathbb{C}$)/SO(8)</td>
<td>$\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$</td>
<td>6</td>
<td>$r = 4$</td>
</tr>
<tr>
<td>SO(10,$\mathbb{C}$)/SO(10)</td>
<td>SL(5,$\mathbb{C}$)/SU(5)</td>
<td>4</td>
<td>$r = 5$</td>
</tr>
<tr>
<td>SO(12,$\mathbb{C}$)/SO(12)</td>
<td>SL(6,$\mathbb{C}$/SU(6)</td>
<td>10</td>
<td>$r = 6$</td>
</tr>
<tr>
<td>SO(14,$\mathbb{C}$)/SO(14)</td>
<td>$\mathbb{H}^3 \times SO(10,$\mathbb{C}$)/SO(10) or SL(7,$\mathbb{C}$)/SU(7)</td>
<td>18</td>
<td>$r = 7$</td>
</tr>
<tr>
<td>SU_{r,r+k}/SU(U_rU_{r+k})</td>
<td>$\mathbb{H}^3 \times SU_{r-2,r-2+k}/SU(U_{r-2}U_{r-2+k})$</td>
<td>$4r + 2k - 9$</td>
<td>$r + 2k &gt; 6$</td>
</tr>
<tr>
<td>SU_{4,5}/SU(U_4U_5)</td>
<td>$\mathbb{H}^3 \times SU_{2,3}/SU(U_2U_3)$ or SL(4,$\mathbb{C}$)/SU(4)</td>
<td>9</td>
<td>$r = 4,$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k = 1$</td>
</tr>
<tr>
<td>Sp_{r,r+k}/Sp_rSp_{r+k}</td>
<td>$\mathbb{H}^5 \times Sp_{r-2,r-2+k}/Sp_{r-2}Sp_{r-2+k}$</td>
<td>$8r + 4k - 17$</td>
<td>$r \geq 4,$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k \geq 1$</td>
</tr>
<tr>
<td>SO^*(4r + 2)/U_{2r+1}</td>
<td>$\mathbb{H}^5 \times SO^*(4r - 6)/U(2r - 3)$</td>
<td>$8r - 15$</td>
<td>$r &gt; 4$</td>
</tr>
<tr>
<td>SO^*(18)/U(9)</td>
<td>$SU^*(8)/Sp(4)$</td>
<td>15</td>
<td>$r = 4$</td>
</tr>
<tr>
<td>E_6/Sp(4)</td>
<td>SL(6,$\mathbb{R}$)/SO(6)</td>
<td>5</td>
<td>$r = 6$</td>
</tr>
<tr>
<td>E_6($\mathbb{C}$)/E_6</td>
<td>SL(6,$\mathbb{C}$)/SU(6)</td>
<td>10</td>
<td>$r = 6$</td>
</tr>
</tbody>
</table>

Table 5.5: Dimension gap after splitting rank ($C_r$-$E_6$)
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$X$ & $Y'$ & Gap & Comments \\
\hline
$E_7^7$/SU(8) & SO$_{6,6}$/SO$_6 \times$ SO$_6$ & 6 & $r = 7$ \\
$E_7$(C)/$E_7$ & SO(12, C)/SO(12) & 12 & $r = 7$ \\
E$_8$/SO(16) & SO$_{7,7}$/SO$_7 \times$ SO$_7$ & 21 & $r = 8$ \\
E$_8$(C)/E$_8$ & SO(14, C)/SO(14) & 42 & $r = 8$ \\
F$_4^4$/Sp(3) \times Sp(1) & $\mathbb{H}^2 \times$ SL(3, $\mathbb{R}$)/SO(3) & 5 & $r = 4$ \\
E$_6^2$/SU(6) \times Sp(1) & SO$_{3,5}$/SO$_3 \times$ SO$_5$ & 3 & $r = 4$ \\
E$_7^{-5}$/SO(12) \times Sp(1) & SO$_{3,7}$/SO$_3 \times$ SO$_7$ & 3 & $r = 4$ \\
E$_8^{-24}$/E$_7$ \times Sp(1) & SO$_{0,11}$/SO$_3 \times$ SO$_1$ & 21 & $r = 4$ \\
F$_4$(C)/F$_4$ & $\mathbb{H}^3 \times$ SL(3, C)/SU(3) & 10 & $r = 4$ \\
\hline
\end{tabular}
\caption{Dimension gap after splitting rank ($E_7$-$F_4$)}
\end{table}

**Proof.** We prove the case of SL($r + 1$, $\mathbb{R}$)/SO($r + 1$) (the first line of Table 5.3). As we see in the proof of Theorem 5.1.3, $Y = SL(i, \mathbb{R}) \times SL(r - i + 1, \mathbb{R})$/SO($r - i + 1$). And the codimension of $Y \times \mathbb{R} \subset X$ is $-i^2 + (r + 1)i$, which attains its minimal codimension $r$ when $i = 1, r$. Now it attains its second minimal value $2r - 2$ when $i = 2, r - 1$ provided $r \geq 3$. In this case, $Y = \mathbb{H}^2 \times$ SL($r - 1$, $\mathbb{R}$)/SO($r - 1$), and the gap is $r - 2$. Again, the remaining cases are analyzed similarly in the Appendix. \hfill \Box

**Lemma 5.1.7.** (Gap) Let $X$ be an irreducible symmetric space of non-compact type. Assume $\dim(X) = n$ and rank($X$) = $r \geq 3$. If $Y \times \mathbb{R}$ is a totally geodesic submanifold whose dimension attains $srk(X)$, and $Y' \times \mathbb{R}$ is another totally geodesic submanifold whose dimension is $< srk(X)$. Then either $Y'$ is irreducible or the gap in dimensions of the two ($\dim(Y) - \dim(Y')$) is at least $r - 2$.  

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Proof. For $r = 3$, the inequality is automatic. For $r \geq 4$, we check that the inequality follows from Table 5.3-5.6 above for all cases except for $\text{SO}_{5,5}/\text{SO}_5 \times \text{SO}_5$, where the gap between the largest ($\text{SO}_{4,4}/\text{SO}_4 \times \text{SO}_4$) and second largest dimension ($\text{SL}(5, \mathbb{R})/\text{SO}(5)$) is 2. However, the space $\text{SL}(5, \mathbb{R})/\text{SO}(5)$ is irreducible. So the gap between $\text{SO}_{4,4}/\text{SO}_4 \times \text{SO}_4$ and any reducible space $Y'$ will be at least 3. This completes the proof.

5.2 The k-th Splitting Rank

Definition 5.2.1. Let $X$ be a rank $r$ symmetric space of non-compact type. For each $k$ ($1 \leq k \leq r$), we define the $k$-th splitting rank of $X$, denoted $srk^k(X)$, to be the maximal dimension of a totally geodesic submanifold $Y \subset X$ which splits off an isometric $\mathbb{R}^k$-factor.

Remark. In the above notion, the first splitting rank is just our previous notion of splitting rank. We can also see that $srk^{k+1}(X) \leq srk^k(X)$ for $1 \leq k \leq r - 1$ and $srk^r(X) = r$. We abuse notation and set $srk^0(X) = \dim(X)$.

Proposition 5.2.2. Suppose $Y \times \mathbb{R}^k \subset X$ has the maximal dimension, that is, $\dim(Y \times \mathbb{R}^k) = srk^k(X)$. Then the corresponding Lie triple system $[p', [p', p']] \subset p'$ has the form $p' = a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V_k) = 0} p_\alpha$, where $a$ is a choice of maximal abelian subalgebra in $p$ that contains the $k$-dimensional Euclidean factor $V_k$.

Proof. The proof is the same as Proposition 5.1.1, just replacing $V$ with $V_k$. □

Proposition 5.2.3. For $1 \leq k \leq r - 1$, we have strict inequality $srk^{k+1}(X) < srk^k(X)$. Therefore, the totally geodesic submanifold $Y \times \mathbb{R}^k$ that has dimension $srk^k(X)$ does not split off any further $\mathbb{R}$-factors.
Proof. Suppose $Y_{k+1} \times \mathbb{R}^{k+1} \subset X$ has the dimension $srk^{k+1}(X)$. According to Proposition 5.2.2, the tangent space of $Y_{k+1} \times \mathbb{R}^{k+1}$ is identified with $p' = a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V_{k+1}) = 0} p_{\alpha}$ for some $a$ that contains the $\mathbb{R}^{k+1}$-factor $V_{k+1}$. We can choose a pair of root vectors $\pm H_{\alpha_0}$ so that it does not lie in the orthogonal complement $V_{k+1}^\perp$. Let $V_k = V_{k+1} \cap H_{\alpha_0}^\perp$ be a $k$-dimensional subspace of $V_{k+1}$, the Lie triple system $p'' = a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V_k) = 0} p_{\alpha}$ strictly contains $p'$ since $\alpha_0(V_k) = 0$ but $\alpha_0(V_{k+1}) \neq 0$. Therefore, $srk^{k+1}(X) = \dim p' < \dim p'' \leq srk^k(X)$. 

Lemma 5.2.4. Let $X$ be an irreducible rank $r$ ($r \geq 2$) symmetric space of non-compact type. Then $srk^k(X) \leq srk(X) - 2(k - 1)$ holds for all $1 \leq k < r$.

Proof. We show this by induction on the rank of the symmetric space. For $r = 2$, the only possible value for $k$ is $k = 1$, and the inequality holds immediately. Suppose we have the inequality for all such irreducible symmetric spaces of rank $l$ ($l \geq 2$), assuming rank($X$) = $l + 1$, we want to show $srk^k(X) \leq srk(X) - 2(k - 1)$ for all $1 \leq k < l + 1$. Notice when $k = 1$, the inequality is trivially true, so we may assume $k \geq 2$.

Let $srk^k(X) = \dim(Y_k \times \mathbb{R}^k)$, where $Y_k$ is described as in Proposition 5.2.2. Let $V_k$ denote the $\mathbb{R}^k$-factor. We inductively define $V_i$ so that it is an $i$ dimensional Euclidean subspace of $V_{i+1}$ and $\ker(V_{i+1}) \cap \Lambda \subset \ker(V_i) \cap \Lambda$. This will give rise to an extending chain of Lie triples $p_k \subset \ldots \subset p_1 \subset p$, corresponding to a totally geodesic chain $Y_k \times \mathbb{R}^k \subset \ldots \subset Y_1 \times \mathbb{R} \subset X$, such that for each $1 \leq i \leq k$, $p_i = a \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V_i) = 0} p_{\alpha}$. The choice of $V_i$ implies that $\dim(p_{i+1}) < \dim(p_i)$ and therefore $Y_i$ does not split off an $\mathbb{R}$-factor, for all $i$. Besides, since $Y_k \times \mathbb{R}^k \subset Y_1 \times \mathbb{R}$ have a common $\mathbb{R}$-factor $V_1$, $Y_k \times \mathbb{R}^{k-1}$ is totally geodesic in $Y_1$.  

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Now if $Y_1$ is irreducible, by the induction hypothesis, we have $\dim(Y_k \times \mathbb{R}^{k-1}) \leq \text{srk}^{k-1}(Y_1) \leq \text{srk}(Y_1) - 2(k - 2)$. According to Corollary 5.1.5, we have $\text{srk}(Y_1) \leq \dim(Y_1) - l \leq \dim(Y_1) - 2$. Hence combining the two inequalities, we conclude $\dim(Y_k \times \mathbb{R}^k) \leq \text{srk}(Y_1) - 2(k - 2) + 1 \leq \dim(Y_1) - 2 - 2(k - 2) + 1 = \dim(Y_1 \times \mathbb{R}) - 2(k - 1) \leq \text{srk}(X) - 2(k - 1)$.

If $Y_1$ is reducible, then $Y_1 \times \mathbb{R}$ can not have the dimension equal to the splitting rank of $X$ by Corollary 5.1.4. So Lemma 5.1.7 implies that $\dim(Y_1 \times \mathbb{R}) \leq \text{srk}(X) - (l + 1 - 2)$.

The increasing chain $p_k \subset \ldots \subset p_1 \subset p$ gives the inequality $\dim(p_k) \leq \dim(p_1) - (k - 1)$. Notice $p_i$ is identified with the tangent space of $Y_i \times \mathbb{R}^i$, so we can then estimate $\dim(Y_k \times \mathbb{R}^k) = \dim(p_k) \leq \text{srk}(X) - (l + 1 - 2) - (k - 1) \leq \text{srk}(X) - 2(k - 1)$.

We have shown in both cases that the inequality holds for all rank $l + 1$ irreducible symmetric spaces. This completes the induction argument and hence the proof of this lemma.

\begin{corollary}
Let $X$ be an irreducible rank $r$ ($r \geq 1$) symmetric space of non-compact type, excluding $\text{SL}(3, \mathbb{R})/\text{SO}(3)$, $\text{Sp}(2, \mathbb{R})/U(2)$, $G_2^2/\text{SO}(4)$ and $\text{SL}(4, \mathbb{R})/\text{SO}(4)$. Then $\text{srk}^k(X) \leq \text{srk}(X) - 2(k - 1)$ holds for all $1 \leq k \leq r$.
\end{corollary}

\begin{proof}
Notice the inequality automatically holds in rank one cases, and in view of Lemma 5.2.4 we only need to consider the case when $k = r$. If $k = r$, the inequality is equivalent to $3r - 2 \leq \text{srk}(X)$. We know that the dimension (= $n$) of an irreducible symmetric space grows roughly quadratically in its rank (= $r$), and actually we can check that $n \geq 3r$ whenever $r \geq 3$. This together with Corollary 5.1.4 implies that $\text{srk}(X) = \dim(Y \times \mathbb{R}) \geq 3(r - 1) + 1 = 3r - 2$ whenever $r \geq 4$. This proves the corollary in all cases where rank $\geq 4$. When $r = 2$, it is equivalent to show $\text{srk}(X) \geq 4$, and according to Table 5.1 and 5.2, this excludes $\text{SL}(3, \mathbb{R})/\text{SO}(3)$, $\text{Sp}(2, \mathbb{R})/U(2)$ and
$G_2^2/\text{SO}(4)$. When $r = 3$, it is equivalent to show $\text{srk}(X) \geq 7$, and by Table 5.1 and 5.2 this only excludes $\text{SL}(4, \mathbb{R})/\text{SO}(4)$.

5.3 Reducible Symmetric Spaces

We intend to generalize Corollary 5.2.5 to all higher rank symmetric spaces of non-compact type. Below is the key lemma that characterizes certain $\mathbb{R}$-split totally geodesic submanifolds in reducible symmetric spaces.

**Lemma 5.3.1.** Let $X$ be a symmetric space of non-compact type, and $Z = Y \times \mathbb{R}^k$ a totally geodesic subspace in $X$ that has dimension equal to the $k$-th splitting rank of $X$. If $X$ splits as a product of $X_1$ and $X_2$, then $Z$ also splits as a product of $Z_1$ and $Z_2$, where $Z_i = Y_i \times \mathbb{R}^{k_i}$ is totally geodesic in $X_i$, for $i = 1, 2$ and some $k_i \geq 0$ satisfying $k_1 + k_2 = k$.

**Proof.** We write $X = G/K$ and fix a basepoint $x \in X$. We form the Cartan decomposition $g = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p}$ can be identified with the tangent space $T_x X$. We denote $V_k \subset \mathfrak{p}$ the $\mathbb{R}^k$ factor of $Z$, and extend $V_k$ to a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Since $X$ splits as a product of $X_1$ and $X_2$, we can write $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, and also the set of roots $\Lambda$ of $X$ decomposes as $\Lambda_1 \cup \Lambda_2$, where $\Lambda_i$ is the set of roots belonging to $X_i$ with respect to $\mathfrak{a}_i$. By Proposition 5.2.2, $Z$ has the Lie triple system $\mathfrak{p}' = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(V_k) = 0} \mathfrak{p}_\alpha$. Let $\mathfrak{p}'_i = \mathfrak{a}_i \oplus \bigoplus_{\alpha \in \Lambda^+_i, \alpha(V_{k,i}) = 0} \mathfrak{p}_{\alpha}$, we have $\mathfrak{p}' = \mathfrak{p}'_1 \oplus \mathfrak{p}'_2$. Notice $\mathfrak{p}'_i \subset \mathfrak{p}_i$ is a Lie triple system. Indeed, $\mathfrak{p}'_i = \mathfrak{a}_i \oplus \bigoplus_{\alpha \in \Lambda^+_i, \alpha(V_{k,i}) = 0} \mathfrak{p}_{\alpha}$ where $V_{k,i}$ is the orthogonal projection of $V_k$ to $\mathfrak{a}_i$. This implies that $V_{k,i}$ is the Euclidean factor of $\mathfrak{p}_i$, therefore $V_k$ splits as $V_{k,1} \oplus V_{k,2}$, which completes the proof. \qed
Corollary 5.3.2. Let \( X \) be a rank \( r \) symmetric space of non-compact type. Assume \( X = X_1 \times X_2 \), where \( \text{rank}(X_i) = r_i \) for \( i = 1, 2 \). Then

\[
srk^k(X) = \text{Max}\{srk^{j_1}(X_1) + srk^{j_2}(X_2) : 0 \leq j_1 \leq r_1, 0 \leq j_2 \leq r_2 \text{ and } j_1 + j_2 = k\}
\]

Remark. This is a direct consequence of Lemma 5.3.1. In the corollary, recall that by definition \( srk^0(X) = \dim(X) \). As a result, \( srk(X_1 \times X_2) = \text{Max}\{srk(X_1) + \dim(X_2), srk(X_2) + \dim(X_1)\} \). Furthermore, if we denote \( si^k(X) := n - srk^k(X) \) the \( k \)-th splitting index of \( X \), then Corollary 5.3.2 simply says

\[
\begin{align*}
si^k(X_1 \times X_2) &= \text{Min}\{si^{j_1}(X_1) + si^{j_2}(X_2) : 0 \leq j_1 \leq r_1, 0 \leq j_2 \leq r_2 \text{ and } j_1 + j_2 = k\} \\
\text{and that } si(X_1 \times X_2) &= \text{Min}\{si(X_1), si(X_2)\}.
\end{align*}
\]

Theorem 5.3.3. Let \( X \) be a rank \( r \) symmetric space of non-compact type. Assume \( X \) has no direct factors isometric to \( \mathbb{H}^2, \text{SL}(3, \mathbb{R})/\text{SO}(3), \text{Sp}(2, \mathbb{R})/\text{U}(2), G_2^2/\text{SO}(4) \) or \( \text{SL}(4, \mathbb{R})/\text{SO}(4) \). Then \( srk^k(X) \leq srk(X) - 2(k - 1) \) for all \( 1 \leq k \leq r \).

Proof. We write \( X \) as a product of irreducible symmetric spaces \( X_1 \times ... \times X_s \). Using the notion of splitting index described in the previous remark, the inequality is equivalent to \( si^k(X) \geq si(X) + 2(k - 1) \). By repeatedly applying Corollary 5.3.2, we can assume \( si^k(X) = \sum_{l=1}^{s} si^{j_l}(X_l) \) for some \( j_l \) satisfying \( 0 \leq j_l \leq r_l \) and \( \sum_{l=1}^{s} j_l = k \) where \( r_l \) is the rank of \( X_l \). For each \( j_l > 0 \), we have \( si^{j_l}(X_l) \geq si(X_l) + 2(j_l - 1) \) by Corollary 5.2.5. Notice \( si^0(X) = 0 \) and \( si^{j_l}(X_l) \) does not contribute to the summation if \( j_l = 0 \). We can further estimate

\[
\begin{align*}
si^k(X) &= \sum_{1 \leq i \leq s, j_l > 0} si^{j_l}(X_l) \geq \sum_{1 \leq i \leq s, j_l > 0} [si(X_l) + 2(j_l - 1)] = 2k + \sum_{1 \leq i \leq s, j_l > 0} (si(X_l) - 2)
\end{align*}
\]

As a consequence of Corollary 5.1.5, we have \( si(X_l) \geq 2 \) as we assume no \( \mathbb{H}^2 \)-factors. Now we apply the inequality \( si(X_l) \geq \text{Min}_{1 \leq i \leq s} si(X_l) = si(X) \) to one of
the $l$ in the summation, and apply $\text{si}(X_l) \geq 2$ to the rest of the $l$. We finally obtain $\text{si}^k(X) \geq \text{si}(X) + 2k - 2$, which completes the proof.

5.4 Bounded Cohomology Revisited

In this section, we generalize the method of Chapter 3, and show the surjectivity of comparison maps in a slightly larger range ($\geq \text{srk} + 2$). The approach is quite similar: Theorem 5.3.3 generalizes Lemma 3.3.5, allowing us to improve Main Theorem 1 to Main Theorem 2.

**Remark.** In the irreducible case of $\text{SL}(r+1, \mathbb{R})/\text{SO}(r+1)$, the splitting rank is exactly $n - r$, so Main Theorem 2 recovers Main Theorem 1. In general, $\text{srk}(X)$ is smaller then $n - r$ (see Corollary 5.1.5) so that Main Theorem 2 produces surjectivity in a larger range than Main Theorem 1.

**Proof of Main Theorem 2:** The proof is similar to that of Main Theorem 1. Notice that in Section 3.3, the problem is reduced to a Hall’s Marriage type combinatorial problem, and the key cardinality estimate is ensured by Lemma 3.3.5. We now replace the lemma by the following modification, hence finishing the proof of Main Theorem 2.

**Lemma 5.4.1.** Let $X = G/K$ be a rank $r \geq 2$ symmetric space of non-compact type, without direct factors isometric to $\mathbb{H}^2$, $\text{SL}(3, \mathbb{R})/\text{SO}(3)$, $\text{Sp}(2, \mathbb{R})/U(2)$, $G_2^2/\text{SO}(4)$, or $\text{SL}(4, \mathbb{R})/\text{SO}(4)$. Fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Assume $\{v_1^*,...,v_r^*\}$ spans $\mathfrak{a}$, and let $Q_i = \bigoplus_{\alpha \in \Lambda^+,\alpha(v_i^*) \neq 0} \mathfrak{p}_\alpha$. Then for any subcollection of vectors $\{v_{i_1}^*,...,v_{i_k}^*\}$, we have $\dim(Q_{i_1} + ... + Q_{i_k}) \geq (2k + n - \text{srk}(X) - 2)$.

**Proof.** Notice $Q_{i_1} + ... + Q_{i_k} = \bigoplus_{\alpha \in \Lambda^+,\alpha(V_k) \neq 0} \mathfrak{p}_\alpha$ where $V_k$ is the span of $v_{i_1}^*,...,v_{i_k}^*$. Its orthogonal complement in $\mathfrak{p}$ is $\mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda^+,\alpha(V_k) = 0} \mathfrak{p}_\alpha$, which has dimension at most
\( \text{srk}^k(X) \), hence according to Theorem 5.3.3 is bounded above by \( \text{srk}(X) - 2k + 2 \).

Therefore, \( \dim(Q_{i_1} + ... + Q_{i_k}) \geq (2k + n - \text{srk}(X) - 2) \). This completes Lemma 5.4.1.

Remark. Notice that Main Theorem 1 required the symmetric space to be irreducible. But the proof of Lemma 3.3.5 was the only step that used irreducibility. The rest of the proof remains valid for reducible symmetric spaces. Thus replacing Lemma 3.3.5 by our Lemma 5.4.1, the actual proof goes through even in the reducible case.

5.5 Discussions

As we have seen, the bound we get in the Main Theorem 2 is almost the optimal possible. We showed the Jacobian is uniformly bounded in degrees \( \geq \text{srk}(X) + 2 \), and we know the Jacobian estimate can not be improved to degrees \( \leq \text{srk}(X) \). Thus the only case left unknown is in degree \( \text{srk}(X) + 1 \). We believe in a negative answer in this degree, as evidenced by the analog in rank one. In rank one cases, Besson, Courtois and Gallot [3] reproved the Mostow rigidity theorem using the barycenter method, where they obtained uniformly bounded Jacobian (same expression as ours) in degrees \( \geq 3 \) (they gave explicit bounds in order to show the Mostow rigidity theorem). Notice the splitting rank of a real rank one symmetric space is identically 1, so our estimate agrees with the original estimate of Besson, Courtois and Gallot. While in degree 2 = \( \text{srk}(X) + 1 \), the Jacobian estimate fails and correspondingly is the fact that the Mostow rigidity theorem is not true in dimension two.
References


Appendix A: Proof of Theorem 5.1.3 and Proposition 5.1.6

In this section, we finish the proof of Theorem 5.1.3 and Proposition 5.1.6. We combine the two proofs as they are both a case by case argument. And we notice the case of $\text{SL}(r+1, \mathbb{R})/\text{SO}(r+1)$ has already been proved in the context.

Case of $\text{SL}(r+1, \mathbb{C})/\text{SU}(r+1)$: the Dynkin diagram is of type $A_r$ and is shown in Figure 5.1, with multiplicities 2 for all simple roots. Since $Y$ is generated by the truncated simple system $\{\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_r\}$ for some $i = 1, \ldots, r$, preserving the same multiplicities and configurations, we have $Y = \text{SL}(i, \mathbb{C})/\text{SU}(i) \times \text{SL}(r-i+1, \mathbb{C})/\text{SU}(r-i+1)$, for some $i = 1, \ldots, r$. The dimension of $Y$ equals $(i-1)(i+1) + (r-i)(r-i+2)$, so the codimension of $Y \times \mathbb{R} \subset X$ is $-2i^2 + 2(r+1)i$, which attains its minimal codimension $2r$ when $i = 1, r$. In both cases we have $Y = \text{SL}(r, \mathbb{C})/\text{SO}(r)$, and hence $\text{srk}(X) = \dim(Y \times \mathbb{R}) = n - 2r$, where $n = \dim(X) = r(r+2)$. The codimension of $Y \times \mathbb{R} \subset X$ attains its second minimal value when $i = 2, r-1$ provided $r \geq 3$. In this case, $Y = \mathbb{H}^3 \times \text{SL}(r-1, \mathbb{C})/\text{SU}(r-1)$ and the codimension of $Y \times \mathbb{R}$ is $4r - 4$, so the gap is $2r - 4$.

Case of $\text{SU}^*(2r+2)/\text{Sp}(r+1)$: the Dynkin diagram is of type $A_r$ and is shown in Figure 5.1, with multiplicities 4 for all simple roots. Hence $Y = \text{SU}^*(2i)/\text{Sp}(i) \times \text{SU}^*(2r+2-2i)/\text{Sp}(r+1-k)$ for some $i = 1, \ldots, r$. The codimension of $Y \times \mathbb{R} \subset X$ is $-4i^2 + 4(r+1)i$, which attains minimal when $i = 1, r$. In both cases $Y = \text{SU}^*(2r)/\text{Sp}(r)$, and $\text{srk}(X) = n - 4r$. The codimension attains its second minimal
Figure A.2: Dynkin diagram of type $B_r$

value when $i = 2, r - 1$ provided $r \geq 3$, in which case $Y = \mathbb{H}^5 \times \text{SU}^*(2r - 2)/\text{Sp}(r - 1)$.

The codimension of $Y \times \mathbb{R}$ is $8r - 8$, so the gap is $4r - 8$.

Case of $E_6^{-26}/F_4$: the Dynkin diagram is of type $A_2$ and is shown in Figure 5.1 where $r = 2$, with multiplicities 8 for both simple roots. Hence $Y$ can only be $\mathbb{H}^9$ so that $\text{srk}(X) = 10$. Notice that $E_6^{-26}/F_4$ is of rank two and it does not satisfy the condition of Proposition 5.1.6.

Case of $\text{SO}_0(r, r + k)/\text{SO}(r) \times \text{SO}(r + k)$: the Dynkin diagram is of type $B_r$ and is shown in Figure A.2, with ordered multiplicities $1, 1, ..., 1, k$. If we remove $\alpha_i$, the remaining diagram (with multiplicity information) will represent $Y_i = \text{SL}(i, \mathbb{R})/\text{SO}(i) \times [\text{SO}(r - i, r - i + k)/\text{SO}(r - i) \times \text{SO}(r - i + k)]$ (Notice $\text{SL}(1, \mathbb{R})/\text{SO}(1)$ and $\text{SO}(0, k)/\text{SO}(0) \times \text{SO}(k)$ are just a point by abuse of notation). Thus we can compute that $Y_i \times \mathbb{R}$ has codimension $-3i^2/2 + (4r + 2k - 1)i/2$ in $X$. It attains a minimum when $i = 1$ provided $r + 2k > 4$, and so $Y = \text{SO}_0(r - 1, r - 1 + k)/\text{SO}(r - 1) \times \text{SO}(r - 1 + k)$, $\text{srk}(X) = n - (2r + k - 2)$. The only space that satisfies $r + 2k \leq 4$ is $\text{SO}_0(2, 3)/\text{SO}(2) \times \text{SO}(3)$, and it has splitting rank 3, corresponding to $\mathbb{H}^2 \times \mathbb{R}$. This agrees with the general formula hence can be absorbed into it.

Now the codimension attains its second minimum when $i = 2$ provided $r + 2k > 7$, and so $Y' = \mathbb{H}^2 \times \text{SO}_0(r - 2, r - 2 + k)/\text{SO}(r - 2) \times \text{SO}(r - 2 + k)$, $\dim(Y' \times \mathbb{R}) = \ldots$
\( n - (4r + 2k - 7) \). Hence the gap is \( 2r + k - 5 \). As we focus on \( r \geq 4 \) in Proposition 5.1.6, the spaces that are excluded by \( r + 2k > 7 \) is \( \text{SO}_0(4, 5)/\text{SO}(4) \times \text{SO}(5) \) and \( \text{SO}_0(5, 6)/\text{SO}(5) \times \text{SO}(6) \). If \( X = \text{SO}_0(4, 5)/\text{SO}(4) \times \text{SO}(5) \) (dim(\( X \)) = 20), then \( Y \) is \( \text{SO}_0(3, 4)/\text{SO}(3) \times \text{SO}(4) \) (dimension 12), and \( Y' \) is \( \text{SL}(4, \mathbb{R})/\text{SO}(4) \) (dimension 9), so the gap is 3. If \( X = \text{SO}_0(5, 6)/\text{SO}(5) \times \text{SO}(6) \), then \( Y = \text{SO}_0(4, 5)/\text{SO}(4) \times \text{SO}(5) \), and \( Y' \) is either \( \mathbb{H}^2 \times \text{SO}(3, 4)/\text{SO}(3) \times \text{SO}(4) \) or \( \text{SL}(5, \mathbb{R})/\text{SO}(5) \) with dimension 14. So the gap is 6.

Case of \( \text{SO}(2r + 1, \mathbb{C})/\text{SO}(2r + 1) \): the Dynkin diagram is of type \( B_r \) and is shown in Figure A.2, with multiplicities 2 for all simple roots. If we remove \( \alpha_i \), the remaining diagram will represent \( Y_i = \text{SL}(i, \mathbb{C})/\text{SU}(i) \times \text{SO}(2r - 2i + 1, \mathbb{C})/\text{SO}(2r - 2i + 1) \) (notice \( \text{SL}(1, \mathbb{C})/\text{SU}(1) \) and \( \text{SO}(1, \mathbb{C})/\text{SO}(1) \) are just a point by abuse of notation). We compute that the codimension of \( Y_i \times \mathbb{R} \) in \( X \) is \(-3i^2 + (4r + 1)i \). It has minimal value \( 4r - 2 \) when \( i = 1 \) (when \( r = 2 \) it takes minimal value on both \( i = 1, 2 \), but they both represent the same subspace \( \mathbb{H}^3 \)). Hence the splitting rank is \( n - (4r - 2) \), corresponding to the subspace \( Y = \text{SO}(2r - 1, \mathbb{C})/\text{SO}(2r - 1) \). Now the codimension takes the second minimal value \( 8r - 10 \) when \( i = 2 \), provided \( r > 5 \). In this case, \( Y' = \mathbb{H}^3 \times \text{SO}(2r - 3, \mathbb{C})/\text{SO}(2r - 3) \) and the gap is \( 4r - 8 \). If \( r = 4 \), then \( X = \text{SO}(9, \mathbb{C})/\text{SO}(9) \) and \( Y = \text{SO}(7, \mathbb{C})/\text{SO}(7) \). The codimension takes its second minimal value 20 when \( Y' = \text{SL}(4, \mathbb{C})/\text{SU}(4) \), hence the gap is 6. If \( r = 5 \), then \( X = \text{SO}(11, \mathbb{C})/\text{SO}(11) \) and \( Y = \text{SO}(9, \mathbb{C})/\text{SO}(9) \). The codimension takes its second minimal value 30 when \( Y' \) is \( \mathbb{H}^3 \times \text{SO}(7, \mathbb{C})/\text{SO}(7) \) or \( \text{SL}(5, \mathbb{C})/\text{SU}(5) \), hence the gap is 12.

Case of \( \text{Sp}(r, \mathbb{R})/U(r) \): the Dynkin diagram is of type \( C_r \) and is shown in Figure A.3, with multiplicities 1 for all simple roots. If we remove \( \alpha_i \), the remaining diagram
will represent $Y_i = \text{SL}(i, \mathbb{R})/\text{SO}(i) \times \text{Sp}(r - i, \mathbb{R})/U(r - i)$ (notice $\text{SL}(1, \mathbb{R})/\text{SO}(1)$ and $\text{Sp}(0, \mathbb{R})/U(0)$ are just a point by abuse of notation). We compute that the codimension of $Y_i \times \mathbb{R}$ in $X$ is $-3i^2/2 + (4r + 1)i/2$. It has minimal value $2r - 1$ when $i = 1$ (when $r = 2$ it takes minimal value on both $i = 1, 2$, but they both represent the same space $\mathbb{H}^2$). Hence the splitting rank is $n - (2r - 1)$ corresponding to the space $Y = \text{Sp}(r - 1, \mathbb{R})/U(r - 1)$. Now the codimension takes the second minimal value $4r - 5$ when $i = 2$, provided $r > 5$. In this case, $Y' = \mathbb{H}^2 \times \text{Sp}(r - 2, \mathbb{R})/U(r - 2)$ and the gap is $2r - 4$. If $r = 4$, then $X = \text{Sp}(4, \mathbb{R})/U(4)$ and $Y = \text{Sp}(3, \mathbb{R})/U(3)$. The codimension takes its second minimal value $10$ when $Y' = \text{SL}(4, \mathbb{R})/\text{SO}(4)$, hence the gap is $3$. If $r = 5$, then $X = \text{Sp}(5, \mathbb{R})/U(5)$ and $Y = \text{Sp}(4, \mathbb{R})/U(4)$. The codimension takes its second minimal value $15$ when $Y'$ is $\mathbb{H}^2 \times \text{Sp}(3, \mathbb{R})/U(3)$ or $\text{SL}(5, \mathbb{R})/\text{SO}(5)$, hence the gap is $6$.

Case of $\text{SU}(r, r)/S(\text{U}(r) \times \text{U}(r))$: the Dynkin diagram is of type $C_r$ and is shown in Figure A.3, with ordered multiplicities $2, 2, ..., 2, 1$. If we remove $\alpha_i$, the remaining diagram will represent $Y_i = \text{SL}(i, \mathbb{C})/\text{SU}(i) \times \text{SU}(r - i, r - i)/S(\text{U}(r - i) \times \text{U}(r - i))$ (notice $\text{SL}(1, \mathbb{C})/\text{SU}(1)$ and $\text{SU}(0, 0)/S(\text{U}(0) \times \text{U}(0))$ are just a point by abuse of notation). We compute that the codimension of $Y_i \times \mathbb{R}$ in $X$ is $-3i^2 + 4ri$. It has minimal value $4r - 3$ when $i = 1$, provided $r > 3$. If $r = 2$, $X$ is isomorphic to
SO_0(2, 4)/SO(2) \times SO(4), which has been solved previously. If \( r = 3 \), the codimension is minimal for both \( i = 1, 3 \), which corresponds to \( SO_0(2, 4)/SO(2) \times SO(4) \) and \( SL(3, \mathbb{C})/SU(3) \). Hence the splitting rank is \( n - (4r - 3) \) corresponding to the space \( Y = \text{Sp}(r - 1, \mathbb{R})/U(r - 1) \) (except for the case \( r = 3 \) where there are two subspaces). Now the codimension takes the second minimal value \( 8r - 12 \) when \( i = 2 \), provided \( r > 6 \). In this case, \( Y' = \mathbb{H}^3 \times SU(r - 2, r - 2)/S(U(r - 2) \times U(r - 2)) \) and the gap is \( 4r - 9 \). If \( r = 4 \), then \( X = SU(4, 4)/S(U(4) \times U(4)) \) and \( Y = SU(3, 3)/S(U(3) \times U(3)) \). The codimension takes its second minimal value \( 16 \) when \( Y' = SL(4, \mathbb{C})/SU(4) \), hence the gap is \( 8 \). If \( r = 5 \), then \( X = SU(5, 5)/S(U(5) \times U(5)) \) and \( Y = SU(4, 4)/S(U(4) \times U(4)) \). The codimension takes its second minimal value \( 25 \) when \( Y' = SL(5, \mathbb{C})/SU(5) \), hence the gap is \( 8 \). If \( r = 6 \), then \( X = SU(6, 6)/S(U(6) \times U(6)) \) and \( Y = SU(5, 5)/S(U(5) \times U(5)) \). The codimension takes its second minimal value \( 36 \) when \( Y' \) is \( \mathbb{H}^3 \times SU(4, 4)/S(U(4) \times U(4)) \) or \( SL(6, \mathbb{C})/SU(6) \), hence the gap is \( 15 \).

Case of \( \text{Sp}(r, \mathbb{C})/\text{Sp}(r) \): the Dynkin diagram is of type \( C_r \) and is shown in Figure A.3, with multiplicities 2 for all simple roots. If we remove \( \alpha_i \), the remaining diagram will represent \( Y_i = \text{SL}(i, \mathbb{C})/SU(i) \times \text{Sp}(r - i, \mathbb{C})/\text{Sp}(r - i) \) (notice \( SL(1, \mathbb{C})/SU(1) \) and \( \text{Sp}(0, \mathbb{C})/\text{Sp}(0) \) are just a point by abuse of notation). We compute that the codimension of \( Y_i \times \mathbb{R} \) in \( X \) is \(-3i^2 + (4r + 1)i\). It has minimal value \( 4r - 2 \) when \( i = 1 \), provided \( r > 2 \). If \( r = 2 \), then \( X \) is isomorphic to \( SO(5, \mathbb{C})/SO(5) \), which has been solved previously. Now the codimension takes the second minimal value \( 8r - 10 \) when \( i = 2 \), provided \( r > 5 \). In this case, \( Y' = \mathbb{H}^3 \times \text{Sp}(r - 2, \mathbb{C})/\text{Sp}(r - 2) \) and the gap is \( 4r - 8 \). If \( r = 4 \), then \( X = \text{Sp}(4, \mathbb{C})/\text{Sp}(4) \) and \( Y = \text{Sp}(3, \mathbb{C})/\text{Sp}(3) \). The codimension takes its second minimal value \( 20 \) when \( Y' = SL(4, \mathbb{C})/SU(4) \), hence
the gap is 6. If \( r = 5 \), then \( X = \text{Sp}(5, \mathbb{C})/\text{Sp}(5) \) and \( Y = \text{Sp}(4, \mathbb{C})/\text{Sp}(4) \). The codimension takes its second minimal value 30 when \( Y' = \mathbb{H}^3 \times \text{Sp}(3, \mathbb{C})/\text{Sp}(3) \) or \( \text{SL}(5, \mathbb{C})/\text{SU}(5) \), hence the gap is 12.

**Case of \( \text{SO}^*(4r)/U(2r) \):** the Dynkin diagram is of type \( C_r \) and is shown in Figure A.3, with ordered multiplicities 4, 4, ..., 4, 1. If we remove \( \alpha_i \), the remaining diagram will represent \( Y_i = \text{SU}^*(2i - 2)/\text{Sp}(i) \times \text{SO}^*(4r - 4i)/U(2r - 2i) \) (notice \( \text{SU}^*(0)/\text{Sp}(1) \) and \( \text{SO}^*(0)/U(0) \) are just a point by abuse of notation). We compute that the codimension of \( Y_i \times \mathbb{R} \) in \( X \) is \(-6i^2 + (8r - 1)i\). It has minimal value \( 8r - 7 \) when \( i = 1 \), provided \( r > 3 \). If \( r = 2 \), then \( X \) is isomorphic to \( \text{SO}_0(2,6)/\text{SO}(2) \times \text{SO}(6) \), which has been solved previously. If \( r = 3 \), then \( X = \text{SO}^*(12)/U(6) \), and the codimension has minimal value 15 when \( i = 3 \). In this case, the splitting rank occurs when \( Y = \text{SU}^*(6)/\text{Sp}(3) \) and is equal to 15. Now the codimension takes the second minimal value 16\(r - 6 \) when \( i = 2 \), provided \( r > 6 \). In this case, \( Y' = \mathbb{H}^5 \times \text{SO}^*(4r - 8)/U(2r - 4) \) and the gap is \( 8r - 19 \). If \( r = 4 \), then \( X = \text{SO}^*(16)/U(8) \) and \( Y = \text{SO}^*(12)/U(6) \). The codimension takes its second minimal value 28 when \( Y' = \text{SU}^*(8)/\text{Sp}(4) \), hence the gap is 3. If \( r = 5 \), then \( X = \text{SO}^*(20)/U(10) \) and \( Y = \text{SO}^*(16)/U(8) \). The codimension takes its second minimal value 45 when \( Y' = \text{SU}^*(10)/\text{Sp}(5) \), hence the gap is 12. If \( r = 6 \), then \( X = \text{SO}^*(24)/U(12) \) and \( Y = \text{SO}^*(20)/U(10) \). The codimension takes its second minimal value 66 when \( Y' = \text{SU}^*(12)/\text{Sp}(6) \), hence the gap is 25.

**Case of \( \text{Sp}(r,r)/\text{Sp}(r) \times \text{Sp}(r) \):** the Dynkin diagram is of type \( C_r \) and is shown in Figure A.3, with ordered multiplicities 4, 4, ..., 4, 3. If we remove \( \alpha_i \), the remaining diagram will represent \( Y_i = \text{SU}^*(2i - 2)/\text{Sp}(i) \times [\text{Sp}(r - i, r - i)/\text{Sp}(r - i) \times \text{Sp}(r - i)] \) (Notice \( \text{SU}^*(0)/\text{Sp}(1) \) and \( \text{Sp}(0,0)/\text{Sp}(0) \times \text{Sp}(0) \) are just a point by abuse of
notation). We compute that the codimension of $Y_i \times \mathbb{R}$ in $X$ is $-6i^2 + (8r + 1)i$. It has minimal value $8r - 5$ when $i = 1$, provided $r > 2$. If $r = 2$, then $X = \text{Sp}(2,2)/\text{Sp}(2) \times \text{Sp}(2)$, and the codimension has minimal value 10 when $i = 2$. So the splitting rank occurs when $Y = \mathbb{H}^5$ and is equal to 6. Now the codimension takes the second minimal value $16r - 22$ when $i = 2$, provided $r > 5$. In this case, $Y' = \mathbb{H}^5 \times [\text{Sp}(r-2,r-2)/\text{Sp}(r-2) \times \text{Sp}(r-2)]$ and the gap is $8r - 17$. If $r = 4$, then $X = \text{Sp}(4,4)/\text{Sp}(4) \times \text{Sp}(4)$ and $Y = \text{Sp}(3,3)/\text{Sp}(3) \times \text{Sp}(3))$. The codimension takes its second minimal value 36 when $Y' = \text{SU}^*(8)/\text{Sp}(4)$, hence the gap is 9. If $r = 5$, then $X = \text{Sp}(5,5)/\text{Sp}(5) \times \text{Sp}(5)$ and $Y = \text{Sp}(4,4)/\text{Sp}(4) \times \text{Sp}(4)$. The codimension takes its second minimal value 55 when $Y' = \text{SU}^*(10)/\text{Sp}(5)$, hence the gap is 20.

Case of $E_7^{25}/E_6 \times U(1)$: the Dynkin diagram is of type $C_3$ and is shown in Figure A.3 where $r = 3$, with ordered multiplicities $8, 8, 1$. Hence $Y$ can only be $\text{SO}_0(2,10)/\text{SO}(2) \times \text{SO}(10)$ (when removing $\alpha_1$), or $\mathbb{H}^9 \times \mathbb{H}^2$ (when removing $\alpha_2$), or $E_6^{26}/F_4$ (when removing $\alpha_3$). Among the three spaces, $E_6^{26}/F_4$ has largest dimension thus $\text{srk}(X) = 27$. Notice that $E_7^{25}/E_6 \times U(1)$ is of rank three and it does not satisfy the condition of Proposition 5.1.6.

Case of $\text{SO}_0(r,r)/\text{SO}(r) \times \text{SO}(r)$: the Dynkin diagram is of type $D_r$ and is shown in Figure A.4, with multiplicities 1 for all simple roots. If we remove $\alpha_i$, the remaining
diagram will represent $Y_i = \text{SL}(i, \mathbb{R})/\text{SO}(i) \times [\text{SO}_0(r-i, r-i)/\text{SO}(r-i) \times \text{SO}(r-i)]$
when $i < r - 2$ (notice $\text{SO}_0(3, 3)/\text{SO}(3) \times \text{SO}(3)$ is the same as $\text{SL}(4, \mathbb{R})/\text{SO}(4)$), and
$Y_i = \mathbb{H}^2 \times \mathbb{H}^2 \times \text{SL}(r-2, \mathbb{R})/\text{SO}(r-2)$ when $i = r - 2$, and $Y_i = \text{SL}(r, \mathbb{R})/\text{SO}(r)$ when
$i = r - 1, r$. We compute that the codimension of $Y_i \times \mathbb{R}$ in $X$ is $-3i^2/2 + (4r - 1)i/2$
for $1 \leq i \leq r - 2$ or $i = r$. It has unique minimal value $2r - 2$ when $i = 1$, provided
$r > 4$. If $r = 4$, the codimension has minimal value 6 when $i = 1, 3, 4$. So the
splitting rank occurs when $Y = \text{SL}(4, \mathbb{R})/\text{SO}(4)$ and is equal to 10. This agrees with
the general result when $r > 4$ hence can be absorbed into it. Now the codimension
takes the second minimal value $4r - 7$ when $i = 2$, provided $r > 7$. In this case,
$Y' = \mathbb{H}^2 \times [\text{SO}_0(r-2, r-2)/\text{SO}(r-2) \times \text{SO}(r-2)]$ and the gap is $2r - 5$. If $r = 4$,
then $X = \text{SO}_0(4, 4)/\text{SO}(4) \times \text{SO}(4)$ and $Y = \text{SL}(4, \mathbb{R})/\text{SO}(4)$. The codimension takes
its second minimal value 9 when $Y' = \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$, hence the gap is 3. If $r = 5$, then
$X = \text{SO}_0(5, 5)/\text{SO}(5) \times \text{SO}(5)$ and $Y = \text{SO}_0(4, 4)/\text{SO}(4) \times \text{SO}(4)$. The codimension
takes its second minimal value 10 when $Y' = \text{SL}(5, \mathbb{R})/\text{SO}(5)$, hence the gap is 2. If
$r = 6$, then $X = \text{SO}_0(6, 6)/\text{SO}(6) \times \text{SO}(6)$ and $Y = \text{SO}_0(5, 5)/\text{SO}(5) \times \text{SO}(5)$. The
codimension takes its second minimal value 15 when $Y' = \text{SL}(6, \mathbb{R})/\text{SO}(6)$, hence the
gap is 5. If $r = 7$, then $X = \text{SO}_0(7, 7)/\text{SO}(7) \times \text{SO}(7)$ and $Y = \text{SO}_0(6, 6)/\text{SO}(6) \times
\text{SO}(6)$. The codimension takes its second minimal value 21 when $Y'$ is either $\mathbb{H}^2 \times
\text{SO}_0(5, 5)/\text{SO}(5) \times \text{SO}(5)$ or $\text{SL}(7, \mathbb{R})/\text{SO}(7)$, hence the gap is 9.

Case of $\text{SO}(2r, \mathbb{C})/\text{SO}(2r)$: the Dynkin diagram is of type $D_r$ and is shown in
Figure A.4, with multiplicities 2 for all simple roots. If we remove $\alpha_i$, the remaining
diagram will represent $Y_i = \text{SL}(i, \mathbb{C})/\text{SU}(i) \times \text{SO}(2r - 2i, \mathbb{C})/\text{SO}(2r - 2i)$ when $i < r - 2$
(notice $\text{SO}(6, \mathbb{C})/\text{SO}(6)$ is the same as $\text{SL}(4, \mathbb{C})/\text{SU}(4)$), and $Y_i = \mathbb{H}^3 \times \mathbb{H}^3 \times \text{SL}(r - 2, \mathbb{C})/\text{SU}(r - 2)$ when $i = r - 2$, and $Y_i = \text{SL}(r, \mathbb{C})/\text{SU}(r)$ when $i = r - 1, r$. We
compute that the codimension of \( Y_i \times \mathbb{R} \) in \( X \) is \(-3i^2 + (4r - 1)i\) for \( 1 \leq i \leq r - 2 \) or \( i = r \). It has unique minimal value \( 4r - 4 \) when \( i = 1 \), provided \( r > 4 \). If \( r = 4 \), the codimension has minimal value 12 when \( i = 1, 3, 4 \). So the splitting rank occurs when \( Y = \text{SL}(4, \mathbb{C})/\text{SU}(4) \) and is equal to 16. This agrees with the general result when \( r > 4 \) hence can be absorbed into it. Now the codimension takes the second minimal value \( 8r - 14 \) when \( i = 2 \), provided \( r > 7 \). In this case, \( Y' = \mathbb{H}^3 \times \text{SO}(2r - 4, \mathbb{C})/\text{SO}(2r - 4) \) and the gap is \( 4r - 10 \). If \( r = 4 \), then \( X = \text{SO}(8, \mathbb{C})/\text{SO}(8) \) and \( Y = \text{SL}(4, \mathbb{C})/\text{SU}(4) \). The codimension takes its second minimal value 18 when \( Y' = \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3 \), hence the gap is 6. If \( r = 5 \), then \( X = \text{SO}(10, \mathbb{C})/\text{SO}(10) \) and \( Y = \text{SO}(8, \mathbb{C})/\text{SO}(8) \). The codimension takes its second minimal value 20 when \( Y' = \text{SL}(5, \mathbb{C})/\text{SU}(5) \), hence the gap is 4. If \( r = 6 \), then \( X = \text{SO}(12, \mathbb{C})/\text{SO}(12) \) and \( Y = \text{SO}(10, \mathbb{C})/\text{SO}(10) \). The codimension takes its second minimal value 30 when \( Y' = \text{SL}(6, \mathbb{C})/\text{SU}(6) \), hence the gap is 10. If \( r = 7 \), then \( X = \text{SO}(14, \mathbb{C})/\text{SO}(14) \) and \( Y = \text{SO}(12, \mathbb{C})/\text{SO}(12) \). The codimension takes its second minimal value 42 when \( Y' \) is either \( \mathbb{H}^3 \times \text{SO}(10, \mathbb{C})/\text{SO}(10) \) or \( \text{SL}(7, \mathbb{C})/\text{SU}(7) \), hence the gap is 18.

**Case of** \( \text{SU}(r, r + k)/\text{S}(\text{U}(r) \times \text{U}(r + k)) \): the Dynkin diagram is of type \((BC)_r\) and is shown in Figure A.5, with ordered multiplicities 2, 2, ..., 2, (2k, 1). If we remove \( \alpha_i \), the remaining diagram will represent \( Y' = \text{SL}(i, \mathbb{C})/\text{SU}(i) \times \text{SU}(r - i, r - i +
\(k)/S(U(r - i) \times U(r - i + k))\) (notice \(\text{SL}(1, \mathbb{C}R)/\text{SU}(1)\) and \(\text{SU}(0, k)/S(U(0) \times U(k))\) are just a point by abuse of notation). We compute that the codimension of \(Y_i \times \mathbb{R}\) in \(X\) is \(-3i^2 + (4r + 2k)i\). It has unique minimal value \(4r + 2k - 3\) when \(i = 1\), provided \(r + 2k > 3\), which holds for higher rank symmetric spaces. So the splitting rank occurs when \(Y = \text{SU}(r-1, r-1+k)/S(U(r-1) \times U(r-1+k))\) and is equal to \(n - (4r + 2k - 3)\).

Now the codimension takes the second minimal value \(8r + 4k - 12\) when \(i = 2\), provided \(r + 2k > 6\). In this case, \(Y' = \mathbb{H}^3 \times \text{SU}(r - 2, r - 2 + k)/S(U(r - 2) \times U(r - 2 + k))\) and the gap is \(4r + 2k - 9\). As we focus on \(r \geq 4\) in Proposition 5.1.6, the only space excluded by \(r + 2k > 6\) is \(\text{SU}(4, 5)/S(U(4) \times U(5))\) \((r = 4, k = 1)\). In this case, \(Y = \text{SU}(3, 4)/S(U(3) \times U(4))\). The codimension takes its second minimal value \(24\) when \(Y'\) is either \(\mathbb{H}^3 \times \text{SU}(2, 3)/S(U(2) \times U(3))\) or \(\text{SL}(4, \mathbb{C})/\text{SU}(4)\), hence the gap is \(9\).

**Case of \(\text{Sp}(r, r+k)/\text{Sp}(r) \times \text{Sp}(r+k)\):** the Dynkin diagram is of type \((BC)_r\) and is shown in Figure A.5, with ordered multiplicities \(4, 4, \ldots, 4, (4k, 3)\). If we remove \(\alpha_i\), the remaining diagram will represent \(Y_i = \text{SU}^*(2i)/\text{Sp}(i) \times [\text{Sp}(r - i, r - i + k)/\text{Sp}(r - i) \times \text{Sp}(r - i + k)]\) (notice \(\text{SU}^*(2)/\text{Sp}(1)\) and \(\text{Sp}(0, k)/\text{Sp}(0) \times \text{Sp}(k)\) are just a point by abuse of notation). We compute that the codimension of \(Y_i \times \mathbb{R}\) in \(X\) is \(-6i^2 + (8r + 4k + 1)i\). It has unique minimal value \(8r + 4k - 5\) when \(i = 1\). So the splitting rank occurs when \(Y = \text{Sp}(r - 1, r - 1 + k)/\text{Sp}(r - 1) \times \text{Sp}(r - 1 + k)\) and is equal to \(n - (8r + 4k - 5)\). Now the codimension takes the second minimal value \(16r + 8k - 22\) when \(i = 2\), provided \(2r + 4k > 11\). In this case, \(Y' = \mathbb{H}^5 \times \text{Sp}(r - 2, r - 2 + k)/\text{Sp}(r - 2) \times \text{Sp}(r - 2 + k)\) and the gap is \(8r + 4k - 17\). As we focus on \(r \geq 4\) in Proposition 5.1.6, the inequality \(2r + 4k > 11\) always holds.

**Case of \(\text{SO}^*(4r + 2)/U(2r + 1)\):** the Dynkin diagram is of type \((BC)_r\) and is shown in Figure A.5, with ordered multiplicities \(4, 4, \ldots, 4, (4, 1)\). If we remove \(\alpha_i\), the
remaining diagram will represent \( Y_i = \text{SU}^*(2i)/\text{Sp}(i) \times \text{SO}^*(4r - 4i + 2)/U(2r - 2i + 1) \) (notice \( \text{SU}^*(2)/\text{Sp}(1) \) and \( \text{SO}^*(2)/U(1) \) are just a point by abuse of notation). We compute that the codimension of \( Y_i \times \mathbb{R} \) in \( X \) is \(-6i^2 + (8r + 3)i\). It has unique minimal value \( 8r - 3 \) when \( i = 1 \). So the splitting rank occurs when \( Y = \text{SO}^*(4r - 2)/U(2r - 1) \) and is equal to \( n - (8r - 3) \). Now the codimension takes the second minimal value \( 16r - 18 \) when \( i = 2 \), provided \( r > 4 \). In this case, \( Y' = \mathbb{H}^5 \times \text{SO}^*(4r - 6)/U(2r - 3) \) and the gap is \( 8r - 15 \). As we focus on \( r \geq 4 \) in Proposition 5.1.6, the only space excluded by the inequality \( r > 4 \) is \( \text{SO}^*(18)/U(9) \). In this special case, \( Y = \text{SO}^*(14)/U(7) \) and the codimension takes its second minimal value 44 when \( Y' = \text{SU}^*(8)/\text{Sp}(4) \), hence the gap is 15.

**Case of \( E_6^{-14}/\text{Spin}(10) \times U(1) \):** the Dynkin diagram is of type \((BC)_2\) and is shown in Figure A.5, with ordered multiplicities 6, (8, 1). Hence \( Y \) can only be \( \mathbb{H}^9 \) or \( \text{SU}(1,5)/S(U(1) \times U(5)) \simeq \mathbb{C}\mathbb{H}^5 \). Comparing the dimensions of the two spaces, we conclude the one that has splitting rank should be \( \mathbb{C}\mathbb{H}^5 \times \mathbb{R} \), and the splitting rank is 11. Notice that \( E_6^{-14}/\text{Spin}(10) \times U(1) \) is of rank two hence it does not satisfy the condition of Proposition 5.1.6.

**Case of \( E_6^6/\text{Sp}(4) \):** the Dynkin diagram is of type \( E_6 \) and is shown in Figure A.6, with multiplicities 1 for all simple roots. If we remove one simple root, the remaining
Figure A.7: Dynkin diagram of type $E_7$

diagram will represent 4 kinds of symmetric spaces: $Y_1 = Y_6 = SO_0(5, 5)/SO(5) \times SO(5)$, $Y_2 = Y_5 = \mathbb{H}^2 \times SL(5, \mathbb{R})/SO(5)$, $Y_3 = \mathbb{H}^2 \times SL(3, \mathbb{R})/SO(3) \times SL(3, \mathbb{R})/SO(3)$ and $Y_4 = SL(6, \mathbb{R})/SO(6)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 26, 17, 13 and 21 respectively. So the splitting rank is 26 and the gap is 5.

*Case of $E_6(\mathbb{C})/E_6$:* the Dynkin diagram is of type $E_6$ and is shown in Figure A.6, with multiplicities 2 for all simple roots. If we remove one simple root, the remaining diagram will represent 4 kinds of symmetric spaces: $Y_1 = Y_6 = SO(10, \mathbb{C})/SO(10)$, $Y_2 = Y_5 = \mathbb{H}^3 \times SL(5, \mathbb{C})/SU(5)$, $Y_3 = \mathbb{H}^3 \times SL(3, \mathbb{C})/SU(3) \times SL(3, \mathbb{C})/SU(3)$ and $Y_4 = SL(6, \mathbb{C})/SU(6)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 46, 28, 20 and 36 respectively. So the splitting rank is 46 and the gap is 10.

*Case of $E_7^7/SU(8)$: the Dynkin diagram is of type $E_7$ and is shown in Figure A.7, with multiplicities 1 for all simple roots. If we remove one simple root, the remaining diagram will represent 7 kinds of symmetric spaces: $Y_1 = SO_0(6, 6)/SO(6) \times SO(6)$, $Y_2 = \mathbb{H}^2 \times SL(6, \mathbb{R})/SO(6)$, $Y_3 = \mathbb{H}^2 \times SL(3, \mathbb{R})/SO(3) \times SL(4, \mathbb{R})/SO(4)$, $Y_4 = SL(7, \mathbb{R})/SO(7)$, $Y_5 = SL(3, \mathbb{R})/SO(3) \times SL(5, \mathbb{R})/SO(5)$, $Y_6 = \mathbb{H}^2 \times SO_0(5, 5)/SO(5) \times SO(5)$ and $Y_7 = E_6^6/Sp(4)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 37, 23, 17, 28, 20, 28 and 43 respectively. So the splitting rank is 43 and the gap is 6.
Case of $E_7(\mathbb{C})/E_7$: the Dynkin diagram is of type $E_7$ and is shown in Figure A.7, with multiplicities 2 for all simple roots. If we remove one simple root, the remaining diagram will represent 7 kinds of symmetric spaces:

- $Y_1 = \text{SO}(12, \mathbb{C})/\text{SO}(12)$,
- $Y_2 = \mathbb{H}^3 \times \text{SL}(6, \mathbb{C})/\text{SU}(6)$,
- $Y_3 = \mathbb{H}^3 \times \text{SL}(3, \mathbb{C})/\text{SU}(3) \times \text{SL}(4, \mathbb{C})/\text{SU}(4)$,
- $Y_4 = \text{SL}(7, \mathbb{C})/\text{SU}(7)$,
- $Y_5 = \text{SL}(3, \mathbb{C})/\text{SU}(3) \times \text{SL}(5, \mathbb{C})/\text{SU}(5)$,
- $Y_6 = \mathbb{H}^3 \times \text{SO}(10, \mathbb{C})/\text{SO}(10)$, and
- $Y_7 = E_6(\mathbb{C})/E_6$.

We compute that the dimensions of $Y_i \times \mathbb{R}$ are 67, 39, 27, 49, 33, 49, and 79 respectively. So the splitting rank is 79 and the gap is 12.

Case of $E_8^8/\text{SO}(16)$: the Dynkin diagram is of type $E_8$ and is shown in Figure A.8, with multiplicities 1 for all simple roots. If we remove one simple root, the remaining diagram will represent 8 kinds of symmetric spaces:

- $Y_1 = \text{SO}_6(7, 7)/\text{SO}(7) \times \text{SO}(7)$,
- $Y_2 = \mathbb{H}^2 \times \text{SL}(7, \mathbb{R})/\text{SO}(7)$,
- $Y_3 = \mathbb{H}^2 \times \text{SL}(3, \mathbb{R})/\text{SO}(3) \times \text{SL}(5, \mathbb{R})/\text{SO}(5)$,
- $Y_4 = \text{SL}(8, \mathbb{R})/\text{SO}(8)$,
- $Y_5 = \text{SL}(4, \mathbb{R})/\text{SO}(4) \times \text{SL}(5, \mathbb{R})/\text{SO}(5)$,
- $Y_6 = \text{SL}(3, \mathbb{R})/\text{SO}(3) \times \text{SO}_0(5, 5)/\text{SO}(5)$, and
- $Y_7 = \mathbb{H}^2 \times E_6^8/\text{Sp}(4)$, and $Y_8 = E_7^7/\text{SU}(8)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 50, 30, 22, 36, 24, 31, 45 and 71 respectively. So the splitting rank is 71 and the gap is 21.

Case of $E_8(\mathbb{C})/E_8$: the Dynkin diagram is of type $E_8$ and is shown in Figure A.8, with multiplicities 2 for all simple roots. If we remove one simple root, the remaining diagram will represent 8 kinds of symmetric spaces:

- $Y_1 = \text{SO}(14, \mathbb{C})/\text{SO}(14)$,
Figure A.9: Dynkin diagram of type $F_4$

$Y_2 = \mathbb{H}^3 \times \text{SL}(7, \mathbb{C})/\text{SU}(7)$, $Y_3 = \mathbb{H}^3 \times \text{SL}(3, \mathbb{C})/\text{SU}(3) \times \text{SL}(5, \mathbb{C})/\text{SU}(5)$, $Y_4 = \text{SL}(8, \mathbb{C})/\text{SU}(8)$, $Y_5 = \text{SL}(4, \mathbb{C})/\text{SU}(4) \times \text{SL}(5, \mathbb{C})/\text{SU}(5)$, $Y_6 = \text{SL}(3, \mathbb{C})/\text{SU}(3) \times \text{SO}(10, \mathbb{C})/\text{SO}(10)$, $Y_7 = \mathbb{H}^3 \times E_6(\mathbb{C})/E_6$, and $Y_8 = E_7(\mathbb{C})/E_7$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 92, 52, 36, 64, 40, 54, 82 and 134 respectively. So the splitting rank is 134 and the gap is 42.

Case of $F_4^4/\text{Sp}(3) \times \text{Sp}(1)$: the Dynkin diagram is of type $F_4$ and is shown in Figure A.9, with multiplicities 1 for all simple roots. If we remove one simple root, the remaining diagram will represent 3 kinds of symmetric spaces: $Y_1 = \text{Sp}(3, \mathbb{R})/U(3)$, $Y_2 = Y_3 = \mathbb{H}^2 \times \text{SL}(3, \mathbb{R})/\text{SO}(3)$, and $Y_4 = \text{SO}_0(3, 4)/\text{SO}(3) \times \text{SO}(4)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 13, 8 and 13 respectively. So the splitting rank is 13 and the gap is 5.

Case of $E_6^2/\text{SU}(6) \times \text{Sp}(1)$: the Dynkin diagram is of type $F_4$ and is shown in Figure A.9, with ordered multiplicities 1, 1, 2, 2. If we remove one simple root, the remaining diagram will represent 4 kinds of symmetric spaces: $Y_1 = \text{SU}(3, 3)/S(U(3) \times U(3))$, $Y_2 = \mathbb{H}^2 \times \text{SL}(3, \mathbb{C})/\text{SU}(3)$, $Y_3 = \mathbb{H}^3 \times \text{SL}(3, \mathbb{R})/\text{SO}(3)$, and $Y_4 = \text{SO}_0(3, 5)/\text{SO}(3) \times \text{SO}(5)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 19, 11, 9 and 16 respectively. So the splitting rank is 19 and the gap is 3.
Case of $E_7^{-5}/SO(12) \times Sp(1)$: the Dynkin diagram is of type $F_4$ and is shown in Figure A.9, with ordered multiplicities $1, 1, 4, 4$. If we remove one simple root, the remaining diagram will represent 4 kinds of symmetric spaces: $Y_1 = SO^*(12)/U(6)$, $Y_2 = \mathbb{H}^2 \times SU^*(6)/Sp(3)$, $Y_3 = \mathbb{H}^5 \times SL(3, \mathbb{R})/SO(3)$, and $Y_4 = SO_0(3, 7)/SO(3) \times SO(7)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 31, 17, 11 and 28 respectively. So the splitting rank is 31 and the gap is 3.

Case of $E_8^{-24}/E_7 \times Sp(1)$: the Dynkin diagram is of type $F_4$ and is shown in Figure A.9, with ordered multiplicities $1, 1, 8, 8$. If we remove one simple root, the remaining diagram will represent 4 kinds of symmetric spaces: $Y_1 = E_7^{-25}/E_6 \times U(1)$, $Y_2 = \mathbb{H}^2 \times E_6^{-26}/F_4$, $Y_3 = \mathbb{H}^9 \times SL(3, \mathbb{R})/SO(3)$, and $Y_4 = SO_0(3, 11)/SO(3) \times SO(11)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 55, 29, 15 and 34 respectively. So the splitting rank is 55 and the gap is 21.

Case of $F_4(\mathbb{C})/F_4$: the Dynkin diagram is of type $F_4$ and is shown in Figure A.9, with multiplicities 2 for all simple roots. If we remove one simple root, the remaining diagram will represent 3 kinds of symmetric spaces: $Y_1 = Sp(3, \mathbb{C})/Sp(3)$, $Y_2 = Y_3 = \mathbb{H}^3 \times SL(3, \mathbb{C})/SU(3)$, and $Y_4 = SO(7, \mathbb{C})/SO(7)$. We compute that the dimensions of $Y_i \times \mathbb{R}$ are 22, 12 and 22 respectively. So the splitting rank is 22 and the gap is 10.
Case of $G_2^2/\text{SO}(4)$: the Dynkin diagram is of type $G_2$ and is shown in Figure A.10, with multiplicities 1 for both simple roots. If we remove one simple root, the remaining diagram will represent the only symmetric space: $\mathbb{H}^2$. So the splitting rank is 3 corresponding to the totally geodesic submanifold $\mathbb{H}^2 \times \mathbb{R}$. Notice this space is of rank two so it does not satisfy the condition of Proposition 5.1.6.

Case of $G_2(\mathbb{C})/G_2$: the Dynkin diagram is of type $G_2$ and is shown in Figure A.10, with multiplicities 2 for both simple roots. If we remove one simple root, the remaining diagram will represent the only symmetric space: $\mathbb{H}^3$. So the splitting rank is 4 corresponding to the totally geodesic submanifold $\mathbb{H}^3 \times \mathbb{R}$. Notice this space is of rank two so it does not satisfy the condition of Proposition 5.1.6.

This verifies all cases, and completes the proofs of both Theorem 5.1.3 and Proposition 5.1.6.