Modeling and Analysis of Human Group Dynamics

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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2016

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Abstract

In this work we provide mathematical and computational tools that allow us to model the “physics” of human group behavior: how the relationship between the individuals’ characteristics, interaction dynamics, and environmental factors translates into changes in the behavior of the group. We characterize the group dynamics for four specific situations, and show the importance of these models to improve the performance of task-solving groups in organizational management, to design strategies that prevent high-risk behaviors, and to motivate collective action and cooperation in situations that lead to social dilemmas.
To my family
Acknowledgments

First, I thank God for give me the strength to complete my doctoral studies and
the opportunity of having an experience not only at the intelectual level but also social
and cultural. I give thanks to my parents and brother for their unconditional love
and support. Also, I give special thanks to my advisor and friend Kevin M. Passino.
His guidance, willingness to discuss ideas, and kindness make him a great model of
an advisor. Also, I want to thank John Clapp for his collaboration and insights,
my colleagues Hugo Gonzalez, Isa Fernandez, and Veronica Badescu for their helpful
discussions, and the department of Electrical and Computer Engineering for providing
funds to support the last years of my studies.
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Fields of Study

Major Field: Electrical and Computer Engineering
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B.1 Profile functions for environments that are protective against (Environment 1) and facilitate (Environment 2) high BAC levels.
Chapter 1: Introduction

We study and model the dynamics of human groups that are in four different situations, in which there is a need for developing mathematical and computational tools for their analysis and to promote our understanding of their behavior. Each chapter in this document addresses one of these situations. Here, we provide a very brief summary of each chapter and the corresponding reference to the corresponding submitted/accepted journal publication.

First, we study those groups whose goal is to solve a task (Chapter 2, [40]). We model relationship between the performance in solving the task, the cohesion of the group, and the patterns of interaction between the group members. We show how the qualitative behavior of the modeled groups is consistent with what has been observed in real human groups. Second, we model the dynamics of groups that are in alcohol-consumption environments. We show how the model can be used to understand the etiology of high-risk drinking and to design intervention strategies that seek to minimize such dangerous behaviors. We study the metabolic and decision-making processes in the individual drinker (Chapter 3, [42]), and extend these results to include the group and environmental influences (Chapter 4, [41]). Third, we study the social dilemma where self-interested individuals choose not to participate in completing a task that can benefit the community they belong to (Chapter 5, [39]). We refer
to this situation a task completion social dilemma. We provide a dynamical model that characterizes this situation, and propose mechanisms that promote participation and cooperation in the community to complete the task. In Chapter 6, we close with a summary of the main contributions in dissertation, and work to be developed in the future.
Chapter 2: Dynamic Task Performance, Cohesion, and Communication in Human Groups

In disciplines ranging from psychology, sociology, social work and engineering, significant efforts have been undertaken to understand the dynamics of human groups, that is, the processes within the group resulting from the actions of its members and the interaction among each other and their environment [51, 97, 32]. Studies suggest that human groups cannot be examined by considering each member in isolation [115]. Elements, such as the situation the group is in, the particular characteristics of each individual, and his or her interactions with other group members, have to be considered for the analysis and understanding of the dynamics of the group. In [70], Lewin provides a theoretical analysis of group dynamics where he states that a group is a dynamical system in which the behavior of a member is guided by the interplay between its individual features and the environment. “Environment” in this context includes other members of the group and the social setting. Lewin summarizes this process using the formula

\[ B = I(P, H) \] (2.1)

which describes a group member’s dynamics \( B \) as a function \( I \) of the interaction between his or her individual attributes \( P \) and the social environment \( H \). In this work, we propose a mathematical characterization of group dynamics that is
conceptually consistent with Lewin’s formula. To model the function $I$ of the interaction between a group member and the social environment he or she is in, we focus on the relationship between three important processes that affect the dynamics of a group: cohesion between the group members, performance of the group while solving a complex task, and the patterns in the flow of information between members.

One of the most well-studied concepts in the analysis of group dynamics is *group cohesion*. Cohesion is defined by Festinger [28] as “the resultant of all the forces acting on the members to remain in the group.” According to Festinger, these forces depend on the attractiveness of each member to other members and to the activities that the group is participating in. Group cohesion then directly depends on the attractions between members, and has an important effect on group uniformity: a highly cohesive group is subject to pressures that result in uniformity among its members, that is, a consensus between the members of the group toward some goal [28, 25, 2]. Another important concept in the analysis of human group dynamics is *group performance*. Individuals work together in a way that their collective effort leads the group to have a good performance. Hence, the success of a task-oriented group is defined by its performance on a task [69]. Evidence suggests that there is a strong relationship between group cohesion and task-performance. In [85, 10], the authors studied the results of different experiments that examined the relation between cohesion and performance in groups, and concluded that there is a bidirectional effect between group cohesion and task-performance: “cohesion makes groups more successful, but groups that succeed also become more cohesive” [32]. Cohesive groups are successful largely because the contributions of each group member are coordinated with those of the other members. At the same time, the success in groups tends to enhance the
cohesiveness between the members of the group [32]. These studies suggest that in the bidirectional relation between performance and cohesion, the impact of performance on cohesion is stronger than the impact of cohesion on performance [85]. The third concept in group dynamics that we consider in our work is the structure of flow of information between members and its impact on performance and cohesion. In a group, members exchange information with other members of the group following a *communication network*. The study of the impact of the communication network topology (i.e., the pattern of interconnections) on the group has focused on the analysis of the effect of the degree of centralization and the number of group members on the performance and cohesion of the group. A group with a high degree of centrality is one in which an individual or few individuals serve as a hub for communications. Studies suggest that groups with communication networks that have a decentralized topology have a better performance than the centralized ones when the group has to solve complex tasks [67, 108, 60]. In these studies, tasks considered as complex had a higher number of task goals, pathways to complete the goals, and load of information than the ones considered as simple [107, 15]. Also, studies have shown that the number of people that interact with an individual affects both cohesion and performance. Too large of a number of individuals interacting with an individual results in a reduction of the performance and cohesion in the group [85].

Here, we propose a mathematical characterization of the dynamic behavior of human groups that is consistent with what has been observed in behavioral studies on cohesion, performance, and communication networks. The development of mathematical models of groups has gained a lot of attention from different disciplines. In social psychology, mathematical descriptions of the behavior of human groups have
been studied. For example, in [47], the authors proposed a gradient-based model of
the dynamics of individuals in a group. They introduce the concept of social force,
which is a vector quantity that drives the behavior of an individual, and it depends on
the interaction with other members of the group and external social influences such
as trends, public opinions, and social norms. The work in [98] studies the modeling
and automatic recognition of human behavior under specific situations using statisti-
cal tools. The main assumption in the model is that the patterns in behavior of the
individuals in a group can be described as the concatenation of multiple prototypical
behaviors. The work in [83] models the behavior of pedestrians that walk in groups.
Here, the motion of the pedestrians is directed by the action of a force that results
from the combination of different components: the pedestrian’s motivation to move
in a desired direction, the repulsion effect to avoid collisions with other pedestrians
and obstacles, and the social interaction with other members of the group. In engi-
neering, there is a well-developed theoretical framework for the modeling and analysis
of systems with multiple interactive agents [35, 90, 71, 130, 126]. It focuses on the
modeling of systems that are composed of agents that have a behavior that depends
on the interaction between them and follow a communication network that can be
constant or changing in time. In that analysis, the main goal is to find the conditions
that make the agents reach an agreement in different scenarios such as changes in the
communication topology. These models have been used in a variety of applications
that include control of cooperative autonomous vehicles [103], distributed learning of
pattern recognition models [31], and synchronization of microgrids in power systems
[38].
Most of these models focus on describing the cohesive behavior of the groups, modeling the attraction forces acting on the individuals, and determining the conditions on the communication network and parameters of the model so that the group achieves certain stability properties. To our knowledge, modeling the relationship between cohesion and performance as observed in human groups has been overlooked. Our aim is to model the dynamics of a group that, given a network of communication between its members, not only tends to be cohesive but at the same time attempts to solve a complex task. Our contribution is two-fold. First, we propose a computational model of the dynamics of human groups that have a communication network and work to solve a task, providing a mathematical formulation of Lewin’s formula [70] that is consistent with the observations in real human groups during task-solving processes. We do not try to propose models to recreate the specific experiments with humans that have been done to study these dynamics. Our aim is to design a mathematical model that characterizes dynamics that are qualitatively consistent with those observations in human groups. This work is conceptually related to the work in [47] [48, Chapter 11], where the behavior of a group is modeled as the result of social forces acting on its members. The main difference is that our work is focused on the development and analysis of a model that captures the relationship between cohesion and performance on the group given the patterns of information flow among its members. Second, we build on and extend the models of multiagent system dynamics in [35, 72, 71, 130]. The work in [35, 72] models the attraction and repulsion dynamics of groups of individuals that are evaluated according to a performance function or “resource environment,” and the work in [71, 130] extends these results to the case where there is a weighted communication topology that defines
the patterns in communication between the individuals. In our work, we formulate a
model of a group of individuals that interact according to a general communication
network topology, where the attraction weights and the commitment to optimize the
performance function change over time (Section 2.1). We show that a key aspect for a
cohesive group to be successful (i.e, having a good performance) is that the attraction
patterns between the members of the group and the commitment to solve the task are
not static but dynamic and develop over time, playing an important role in shaping
the behavior of the group. We provide a mathematical analysis of group cohesiveness
that is an extension of the analysis developed in [72, Theorem 1], where we study the
conditions in our model that allow the group to be cohesive (Section 2.2). Moreover,
we show through simulations (Section 2.3) the impact of the parameters of the model
and the communication networks on the behavior of the group in terms of group
performance and group cohesion when the group is required to solve a complex task.

2.1 Mathematical Model

We propose a mathematical formalization of Lewin’s formula (2.1) that charac-
terizes the dynamics of the behavior of each group member. We model the group as
a network of interconnected dynamical systems in which a group member’s variables
are driven by his/her personal characteristics, interaction with the other members,
and performance in solving a given task. One of the main features of our model is
that the interaction patterns of an individual and his/her commitment to solve a task
are not static, but develop over time.
2.1.1 Model Formulation

The network that describes how the group members are interconnected is represented by a directed graph \(G = (\mathcal{V}, \mathcal{E})\), where \(\mathcal{V} = \{1, 2, \ldots, N\}\) denotes the set of labels of the \(N\) individuals of the group, and \(\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}\) is the set of edges of the graph. Edge \((i, j)\) in \(\mathcal{E}\) indicates that individual \(i\) gets information from individual \(j\). All group members that supply information to individual \(i\) are \(\mathcal{N}_i = \{j : (i, j) \in \mathcal{E}\}\). Hence, \(G\) defines the communication patterns between the members of the group. The arrows indicate the direction of the flow of information between individuals. Let \(x^i \in \mathbb{R}^p\) be a vector of measurable variables that indicate the “state” of the individual in the task-solving process. In models of human decision processes, these variables typically include preferences of the individual, and attributes of the alternatives that are involved in the task [105, 119]. We refer to \(x^i(t)\) as the position of individual \(i\) at given time \(t\), and motion refers to a change in position with respect to time. “Position” here is not necessarily physical location, but can represent “viewpoint” or “opinion.”

Let \(f : \mathbb{R}^p \rightarrow \mathbb{R}\) be a continuously differentiable function that evaluates the position of an individual according to the task assigned to the group and quantifies its performance. We assume that lower values of \(f\) correspond to better positions. The behavior of an individual in the group is assumed to be directly influenced by his or her characteristics, performance during the task-solving process, and interaction with other members of the group. We refer to these influences as forces that determine the individual’s behavior. “Force” is not necessarily the one from physics with units of Newtons, but simply represents “influence” of some sort. To provide a mathematical formalization of Lewin’s formula for behavior, we characterize the interaction function
\[ I \] in Equation (2.1) from the perspective of Newton’s second law: forces acting on
the group and driving its behavior. We assume that the behavior of individual \( i \) can
be described by

\[
\begin{align*}
\dot{x}^i(t) &= v^i(t) \\
\dot{v}^i(t) &= \frac{1}{M^i} u^i(t)
\end{align*}
\] (2.2)

where \( t \in \mathbb{R}_{\geq 0} \) is the time variable, \( \dot{x}^i(t) \) denotes the derivative of \( x^i \) with respect to
\( t \), and \( v^i(t) \) is its velocity at time \( t \). The acceleration of the individual is \( \dot{v}^i(t) \), and
\( u^i(t) \) is the force vector that drives the motion of the individual. The variable \( M^i \)
then corresponds to the mass of individual \( i \) if it has mass, or more generally, it is
a parameter that affects the influence of \( u^i \) on the dynamics of group member \( i \). It
represents a parameter that affects the tendency of viewpoint or opinion to change.

The force \( u^i \) is defined by four different components

\[
\begin{align*}
\dot{u}^i(t) &= p^i(t) + a^i(t) + r^i(t) - \zeta_i v^i(t)
\end{align*}
\] (2.3)

where \( p^i(t) \) is the force component that provides the direction that optimizes the
performance of individual \( i \) at time \( t \). Since lower values of \( f(x^i(t)) \) indicate a better
performance of \( i \), we define \( p^i(t) \) such that it points toward positions where \( f \) is
reduced. Hence, we choose \( p(t) \) to move along the (scaled) negative gradient of \( f \) evaluated at \( x^i(t) \)

\[
p^i(t) = -\eta^i(t) \nabla f(x^i(t))
\] (2.4)

where \( \nabla f(x^i(t)) \) is the gradient of \( f \) evaluated at \( x^i(t) \), and \( \eta^i(t) \) is the magnitude of
the influence of the performance on the individual’s motion. Later, we will define
the dynamics of \( \eta^i(t) \) and show that \( \eta^i(t) \geq 0 \) for all \( t \geq 0 \) if \( \eta^i(0) \geq 0 \), \( i = 1, \ldots, N \).
The second component of $u^i$ in Equation (2.3) is the attraction of $i$ to every $j \in \mathcal{N}_i$ in position and velocity, and it is defined as

$$a^i(t) = -\frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \left[w^{ij}(t)(x^i(t) - x^j(t)) + bw^{ij}(t)(v^i(t) - v^j(t))\right] \quad (2.5)$$

where $|\mathcal{N}_i|$ is the number of individuals that communicate with $i$, and $w^{ij}(t) \geq 0$, $t \geq 0$, is the attraction strength of individual $i$ to individual $j \in \mathcal{N}_i$, with $b \geq 0$. We have that $-w^{ij}(t)(x^i(t) - x^j(t))$ is the component of $a^i(t)$ associated with the attraction in position of $i$ to some $j \in \mathcal{N}_i$. It indicates that a larger separation between individuals implies a larger attraction of $i$ in the direction of $j$. A similar statement can be made for the attraction in velocity. Note that the attraction strengths $w^{ij}$ can be represented as weights associated to the edges of the graph $\mathcal{G}$.

The next force component in Equation (2.3) is the short-range repulsion between the group members. We allow the model to include behaviors where a group member tries to avoid having the same position as the other members he or she communicates with to allow some degree of variability during the process of solving the task. Let $r^i(t)$ be the repulsion of $i$ from the individuals in $\mathcal{N}_i$ on short distances given by

$$r^i(t) = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \kappa^{ij} \exp\left(-\frac{1}{2\beta^{ij}_2}\|x^i(t) - x^j(t)\|^2\right)(x^i(t) - x^j(t)) \quad (2.6)$$

where $\kappa^{ij} \geq 0$ and $\beta^{ij} \geq 0$. Term $\kappa^{ij} \exp\left(-\frac{1}{2\beta^{ij}_2}\|x^i(t) - x^j(t)\|^2\right)(x^i(t) - x^j(t))$ corresponds to the component of the force that makes individual $i$ be repelled by individual $j$, where $\kappa^{ij}$ is the repulsion strength and $\exp\left(-\frac{1}{2\beta^{ij}_2}\|x^i(t) - x^j(t)\|^2\right)$ defines the radius of action of the repulsion. We also have the force component $-\zeta^i v^i(t)$, $\zeta^i > 0$, which points towards the opposite direction of the velocity vector $v^i$. This term produces a reduction in the velocity of the individual.
A key aspect in human group processes is that the attraction patterns among the group members and their commitment to solving a task are not static but dynamic and develop over time [32]. In our model, we characterize the dynamics of the attraction and performance strengths defined in Equations (2.4) and (4.3) based on the observations that (i) individuals tend to be more attracted to those group members that are performing better during the task-solving process [32], and (ii) group dynamics tend to reach a consensus among members to move toward some goal [28]. We first describe the dynamics of the attraction strengths using a growth model that makes the individuals be more attracted to those that have better relative performance in solving the given task as follows

\[ \dot{w}_{ij}(t) = \alpha_w (w_{ij}(t) - C_{w}^l)(C_{w}^u - w_{ij}(t))\Delta f_{ij}(t) \]  

(2.7)

where \( \alpha_w > 0 \) is the growth rate, and \( \Delta f_{ij}(t) = (f(x^i(t)) - f(x^j(t))) / \sum_{k \in N_i} f(x^k(t)) \) is the scaled difference between the performance function of individuals \( i \) and \( j \). The strength of attraction of \( i \) toward \( j \) increases when \( f(x^i) > f(x^j) \), i.e., when individual \( j \) is performing better than \( i \), and decreases when \( f(x^i) < f(x^j) \). Recall that lower values of \( f \) correspond to better performance of the individual. Note that \( C_{w}^l \) and \( C_{w}^u \) are equilibrium points of the system (2.7) where \( 0 \geq C_{w}^l \geq C_{w}^u \), and \( \dot{w}_{ij}(t) > 0 \) when \( \Delta f_{ij}(t) > 0 \) and \( \dot{w}_{ij}(t) < 0 \) when \( \Delta f_{ij}(t) < 0 \), for all \( w_{ij}(t) \in (C_{w}^l, C_{w}^u) \). This means that the trajectories of \( w_{ij} \) are bounded below by \( C_{w}^l \) and above by \( C_{w}^u \) for all \( t \geq 0 \) if \( w_{ij}(0) \in [C_{w}^l, C_{w}^u] \), and their rate of change and direction are affected by \( \alpha_w \Delta f_{ij}(t) \).

The strength of the force component associated with performance optimization has dynamics given by the growth model

\[ \dot{\eta}^i(t) = \alpha_\eta \eta^i(t)(C_\eta - \eta^i(t)) \left( B_\eta - \frac{1}{|N_i|} \sum_{j \in N_i} \|x^i(t) - x^j(t)\|^2 \right) \]  

(2.8)
where $\alpha_\eta > 0$ is the growth rate, and $B_\eta > 0$ is a threshold parameter that determines the sign of the growth rate. The strength of the performance optimization component of the motion’s force of individual $i$ decreases when $|\mathcal{N}_i|^{-1} \sum_{j \in \mathcal{N}_i} \|x^j - x^i\|^2$ is larger than $B_\eta$, that is, when the group is not cohesive. This dynamical model allows force $u^i$ in Equation (2.3) to be more influenced by the attraction between the members of the group than the commitment to solve the task when the group is not cohesive enough. The trajectories of $\eta^i$ are bounded below by zero and above by $C_\eta$ when $\eta^i(0) \geq 0$.

In Section 2.3 we show that Equations (2.7) and (2.8) play a fundamental role in modeling dynamics that are consistent with the observations in human groups during a task-solving process in terms of cohesion and performance.

### 2.1.2 Optimization Point of View for the Model

We can analyze the role of each one of the components of the driving force $u^i$ from an optimization point of view. The dynamics characterized by the model in Equations (2.2) and (2.3) can be viewed as the result of minimizing a set of cost functions that are involved in the task-solving process by the group. Let $L^a_j(x^i(t), v^i(t)) \in \mathbb{R}$ be a function of $x^i(t)$ and $v^i(t)$ that measures how far individual $i$ is to individual $j \in \mathcal{N}_i$ in position and velocity defined as

$$L^a_j(x^i(t), v^i(t)) = \frac{1}{2} w^{ij}(t) \|x^i(t) - x^j(t)\|^2 + \frac{1}{2} bw^{ij}(t) \|v^i(t) - v^j(t)\|^2$$  \hspace{1cm} (2.9)

This function can be seen as a weighted squared error between the velocity and position of $j$ with respect to a given $i$. Let $L^r_j(x^i(t)) \in \mathbb{R}$ be a function of $x^i(t)$ defined as

$$L^r_j(x^i(t)) = \frac{1}{2} \hat{\kappa}^{ij} \exp \left( -\frac{1}{2 \beta_0^2} \|x^i(t) - x^j(t)\|^2 \right)$$  \hspace{1cm} (2.10)
This is a Gaussian function of $x^i(t)$ centered at $x^j(t)$, with amplitude and width controlled by $\hat{\kappa}^{ij} \geq 0$ and $\beta_{ij}$. It has its highest value at $x^i(t) = x^j(t)$, and decreases as $x^i(t)$ moves away from $x^j(t)$. Let

$$q_v(v^i(t)) = \frac{1}{2}||v^i(t)||^2 \tag{2.11}$$

be a function of $v^i(t)$ with a unique minimum at $v^i(t) = 0$. Using Equations (2.9), (2.10), and (2.11) we define the cost function

$$L(x^i(t), v^i(t)) = \eta^i f(x^i(t)) + \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \nabla_x L_a^j(x^i(t), v^i(t)) + \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \nabla_x L_r^j(x^i(t)) + \zeta_i q_v(v^i(t)) \tag{2.12}$$

Let $\nabla_x$ and $\nabla_v$ denote the gradient operator with respect to the position and velocity vectors. The gradient of $L$ with respect to $x^i(t)$ and $v^i(t)$ is given by

$$\nabla_x L(x^i(t)) = \eta^i(t) \nabla_x f(x(t)) + \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \nabla_x L_a^j(x^i(t)) + \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \nabla_x L_r^j(x^i(t)) \tag{2.13}$$

$$\nabla_v L(v^i(t)) = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \nabla_v L_a^j(v^i(t)) + \zeta_i \nabla_v q_v(v^i(t)) \tag{2.14}$$

Using Equations (2.13) and (2.14) we can rewrite force $u^i(t)$ in Equation (2.3) as

$$u^i(t) = -\nabla_x L(x^i(t)) - \nabla_v L(v^i(t)) \tag{2.15}$$

This expression shows us that the force $u^i(t)$ points to the opposite direction of the gradient of the cost function $L$, that is, the motion of individual $i$ tends to minimize $L$ with respect to the position and velocity. According to this, we have that the attraction term in Equation (4.3) can be rewritten as

$$a^i(t) = -\frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \left[\nabla_x L_a^j(x^i(t)) + \nabla_v L_a^j(v^i(t))\right]$$
which is the vector that points to the direction that minimizes the error in position and velocity between $i$ and $j$. Also, the repulsion term in Equation (2.6) can be rewritten as the vector that points to the direction that minimizes the Gaussian function $L_r^j$, for every $j \in \mathcal{N}_i$ as follows

$$r^i(t) = -\frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \nabla_x L_r^j(x^i(t))$$

The velocity damping term in $u^i(t)$ corresponds to the vector $-\zeta_i \nabla_v q_v(v^i(t))$, which points to the direction in which the velocity is minimized.

Equation (2.15) shows that the dynamics of each member of the group can be seen as the result of a continual optimization process with respect to his or her position and velocity, where the objective function changes over time and is interconnected with those of the other members of the group. Note that depending on the selection of the parameters of the cost function $L$, the descent direction will be biased toward the minimization of those components with higher magnitude. In Section 2.3 we will show that the cost function changes over time such that there is an interaction between the performance optimization component $f$ and attraction component $L_a^j$ that allows the minimization process to achieve a solution that involves a tradeoff between them.

### 2.2 Conditions for Cohesive Behavior of the Group

In a cohesive group, each individual behaves in a way that its position and velocity during the task-solving process ultimately are close to those of the other members of the group. Here, we analyze the conditions that allow the group to exhibit cohesive behaviors while solving a task and under an assumed communication network topology. Without loss of generality, we assume that $M^i = 1$, $\zeta^i = \zeta$, $\zeta > 0$, for $i = 1, \ldots, N$. 

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Also, to facilitate the notation in this analysis, we do $\tilde{w}^{ij}(t) = w^{ij}(t)/|N_i|$ for every $i = 1, \ldots, N$ and $j \in N_i$. We define the error in position and velocity between individuals $i$ and $j$

$$e^{ij}_x = x^i - x^j$$
$$e^{ij}_v = v^i - v^j$$

In a cohesive group, the error in position and velocity reaches a point where every two members of the group are considered close to each other. To analyze the cohesion of the group we study the dynamics of the error in position and velocity of the individuals in the group. The error dynamics are given by

$$\dot{e}^{ij}_x = \dot{x}^i - \dot{x}^j = e^{ij}_v$$
$$\dot{e}^{ij}_v = \dot{v}^i - \dot{v}^j$$

$$= - \sum_{k \in N_i} \tilde{u}^i_k(t) e^{ik}_x - \sum_{k \in N_i} (b \tilde{u}^i_k(t) + \zeta) e^{ik}_v$$
$$+ g^{ij} + \delta^{ij} + \phi^{ij}$$

where

$$\phi^{ij} = \sum_{l \in N_j} \tilde{u}^{jl}(t)(e^{jl}_x + b e^{jl}_v)$$
(2.16)

$$g^{ij} = - \eta^i(t) \nabla f(x^i(t)) + \eta^j(t) \nabla f(x^j(t))$$
(2.17)

$$\delta^{ij} = \frac{1}{|N_i|} \sum_{k \in N_i} \kappa^{ik}$$

$$\exp \left( - \frac{1}{2 \beta_{ik}^2} \|x^i(t) - x^k(t)\|^2 \right) (x^i(t) - x^k(t))$$

$$- \frac{1}{|N_j|} \sum_{l \in N_j} \kappa^{jl}$$

$$\exp \left( - \frac{1}{2 \beta_{jl}^2} \|x^j(t) - x^l(t)\|^2 \right) (x^j(t) - x^l(t))$$
(2.18)
Let \( E^{ij} = [e^{ij}_x, e^{ij}_v]^{\top} \). The system can be written in matrix form as

\[
\dot{E}^{ij} = \sum_{j \in \mathcal{N}_i} \left[ \begin{array}{ccc}
A_{ij}(t) & I_p \\
0_p & -\bar{w}^{ij}(t)I_p - (b\bar{w}^{ij}(t) + \zeta) I_p
\end{array} \right] E^{ij} + \left[ \begin{array}{c}
0_p \\
I_p
\end{array} \right] \left( g^{ij} + \delta^{ij} + \phi^{ij} \right)
\] (2.19)

where \( I_p \) and \( 0_p \) are the \( p \times p \) identity and zero matrices. Note that \( A_{ij}(t) \) in Equation (2.19) is a time-varying matrix that has eigenvalues given by the roots of the polynomial \((\lambda^2 + (b\bar{w}^{ij}(t) + \zeta)\lambda + \bar{w}^{ij}(t))^p\). Since we assume that \( \bar{w}^{ij}(t) > 0 \) for all \( t \geq 0 \), then the eigenvalues of \( A_{ij}(t) \) have negative real part for all \( t \geq 0 \). The following theorem specifies conditions that are sufficient for the group to reach a state in which the group is considered cohesive.

**Theorem 1.** Consider the group whose dynamics are characterized by the model in Equations (2.2)-(2.8). Assume that \( C^l_w < w^{ij}(0) < C^u_w \), for all \( i = 1, \ldots, N, j \in \mathcal{N}_i \). Also, assume that the difference in performance between two members of the group in Equation (2.7) and the component of the force \( u_i \) in (2.3) that seeks to optimize the performance of the individuals in the group are bounded during the task-solving process, i.e., there exist constants \( \psi \geq 0 \) and \( \psi_g \geq 0 \) such that

\[
|\Delta f^{ij}(t)| \leq \psi, \quad \forall t \geq 0
\]

(2.20)

\[
\|\nabla f(x)\| \leq \psi_g
\]

(2.21)

for all \( x \in \mathbb{R}^p, i = 1, \ldots, N, \) and \( j \in \mathcal{N}_i \). Let the parameters of the model be such that, for all \( t \geq 0 \),

\[
\dot{c}^{ij}_1 > 0
\]

(2.22)
and

\[
\frac{c^i_j |N_j| + \sum_{l \in \Gamma_i} c^l_j}{2(1 - \theta_{ij})} < c^i_1
\]

for some \( \theta_{ij} \in (0, 1) \), \( i = 1, \ldots, N, j \in N_i \), where \( \Gamma_i \) is the set of members of the group that has \( i \) as a neighbor in the communication network, and

\[
c^i_1 = \max_{\tilde{w}^{ij} \in \left[ \frac{C_l}{|N_i|}, \frac{C_u}{|N_i|} \right]} 2\tilde{w}^{ij} - \left[ b + \frac{\zeta(1 + 2\tilde{w}^{ij}) + b(\tilde{w}^{ij})^2}{(\zeta + b\tilde{w}^{ij})^2} \right]
\]

\[
c^i_3 = \max_{\tilde{w}^{ij} \in \left[ \frac{C_l}{|N_i|}, \frac{C_u}{|N_i|} \right]} \left[ \frac{(\zeta + b\tilde{w}^{ij})^2 + (\tilde{w}^{ij} + 1)^2}{\zeta + b\tilde{w}^{ij}} \right] + \sqrt{2(\tilde{w}^{ij})^2 + (\zeta + b\tilde{w}^{ij})^2 + \frac{(\tilde{w}^{ij})^2 - 1)^2}{(\zeta + b\tilde{w}^{ij})^2}} \right] \sqrt{1 + b}
\]

Then, the trajectories of the error system in Equation (2.19) for every \( i = 1, \ldots, N \) and \( j \in N_i \) are uniformly ultimately bounded.

Furthermore, if the topology of the communication network in the group is strongly connected, i.e., there is a directed path between every two nodes, then the trajectories of the error between every pair of individuals in the group are uniformly ultimately bounded.

**Proof.** The Lyapunov function chosen to do the cohesiveness analysis of the system in Equation (2.19) is

\[
V(t, E) = \sum_{i=1}^{N} \sum_{j \in N_i} V^{ij}(t, E)
\]

where \( E \) is the vector containing all the errors \( E^{ij} \). The functions \( V^{ij}(E) \) are given by

\[
V^{ij}(t, E) = E^{ij \top} P_{ij}(t) E^{ij}
\]

where \( P_{ij}(t) \) is a \( 2p \times 2p \) time-varying matrix such that \( P_{ij}(t) = P_{ij}(t) \top > 0 \) for all \( t \geq 0 \). The derivative of \( V^{ij}(t, E) \) along the trajectories of the system in (2.19) is
given by

\[
\dot{V}^{ij}(t, E) = E^{ij\top} \dot{P}_{ij}(t) E^{ij}
+ E^{ij\top} (A_{ij}^\top P_{ij}(t) + P_{ij}(t) A_{ij}) E^{ij}
+ \left[ B(g^{ij} + \delta^{ij} + \phi^{ij}) \right] \top P_{ij}(t) E^{ij}
+ E^{ij\top} P_{ij}(t) B(g^{ij} + \delta^{ij} + \phi^{ij})
\]

\[(2.27)\]

From [63, Theorem 4.6] and [131], given a matrix \(Q_{ij}(t) > 0\) and a matrix \(A_{ij}(t)\) that has all its eigenvalues with negative real part for all \(t \geq 0\), we can find \(P_{ij}(t)\) such that

\[-Q_{ij}(t) = P_{ij}(t) A_{ij}(t) + A_{ij}(t) \top P_{ij}(t),\]

for all \(t \geq 0\). Using this result, we can rewrite (4.12) as

\[
\dot{V}^{ij}(t, E) = E^{ij\top} \left[ \dot{P}_{ij}(t) - Q_{ij}(t) \right] E^{ij}
+ \left[ B(g^{ij} + \delta^{ij} + \phi^{ij}) \right] \top P_{ij}(t) E^{ij}
+ E^{ij\top} P_{ij}(t) B(g^{ij} + \delta^{ij} + \phi^{ij})
\]

\[(2.28)\]

If we choose \(Q_{ij}(t) = 2\tilde{w}^{ij}(t)I_{2p}\), the matrix \(P_{ij}(t)\) that satisfies the above relation is

\[
P_{ij}(t) = \left[ \begin{array}{cc}
\frac{(b\tilde{w}^{ij}(t)+\zeta)^2 + \tilde{w}^{ij}(t)(1+\tilde{w}^{ij}(t))}{b\tilde{w}^{ij}(t)+\zeta} & I_p \\
I_p & \frac{(1+\tilde{w}^{ij}(t))}{b\tilde{w}^{ij}(t)+\zeta} I_p \\
\end{array} \right]
\]

Its maximum eigenvalue satisfies \(\lambda_{\text{max}}[P_{ij}(t)] \leq c^{ij}_3 |N_i|/(2C_w^u \sqrt{1+b^2})\) for all \(t \geq 0\), where \(c^{ij}_3\) is defined in Equation (2.25). Matrix \(P_{ij}(t)\) is a time-varying matrix whose entries depend on \(\tilde{w}^{ij}(t)\), where its derivative with respect to \(t\) is

\[
\dot{P}_{ij} = \tilde{w}^{ij} \frac{\partial P_{ij}}{\partial \tilde{w}^{ij}}
= \tilde{w}^{ij} \left[ \begin{array}{cc}
b + \frac{\zeta(1+2\tilde{w}^{ij}(t)+b\tilde{w}^{ij}(t))^2}{(\zeta + b\tilde{w}^{ij}(t))^2} & I_p \\
I_p & 0_p \\
\end{array} \right]
\]

where \(\tilde{w}^{ij}\) is defined in Equation (2.7). The maximum eigenvalue of the matrix \(\dot{P}_{ij}(t) - Q_{ij}(t)\) satisfies \(\lambda_{\text{max}}[\dot{P}_{ij}(t) - Q_{ij}(t)] \leq -c^{ij}_1\), for all \(t \geq 0\), where \(c^{ij}_1\) is defined
in Equation (2.24). This information will be useful to obtain an upper bound for 
\( \dot{V}^{ij}(t, E) \) in Equation (4.13). Now we study the remaining elements in Equation (4.13). It can be shown that \( \exp(-1/2)\beta \) is the unique maximum value of the function 
\( h(y) = \exp(-1/2\beta^2||y||^2) y \), for all \( y \in \mathbb{R}^p \). Using this in Equation (2.18) we obtain

\[
||\delta^{ij}|| \leq \exp(-1/2) \left( \frac{1}{|\mathcal{N}_i|} \sum_{k \in \mathcal{N}_i} \kappa^{ij}_{ik} \beta \right) + \frac{1}{|\mathcal{N}_j|} \sum_{l \in \mathcal{N}_j} \kappa^{jl}_{jl} \beta
\]

(2.29)

\[
= \bar{\delta}^{ij}
\]

From Equation (2.16) we have

\[
||\phi^{ij}|| \leq \sum_{l \in \mathcal{N}_j} \tilde{w}^{jl}(t) \sqrt{1 + b^2} ||E^{jl}||, \quad j \in \mathcal{N}_i
\]

\[
\leq C_w \sqrt{1 + b^2} \sum_{l \in \mathcal{N}_j} ||E^{jl}||
\]

(2.30)

where we used the fact that \( \tilde{w}^{jl} < C_w/|\mathcal{N}_j| \), and \( |\mathcal{N}_j| \geq 1 \). From the assumption in (2.21), and since we know from (2.8) that \( \eta^i \geq C_\eta^i \), we obtain

\[
||g^{ij}|| \leq 2C_\eta^i \psi_g
\]

(2.31)

Assume that (2.22) is satisfied. Then, putting (2.29), (2.30), and (2.31) together in (4.13), and since \( ||P_iB|| \leq \lambda_{\max}[P_i] \leq c^{ij}_{ij} |\mathcal{N}_i|/(2C_w \sqrt{1 + b^2}) \) and

\[
E^{ij}^\top \left[ \dot{P}_{ij}(t) - Q_{ij}(t) \right] E^{ij} \leq \lambda_{\max}[\dot{P}_{ij}(t) - Q_{ij}(t)] ||E^{ij}||^2
\]

\[
\leq -c^{ij}_1 ||E^{ij}||^2
\]

we obtain

\[
\dot{V}^{ij}(E) \leq -c^{ij}_1 ||E^{ij}||^2 + c^{ij}_2 ||E^{ij}||^2 + c^{ij}_3 ||E^{ij}|| \sum_{l \in \mathcal{N}_j} ||E^{jl}||
\]

(2.32)
where $c^{ij}_2 = [c^{ij}_3|N_i|/(C_w\sqrt{1+b^2})](\delta^{ij} + 2C_u\psi_u)$. Note that, for any $0 < \theta^{ij} < 1$, Equation (4.14) can be written as

$$-c^{ij}_1||E^{ij}||^2 + c^{ij}_2||E^{ij}|| \leq -(1 - \theta^{ij})c^{ij}_1||E^{ij}||^2$$

$$\forall ||E^{ij}|| > r^{ij}$$

$$= -\sigma^{ij}||E^{ij}||^2$$  \hspace{1cm} (2.33)

where $r^{ij} = c^{ij}_2/(c^{ij}_1\theta^{ij})$, and $\sigma^{ij} = (1 - \theta^{ij})c^{ij}_1 > 0$. This indicates that the first two terms of the right-hand side of Equation (4.14) are negative when $||E^{ij}|| > r^{ij}$.

To find the conditions that make $\dot{V} < 0$ in Equation (2.26), we define the sets $\Pi^i_O$ and $\Pi^i_I$

$$\Pi^i_O = \{ j : ||E^{ij}|| \geq r^{ij}, j \in N_i \}$$

$$\Pi^i_I = \{ j : ||E^{ij}|| < r^{ij}, j \in N_i \}$$

The sets $\Pi^i_O$ and $\Pi^i_I$ contain the indices of the neighbors of $i$ whose error $E^{ij}$ at a given time satisfy $||E^{ij}|| \geq r^{ij}$ and $||E^{ij}|| < r^{ij}$, respectively. The size of the sets is $N^i_O = |\Pi^i_O|$ and $N^i_I = |\Pi^i_I|$. We first analyze the case when $1 \leq N^i_O \leq |N_i|$, and later we discuss the case when $N^i_O = 0$. In the first case, since $N_i = \Pi^i_O \cup \Pi^i_I$, we can set bounds on the components in $\dot{V}_{ij}$ that depend on the $||E^{ij}||$ with $j \in \Pi^i_I$. Hence, from
Equations (2.26), (4.14) and (2.33), we obtain

$$
\dot{V}(E) = \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{i}} \dot{V}^{ij}(E)
$$

$$
\leq \left[ \sum_{i=1}^{N} \sum_{j \in \Pi_{O}^{i}} \left( -\sigma^{ij} ||E^{ij}||^{2} + c_{3}^{ij} ||E^{ij}|| \sum_{l \in \Pi_{O}^{j}} ||E^{jl}|| \right) \right]
$$

$$
+ \left[ \sum_{i=1}^{N} \left( \sum_{j \in \Pi_{O}^{i}} c_{3}^{ij} K_{1}^{ij} ||E^{ij}|| \right) \right]
$$

$$
+ \left[ \sum_{i=1}^{N} \left( K_{3}^{i} \sum_{k \in \mathcal{N}_{i}} \sum_{l \in \Pi_{O}^{k}} c_{3}^{ik} ||E^{kl}|| \right) \right]
$$

$$
+ K_{2}
$$

$$
(2.34)
$$

Here, we use the fact that for the $||E^{ij}|| < r^{ij}$ with $j \in \Pi_{i}^{j}$ and $i = 1, \ldots, N$, there exist positive constants $K_{1}^{ij}, K_{2}, K_{3}^{i}$ that satisfy [72, Theorem 1]

$$
K_{1}^{ij} \geq \sum_{l \in \Pi_{j}^{i}} ||E^{jl}||, \quad j \in \mathcal{N}_{i}
$$

$$
K_{2} \geq \sum_{i=1}^{N} \sum_{j \in \Pi_{i}^{j}} \left[ -c_{1}^{ij} ||E^{ij}||^{2} + c_{2}^{ij} ||E^{ij}|| \right]
$$

$$
+ c_{3}^{ij} ||E^{ij}|| \sum_{l \in \Pi_{j}^{i}} ||E^{jl}||
$$

$$
K_{3}^{i} \geq \sum_{j \in \Pi_{i}^{j}} ||E^{ij}||
$$

The first term in brackets on the right-hand side of (4.15) is a quadratic form of the variables $||E^{ij}||, \quad i = 1, \ldots, N, \quad j \in \Pi_{O}^{i}$. Let $y$ be a column vector of dimension $\sum_{i=1}^{N} |\Pi_{O}^{i}|$ with entries given by all the $||E^{ij}||$ for $i = 1, \ldots, N, \quad j \in \Pi_{O}^{i}$. The quadratic form can be written as $y^{\top} Sy$, where $S$ is a squared matrix whose diagonal elements
are given by \(-\sigma^{ij}\), and the off-diagonal elements depend on the variables \(c^{ij}_3\). We know that \(\mathbf{y}^T \mathbf{S} \mathbf{y} < 0\) if and only if the eigenvalues of its symmetric part \((\mathbf{S} + \mathbf{S}^T)/2\) are negative. According to the Gershgorin circle theorem [43, Chapter 7], all its eigenvalues \(\lambda\) satisfy

\[
|\lambda + \sigma^{ij}| \leq \frac{1}{2} \left( c^{ij}_3 |\Pi^i_O| + \sum_{l \in \Gamma_i} c^{il}_3 \right) \tag{2.35}
\]

where \(\Gamma_i = \{j : j \in \{1, \ldots, N\}, i \in \mathcal{N}_j\}\), that is, \(\Gamma_i\) is the set of nodes that has node \(i\) as a neighbor. The right-hand side of Equation (2.35) corresponds to the sum of the elements of the row of \((\mathbf{S} + \mathbf{S}^T)/2\) associated with the variable \(||E^{ij}||\), where \(c^{ij}_3 |\Pi^i_O|\) and \(\sum_{l \in \Gamma_i} c^{il}_3\) are the sum of the elements of the \(i\)-th row of \(\mathbf{S}\) and \(\mathbf{S}^T\), respectively.

Then, a sufficient condition for the quadratic form to be negative definite is

\[
-\sigma^{ij} + \frac{1}{2} \left( c^{ij}_3 |\Pi^i_O| + \sum_{l \in \Gamma_i} c^{il}_3 \right) < 0 \tag{2.36}
\]

Note that this is satisfied if (2.22) and (2.23) are satisfied. Assume that these conditions are satisfied, and the first term in brackets on the right-hand side of (4.15) is negative. Then, \(\lambda_{\text{max}}[(\mathbf{S} + \mathbf{S}^T)/2] < 0\), and

\[
\dot{V}(E) \leq \left[ \lambda_{\text{max}}[(\mathbf{S} + \mathbf{S}^T)/2] \sum_{i=1}^N \sum_{j \in \Pi^i_O} ||E^{ij}||^2 \right] \tag{2.37}
\]

\[
+ \left[ \sum_{i=1}^N \sum_{j \in \Pi^i_O} c^{ij}_3 K_1^{ij} ||E^{ij}|| \right]
\]

\[
+ \left[ \sum_{i=1}^N K_3^i \sum_{k \in \mathcal{N}_i} \sum_{l \in \Pi^k_O} c^{ik}_3 ||E^{kl}|| \right] + K_2
\]

Then, from (2.37), when \(||E^{ij}||, i = 1, \ldots, N, j \in \Pi^i_O\), is sufficiently large, the term

\[
\lambda_{\text{max}}[(\mathbf{S} + \mathbf{S}^T)/2] \sum_{i=1}^N \sum_{j \in \Pi^i_O} ||E^{ij}||^2
\]
determines the sign of $\dot{V}(E)$ and therefore, $\dot{V}(E) < 0$. In the case when $N_O = 0$, the trajectories satisfy $||E^{ij}|| < r^{ij}$, for all $i = 1, \ldots, N$, and $j \in N_i$. Therefore, we can take $\max_{i,j} r^{ij}$ as the ultimate bound.

So far we have shown the conditions to achieve uniform ultimate boundedness for each $E^{ij}$. This result shows that if the difference in position and velocity of between an individual and his or her neighbors is sufficiently large, then the trajectories of position and velocity will behave such that this difference decreases until it reaches a boundary. To complete the proof, we have to show that the error in position and velocity between every two individuals in the group is bounded when $||E^{ij}||$, for all $i,j \in \{1, \ldots, N\}$, is bounded. Assume that the uniform ultimate boundedness has been achieved in the trajectories of the error system (2.19), and $||E^{ij}|| < r^i$ for all $i \in \{1, \ldots, N\}$ and $j \in N_i$. Let $i \rightarrow j$ denote that individual $i$ is a neighbor of $j$ in the communication network, that is, $i \in N_j$. Let $l_1 \rightarrow l_2 \rightarrow \cdots \rightarrow l_M$ denote the directed path starting from $l_1$ and ending in $l_M$, where $l_i \in \{1, \ldots, N\}$. We know that for $l_1 \rightarrow l_2$ and $l_2 \rightarrow l_3$, it is true that $||E^{l_2l_1}|| < r^{l_2l_1}$ and $||E^{l_3l_2}|| < r^{l_3l_2}$. Since the error term $||E^{ij}||$ is a weighted Euclidean distance in position and velocity between individuals $i$ and $j$, we have that $||E^{l_3l_1}|| \leq ||E^{l_2l_1}|| + ||E^{l_3l_2}|| < r^{l_3l_1} + r^{l_3l_2}$. Applying sequentially this relation to the directed path $l_1 \rightarrow l_2 \rightarrow \cdots \rightarrow l_M$, we obtain

$$||E^{l_Ml_1}|| < \sum_{j=1}^{M-1} r^{l_j+1l_j} = r^{l_Ml_1}$$

(2.38)

Therefore, from the assumption that there is a directed path between every two nodes, if the trajectories of the error system (2.19) achieve uniform ultimate boundedness, then there exists a bound for $||E^{ij}||$ for every $i,j \in \{1, \ldots, N\}$. \qed
Remark 1. This result shows that, when the assumptions on the parameters of the model and the communication topology are satisfied, if the $||E_{ij}||$ are large enough, then the trajectories of the error system will achieve uniform ultimate boundedness. This means that if the individuals in the group are sufficiently separated in position and velocity from the other members, they will move closer to them in a way that the uniform ultimate bound for position and velocity is satisfied.

Remark 2. This analysis provides information about conditions that are sufficient to have cohesiveness in the group, but it does not state any condition on achieving local optimality of the performance function.

Remark 3. The assumptions on the boundedness of the difference in performance and the gradient of the performance function in Equations (2.20) and (2.21) are reasonable assumptions that can be satisfied by realistic performance functions such as a mixture of Gaussian functions and a plane, and hence they do not represent a restrictive condition for our analysis.

Remark 4. The assumption that the communication network is represented by a strongly connected graph indicates that every individual in the group influences any other group member either by a direct connection, or indirectly by a “chain effect.” The latter means that, since there is a direct path in communication between every two members of the group, every individual can indirectly affect the dynamics of every other group member as Equation (2.38) indicates. If this assumption is not satisfied, from the proof of Theorem 1, cohesiveness can only be guaranteed between an individual and his or her direct neighbors.

Remark 5. Constant $c_{ij}^1$ in Equation (4.14) gives us information about the strength in attraction from individual $i$ to $j$ with respect to the influence of the other components.
that drive the dynamics of \( i \). On the other hand, \( c_2^{ij} \) and \( c_3^{ij} \) corresponds to the strength of the other components that influence the individual’s behavior such as the task to optimize, and repulsions and attractions to other individuals. Then, Equations (2.22) and (2.23) indicate that if the influence of the attraction from \( i \) to \( j \) represented in \( c_1^{ij} \) is stronger than the influence of the rest of the components of the behavior of individual \( i \) represented in \( c_3^{ij} \), then it is guaranteed that there is a bound of the error in position and velocity between \( i \) and \( j \) that will be satisfied at some finite time.

**Remark 6.** Note that from Equations (2.22) and (2.23), the conditions for ultimate uniform boundedness do not depend on the parameters of the component of the force \( u^i \) in Equation (2.3) that is associated with repulsion between individuals. However, from Equation (2.33), we see that the parameters of the repulsion term do affect the size of the ultimate bound through the constant \( c_2^{ij} \).

**Remark 7.** Note that when the repulsion and the optimization of the performance function do not have any influence in the computation of the force \( u^i \) in Equation (2.3) (i.e., \( r(t) = p(t) = 0 \)), from Equation (2.33) we have that \( c_2^{ij} = 0 \), and \( \Pi^I_i = N_i \). Then, if the condition (2.36) is satisfied for all \( t \geq 0 \), the uniform ultimate bound will be as tight as possible in this setting. If we additionally assume that the attraction weights \( w^{ij} \) and the performance optimization parameters \( \eta^i \) are constant, this particular case of our model is known as a “consensus algorithm” for double-integrator dynamics [103, Chapter 4].

### 2.3 Dynamics of the Group in Complex Task Environments

We explore the dynamics of the group when its members have to solve a given task, which is defined by \( f \). Recall that \( N \) denotes the number of members in the
group. We choose $x^i \in \mathbb{R}^2$ for all $i = 1, \ldots, N$, and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ to be a continuously differentiable function. We consider two different complex task environments: first, we choose a function that has 3 local minima, 3 local maxima, one global minimum $x^*$ such that $f(x^*) = 0$, and flat regions. The equation of this function is

$$f(x_1, x_2) = 3(1 - x_1)^2 \exp \left[ -(x_1^2) - (x_2 + 1)^2 \right] - 10 \leftcheck{\frac{x_1}{5} - x_3 - x_5} \exp \left( -x_1^2 - x_2^2 \right) - \frac{1}{3} \exp \left[ -(x_1 + 1)^2 - x_2^2 \right] + 6.5511$$

Figure 2.1 shows a contour plot of this function. This function allows us to study the behavior of the group when the task-solving process can lead to different solutions.

In the second scenario, we choose a function defined by a plane. This function allows us to study the dynamics of the group in terms of performance and cohesion when the task-solving process is continual, and does not have a fixed solution point. To quantify the group cohesion and performance at a given time $t$, we define three measurements: Group cohesion in position, defined as

$$\gamma(t) = -\frac{1}{N(N-1)/2} \sum_{i,j \in V} \|x^i(t) - x^j(t)\|^2$$

(2.39)

is the negative sum of the pairwise distances between the individuals at time $t$. This function allows us to quantify how close the group members are to each other at a given time. We have $\gamma(t) = 0$ when all the individuals are at the same position, and negative values of $\gamma(t)$ indicate separation between the group members. Similarly, we have the group cohesion in velocity, defined as

$$\gamma_v(t) = -\frac{1}{N(N-1)/2} \sum_{i,j \in V} \|v^i(t) - v^j(t)\|^2$$

(2.40)
In addition to cohesion measurements, we define Group performance as

\[ J(t) = -\frac{1}{N} \sum_{i=1}^{N} f(x^i(t)) \]  

(2.41)

which is the negative sum of the individual performance of each group member. This function allows us to measure how the group is performing in solving the task at time \( t \). Below, we present the results of the simulations of the dynamics of the group under different scenarios in the topology of the communication network, parameters of the model, and flow of information.

2.3.1 Trajectories and Attraction Patterns in the Group

To illustrate the group dynamics that can be generated using the proposed model, we first study the performance and cohesion of the group and the trajectories of each individual given a set of initial conditions and parameters of the model and the performance function \( f \) that has multiple solutions. In this case, the group is composed of \( N = 6 \) individuals, whose initial position is randomly selected and initial velocities are set to 0. Also, \( w_{ij}(0) = 1/|N_i| \) for every \( i \in V \) and \( j \in N_i \). The parameters of the model are chosen such that at \( t = 0 \) the contribution of the term associated with the gradient of the function \( \nabla f(x^i(t)) \) in Equation (2.3) is large compared to the one associated with the attractions \( a^i(t) \), and the short-range repulsion in position between individuals is very small. To facilitate the analysis of the attraction patterns of the group, in this simulation there is no attraction in velocity, that is, \( b = 0 \). The communication network between the group members follows a symmetric ring topology (each node has two neighbors). This means that each group member communicates with his or her contiguous group members via a bidirectional flow of information. The discretization of the differential equations is done using Euler's approximation
method with sampling time $T = 0.1$. Figure 2.1 shows the behavior of the individuals at different time steps. From the trajectory of the agents in Figure 2.1a we can observe that initially the individuals tend to go to their closest local minimum or saddle point. This is due to the fact that initially the force that drives each individual is more oriented toward solving the task than being attracted to the rest of the group. However, individuals start increasing their attraction to those that are connected to them and have a better relative performance as Figure 2.1b illustrates. There is a point where the attraction weights become large enough that the force driving the individuals points toward those individuals with better relative performance, even if it implies moving in a direction that locally does not optimize its performance. Note that, since the agents are not fully interconnected, some individuals sequentially jump to minima until eventually they reach the global one. This behavior is evident in the Figure 2.1c, where the group performance decreases at some points where group cohesion increases.

Figure 2.1: (a) Contour of the cost function $f(x)$ and trajectory of the members of the group (black solid lines), where symbol 'o' indicates the initial position, and the arrows represent the direction of their trajectory. (b) Contour of the performance function and communication network in the group, where the arrows indicate the flow of the information, the thickness of the line indicates the strength in attraction between members, and the symbol 'o' marks the location of the individual at time step 150. (c) Cohesion $\gamma(t)$ and performance $J(t)$ of the group.
A further interpretation of the mathematical model in Equations (2.2)-(2.8) allows us to explain the attraction patterns in the dynamics observed in Section 2.3. Using simple algebraic operations, we can rewrite the component associated with the attraction in position and velocity in Equation (4.3) as

$$a^i(t) = -\bar{w}^i(t)(x^i(t) - \bar{x}^i(t)) - b\bar{w}^i(t)(v^i(t) - \bar{v}^i(t))$$  \hspace{1cm} (2.42)$$

where $\bar{w}^i(t) = \sum_{j\in\mathcal{N}_i} w^{ij}(t)$ is the sum of the attraction in position strengths. We have that

$$\bar{x}^i(t) = \bar{w}^i(t)^{-1} \sum_{j\in\mathcal{N}_i} w^{ij}(t)x^j(t)$$

corresponds to the weighted average of the position of the individuals in $\mathcal{N}_i$. Here, $\bar{x}^i(t)$ is located closer to those individuals in $\mathcal{N}_i$ with higher attraction strength. In a similar way we can compute $\bar{v}^i(t)$. During the process of solving the task, from Equation (2.7) we have that an individual tends to be more attracted to the ones that are performing better than him or her, that is, have a higher contribution in the computation of the weight average $\bar{w}^i(t)$. As some of the $w^{ij}$ increase, term $\bar{w}^i(t)$ in Equation (2.42) becomes larger and the influence of the attraction component in force $u^i(t)$ becomes significantly larger than the component associated with performance optimization. In Figure 2.1 we can observe that group members that initially move to local minima or saddle points later move toward those individuals that have better performance. For example, individual 2 moves to a saddle point, but later it gets attracted to 1, who is performing better. Also, individual 3 reaches a local minimum, but it later climbs up the valley because it gets attracted to individual 2 who is performing better (because 2 is being attracted to 1). These dynamics allow the group to eventually reach the global minimum and remain cohesive.
2.3.2 Topology of the Communication Network

In the next set of simulations, we explore the behavior of the cohesion and performance of the group to solve the given task under different communication topologies. We estimate group cohesion $\gamma(t)$ and group performance $J(t)$ in Equations (2.39) and (2.41) using the Monte Carlo simulation method by repeating the simulations 1000 times with different initial conditions each time. This number of repetitions allows us to obtain a good estimate of the means and standard deviations of $\gamma(t)$ and $J(t)$. We create a group with $N = 10$ members, whose initial conditions in position are randomly selected, and initial velocities are set to zero. The performance function is the one that has several local minima. The discretization of the differential equations is done using Euler’s approximation method with sampling time $T = 0.01$.

We simulate the model under different topologies of the network that define how the members of the group communicate between each other. First, we compute group cohesion and performance under three topologies: a wheel, in which most of the members of the group communicate with only one individual. A ring, in which every individual communicates with only two other group members. And a line, in which is similar to the ring topology but two of the group members communicate with only one individual. For each topology, the parameters of the model are chosen under three different scenarios: (i) as in the simulation in Figure 2.1, where the performance optimization component of force $u^i$ is initially stronger than the attraction ones. (ii) Dynamics for the attraction weights and performance optimization strength (i.e., $\alpha_w = 0$ and $\alpha_\eta = 0$), in which the attraction strength is significantly larger than that of performance optimization; and (iii) again no dynamics for the attraction weights and performance optimization strength (i.e., $\alpha_w = 0$ and $\alpha_\eta = 0$), but in this case
the attraction and performance optimization strengths have a similar influence on the force $u^i$.

Figures 2.2a-2.2i show the results for each case. In the first scenario, where the dynamics of the attraction weights and the strength in performance optimization is given by Equations (2.7) and (2.8), we can observe that the average group cohesion $\gamma(t)$ and average group performance $J(t)$ reach a value close to zero, which means that all the members of the group reach almost the same position, and they achieve the global optimum. However, depending on the topology, there are significant differences in the trajectories that the group follows. It can be observed that the group in a wheel-type communication network (Figure 2.2a), compared to the other two network topologies, takes longer to reach a state in which the group can be considered as cohesive and has larger variability in the trajectories of $\gamma(t)$ and $J(t)$ for different initial conditions. The communication network that with better performance is the one with a ring-type topology (Figure 2.2b). The time the group takes to achieve a steady state and the variability for different initial conditions in performance and cohesion is considerably shorter than the other two topologies. These results are consistent with observations of human groups solving complex tasks [108, 32]. It has been reported that groups tend to perform better in decentralized communication networks (ring) than centralized ones (wheel).

In the second scenario, the parameters in attraction and performance optimization have no dynamics, and the contribution of the attraction term to the force that drives the members of the group is higher than the contribution of the performance optimization. In this case, the group reaches an agreement. However, the position that the group agrees at has a large variability, and on average, it does not correspond
Figure 2.2: Mean (solid line) and standard deviation (shaded region) of group cohesion $\gamma(t)$ and group performance $J(t)$ estimated using Monte Carlo simulations for a group with 10 members under (a,d,g) wheel, (b,e,h) ring, and (c,f,i) line communication networks. Simulations are performed in three different scenarios: (a,b,c) dynamics on the attraction and performance weights, (d,e,f) constant and equal attraction weights, where the attraction term has a predominance in the behavior of the individuals, and (g,h,i) constant and equal attraction weights, where both the attraction term and gradient of the cost function affect the behavior of the individuals.
to the global minimum. In the third scenario, the attraction and performance optimization parameters have no dynamics, but in this case they have equal contribution to the force $u^i$. In this case, on average, neither $\gamma(t)$ nor $J(t)$ are maximized.

### 2.3.3 Density of the Communication Network

To observe the effect of the density of the topology on the cohesion and performance of the group, we used the Monte Carlo simulation method with 1000 repetitions to estimate group cohesion and group performance in a group with $N = 50$ members. The parameters are chosen as in the simulations in Section 2.3.1. The selected performance function is the one that has several local minima. The density of a network topology is defined as the number of connections divided by the total possible number of connections between the members of the group. A density of 1 corresponds to a fully connected network. Figure 2.3 shows the results of the simulation for a network with a ring topology, where each node has 25 neighbors, and a network that is fully connected. This corresponds to networks that have densities 0.4898 and 1, respectively. In Figure 2.3, we can observe that as the number of neighbors increases, the time it takes to the members of the group to be considered cohesive increases, and the performance of the whole group decreases. This behavior can be noted from Equations (4.3) and (2.6). Here, the attraction, repulsion, and performance optimization terms are scaled by $|N_i|^{-1}$. When the number of neighbors increases, then the force that directs the motion of the individuals is mostly influenced by the term $-\zeta^i v^i$, $i = 1, \ldots, N$, which makes the velocities of the individuals decrease. This behavior has been observed in human groups [85, 32]. It has been shown that the size of the
group significantly affects both cohesion between members and the performance during the task-solving process. In the model, term $|\mathcal{N}_i|^{-1}$ in the social force components accounts for this effect. When the number of neighbors increases, then the social influence on the dynamics of the individual decreases. As the cohesive behavior of the group decreases, the overall performance of the group also is affected.

2.3.4 Group Dynamics in a Task Environment Represented by a Plane

A task environment $f$ given by a plane with respect to the position of the individuals in the group allows the group to have a continual optimization process of its performance during the selected time interval of analysis. In this case, the component of the force $u^j$ associated with the performance optimization is constant and different from zero. In this set of simulations we measure the cohesion of the group in position and velocity and its performance for the wheel, ring, and line communication network.
topologies used in the previous set of simulations (see Figure 2.2). The communication network has a ring topology, and the parameters are chosen such that there is an interaction between the attraction and performance components of the force that drive the motion of the group as in Section 2.3.1. However, in this case the attraction term depends not only on position but also in velocity. To study the behavior of cohesion and performance in the group, we used the Monte Carlo simulation method with 1000 repetitions, and estimated the mean and standard deviation of the group cohesion in position and velocity, and group performance, measured by $\gamma(t)$, $\gamma_v(t)$, and $J(t)$, respectively. Figure 2.4 shows these measurements for different communication network topologies. Note that the group in a wheel topology reaches significantly faster cohesiveness in position than the ring and line topologies. However, there is a larger variability in the initial behavior of the velocity vector and in the performance of the group. The groups working in ring and line topologies take longer to reach a state considered as cohesive, but their performance has less variability and increases faster than the one obtained by the groups working in a wheel topology.

### 2.4 Discussion

We developed a model of a group of individuals that interact according to a communication network, and whose dynamics are driven by a component that seeks to optimize a task and the influence of the interactions of the group members. The key aspect in our model, which is the main difference with other models of group dynamics and consensus, is that the strength of the social influence and the commitment to solve the task are not static but dynamic, and depend on the relation between cohesion and performance of the group. Simulation results show that the relation between
Figure 2.4: Mean and standard deviation of group cohesion in (a) position and (b) velocity, and (c) performance of the group for different communication topologies and when the performance function is characterized by a plane. The solid line and the shaded region correspond to the mean and standard deviation estimated using Monte Carlo simulations.

cohesion and performance, described in Equations (2.7) and (2.8), allows the model to characterize dynamics that are qualitatively consistent with the results observed in human groups. It is shown that, in average, decentralized communication network topologies outperform the centralized ones when solving a complex task, and a larger number of connections between individuals decreases cohesion and task performance. Also, a theoretical analysis of the model shows that, under the appropriate conditions on the parameters and topology of the network, the group reaches a point where it is considered as cohesive.
This model represents a tool for the analysis and design of communication networks of task-solving groups. It enables the social scientist to understand human groups as a dynamical system, to explore the group behaviors under different conditions in the parameters and network topology, and to design new methodologies for the development of experiments in real human groups whose aim is to solve complex tasks.
Chapter 3: Dynamics of Metabolism and Decision Making during Alcohol-Consumption: Modeling and Analysis

High-risk alcohol consumption is considered a large problem in public health. For example, in the United States, around 88,000 people die annually from alcohol-related causes [110], and it is considered the third major preventable determinant of death in the United States [1]. Also, over 40% of all college students report getting drunk in the past month [54] resulting in an estimated 1800 deaths per year [49]. A drinking event is defined as the situation where an individual or a group of people naturally participate in activities related to alcohol-consumption during a specified period of time. Research has focused on comprehending the etiology of heavy drinking at the event level, since this understanding is crucial to identifying potential "leverage points" in the event to intervene and alter heavy drinking or related problems [111].

Several field studies have examined drinking events in situ. These studies have found that individual characteristics (e.g., motivations for drinking, and drinking history) [114, 102], event-level factors (e.g., duration of drinking) [22], group activities (e.g., drinking games) [2, 123], and environmental factors (e.g., drink specials, and dancing) [128, 21] impact intoxication. These findings rely on data that are obtained through questionnaires, breathalyzer readings of blood alcohol content (BAC), and observations of the environment. Their interpretation is mainly based on statistical
analyses that try to uncover correlations between these factors and high-risk drinking [22]. Although they represent a very important contribution to better understand heavy alcohol consumption, there are still major gaps in our understanding of drinking events. The available information and the tools employed in the above analyses are limited. For example, the behavior of the BAC in an individual has been measured only before and after the drinking event, but not throughout it. Also, the statistical tools that have been used for their analysis only capture “snapshots” of the underlying dynamics of the drinking event, not its time evolution. Studying these elements is crucial to understand the dynamic and complex nature of high-risk drinking at the event level. In this work we set a starting point to fill these gaps.

There is a growing need for developing dynamical system models to better understand and address problems related to alcohol and, in general, to public health [52, 75, 96]. These models enable a comprehensive analysis of the problem and the elements involved in it, to design more effective interventions and assessment methodologies, and to formulate large-scale field experiments. Several dynamical models have been proposed to study high-risk drinking [77, 84]. These characterize how alcohol use in large groups or populations changes between different categories such as “non-drinkers,” “social-drinkers,” and “heavy-drinkers.” Also, several simulation strategies have been proposed to recreate drinking scenarios [36, 30]. However, as far as we know, there has not yet been a model constructed that characterizes how the BAC of an individual varies over time throughout the drinking event, taking into account the dynamical interaction between physiological factors and decision-making processes, and that allows us to conduct both computational and mathematical analyses for a more complete understanding of the modeled behaviors.
The contribution of this chapter is two-fold. First, we provide a model of a system that, based on observations and up-to-date collected data of behaviors in drinking events, we believe characterizes the BAC dynamics of a drinker throughout the event. Second, we provide a mathematical analysis of the modeled behaviors, and show how this model can complement empirical research. This model is constructed as the interconnection of a component that accounts for the dynamics of the metabolic process of alcohol, and a component that describes the decision-making process that drives the individual’s alcohol consumption. In the same way that physics attempts to describe how forces translate into changes in motion, our main objective is to propose a model that explains the “physics” of an individual’s BAC during a drinking event: how the output of a decision-making process translates into BAC variations and how BAC variations in turn affect decision-making. We propose a methodology to construct the model using data collected in situ, and to find the parameters of the decision-making process that generate the BAC trajectories that fit the available data.

We envision this model to be the starting point of a body of research that looks for a more complete characterization of the dynamics of social drinking events. Technological advances have made available via portable and reliable devices for real-time data collection on drinking events that include transdermal alcohol sensors for BAC monitoring [68, 8, 74], and devices to estimate relative proximity between individuals and their geographical location to make inferences about their social interactions and environment [18, 93, 96]. These sources of real-time data collection will eventually be used, along with survey and observational data, to build on our model and develop more accurate descriptions of the dynamics of drinking events. In the same way that
modern control engineering derives mathematical descriptions of systems (such as electrical, mechanical, chemical, economic, and human systems) to design strategies that control them such that they have a desired behavior [89, 5], our medium-term goal is to have a mathematical description of the dynamics of a drinking event to design control strategies (i.e., interventions) at the individual, group, and environment level that manipulate them such that high-risk behaviors are minimized [44]. Current technologies, such as smartphones, could possibly assist with interventions at the individual and group level and help minimize high-risk situations [127, 104, 27]. Figure 3.1 illustrates this concept. This work is a step toward that goal.

Figure 3.1: Block diagram of an intervention strategy that seeks to minimize high-risk drinking behaviors. The network in the diagram represents the drinking group where each individual (small circles) interacts with other individual’s behaviors.

The proposed model follows a standard mathematical framework for the analysis of linear and nonlinear systems [63, 4, 23]. Using this framework, we are able to model the dynamics of metabolic and decision-making processes of an individual in a drinking event. We characterize, based on data collected in laboratory settings and through a set of differential equations, the dynamics of the BAC resulting from the metabolic process of alcohol given arbitrary alcohol intake patterns, and provide
its causal-loop representation (Section 3.1.1). Also, we propose a description of the mechanisms that drive the decision-making process of a drinker during a drinking event based on data collected in situ and observations about the relationship between the subjective level of intoxication, BAC, and BAC rate of change (Section 3.1.2). Using computer simulations and Lyapunov stability analysis of non-smooth systems [63, 109] we analyze the modeled behaviors (Sections 3.1.3 and 3.2). We finish the chapter with a discussion about the utility of these results in the future design of intervention strategies for the prevention/minimization of high-risk behaviors in drinking events.

3.1 Model Formulation

The dynamics of alcohol consumption of an individual in a drinking event are modeled as the result of the interaction between a decision-making process, where the individual manipulates his/her drinking rate to reach a desired level of alcohol intoxication, and the metabolic process of the consumed alcohol that drives the BAC. Figure 3.2 shows a block diagram that represents the functional relationships of the components of the dynamical system that characterizes the drinking behaviors of an individual during a drinking event. This system has a decision making component that, based on the current BAC and BAC rate of change, adapts the rate of alcohol intake to achieve the desired level of intoxication. The consumed alcohol is then metabolized by the body following a dynamical process that is characterized by a component whose output is the BAC and BAC rate of change. The behavior of this system might be affected by external influences such as food consumption. A system with this structure of the interconnection between components is a feedback control
system [5]. A feedback control system is typically represented by the interconnection of two basic elements: a process (“plant”), which describes the relation between the variables to be controlled and the control variables (“outputs”), and the controller, which compares the outputs with the desired state to be achieved, and manipulates the control variables accordingly. In the system represented in Figure 3.2, the process and controller are the metabolic and the decision-making processes, respectively. The variables to be controlled are the BAC and BAC rate of change, and the control variable is the alcohol intake rate. Next, we present models that characterize the dynamics of the metabolism and decision-making components of the feedback system, and show through a mathematical and computational analysis that the behaviors that they describe are consistent with observations of alcohol-consumption behaviors in individuals who are in a drinking event.

3.1.1 Metabolism

Research on pharmacokinetics of alcohol has shown that, after a single dose of alcohol has been administered, the human body metabolizes it throughout different stages such that the BAC exhibits a profile as shown in Figure 3.5. Starting from
an initial condition, the BAC rises and reaches a peak concentration, followed by a declining phase where the BAC reaches zero \([129, 55]\). Different mathematical models have been proposed to describe this behavior of the BAC. Among the most relevant models are the Widmark and Michaelis-Menten models, which characterize the alcohol elimination process after the peak in BAC has been reached \([3, 76]\). Also, models based on the physiology of the human body have been formulated, taking into account the interaction between organs such as the liver, kidney, and stomach \([86, 118, 100]\). Although they represent a very important contribution to the understanding of the metabolic process of alcohol, they either are nonlinear or describe only some of the stages in this process. Our aim in this work is to formulate a model that characterizes the dynamics of all the stages in the pharmacokinetics of alcohol and at the same time it must be simple enough so that a mathematical and computational analysis can be conducted once the model is integrated into the feedback system in Figure 3.2.

In the theory of dynamical systems, the impulse response of a system refers to the output when there is a very short-duration input signal whose integral has a given value. If we model the alcohol metabolic process as a system whose input is the alcohol intake rate and the output is the BAC level, then the behavior shown in Figure 3.5 approximates the impulse response of a second-order dynamical system \([89, \text{Ch } 5]\), assuming that the system is linear and time invariant. This means that the dynamics of the system can be characterized as the result of the interaction between an external input and two variables that describe the intrinsic state of the system. In our case, these two variables are the BAC level and BAC rate of change.

The mathematical formulation is as follows. Let \(x(t) \geq 0\) be the BAC level at time \(t \geq 0\), and let \(v(t) = \dot{x}(t)\) be the BAC rate of change, where \(\dot{x}\) denotes the
derivative of $x(t)$ with respect to $t$. Let $u(t) \geq 0$ be the alcohol intake (consumption) rate. Note that the amount of alcohol consumed from time 0 to time $t > 0$ is given by $\int_0^t u(\tau)d\tau$. The set of differential equations that represents a second-order dynamical system whose impulse response follows the one in Figure 3.5 is

$$
\dot{x}(t) = v(t) \\
\dot{v}(t) = -\beta^2 x(t) - 2\zeta \beta v(t) + \beta^2 \eta u(t)
$$

(3.1)

where $\eta$, $\beta$, and $\zeta$ are the parameters of the model. This dynamical system can be seen from the viewpoint of Newton’s second law, where there are forces that define the BAC’s acceleration $\dot{v}$. The term $-\beta^2 x$ is one that forces the BAC to decrease to zero, and $-2\zeta \beta v$ affects how quickly the BAC changes. The term $\beta^2 \eta u(t)$ is the scaled external input to the system, that in this case is related to the alcohol consumption rate throughout the drinking event. Figure 3.3 shows the causal-loop diagram of the dynamical system in Equation (3.1).

![Causal-loop diagram of the dynamical system that characterizes the metabolic process of alcohol in Equation (3.1).](image)

Next, we provide a theorem that characterizes the behavior of the trajectories generated by the system in (3.1), and show how the model fits data collected in laboratory settings.

**Theorem 2.** Let the parameters of the model in (3.1) be such that $\eta > 0$, $\beta > 0$, $\zeta > 1$. Then, a solution trajectory of the BAC for a given alcohol intake rate $u(t)$,
\[ t \geq 0, \text{ is} \]
\[ x(t) = g_0(t) + \frac{\beta \eta}{2\sqrt{\zeta^2 - 1}} \int_0^t \left[ e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)} \right] u(s)ds \quad (3.2) \]

where
\[ g_0(t) = \frac{1}{2\beta \sqrt{\zeta^2 - 1}} \left[ (\lambda_2 x_0 - v_0)e^{\lambda_1 t} - (\lambda_1 x_0 - v_0)e^{\lambda_2 t} \right] \]

and \( x_0 = x(0) \), \( v_0 = v(0) \), \( \lambda_1 = -\beta \zeta - \beta \sqrt{\zeta^2 - 1} \) and \( \lambda_2 = -\beta \zeta + \beta \sqrt{\zeta^2 - 1} \).

**Corollary 2.1.** In addition to the assumptions in Theorem 2, assume that that \( u(t) \in [0, +\infty) \) for all \( t \geq 0 \) and \( x_0 = x(0) \geq 0 \). Then, the trajectories generated by Equation (3.1) satisfy the following properties:

(i) \( x(t) \geq 0 \) for all \( t \geq 0 \).

(ii) Let the alcohol intake rate be an impulse with amplitude \( C > 0 \) (i.e., a single dose administered at \( t = 0 \) with amount of alcohol \( C \)), and let the initial conditions be \( x(0) = 0 \) and \( v(0) = 0 \). Then, the impulse response of the system for \( t \geq 0 \) is given by
\[ x_\delta(t) = \frac{C \beta \eta}{2\sqrt{\zeta^2 - 1}} \left( e^{\lambda_2 t} - e^{\lambda_1 t} \right) \quad (3.3) \]

(iii) \( \lim_{t \to \infty} x_\delta(t) = 0 \).

(iv) The impulse response \( x_\delta(t), t \geq 0 \), has a peak that occurs at time
\[ t_p = \frac{1}{2\beta \sqrt{\zeta^2 - 1}} \log \left( \frac{-\zeta + \sqrt{\zeta^2 - 1}}{-\zeta - \sqrt{\zeta^2 - 1}} \right). \quad (3.4) \]

The proofs are based on basic concepts of linear systems and optimization [4], and they are presented in Appendix A.1 in the supplementary file. Theorem 2 shows that the behavior of the BAC level is modeled as the integral of the alcohol intake rate whose amplitude is modulated by the subtraction of two exponentials with different
decay rates. It characterizes the delayed response of the BAC when alcohol in consumed and the decreasing behavior of the BAC when there is no consumption. The term $g_0$ is associated with dynamics that result from initial conditions that are different from zero (e.g., when someone has consumed alcohol before the time reference of $t = 0$). Some useful properties of this family of trajectories are presented in Corollary 2.1: Property (i) indicates that the BAC cannot take negative values as long as the initial conditions are nonnegative. Property (ii) shows that the impulse response of the system corresponds to the subtraction of two exponential curves that have different decay rates. This result is consistent with the models describing the BAC profile in Figure 3.5 presented in [37, 121], which are referred as “one-compartment models.” As shown in Appendix A, the conditions on the parameters that are necessary to describe this behavior are that $\beta$ and $\eta$ have to be positive, and $\zeta > 1$, which guarantee that $\lambda_1 < \lambda_2 < 0$. Property (iii) shows that the BAC decreases and tends to zero asymptotically when the input is an impulse signal, behavior resulting from the elimination stage of the metabolic process. Property (iv) shows that the modeled impulse response has a peak whose occurrence time depends on parameters $\zeta$ and $\beta$, but not on $\eta$. From Equation (3.4), the peak value of $x_\delta$ is

$$x_\delta(t_p) = \frac{C \beta \eta}{2\sqrt{\zeta^2 - 1}} \left( e^{\frac{\sigma_2}{\sigma_1}} \log\left(\frac{\sigma_2}{\sigma_1}\right) - e^{\frac{\sigma_1}{\sigma_1}} \log\left(\frac{\sigma_2}{\sigma_1}\right) \right)$$

where $\sigma_2 = -\zeta - \sqrt{\zeta^2 - 1}$ and $\sigma_1 = -\zeta + \sqrt{\zeta^2 - 1}$. This corresponds to maximum value of the BAC that marks the beginning of the ethanol elimination stage. Note that parameter $\eta$ is associated with the height of the peak but not with its occurrence time. Figure 3.4 shows the relationship between the parameters $\zeta$ and $\beta$ on the time $t_p$ when the peak occurs and its amplitude $x_\delta(t_p)$. Observe that $t_p$ decreases as $\zeta$ and $\beta$ increase, as opposed to $x_\delta(t^*)$, which increases as as $\zeta$ and $\beta$ increase.
Figure 3.4: (a) Time \((t^*)\) and (b) amplitude \((x_\delta(t^*))\) of the maximum value that the BAC reaches during the metabolic process for different values of the parameters \(\zeta\) and \(\beta\) when the input alcohol intake rate corresponds to an impulse.

Figure 3.5 shows the average BAC samples collected when a dose of alcohol was administered to participants under two different situations: fasted condition and fed condition. The trajectories generated by the model were computed using Equation (3.1) after finding the parameters that provided the best fit according to a least squares criterion for each case (fasted and fed conditions). Note that the proposed system provides a good approximation of the measured BAC response. We also simulate the the case when the alcohol intake rate is not an impulse. Figure 3.6 shows the solution trajectory of the BAC when the alcohol intake rate is a signal containing short duration pulses that simulate an individual taking sips of alcohol. Details of the implementation are presented in Appendix A.3 in the supplementary file.

Although Equation (3.2) provides an expression for the BAC trajectories for any alcohol intake rate function \(u(t)\) with respect to time, the formulation of the system
Figure 3.5: Average of BAC samples measured on 12 individuals when a dose of 0.8 g ethanol/kg of weight was administered under two different situations: when they were in a fasted condition (symbol 'x'), and when they were in a fed condition (symbol 'o') (measurements taken from [56]). The solid and dashed lines are the estimated trajectories using the model in Equation (3.1) and a least-squares curve fit. The values of the parameters for the fasted condition were $\zeta^* = 1.4$, $\beta^* = 0.0107$, and $\eta^* = 39.23$, and for the fed condition were $\zeta^* = 1.0003$, $\beta^* = 0.0096$, and $\eta^* = 21.21$. The methodology used to fit the data is provided in Appendices A.3 in the supplementary file.

Figure 3.6: Trajectory of the BAC when the alcohol intake rate is a signal that simulates a drinker taking sips whose alcohol concentration changes around 0.02 grams of alcohol per kg of weight. The parameters of model associated with the metabolic process are the ones estimated in Figure 3.5 in the fasted condition case.
through the differential equations in (3.1) facilitates a mathematical analysis once it is integrated into the feedback system in Figure 3.2 as we will show below.

### 3.1.2 Decision Making

Research that studies the factors that affect the behavior of drinkers in a drinking event has shown that the individual’s alcohol intake rate is correlated with the desired level of intoxication that he/she wants to achieve. The desired level of intoxication of an individual in a drinking event is influenced by elements that depend on the characteristics of the drinker and the social and environmental influences. Research suggests that the individual’s characteristics such as desired body reaction to the BAC, history of drinking, and drinking motives are individual factors that affect the alcohol intoxication level at the drinking event [114, 102]. At the group level, peer interactions have a significant impact on the behavior of a person in a drinking group. Perception of social norms [99], the way that people communicate in the group, and the strength of the interpersonal relations influence drinking behaviors [2, 123]. Also, data that have been collected in situ indicate that environmental factors affect consumption patterns. For example, drinking games and alcohol price “specials” can promote higher consumption of alcohol, as opposed to scenarios that are crowded, which can result in lower alcohol consumption [128, 21]. The interactions between the influences at the individual, group, and environmental levels sets the level of intoxication that the individual wants to achieve during the drinking event. In this work we do not model the dynamics of such interactions. We focus our analysis on the dynamics of the decision-making and metabolism processes as shown in the feedback
system in Figure 3.2 given the desired level of intoxication, where this desired level might change over time. That is, we focus on the individual.

Following these observations, we hypothesize that an individual manipulates his/her alcohol intake rate to reach the intended intoxication level based on his/her subjective feeling of intoxication. It has been shown that both the current state of the BAC and the BAC rate are important determinants of how a drinker perceives his/her own level of intoxication [78, 41]. The study in [78] found that the subjective intoxication level changes depending on the BAC rate of change: an individual tends to perceive levels of intoxication that are higher (lower) than the one associated with his/her current BAC when the BAC rate of change is larger (lower, respectively). Hence, we define the subjective intoxication at time $t$ as

$$ y(t) = x(t) + k_d v(t) $$

where $k_d \geq 0$ is a scaling factor that quantifies the influence of $v$ on the perceived intoxication level. We assume that the mechanisms that an individual uses in a drinking event to manipulate his/her alcohol intake are based on his/her desire to reach the intended level of intoxication and his/her subjective level of intoxication $y(t)$. We model the decision-making process of a drinker such that the magnitude of his/her alcohol intake rate decreases when the perceived BAC $y(t)$ gets closer to the desired level of intoxication. Also, it is positive when is below the level of intoxication, and zero if it is above. A formulation that satisfies this conditions is

$$ u(t) = [k_p (x^* - y(t))]_+ $$

where $x^* \geq 0$ is the desired level of intoxication and $k_p > 0$ is a scaling factor that quantifies the commitment of the individual to reach the desired intoxication level $x^*$. 
The operator \([z]_+ = \max\{z, 0\}\) guarantees that \(u(t) \geq 0\) for all \(t \geq 0\). This expression can be rewritten as

\[
  u(t) = [k_p(x^* - x(t)) - k_p k_d v(t)]_+
\]

Interestingly, this decision-making strategy is typically known in feedback control systems theory as proportional-derivative control [5, Ch 10]. The alcohol intake rate has a component that changes proportional to the error between the desired level of intoxication and the current BAC level, and a component that changes in the opposite direction of the BAC rate of change (derivative of BAC). This last component characterizes the opposition of the drinker to quick changes in the BAC.

It is important to note that the alcohol intake rate of a drinker typically has a shape over time that resembles a sequence of pulses (due to sipping and gulping), as shown in Figure 3.6, whose frequency and amplitude change over time. In our model, \(u(t)\) represents the resultant average alcohol intake rate that an individual can have given his/her desired level of intoxication, current BAC, and BAC rate of change (from the observations explained above).

Even though the decision-making strategy in (3.5) captures the basic behavior of an individual trying to achieve his/her desired level of intoxication, the assumption that the individual has a unique desired BAC level is restrictive. It can be the case that a drinker desires to achieve values of BAC that are within an interval such that they produce a desired influence on their body. For instance, a drinker who wants to have a minimal behavioral change will seek to reach values of the BAC that are between 0 and 0.04 g/100 mL [58, Ch 13]. In order to generalize the formulation in (3.5) to take into account these cases, we introduce a personal preference function, \(f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\), that is a continuous and differentiable convex function that has a
unique minimum such that, without loss of generality, \( \min_y f(y) = 0 \). This function quantifies the individual’s preferences with respect to the BAC. Lower values of \( f(y) \) correspond to more preferable levels of the BAC. The most preferable BAC levels satisfy \( f(y) = 0 \). The most useful property of this function in our problem is that the scaled negative derivative of \( f \) with respect to \( y \) satisfies the properties of the decision-making strategy: its magnitude decreases as the BAC gets closer to the most desirable levels, and it is positive (negative) if the current BAC is lower (higher) than the most desirable levels. In Equation (3.5) it is implied that the preference function is \( f(y(t)) = \frac{1}{2}(y(t) - x^*)^2 \), where \(-k_p df(y(t))/dy = k_p(x^* - y(t))\). Equation (3.5) can be generalized as

\[
  u(t) = [-k_p g(y(t))]_+ \tag{3.6}
\]

where

\[
g(y) = df(y)/dy
\]

and \( y(t) = x(t) + k_d v(t) \). The benefit of using the preference function in our formulation is that different profiles can be designed depending on the desired level of intoxication the drinker is assumed to have. Figure 3.12 shows examples of profile functions. These preference functions have an interval of desired BAC levels rather than one single value. Also, they are steeper for BAC levels lower than the desired ones (when \( f(x) = 0 \)) than those levels that are higher than the desired ones.

### 3.1.3 Combining the Metabolism and Decision-Making Processes

The model for the feedback interconnection between the metabolic and decision-making systems in Figure 3.2 is obtained by substituting \( u(t) \) from Equation (3.6) in
Equation (3.1), to get
\[
\dot{x}(t) = v(t)
\]
\[
\dot{v}(t) = -\beta^2 x(t) - 2\zeta \beta v(t) + \beta^2 \eta [-k_p g(x(t)) + k_d v(t)]
\]  
(3.7)

This captures, in one set of differential equations, the dynamics that result of the interaction between the metabolic process of alcohol and the decision-making process that drives the alcohol intake rate. The following simulations and mathematical analysis show some of the qualitative properties of the BAC dynamics for (3.7).

Figure 3.7 shows the response of the feedback system for different values of the parameters \(k_p\) and \(k_d\). The case in Figure 3.7a illustrates the effect of \(k_d\) on the trajectories. Here, \(k_p\) and the initial condition are kept fixed. We can observe that as \(k_d\) increases (in the figure, \(k_{d1} < \cdots < k_{d5}\)), the overshoot in the BAC trajectories tends to decrease. This is due to the effect of \(v(t)\) on the perception of the intoxication level. Individuals with a strong awareness of the BAC rate of change will modulate their alcohol intake rate such that their BAC trajectories increase at a slow rate. On the other hand, individuals with a small awareness of the BAC rate of change will be affected by the delayed response of the metabolic process, and therefore will have trajectories that exhibit overshoot. Figure 3.8 shows the alcohol intake rate over time for each case. Observe that each individual starts with a high rate of alcohol consumption that decreases as the BAC gets closer to the desired level of intoxication. Once the BAC is around \(x^*\), it manipulates \(u(t)\) so that the BAC stays very close to the desired level.

The case in Figure 3.7b explores the effect of \(k_p\) on the trajectories. Here, \(k_d\) and the initial condition are kept fixed. It can be observed that as \(k_p\) increases (in
the figure, \( k_{p1} < \cdots < k_{p5} \), the error between the desired intoxication level and the BAC decreases and the overshoot increases. The parameter \( k_p \) can be seen as the commitment of the individual to achieve his/her desired level of intoxication. Different combinations of these two parameters, \( k_p \) and \( k_d \), provide a rich variety of responses that an individual can have during a drinking event.

![Diagram](image)

**Figure 3.7**: (a) Trajectories of the BAC for different values of the parameter \( k_d \) and a fixed \( k_p \). In this case, \( k_{d1} < k_{d2} < \cdots < k_{d5} \). (b) Trajectories of the BAC for different values of the parameter \( k_p \) and a fixed \( k_d \). In this case, \( k_{p1} < k_{p2} < \cdots < k_{p5} \). The dashed line indicates the desired level of intoxication.

In order to study the effect of the metabolism parameters \( \zeta, \beta, \) and \( \eta \), on the decision-making process, we generate different sets of metabolism parameters that produce responses that vary in between the ones for the fasted and fed conditions in Figure 3.5. Let \( \beta_0, \zeta_0, \) and \( \eta_0 \) the parameters associated with a metabolic process in a fasted condition, and \( \beta_1, \zeta_1, \) and \( \eta_1 \) the parameters associated with a metabolic process in a fed condition. A set of parameters associated with a metabolic process in between these two conditions is defined as \( \beta_\alpha = (1-\alpha)\beta_0 + \alpha\beta_1, \zeta_\alpha = (1-\alpha)\zeta_0 + \alpha\zeta_1, \)
Figure 3.8: Trajectories of the alcohol intake rate $u(t)$ for the cases shown in Figure 3.7a.

and $\eta_\alpha = (1 - \alpha)\eta_0 + \alpha\eta_1$, where $\alpha \in [0, 1]$. When $\alpha = 0$, we obtain the set of parameters associated with the fasted condition, and when $\alpha = 1$, we obtain the set of parameters associated with the fed condition. For $\alpha \in (0, 1)$ we get cases in between these two conditions. Figure 3.9a shows the impulse responses of the metabolic process for different values of $\alpha$. Let $x^\alpha(t)$ the trajectory generated for the feedback system in Equation (3.7) using the set of metabolism parameters $\beta_\alpha$, $\zeta_\alpha$, and $\eta_\alpha$. Let the accumulated error be defined as

$$e_\alpha = \int_0^T [x^\alpha(t) - x^0(t)]^2 dt$$

(3.8)

where $T > 0$ is the duration time of the drinking event. This corresponds to the error between the trajectory generated by the system with parameters $\beta_\alpha$, $\zeta_\alpha$, and $\eta_\alpha$, and the baseline trajectory $x^0$ (the one generated by the system with parameters $\beta_0$, $\zeta_0$, and $\eta_0$). Figure 3.9b shows the result of Monte Carlo simulations for different values of $\alpha$ with 1000 runs each, where $k_p$, $k_d$, and $x^*$ (desired intoxication level) have fixed values. The initial conditions are randomly chosen following a uniform distribution on
the interval $[0, 2x^*]$. Note that the difference between the trajectories increases as $\alpha$ increases. Even though the median is very small, the range where the error lies seems to increase exponentially with a linear increment of $\alpha$. This indicates that, according to the proposed model, the decision-making process can lead to very different BAC trajectories when the metabolic process changes.

Figure 3.9: (a) Impulse response of metabolic processes associated with different parameters $\zeta$, $\beta$, and $\eta$. The BAC trajectory associated with $\alpha = 0$ corresponds to the response of a metabolic process in the fasted condition in Figure 3.5, and the one associated with $\alpha = 1$ corresponds to the response of the metabolic process in the fed condition. Values of $\alpha$ inside the interval $(0, 1)$ are associated with responses of metabolic processes in between these two conditions. (b) Box plot of the squared error $e_{\alpha}$ defined in Equation (3.8). The desired level of intoxication $x^*$ and parameters $k_p$ and $k_d$ are fixed, and the initial conditions are chosen randomly.

The next theorem illustrates the impact of the model parameters on the resulting BAC trajectories.

**Theorem 3.** Let the parameters of the model in (3.7) be such that $\eta, \beta, k_p, k_d > 0$, and $\zeta > 1$. Assume that the gradient of the preference function $f$ is Lipschitz continuous with Lipschitz constant $L > 0$. Also, assume that

$$\frac{Lk_pk_d}{2} < \frac{\zeta}{\beta \eta}$$

(3.9)
Then, the trajectories of the system in Equation (3.7) are uniformly ultimately bounded around the point \((x = x^*, v = 0)\), with ultimate bound \(\gamma\) given by

\[
\gamma = \max \{\gamma_1, \gamma_2\}
\]

where

\[
\gamma_1 = \frac{Mx^*}{\sqrt{2} \max \left\{ \left( \frac{\zeta}{\beta} - \frac{\eta}{2}k_pk_dL \right), \beta \eta k_p L \right\}},
\]

\[
\gamma_2 = \frac{Mx^*}{\zeta/\beta},
\]

and \(M = \sqrt{\frac{\max\{1, \beta^2\}}{\min\{1, \beta^2\}}}\).

The proof of this theorem is based on Lyapunov stability analysis of non-smooth systems [63, 109], and it is presented in Appendix A.2 in the supplementary file. This theorem states that, under the given assumptions and for any given initial condition, it is guaranteed that the trajectories of the BAC will eventually reach values that are inside a ball centered at \((x = x^*, v = 0)\), and that has a radius (ultimate bound \(\gamma \geq 0\)) that depends on the parameters of the model. Saying that the BAC trajectories are “ultimately bounded” means that the individual’s goals with respect to his/her BAC are the “approximately met”, that is, BAC will remain closer to \(x^*\) and the BAC rate of change will remain closer to zero. A small radius implies that the trajectories of the BAC are guaranteed to reach a more localized region.

Figure 3.10 shows the behavior of the ultimate bound \(\gamma\) in Equation 3.10 for different values of \(k_p\) and \(k_d\), where \(\zeta^* = 1.4\), \(\beta^* = 0.0107\), and \(\eta^* = 39.23\), and \(L = 0.08\). The shaded area in the \((k_p, k_d)\) plane shows the region where \(k_p\) and \(k_d\) satisfy the condition in (3.9). Observe that \(\gamma_2\) dominates the max operation in \(\gamma\) for those values of \(k_p\) and \(k_d\) that are far from the boundary. This means that the
achieved precision of the individual in reaching the desired level of intoxication will depend mostly on his/her metabolism. Also, note that $\gamma$ can increase for larger values of $k_p$. This is the situation in which the individual has aggressive alcohol consumption patterns and does not take into account the delayed response of his/her metabolism to process alcohol. Larger values of $k_d$ allow the individual to reach values of the BAC that are close to the desired ones (i.e., small values of $\gamma$). For those values of $k_p$ and $k_d$ that do not satisfy the condition in (3.9) we are not able to guarantee, using our mathematical analysis, that the BAC trajectories will eventually reach a region close to the desired level of intoxication. An interpretation of this condition can be that if the subjective intoxication level is too sensitive to changes in the BAC rate of change and there is an aggressive commitment to reach the desired level of intoxication (large values of $k_d$ and $k_p$, respectively), then we cannot have guarantees that the individual will take into account the delayed response of the metabolic process and have BAC trajectories that will tend to reach the desired BAC.

3.2 Estimation from Field Data

We inform the model in Equation (3.7) using current available data that was collected \textit{in situ}, and then use it to generate possible trajectories of the BAC during the drinking event under some assumptions on the parameters. We show how the model parameters $k_p$ and $k_v$ allow us to estimate information from the available data on BAC dynamics during drinking events. The details of the implementation are given in Appendix A.4 in the supplementary file.

We use the field data in [20, 22], which contains information from 1,040 participants attending 30 bars. Patrons were randomly selected and interviewed, and
breath alcohol samples were taken upon entering and exiting the bar. Participants were asked about their drinking plans and motivations for the night. Before entering the bar, they reported the intended level of intoxication that they wanted to reach during the drinking event. They chose one of the following categories: “not drinking,” “not enough to get buzzed,” “slight buzz,” and “very drunk.” Also, the time that each participant spent at the bar was reported. There is no information about the social interactions among participants. Using these data we construct our model to generate the BAC trajectories that start at the BAC levels when entering the bar and go through the BAC levels when exiting, having the reported duration of the drinking event. Since we have limited information to estimate all the model parameters, we make assumptions on those related to the metabolic process (parameters $\beta$, $\eta$, and $\zeta$) to estimate those related to the decision-making process (parameters $k_p$ and $k_d$).
First, we construct the profile of the preference functions based on the desired level of intoxication that the participants reported before entering the bar. Figure 3.11 shows the histogram of the BAC level measured from the participants when exiting the bar given the intended level of intoxication. Note that the distribution of the BAC samples changes depending on the stated intentions before entering the bar. For each category of the intended level of intoxication we select the BAC interval that contains approximately 60% of the distribution. Category “not enough to get buzzed” is then associated with the interval [0, 0.053], “slight buzz” with [0.042, 0.093], “very drunk” with [0.097, 0.3], and “not drinking” with 0. The preference function per category is constructed such that it is zero when the BAC is within the corresponding interval.

Figure 3.11: Histogram of the BAC measured after exiting the bar. Each plot is associated with the preferred level of alcohol intoxication (a) “not drinking,” (b) “not drinking enough to get buzzed,” (c) “slight buzz,” and (d) “very drunk.”
and it is steeper for BAC levels that are less than this interval than for BAC levels that are greater than it. Figure 3.12 shows the profile functions for each category of the intended level of intoxication.

Second, we define the strategy to find the parameters $k_p$ and $k_b$ such that the generated trajectory reaches the measured final BAC at the specified time (duration of the drinking event), given the initial BAC, the parameters associated with the metabolic process, and the intended level of intoxication. The parameters associated with the metabolism process are assumed to be the same for all the participants and have the same values obtained in Figure 3.5 for the curve approximation in the fasted condition case. Let $x_0$ and $x_f$ be the initial and final BAC levels measured from a participant, and let $T > 0$ be the reported duration time of the drinking event. Let $x(t; x(0), k_p, k_d)$ denote the solution trajectory of the BAC generated by the model in Equation (3.7) evaluated at time $t \geq 0$ with initial condition $x(0)$ and parameters $k_p$. 

Figure 3.12: Profile of the preference function for the four categories of the intended level of intoxication reported by the participants: “not drinking,” “not drinking enough to get buzzed,” “slight buzz,” and “very drunk.”
and $k_d$. The problem of finding $k_p$ and $k_d$ is formulated as the optimization problem

$$(k_p^*, k_d^*) = \arg \min_{k_p, k_d > 0} (x_f - x(T; x_0, k_p, k_d))^2$$

(3.11)

This is a search problem that tries to find those parameters $k_p > 0$ and $k_d > 0$ such that the BAC level of the trajectory generated by the model at time $T$ is as close as possible to the measured BAC at the exit of the bar.

Figure 3.13 shows the estimated trajectories for five samples taken from participants that reported “slight buzz” as their desired level of intoxication to reach during the drinking event. Trajectory 1 in the figure was generated for a value of $k_d^*$ that is significantly larger than that of $k_p^*$. Since the BAC and the initial and final BAC levels are almost zero, and the duration of the drinking event is short, large values of $k_d$ are required to keep the trajectories from increasing to BAC levels that take longer to metabolize. Recall that $k_d$ is the strength of the opposition to change in the BAC. For trajectories 3 and 5, parameters $k_p^*$ and $k_d^*$ have similar values. The final BAC is below or within the desired interval, and the duration of the drinking event is not as short as the other cases. On the other hand, trajectories 2 and 4 were generated for values of $k_p^*$ significantly larger than that of $k_d^*$. The initial BAC level is below the preferred interval of intoxication and the final BAC is within or above it. In this case, to reach the final BAC in the specified period of time, a small awareness of how quickly the BAC is changing (i.e., lower values of $k_d$) is required to generate a quick response.

Figure 3.14 shows the relationship between the parameters $k_p$ and $k_d$ estimated using Equation (3.11) and the initial and final BAC measured during the drinking event for individuals that reported “slight buzz” as the desired intoxication level. The horizontal dashed lines indicate the individual’s desired BAC interval, and the
Figure 3.13: Trajectories (solid lines) estimated using the proposed model for six participants that reported “slight buzz” as the intended level of intoxication to reach (dashed lines). Symbol ‘x’ marks the end of the trajectory.

size of the markers is proportional to the parameter $k_p$ for Figure 3.14a and to the parameter $k_d$ for Figure 3.14b. From Figure 3.14a observe that the model predicted that those individuals that had a low BAC before entering the bar and a relatively high BAC when they left it, required higher values of $k_p$. This indicates that those individuals had a stronger commitment to reach the desired intoxication level, which is the region between the dashed lines. On the other hand, those individuals whose BAC before entering the bar was within, or higher than the desired BAC interval did not have to have high values of $k_p$ to remain at that state. Also, from Figure 3.14b observe that the largest values of $k_d$ are associated with low values of the BAC at the entrance of the bar. This indicates that those individuals who want to remain outside the desired interval of intoxication must have a high sensitivity to the BAC rate of change, that is, their subjective level of intoxication $y(t) = x(t) + k_d v(t)$ easily reaches the desired level. We show in Figure 3.14c the relationship between the product of
$k_p$ and $k_d$ and the initial and final BAC levels. Observe that $k_p$ and $k_d$ tend to have relatively large values when the individual has initial and final BAC levels closer to the desired interval of intoxication. In this case, individuals have to be sensitive to decreasing variations of the BAC and have a response such that it keeps their BAC level within the desired interval. A similar analysis can be done for the other intervals of intoxication.

Additionally, we show in Figure 3.15 the values of $k_d$ versus $k_p$. Each point is associated with an individual, and the size of the markers is proportional to the difference between the initial BAC and the closest point of the desired interval. Samples with the ‘x’ (red) marker indicate that the initial BAC is above the target BAC, and samples with the ‘.’ (blue) marker indicate that the initial BAC is below the target BAC. The shaded region corresponds to the values of $k_p$ and $k_d$ that satisfy the condition in Equation (3.9). Observe that there is a tendency of the values of $k_p$ and $k_d$ to follow the shaded region. There is not an exact match since the condition in Equation (3.9) is derived from a qualitative analysis of the long-term behaviors of the modeled trajectories. The results shown in Figure 3.15 are computed for limited durations of the drinking event. Note that the points that do not follow this tendency are those that are associated with initial BAC values that are above the desired interval of intoxication. These values are also associated with values of the final BAC that are outside this interval. It seems that the reported desired level of intoxication changed during the drinking event. In these situations the proposed algorithm gives results that do not follow a consistent pattern.
Figure 3.14: (a) BAC measured on subjects before entering the bar (horizontal axis) and after leaving the bar (vertical axis). Each dot corresponds to the samples collected from an individual. The size of the marker is proportional to the value of $k_p$ computed using the parameter estimation method in Equation (3.11). Each individual reported “slight buzz” as the intended level of intoxication to achieve (dashed lines). The dotted line is a 45-degree line for reference. (b) Similar description as in (a), where the size of the marker is proportional to $k_d$. (c) Similar description as in (a), where the size of the marker is proportional to $k_p k_d$.

3.3 Conclusion

We developed a model that characterizes the BAC dynamics of an individual that is part of a drinking event. This model is composed of two main components:
Figure 3.15: Resulting values of $k_p$ versus $k_d$. Each point is associated with an individual. The size of the marker is proportional to the difference between the initial BAC and the closest point of the reported desired interval. The ‘x’ (red) marker indicates that the initial BAC is above the target BAC, and the ‘.’ (blue) marker indicates that the initial BAC is below the target BAC. The shaded region corresponds to the values of $k_p$ and $k_d$ that satisfy the condition in Equation (3.9).

a component that explains the metabolic process of alcohol in the individual for arbitrary alcohol intake patterns, and a decision-making component that manipulates the alcohol intake rate based on the desired level of intoxication and the current state of the BAC and BAC rate of change. We present computer simulations and a mathematical analysis of the impact of the parameters on the modeled dynamics, and propose a methodology to find the parameters of the decision-making process that allows the model to generate BAC trajectories that fit the available real data.

We showed how the model of the drinker’s decision-making process, which was formulated following the observations from field studies, resembled the proportional-derivative controller from feedback control system engineering. This result provides useful insights into the mechanisms that drive the decision-making process of an individual in a drinking event. It might be the case that there are additional mechanisms
that have not been uncovered yet, but are already known in the theory of feedback systems. An example is the proportional-integral-derivative controller, which in addition to the components explained in the previous sections, has a component (the integrator) that allows the system to accurately reach the desired state. This can be explored in our future work, along with personal drinking advisor development.

Currently, we are in the process of collecting real-time drinking event data that includes individual, group, and environmental level variables [68, 74, 18]. Our objective is to study these data using tools in statistics and dynamical systems theory, and conduct a model identification process from the measured data [73]. These results will help us to validate and update our hypotheses on the modeled drinking event dynamics. Our aspiration is to gain enough understanding to start interventions at both the individual and group event levels.
Chapter 4: Modeling and Analysis of Group Dynamics in Alcohol-Consumption Environments

Heavy alcohol consumption is the cause of approximately 1800 deaths per year among college students, and is considered a major public health issue in the United States [49]. During the last twenty years, researchers have tried to understand the etiology of heavy drinking among this population and design strategies to intervene to reduce heavy drinking and its consequent problems [111]. The main goal is to find the “leverage points” of the heavy drinking event, that is, the places and times in the event that are crucial for intervention.

Several studies have been conducted to examine the behavior of groups that are in naturally occurring drinking events \textit{in situ} [20, 113, 74]. These studies have found that the dynamics of alcohol consumption are affected by the characteristics of the individual (e.g., drinking motivations and history), event-level factors (e.g., duration of the drinking event, or playing drinking games), and environmental factors (e.g., dancing, food, or drink specials). Statistical tools have been employed to analyze how these factors are correlated and to determine their significance on drinking behaviors [22]. Even though these studies are very important for better understanding high-risk drinking, they are very expensive, difficult to conduct, and they have major gaps. For example, the impact of the interaction between the group network and
the individual motivations in the dynamic environment has not been measured or modeled. The social interactions play an important role during the drinking activity [13, 22, 65], and studying them is critical to understanding the etiology of high-risk alcohol consumption. There is a need to develop methodologies to not only identify the most important factors that affect the alcohol-consumption patterns, but also to determine how they influence the dynamics of the group throughout the drinking event. This is done here.

This need has led to an increasing interest in developing dynamical system models as tools to complement empirical research that addresses not only alcohol-related problems but also public health issues in general [75, 96]. Models of dynamical systems provide a way to analyze comprehensively the problem settings, to develop more effective intervention designs and evaluation methods, and to plan large-scale field studies. As part of the research in public health that works to develop strategies for the effective design of interventions that reduce high-risk alcohol consumption, dynamical models have been proposed to characterize how drinking patterns are affected by the social interactions in large groups [77, 84]. These models describe how the alcohol usage in large populations changes between categories such as “heavy-drinkers,” “social-drinkers,” and “non-drinkers.” The formulation of these models is closely related to the ones employed to describe the dynamics of infectious diseases that spread in a population. Also, several simulation methodologies have been designed to recreate drinking scenarios [36, 30]. However, to our knowledge, building a model that describes how the BAC level in a group changes over time during the drinking event and permits analysis not only through simulations but also at the
mathematical level for a comprehensive understanding of the modeled behaviors has not been studied yet.

Here, we construct, based on current observations and empirical data on drinking groups, a model of a system that characterizes how the dynamics of the social interactions, individual characteristics, and environment translate into changes in the drinking patterns of individuals measured through the BAC level. We derive a formulation of the model and a mathematical analysis of the behaviors that can be characterized, and show how this model could complement empirical research by informing theory and testing constructs. In Figure 4.1, we present a conceptual diagram of how the theoretical analysis of drinking groups interact with the empirical research and mathematical and computational system models.

![Figure 4.1: Conceptual diagram describing the interplay between theory, empirical evidence, and mathematical and computational models in the study of the dynamics of social systems. In that spirit, this work uses previously collected field data on drinking, theoretical assumptions of group behavior, and a model of a dynamical system along with computer simulations to advance our understanding of the etiological ecology of drinking events.](image)

In the theoretical analysis of groups in social psychology, the behavior of the group is assumed to be influenced by the environment and the mutual interactions between the group members. This is described by Lewin, who is considered the father of social psychology, through the formula $B = f(P, H)$, where the individual’s behavior
$B$ is a function $f$ of the personal characteristics and preferences $P$, and external influences that include the environment and other people $H$ [70][32, p 17]. Our previous field studies on drinking events are consistent with this description of group dynamics. These studies have shown that the personal preferences on drinking is not the only mechanism that drives the behavior of an individual in a drinking group. For example, using a portal design [124], we studied the group drinking behavior in bars [20]. It was observed that college students usually drink in settings that vary in risk and protective factors from heavy drinking [22, 114]. Patrons were interviewed and breath alcohol samples were taken upon entering and exiting the bar. When bar patrons are asked their intended level of intoxication when entering a bar, their level of intoxication often failed to match their previously stated intentions (e.g., to get very drunk, etc.) once they exited the bar. Figure 4.2 shows the measured Blood Alcohol Content (BAC) using a breathalyzer when entering and exiting the bar for four different categories of the intended intoxication level. Although the average BAC does tend to change across categories, it is clear that the BAC that is reported when exiting the bar does not always match the intended level of intoxication. These observations suggest that the behavior of individuals in a group during the drinking event is affected by additional basic mechanisms. We want to propose then a model that describes the dynamics of groups in drinking environments that is consistent with these observations and the theoretical analysis of group dynamics.

Current studies where data is collected in situ during the drinking events, although they are very informative, they fall short of describing the drinking patterns throughout the drinking event. For example, in the study reported in [20], the BAC
Figure 4.2: Measured BAC on subjects before entering the bar (horizontal axis) and after they leave it (vertical axis). Each dot corresponds to the BAC sample obtained from an individual, and the dashed line corresponds to a 45-degree line for reference. The desired level of alcohol that the subjects reported before entering the bar is categorized as (a) “not drinking,” (b) “not drinking enough to get buzzed,” (c) “slight buzz,” and (d) “drunk.”

level is measured only before and after the drinking event, but there is no information of how the BAC levels change in between. However, current technology has now made accessible new types of information that helps us overcome several of these limitations. It allows us to have a real-time monitoring of the BAC level [9], social interactions [18, 133], and location in space [24]. Our medium-term goal then is to design and conduct new studies where observational and self-report survey data are collected along with measurements that quantify the dynamics of the drinking group. The contribution of this work is to provide a mathematical model that, based on up-to-date observations and empirical data of drinking groups, we hypothesize captures the dynamics of the BAC level given the mechanisms that drive the behavior of the group, and can be updated once we collect data for this purpose. In the same way
conventional physics tries to explain how force translates into changes in motion, our aim is to create a model that explains the “physics” of the drinking event: how the influences from the individual’s personal preferences (e.g., desired effect of the BAC level on his/her body), other members of the group, and the environment translate into changes in the BAC level. We do not claim that this is a perfect model. This work is just a step in the cyclic process depicted in Figure 4.1: given current empirical evidence and theory of group dynamics, we propose a model that improves our understanding of the drinking event and allows us to design better methodologies for the collection of new evidence. This evidence will eventually be used to evaluate the model and make the corresponding improvements/changes on the hypotheses concerning the mechanisms that drive the event dynamics and the construction of intervention strategies.

The proposed model follows a well-developed mathematical framework used in engineering to study the dynamics of multiple interactive agents and their stability properties [34, 45, 40]. Under this framework, we are able to incorporate in a dynamical system the relationship between the BAC level and the individuals’ personal characteristics, the environment they are in, and the influence network between the members of the drinking group (Section 4.1). Using computer simulations and Lyapunov stability theory [63] we present a computational and mathematical analysis of how the parameters affect the modeled dynamics (Sections 4.2 and 4.3), and a discussion of the importance of these results in the further design of interventions and methodologies for the prevention of high-risk alcohol consumption. (Section 5.4).
4.1 Construction of The Model

Our goal is to model how an individual regulates his/her own Blood Alcohol Content (BAC) given the current state of the BAC and the influence of his/her personal motivations, other individuals, and the environment the individual is in. We explain step by step how the model is constructed, starting from the simpler case where the individual is only influenced by his/her own motivations, and extend it to the case where the group and the environment affect the dynamics of the individuals.

4.1.1 Individual Influences on Behavior

For the construction of the model, we quantify the behavior of an individual using his/her BAC level and its rate of change varying over time. Let $x_i(t) \geq 0$ denote the BAC level of individual $i$ at time $t \geq 0$, and let $v_i(t) \in \mathbb{R}$ denote its rate of change. We start by assuming that there is no environmental and group pressures, and that individual $i$ has a unique desired BAC level $x_i^*$. Research suggests that individual factors such as drinking motives (e.g., desired outcome and alcohol level) and drinking history (frequency of heavy drinking) influence the level of intoxication in the individual at the event-level [114, 102]. We assume therefore that $x_i^*$ is chosen by the individual in accordance with his/her personal motives and characteristics.

The model then should describe the behavior of an individual such that he/she tends to regulate his/her own BAC to reach the desired level. This means that if the actual BAC is below $x_i^*$, then the dynamics of the individual (e.g., amount of alcohol consumed) should be such that there is an increase in his/her BAC level. On the other hand, if the actual BAC is above $x_i^*$, then the dynamics of the individual should be such that there is a decrease his/her BAC level. An initial model that captures such
dynamics is shown in the causal loop diagram in Figure 4.3. In this model it is assumed that the individual regulates his/her change rate of the BAC proportional to the difference between the desired level and actual level. The constant equilibrium point of the dynamical system is \( x_i^* \). The differential equation associated with the diagram in Figure 4.3 is given by \( \dot{x}_i(t) = \eta_p^i (x_i^* - x(t)) \) where \( \dot{x}_i \) is the derivative of \( x_i \) with respect to \( t \), \( v_i(t) = \dot{x}_i(t) \), and \( \eta_p^i > 0 \) is the proportional action parameter, which determines how quickly the individual changes his/her BAC, and can be seen as the commitment of individual \( i \) to reach the desired level.

![Causal loop diagram of the dynamics of the BAC modeled using a first order system.](image-url)

**Figure 4.3:** Causal loop diagram of the dynamics of the BAC modeled using a first order system. The action proportional to error corresponds to the product of the difference in BAC and a scaling factor that determines how quickly the individual reaches the desired BAC level. In this model it is assumed that there are no group and environmental pressures. The causal links that have no sign are assumed to have positive polarity.

Even though this model captures the basic behavior of an individual that wants to reach his/her desired BAC level, it is restrictive with respect to the dynamics that can be represented. An example is the fact that the BAC trajectories modeled using the system in Figure 4.3 will never exhibit an overshoot with respect to the desired BAC level, a situation that is actually possible in drinkers who are not able to regulate accurately the alcohol consumed during the drinking event. A model formulation that generates a richer set of trajectories of the BAC level, including the possibility of modeling overshoot, is presented in Figure 4.4, which is an extension of the one in Figure 4.3 where now the dynamics of the individual are such that the difference
between the desired level and the actual BAC affects the acceleration of the BAC, that is, it directly affects how \( \dot{v}_i \) changes over time. According to the diagram in Figure 4.4, an individual tends to accelerate his/her BAC when the actual BAC is below the reference, and decelerate when it is above. Also, an individual is assumed to restrain the BAC acceleration depending on his/her perception on how quickly the BAC level is changing. This last component of the model is the one that shapes the trajectory, and allows the behavior of the BAC level to have overshoot or not. The set of differential equations that represents this dynamical system is given by

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \eta^p_i (x^*_i - x_i(t)) - \zeta_i v_i(t)
\end{align*}
\]

(4.1)

where \( \zeta_i > 0 \). Note that an individual accelerates or decelerates his/her BAC depending on the current value of \( v_i(t) \) and where the actual BAC level is with respect to the desired one. Parameter \( \eta^p_i \) determines how strong the commitment of the individual to reach the desired BAC level is (i.e., commitment to make \( x^*_i - x_i(t) = 0 \)), and parameter \( \zeta_i \) can be seen as the strength of the opposition of individual \( i \) to quick variations of the BAC level. Figure S1 in the Supplementary File shows instances of BAC trajectories that can be generated for different values of \( \eta^p_i \) and \( \zeta_i \), given \( x^*_i \).

Figure 4.4: Causal loop diagram of second order system that models the dynamics of the BAC level. It is assumed that there are no group and environmental pressures.
Although the dynamical system in Figure 4.4 with differential equation in (4.1) allows for modeling a wide variety of behaviors, the assumption that the individual has a unique desired BAC level is restrictive. It can be the case that the objective of an individual is to reach any BAC level within an interval that produces certain body reactions. For example, a person who wants to have a “slight buzz” effect will typically reach BAC levels that are between 0.02 and 0.04 [57, Ch 4]. To be able to generalize our formulation to include these cases in the model, we introduce the personal preference function. Let \( f^p_i : \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function that quantifies how preferable a given BAC is for individual \( i \). We assume that \( f^p_i \) has a unique minimum, and without loss of generality, \( \min_x f^p_i(x) = 0 \). Lower values of \( f^p_i(x_i) \) indicate more preferable BAC levels for the individual. For example, if individual \( i \) wants to get drunk, then higher values of \( x_i \) will correspond to lower values of \( f^p_i(x_i) \). The influence of an individual’s personal preference on his/her behavior is then given by the negative derivative of function \( f^p_i(x) \) with respect to \( x \), since it points to the direction where \( f^p_i(x) \) decreases. This means that \(-d f^p_i(x_i(t))/dx_i\) indicates whether the individual should increase or decrease his/her BAC in order to reach the desired level. It is implicit in the system in Figure 4.4 and Equation (4.1) that individual \( i \) has a function \( f^p_i(x_i(t)) = \frac{1}{2}(x_i(t) - x_i^*)^2 \), where Equation (4.1) can be rewritten as

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= -\eta^p_i \frac{df^p_i}{dx_i}(x_i(t)) - \zeta_i v_i(t)
\end{align*}
\]

In the specific case of the system in Equation (4.1), the negative derivative of the preference function with respect to \( x_i \) is given by \(-df^p_i(x_i(t))/dx_i = x_i^* - x_i(t)\), which corresponds to the difference in BAC part of the model. Note that in this case
has a unique minimum at $x_i^*$ (the most desirable BAC level, where the
derivative is zero) and is symmetric around this point. Instead of modeling the
influence of the individual on his/her own behavior to reach the desired BAC level
through the term $\eta_i^p(x_i^* - x_i(t))$, we model it in a more general way using the negative
derivative of the preference function, where now the goal is to reach a BAC level such
that $-df_i^p(x_i(t))/dx_i$ is zero.

Using the concept of preference function, we can take empirical evidence that
categorizes the effect of BAC level on an individual’s body to construct different
function profiles that describe how an individual tends to regulate his/her BAC to
reach the desired effects. Figure 4.5a shows the preference profiles for four categories
of alcohol intoxication: “not enough to get buzzed,” “slight buzz,” “drunk,” and “no
drinking.” The mathematical formulation of these functions is presented in Table S1
in the Supplementary File. Note that the minimum values of these profiles correspond
to the intervals of the BAC level that have been reported to produce the respective
effects. BAC levels in the interval $[0, 0.02]$ are associated with category “not enough
to get buzzed,” interval $[0.02, 0.04]$ with “slight buzz,” values greater than 0.04 with
“drunk,” and 0 with “no drinking” [57, Ch 4]. The slope of the functions is chosen
to be larger for values of the BAC that are before the interval where the function
is minimum than those that are after the interval, implying that the individuals are
more committed to increase their BAC to reach the desired effect than to decrease
it. Figure 4.5b shows example trajectories modeled using Equation 4.2 of individuals
whose preferred effect of the BAC level is “slight buzz.” All the trajectories converge
to the interval of the BAC level $[0.02, 0.04]$, instead of converging to a single BAC
level, due to the appropriate choice of the preference function.
Figure 4.5: (a) Preference functions associated with the categories of the body’s reaction to the BAC level: “not enough to get buzzed,” “slight buzz,” “drunk,” “No drinking.” The formulation of these functions is presented in Table S1 in the Supplementary File. (b) Trajectories of the BAC level for different initial conditions and values of $\eta^p_i$ and $\zeta_i$. In this case the individual’s desired effect is “slight buzz,” which implies that the final BAC level is within the interval $[0.02, 0.04]$, which is where the derivative of the corresponding preference function in (a) is zero.

The dynamical system in Equation (4.2) enables us to model a variety of trajectories that the BAC level of an individual can have in a drinking event. It has the flexibility to model the commitment of the individual to approach the desired levels, his/her own perception of the change rate to restrain quick changes on his BAC, and also through the personal preference function we can define intervals of desired levels that are associated with effects on the individual’s body. The dynamical system in Equation (4.2) can be seen from a physics perspective. As in Newton’s second law, there are “forces” acting on the individual’s acceleration. The concept of forces is not necessarily the one measured in Newtons, but it refers to influences that cause a change in the individual’s BAC level, which in this case correspond to the influence of the individual to reach the desired BAC levels and the influence that restrains the individual to quick changes in his/her BAC according to his/her perception of the
BAC rate of change. We will prove in Section 4.3.1 that this last component (perception of the BAC rate of change) is necessary in the model for the convergence of the individual’s BAC level to the desired interval.

4.1.2 Adding Social and Environmental Influences

In the study of human group dynamics it has been observed that the behavior of an individual is determined not only by his/her own personal characteristics but also by the influences of the group and environment he/she is in [70][32, p 17]. This has been already observed in the particular case where the group is in an alcohol-consumption environment. It has been shown that the social interactions influence the dynamics of the individuals, where the patterns of communication between group members and the strength of the influence of other members on an individual play an important role in shaping the behavior of the whole group [2, 123]. Also, data collected \textit{in situ} provide evidence that environmental factors have a significant effect on the dynamics of the drinking groups [128, 21]. For example, an environment where there are large crowds has an impact on the behavior such that there is a tendency to restrain increases in the BAC. On the other hand, environments with drinking games promote higher BAC levels.

In the same way the personal preference function is used to include in the model the influence of the individual on his/her own behavior to reach the desired BAC levels, we define functions to describe how the environment and the group affect the behavior of individual $i$. First, to model the influence of the group on an individual’s behavior, we consider evidence that suggests that in a group of people there are mutual attractions that lead the group toward consensus in drinking behavior [2][28].
This means that, in the context of our model, an individual is attracted to the BAC level and its change rate of other individuals. Since an individual might interact with only a subset of the group and the impact of those interactions can vary in strength between people, we need to formally define an influence network in the drinking group. Assume that the drinking group has \( n \) members. The structure of the network is given by \( G = (V, E, W) \), where \( V = \{1, \ldots, n\} \) is the set of labels for each individual in the group, and \( E \subset V \times V \) is the set of directed links that connect the individuals. Link \((i, j) \in E\) indicates that individual \( j \) influences individual \( i \). It might be the case where \((j, i) \in E\) does not exist, meaning that there is no influence from \( i \) to \( j \). Let \( N_i = \{j \in V : (i, j) \in E\} \) be the set of all the group members that have some influence on individual \( i \). Each link \((i, j)\) is associated with a weight \( w_{ij} > 0 \) that corresponds to the strength of the influence of \( j \) on \( i \). We have that \( W = \{w_{ij} : i \in V, j \in N_i\} \) is the set of all the weights associated with the links in \( E \).

The mathematical formulation of the social interactions in this context is in terms of attractions. We say that individual \( i \) is influenced by \( j \) if there is a tendency from \( i \) to follow the BAC level and/or its rate of change of individual \( j \). It is assumed that each individual is able to estimate the BAC level and its rate of change of the group members that influence him/her (i.e., those that are in the set \( N_i \)). The function associated with the group influence is formulated as

\[
f^g_i(x_i(t), v_i(t)) = \frac{1}{2} \sum_{j \in N_i} w_{ij} \left[ (x_i(t) - x_j(t))^2 + b(v_i(t) - v_j(t))^2 \right]
\]

where \( b \geq 0 \) is a parameter associated with the influence on the rate of change of the individual’s BAC level. This function is minimized when individual \( i \) tends to follow his/her neighbors’ BAC level and its rate of change. Larger values for \( w_{ij} \) imply a larger attraction of individual \( i \) to follow \( j \). The influence component associated with
the social pressures will be then given by the negative derivative of $f^g_i$ with respect to both $x_i$ and $v_i$

$$a_i(t) = -\frac{df^g_i}{dx_i}(x_i(t)) - \frac{df^g_i}{dv_i}(v_i(t))$$

$$= -\sum_{j \in N_i} w_{ij} [(x_i(t) - x_j(t)) + b(v_i(t) - v_j(t))]$$

(4.3)

Term $-w_{ij} [(x_i(t) - x_j(t)) + b(v_i(t) - v_j(t))]$ is a value that points to the direction where $x_j(t)$ and $v_j(t)$ are with respect to $x_i(t)$ and $v_i(t)$, and describes the influence of $j$ on the behavior of $i$. This part of the model is the one that allows us to characterize the drinking group as an interconnection of dynamical systems, that is, a dynamical social system.

To define the influence of the environment on the group’s drinking behavior, we define a continuously differentiable function $f^e : \mathbb{R} \rightarrow \mathbb{R}$ that is associated with the environment the group is in. The environment promotes BAC levels where the function $f^e$ takes lower values. For example, an environment where there are large crowds will have a function profile with lower values at lower BAC levels. The influence of the environment on individual $i$ is given by the scaled negative derivative of this function

$$h_i(t) = -\eta^e_i \frac{f^e(x_i(t))}{dx_i}$$

(4.4)

where $\eta^e_i \geq 0$ is the strength of the influence of the environment on individual $i$. Equation (4.4) characterizes the environmental pressures that affect the behavior of the drinking group.

Using the results in Equations (4.3) and (4.4) along with Equation (4.2) we construct a model that characterizes how influences of the group, environment, and individual preferences on the BAC effects translate into changes in a person’s BAC level. To complete the model, we assume that the action of these three components...
on behavior is additive. The complete model is illustrated in Figure 4.6 and defined in the set of equations

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= p_i(t) + a_i(t) + h_i(t) + d_i(t)
\end{align*}
\] (4.5)

where \( p_i(t) = -\eta_i^P df_i^P(x_i(t))/dx_i \), and \( d_i(t) = -\zeta_i v_i(t) \).

Figure 4.6: Causal loop diagram of the final model that accounts for the effect of the individual characteristics, group, and environment on changes in the BAC level.

4.2 Simulation Results

The mathematical formulation of the group dynamics in (4.5) allows us to characterize behaviors that have been observed in human groups involved in alcohol-consumption activities. Next, through simulations, we show the dynamics that can result for different choices of the desired effect of the alcohol intoxication and the structure of the influence network in the drinking group. Also, we study data from real drinking groups that were collected in situ. We use this information to inform our model and make predictions about the behavior of the group given some assumptions on the parameters of the model.
4.2.1 Group Dynamics Under Different Conditions on The Personal Preferences and Influence Networks

In the first set of simulations, we study the dynamics of a group of six people in three different scenarios. Individuals 1, 2, and 3 have “slight buzz” as the preferred effect of the BAC level, meaning that these individuals prefer values of \( x \) within the interval \([0.02, 0.04]\). On the other hand, “Not drinking” is the preference for individuals 4, 5, and 6, which means that they prefer having values of \( x \) that tend to zero. The profiles of the preference functions for these two subgroups are shown in Figure 4.5a. The implementation details and source code of the simulations that we present in this Section are given in Sections C and F of the Supplementary File.

In the first case, the group has a network structure shown in Figure 4.7a. Each arrow indicates the direction of the influence between a pair of individuals, and its thickness is proportional to the strength of the influence (i.e., \( w_{ij} \) in Equation (4.3)). The only interaction between subgroups occurs between 3 and 4. The influence of 3 on 4 is stronger than the influence in any other pair of individuals. There is no influence of 4 on 3. Figure 4.7b shows the dynamics of the group given the initial conditions and model parameters. Note that the individuals whose preference is “no drinking” initially tend to have lower values of the BAC level even though some of them start at high values. People in the group whose preferred effect of the BAC level is “slight buzz” tend to increase their BAC. However, the strong influence of 3 on 4 makes 4 change his/her behavior in a way that his/her BAC tends to reach BAC levels in the interval that produces the “slight buzz” effect. The mutual influences of 4 on 5 and 6 are not strong enough to change significantly their behavior.
Figure 4.7: (a) Influence network in a group where individuals 1, 2, and 3 have “slight buzz” as desired effect of the BAC level while individuals 4, 5, 6 prefer being in the category “No drinking.” (b) Trajectories of the simulated BAC level for each one of the individuals in the group.

In the second case, individuals 5 and 6 influence each other, as shown in Figure 4.8a. Now, the influence of 4 on 5 and 6 is stronger than the one in the first case in Figure 4.7a. The trajectories of the BAC through time in Figure 4.8b show that individuals 4, 5, and 6 are affected by the influence of individual 3 on 4, where their BAC tends to increase even though their personal preference is “No drinking.”

### 4.2.2 Estimation of Group Dynamics From Field Data

The information in [20] contains field data from 1024 people surveyed at 30 different bars. Data collected from each subject include the BAC measurements before entering the bar and after leaving it, duration time in the bar, whether the subject is alone or not, the amount of money available to spend on food, and the amount of money available to spend on alcohol. The subjects reported the level of alcohol intoxication that they desired to reach during the drinking activity by choosing one of the following categories: “not drinking,” “not drinking enough to get buzzed,”
Figure 4.8: (a) Influence network in a group where individuals 1, 2, and 3 have “slight buzz” as preferred effect of the BAC level, and individuals 4, 5, 6 prefer “No drinking.” The difference of this network with respect to the one in Figure 4.7a is that now individuals 5 and 6 influence each other, and the influence of 4 on 5 and 6 is stronger. (b) Trajectories of the simulated BAC for each one of the individuals in the group where the influence network is given in (a).

“slight buzz,” and “drunk.” In addition to this information, there is observational data that report how crowded the bar was, and whether the subjects were exposed to drinking games and alcohol and food specials. Despite of the large amount of data collected from drinking events at different places, the only available information about the individual’s drinking patterns is the BAC measured at the entrance and exit of the bar. We show in this section how we can have a tentative estimation of the trajectory of the BAC level during the drinking event given the available field data and some assumptions on the dynamics of the group.

First, we define the personal preference and environment functions. Since the subjects reported the desired effect of the BAC level that they wanted to reach during the drinking event, we use the preference profiles shown in Figure 4.5a. The environment function is constructed as the linear combination of influences that either protect against or favor increases in the alcohol level. The amount of money available
to spend on food, whether food specials are offered or not, and how crowded the bar was, is information used to construct the component of the environment profile that promotes lower alcohol levels. On the other hand, the presence of drinking games, the amount of money to spend on alcohol, and alcohol specials represent information useful to construct the component of the environment profile that promotes higher BAC. The perception of the BAC rate of change \( \zeta_i \) in Equation (4.5) is assumed to be the same for all the individuals.

The only information available about the social interaction is whether the subject is alone or not. Since there are no reported observations of the social interactions between subjects, we assume in our model that there exists a non-observed social component that allows the subject to achieve the reported final BAC level, given the preferred effect of the BAC level, environment profile, initial BAC level, and the duration time of the drinking activity. Assuming that parameter \( b = 0 \), for those that reported that were not alone during the drinking event Equation (4.3) can be rewritten as

\[
a_i(t) = -\bar{w}_i(x_i(t) - \bar{x}_i(t))
\]

where \( \bar{w}_i = \sum_{j \in N_i} w_{ij} \) is the total influence strength acting on \( i \), and \( \bar{x}_i = \frac{1}{\bar{w}_i} \sum_{j \in N_i} w_{ij} x_j \) is the weighted average of the BAC level of those that influence individual \( i \). Here, we assume that \( \bar{x}_i \) points to the reported final BAC level. Parameter \( \bar{w}_i > 0 \) is computed to be the smallest scalar such that the trajectory of the BAC level generated using Equation (4.3) along with (4.6) and the previous assumptions reaches the measured final BAC level during the reported time of the drinking activity and starting at the measured initial BAC level. Low values of \( \bar{w}_i \) do not imply that there was not social interactions during the drinking event. It implies that, under the assumptions
on the parameters of the model and the reported information, the unobserved social pressures are not significantly stronger than other influences to modify the drinking patterns of the individual. The details of the implementation, including the construction of the environment profile, can be found in Section D of the Supplementary File.

Figure 4.9: (a) Estimated trajectories of the BAC provided by the model for ten randomly selected subjects that reported “drunk” as the preferred level of alcohol intoxication, where symbol ‘x’ marks the BAC measured when the subject exited the bar. (b) Samples of the BAC level when entering the bar (horizontal axis) and after leaving it (vertical axis). The dashed line is a 45-degree line for reference. The thickness of the lines in (a) and the size of the markers in (b) are proportional to the strength of the social influence acting on the subjects.

Figure 4.9 shows the results obtained using our model following the methodology and assumptions described above on the data collected in situ from individuals that reported “drunk” as desired effect of BAC level to reach during the drinking event. Using the corresponding preference function shown in Figure 4.5a, the environment function constructed from the information in the field data, the initial and final BAC, and the time spent in the bar, we show in Figure 4.9a the estimated trajectories followed by ten randomly selected subjects during the drinking activity (solid
lines), and the estimated strength of the social influence on the subjects (thickness of the lines), which is proportional to $\bar{w}_i$ in Equation (4.6). Note that the thickest line corresponds to the trajectory of a subject that maintains a relatively low level of alcohol with respect to his/her preference of getting drunk. This person reported that he/she did not get food, the bar was not crowded, there were not food specials, and there were alcohol specials and drinking games. Under the assumptions on the parameters of the model and the reported information, the dynamics estimated from the model suggest that there were strong social pressures on the subject that influenced his/her behavior.

Also, we show in Figure 4.9b a plot of the BAC when the subjects exited the bar versus the BAC before they entered it. The size of the markers is proportional to the estimated strength of the social influence on the individuals. Note that individuals who maintain lower levels of the BAC tend to have larger social pressures acting on them than those that have larger levels of the BAC. These results are consistent with the fact that this set of subjects had “drunk” as the desired level of intoxication to reach while they were in the bar, hence additional pressures on the behavior are required to maintain lower values of the BAC. The sample pointed by the arrow corresponds to an individual that was not exposed to drinking games and had little money to spend in alcohol. In this case, the results given by the model suggest that the social pressures were not significantly larger than the other influences to change the dynamics of the individual.
4.3 Mathematical Analysis of The Group Dynamics

In the previous Section we provided computer simulations to show the dynamics that the proposed model is able to describe and that are consistent with the observations of groups that are in a alcohol-consumption environment for a given set of parameters and initial conditions. Now, through a mathematical analysis of the model, we study how the model parameters affect all possible trajectories of the BAC level that can be described by the system in Equation (4.5) given any initial condition. This analysis is based on Lyapunov stability theory [63], which equips us with mathematical tools to derive statements on the dynamics of the group given specific parameter settings. First, we study the case when the influence strength of the social interactions and the environment is zero and the behavior of the individual is driven mainly by his/her desire to reach a specific effect of the BAC level. Theorem 4 shows that each individual reaches the desired effect in the drinking activity when there is no influence of other members of the group and the environment. Then, through Theorem 5, we show how the dynamics of the group are affected when there are social interactions and also environmental pressures on the group. We also discuss the implications of these mathematical results.

4.3.1 No Social and Environmental Pressures

In this analysis, we use the concept of asymptotic stability to show that when there are no social and environmental pressures, the individuals achieve their personal preferences in the long term. We start by assuming that $f^p_i$ (personal preference function of individual $i$) is a strictly convex function that has a unique minimum at $x^*_i$, for $i = 1, \ldots, n$. Without loss of generality, $f^p_i(x^*_i) = 0$ for every $i \in \{1, \ldots, n\}$. 
Let $\tilde{x}_i = x_i - x^*_i$ be the relative BAC level of individual $i$ with respect to his/her desired BAC level. Assume that there is no influence from the other members of the group and the environment on the individual. Then, the dynamics of $i$ can be written as

$$\dot{\tilde{x}}_i = v_i$$

$$\dot{v}_i = -\eta^p_i \frac{df^p_i}{dx_i}(\tilde{x}_i + x^*_i) - \zeta_i v_i, \quad i = 1, \ldots, n$$

(4.7)

The following theorem shows that an individual modulates his/her own behavior such that his/her BAC level reaches the desired one when there is no pressures from the social interactions and the environment the group is in, and that the concept of perception of the BAC rate of change affecting the BAC level has to be included in the model to guarantee convergence of the trajectories of the BAC level.

**Theorem 4.** Consider the dynamics of the individuals in the group characterized in Equation (4.7). The point $\tilde{x}_i = 0$ and $v_i = 0$ is a globally asymptotically stable equilibrium.

**Proof.** Let $V_i(\tilde{x}_i, v_i) = \eta^p_i f^p_i(\tilde{x}_i + x^*_i) + \frac{1}{2} |v_i|^2$. From the assumptions that $f^p_i(x^*_i) = 0$ and that $f^p_i$ is strictly convex and has a unique minimum at $x^*_i$, we have that $f^p_i(\tilde{x}_i + x^*_i) > 0$ for all $\tilde{x}_i \neq 0$. Since $|v_i|^2 > 0$ for all $v_i \neq 0$, it implies that $V_i(\tilde{x}_i, v_i) = 0$ only when $\tilde{x}_i = 0$ and $v_i = 0$, and $V_i(\tilde{x}_i, v_i) > 0$ otherwise. We use then $V_i$ as our Lyapunov function candidate. Its derivative with respect to time is

$$\dot{V}_i(\tilde{x}_i, v_i) = \eta^p_i \frac{df^p_i}{dx_i}(\tilde{x}_i + x^*_i) \dot{\tilde{x}}_i + \frac{1}{2} v_i \dot{v}_i$$

$$= -\zeta_i |v_i|^2 \leq 0$$

$\dot{V}_i(\tilde{x}_i, v_i)$ is negative semidefinite. However, from Equation (4.7), we can see that if $v_i(t) = 0$ for a given $t$, variable $\dot{v}_i(t)$ is not zero unless $\tilde{x}_i(t) = 0$. This is consistent
with LaSalle’s invariance principle [63, Theorem 4.4]. Let \( S = \{ \tilde{x}, v : \dot{V}(\tilde{x}, v) = 0 \} \). From (4.7), we have that no solution can stay identically in \( S \) other than \( \tilde{x}_i = 0 \) and \( v_i = 0 \). Therefore, from LaSalle’s invariance principle, the solution \( \tilde{x}_i = 0 \) and \( v_i = 0 \) is globally asymptotically stable.

\[ \Box \]

Remark 8. Since there is no effect of the environment on the group and there is no coupling between its members, the force that drives the individual’s behavior is his/her own preferences. Hence, the dynamics of an individual in the drinking group modeled in Equation (4.7) are the result of an optimization process driven by the cost function \( f^p_i \).

Remark 9. The fact that \( \zeta_i \) is greater than zero guarantees that the individual approaches the point \( \tilde{x}_i = 0 \) and \( v_i = 0 \) as \( t \) tends to infinity. The term \(-\zeta_i v_i\) in Equation (4.7) causes a gradual deceleration of the trajectories of \( \tilde{x}_i \) as \( \tilde{x}_i \) approaches zero, i.e., when \( x_i \) approaches \( x^*_i \).

### 4.3.2 Influence of The Environment and Social Interactions on the Group Dynamics

To examine the influence of the environmental and social pressures on the dynamics of the drinking group, we now think of the group as a dynamical system that is a perturbation of the “nominal” system given in Equation (4.7), where the perturbations are given by the components of the force associated with the environment and social interactions in Equations (4.3) and (4.4). In this case, we study the dynamics of the system through the concept of uniform ultimate boundedness [63, Ch 4.8], which allows us to characterize how the resultant dynamics of the BAC level in the group members deviate from their desired level in terms of the model parameters.
Assume that $f^e$ (environment function) is a strictly convex function and has a unique minimum $x_e \in \mathbb{R}$, and that its derivative is Lipschitz continuous, that is, for any $z,y \in \mathbb{R}$ there exists a constant $M^e > 0$ such that $|\frac{df^e}{dx}(z) - \frac{df^e}{dx}(y)| \leq M^e|z - y|$. Also, as in Section 4.3.1, we assume that $f^p_i$ is strictly convex and has a unique minimum at $x^*_i$ with $f^p_i(x^*_i) = 0$, and that its derivative is also Lipschitz continuous with constant $M^p_i > 0$, for every $i = 1, \ldots, n$. Let $x = [x_1, \ldots, x_n]^\top$, $x^* = [x^*_1, \ldots, x^*_n]^\top$, $\tilde{x} = [\tilde{x}_1, \ldots, \tilde{x}_n]^\top$, and $v = [v_1, \ldots, v_n]^\top$. Also, let

$$F_p(x) = \left[\eta^p_1 \frac{df^p_1}{dx}(x_1), \ldots, \eta^p_n \frac{df^p_n}{dx}(x_n)\right]^\top$$

$$F_e(x) = \left[\eta^e_1 \frac{df^e}{dx}(x_1), \ldots, \eta^e_n \frac{df^e}{dx}(x_n)\right]^\top$$

We define matrix $L$ such that its entry at the $ij$ position is given by

$$[L]_{ij} = \begin{cases} 0 & \text{if } i \neq j \text{ and } j \notin N_i \\ -w_{ij} & \text{if } i \neq j \text{ and } j \in N_i \\ \sum_{l \in N_i} w_{il} & \text{if } i = j \end{cases}$$

where $w_{ij} > 0$ is the strength of the influence of $j$ on $i$ defined in Equation (4.3). This matrix corresponds to the Laplacian of the graph $\mathcal{G}$ that defines the influence network in the drinking group. Then, the model of the drinking group in Equation (4.5) can be written as

$$\dot{\tilde{x}} = v$$

$$\dot{v} = -L(\tilde{x} + x^*) - bLv - F_p(\tilde{x} + x^*) - F_e(\tilde{x} + x^*) - Dv$$

where $D$ is a $n \times n$ diagonal matrix defined as $D = \text{diag}(\zeta_1, \ldots, \zeta_n)$. The following theorem shows how the social and environmental pressures affect the behavior of each individual in a way that it deviates from their personal preferences.
Theorem 5. Let the social pressures in the drinking group characterized in Equation (4.8) be such that the strength of the attractions satisfy

\[ b \sum_{j \in N_i} w_{ij} + 2\zeta_i > b \sum_{j \in \Gamma_i} w_{ji}, \quad i = 1, \ldots, n \]  

(4.9)

where \( \Gamma_i = \{ j \in V : i \in N_j \} \) is the set of all members of the group that are influenced by individual \( i \). Then, the trajectories of the social system in (4.8) are uniformly ultimately bounded with ultimate bound \( \gamma \) given by

\[ \gamma = \beta \max_i \left[ \frac{1}{\delta_i \sqrt{2n}} \sum_{j \in N_i} w_{ij} |x_i^* - x_j^*| + \eta_i^e M^e |x_e - x_i^*| \right] \]

(4.10)

where \( \delta_i = \max \left[ c\theta_i, \sum_{j \in \Gamma_i} w_{ji} + \eta_i^e M^e \right] \), and constants \( \beta > 0, M^e > 0, \) and \( \theta_i \in (0, 1) \) for \( i = 1, \ldots, n \). Constant \( c \) is the minimum eigenvalue of the matrix \( D + b/2(L+L^\top) \).

Proof. Let \( L^o \) be a \( n \times n \) matrix whose \( ij \)-th entry is given by

\[ [L^o]_{ij} = \begin{cases} 0 & \text{if } i \neq j \text{ and } j \notin \Gamma_i \\ -w_{ji} & \text{if } i \neq j \text{ and } j \in \Gamma_i \\ \sum_{l \in \Gamma_i} w_{li} & \text{if } i = j \end{cases} \]

This corresponds to the column Laplacian matrix of the graph that defines the structure of the influence network \( G \). Let

\[ V(\tilde{x}, v) = \frac{1}{2} \tilde{x}^\top (L + L^o) \tilde{x} + \sum_{i=1}^n \eta_i^e f_i^p(\tilde{x}_i + x_i^*) + \frac{1}{2} v^\top v \]

(4.11)

From the assumption that \( f_i^p \) is strictly convex and has a unique minimum at \( x_i^* \) and \( f_i^p(x_i^*) = 0 \) for all \( i = 1, \ldots, n \), we have that \( \sum_{i=1}^n \eta_i^e f_i^p(\tilde{x}_i + x_i^*) \) is positive definite with respect to \( \tilde{x} \). Also, \( v^\top v \) is positive definite with respect \( v \). From [132, Lemma 5], we have that \( \tilde{x}^\top (L + L^o) \tilde{x} = \sum_{i=1}^n \sum_{j \in N_i} w_{ij} |\tilde{x}_j - \tilde{x}_i|^2 \) is positive semidefinite with respect to \( \tilde{x} \). Hence, \( V(\tilde{x}, v) = 0 \) only when \( \tilde{x} = 0 \) and \( v = 0 \), and \( V(\tilde{x}, v) > 0 \)
otherwise. We select then $V$ as our Lyapunov function candidate. The derivative of $V$ with respect to time is

$$
\dot{V}(\bar{x}, v) = -v^T (D + bL)v + v^T (L^o \bar{x} - Lx^*)
$$

$$
- \sum_{i=1}^{n} v_i^T \eta_i^e \frac{df^e}{dx}(\bar{x}_i + \bar{x}_i^*)
$$

$$
= -v^T (D + bL)v - \sum_{i=1}^{n} v_i^T \left[ \sum_{j \in N_i} w_{ij}(x_i^* - x_j^*) + \sum_{j \in \Gamma_i} w_{ji} (\bar{x}_i - \bar{x}_j) + \eta_i^e \frac{df^e}{dx}(\bar{x}_i + x_i^*) \right]
$$

(4.12)

In Equation (4.12), term $v^T (D + bL)v$ is positive definite with respect to $v$ if and only if the eigenvalues of the matrix $D + b/2(L + L^T)$ are positive. From Gershgorin’s circle theorem [43, Theorem 7.2.1], every eigenvalue $\lambda$ of this matrix is in the region

$$
|\lambda - b \sum_{j \in N_i} w_{ij} - \zeta_i| \leq b \left[ \sum_{j \in N_i} w_{ij} + \sum_{j \in \Gamma_i} w_{ji} \right]
$$

From the assumption (4.9), we have that all the eigenvalues of this matrix are positive.

Let $c = \lambda_{\min}[D + b/2(L + L^T)]$ be the minimum eigenvalue of the matrix $D + b/2(L + L^T)$. Then, using the fact that $v^T (D + bL)v = v^T (D + b/2(L + L^T)v$, and from Rayleigh quotient [66, Theorem 10.13], we have that $-v^T (D + bL)v \leq -c \sum_{i=1}^{n} |v_i|^2$.

Also, since $\frac{df^e}{dx}$ is assumed to be Lipschitz continuous, we know that there exists a constant $M^e > 0$ such that $|\frac{df^e}{dx}(\bar{x}_i + x_e) - \frac{df^e}{dx}(x_e)| = |\frac{df^e}{dx}(\bar{x}_i + x_e)| \leq M^e |\bar{x}_i - x_e^*|$, where $x_e^* = x_e - x_i^*$. Using this result in (4.12), we obtain

$$
\dot{V}(\bar{x}, v) \leq \sum_{i=1}^{n} \left[ -c |v_i|^2 + |v_i| \left( \sum_{j \in N_i} w_{ij} |x_i^* - x_j^*| 
+ \sum_{j \in \Gamma_i} w_{ji} |\bar{x}_i - \bar{x}_j| + \eta_i^e M^e |\bar{x}_i - x_e^*| \right) \right]
$$

(4.13)
For a constant $\theta_i \in (0, 1)$, we have that $-c|v_i|^2 = -(1 - \theta_i)|v_i|^2 - c\theta_i|v_i|^2$. Then, we can rewrite the inequality in (4.13) as

$$\dot{V}(\tilde{x}, v) \leq \sum_{i=1}^{n} -c(1 - \theta_i)|v_i|^2$$

for all

$$c\theta_i|v_i| - \left( \sum_{j \in \Gamma_i} w_{ji} + \eta_i^e M^e \right) |\tilde{x}_i| - \sum_{j \in \Gamma_i} w_{ji} |\tilde{x}_j| \geq \sum_{j \in N_i} w_{ij} |x_i^* - x_j^*| + \eta_i^e M^e |x_e - x_i^*|, \quad i = 1 \ldots, n$$

Equation (4.14) can be written in a compact yet more conservative way

$$\dot{V}(\tilde{x}, v) \leq \sum_{i=1}^{n} -c(1 - \theta_i)|v_i|^2$$

for all $\|[\tilde{x}, v]^\top\| \geq \mu$ (4.15)

where

$$\mu = \max_i \left[ \frac{1}{\delta_i \sqrt{2n}} \sum_{j \in N_i} w_{ij} |x_i^* - x_j^*| + \eta_i^e M^e |x_e - x_i^*| \right]$$

(4.16)

and

$$\delta_i = \max \left[ c\theta_i, \sum_{j \in \Gamma_i} w_{ji} + \eta_i^e M^e \right]$$

This expression was obtained using the norm inequality [63, p 648]

$$\sum_{i=1}^{n} |\tilde{x}_i| + |v_i| = \| [\tilde{x}^\top, v^\top]^\top \|_1 \leq \sqrt{2n} \| [\tilde{x}^\top, v^\top]^\top \|_2$$

Equation (4.15) indicates that $\dot{V}$ is negative semidefinite for all $\| [\tilde{x}^\top, v^\top]^\top \|_2 \geq \mu$.

However, similar to the proof of Theorem 4, from Equation (4.8) we know that if $v(t) = 0$ for a given $t$, variable $\dot{v}(t)$ is not zero unless additional conditions on both $\tilde{x}(t)$ and $v(t)$ are satisfied. Hence, $\dot{V}$ is negative as long as the bound in Equation (4.15) is satisfied. Thus, the trajectories of the social system in (4.8) are uniformly ultimately bounded. From now on, unless it is indicated, the vector norm $\| \cdot \|$ corresponds to the $L_2$ vector norm.
To compute the ultimate bound, we need to find strictly increasing functions $\alpha_1$ and $\alpha_2$, with $\alpha_1(0) = 0$ and $\alpha_2(0) = 0$, such that

$$\alpha_1 \left( \left\| [\tilde{x}^T, v^T]^T \right\| \right) \leq V(\tilde{x}, v) \leq \alpha_2 \left( \left\| [\tilde{x}^T, v^T]^T \right\| \right)$$

According to [63, Theorem 4.18], the ultimate bound will be given by $\gamma = \alpha_1^{-1}(\alpha_2(\mu))$.

From Equation (4.11), we know that $V(\tilde{x}, v)$ satisfies

$$V(\tilde{x}, v) \geq \frac{1}{2} \lambda_{\min} [L + L^0] \| \tilde{x} \|^2 + \frac{1}{2} \| v \|^2$$

$$\geq \beta_1 \left[ \| \tilde{x} \|^2 + \| v \|^2 \right] = \alpha_1 \left( \left\| [\tilde{x}^T, v^T]^T \right\| \right)$$

(4.17)

where $\beta_1 = \min \left\{ \frac{1}{2} \lambda_{\min} [L + L^0], \frac{1}{2} \right\}$.

To find the upper bound of $V(\tilde{x}, v)$, we use the assumption of Lipschitz continuity on $\frac{df^p_i}{dx}$ for $i = 1, \ldots, n$. Here, from [11, Proposition A.24], we have that there exists a constant $M^p_i > 0$ such that $f^p_i$ satisfies $f^p_i(\tilde{x}_i + x_i^*) \leq M^p_i \| \tilde{x}_i \|^2 / 2$. Using this property we obtain

$$V(\tilde{x}, v) \leq \frac{1}{2} \lambda_{\max} [L + L^0] \| \tilde{x} \|^2 + \sum_{i=1}^{n} \eta_i^p M^p_i \| \tilde{x}_i \|^2 + \frac{1}{2} \| v \|^2$$

(4.18)

Equation (4.18) then can be bounded as

$$V(\tilde{x}, v) \leq \sum_{i=1}^{n} \beta_{2i} \left[ \| \tilde{x}_i \|^2 + \| v_i \|^2 \right]$$

$$\leq \beta_2 \left\| [\tilde{x}^T, v^T]^T \right\|^2 = \alpha_2 \left( \left\| [\tilde{x}^T, v^T]^T \right\| \right)$$

(4.19)

where

$$\beta_{2i} = \max \left[ \frac{1}{2} \lambda_{\max} [L + L^0], \frac{\eta_i^p M_i^p}{2}, \frac{1}{2} \right]$$

and

$$\beta_2 = \max_{i=1,\ldots,n} \beta_{2i}$$
From Equations (4.17) and (4.19), we can obtain an expression for the ultimate bound

\[ \gamma = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\alpha_2(\mu)}{\beta_1}} = \sqrt{\frac{\beta_2}{\beta_1}} \mu = \beta \mu \]

where \( \mu \) is defined in Equation (4.16), and \( \beta = \sqrt{\beta_2/\beta_1} \).

**Remark 10.** Theorem 5 indicates that the social interactions and the environment add dissipative elements in the dynamics that make the individuals deviate from their personal preferences. Note that the ultimate bound in (4.10) mainly depends on the social influence and environment parameters.

**Remark 11.** If the ultimate bound \( \gamma \) defined in (4.10) is zero, it means that every individual follows his/her personal preferences. There are two cases when this situation happens. First, when there is no social interaction and no influence of the environment (i.e., \( w_{ij} = 0, \eta^p_i = 0 \) for all \( i \in V, \) and \( j \in N_i \)), which is the case described in Theorem 4. The other case is when \( x_i^* = x_j^* \), and \( x_i^* = x_e \), for all \( i \in V, j \in N_i \). It means that, given the initial conditions of the variables \( x \) and \( v \), in the long term all the individuals will reach their desired BAC level even though there are some force components that pressure the individuals to have the same BAC and its rate of change during the drinking activity.

**Remark 12.** The ultimate bound in (4.10) tends to zero as the strength of the social interactions and/or the difference between the desired positions of the individuals decrease.

**Remark 13.** In Theorem 5, Lipschitz continuity in the gradient of the personal preference and environment functions is a reasonable assumption that implies some convenient properties for the functions that allow us to obtain results that are easily interpretable.
Remark 14. Note that there are no assumptions on the connectivity of the structure of the influence network $\mathcal{G}$. The only related assumption is the one in (4.9), which guarantees that, in the long term, the rate of change in BAC level of the individuals $v$ will be zero. There are special cases when this assumption can be satisfied: when the network $\mathcal{G}$ is balanced, that is, $\sum_{j \in \mathcal{N}_i} w_{ij} = \sum_{j \in \mathcal{L}_i} w_{ji}$ for all $i \in \mathcal{V}$ (i.e., the total strength of the influence of individuals in the group on $i$ equals the total strength of the influence of individual $i$ on others); and when there are no attractions on the rate of change in BAC in the group, that is, $b = 0$.

The results given in Theorem 4 suggest that there must exist an influence acting on the individual’s behavior that depends on his/her perception on how quickly his/her own BAC level is changing and modulates the dynamics of the BAC accordingly. Also, the ultimate bound provided in Theorem 5 shows that the compound action of the environment (with strength proportional to $\eta_i^e$) and social influences (with strength proportional to the $w_{ij}$) can have a significant impact on the BAC level trajectories at the individual level, as expected. It is important to note that the design of interventions has focused mainly on affecting the conditions in the environment through individual incentives to prevent problems like heavy alcohol consumption. However, as shown in the ultimate bound from Theorem 5, an individual’s drinking behavior can be impacted by having either a large influence coming from the environment, or small social influences that add up together. This observation suggests that intervention designs at the social level could lead to promising outcomes in terms of prevention of unhealthy drinking behaviors. For example, a recent study has shown that using social pressures instead of individual incentives to increase physical activity levels in a community provided significantly better results [96, Ch 4].
4.4 Conclusion

Methodologically, this chapter illustrates how field data and computational and mathematical modeling complement each other. In our original field studies, we did not have the ability to directly measure or model the influence of groups on drinking behavior. The combination of these approaches represents a way to maximize the data collected in large studies. Further, measuring dynamical processes using traditional social science methods is often not possible or extremely difficult [82]. As part of the cyclic process in Figure 4.1, our model along with the mathematical analysis and simulations presented above will help us refine our future field studies, especially as they relate to the interplay between individual, group, and environment and the relationship to alcohol intoxication. Also, as recent technological advances improve our ability to collect real-time data, we will better inform the empirical specification of the proposed model [74].

Theoretically, through the model presented above and its analysis through stability theory and simulations, we hope to inform our understanding of group dynamics as they relate to drinking behavior. Traditional social psychological models (e.g., Lewin [70]) have given us the foundation from which to build more sophisticated complex dynamical models. Understanding group influence in the context of environment, network relationships, and individual preference – taking into account for the influence of psychoactive substances – affords a richer etiological understanding of real world phenomena like drinking behavior. In turn, understanding the influences at different levels, how environment can moderate personal in-group influence for example, may ultimately help guide applied preventive solutions to problems like heavy alcohol consumption and the problems that flow from that consumption.
4.4.1 Future Directions

On the modeling side of our work, our next steps include developing models that incorporate additional information about the individual, such as gender or weight. In this way, the model can be coupled with Wegner’s equations of alcohol content [125] to characterize the trajectories of the BAC given the number and type of drinks per time unit. On the empirical side of our work, we are designing a real-time data collection process at drinking events that includes measurements of variables at the individual, group, and environmental level [9, 18, 24]. Our aim is to study these data using tools from both statistics and dynamical system theory [63]. We plan to do system identification [73] to find the parameters that allow the model to have the closest approximation to the measured behaviors. Depending on the results, we will validate and improve our hypotheses on the mechanisms that drive behavior during the drinking event. We hope that this subsequent round of model and field validations will have contributed to our understanding sufficient to engage in a series of interventions at the event level.
Human cooperation is the process of people acting collectively toward a common end. People that decide to cooperate usually have to pay costs associated with their individual contribution during this process, but this results in them gaining greater benefits, ones that result from collective coordination and action [101]. A social dilemma is the situation that arises when the individual interests are not aligned with the collective ones. The costs and problems involved in cooperation can make individuals in this dilemma behave in a way that they opt not to participate in the collective action. In this case, “individual rationality leads to collective irrationality” [64, p 183]. It is very important then to understand the conditions that cause social dilemmas, and how to motivate people to cooperate and enjoy the benefits that result from their collective action.

Many important situations have the characteristics of a social dilemma. For example, the situation where individuals can decide to commute by either automobile or public transportation. Commuting by automobile might provide individual benefits that include shorter travel times and flexibility. However, as more individuals choose this option, consequences such as air pollution and traffic congestion increase as well, negatively affecting the whole community [122]. The social dilemma arises
when the individual interests are in favor of maximizing their own immediate benefits, such as travel convenience, leading to decisions that are detrimental to the community. Similar situations occur in other domains and scales, such as household energy consumption and conservation [33], provision in the welfare state [106], and the relation between our attitude toward nature and dangerous climate change [81, 53]. The common pattern that characterizes these cases is that the shortsighted actions of the individuals prevent the entire community from obtaining long-term benefits.

In this work, we study those situations where the individuals in a community have the choice to contribute toward the completion of a common task. The process of task completion might involve only costs and no real benefits to the community members while they are participating. However, the completion of the task results in changes that substantially benefit the community and exceeds the participation efforts. An interesting situation that conceptually exemplifies this is “barn raising” [61, Ch 6]. This practice entails building a barn that will be either owned by an individual or shared by the community. The individuals that decide to collaborate do not get any payment, and the process of completing the task can be time consuming and requires significant effort. The result of completing the task, that in this case is building the barn, includes the functional benefits provided by the barn in the community and the strengthening of the social bonds between the community members. We do not focus on studying the development of trust as the result of the iterated completion of tasks, as has been studied before [14]. We focus our analysis on the process of promoting participation and engaging people in the community in completing the task.

Even though the results of completing the task can be highly desirable, self-interested individuals easily turn this situation into a social dilemma: the short-term
costs associated with participation can dissipate any motivation to contribute to the completion of the task and therefore to obtain the benefits that it provides in the long run. We call this situation a \textit{task completion social dilemma}. Some of the basic features that characterize the task completion problem are:

- The community does not get benefits until the task is completed.

- Each individual knows the current state of the task.

- The task is completed by the continued contributions from the individuals in the community.

- Each individual pays participation costs during the process of task completion.

This problem is similar to the public goods problem \cite{91, 79} in that individuals are able contribute an amount of resources to generate a collective good for the benefit of the whole community by paying costs associated with their individual contribution. However, the facts that the benefits of the collective good will be available only when the task has been completed, and that the community knows the current progress of the task that is being completed, are characteristics that, taken together, differentiate the task completion problem from the other types of problems studied in the context of social dilemmas.

Due to their importance, a large body of research has been conducted to understand different aspects of cooperation in social dilemmas. Work in several disciplines ranging from evolutionary biology \cite{6, 87} and ecology \cite{80, 19, 112} to social phycology \cite{91, 79, 117, 120} and political sciences \cite{92, 26}, has provided models and theoretical insights that try to explain the conflict between the individual and the group interests,
and the mechanisms that allow for the emergence of cooperation. Inspired by some of
these developments, we present in this paper a mathematical model that captures the
relationship between the task to be completed and the contribution provided by the
individuals in the community. We propose basic mechanisms that promote partici-
pation and cooperation during the task completion process. This relationship, and
the ways to promote an active participation by the community members, represent a
complex problem [61, 50, 94], where the different factors to be considered can grow
considerably in number. However, our work focuses on those elements that we have
identified to be key to describe and study the task completion social dilemma and
the conditions that are suitable for the emergence of cooperation.

Different mathematical tools have been employed to study cooperation, including
concepts commonly used in game theory [6, 46, 17], theory of dynamical systems
[29, 40], and optimization [79, 95]. Due to the nature of the problem, we charac-
terize and analyze the process of task completion as a discrete dynamical system.
In this model, the task is completed by the iterated contribution of the community
members to the task (Section 5.1.1). We refer to participation load as the degree of
involvement of the individual in the task. Then, the contribution of an individual
depends on the participation load that he/she takes and a production function. The
concept of production function has been previously used in the analysis of collective
action and common goods to describe the relationship between the resources provided
by the individual and the amount of collective good that is generated by those re-
sources [79, Ch 4][46]. In our problem, the production function is used to model the
relationship between the participation load and the contribution of this load to the
completion of the task. The dynamics of participation (Section 5.2) are modeled to
include mechanisms that promote participation in a way that the agents share their participation load and costs between them. We model the community so that the individuals are able to communicate between each other via an interaction network. It has been shown that communication is a mechanism that is crucial for the emergence of cooperation [62, 64, 7]. First, in our dynamical model, communication allows an agent to build its relative reputation (Section 5.1.3), defined as the reputation of an agent from the viewpoint of another one, as a mean to promote participation in the community through reciprocity [80, 88]. Second, the ability to locally interact enables the agents to dynamically distribute the participation load between them based on the individual costs of participation and the potential benefit that each agent takes into consideration from completing the task (Section 5.1.2).

The mathematical model that we propose can be seen as an extension of the work in [79] in the context of collective goods. The authors in [79] model the gain of a community member based on the benefits obtained by the current level of the collective good and the costs associated with the individual contributions, and study the effect of different forms of production functions on the generation of the collective good. Although their work is seminal in the study of cooperation and social dilemmas, they recognize that their models are essentially static and that there is a need for the development of dynamical models that build on their work [79, p. 190]. In our work, we formulate a dynamical system that characterizes the evolution of the task completion process and participation dynamics, allowing us to study mechanisms that promote participation and cooperation in the community. We use concepts of stability analysis of discrete nonlinear systems and Monte Carlo simulations to analyze the qualitative and quantitative behavior of the modeled community (Section 5.3). The
paper closes with a discussion on the utility of these results and future developments of the model (Section 5.4).

5.1 Formulation of the Model

5.1.1 Task Completion Dynamics

From now on we refer to the individuals in the community as agents. The amount of participation load taken by an agent to solve the task is denoted by $p_i \in \mathbb{R}_{\geq 0} = [0, \infty)$. We assume that the task to be completed allows the community to take a participation load up to $P > 0$. It means that the agents cannot take an arbitrary participation load, but it must be distributed so that

$$\sum_{i=0}^{n} p_i \leq P$$

where $n \geq 1$ is the number of agents.

The contribution of the all agents toward the completion of the task given their participation load $p = [p_1, \ldots, p_n]^\top$ is characterized by the production function $h(p) : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}$. Let $z \geq 0$ be the variable that quantifies the task completion level. A value of $z = 0$ indicates that the task has not been started, and $z = \bar{z} > 0$ indicates that the task has been completed. The dynamics of task completion describe how $z$ gradually changes depending on the agents’ participation patterns, and their contribution described by the production function. We characterize the dynamics of task completion via

$$z(t+1) = z(t) + \phi h(p(t)) (\bar{z} - z(t)) \quad (5.1)$$

where $z(t)$ and $p(t)$ are the task completion and participation load variables at time step $t \in \mathbb{N}$, and $\phi \in (0,1]$ is a parameter that scales the action of the production function.
function on the completion of the task. Note that the production function \( h(p) \) drives the rate of change of the task variable. According to this equation, the task is at an *equilibrium point* when either the agents’ participation pattern is such that there is no contribution to complete the task, or the task has been already completed. In other words, we have that \( z(t+1) = z(t) \) when either \( h(p(t)) = 0 \) or \( z(t) = \bar{z} \).

Different forms of the production function can be used to capture different situations in the relationship between the participation load and their contribution to the task completion. In general, we assume that the production function: (i) is the result of the additive contribution of each agent, (ii) is monotonic increasing with respect to each agent’s participation load, and (iii) satisfies \( h(0) = 0 \). This means that the contribution to the completion of the task increases as the agents increase their participation, and there is no contribution if no agent in the community takes any participation load. These assumptions on the production function imply that \( h(p(t)) \geq 0 \) for all \( t \geq 0 \).

In the context of the public goods problem, several types of production functions have been proposed to describe the relationship between the number of agents cooperating in the production of the public goods and the amount of public goods that are produced [79, Ch 4][46, 64]. In the task completion problem, we introduce production functions with a structure that can be considered an extension of the ones presented in the problem of providing public goods, where we characterize the relationship between each agent’s participation load and their contribution to the completion of the task. We introduce the family of functions defined by

\[
h(p(t)) = \sum_{i=1}^{n} \beta_i p_i^\alpha_i(t) \tag{5.2}
\]
Figure 5.1: Production function in Equation (5.2) when \( n = 2, \alpha = \alpha_1 = \alpha_2 = \alpha_3 \in \{0.3, 1, 3\} \) and \( \beta = \beta_1 = \beta_2 = \beta_3 = 0.5 \).

where \( \beta_i \geq 0 \) and \( \alpha_i > 0 \) are parameters associated with agent \( i \). This family of functions describes the collective action of the community as the additive contribution of each agent’s participation load to the task completion. When \( \alpha_i \in (0, 1) \), the contribution of each agent increases quickly as its participation load increases, especially for those values closer to zero. On the other hand, when \( \alpha_i > 0 \), the agent’s contribution to the task completion has a significant effect on the task completion when the participation load is higher. A linear relation between the agent’s contribution to task completion and its participation load is produced when \( \alpha_i = 1 \). FIG. 5.1 shows an example of these three cases.
Convergence Analysis

We provide some sufficient conditions on the parameters of the model that guarantee the completion of the task, and show through simulations the qualitative properties of the task convergence for different values of the parameters of the production function and participation load variables.

The next theorem states sufficient conditions on $\phi$ and $h$ so that $z$ converges to $\bar{z}$. The proof of this theorem is in the supplemental material.

**Theorem 6.** Let $0 \leq z(0) \leq \bar{z}$. Assume that $h(p) \in [0, 1)$ for all $p \in [0, P]^n$. Also, assume that $\phi h(p(t))$ is such that, for all $t \geq 0$,

$$h(p(t)) |\bar{z} - z(t)| \geq \psi(|\bar{z} - z(t)|)$$

(5.3)

where $\psi$ is a strictly increasing function that satisfies $\psi(0) = 0$. Then, $z(t)$ in Equation (5.1) satisfies $0 \leq z(t) \leq \bar{z}$ for all $t \geq 0$ and converges to $\bar{z}$.

**Corollary 6.1.** Assume that $h(p(t))$ is fixed for all $t \geq 0$. Then, $z$ converges to $\bar{z}$ exponentially, and its solution trajectory is

$$z(t) = \bar{z} - [1 - \phi h(p(t))]^t (\bar{z} - z(0))$$

(5.4)

The assumption that $h(p) \in [0, 1)$ does not make the formulation less general. Since the agents’ participation load $p(t)$ is bounded for every $t \geq 0$, the range of the production function is bounded below by zero and above by $P$, and therefore it is generally easy to formulate a production function that satisfies $h(p(t)) \in [0, 1)$ for all $t \geq 0$. In general, the assumption in Equation (5.3) implies that a task that is modeled using Equation (5.1) can be completed when the agents behave such that $h(p(t)) > 0$ for all $t \geq 0$. However, interestingly, this assumption also implies that the
task can be also completed if the participation variables are functions of $|\bar{z} - z(t)|$. This means that the participation load taken by an agent can decrease proportional to the difference between the final and current states of the task, and the task will be eventually completed. Later, we will use this fact to design a participation strategy that guarantees the completion of the task and minimizes costs of participation.

From Corollary 6.1, we have that if the participation variables are assumed to be fixed for all $t$, the task completion variable will decrease exponentially with a convergence rate $1 - \phi h(p(t))$. The task is completed when the expression $([1 - \phi h(p(t))]^t$ is zero. Next, we analyze the effect of the production function and the participation load on the convergence of the task. Assume that the agents take a fixed participation load $p(t) = p^* \in [0, P]^n$ for all $t \geq 0$, with $\sum_{i=1}^{n} p^*_i \leq P$, and let $\sigma = [1 - \phi h(p^*)]^t$, where $1 - \sigma$ is the proportion of the task that has been already completed after $t^*$ iterations. Then, the minimum number of iterations taken to have a proportion $1 - \sigma$ of the task being completed is given by

$$t^* = \left\lceil \frac{\log(1 - \sigma)}{\log (1 - \phi h(p^*))} \right\rceil$$

(5.5)

where $\lceil \cdot \rceil$ denotes the ceiling function. As expected, the minimum number of time steps required to complete a proportion $1 - \sigma$ of the task decreases as the contribution provided by the agents increases. Figures 5.2 and 5.3 show the minimum number of iterations $t^*$ needed to complete 95% of the task, i.e., $\sigma = 0.05$. We choose $\phi = 0.1$, $P = 1$, and the production function in (5.2) with $\beta_i = 1/n$ for all $i = 1, \ldots, n$. Figure 5.2 shows the contour plot of $t^*$ as a function of $p^*_i$ and $\alpha_i$ in Equation (5.2). Lower values of $\alpha_i$ are associated with smaller convergence times $t^*$. For this specific production function, agents provide a more significant contribution for low participation loads when $\alpha_i$ is smaller. Figure 5.3 shows the behavior of the
convergence time $t^*$ as a function of $p_1^*$ and $p_2^*$ when $\alpha = \alpha_1 = \alpha_2 \in \{0.1, 1, 5\}$, where $p_1^* + p_2^* \leq 1$. When $\alpha = 1$, the contribution of each agent is proportional to its participation load. When $\alpha = 0.1$, the agents can decrease their participation load and keep the same convergence time as long as both agents share the participate load during the process of task completion. The opposite situation occurs when $\alpha = 5$. The convergence rate will increase if both agents decide to participate and share the participation load.

![Contour plot of the convergence time $t^*$ in Equation (5.5) as a function of the production function parameter $\alpha$ and the participation load $p_i^*$, when only one agent participates in the task completion process.](image)

Figure 5.2: Contour plot of the convergence time $t^*$ in Equation (5.5) as a function of the production function parameter $\alpha$ and the participation load $p_i^*$, when only one agent participates in the task completion process.

### 5.1.2 Potential Benefits and Participation Costs

In the task completion problem, the agents obtain a benefit from the task once the task has been completed, and have to pay a cost proportional to their participation load if they want to contribute to the task completion. Therefore, an agent that decides to participate has to have the ability to see the potential benefit of solving the task. Hence, similar to the concept of potential energy in physics, we define the
potential benefit $b(t) \geq 0$ associated with the task as the benefit in terms of the task’s current state relative to the completed state of the task at time step $t$. If $c_i > 0$ is the cost associated with participation for agent $i$, then the net value to an agent that has a participation load $p_i(t)$ of the task completion action is its “net gain”

$$\gamma_i b(t) - c_i p_i(t)$$

where $\gamma_i \in [0, \bar{\gamma}]$ is the initiative of the agent $i$ to participate in the completion of the task, with $\bar{\gamma} > 0$. Expression $c_i p_i(t)$ is the cost of taking a participation load $p_i(t)$. Parameter $\gamma_i$ quantifies the ability of agent $i$ to take into consideration the potential benefit of the task. If $\gamma_i = 0$, it means that the agent has no interest in, or is unable to foresee, the benefits associated with task completion. The net gain is nonnegative when the initiative is greater or equal the cost-to-benefit ratio of the participatory action at a given time step $t$, i.e., $\gamma_i \geq c_i p_i(t)/b(t)$. Note that Equation (5.6) has a similar structure to the net gain presented in the context of collective good models (Equation (1) in [79]).

An agent whose strategy seeks to maximize the net gain will choose a participation load $p_i(t) = 0$ for all $t \geq 0$, and therefore there will be no contribution to complete
the task. This state where no agent participates is called a deficient equilibrium. “It is deficient in that there is at least one other outcome in which everyone is better off, and it is an equilibrium in that no one has an incentive to change their behavior” [64, p 184]. Later in Section 5.2 we will present strategies that allow the agents to promote participation, and avoid the deficient equilibrium.

5.1.3 Communication and Relative Reputation

We assume that the $n$ agents in the community are able to communicate across a network whose topology is represented by an undirected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ is the set of nodes and $E \subset V \times V$ is the set of edges. Each node is associated with an agent, and each edge is associated with the interaction between two agents. Edge $(i, j)$ indicates that agents $i$ and $j$ share information about their participation and net gain. The agents that interact with agent $i$ are its neighbors $N_i = \{j \in V : (i, j) \in E\}$.

According to Equation (5.6), the net value to an agent of the participatory action depends on the agent’s initiative to complete the task. An agent with no initiative will not take any participation load since it does not take into consideration the potential benefit of completing the task and therefore any participation load will incur only costs. A mechanism that can help promote participation in the community is a form of indirect reciprocity [88]. An agent can be encouraged to consider the potential benefit of completing a task if its neighbors also participate in the task completion process. In order to model this mechanism, we first introduce the concept of relative reputation. Let $j \in N_i$. We define the relative reputation of agent $j$ from agent $i$’s viewpoint as the agent $i$’s recognition of agent $j$’s involvement in completing the task.
The process of $j$ building a reputation from $i$’s viewpoint can be characterized using the update rule:

$$r_{ij}(t + 1) = (1 - \lambda_{ij}) r_{ij}(t) + \lambda_{ij} \frac{p_j(t)}{P} \bar{r} \tag{5.7}$$

where $\lambda_{ij} \in [0, 1]$ is the rate of change in reputation, and $\bar{r} \geq 0$ is the upper bound of reputation. A value of $\lambda_{ij}$ close to one implies that the reputation is highly influenced by the participation load, and develops according to it. Assuming that $r_{ij}(0) \in [0, \bar{r}]$, $\lambda_{ij} > 0$, and that the participation load $p_j(t) = p^*_j$ is fixed for all $t \geq 0$, we have that

$$\lim_{t \to \infty} r_{ij}(t) = \bar{r} \frac{p^*_j}{P}$$

When $\lambda_{ij} = 1$, the relative reputation reaches its maximum value $\bar{r} p^*_j / P$ at one iteration. On the other hand, when $\lambda_{ij} = 0$, the relative reputation does not change and remains at its initial value $r_{ij}(0)$. The term $(1 - \lambda_{ij}) r_{ij}(t)$ in Equation (5.7) can be seen as a “forgetting factor:” if the current participation load taken by agent $j$ is greater than zero, agent $i$ forgets past participation patterns and adjusts $j$’s reputation accordingly. If the participation load taken by agent $j$ is zero, then its reputation with respect to $i$’ viewpoint will decrease and reach zero asymptotically.

The reciprocity of agent $i$ to participate in the task is then quantified as the linear combination of the reputation of its neighbors, i.e., for $i = 1, \ldots, n$,

$$r_i(t) = \sum_{j \in N_i} \delta_{ij} r_{ij}(t),$$

with $\delta_{ij} \in [0, 1]$. If $\delta_{ij} = 0$ ($> 0$), then $i$ does not (does, respectively) consider $j$ for reciprocal participatory actions. Reciprocity acts as a catalyst for participation. An agent takes into consideration the potential benefit of completing the task based not only on its initiative but also on its reciprocity to its neighbors’ actions. The net
gain to an agent that has a participation load $p_i(t)$ of the task completion action in Equation (5.6) can be generalized to include reciprocity as

$$g_i(t) = (\gamma_i + r_i(t)) b(t) - c_i p_i(t)$$  (5.8)

Recall that the community of agents can take a total participation load up to $P$. Next, we will show how communication also allows the agents to collaborate by distributing the available participation load $P$ between them based on their individual participation costs and the potential benefits that they perceive from task completion.

### 5.2 Collective Action

Instead of trying to maximize their individual gain, which leads to a social dilemma, a strategy that promotes cooperation is one where the agents seek to distribute the available participation load such that the relation between costs and benefits is the same for all the agents. Agents with higher initiative and reciprocity will be willing to assume higher costs of participation, which implies taking larger participation loads than those with lower initiative and reciprocity. Figure 5.4 illustrates this idea. It shows an example of the net gain for three different agents as function of their participation variable. We want to define a mechanism that tries to find a distribution of the participation loads $p_1^*$, $p_2^*$, and $p_3^*$ such that the loads are as large as possible, (i.e., find a $g^* \geq 0$ as small as possible) such that $g^* = g_1 = g_2 = g_3$ and $\sum_{i=1}^{3} p_i^* \leq P$.

Assuming that the agents are able to communicate across the network, we design a cooperation strategy that locally distributes the available participation load. Each agent only knows the information from its neighbors, and the community acts in a
way that the agents share the participation load and seek to balance their individual net gain between them during the process.

Let $Q(t) \in [0, P]$ be such that $\sum_{i=1}^{n} p_i(t) = P - Q(t) \leq P$. This “slack variable” quantifies the participation load that the agents have not taken from the total $P$. If $Q(t) = 0$, it means that the agents are taking all the participation load $P$, and $Q(t) = P$ means that none of the agents is participating to complete the task. We define the distribution dynamics of participation such that each agent can take (pass) an amount of load from (to) one of its neighbors. Also, the agents are allowed to take load from or pass load to the $Q(t)$. In this case, $Q(t)$ can be seen as a variable associated with an additional node in the network. Let $q$ be the label of this additional node. These dynamics are characterized by

$$p_i(t+1) = p_i(t) - L_i^{j^*}(t) + \sum_{\{l: i \in N_l\} \cup \{q\}} L_i^l(t), \quad (5.9)$$

where $L_i^{j^*}(t)$ denotes the participation load that agent $i$ is passing to $j^*$. Subscript $j^*$ refers either to an agent in $N_i$ or to $q$. The latter means that agent $i$ is releasing some
of the participation load that it currently holds. Also, $L^i_q(t)$ denotes the participation load that agent $i$ is taking from the available load $Q(t)$, and $L^i_l(t)$ denotes the participation load that $i$ receives from agent $l$, with $i \in N_l$. We formulate the participation dynamics in Equation (5.9) so that the agents tend to equalize their individual net gain based on information from their neighbors. An agent that has neighbors with larger gains will pass some amount of participation load to them, since they are more willing to participate in the task. Also, the participation dynamics in Equation (5.9) are designed so that the participation load taken by the agents is as large as possible for each agent. Agents will participate in a way that their net gain tends to zero, that is, such that the benefit that they foresee compensates the costs of their participation.

A cooperation policy that has these characteristics can be summarized by the following rules:

1. If agent $i$ and all its neighbors have a nonpositive net gain, then agent $i$ releases a participation load proportional to its gain, i.e., passes it to $Q(t)$ ($L^i_q(t)$ in Equation (5.9)).

2. Agents with the greatest gain with respect to their neighbors can request a participation load from $Q(t)$ proportional to their net gain ($L^i_l(t)$ in Equation (5.9)).

3. If agent $i$ has at least one neighbor $j$ whose net gain is greater than $i$’s gain, then agent $i$ passes a participation load to $j$ proportional to the difference in gains ($L^i_j(t)$ in Equation (5.9)) to some of these neighbors.

4. An agent cannot pass a participation load that is greater than the one that it currently has.
From rule 1, each agent can release an amount of participation load and make it available to other agents, and rule 2 indicates that the agents can take an amount of participation load that has not been taken by any other agent. Rule 3 makes the agents try to equalize their net gain, and rule 4 guarantees that each agent passes at most the participation load that it currently has.

To provide a mathematical formulation of this set of rules, we first define the sets $S_i$ and $S_q$ as

$$S_i = \{ j \in N_i : [g_j(p_j(t))]_+ > [g_i(p_i(t))]_+, \\
[g_j(p_j(t))]_+ \geq [g_i(p_i(t))]_+, \forall l \in N_i \}$$

$$S_q = \{ i \in V : [g_i(p_i(t))]_+ > 0, \\
[g_i(p_i(t))]_+ \geq [g_j(p_j(t))]_+, \forall j \in N_i \}.$$

where $[\cdot]_+ = \max\{\cdot, 0\}$. The set $S_i$ contains agent $i$’s neighbors with net gain larger than $i$’s net gain and is largest net gain among $i$’s neighbors. The set $S_q$ contains all the agents that have the largest positive net gain with respect to the one from its neighbors. These two sets identify those agents who need to take a participation load either from other agents ($S_i$) or from $q$ ($S_q$) so that their net gain decreases. Below, we gives the algorithm with the mathematical description of the steps that the agents follow to compute the participation loads to be passed at each time.

**Algorithm: Distribution of participation load.** Given the interaction network $G$, the current participation load $p(t)$, the available load $Q(t)$, the load passing rate $\theta$, and the agents’ net gain, the algorithm to compute $L^j_i(t)$, $L^q_i(t)$, and $L^L_i(t)$ in Equation (5.9) is:

$$121$$
1: **Input:** $G$, $p(t)$, $Q(t)$, $\theta \in (0, 1]$, and $g_i$ for all $i \in V$.

2: **Output:** $L^i_j(t)$ for all $i \in V$ and $j \in N_i$, and $L^q_j(t)$ for all $j \in V$.

3: Let all the passing loads $L^i_j(t) = 0$ and $L^q_j(t) = 0$ for all $i, j$, and $q$.

4: **for each** $i \in V$ **do**

5:  

6:  

7:  

8:  **else**

9:  

10:  **end if**

11: **end for**

12: Choose one $k^* \in \mathcal{S}_q$ if possible

13: 

14: 

15: $L^i_j(t) = \min\{\tilde{L}^i_j(t), p_i(t)\}$, for all $i \in V$

16: $L^q_k(t) = \min\{\tilde{L}^q_k(t), Q(t)\}$

Note that the net gain decreases linearly with the participation load. If an agent and all its neighbors have a nonpositive net gain, then, from lines 5 to 7 in the algorithm, it can release an amount of participation load (rule 1). If $\mathcal{S}_i$ in Equation (5.10) is not empty, then it means that agent $i$ can share participation load with a neighbor and try to equalize their net gain. This action is stated in lines 9 and 10 in the algorithm (rule 3). From lines 13 and 14 in the algorithm, if $\mathcal{S}_q$ is not empty, it means that
an agent can take participation load from $Q(t)$ (rule 2). Lines 15 and 16 state that an agent cannot pass more than its current participation load, or take more from the available one (rule 4).

Note that, in the distribution algorithm, the agents only interact with its neighbors. Each agent knows the current net gain and the costs of participation of its neighbors. The next theorem states that if the benefits in the net gain $g_i$ are assumed to be fixed, under the proper conditions, this distribution algorithm along with Equation (5.9) lead to a distribution of the participation load such that the net gains of the agents are equalized.

**Theorem 7.** Let $b(t) = b$ and $r_i(t) = r_i$ in Equation (5.8) be fixed for all $t \geq 0$. Assume that the load passing rate satisfies $\theta \in (0, 1]$ and

\[
\frac{n \left( \max_i \{\gamma_i\} + n\bar{r} \right) b}{\min_i \{c_i\}} \leq P \tag{5.12}
\]

Let $Q(t), p_i(0) \geq 0$ be such that $Q(0) + \sum_{i=1}^{n} p_i(0) = P$, and $p_i(0) \leq b(\gamma_i + r_i)/c_i$ for all $i \in V$. Let

$$\mathcal{P}_c = \{ p(t) \in \mathcal{P} : g_i(p_i(t)) = 0, \text{ for all } (i, j) \in E \},$$

where $\mathcal{P} = \{ p(t) \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^{n} p_i(t) \leq P \}$. Assume that the network is connected. Then, the participation load after the iterative computation of the distribution algorithm and Equation (5.9) is such that $\mathcal{P}_c$ is invariant and exponentially stable in the large.

This theorem states that the agents that have the ability to communicate will tend to equalize their net gains. If all the assumptions are satisfied, their net gain will be zero. This implies that those agents that are more willing to participate will
eventually take more participation load in the task completion process. Figure 5.5 shows an example of the trajectories of $p_i$ and $g_i$ throughout the iterative process. The three agents interact according to a network with a line topology (one agent with two neighbors, the other two agents have one neighbor). The total participation load is $P = 1$, and the load passing load is $\theta = 0.4$. As they interact locally according to the network and pass the participation load, the net gain for all the agents tends to equalize during the process and go to zero as the participation load taken by the agents increases. Note that the participation load satisfies $\sum_{i=3}^{n_i} p_i(t) \leq P$ for all $t \geq 0$.

![Figure 5.5: Example of the net gain balancing between three agents that interact according to a line topology. The net gain (top plot) and the participation load (bottom plot) for each one of the agents is shown throughout the iterative process in Equation (5.9) and the load distribution algorithm. The Matlab code is in the supplemental material.](image)

Now, in order to study the conditions that guarantee the completion of tasks when the agents distribute their participation load following the algorithm above, we have to define the behavior of $b(t)$. As we mentioned before, similar to the concept of
potential energy in physics, \( b(t) \) is the benefit in terms of the task’s current state relative to the completed state of the task at time step \( t \). We will show in Theorem 8 sufficient conditions on \( b \), as a function of the difference between \( \bar{z} \) and \( z(t) \), so that the convergence of \( z(t) \) to \( \bar{z} \) is guaranteed.

**Theorem 8.** Assume that at least one agent has initiative \( \gamma_i \) different from zero. Let \( p_i(0) = 0 \) for all \( i \in V \), and let the production function be such that \( h(p) \in [0,1) \) for all \( p \in [0,P]^n \). Assume that the agents distribute their participation load following Equation (5.9) and the algorithm for participation load distribution. Let the potential benefit \( b \) be a function of \( \bar{z} - z(t) \) such that either, for all \( 0 \leq z \leq \bar{z} \),

(i) \( b \) is a function bounded below and above by positive numbers, or

(ii) \( b \) is an increasing and non-negative function.

Then, \( z(t) \) in Equation (5.1) converges to \( \bar{z} \).

In this theorem, the conditions on the potential benefit are a consequence of the result in Theorem 6. There is a large variety of families of functions that satisfy these conditions. For example, if during the task completion process, participation is considered less influential as the task gets closer to be completed, then a potential benefit function that can capture this behavior is an increasing function with respect to \( \bar{z} - z \). This type of function is useful in cases when the accumulated costs of participation are sought to be minimized. On the other hand, if the agents’ perception of the importance of participation is the same during all the task completion process, then a potential benefit that is constant for all \( t \) can characterize this behavior. From Theorem 8, the task is guaranteed to be completed in both examples. However,
the dynamics of interaction and the time that the task takes to converge could vary depending on the choice of the potential benefit function.

5.3 Analysis of Participation Dynamics

The participation dynamics in the community depend on the potential benefit function $b(t)$. In this section, we assume that participation is perceived as less indispensable when the task gets closer to be completed. A simple function that can characterize this behavior and guarantees that the task is completed is $b(t) = \bar{z} - z(t)$. According to this function, the potential benefit decreases as $z(t)$ gets closer to $\bar{z}$, and it is zero when the task has been completed. In this way, we can observe the dynamics that result from the interaction between the load distribution algorithm, reciprocity, production function, and topology of the network when the potential benefit changes during the process.

We study the dynamics of the task completion problem in two simulation scenarios. First, we take a scenario with five agents and a specific selection of the model parameters and topology of the interaction network (Section 5.3.1). We show how the participation, task completion, net gain, and relative reputation variables change over time. Second, using Monte Carlo simulations, we show the behavior of the community for different production functions and topologies when some of the model parameters are randomly sampled (Section 5.3.2). The details of implementation are provided in the Supplemental Material.

5.3.1 Participation Patterns in the Community

We illustrate the evolution of the variables involved in the task completion dynamics for a specific example. The community has $n = 5$ agents who communicate
following the network shown in Figure 5.6a. In this example, agent 1 was the only one that had an initiative different from zero, and agent 3 had the highest cost of participation. The task completion dynamics involved the production function in Equation (5.2) with $\alpha_i = 0.25$ and $\beta_i = 1/P^{\alpha_i}$, for all $i = 1, \ldots, 5$. The total participation load was $P = 2$, and all the agents started with a participation load and reputation of zero. We assumed that all the neighbors contributed equally to the reciprocity of each agent.

Figure 5.6b shows that during the first iterations of the participatory process, agent 1 increases its participation load, taking almost all the available load $P$. Since its initiative $\gamma_i$ is different from zero, agent 1 takes into consideration the potential benefit of completing the task. Since agent 2 communicates with 1, its reciprocity toward participation develops as well as its participation load. Note that the task completion variable increases its growth rate when agent 2 starts participating (at around $t = 7$). Even though 1 shares its load with 2, from the choice of the production function, we have that two agents whose participation load sum up $\bar{p}$ are more productive than only one agent with the same load $\bar{p}$ (since, in general, $(a + b)^\alpha \leq a^\alpha + b^\alpha$, for $a, b > 0$ and $\alpha \in (0, 1)$). Also, agent 3 increases its reciprocity. However, it initially does not participate since its benefits from reciprocity are not high enough and its costs of participation are larger than the costs of agent 1, who is its neighbor. As the task is being completed, the participation load taken by the agents decreases proportionally. Once the net gain of the agents is equalized at zero, agent 3 takes a small participation load during the last stage of the task completion process. Agents 4 and 5 take a participation load close to zero since the relative reputation of 3 is very low during most of the observed time interval.
Figure 5.6: (a) Topology of the interaction network, where each node is associated with an agent, and each link denotes interaction between two agents. (b) Trajectories of the agents’ participation load variables $p_i(t)$, net gain $g_i(t)$, and the task completion variable throughout the iterative process.

Figure 5.7 shows the evolution of the relative reciprocity of the agents during the task completion process. The relative reputation $r_{ij}(t)$ in Equation (5.7) (reputation of $j$ from $i$’s point of view) is represented as the edge of a graph that connects nodes $i$ and $j$ at time step $t$ in the direction $j \rightarrow i$, and whose thickness is proportional to $r_{ij}(t)$. At time $t = 0$ none of the agents has developed any reputation (Figure 5.7a). Note that agent 1 builds its reputation from the viewpoint of agents 2 and 3 since it has an early involvement in the task completion process (Figure 5.7b). Agent 2 develops its reputation from 1’s point of view as it takes an amount of participation load (Figure 5.7c). Agents 3, 4, and 5 do not build a significant amount of reputation during the process. Due to the forgetting factor in Equation (5.7), the agents’ relative reputation decreases as the participation load decreases as well (Figure 5.7d).

This set of simulations shows the effect of reputation on the dynamics of participation during the task completion process. Although only one agent has the initiative
Figure 5.7: Evolution of the relative reputation $r_{ij}(t)$ in Equation (5.7) for time steps (a) $t = 0$, (b) $t = 10$, (c) $t = 50$, and (d) $t=100$. Each node is associated with an agent, and each edge $(i, j)$ (arrow going from $i$ to $j$) is associated with $r_{ij}(t)$. The thickness of the edges is proportional to $r_{ij}(t)$. Only agent 1 has initiative $\gamma_1$ different from zero.

to see the potential benefit of completing the task, reciprocity through the relative reputation helps to promote participation. The effect of reciprocity is not immediate, since building the relative reputation is a dynamical process as well that takes time [88]. The load distribution algorithm allows the agents to interact and distribute the available participation load according to the potential benefit that they take into consideration and their participation costs. As reciprocity increases, the distribution of the participation load changes as more agents are willing to participate. Since the network in the example is connected, the initiative of agent 1 (who is the only agent with initiative different from zero) encourages reciprocal participation across the entire network following a “chain reaction.”
5.3.2 Monte Carlo Simulations

In this scenario, at each simulation run, only one agent in the community is randomly chosen to have an initiative $\gamma_i$ different from zero. Also, the cost of participation $c_i$ and the rate of increase in reputation $\lambda_{ij}$ are randomly chosen from a uniform distribution. We assumed that the parameters $\alpha_i$ and $\beta_i$ in the production function in Equation (5.2) were the same for all the agents, and that the neighbors contributed equally to the computation of reciprocity. We conducted 10000 Monte Carlo runs in total, ensuring that the estimated median, and 25% and 75% percentiles converged. We explored situations with different production functions, topologies of the interaction network, and size of the community.

![Figure 5.8](image.png)

**Figure 5.8:** Evolution of the median (line) and 25% and 75% percentiles (lower and upper boundaries of the shaded region) of the task variable $z(t)$ estimated from 10000 Monte Carlo runs for a (a) fully connected and (b) line topologies of the interaction network. The parameter of the production function $\alpha$ in Equation (5.2) takes the values 0.1, 0.25, 0.5, 0.75, 1, 2, and 3, corresponding to the bottom ($\alpha = 0.1$) to top ($\alpha = 3$) lines, respectively.
Effect of the Production Function

In the first set of simulations we tested the behavior of the agents when they followed a fully connected and a line topology, and for different values of the parameter $\alpha = \alpha_i, \ i = 1, \ldots, n$, in the production function in Equation (5.2). Recall that $\alpha$ determines the shape of the production function, and therefore the relationship between the participation load taken by the agents and their contribution to the completion of the task. Figure 5.8 shows the median and the 25% and 75% percentiles of the task completion variable $z(t)$ estimated from the Monte Carlo runs. Note that, as $\alpha$ increased, the agents took longer to complete the task and their variability to changes in the parameters increased, for both network topologies. As shown in Figure 5.1, a production function with large values of $\alpha$ implies that the participation load taken by an agent has to be large in order to have a significant contribution to the completion of the task. In this set of simulations, the initial contribution of the agents is enough to complete those tasks associated with lower values of $\alpha$, opposite to the cases with larger values of $\alpha$. From these results, observe that a fully connected network allows the agents to have a faster completion of the task compared to a community connected following a line topology. Recall that only one agent has initiative different from zero. Here, a fully connected topology allows for a faster spread of reputation and therefore reciprocal participation in the task completion process than the line topology case, as shown in Figure 5.9. This figure shows the median and the 25% and 75% percentiles of the average reputation variable $\hat{r}(t)$ estimated from the Monte Carlo runs, where $\hat{r}$ is defined as

$$\hat{r}(t) = \frac{1}{nN_i} \sum_{i=1}^{n} \sum_{j \in N_i} r_{ij}(t)$$
Since the convergence times are faster for smaller values of $\alpha$, observe that the relative reputation in these cases develops until the task is completed and decreases once the task has been completed. As $\alpha$ increases, the task takes longer to be completed, and therefore, the relative reputation is able to increase significantly. The case with the fully connected network allows larger values of the reciprocity values than the case with the line network, since in the first case each agent has more neighbors than the agents in the second case.

**Effect of the Number of Agents**

In the second set of simulations, we studied the behavior of communities that varied in size. The total participation load $P$ and the production function remained the same in all the simulations. We define the convergence time as the time step where the task variable $z(t)$ first reaches 95% of its completion state $\bar{z}$. Figure 5.10 shows the
behavior of the convergence time versus the number of agents in the community after 10000 Monte Carlo runs for different combinations of the topology of the interaction network and the parameter $\alpha$ of the production function. These results show that, in general, the fully connected topology has faster convergence times than the line topology as it was shown in the previous set of simulations. Also, a community that communicates with a fully connected topology tends to decrease their convergence time and its variance as the number of agents increase up to a point where the size of the community does not give any additional contribution to decrease the convergence time. The community increases in size, but the available load remains the same. At some point, even if more agents are willing to participate, their contribution will not represent a significant change in the completion of the task. However, since more agents are willing to participate in the task, the variations in the individual costs of participation and the rates of change in reputation have a lower impact in the convergence times. Also, note that the parameter $\alpha$ affects the convergence time of the task in both topologies. Larger values of $\alpha$ are associated with longer convergence times. This is consistent with the observations in Section 5.3.2.

5.4 Conclusion

We defined the task completion social dilemma as the situation where self-interested individuals in a community choose not to participate and contribute to a task whose completion can bring significant benefits for the whole community, including themselves. We characterized the dynamical process of completing a task as the result of the repeated contribution of participating agents in a community. The relationship between the participatory action and the contribution toward the task completion is
described by a production function. The process of completing the task involves only costs associated with participation. Therefore, an agent that decides to participate takes into consideration the potential benefit of solving the task. The ability of the agents in the community to communicate allows for the emergence of cooperation. We modeled the development of relative reputation, defined as an agent’s reputation with respect to another agent’s point of view, leading to participation through reciprocity. Also, we propose an interaction strategy where the available participation load is locally distributed between the agents. This strategy seeks to distribute the participation load so that the agents have the same net gain from participation. Agents with higher consideration of the potential benefit of completing the task are more likely to take higher loads of participation.
We used concepts of stability analysis in dynamical system theory to do a mathematical analysis of the proposed model, and also we conducted simulations to observe the qualitative behavior of the community for different values of the parameters. We proved that the cooperation strategies that we proposed guaranteed the completion of the task in the long term, and also we showed through simulations that the choice of the potential benefit function, production function, the topology of the interaction network, and the number of agents affect the dynamics of the task completion process. The results of the simulations showed that communities that are well connected have participation dynamics that allowed for faster convergence times of the task for a variety of production functions than those whose members are not well connected. In these cases, reputation and therefore reciprocity developed faster and stronger, promoting participation in the community.

The model presented in this paper can be extended to study additional situations that arise in the study of the dynamics of cooperation. For example, the situation where the members of the community have to distribute their participation level among several tasks. Also, the behavior of the community for different production functions, topologies of the interaction network, and larger scales of the community can be explored. Concepts from role theory [12] and the theory of collective behavior [116] can be connected to the concept of the production function for the analysis of additional mechanisms that solve task completion and collective goods social dilemmas.
Chapter 6: Summary and Future Work

6.1 Summary and Contributions

Dynamic Task Performance, Cohesion, and Communication in Human Groups: In Chapter 2, we propose a mathematical characterization of the dynamic behavior of human groups that is consistent with what has been observed in behavioral studies on cohesion, performance, and communication networks. Most of these models focus on describing the cohesive behavior of the groups, modeling the attraction forces acting on the individuals, and determining the conditions on the communication network and parameters of the model so that the group achieves certain stability properties. To our knowledge, modeling the relationship between cohesion and performance as observed in human groups has been overlooked. Our aim is to model the dynamics of a group that, given a network of communication between its members, not only tends to be cohesive but at the same time attempts to solve a complex task. Our contribution is two-fold. First, we propose a computational model of the dynamics of human groups that have a communication network and work to solve a task, providing a mathematical formulation of Lewin’s formula [70] that is consistent with the observations in real human groups during task-solving processes. We do not try to propose models to recreate the specific experiments with humans that have been
done to study these dynamics. Our aim is to design a mathematical model that characterizes dynamics that are qualitatively consistent with those observations in human groups. This work is conceptually related to the work in [47] [48, Chapter 11], where the behavior of a group is modeled as the result of social forces acting on its members. The main difference is that our work is focused on the development and analysis of a model that captures the relationship between cohesion and performance on the group given the patterns of information flow among its members. Second, we build on and extend the models of multiagent system dynamics in [35, 72, 71, 130]. The work in [35, 72] models the attraction and repulsion dynamics of groups of individuals that are evaluated according to a performance function or “resource environment,” and the work in [71, 130] extends these results to the case where there is a weighted communication topology that defines the patterns in communication between the individuals. In our work, we formulate a model of a group of individuals that interact according to a general communication network topology, where the attraction weights and the commitment to optimize the performance function change over time. We show that a key aspect for a cohesive group to be successful (i.e., having a good performance) is that the attraction patterns between the members of the group and the commitment to solve the task are not static but dynamic and develop over time, playing an important role in shaping the behavior of the group. We provide a mathematical analysis of group cohesiveness that is an extension of the analysis developed in [72, Theorem 1], where we study the conditions in our model that allow the group to be cohesive. Moreover, we show through simulations the impact of the parameters of the model and the communication networks on the behavior of the group in terms of
group performance and group cohesion when the group is required to solve a complex task.

**Dynamics of Metabolism and Decision Making during Alcohol-Consumption:**

**Modeling and Analysis:** In Chapter 3, we address the problem of modeling individuals in alcohol-consumption scenarios, where the goal is to better understand the etiology of high-risk drinking and find its “leverage points” for the design of intervention strategies that minimize such dangerous drinking patterns. We construct a model that characterizes how the Blood Alcohol Content (BAC) of an individual varies over time throughout the drinking event, taking into account the dynamical interaction between physiological factors and decision-making processes, and that allows us to conduct both computational and mathematical analyses for a more complete understanding of the modeled behaviors. As far as we know, such a model has not been constructed previously. The contribution of this chapter is two-fold. First, we provide a model of a system that, based on observations and up-to-date collected data of behaviors in drinking events, we believe characterizes the BAC dynamics of a drinker throughout the event. Second, We provide a mathematical analysis of the modeled behaviors, and show how this model can complement empirical research. This model is constructed as the interconnection of a component that accounts for the dynamics of the metabolic process of alcohol, and a component that describes the decision-making process that drives the individual’s alcohol consumption. In the same way that physics attempts to describe how forces translate into changes in motion, our main objective is to propose a model that explains the “physics” of an individual’s BAC during a drinking event: how the output of a decision-making process translates
into BAC variations and how BAC variations in turn affect decision-making. We propose a methodology to construct the model using data collected in situ, and to find the parameters of the decision-making process that generate the BAC trajectories that fit the available data.

**Modeling and Analysis of Group Dynamics in Alcohol-Consumption Environments:** In Chapter 4, we extend the results in Chapter 3 to include social and environmental influences in the drinking patterns. We construct, based on current observations and empirical data on drinking groups, a model of a system that characterizes how the dynamics of the social interactions, individual characteristics, and environment translate into changes in the drinking patterns of individuals measured through the BAC level. We derive a formulation of the model and a mathematical analysis of the behaviors that can be characterized, and show how this model can complement empirical research by informing theory and testing constructs. The contribution of this work is to provide a mathematical model that, based on up-to-date observations and empirical data of drinking groups, we hypothesize captures the dynamics of the BAC level given the mechanisms that drive the behavior of the group, and can be updated once we collect data for this purpose. In the same way conventional physics tries to explain how force translates into changes in motion, our aim is to create a model that explains the “physics” of the drinking event: how the influences from the individual’s personal preferences (e.g., desired effect of the BAC level on his/her body), other members of the group, and the environment translate into changes in the BAC level.
Dynamics of Cooperation in the Task Completion Social Dilemma: In Chapter 5, we study human cooperation and social dilemmas. Specifically, we study those situations where the individuals in a community have the choice to contribute to the completion of a common task. The process of task completion might involve only costs and no real benefits to the community members while they are participating. However, the completion of the task results in changes that substantially benefit the community and exceeds the participation costs. We do not focus on studying the development of trust as the result of the iterated completion of tasks, as has been studied before [14]. We focus our analysis on the process of promoting participation and engaging people in the community in completing the task. Even though the results of completing the task can be highly desirable, self-interested individuals easily turn this situation into a social dilemma: the short-term costs associated with participation can dissipate any motivation to contribute to the completion of the task and therefore to obtain the benefits that it provides in the long run. We call this situation a task completion social dilemma. This problem is similar to the public goods problem [91, 79] in that individuals are able contribute an amount of resources to generate a collective good for the benefit of the whole community by paying costs associated with their individual contribution. However, the facts that the benefits of the collective good will be available only when the task has been completed, and that the community knows the current progress of the task that is being completed, are characteristics that, taken together, differentiate the task completion problem from the other types of problems studied in the context of social dilemmas. We present in Chapter 5 a mathematical model that captures the relationship between the task to be completed and the contribution provided by the individuals in the community.
We propose basic mechanisms that promote participation and cooperation during the task completion process. This relationship, and the ways to promote an active participation by the community members, represent a complex problem [61, 50, 94], where the different factors to be considered can grow considerably in number. However, our work focuses on those elements that we have identified to be key to describe and study the task completion social dilemma and the conditions that are suitable for the emergence of cooperation.

The mathematical model that we propose can be seen as an extension of the work in [79] in the context of collective goods. The authors in [79] model the gain of a community member based on the benefits obtained by the current level of the collective good and the costs associated with the individual contributions, and study the effect of different forms of production functions on the generation of the collective good. Although their work is seminal in the study of cooperation and social dilemmas, they recognize that their models are essentially static and that there is a need for the development of dynamical models that build on their work [79, p. 190]. In our work, we formulate a dynamical system that characterizes the evolution of the task completion process and participation dynamics, allowing us to study mechanisms that promote participation and cooperation in the community. We use concepts of stability analysis of discrete nonlinear systems and Monte Carlo simulations to analyze the qualitative and quantitative behavior of the modeled community.
6.2 Future Work

Dynamic Task Performance, Cohesion, and Communication in Human Groups: This model represents a tool for the analysis and design of communication networks of task-solving groups. It enables the social scientist to understand human groups as a dynamical system, to explore the group behaviors under different conditions in the parameters and network topology, and to design new methodologies for the development of experiments in real human groups whose aim is to solve complex tasks. Future work includes the analysis of groups for different topologies, tasks, and size of the group, to determine those conditions that allow the group to improve their performance.

Dynamics of Metabolism and Decision Making during Alcohol-Consumption: Modeling and Analysis: We showed how the model of the drinker’s decision-making process, which was formulated following the observations from field studies, resembled the proportional-derivative controller from feedback control system engineering. This result provides useful insights into the mechanisms that drive the decision-making process of an individual in a drinking event. It might be the case that there are additional mechanisms that have not been uncovered yet, but are already known in the theory of feedback systems. An example is the proportional-integral-derivative controller, which in addition to the components explained in the chapter, has a component (the integrator) that allows the system to accurately reach the desired state. This can be explored in our future work, along with personal drinking advisor development. Currently, we are in the process of collecting real-time drinking event data that includes individual, group, and environmental level variables.
Our objective is to study these data using tools in statistics and dynamical systems theory, and conduct a model identification process from the measured data [73]. These results will help us to validate and update our hypotheses on the modeled drinking event dynamics. Our aspiration is to gain enough understanding to start interventions at both the individual and group event levels.

Modeling and Analysis of Group Dynamics in Alcohol-Consumption Environments: On the modeling side of our work, our next steps include developing models that incorporate additional information about the individual, such as gender or weight. In this way, the model can be coupled with Wegner’s equations of alcohol content [125] to characterize the trajectories of the BAC given the number and type of drinks per time unit. On the empirical side of our work, we are designing a real-time data collection process at drinking events that includes measurements of variables at the individual, group, and environmental level [9, 18, 24]. Our aim is to study these data using tools from both statistics and dynamical system theory [63]. We plan to do system identification [73] to find the parameters that allow the model to have the closest approximation to the measured behaviors. Depending on the results, we will validate and improve our hypotheses on the mechanisms that drive behavior during the drinking event. We hope that this subsequent round of model and field validations will have contributed to our understanding sufficient to engage in a series of interventions at the event level.

Dynamics of Cooperation in the Task Completion Social Dilemma: The model presented in this chapter can be extended to study additional situations that arise in the study of the dynamics of cooperation. For example, the situation where
the members of the community have to distribute their participation level among several tasks. Also, the behavior of the community for different production functions, topologies of the interaction network, and larger scales of the community can be explored. Concepts from role theory [12] and the theory of collective behavior [116] can be connected to the concept of the production function for the analysis of additional mechanisms that solve task completion and collective goods social dilemmas.
Appendix A: Proofs and Implementation Details Chapter 3

A.1 Proofs Section 3.1.1: Metabolism

Proof Theorem 2  Theorem 2 presents the solution trajectory of the system (3.1).
To obtain this solution we use standard properties of linear systems and the matrix
exponential [4]. The trajectories of the BAC and BAC rate for \( t \geq 0 \) and any input
\( u(t) \geq 0 \) are given by [4, Ch 2]

\[
z(t) = e^{At}z(0) + \int_{0}^{t} e^{A(t-s)}Bu(s)ds
\]  \hspace{1cm} (A.1)

where

\[
z(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\beta^2 & -2\zeta\beta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \eta\beta^2 \end{bmatrix}
\]

The eigenvalues of \( A \) are \( \lambda_1 = -\beta\zeta - \beta\sqrt{\zeta^2 - 1} \) and \( \lambda_2 = -\beta\zeta + \beta\sqrt{\zeta^2 - 1} \). Since
\( \beta > 0 \) and \( \zeta > 1 \), eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are real, negative, and distinct. Using
the properties of the matrix exponential when \( A \) has distinct and real eigenvalues[4,
Ch 3], we have that \( e^{At} = A_1e^{\lambda_1t} + A_2e^{\lambda_2t} \), where

\[
A_1 = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ \lambda_1\lambda_2 & -\lambda_1 \end{bmatrix}
\]

and

\[
A_2 = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_1\lambda_2 & \lambda_2 \end{bmatrix}
\]
Using this result in Equation (A.1), after some manipulations we obtain Equation (3.2).

**Proof Corollary 2.1.** We derive each of the properties in this corollary from Equation (3.2). To prove Property (i), we first prove that $g_0 \geq 0$. Since $2\beta \sqrt{\zeta^2 - 1} > 0$, then it is enough to show that $(\lambda_2 x_0 - v_0)e^{\lambda_1 t} - (\lambda_1 x_0 - v_0)e^{\lambda_2 t} \geq 0$. This implies that it must satisfy

$$\frac{-\lambda_1 x_0 + v_0}{-\lambda_2 x_0 + v_0} e^{(\lambda_2 - \lambda_1)t} \geq 1, \quad t \geq 0$$

Since $\lambda_2 > \lambda_1$ and $x_0 \geq 0$, then terms $e^{(\lambda_2 - \lambda_1)t}$ and $\frac{-\lambda_1 x_0 + v_0}{-\lambda_2 x_0 + v_0}$ are always greater or equal than one. Therefore, $g_0 \geq 0$.

In the second term of the right-hand side of Equation (3.2), since $\lambda_1 < \lambda_2 < 0$, we have that $0 \geq e^{\lambda_2 t} \geq e^{\lambda_1 t}$ for all $t \geq 0$. From the assumption that $u(t) \geq 0$, we have that the integration is always positive. Therefore, $x(t) \geq 0$ for all $t \geq 0$ (Property (i)). Now, let the alcohol intake rate be represented by a very short-duration pulse at time $t = 0$ such that the amount of alcohol consumed is $C > 0$. This is typically represented by $u(t) = C\delta(t)$, where $\delta(t)$ is the Dirac delta. The Dirac delta satisfies $\int_0^t f(\tau)\delta(\tau)d\tau = f(0)$ for any continuous function $f(t)$ defined in $t \geq 0$. The resultant trajectory is then the impulse response of the system $x_\delta(t)$ given by Equation (3.3) (Property (ii)). Since $\lambda_1 < \lambda_2 < 0$, trajectory $x_\delta(t)$ tends to zero as $t$ tends to infinity (Property (iii)). The peak of the impulse response occurs when $\lambda_1 e^{\lambda_1 t} = \lambda_2 e^{\lambda_2 t}$, that is, when $t = t_p$, where $t_p$ is given in Equation (3.4) (Property (iv)).
A.2 Proof Section 3.1.3: Combined Metabolism and Decision-Making

**Proof Theorem 3** This theorem shows that there exists a time such that the trajectories of $x$ and $v$ are guaranteed to enter a ball (and remain there) with center at $(x = x^*, v = 0)$ and a radius that depends on the parameters. This radius is referred as a ultimate bound, and the trajectories are said to be uniformly ultimately bounded [63, Ch 4.8]. Since the system in (3.7) uses the function $\lfloor \cdot \rfloor_+$, we need to use the Lyapunov stability theory of nonsmooth systems [109]. The proofs that use to show uniformly ultimate boundedness are identical to the smooth counterpart, with the difference that now some relations hold “almost everywhere” rather than “everywhere.” In our specific problem, the Lyapunov function candidate $V$ is smooth, and the vector field is continuous. Hence, our proof involves showing that the set-valued map $\dot{V}$ is negative definite when the result of applying $\lfloor \cdot \rfloor_+$ is positive, and when the result of applying $\lfloor \cdot \rfloor_+$ is zero.

The assumption of Lipschitz continuity in the derivative of the preference function $f$ with respect to the BAC level $x$ limits how fast it can change with respect to $x$, and it is required in order to have a closed form of the ultimate bound in this theorem. It means means that the slope of the derivative of the preference function is bounded, which is a reasonable assumption in reality that allows us to provide results that are interpretable.

To facilitate the mathematical analysis of the impact of the parameters on the solutions obtained using the proposed model, we assume that the preference function $f$ has a unique minimizer $x^* \geq 0$. Let $\tilde{x} = x - x^*$. The system in (3.7) can be
rewritten in terms of $\tilde{x}$ as
\[
\dot{x}(t) = v(t)
\]
\[
\dot{v}(t) = -\beta^2 \tilde{x}(t) - \beta^2 x^* - 2\zeta \beta v(t)
+ \beta^2 \eta [-k_p g (\tilde{x}(t) + x^* + k_d v(t))]_+
\] (A.2)

Let
\[
V(\tilde{x}, v) = \frac{\beta^2}{2} \tilde{x}^2 + \frac{1}{2} v^2
\] (A.3)

Since $V(0, 0) = 0$ and $V(\tilde{x}, v) > 0$ otherwise, we use $V$ as our Lyapunov function candidate. Let $Z(\tilde{x}, v)$ be the vector field in (A.2). Since the $Z$ is not smooth due to the operator $[\cdot]_+$, we formulate the lie derivative of $V$ with respect to the Filippov’s set-valued map $\mathcal{K}[Z](\tilde{x}, v)$ [109, 23] as
\[
\dot{V} = \left[ \frac{\partial V}{\partial \tilde{x}}, \frac{\partial V}{\partial v} \right]^{\top} \mathcal{K}[Z](x, v)
\subset -\beta^2 x^* v - 2\zeta \beta v^2 + \beta^2 \eta v \mathcal{K}[-k_p g (\tilde{x} + x^* + k_d v)]_+
\] (A.4)

We need to find sufficient conditions on the trajectories of $\tilde{x}$ and $v$ such that $\dot{V}$ is negative. From the right-hand side of Equation (A.4), we have to consider two different cases: when $[r]_+ = r$ and when $[r]_+ = 0$, where $r = -k_p g (\tilde{x} + x^* + k_d v)$, or in other words, when $r > 0$ and when $r \leq 0$. For the first case, we define function $\dot{V}_1$ of $\tilde{x}$ and $v$ as the right-hand side of (A.4) when $-g (\tilde{x} + x^* + k_d v) > 0$ and
\[
\dot{V}_1 = -\beta^2 \eta v k_p g (\tilde{x} + x^* + k_d v) - \beta^2 x^* v - 2\zeta \beta v^2
\]

After some manipulations, we obtain
\[
\dot{V}_1 \leq \beta^2 x^* |v| - \zeta \beta |v|^2 + \beta^2 \eta k_p |g(\tilde{x} + x^* + k_d v)|
\]

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Using the assumption that $g$ is Lipschitz continuous with Lipschitz constant $L$, we have that $|g(\bar{x} + x^* + k_d v)| \leq L|\bar{x} + k_d v| \leq L|\bar{x}| + Lk_d |v|$. Then,

$$
\dot{V}_1 \leq \beta^2 x^* |v| + \beta^2 \eta k_p L|\bar{x}| |v| - \delta |v|^2
$$

where $\delta = 2\zeta \beta - \beta^2 \eta k_p Lk_d$. From the assumption in (3.9), we have that $\delta \geq 0$. For a constant $\theta_1 \in (0, 1)$, we have that $-|v|^2 = -(1 - \theta_1)|v|^2 - \theta_1 |v|^2$. Then, we can rewrite the previous inequality as

$$
\dot{V}_1 \leq -\frac{\delta}{2} |v|^2, \quad \text{for all } \frac{\delta}{2} |v| \geq \beta^2 x^* + \beta^2 \eta k_p L|\bar{x}|
$$

where $\theta_1 = 0.5$. This expression can be can be rewritten in a compact yet more conservative way

$$
\dot{V}_1 \leq -\frac{\delta}{2} |v|^2 \text{ for all } \|\bar{x}, v\|^T \geq \mu_1 \quad (A.5)
$$

where

$$
\mu_1 = \frac{x^*}{\sqrt{2} \max \left\{ \frac{1}{2} \left( \frac{2\xi}{\beta} - \eta k_p k_d L \right), \beta \eta k_p L \right\}}
$$

and $\|\cdot\|_2$ denotes the $L_2$ vector norm. This expression was obtained using the norm inequality [63, p 648]

$$
|v| \leq |\bar{x}| + |v| = \|\bar{x}, v\|^T \|_1 \leq \sqrt{2} \|\bar{x}, v\|^T \|_2
$$

For the second case, we define the function $\dot{V}_2$ of $\bar{x}$ an $v$ as the right-hand side of (A.4) when $-g(\bar{x} + x^* + k_d v) \leq 0$ as

$$
\dot{V}_2 = -\beta^2 x^* v - 2\zeta \beta v^2 + \beta^2 x^* |v| - 2\zeta |v|^2
$$

For a constant $\theta_2 \in (0, 1)$, we have that $-|v|^2 = -(1 - \theta_2)|v|^2 - \theta_2 |v|^2$. Then, we can rewrite the previous inequality as

$$
\dot{V}_2 \leq -2\zeta \frac{\beta}{2} |v|^2, \quad \text{for all } \beta^2 x^* |v| - 2\zeta \frac{\beta}{2} |v|^2 < 0
$$

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where $\theta_2 = 0.5$. The previous equation can be written in a more compact way

$$
\dot{V}_2 \leq -2\zeta \frac{\beta}{2}|v|^2, \quad \text{for all } \| [\tilde{x}, v]^T \|_2 \geq \mu_2 \tag{A.6}
$$

where

$$
\mu_2 = \frac{x^*}{\beta}
$$

However, from Equation (A.2) we know that $\dot{v}$ is not zero unless additional conditions on $\tilde{x}$ and $v$ are satisfied. Hence, $\dot{V}_1$ is negative as long as the bound in (A.5) and $-g(\tilde{x} + x^* + k_d v) > 0$ are satisfied. Similarly, $\dot{V}_2$ is negative as long as the bound in (A.6) and $-g(\tilde{x} + x^* + k_d v) \leq 0$ are satisfied. Hence, the trajectories of the dynamical system in (A.2) are uniformly ultimately bounded [63, Ch 4.8].

Equations (A.5) and (A.6) indicate that $\dot{V}_1$ and $\dot{V}_2$ are negative definite for all $\| [\tilde{x}, v]^T \|_2 \geq \mu_1$ and $\| [\tilde{x}, v]^T \|_2 \geq \mu_2$, respectively. This means that a sufficient condition for the set valued map $\dot{V}$ to have negative elements is

$$
\| [\tilde{x}, v]^T \|_2 \geq \mu
$$

where $\mu = \max \{\mu_1, \mu_2\}$. To compute the ultimate bound, we need to find strictly increasing functions $\alpha_1, \alpha_2: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\alpha_1(0) = 0$ and $\alpha_2(0) = 0$ such that

$$
\alpha_1(\| [\tilde{x}, v]^T \|) \leq V(\tilde{x}, v) \leq \alpha_2(\| [\tilde{x}, v]^T \|)
$$

According to [63, Theorem 4.18], the ultimate bound will be given by $\gamma = \alpha_1^{-1}(\alpha_2(\mu))$.

From Equation (A.3) we know that $V(\tilde{x}, v)$ satisfies

$$
V(\tilde{x}, v) = \frac{\beta^2}{2}|\tilde{x}|^2 + \frac{1}{2}|v|^2 \\
\geq M_1 (|\tilde{x}|^2 + |v|^2) = \alpha_1 (\| [\tilde{x}, v]^T \|_2) \tag{A.7}
$$
where $M_1 = \min \left\{ \frac{1}{2}, \frac{\beta}{2} \right\}$. To find the upper bound on $V(\bar{x}, v)$, we use the fact that $V$ can be bounded as

$$V(\bar{x}, v) \leq M_2 \left( \left\| [\bar{x}, v]^T \right\|^2 \right) = \alpha_2(\left\| [\bar{x}, v]^T \right\|)$$

(A.8)

where $M_2 = \max \left\{ \frac{1}{2}, \frac{\beta^2}{2} \right\}$. From Equations (A.7) and (A.8) we can obtain an expression for the ultimate bound

$$\gamma = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\alpha_2(\mu)}{M_1}} = \sqrt{\frac{M_2}{M_1}} \mu = M \mu$$

where $M = \sqrt{M_2/M_1}$. This proves Equation (3.10).

### A.3 Implementation Details of Simulations in Section 3.1.1

The parameters of the model that generated the trajectories in Figure 3.5 where computed using a least square approximation. Let $\hat{x}_i$ be $i$-th measurement of BAC at time $t_i$ obtained after administering a dose of alcohol to participants (points marked with symbols 'x' and 'o' in the figure). The problem of finding the model parameters that approximate the collected samples $\{t_i, \hat{x}_i\}_{i=1}^n$ is formulated as the optimization problem

$$(\zeta^*, \beta^*, \eta^*) = \arg \min_{\zeta > 0, \beta > 0, \eta > 0} \sum_{i=1}^n (\hat{x}_i - x_{\delta}(t_i; \zeta, \beta, \eta))^2$$

where $x_{\delta}$ is defined in Equation (3.3). Since this problem may have multiple local minima, we run the optimization algorithm for different initial conditions of the parameters and choose the solution with the minimum value of the cost function. The values of the parameters for the fasted condition were $\zeta^* = 1.4$, $\beta^* = 0.0107$, and $\eta^* = 39.23$, and for the fed condition $\zeta^* = 1.0003$, $\beta^* = 0.0096$, and $\eta^* = 21.21$. 
A.4 Implementation Details of Simulations in Section 3.2

The model parameters associated with the metabolic process were chosen to be the same ones obtained for the fasted condition: $\zeta^* = 1.4$, $\beta^* = 0.0107$, and $\eta^* = 39.23$. The profiles of the preference function for each category of the intended level of intoxication are given in Table B.1.

<table>
<thead>
<tr>
<th>Category</th>
<th>Function Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>“No drinking”</td>
<td>$f(x) = \frac{1}{2}x^2$ for $x \geq 0$</td>
</tr>
<tr>
<td>“Not enough”</td>
<td>$f(x) = \begin{cases} 0 &amp; \text{if } 0 \leq x \leq 0.053 \ \frac{1}{2}(x - 0.053)^2 &amp; \text{if } x &gt; 0.053 \end{cases}$</td>
</tr>
<tr>
<td>“Slight buzz”</td>
<td>$f(x) = \begin{cases} 4(x - 0.042)^2 &amp; \text{if } 0 \leq x \leq 0.042 \ 0 &amp; \text{if } x \in (0.042, 0.093] \ \frac{1}{2}(x - 0.093)^2 &amp; \text{if } x &gt; 0.093 \end{cases}$</td>
</tr>
<tr>
<td>“Very drunk”</td>
<td>$f(x) = \begin{cases} 4(x - 0.097)^2 &amp; \text{if } 0 \leq x \leq 0.097 \ 0 &amp; \text{if } x &gt; 0.097 \end{cases}$</td>
</tr>
</tbody>
</table>

Table A.1: Functions associated with the category of intended level of intoxication.
Appendix B: Implementation Details Chapter 4

B.1 Personal Preference Function For Different Categories of The Desired Effect of The BAC Level

The following table corresponds to the proposed preference functions shown in Figure 4.5a.

<table>
<thead>
<tr>
<th>Category</th>
<th>Function $f_i^p(x_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;No drinking&quot;</td>
<td>$f_i^q(x_i) = \frac{1}{2}x_i^2$ for $x_i \geq 0$</td>
</tr>
<tr>
<td>&quot;Not enough&quot;</td>
<td>$f_i^p(x_i) = \begin{cases} 0 &amp; \text{if } 0 \leq x_i \leq 0.02 \ \frac{1}{2}(x_i - 0.02)^2 &amp; \text{if } x_i &gt; 0.02 \end{cases}$</td>
</tr>
<tr>
<td>&quot;Slight buzz&quot;</td>
<td>$f_i^p(x_i) = \begin{cases} 4(x_i - 0.02)^2 &amp; \text{if } 0 \leq x_i \leq 0.02 \ 0 &amp; \text{if } x_i \in (0.02, 0.04] \ \frac{1}{2}(x_i - 0.04)^2 &amp; \text{if } x_i &gt; 0.04 \end{cases}$</td>
</tr>
<tr>
<td>&quot;Drunk&quot;</td>
<td>$f_i^p(x_i) = \begin{cases} 4(x_i - 0.04)^2 &amp; \text{if } 0 \leq x_i \leq 0.04 \ 0 &amp; \text{if } x_i &gt; 0.04 \end{cases}$</td>
</tr>
</tbody>
</table>

Table B.1: Functions associated with the preferred effect of the BAC level.
B.2 Implementation Details of Simulations in Section 4.2.1

The parameters of the model are \( \zeta_i = 3, \eta_i^p = 0.5 \) for \( i = 1, \ldots, 6 \). The weights that quantify the influence between group members are given by

\[
W_1 = \begin{bmatrix}
0 & 0 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0.2 & 0 & 0 & 0 \\
0.2 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0.1 & 0.1 \\
0 & 0 & 0 & 0.2 & 0 & 0.2 \\
0 & 0 & 0 & 0.2 & 0 & 0
\end{bmatrix}, \quad W_2 = \begin{bmatrix}
0 & 0 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0.2 & 0 & 0 & 0 \\
0.2 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0.1 & 0.1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix},
\]

where \( W_1 \) and \( W_2 \) are the weighted adjacency matrices for cases 1 and 2. Parameter \( b = 0 \) to facilitate the analysis of the simulation results. The differential equations are solved using Euler’s approximation with \( T = 0.01 \) and 1000 iterations steps, where the time axis is scaled to be between 0 and 4. The personal preference functions used for individuals with “No drinking” and “Slight buzz” as desired effect of the BAC level are the ones in Table B.1.

B.3 Implementation Details of Simulations in Section 4.2.2

In this set of simulations, it is assumed that \( \zeta_i = 2, \eta_i^p = 5, \) and \( \eta_i^e = 1 \), in Equation (4.5). Since all the individuals have a desired effect of the BAC level to be “Drunk,” the corresponding profile in Table B.1 for simulation. The environment profile is constructed as a linear combination of components that belong to two different types: environments that protect against increments of the BAC level (Environment 1), and environments that promote high BAC levels (Environment 2). Table B.2 shows the expressions for each one of the environment types (\( f_1^e \) for Environment 1 and \( f_2^e \) for Environment 2) and Figure B.1 shows their corresponding plots.
Environment 1  \[ f_1^e(x_i) = \frac{1}{2}x_i^2 \]

Environment 2  \[ f_2^e(x_i) = \begin{cases} 
0 & \text{if } 0 \leq x_i \leq 0.02 \\
\frac{1}{2}(x_i - 0.02)^2 & \text{if } 0 \leq x_i \leq 0.02
\end{cases} \]

Table B.2: Functions associated with the environments that are protective against (Environment 1, \( f_1^e \)) and promote (Environment 2, \( f_2^e \)) high BAC levels.

![Profile functions for environments that are protective against (Environment 1) and facilitate (Environment 2) high BAC levels.](image)

Figure B.1: Profile functions for environments that are protective against (Environment 1) and facilitate (Environment 2) high BAC levels.

The resultant expression for the environment profile is given by the equation

\[ f^e(x_i) = \sum_{j=1}^{4} \alpha_{1j}^i f_1^e(x_i) + \sum_{j=1}^{3} \alpha_{2j}^i f_2^e(x_i) \]

where \( \alpha_{11}^i, \ldots, \alpha_{13}^i \in \{0, 1\} \) are 1 when individual \( i \) reported that there was food available, the bar was crowded, and there were food specials, respectively, and 0 otherwise. Parameter \( \alpha_{14}^i \in [0, 1] \) is proportional to the amount of money reported to be spent in food. On the other hand, \( \alpha_{21}^i, \alpha_{22}^i \in \{0, 1\} \) are 1 when individual
reported that there were drinking games and alcohol specials, respectively, and 0 otherwise. Parameter $\alpha_{14}^i \in [0, 1]$ is proportional to the amount of money reported to be spent in alcohol.
Appendix C: Proofs Chapter 5

C.1 Proof Theorem 6

This proof is based on concepts from Lyapunov stability theory in the analysis of discrete nonlinear systems. We will provide sufficient conditions such that the trajectories of \( z \) converge to \( \bar{z} \) in Equation (5.1).

First, we define the variable \( \tilde{z}(t) = \bar{z} - z(t) \). Equation (5.1) can be rewritten as

\[
\tilde{z}(t+1) = \left[1 - \phi h(p(t))\right] \tilde{z}(t)
\]  

where \( 0 \leq \tilde{z}(0) \leq \bar{z} \). Since \( \phi h(p(t)) \in [0, 1) \), we know that \( 0 \leq z(t) \leq \bar{z} \) for all \( t \geq 0 \) if \( z(0) \leq \bar{z} \). From [59, Theorem 1], if there exists a strictly increasing function \( \psi \), with \( \psi(0) = 0 \), such that \( V(\tilde{z}(t+1)) - V(\tilde{z}(t)) \leq -\psi(|\tilde{z}|) \) for all \( t \geq 0 \), with \( V(\tilde{z}(t)) = |\tilde{z}(t)| \) (Lyapunov function candidate), then the point \( \tilde{z} = 0 \) is asymptotically stable. That is, the trajectories of \( \tilde{z} \) with initial conditions \( 0 \leq \tilde{z}(0) \leq \bar{z} \) will asymptotically converge to 0. Using Equation (C.1) we obtain

\[
V(\tilde{z}(t+1)) - V(\tilde{z}(t)) = |1 - \phi h(p(t))||\tilde{z}| - |\tilde{z}|
\]

\[
= |(1 - \phi h(p(t)) - 1)|\tilde{z}|
\]

From the assumption that \( h(p(t)) \in [0, 1) \) for all \( t \geq 0 \), we have

\[
V(\tilde{z}(t+1)) - V(\tilde{z}(t)) = -\phi h(p(t)) |\tilde{z}|
\]
Then, if we can find such function \( \psi \) such that \(-\phi h(p(t))|\tilde{z}| \leq \psi(|\tilde{z}|) \) for all \( t \geq 0 \), then \( \tilde{z} \) will converge to 0.

\section*{C.2 Proof Corollary 6.1}

Let \( p^* \) the fixed participation load taken by the agents. From the previous proof, we have that

\[ V(\tilde{z}(t + 1)) - V(\tilde{z}(t)) = -\phi h(p^*) |\tilde{z}| \]

Since \( V(\tilde{z}(t)) = |\tilde{z}(t)| \), from [16, Theorem 3], we have then that \( \tilde{z} \) converges exponentially to zero.

From Equation (C.1), given the initial condition \( \tilde{z}(0) \), the trajectory of \( \tilde{z}(t) \) is given by

\[ \tilde{z}(t) = [1 - \phi h(p^*)]^t \tilde{z}(0) \]

Substituting \( \tilde{z} \) by \( \tilde{z} - z \), we obtain Equation (5.4).

\section*{C.3 Proof Theorem 7}

We show that the load distribution algorithm satisfies the conditions presented in [29] under which a load is distributed across the nodes in a network such that the net gain of any pair of nodes, quantified in this case by \( g_i(p_i(t)) \), is equalized.

Let \( b_i = (\gamma_i + r_i)b \). Note that the net gain defined in Equation (5.8) decreases linearly with the participation load. In the load distribution algorithm, lines 6 and 7 guarantee that the participation load does not remain outside \( b_i/c_i \). Therefore, we know that the assumption in [29, Equation (1)] is satisfied, where \( s_i(x) - s_i(y) = -c_i(x-y) \) for all \( x, y \leq b_i/c_i \) and \( x \neq y \). This corresponds to the case when \( s_i(p_i(t)) = 0 \) and \( s_j(p_j(t)) = 0 \) for all \( j \in N_i \), then the fictitious agent is the only neighbor that
has suitability greater than agent $i$’s suitability. Then, agent $i$ shares participation load with the fictitious agent, increasing in that way its net gain. This action is stated in lines 15 and 16, and also satisfies conditions (i), (ii) and (iii) in [29, p 1428].

We assume that there is a fictitious agent with load $Q(t)$ that is connected to all the other $n$ agents and that has a suitability $\epsilon > 0$ that is infinitesimally small. If $S_i$ in Equation (5.10) is not empty, then it means that $i$ can share participation load with a neighbor. This action is stated in lines 9 and 10 in the distribution algorithm, which satisfies conditions (i), (ii) and (iii) in [29, p 1428]. If $S_q$ is not empty, it means that the fictitious agent can give participation load to those agents that need it. This is done in lines 13 and 14, satisfying conditions (i), (ii) and (iii) in [29, p 1428].

This strategy and the assumption in Equation (5.12), from [29, Theorem 3.4] we have that, during this process, the agents share their participation load in a way that every pair of agents in the network tries to equalize their net gain. Since the fictitious agent also participates in this process and its net gain is fixed and infinitesimally small, the other agents try to increase their participation load as much as possible in order to decrease their net gain and equalize it with that of the fictitious agent. Since it is assumed that $P \geq \sum_{i=1}^{n} b_i/c_i$, the participation loads will eventually be $p_i(t) = b_i/c_i$. At this point, the net gains are zero, which corresponds to an invariant set since the load that is transferred from one agent to another is proportional to the net gain. From [29, Theorem 3.4], this set is exponentially stable.

C.4 Proof Theorem 8

We have that that $p_i(0) = 0$ for all $i \in V$. At this step, the participation load to assigned to the agents is proportional to $b(t)$ for those agents whose initiative
or reciprocity is different from zero. Since the load to be passed at each iteration is proportional to the net gain, and at the same time the net gain is a linear combination of the participation load and \( b(t) \), we have that the load to be passed at each iteration is also proportional to \( b(t) \) during the iterative process. Therefore, the participation load has the form \( p_i(t) = \eta_i(t)b(t) \), where \( \eta_i(t) \geq 0 \) for all \( t > 0 \). Let \( \tilde{\eta}_i = \min_{t>0} \eta_i(t) \). Since we assumed that at least there is one agent with initiative different from zero, we know that at least there is one \( \tilde{\eta}_i > 0 \).

Assume that \( b(t) = b^* \) satisfies \((i)\). Then, we have that

\[
\begin{align*}
h(p(t))|\bar{z} - z(t)| &\geq h(\tilde{\eta}_1b^*, \ldots, \tilde{\eta}_n b^*) |\bar{z} - z(t)| \\
\end{align*}
\]

The right-hand side of this equation is a strictly increasing function of \(|\bar{z} - z(t)|\).

Then, from Theorem 6, we have that \( z \) converges to \( z(t) \).

Now assume that \( b(t) \) satisfies \((ii)\). Since the production function is assumed to be strictly increasing with respect to each participation variable, and the contribution of each agent is additive in the production function, we have that

\[
\begin{align*}
h(p(t))|\bar{z} - z(t)| &\geq h(\tilde{\eta}_1 b(|\bar{z} - z(t)|), \ldots, \tilde{\eta}_n b(|\bar{z} - z(t)|)) |\bar{z} - z(t)| \\
\end{align*}
\]

The right-hand side of this equation is also a strictly increasing function with respect to \(|\bar{z} - z(t)|\). Then, from Theorem 6, we have that \( z \) converges to \( z(t) \).
Bibliography


