Step-response of discontinuous non-linear torsional systems:

Experimental and parameter estimation studies

DISSECTATION

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Abstract

This study examines the step-response of discontinuous non-linear torsional systems that contain clearances, multi-staged torsional springs, stopper or pre-load features, and multi-staged dry friction elements. The primary goal of this work is to develop two new laboratory experiments to provide benchmark data and to propose time domain estimation methods for elastic and dissipative parameters as well as for time-varying oscillatory periods. First, a new laboratory experiment, incorporating a multi-staged clutch damper, is designed, built, and instrumented. Typical measured responses under a step-like torque loading exhibit rich non-linear behavior, such as vibro-impact phenomena with distinct regimes (i.e. single and double-sided impacts) and time-varying oscillatory periods. Predictions from a minimal order non-linear model verify that the experiment performs well and as designed. Second, an original laboratory experiment is developed as a more scientific version of the first (component driven) experiment, with focus on stiffness non-linearity. It is excited by a step-like torque in no-clearance, single-clearance, and dual-clearance configurations, each described by minimal order positive-definite non-linear models. The feasibility of this experiment is verified by comparing measurements with predictions from analogous models. Both hardening and softening trends in the time-varying oscillatory periods are found in the experiments and are discussed in the context of double and single-sided impact regimes.
Next, a refined method to estimate the time-varying oscillatory periods of the step-response of a non-linear torsional device is developed by extending the prior work on the stochastic linearization techniques. Subsequently, an instantaneous expected value operator and the concept of instantaneous effective stiffness are formulated. The proposed method is rigorously illustrated via two computational example cases and the necessary digital signal processing parameters for time domain analysis are investigated. The feasibility of the proposed method is demonstrated by estimating trends in measured time-varying oscillatory periods. Finally, a refined time domain method to estimate elastic and dissipative parameters of a practical non-linear component is formulated. Since prior estimation methods have utilized either linearization or assumed that non-linearities are differentiable, the proposed method is developed specifically for devices that contain discontinuous non-linear features. Elastic parameters are first estimated through an instantaneous stochastic linearization technique. Then, the energy balance principle is employed to estimate a combination of viscous and Coulomb damping parameters for seven local (stage-dependent) and global (single-staged) damping formulations. The proposed method is validated by comparing time domain predictions from non-linear models to measurements. Non-linear models that utilize the proposed stage-dependent damping formulations are found to be superior to those that solely rely only on parameters from a quasi-static experiment. On a more fundamental note, both experiments overcome the general void in the field of experimental non-linear dynamics. The proposed experimental methods, benchmark data, and parameter estimation techniques should be equally valuable to the scientific literature and engineering practice.
Dedication

To Laura and my family
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Table of Contents

Abstract ............................................................................................................................... ii

Acknowledgments............................................................................................................... v

Vita..................................................................................................................................... vi

List of Tables ..................................................................................................................... xi

List of Figures .................................................................................................................. xiv

Chapter 1:  Introduction ...................................................................................................... 1

1.1  Motivation ................................................................................................................ 1

1.2  Literature review ...................................................................................................... 2

1.3  Problem formulation ................................................................................................ 4

1.3.1  Unresolved research issues ................................................................................ 4

1.3.2  Scope, assumptions, and objectives ................................................................... 5

References for Chapter 1 ............................................................................................... 11

List of symbols for Chapter 1 ........................................................................................ 13

Chapter 2:  Step-response of a torsional device with multiple discontinuous non-
linearities: Formulation of a vibratory experiment ........................................................... 14
Chapter 4: Asymptotic trends in time-varying oscillatory period for a dual-staged torsional system

4.1 Introduction ............................................................................................................ 88
4.2 Analytical formulation of the problem ................................................................... 89
4.3 Time-varying oscillatory period ............................................................................. 92
4.4 Concept of instantaneous effective stiffness .......................................................... 96
4.5 Equivalent linear system with instantaneous stiffness .......................................... 102
4.6 Validation of estimation method using measurements from experiments ........... 111
4.7 Conclusion ............................................................................................................ 115

References for Chapter 4 ............................................................................................. 116

List of symbols for Chapter 4 ...................................................................................... 118

Chapter 5: Correlation between quasi-static and dynamic experiments for a practical torsional device with multiple discontinuous non-linearities ................................................. 121

5.1 Introduction ........................................................................................................... 121
5.2 Example case and literature review ...................................................................... 122
5.3 Analytical formulation of problem ....................................................................... 125
5.4 Characterization under quasi-static loading ......................................................... 129
5.5 Characterization under dynamic loading ............................................................... 134
List of Tables

Table 1.1 Summary of Objectives 1-4 with corresponding chapters. ........................................ 10

Table 2.1 Sampling errors in peak-to-peak velocity (measured) and acceleration (calculated) for laser vibrometer measurement of a harmonic shaker. ........................................................................... 25

Table 2.2 Mean (μ) and standard deviation (σ) of peak-to-peak $\overline{\theta}(T)$ of the measured responses for $\overline{\theta}_o \approx 26$ over three assemblies (with dual clutch damper configuration). ......................... 32

Table 2.3 Analysis of measurements in terms of the mean (μ) and standard deviation (σ) of response regime transition times and relative displacement between initial and final operating points for $\overline{\theta}_o \approx 26$ over three assemblies (with dual clutch damper configuration)...................... 32

Table 2.4 Comparison of measured and predicted motions in terms of the mean (μ) and standard deviation (σ) of: (a) peak-to-peak $\overline{\theta}(T)$ and (b) response regime transition times for $\overline{\theta}_o \approx 23$ (with single clutch damper configuration). .......................................................... 39

Table 3.1 Summary of the necessary system parameters (including the sequence, method of estimation, and corresponding figure) for non-linear models of the proposed experiment (X1)... 65

Table 3.2 Measured and predicted peak to peak values for oscillatory periods in $\overline{\theta}_d(T)$ of the no-clearance configuration (X1-0) given step excitation. ................................................................. 70

Table 3.3 Measured and predicted peak to peak values for selected impacts: (a) in $\overline{\theta}_d(T)$; and (b) in $\overline{\theta}_a(T)$ of the single-clearance configuration (X1-1a) given step excitation. 73
Table 3.4  Measured and predicted times of occurrence selected impacts: (a) in $\bar{\theta}_A(T)$; and (b) in $\bar{\theta}_B(T)$ of the single-clearance configuration (X1-1a) given step excitation. ................................................. 73

Table 3.5  Measured and predicted peak to peak values for selected impacts: (a) in $\bar{\theta}_A(T)$; (b) in $\bar{\theta}_B(T)$; and (c) in $\bar{\theta}_C(T)$ of the dual-clearance configuration (X1-2) given step excitation. ....... 75

Table 3.6  Measured and predicted times of occurrence for selected impacts: (a) in $\bar{\theta}_A(T)$; (b) in $\bar{\theta}_B(T)$; and (c) in $\bar{\theta}_C(T)$ of the dual-clearance configuration (X1-2) given step excitation. ....... 75

Table 4.1  Regime-dependent asymptotic trends of $\tau(T)$ for $\alpha = 0.1$ and $\alpha = 10$. ...................... 96

Table 4.2  Maximum values of estimation metrics $\Pi_{rr_{best}}$ using uniform window $w_{shp}$. ........... 105

Table 4.3  Normalized window length $w_{rr_{best}}$ necessary for $\Pi_{rr_{best}}$ using uniform window $w_{shp}$. ............................................................................................................................................. 105

Table 4.4  Window length coefficient $\beta_{rr_{best}}$ necessary for $\Pi_{rr_{best}}$ using adaptive window $w_{shp_{adp}}$. ............................................................................................................................................. 110

Table 4.5  Summary of $\Pi_{ov_{best}}$ for uniform and adaptive windows, $\Pi_{ov_{best}}(w_{shp})$ and $\Pi_{ov_{best}}(w_{shp_{adp}})$, respectively. ..................................................................................................... 111

Table 5.1  Summary of proposed formulations for dissipative torque $\Psi_D(\theta, \dot{\theta})$ corresponding to Fig. 5.11. ................................................................................................................................................. 143

Table 5.2  Intermediate parameters used in the estimation of $c_{D3j\pm}$ and $h_{D3j\pm}$ for dissipative formulation D3.1-2. .................................................................................................................................................. 148
Table 5.3 Summary of the response metrics (R1-R6) for several dissipation formulations (D1-D3): (a-c) normalized geometric norm of error between measurement and prediction; and (d) absolute difference between measured and predicted response regime periods. The ideal value for each metric is 0 (zero) and the worst case is associated with a very high positive value; lowest values (“best”) per metric are **emboldened**. 

........................................................................................... 155
List of Figures

Figure 1.1 Example illustration (with cut-away view) of a vehicle clutch damper containing multi-staged torsional springs and dry friction elements (imaged extracted from [1.14]). Key: A – flywheel/pressure plate interface; B – multi-staged coil springs; C – inner hub with shaft spline connection; D – multi-staged dry friction elements; and E – out hub (rigidly connected to A). ..... 3

Figure 1.2 Typical path-dependent torque transmission $\Gamma(\theta, \dot{\theta})$ through a four-staged symmetric vehicle clutch damper. Here, $\{\theta, \dot{\theta}\}$ are the relative angular displacement and velocity (between the inner and outer hubs, see Fig. 1.1), respectively. Key (▬▬) – $\dot{\theta} > 0$; and (▬) - $\dot{\theta} < 0$. Arrows indicate path direction................................................................. 3

Figure 1.3 Path-dependent torque transmission $\Gamma(\theta, \dot{\theta})$ through a dual-staged asymmetric clearance element. Here, $\Phi(\theta, \dot{\theta})$ and $\Psi(\theta, \dot{\theta})$ are the elastic and dissipative torques, respectively, $\{\theta, \dot{\theta}\}$ are the relative angular displacement and velocity, respectively, subscripts {I,II} and $\pm$ denote stages, $k$ is the torsional stiffness, and $\Theta$ is the angular stage transition. Key: (▬▬) – $\Gamma(\theta, \dot{\theta} > 0)$; (▬▬) – $\Gamma(\theta, \dot{\theta} < 0)$; and (▪▪▪▪) – $\Phi(\theta,0)$ ................................................................. 6
Figure 1.4  Minimal order non-linear models conceptually that illustrate the proposed experiments (dissipative elements are omitted for the sake of simplicity): (a) Singe degree of freedom non-linear system used to achieve Objective 1; (b) Two degree of freedom non-linear system used to achieve Objective 2 (single-clearance configuration); and (c) Three degree of freedom non-linear system used to achieve Objective 2 (dual-clearance configuration). Here, $\theta$ is the angular displacement, $\{A, B, C\}$ are coordinate and element indices, $k$ is the torsional stiffness, $J$ is the torsional inertia, and $T$ is the external torque.

Figure 2.1  Typical quasi-static performance curve for a four-stage, asymmetric clutch damper displaying relations for: (a) transmitted torque $\Gamma(\theta, \dot{\theta})$; (b) elastic torque $\Phi(\theta)$; and (c) dissipative $\Psi(\theta, \dot{\theta})$. Here, $\theta$ and $\dot{\theta}$ are relative angular displacement and velocity. Key: (▬▬) – $\theta > 0$; (▬) – $\theta < 0$; $k$ – torsional stiffness; $h$ – Coulomb hysteresis amplitude; and $\Theta$ – stage transition (angular). Stages are denoted by subscripts $\{0, I, \ldots IV\}$ the drive side ($\theta > 0$) by subscript $+$, and the coast side ($\theta < 0$) by -.

Figure 2.2  Conceptual illustration of the proposed non-linear experiment and excitation: (a) SDOF non-linear system; (b) system operating points on the clutch damper performance curve; and (c) external torque $T(t)$ in time domain. Key: A – torsion arm and shaft inertia $J$; B – viscous damping $c$; D – two parallel multi-staged clutch dampers of stiffness $k(\theta)$ and Coulomb hysteresis amplitude $h(\theta)$.

Figure 2.3  Physical display of the proposed non-linear experiment (using solid model). Key: A – flywheel (ground); B – clutch assembly (sleeve bearing and two parallel multi-stage clutch dampers); C – clutch shaft; D – ball bearing; E – torsion arm; F – pneumatic cylinders with a mechanical quick release; G – steel support structures (ground); and H – steel bed plate (ground).
**Figure 2.4** Illustrations of the torsion arm and measurement of angular motion using a laser vibrometer and accelerometers: (a) torsion arm geometry; and (b) instrumentation schematic. Key: A – shaft; B – shaft axis; C – torsion arm; D – laser vibrometer; E – laser; F – laser point; G – translational accelerometer V; H – finished block; and I – translational accelerometer W.

**Figure 2.5** Normalized measured motions for $\theta_0 \approx 26$ and demonstration of non-linear response regimes (with dual clutch damper configuration). Key: (▬▬) – measured motion; ( | ) – regime transition; (− ∙ −) – stage and clearance transitions; di – double-sided impact regime; si – single-sided impact regime; and ni – no-impact regime.

**Figure 2.6** Comparison of acceleration measurements of $\ddot{\theta}(T)$ from laser vibrometer, $\ddot{\theta}_V(T)$ from accelerometer V, and $\ddot{\theta}_W(T)$ from accelerometer W for $\theta_0 \approx 26$ (with dual clutch damper configuration).

**Figure 2.7** Comparison of normalized measured motions for $\theta_0 \approx 21$ and $\theta_0 \approx 33$, and demonstration of non-linear response regimes (with dual clutch damper configuration). Key: (▬▬) – measured motion for $\theta_0 \approx 21$; (▬▬) – measured motion for $\theta_0 \approx 33$; ( | ) – response regime transition for $\theta_0 \approx 21$; ( | ) – response regime transition for $\theta_0 \approx 33$; and (− ∙ −) – stage transitions.

**Figure 2.8** Comparison of measured and predicted motions for $\theta_0 \approx 23$ (with single clutch damper configuration). Key: (▬▬) – measured motion; (▬▬) – predicted motion using SDOF model; ( | ) – regime transition for the measured motion; ( | ) – regime transition for the simulated motion; (− ∙ −) – stage and clearance transitions.
Figure 3.1 Illustration of an angular symmetric clearance element $k(\theta)$. Here, elastic torque transmission is given by $\Phi(\theta)$, $\theta$ is the relative angular displacement, $\Theta_1$ is the angular transition between stage I (clearance with torsional stiffness $k_1 = 0$) and II, $k_{II}$ is the torsional stiffness of stage II, $(\theta_i, \phi_i)$ is the initial operating point, and $(\theta_f, \phi_f)$ is the final operating point.

Figure 3.2 Conceptual illustration of the proposed experiment X1 with clearances as defined in Fig. 3.1: (a) dual-clearance configuration (X1-2); (b) single-clearance configuration (X1-1a); and (c) no-clearance configuration (X1-0). Here, $\{\theta, \dot{\theta}\}$ is the angular displacement and velocity (respectively), $J$ is the torsional inertia, $k$ is the torsional stiffness, $h$ is the Coulomb friction, and $T$ is the external torque. Subscripts $\{A, B, C\}$ are element and coordinate indices. Key: 1 – torsion arm, shaft, and disks; 2 – key and keyway (clearance); 3 – coupling hub; 4 – torsional spring (linear); and 5 – shaft and bearing interface.

Figure 3.3 Solid model of the proposed experiment X1 as conceptually described in Fig. 3.2. Key: 1 – torsion arm, shaft, and disks ($J_A$); 2 – bearing; 3 – coupling hub ($J_B$); 4 – coil spring sets between coupling jaws; 5 – coupling hub ($J_C$); 6 – electromagnet mass drop; and 7 – base plate.

Figure 3.4 Instrumentation schematic for experiment X1. Here, $\ddot{\theta}$ is the calculated angular acceleration, $\ddot{q}$ is the measured translational acceleration, $J$ is the torsional inertia, and $R$ is the radial distance from the axis of rotation to the translational accelerometers [3.19]. Key: 1 – torsion arm, shaft, and disks; 2 – coupling hub; 3 – coupling hub; 4 – translational accelerometer; 5 – data acquisition device (DAQ) [3.20-21]; and 6 – computer (CPU) [3.22].

Figure 3.5 Measured (normalized) angular acceleration (a) $\overline{\theta_A(t)}$ for the no-clearance configuration of Fig. 2c (X1-0), given (b) step-like excitation $\overline{T_A(t)}$. 

xvii
Figure 3.6  Measured angular accelerations for the single-clearance configuration of Fig. 3.2b (X1-1), given step-like excitation $\vec{\ddot{\theta}}_A(\tau)$ of Fig. 3.5b: (a) $\vec{\ddot{\theta}}_A(\tau)$; and (b) $\vec{\ddot{\theta}}_B(\tau)$. ................................. 62

Figure 3.7  Measured angular accelerations for the dual-clearance configuration of Fig. 3.2a (X1-2), given step-like excitation $\vec{\ddot{\theta}}_A(\tau)$ of Fig. 3.5b: (a) $\vec{\ddot{\theta}}_A(\tau)$; (b) $\vec{\ddot{\theta}}_B(\tau)$; and (c) $\vec{\ddot{\theta}}_C(\tau)$. ................................. 63

Figure 3.8  Sub-experiments of X1 utilized for parameter estimation: (a) sub-experiment X1-P1 (estimation $J_B$ and $h_{bc}$); (b) sub-experiment X1-P2 (estimation $h_{ab}$); and (c) sub-experiment X1-P3 (estimation of $\Theta_{AB.1}$ and $k_{AB..u}$).  Here, $\{\theta, \dot{\theta}\}$ is the angular displacement and velocity (respectively), $J$ is the torsional inertia, $k$ is the torsional stiffness, $h$ is the Coulomb friction, and $T$ is the external torque. Subscripts $\{A,B,C\}$ are element and coordinate indices. Key: 1 – torsion arm, shaft, and disks; 2 – key and keyway (clearance); 3 – coupling hub; 4 – torsional spring (linear); and 5 – shaft and bearing interface. ............................................................................... 67

Figure 3.9  Measured and predicted angular acceleration $\vec{\ddot{\theta}}_A(\tau)$ (normalized) step-response of the no-clearance configuration (X1-0).  Key: (▬▬) – measurement; and (▬▬) – prediction. Here, ($j^{th}$) marks are the peak to peak values of their oscillatory periods.................................................. 70

Figure 3.10  Measured and predicted angular accelerations $\vec{\ddot{\theta}}_A(\tau)$ and $\vec{\ddot{\theta}}_B(\tau)$ (normalized) step-response of the single-clearance configuration (X1-1a).  Key: (▬▬) – measurement; and (▬▬) – prediction. Here, ($j^{th}$) impacts are marked................................................................. 72

Figure 3.11  Measured and predicted angular accelerations (normalized) for step-response of the dual-clearance configuration (X1-2): (a) $\vec{\ddot{\theta}}_A(\tau)$; (b) $\vec{\ddot{\theta}}_B(\tau)$; and (c) $\vec{\ddot{\theta}}_C(\tau)$;  Key: (▬▬) – measurement; and (▬▬) – prediction. Here, ($j^{th}$) impacts are marked.................................................. 74
Figure 3.12 Characteristics of two multi-staged clutch dampers (denoted C1 and C2) that are utilized in a prior large-scale experiment (denoted X2) [3.8]: (a) C1 (where stage I is a very compliant spring); and (b) C2 (where stage I is a spline clearance). Here, $\Phi(\theta)$ is the elastic torque transmission; see Fig. 3.1 for other symbols.

Figure 3.13 Measured step-response of clutch damper C1 (X2-C2) from the prior experiment X2 [3.8] with: (a) angular acceleration $\overline{\theta}(\overline{T})$; and (b) impulse period $\overline{\tau}_{osc}^{(j)}$ (for double (di) and single-sided (si) impact regimes only). Here, $\overline{t}_{di}$ is the time of transition from double to single-sided impact regimes and $\overline{t}_{si}$ is the time of transition from the single to no-impact regime. Key: (▬▬) – $\overline{\theta}(\overline{T})$ and (▬x▬) – $\overline{\tau}_{osc}^{(j)}$.

Figure 3.14 Measured step-response of clutch damper C2 (X2-C2) from the prior experiment [3.8] with: (a) angular acceleration $\overline{\theta}(\overline{T})$; and (b) impulse period $\overline{\tau}_{osc}^{(j)}$ (for double (di) and single-sided (si) impact regimes only). Here, $\overline{t}_{di}$ is the time of transition from double to single-sided impact regimes and $\overline{t}_{si}$ is the time of transition from the single to no-impact regime. Key: (▬▬) – $\overline{\theta}(\overline{T})$ and (▬x▬) – $\overline{\tau}_{osc}^{(j)}$.

Figure 3.15 Measured step-response of the single-clearance configuration (X1-1a) of the proposed experiment: (a) angular acceleration $\overline{\theta}_{\alpha}(\overline{T})$; and (b) impulse period $\overline{\tau}_{osc}^{(j)}$. Key: (▬▬) – $\overline{\theta}_{\alpha}(\overline{T})$ and (▬x▬) – $\overline{\tau}_{osc}^{(j)}$. 

\[ \text{xix} \]
**Figure 4.1** Conceptual illustration of the non-linear torsional system: (a) single-degree of freedom model; (b) elastic torque $\Phi(\theta)$; and (c) examples of near backlash ($\alpha = 0.1$) and pre-load ($\alpha = 10$) non-linearities examined in this article. Here, $\theta$ is the angular displacement, $J$ is the torsional inertia, $h$ is a Coulomb friction element, $k(\theta)$ is a non-linear torsional stiffness element, and $T(t)$ is the external torque.

**Figure 4.2** Predicted angular motions (normalized) for a backlash type non-linear system ($\alpha = 0.1$). Here, $\bar{\theta}$ is the angular displacement, $\bar{\theta}$ is the angular velocity, $\bar{\theta}$ is the angular acceleration, $\tau^{(j)}$ is oscillatory period $j$, $t_{oi}$ is the time of transition from the double-sided to single-sided impact regimes, $t_{oi}$ is the time of transition from the single-sided to no-impact regimes, and $\Theta_l$ is the angular transition from stage I to II. Key: (▬▬) – angular motion; (– – –) – $\Theta_l$; and ( | ) – $t_{oi}$, $t_{oi}$.  

**Figure 4.3** Predicted angular motions (normalized) for a pre-load type non-linear system ($\alpha = 10$). Here, $\bar{\theta}$ is the angular displacement, $\bar{\theta}$ is the angular velocity, $\bar{\theta}$ is the angular acceleration, $\tau^{(j)}$ is oscillatory period $j$, $t_{oi}$ is the time of transition from the double-sided to single-sided impact regimes, $t_{oi}$ is the time of transition from the single-sided to no-impact regimes, and $\Theta_l$ is the angular transition from stage I to II. Key: (▬▬) – angular motion; (– – –) – $\Theta_l$; and ( | ) – $t_{oi}$, $t_{oi}$. 
Figure 4.4  Calculated oscillatory period $\overline{\tau}(\overline{T})$ (normalized) for a torsional system with: (a) backlash type non-linearity ($\alpha = 0.1$); and (b) pre-load type non-linearity ($\alpha = 10$). Here, $\overline{\tau}_{\text{II}}$ is the natural period of stage II, $\overline{\tau}_{\text{II}}$ is the time of transition from double-sided to single-sided impact regime, and $\overline{\tau}_{\text{II}}$ is the time of transition from the single-sided to no-impact regime. Key: (▬▬) – calculated from $\overline{\theta}(\overline{T})$; (▬▬) – calculated from $\overline{\theta}(\overline{T})$; (▪▪▪▪▪) – calculated from $\overline{\theta}(\overline{T})$; (— --------) – $\overline{\tau}_{\text{II}}$; and ( | ) – $\{\overline{\tau}_{\text{II}}, \overline{\tau}_{\text{II}}\}$. .......................................................... 95

Figure 4.5 Uniform windowing functions $w_{\text{shp}}$ used for estimations: (a) box-car, $w_{\text{box}}$; (b) triangular, $w_{\text{tri}}$; (c) saw-tooth, $w_{\text{saw}}$. Here, $t'$ is an arbitrary time and $\tau_{w}$ is the window length............................................................................................................................... 98

Figure 4.6 Instantaneous effective stiffness $\overline{k}(\overline{T})$ (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with windowing function $w_{\text{box}}$ and uniform window lengths of: (a) $\overline{\tau}_{w} = 0.5$; (b) $\overline{\tau}_{w} = 1$; and (c) $\overline{\tau}_{w} = 2$. Here, $\overline{T}^{(1)}$ and $\overline{T}^{(P)}$ are the domain limits of $\overline{T}(\overline{T})$, $\overline{\tau}_{\text{II}}$ and $\overline{\tau}_{\text{II}}$ are regime transition times, and $\overline{k}_{I}$ and $\overline{k}_{\text{II}}$ are the torsional stiffness values for stages I and II, respectively. Key: (▬▬) – $\overline{k}(\overline{T})$; ( | ) – $\{\overline{T}^{(1)}, \overline{T}^{(P)}, \overline{\tau}_{\text{II}}, \overline{\tau}_{\text{II}}\}$; and (— --------) – $\{\overline{k}_{\text{II}}, \overline{k}_{\text{II}}\}$. .......................................................... 99

Figure 4.7 Instantaneous effective stiffness $\overline{k}(\overline{T})$ (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with uniform window length $\overline{\tau}_{w} = 1$ and the following windowing functions: (a) $w_{\text{box}}$; (b) $w_{\text{tri}}$; and (c) $w_{\text{saw}}$. Here, $\overline{T}^{(1)}$ and $\overline{T}^{(P)}$ are the domain limits of $\overline{T}(\overline{T})$, $\overline{\tau}_{\text{II}}$ and $\overline{\tau}_{\text{II}}$ are regime transition times, and $\overline{k}_{I}$ and $\overline{k}_{\text{II}}$ are the torsional stiffness values for stages I and II, respectively. Key: (▬▬) – $\overline{k}(\overline{T})$; ( | ) – $\{\overline{T}^{(1)}, \overline{T}^{(P)}, \overline{\tau}_{\text{II}}, \overline{\tau}_{\text{II}}\}$; and (— --------) – $\{\overline{k}_{\text{II}}, \overline{k}_{\text{II}}\}$. ............ 101

xxi
Figure 4.8 Instantaneous effective $\tilde{k}(\bar{t})$ and mean $\tilde{k}_\mu(\bar{t})$ stiffness values (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with uniform window length $\bar{w}_w = 1$, and windowing function $w_{\text{box}}$. Here, $\bar{T}^{(i)}$ and $\bar{T}^{(p)}$ are the domain limits of $\bar{t}$, $\bar{t}_d$ and $\bar{t}_s$ are regime transition times, and $\bar{k}_{II}$ is the torsional stiffness of stage II. Key: (▬▬) – $\tilde{k}(\bar{t})$; (▬▬) – $\tilde{k}_\mu(\bar{t})$; ( | ) – $\{\bar{T}^{(i)}, \bar{T}^{(p)}, \bar{t}_d, \bar{t}_s\}$; and (— — —) – $\bar{k}_{\text{all}}$. ......................................................................................................................... 102

Figure 4.9 Effective undamped linear time-invariant system with instantaneous effective stiffness $\tilde{k} \big|_{t=t'}$ at time $t = t'$. Here, $\theta$ is the angular displacement and $J$ is the torsional inertia. ........... 104

Figure 4.10 Calculated $\bar{t}$ and estimated $\tilde{\bar{t}}(\bar{t})$ oscillatory periods (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with uniform window length $\bar{w}_w = 2$ and the following windowing functions: (a) $w_{\text{box}}$; (b) $w_{\text{tri}}$; and (c) $w_{\text{saw}}$. Here, $\bar{T}^{(i)}$ and $\bar{T}^{(p)}$ are the domain limits of $\bar{t}$, and $\bar{t}_d$ and $\bar{t}_s$ are regime transition times. Key: (▬▬) – $\bar{t}$; (▬▬) – $\tilde{\bar{t}}(\bar{t})$; ( | ) – $\{\bar{T}^{(i)}, \bar{T}^{(p)}, \bar{t}_d, \bar{t}_s\}$; and (— — —) – $\bar{t}_{\text{all}}$. ......................................................................................................................... 106

Figure 4.11 Regime-dependent metrics $\Pi_{rr}$ for a backlash type non-linear system ($\alpha = 0.1$) with uniform window $w_{\text{slp_slp}}$. Here, $\bar{w}_w$ is the uniform window length (normalized), $d_i$ is the double-sided impact regime, $s_i$ is the single-sided impact regime, $o_v$ is the entire response, and $R[\bar{t}_{rr}]$ is the range of $\bar{t}$ (observed) in regime $rr$. Key: (▬▬) – $w_{\text{box}}$; (▬▬) – $w_{\text{tri}}$; (▬▬ ▬) – $w_{\text{saw}}$; and ( | ) – $R[\bar{t}_{rr}]$. ......................................................................................................................... 107
Figure 4.12 Regime-dependent metrics $I_n$ for a backlash type non-linear system ($\alpha = 0.1$) with adaptive window $w_{shp, adp}$. Here, $\beta$ is the window length factor, $d_i$ is the double-sided impact regime, $s_i$ is the single-sided impact regime, $o_v$ is the entire response, and $R[I]$ is the range of $\tau(t)$ (observed) in regime $rr$. Key: (□□□□□) -- $w_{box, adp}$; (□□□□□) -- $w_{tri, adp}$; (□□□□□) -- $w_{saw, adp}$; and (□□□□□) -- $R[I]$. ........................................................................................................................................ 109

Figure 4.13 Illustration of multi-staged elastic torque curves $\Phi(\theta)$, and initial $(\theta_i, T_o)$ and final $(\theta_f, T_f)$ operating points for two experiments used to validate estimation methods: (a) experiment X1 [3.18]; and (b) experiment X2 [3.8]. Here, $\theta$ is the angular displacement, $T$ is external torque, $\Theta_j$ are angular stage transitions, and subscript $\{I, II, \ldots\}$ denote stages. ................................................. 112

Figure 4.14 Calculated $\tau(t)$ and estimated $\hat{\tau}(t)$ oscillatory periods (normalized) for experiment X1 [3.18] using adaptive windowing function $w_{tri, adp}$ with window length factor $\beta = 2$ and initial window length $\tau_{wo} = 1$. Here, $\tau^{(i)}$ and $\tau^{(p)}$ are the domain limits of $\tau(t)$. Key: (□□□□□) -- $\tau(t)$; (□□□□□) -- $\tau(t)$; (□□□□□) -- $\tau(t)$; (□□□□□) -- $\tau(t)$; and (□□□□□) -- $\tau(t)$. ........................................................................................................... 113

Figure 4.15 Calculated $\tau(t)$ and estimated $\hat{\tau}(t)$ oscillatory periods (normalized) for experiment X2 [3.8] using adaptive windowing function $w_{tri, adp}$ with window length factor $\beta = 2$ and initial window length $\tau_{wo} = 1$. Here, $\tau^{(i)}$ and $\tau^{(p)}$ are the domain limits of $\tau(t)$. Key: (□□□□□) -- $\tau(t)$; (□□□□□) -- $\tau(t)$; (□□□□□) -- $\tau(t)$; (□□□□□) -- $\tau(t)$; and (□□□□□) -- $\tau(t)$. ........................................................................................................... 114

xxiii
**Figure 5.1** Illustration of a multi-staged vehicle clutch damper: (a) schematic with parts labeled; (b) photograph of a typical production component; and (c) measured (normalized) quasi-static performact curve (arrows indicate path direction). Here, $\Gamma_Q$ is the normalized torque transmitted through the device, $\theta$ is the normalized relative angular displacement ($\theta = \theta_{o-hub} - \theta_{i-hub}$), and $\Theta$ is the normalized relative angular velocity. Key for (a): A – flywheel and pressure plate interface; B – rivet; C – multi-staged coil spring; D – outer hub (angular displacement $\theta_{o-hub}$); E – inner hub (angular displacement $\theta_{i-hub}$); and F – multi-staged Coulomb friction element. ........................ 123

**Figure 5.2** Conceptual illustration of torque transmission through a symmetric, two-stage vehicle clutch damper under quasi-static loading conditions. Here, $\Gamma_Q(\theta, \dot{\theta})$ is transmitted torque, $\theta$ is angular displacement, and $\dot{\theta}$ is angular velocity. Stages are denoted by subscripts I and II, torsional stiffness is denoted by $k$, and angular stage transitions are denoted by $\Theta$. Key: (▬►▬) – transmitted torque $\Gamma_Q(\theta, \dot{\theta})$; (▬▬) – elastic torque component $\Phi_Q(\theta)$; and (−∙−) – angular stage transition $\Theta$. ....................................................................................................................... 126

**Figure 5.3** Quasi-static and dynamic experiments for a vehicle clutch damper: (a) conceptual illustration for quasi-static loading (denoted X-Q); (b) conceptual and (c) physical illustrations for dynamic loading (denoted X-D). Here, $\theta$ is angular displacement, $\dot{\theta}$ is angular velocity, $\Phi$ is the elastic torque transmission, $\Psi$ is the dissipative torque transmission, $\Gamma$ is the total torque transmission ($\Gamma = \Psi + \Phi$), $T$ is external torque, and $J$ is torsional inertia. Key for (c): A – flywheel (ground); B – clutch assembly (houses one clutch damper); C – shaft; D – bearing; E – torsion arm; F – pneumatic actuators; G – structural supports (ground); and H – bed plate (ground). ...................................................................................................................................... 128
Figure 5.4  Typical measurements in time and physical domains from quasi-static experiment X-Q. Here, $\bar{\theta}$ is the normalized angular displacement, $\bar{\dot{\theta}}$ is the normalized angular velocity, $\bar{\Gamma}_0$ is the normalized transmitted torque, and $\bar{T}$ is the normalized time. Key: (▬▬) – path P1; (▬) – path P2; (▬ ▬) – path P3; (▬ ∙ -) – angular stage transitions $\Theta_{jz}$; \{0, I, … IV\} – stage indices; (+) – drive side; and (-) – coast side. ........................................................................................... 131

Figure 5.5  Predicted elastic and dissipative torques in quasi-static experiment X-Q. Here, $\phi(\bar{\theta})$ is the normalized elastic torque and $\Psi(\bar{\theta}, \bar{\dot{\theta}})$ is the normalized dissipative torque. Key: (▬▬) – path P1; (▬) – path P2; (▬ ▬) – path P3; (▬ ∙ -) – angular stage transitions $\Theta_{jz}$; \{0, I, … IV\} – stage indices; (+) – drive side; and (-) – coast side. ........................................................................................................ 133

Figure 5.6  Predicted and measured (normalized) torque transmission $\bar{\Gamma}_0$ for quasi-static experiment X-Q. Here, $\bar{\Gamma}_0$ is the time derivative of $\Gamma_0$. Key: (▬▬) – measurement; (▬▬) – prediction; (▬ ∙ -) – angular stage transitions $\Theta_{jz}$; \{0, I, … IV\} – stage indices; (+) – drive side; and (-) – coast side. ........................................................................................................ 133

Figure 5.7  Typical measured motions from dynamic experiment X-D. Here, $\bar{\theta}$ is the normalized angular displacement, $\bar{\dot{\theta}}$ is the normalized angular velocity, and $\bar{\ddot{\theta}}$ the is normalized angular acceleration. Key: (▬▬) – measured motion; (▬ ∙ -) – angular stage transitions $\Theta_{jz}$; (|) – response regime transition; (di, si, ni) – double-sided, single-sided, and no-impact regimes; \{0, I, II\} – stage indices; (+) – drive side; and (-) – coast side. ........................................................................................................ 135

Figure 5.8  Effect of window length on the estimated dynamic torque transmission $\bar{\Gamma}_0(\bar{T})$ for dynamic experiment X-D in time domain. Key: (▬▬) – $\tau_g = 0.05 \tau_s$; and (▬) – $\tau_g = 0.4 \tau_s$. 138
Figure 5.9 Comparison of quasi-static $F_Q(t)$ and estimated dynamic $\Gamma_D(t)$ torques for dynamic experiment X-D in time domain. Key: $(\quad)$ – $F_Q(t)$; $(\quad)$ – $\Gamma_D(t)$; and $(\quad)$ – limits of period $T^{(i)}$.

Figure 5.10 Comparison of quasi-static $F_Q(\theta, \dot{\theta})$ and estimated dynamic $\Gamma_D(\theta, \dot{\theta})$ torque for experiment X-D in physical domain over the oscillatory period $T^{(i)}$. Key: $(\quad)$ – $F_Q(\theta, \dot{\theta} > 0)$; $(\quad)$ – $F_Q(\theta, \dot{\theta} < 0)$; $(\quad)$ – $\Gamma_D(\theta, \dot{\theta} > 0)$; $(\quad)$ – $\Gamma_D(\theta, \dot{\theta} < 0)$; $(\quad)$ – angular stage transitions $\Theta_{j \pm}$; $\{0, I, II\}$ – stage indices; $(\quad)$ – drive side; and $(\quad)$ – coast side.

Figure 5.11 Comparison of quasi-static elastic $Q_{\theta}(\theta)$ and dynamic $\Phi_{\theta}(\theta, 0)$ elastic torques for experiment X-D in physical domain at $\theta = 0$. Key: $(\quad)$ – $Q_{\theta}(\theta)$; $(\quad)$ – $\Phi_{\theta}(\theta, 0)$; $(\quad)$ – angular stage transitions $\Theta_{j \pm}$; $\{0, I, II\}$ – stage indices; $(\quad)$ – drive side; and $(\quad)$ – coast side.

Figure 5.12 Conceptual illustration of formulations D0-D3 for the dissipative torque $\Psi_D(\theta, \dot{\theta})$. Here, $\theta$ is the angular displacement, $c$ denotes torsional viscous damping, and $h$ denotes Coulomb friction amplitude.

Figure 5.13 Comparison between measured and predicted (with formulation D0) motions for experiment X-D. Key: $(\quad)$ – measured motion; $(\quad)$ – predicted motion; $(\quad)$ – angular stage transitions $\Theta_{j \pm}$; $(\quad)$ – regime transitions of the measured response; $\{d, s, n\}$ – double-sided, single-sided, and no-impact regimes of the measured response; $\{0, I, II\}$ – stage indices; $(\quad)$ – drive side; and $(\quad)$ – coast side.
**Figure 5.14** Comparison between measured and predicted (with formulation D3.2) motions for experiment X-D. Key: (▬▬) – measured motion; (▬▬) – predicted motion; (−∙−) – angular stage transitions \( \Theta_j \); (|) – regime transitions of the measured response; \{di, si, ni\} – double-sided, single-sided, and no-impact regimes of the measured response; \{0, I, II\} – stage indices; (+) – drive side; and (-) – coast side. 153

**Figure 5.15.** Comparison between measured and predicted (with formulations D0, D2.1, and D3.2) \( \ddot{\theta}(t) \) for experiment X-D. Key: (▬▬) – measurement; and (▬▬) – prediction. 154

**Figure C.1** Comparison between measured and predicted (with formulation D1.1) motions for dynamic experiment X-D. Here, \( \bar{\theta} \) is the normalized angular displacement, \( \bar{\theta} \) is the normalized angular velocity, and \( \bar{\dot{\theta}} \) is the normalized angular acceleration. Key: (▬▬) – measured motion; (▬▬) – predicted motion; (−∙−) – angular stage transitions \( \bar{\Theta}_j \); (|) – regime transitions of the measured response; \{di, si, ni\} – double-sided, single-sided, and no-impact regimes of the measured response; \{0, I, II\} – stage indices; (+) – drive side; and (-) – coast side. 176

**Figure C.2** Comparison between measured and predicted (with formulation D1.2) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption. 177

**Figure C.3** Comparison between measured and predicted (with formulation D2.1) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption. 178

**Figure C.4** Comparison between measured and predicted (with formulation D2.2) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption. 179

**Figure C.5** Comparison between measured and predicted (with formulation D3.1) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption. 180
Chapter 1: Introduction

1.1 Motivation

Many real-life devices, such as vibration isolators [1.1] and torque transmission components [1.2-13], contain discontinuous non-linear features by design or otherwise (e.g. assembly or manufacturing error and wear). Ground vehicle drivelines and their components [1.3-13] are representative examples of this family. For instance, consider a multi-staged vehicle clutch damper (illustrated in Fig. 1.1 [1.14]) that must transmit a wide range of mean operating torque (say from idling to driving conditions) while attenuating pulsations from the engine to the vehicle transmission [1.8]. Accordingly, clutch dampers intentionally contain multi-staged torsional springs, clearances, stopper elements, pre-load features, and multi-staged dry friction elements [1.3-5, 1.8-10]. Typical characteristics are illustrated in Fig. 1.2 where \( \Gamma'(\theta, \dot{\theta}) \) is the path-dependent torque transmission through the device and \( \{\theta, \dot{\theta}\} \) are the relative angular displacement and velocity, respectively.

Although the mentioned discontinuous features are necessary, they pose significant challenges in both research work and engineering practice. Foremost, closed-form solutions are not possible, thus smoothing approximations [1.15] in computational methods are typically employed [1.3, 6, 9-13]. Additionally, characterization of the device’s full operating range under dynamic conditions is hindered by strong non-linearities and high
torque capacities. As a result, parameter values are usually extracted from laboratory measurements [1.17] under quasi-static conditions only [1.3-5, 8-10]. Furthermore, such features exhibit rich non-linear behavior such as vibro-impact phenomena (gear rattle [1.3, 9-10, 12-13] and vehicle driveline clunk [1.3-7, 11]) with distinct regimes (double and single-sided impact [1.12]) and time-varying oscillatory periods [1.3-7]. Very few laboratory experiments are reported in the literature [1.3-7]. More experimental studies are obviously needed to: 1) provide physical insight, 2) validate non-linear simulation models, and 3) estimate and select parameters for quasi-static or dynamic conditions. This is the subject of this dissertation.

1.2 Literature review

Literature on the experimental study of discontinuous non-linear torsional systems is sparse [1.3-7]. Limited laboratory experiments in the context of vehicle drivelines [1.3-7] are large-scale non-linear systems that typically accommodate production components (though only a few contain clutch dampers [1.3-5]). These usually demonstrate the dynamic response to a step-like torque excitation (to simulate the vehicle driveline clunk). The corresponding non-linear models [1.3-5], which are utilized for design and simulation, make the critical assumption that elastic and dry friction parameters can be estimated from a separate quasi-static experiment [1.17]. Additionally, assumed modal damping schemes [1.3-7] are employed despite the non-linear nature of such systems. The measured and predicted motions for all experiments [1.3-7] exhibit
Figure 1.1 Example illustration (with cut-away view) of a vehicle clutch damper containing multi-staged torsional springs and dry friction elements (imaged extracted from [1.14]). Key: A – flywheel/pressure plate interface; B – multi-staged coil springs; C – inner hub with shaft spline connection; D – multi-staged dry friction elements; and E – out hub (rigidly connected to A).

Figure 1.2 Typical path-dependent torque transmission $\Gamma(\theta, \dot{\theta})$ through a four-staged symmetric vehicle clutch damper. Here, \{\theta, \dot{\theta}\} are the relative angular displacement and velocity (between the inner and outer hubs, see Fig. 1.1), respectively. Key (---) -- $\dot{\theta} > 0$; and (-----) - $\dot{\theta} < 0$. Arrows indicate path direction.
the intended vibro-impact phenomena; these also illustrate time-varying oscillatory periods with regime-dependent (i.e. single or double-sided impact regimes) symptoms. Though trends in time-varying oscillatory periods [1.3-7] have not been examined, similar behavior is discussed in prior work by Wallaschek [1.17] and Rook and Singh [1.13]. Both articles [1.13, 18] utilize stochastic linearization techniques to estimate the effective natural frequencies (as a function of the operating range) of devices that contain discontinuous elastic elements, such as idler gear sets [1.13] and shock absorbers [1.18]. While the reported experiments [1.3-7] have several intrinsic advantages and offer much-needed system-level insight [1.16], they fail to provide benchmark datasets for individual components. More detailed and critical literature reviews will be provided in Chapters 2-5.

1.3 Problem formulation

1.3.1 Unresolved research issues

Some of the unresolved research issues regarding the step-response of discontinuous non-linear torsional systems [1.3-7] are summarized as follows: 1) There is a lack of time domain benchmark datasets for components that contain a) both known and unknown discontinuous non-linear features, and as well b) only known and well-controlled discontinuous non-linear features; 2) Trends in time-varying oscillatory periods have not been adequately examined within the context of existing stochastic linearization techniques [1.13, 18]; and 3) Precise correlations between parameters estimated from quasi-static [1.17] and dynamic experiments [1.3-5] have not been evaluated. Accordingly, the goal of
this research is to develop two new laboratory experiments that provide benchmark
datasets, and propose time domain estimation methods for time-varying oscillatory periods
as well as elastic and dissipative parameters.

1.3.2 Scope, assumptions, and objectives

The scope of this dissertation is limited to the step-response of positive-definite
discontinuous non-linear torsional systems. All analyses are conducted in the time and
physical domains, and only low-dimensional (1, 2, or 3 degrees of freedom) non-linear
models are considered, consistent with prior work [1.3-7, 9-13]. For the sake of simplicity
(and to develop tractable models), all discontinuous non-linearities are defined by
piecewise linear formulations with well-known static/quasi-static properties. Additionally,
contact mechanics is not used to calculate elastic or dissipative parameters. For example,
consider the path-dependent torque \( \Gamma(\theta, \dot{\theta}) = \Phi(\theta, \dot{\theta}) + \Psi(\theta, \dot{\theta}) \), where \( \Phi(\theta, \dot{\theta}) \) and
\( \Psi(\theta, \dot{\theta}) \) are the elastic and dissipative torques, respectively, transmitted through a dual-
staged asymmetric clearance element, as illustrated in Fig. 1.3. Here, Roman numerals and
\( \pm \) in subscripts denote stages, \( k_{j\pm} \) is the torsional stiffness of stage \( j \pm \), and \( \Theta_{j\pm} \) is the
angular transition from stage \( j \pm \) to \((j+1) \pm \). (Also, refer to the List of Symbols at the
end of this chapter for nomenclature). The elastic torque near zero angular velocity \( \Phi(\theta,0) \)
is defined by the following equation where \( \Xi(\theta) \) is the ideal unit step function:

\[
\Phi(\theta,0) = \begin{cases} 
(k_{II} - k_1)(\theta - \Theta_{1})\Xi(\theta - \Theta_{1}) + ... \\
k_1\theta + ... \\
(k_{II} - k_1)(\theta - \Theta_{1})\Xi(\theta - \Theta_{1}) 
\end{cases} 
\]  

(1.1)
The limiting cases of $\Phi(\theta,0)$ are summarized as follows: i) a backlash non-linearity for $k_i = 0$; ii) a pre-load feature for $k_i \rightarrow \infty$; iii) a stopper element for $k_{II+}, k_{II-} \rightarrow \infty$; and iv) a linear spring for $k_i = k_{II+} = k_{II-}$ [1.19].

Figure 1.3 Path-dependent torque transmission $\Gamma(\theta,\dot{\theta})$ through a dual-staged asymmetric clearance element. Here, $\Phi(\theta,\dot{\theta})$ and $\Psi(\theta,\dot{\theta})$ are the elastic and dissipative torques, respectively, $\{\theta,\dot{\theta}\}$ are the relative angular displacement and velocity, respectively, subscripts {I,II} and ± denote stages, $k$ is the torsional stiffness, and $\Theta$ is the angular stage transition. Key: (▬▬) – $\Gamma(\theta,\dot{\theta} > 0)$; (▬▬▬) – $\Gamma(\theta,\dot{\theta} < 0)$; and (▪▪▪▪) – $\Phi(\theta,0)$.
The key assumptions for this work are as follows: 1) Dry friction is approximated by a hyperbolic tangent Coulomb model [1.10, 15]; 2) Discontinuous steps are smoothened by hyperbolic tangent functions [1.15], when necessary; 3) Angular motions are calculated from translational motion measurements; 4) Static/quasi-static elastic and dissipative parameters are utilized when dynamic parameters are unknown; 5) All torque transmission elements are described by a parallel configuration of elastic and dissipative elements; 6) All findings of this dissertation could be extended to semi-definite torsional systems.

The specific objectives of this dissertation are outlined below along with sub-objectives. The objectives are organized to parallel Chapters 2 to 5.

**Objective 1** (Addressed by Chapter 2)

Propose a new laboratory experiment that demonstrates the step-response of a component which contains known and unknown discontinuous non-linear features.

1a) Discuss and resolve practical mechanical design, instrumentation, and signal processing issues such as system actuation, usage of accelerometers, and time domain analysis;

1b) Verify that the experiment is operating as designed by comparing measurements to prediction from a simplified single degree of freedom non-linear model (conceptually illustrated in Fig. 1.4a).
Figure 1.4 Minimal order non-linear models conceptually that illustrate the proposed experiments (dissipative elements are omitted for the sake of simplicity): (a) Single degree of freedom non-linear system used to achieve Objective 1; (b) Two degree of freedom non-linear system used to achieve Objective 2 (single-clearance configuration); and (c) Three degree of freedom non-linear system used to achieve Objective 2 (dual-clearance configuration). Here, $\theta$ is the angular displacement, \{A,B,C\} are coordinate and element indices, $k$ is the torsional stiffness, $J$ is the torsional inertia, and $T$ is the external torque.
Objective 2 (Addressed by Chapter 3)

Propose a new laboratory experiment that demonstrates the step-response of a component which contains only known and well-controlled discontinuous non-linear features.

2a) Design the proposed experiment to accommodate single and dual-clearance configurations (conceptually illustrated in Figs. 1.4b-c, respectively);
2b) Verify the feasibility of the proposed experiment by comparing typical measurements with predictions from non-linear models;
2c) Demonstrate the utility of the proposed experiment through comparative studies between the single-clearance configuration and a large-scale laboratory experiment (see Objective 1).

Objective 3 (Addressed by Chapter 4)

Extend prior work to develop a time domain method to estimate the time-varying oscillatory period of the step-response of a non-linear torsional device.

3a) Propose an instantaneous effective stiffness concept using stochastic linearization techniques [1.13, 18];
3b) Propose the necessary digital signal processing parameters for related time domain analyses;
3c) Validate the proposed method by estimating the asymptotic trends in the time-varying oscillatory period of two laboratory experiments (see Objectives 1 and 2).
Objective 4 (Addressed by Chapter 5)

Formulate a time domain method to estimate elastic and dissipative parameters of a component that contains known and unknown discontinuous non-linear feature (see Objective 1) and address the following hypothetical questions:

4a) Are elastic and dissipative parameters estimated from a quasi-static experiment equivalent to those estimated from a dynamic experiment?

4b) Which dissipative mechanism (viscous damping or Coulomb friction) is dominant under dynamic conditions?

4c) Given a dynamic response in which the mean operating point rapidly crosses multiple stages, is the dissipative torque $\Psi(\theta, \dot{\theta})$ best described in a global (like prior work [1.3-7, 9-11]) or local fashion?

Finally, each chapter is self-sufficient, and the content of each is summarized in Table 1.1 to illustrate the organization of this dissertation.

Table 1.1 Summary of Objectives 1-4 with corresponding chapters.

<table>
<thead>
<tr>
<th>Topic</th>
<th>Objective 1 Chapter 2</th>
<th>Objective 2 Chapter 3</th>
<th>Objective 3 Chapter 4</th>
<th>Objective 4 Chapter 5</th>
</tr>
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<tbody>
<tr>
<td>Propose new experiment</td>
<td>✓</td>
<td>✓</td>
<td></td>
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<tr>
<td>Develop non-linear models</td>
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<td>✓</td>
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<tr>
<td>Examine time-varying oscillatory period</td>
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<tr>
<td>Estimate parameters under dynamic conditions</td>
<td></td>
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<td></td>
<td>✓</td>
</tr>
</tbody>
</table>
References for Chapter 1


List of symbols for Chapter 1

\( J \)  \( \) torsional inertia  
\( k \)  \( \) torsional stiffness  
\( T \)  \( \) external torque  
\( \Gamma \)  \( \) total torque transmission  
\( \theta, \dot{\theta} \)  \( \) angular displacement and velocity  
\( \Theta \)  \( \) angular stage transition  
\( \Xi \)  \( \) ideal unit step function  
\( \Phi \)  \( \) elastic torque  
\( \Psi \)  \( \) dissipative torque  

Subscripts

\( A,B,C \)  \( \) element and coordinate indices  
\( I,II,... \)  \( \) stage indices  
\( + \)  \( \) drive side \( (\theta > 0) \)  
\( - \)  \( \) coast side \( (\theta < 0) \)
Chapter 2: Step-response of a torsional device with multiple discontinuous non-linearities: Formulation of a vibratory experiment

2.1 Introduction

Practical multi-staged torsional stiffness and friction elements, such as clutch dampers for vehicle drivelines, are designed to transmit variable torque loads while mitigating clearance-induced impact phenomena in gearboxes and other driveline elements [2.1-7]. Such devices are discontinuously non-linear by nature (intentional or otherwise) as they contain multiple spring and variable friction elements that are spread over several stages depending on the performance requirements of the car or truck powertrain subsystem [2.1]. Intentional discontinuous non-linear features include clearances, abrupt changes in stiffness, stoppers, pre-load features, and dry friction elements. A typical torque-relative displacement \( \Gamma(\theta, \dot{\theta}) \) curve is shown schematically in Fig. 2.1a where stage transitions are denoted by \( \Theta \) and stages are indexed by Roman numeral subscripts. In this example, \( \Gamma(\theta, \dot{\theta}) \) is described by a sum of piecewise elastic \( \Phi(\theta) \) and dissipative torque \( \Psi(\theta, \dot{\theta}) \) (as shown in Fig.2.1b-c). The drive side (subscript +) and coast side (subscript -) are often designed to be asymmetric [2.1, 2]. In high load applications, a multi-staged clutch damper may have a torque capacity of the order of 1000 Nm and stiffness ratios between adjacent stages might be as high as 100 or 1000.
Figure 2.1 Typical quasi-static performance curve for a four-stage, asymmetric clutch damper displaying relations for: (a) transmitted torque $\Gamma(\theta, \dot{\theta})$; (b) elastic torque $\Phi(\theta)$; and (c) dissipative $\Psi(\theta, \dot{\theta})$. Here, $\theta$ and $\dot{\theta}$ are relative angular displacement and velocity. Key: ( ) – $\dot{\theta} > 0$; ( ) – $\dot{\theta} < 0$; $k$ – torsional stiffness; $h$ – Coulomb hysteresis amplitude; and $\Theta$ – stage transition (angular). Stages are denoted by subscripts {0, I, II, III} the drive side ($\theta > 0$) by subscript + , and the coast side ($\theta < 0$) by -.
It is evident that linear models cannot fully describe the dynamics of multi-staged clutch dampers, especially when considering transient phenomena such as driveline clunk [2.4]. Development of the appropriate non-linear models requires validation through benchmark time-domain measurements that are often not available. Previous articles have analyzed discontinuous non-linear systems mostly by using mathematical or computational models [2.2-18]; however, few papers use experimental methods [2.2, 14-18] and even fewer focus on transient events [2.2, 17, 18]. For instance, Gurm et al. [2.17] proposed a step-response experiment to study the clunk behavior of a two degree of freedom driveline system given a single clearance and dry friction element. Crowther et al. [2.18] suggested a similar experiment to study the step-response of a multiple degree of freedom driveline system with multiple clearances. Both prior experiments [2.17, 18] considered non-rotating systems that were excited by an external step-like torque provided by a variable mass drop (using magnets); however, neither included multi-staged stiffness or friction elements.

Additionally, characterization of dissipative elements within multi-staged clutch dampers, such as dry friction, is critical to understanding transient phenomena. In particular, dry friction in translational mechanical systems has been a thoroughly investigated topic; see Wojewoda et al. [2.19] for a review of prior work. Parameter estimation for refined models requires a controlled laboratory experiment that provides accurate measurement of friction force and interfacial velocity, such as the translational system proposed by Wojewoda et al. [2.19]. However, measurement of these signals in a
practical system that utilizes a torsional device is very difficult due to the device’s inherent complexity (i.e. known and unknown non-linear features) and limited instrumentation.

Therefore, the chief goal of this article is to propose a refined vibratory (non-rotating) experiment that exhibits the step-response of a torsional device with multi-staged stiffness and friction elements. Additional objectives include the following: 1) discuss and resolve practical mechanical design, instrumentation, and signal processing issues such as system actuation, usage of accelerometers, and time domain analysis; and 2) verify that the experiment is operating as designed by comparing measurements to predictions from a simplified single degree of freedom (SDOF) non-linear model. Characterization of specific dissipative elements within the torsional device will be left to future studies.

2.2 Problem Formulation

The proposed non-rotating experiment is conceptually illustrated in Fig. 2.2a. For design purposes the experiment is simply described as a SDOF non-linear torsional system. An inertia element (consisting of a shaft and torsion arm) is coupled to ground through a shaft spline in series with two parallel practical multi-staged torsional springs of stiffness \( k(\theta) \) and Coulomb hysteresis elements \( h(\theta) \) that are assumed from quasi-static performance curves [2.5]; also, refer to the end of this chapter for the full list of symbols. It is assumed that the multi-staged springs and hysterisis elements are identical, synchronous, and have negligible torsional inertia. Additional dissipation sources could be modeled by a viscous damping element \( (c) \) located between \( J \) and ground; further
system modeling and estimation of parameters are discussed later. Specific experimental objectives are established as follows.

1) Select production components, such as multi-staged clutch dampers, clutch assembly, clutch shaft, and flywheel, that provide $k(\theta)$ and $h(\theta)$; the clutch damper must have at least three stages, say I, II and III, where II is a pre-load feature.

2) Select components such that the corresponding linear system (with stiffness $k_{III} = (\Phi_{III} - \Phi_{II}) / (\Theta_{III} - \Theta_{II})$ and torsional inertia $J$) has a natural period $(\tau_n = 2\pi / (k_{III} / J)^{0.5})$ of about 100 ms; the step response of most real-world vehicle drivelines is closely related to the torsional surge mode which has a natural frequency between 5 and 15 Hz and is often estimated using a SDOF approximation [2.12-13, 17-18].

3) Select the first relevant flexural mode of the torsion arm to be greater than 100 Hz to ensure minimal interaction with angular motion $\theta(t)$.

4) Excite the system with a step-like torque $T(t)$ that is approximated by a ramp function initiated at $t = 0$ with duration $t_e$ and is defined by initial $(\theta_o, T_o)$ and final $(\theta_f, T_f)$ operating points of $J$; the intent here is to ensure a non-linear response by allowing the points to lie on different stages of the clutch dampers that are defined by the following and illustrated in Fig. 2.2b-c

\[
\theta_o \in [\Theta_{III}, \Theta_{II}], \quad \theta_f \in [\Theta_{III}, \Theta_{II}], \quad (2.1a-b) \\
T_o \in [\Phi_{III}, \Phi_{II}], \quad T_f \in [0, \Phi_{II}], \quad (2.2a-b)
\]
5) Choose the sampling period \( (\tau_s) \) to be at most 0.1 ms to prevent aliasing of measured signals.

6) Measure velocity by a direct mean to improve the numerical estimation of angular displacement \( \dot{\theta}(t) \) and acceleration \( \ddot{\theta}(t) \).

**Figure 2.2** Conceptual illustration of the proposed non-linear experiment and excitation: (a) SDOF non-linear system; (b) system operating points on the clutch damper performance curve; and (c) external torque \( T(t) \) in time domain. Key: A – torsion arm and shaft inertia \( J \); B – viscous damping \( c \); D – two parallel multi-staged clutch dampers of stiffness \( k(\theta) \) and Coulomb hysteresis amplitude \( h(\theta) \).
2.3 Physical design of the non-linear experiment

The proposed non-rotating experiment is illustrated in Fig. 2.3 where the flywheel is grounded to a massive steel structure and bedplate for the sake of simplicity. The clutch assembly is bolted to the flywheel at one end and the clutch is engaged (i.e. no slip between the flywheel and clutch dampers). The clutch assembly houses a greased sleeve bearing and two parallel multi-staged clutch dampers. The clutch shaft contacts the two clutch dampers through a spline and is radially supported by the sleeve bearing and a ball bearing that is exterior to the clutch assembly.

Figure 2.3 Physical display of the proposed non-linear experiment (using solid model). Key: A – flywheel (ground); B – clutch assembly (sleeve bearing and two parallel multi-stage clutch dampers); C – clutch shaft; D – ball bearing; E – torsion arm; F – pneumatic cylinders with a mechanical quick release; G – steel support structures (ground); and H – steel bed plate (ground).
A torsion arm is rigidly attached to the clutch shaft like previous experiments [2.17-18]. This arm is relevant to three design objectives: natural frequency, structural modes, and $T_f$. For simplicity, the torsion arm is constructed of bar stock of length $L_x$, height $L_y$, width $L_z$, and density $\rho_A$, as illustrated in Fig. 2.4a. The effective length ($L_A$) of the torsion arm is measured from the shaft axis to the free end of the arm. The torsion arm’s mass moment of inertia about the shaft axis ($J_A$) is defined as follows

$$J_A = \rho_A L_x L_y L_z \left( \frac{L_x^2}{3} + \frac{L_y^2}{12} + \frac{L_z^2}{12} - L_A L_x \right).$$

(2.3)

Inertia $J_A$ must also satisfy the following relation where $J_B$ is the mass moment of inertia of the shaft about its axis and $\tau_n = 100$ ms

$$J_A = k_{\text{III}} (\tau_n / 2\pi)^2 - J_B.$$

(2.4)

Torque $T_f$ is equivalent to the torque due to the overhanging mass of the torsion arm $T_A$, which is defined as the following where $g$ is the gravitational acceleration; the limits of $T_A$ are determined by Eq. (2.2b)

$$T_A = 0.5 g \rho_A L_A^2 L_y L_z.$$

(2.5)

The first flexural vibration mode of the torsion arm (within the plane of rotation) must occur beyond 10 ms. To estimate this, the torsion arm is approximated as a beam with fixed (shaft end) and free (load end) boundaries. The beam, with length $L_A$, height $L_y$, width $L_z$, density $\rho_A$, and modulus of elasticity $E_A$, is assumed to be nearly slender
The $r$th natural period $\tau_{nr}$ is defined using Euler-Bernoulli beam theory by the following where $\beta_r$ is the $r$th wave number [2.20]

$$\tau_{nr} = 2\pi/\beta_r^2 \left( E_A L_y^2 / 12 \rho_A \right)^{0.5}$$  \hspace{1cm} (2.6)

For the first mode ($r = 1$) the wave number is $\beta_1 \approx 1.875$. Using Eqs. (2.2b-6), material and geometry of the torsion arm are selected to satisfy the natural period, flexural modes, and $T_f$.

Torque $T_o$ is the sum of $T_A$ and external torque $T_e$, where torque $T_A$ is assumed to be constant with respect to $\theta$ due to small angular displacements. In previous experiments [2.17-18], $T_e$ is applied through a variable electromagnet mass drop. The employment of that same method for the current experiment would require a mass drop of approximately 300 kg, which presents significant difficulties. First, it would be unsafe to lift and suddenly drop a large mass. Second, a large electromagnet would have to be attached to the torsion arm, which would lower the first flexural frequency of the arm. Increasing the length of the torsion arm could reduce the drop mass and size of an electromagnet, but spatial restrictions prevent a longer arm. To overcome these issues, $T_e$ is provided by a pair of pneumatic cylinders with a mechanical quick release as shown in Fig. 2.3.

The pneumatic cylinders are characterized by effective diameter $d$ and translational throw $l$ and are loaded by gauge pressure $p$. Using the upper limit defined by Eq. (2.1a) and given $L_A$, the following requirement is defined for $l$

$$l > 2L_A \sin(\Theta_{\alpha\alpha^*})$$  \hspace{1cm} (2.7)
Torque $T_e$ is defined by the following

$$T_e = 0.5\pi d^2 p L_A$$  \hspace{1cm} (2.8)

Using Eq. (2.2a) and correcting for $T_A$, the range of $T_e$ is as follows:

$$T_e \in [\Phi_{II+} - T_A, \Phi_{III+} - T_A].$$  \hspace{1cm} (2.9)

The gauge pressure ($p_{max}$) needed to achieve the upper limit of $T_e$ is found from:

$$p_{max} = 2(\Phi_{III+} - T_A)/(\pi d^2 L_A).$$  \hspace{1cm} (2.10)

For the use of shop air, $p_{max}$ must be less than 690 kPa. Using Eqs. (2.7-10) and given $L_A$, $T_A$, $d$ and $l$ are selected to meet $(\theta_o, T_o)$ and $p_{max}$.

### 2.4 Instrumentation system and signal processing

Previous experiments in the literature [2.17-18] directly measured acceleration and strain; velocities were estimated by numerically integrating acceleration signals, and strain was considered analogous to displacement [2.17-18]. However, in the current study $\theta(t)$ and $\ddot{\theta}(t)$, which are difficult to measure directly, are of interest. It is not possible to place sensors (e.g. angular accelerometers, rotary encoders, strain gages) on the clutch shaft due to spatial restrictions. Translational acceleration of the torsion arm can however be directly measured as in previous experiments [2.17-18], but this presents several undesirable issues: 1) The impulsive nature of the response would induce “ringing” in the signals [2.21]; 2) Fidelity of the estimated displacements would be reduced by multiple integrations of measured acceleration signals; and 3) Accuracy of the accelerometers would diminish.
when measuring small time-scale dynamics [2.21], such as those induced by impacts [2.17-18]. To overcome these complications, a laser vibrometer [2.22] is chosen to directly measure the velocity. The vibrometer [2.22] and its paired data acquisition system [2.23] can sample at a minimum period \((\tau_s)\) of 1 \(\mu\)s and measure translational velocities up to 10 m/s; also, refer to Appendix A for the specifications of the instrumentation system. To quantify the error of the vibrometer system [2.22-23], the velocity of a harmonic shaker is measured without using a filter. The shaker is calibrated to provide 9.81 m/s² RMS acceleration at a frequency of 159.2 Hz [2.24], which corresponds to a 27.7 m/s² peak-to-peak acceleration and 2.77 cm/s peak-to-peak velocity. The error of measured peak-to-peak velocity for selected values of \(\tau_s\) is shown in Table 2.1; for \(3.9 \mu\)s \(\leq \tau_s \leq 1\) ms the average error is 6.7%. Acceleration is then calculated by numerically differentiating the velocity signal using forward and central difference methods; the error is 125% at \(\tau_s = 3.9\) \(\mu\)s and ±10% at \(\tau_s = 1.0\) ms. It is assumed that signal noise is prominent at small values of \(\tau_s\). Since the truly impulsive nature of the torsional system response is not the primary focus of this study, a very small value of \(\tau_s\) is not necessary. Therefore, a sampling period of 0.781 ms is chosen to ensure accurate estimation of acceleration and to reduce aliasing.

For measurement of the system response, the laser vibrometer is located a sufficient distance from the experiment to mitigate ground-transmitted vibrations. A clean, flat, reflective surface for the laser point is provided by a square, finished block of negligible torsional inertia; the block is rigidly mounted to the top of the torsion arm such that the measurement surface is in-plane with the shaft axis, as illustrated in Fig.2.4b. The
translational velocity of the laser point is denoted by \( \dot{u}(t) \), the radial distance from the shaft axis to the laser point by \( L_u(t) \), and the angular displacement of the torsion arm with respect to the \( x \)-axis by \( \theta'(t) \). For the purpose of comparison, two translational accelerometers [2.25], say \( V \) and \( W \), are located on the block and torsion arm; the radial distances from the shaft axis to \( V \) and \( W \) are denoted by \( L_v \) and \( L_w \), respectively. Measured translational accelerations \( \ddot{v}(t) \) and \( \ddot{w}(t) \) are assumed to be tangential to the angular motion of the arm.

Prior to excitation, \( \theta'_o \) and \( u_o \) are measured with a digital level [2.26] and a ruler, respectively; \( u_o \) is calculated from \( u_o = L_u \sin(\theta'_o) \). Near \( t \approx 0 \), the step-like torque is applied to \( J \) and measurement is triggered by a 98.1 m/s\(^2\) falling value of \( \ddot{w}(t) \); this triggering value is empirically determined and is valid over the range of \((\theta_o, T_o)\). The pre and post-triggering periods are approximately 100 and 4000 ms, respectively.

**Table 2.1** Sampling errors in peak-to-peak velocity (measured) and acceleration (calculated) for laser vibrometer measurement of a harmonic shaker.

<table>
<thead>
<tr>
<th>( \tau_o ) (ms)</th>
<th>Peak-to-peak velocity error</th>
<th>Peak-to-peak acceleration error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Forward difference method</td>
<td>Central difference method</td>
</tr>
<tr>
<td>0.0039</td>
<td>7.3%</td>
<td>126%</td>
</tr>
<tr>
<td>0.0156</td>
<td>7.2%</td>
<td>122%</td>
</tr>
<tr>
<td>0.0625</td>
<td>6.7%</td>
<td>88%</td>
</tr>
<tr>
<td>0.2500</td>
<td>6.6%</td>
<td>35%</td>
</tr>
<tr>
<td>1.0000</td>
<td>5.8%</td>
<td>8.0%</td>
</tr>
</tbody>
</table>
Figure 2.4  Illustrations of the torsion arm and measurement of angular motion using a laser vibrometer and accelerometers: (a) torsion arm geometry; and (b) instrumentation schematic. Key: A – shaft; B – shaft axis; C – torsion arm; D – laser vibrometer; E – laser; F – laser point; G – translational accelerometer V; H – finished block; and I – translational accelerometer W.

The measured response is contained within the first 1 to 2 s of the signal, and the latter 2 s provides corrections for numerical integration and quantification of signal noise. After the system reaches equilibrium, $\theta'_f$ is measured using the digital level [2.26].
Translational displacement $u(t)$ is calculated by numerically integrating signal $\dot{u}(t)$, and $\theta(t)$ is calculated by the following where $\theta_f$ is determined from the static balance of $T_f$

$$L_o(t) = \left[ L_o^2 + \left( u_o - u(t) \right)^2 - 2L_o \left( u_o - u(t) \right) \sin(\theta_o') \right]^{0.5},$$  

(2.11)

$$\theta(t) = \theta_f + \theta_o' - \theta_f' - \sin^{-1}\left( \frac{(u_o - u(t))\cos(\theta_o')}{L_o(t)} \right).$$  

(2.12)

Angular velocity $\dot{\theta}(t)$, acceleration $\ddot{\theta}(t)$, and jerk $\dddot{\theta}(t)$ are then calculated by numerically differentiating $\theta(t)$ by a forward differencing method. Angular accelerations estimated from signals $\dddot{v}(t)$ and $\dddot{w}(t)$ are defined by the following

$$\dddot{v}(t) = \dddot{v}(t)/L_v, \quad \dddot{w}(t) = \dddot{w}(t)/L_w$$  

(2.13a-b)

All angular motion signals are then synchronized such that $|\dddot{\theta}(0)|$ is maximum.

For this experiment, signal analysis in the frequency domain is hindered by the inherently non-linear and transient nature of the responses [2.27]. Time-frequency domain analysis (though not included in the article) shows that the responses are dominated by lower frequencies, which is evident from the time history alone. Auto and cross-correlations do not provide significant insight due to a very short duration of the response and close spatial proximity of measurement locations [2.28]. Accordingly, all signals are examined solely in time domain. It is consistent with literature [2.2-6, 12-13, 17-18] where the step responses of highly non-linear systems have been studied primarily in time domain.
2.5 Analysis of measured motions

Typical responses measured by the laser vibrometer (with dual clutch damper configuration) are shown in Fig. 2.5; all motions and time are normalized as follows to highlight the generic nature of a non-linear device

\[ \bar{\tau} = t/\tau_n, \; \bar{\theta}(\bar{\tau}) = \theta(\bar{\tau})/\theta_\theta, \; \bar{\dot{\theta}}(\bar{\tau}) = \dot{\theta}(\bar{\tau})\tau_n/\theta_\theta, \; \bar{\ddot{\theta}}(\bar{\tau}) = \ddot{\theta}(\bar{\tau})\tau_n^2/\theta_\theta. \]  

(2.14a-d)

Upon inspection, it can be seen that \( \bar{\theta}_o \) and \( \bar{\theta}_f \) lie within the desired clutch damper stages. It is also evident that there are three distinct non-linear response regimes: double-sided impact (di), single-sided impact (si), and no impact (ni) [2.29]. The di regime \((0 < \bar{\tau} < 0.8)\) is characterized by \( J \) losing and regaining contact with stage II on both the drive and coast sides at \( \bar{\theta}_i \) and \( \bar{\theta}_f \) with significant peak positive and negative \( \bar{\ddot{\theta}}(\bar{\tau}) \). The si regime \((0.8 < \bar{\tau} < 10)\) is characterized by \( J \) losing and regaining contact with stage II on the drive side at \( \bar{\theta}_i \) with significant peak negative \( \bar{\ddot{\theta}}(\bar{\tau}) \) only. In the ni regime \((\bar{\tau} > 10)\), \( J \) maintains contact with stage II on the drive side \((\bar{\theta}_i \leq \bar{\theta}(\bar{\tau}) \leq \bar{\theta}_f)\); the response remains non-linear and appears nearly harmonic. For this measured response, the time of transition from the di to si regime is defined by the last \( \bar{\tau} \) at which \( \bar{\theta}(\bar{\tau}) \leq \bar{\theta}_i \) and from si to ni by the last \( \bar{\tau} \) at which \( \bar{\theta}(\bar{\tau}) \leq \bar{\theta}_i \).

Signal \( \bar{\ddot{\theta}}(\bar{\tau}) \) from the laser vibrometer is verified by comparing it to \( \bar{\ddot{\theta}}_v(\bar{\tau}) \) and \( \bar{\ddot{\theta}}_w(\bar{\tau}) \) from accelerometers V and W, all of which are shown in Fig. 2.6. Peak locations and peak-to-peak amplitudes roughly agree across all signals, which gives confidence in...
The “ringing” phenomena is clearly present in accelerometer signal $\bar{\theta}(\bar{r})$ but nearly absent in $\bar{\theta}_c(\bar{r})$. The close proximity of accelerometer V (point G in Fig.2.4b) to impact locations and the complex structures of the production components makes it more susceptible to higher frequency dynamics, whereas accelerometer W (point I in Fig. 2.4b) is located near the free end of the torsion arm.

To demonstrate the repeatability and quantify variance of measurements, 12 response sets are recorded, and metrics are compared. The experiment (with dual clutch damper configuration) is partially disassembled and re-assembled three times, designated ASM1, ASM2, and ASM3. Four measurements per assembly are recorded under similar loading conditions ($\bar{\theta}_o \approx 26$). The arithmetic mean ($\mu$) and standard deviation ($\sigma$) of regime transition times and peak-to-peak $\bar{\theta}(\bar{r})$ are calculated for each assembly and all 12 measurements; these are listed in Tables 2.2 and 2.3. The peak-to-peak $\bar{\theta}(\bar{r})$ is fairly consistent within each assembly and overall. However, the response regime transition times show more variation, which is most likely due to small differences in shaft alignment (friction) and system actuation. Additionally, the calculation of $\bar{\theta}(\bar{r})$ is verified by comparing $\bar{\theta}_o - \bar{\theta}_f$ to the measured values $\bar{\theta}_o - \bar{\theta}_f$ (tolerance: ±0.4) as shown in Table 2.3; all values of $\bar{\theta}_o - \bar{\theta}_f$ fall within the tolerance of $\bar{\theta}_o - \bar{\theta}_f$. 

29
Figure 2.5 Normalized measured motions for $\bar{\theta}_0 \approx 26$ and demonstration of non-linear response regimes (with dual clutch damper configuration). Key: (▬▬) – measured motion; ( | ) – regime transition; (− ∙ −) – stage and clearance transitions; di – double-sided impact regime; si – single-sided impact regime; and ni – no-impact regime.
Figure 2.6 Comparison of acceleration measurements of $\ddot{\theta}(\bar{t})$ from laser vibrometer, $\ddot{\theta}_V(\bar{t})$ from accelerometer V, and $\ddot{\theta}_W(\bar{t})$ from accelerometer W for $\bar{\theta}_e \approx 26$ (with dual clutch damper configuration).
Table 2.2 Mean ($\mu$) and standard deviation ($\sigma$) of peak-to-peak $\bar{\theta}(t)$ of the measured responses for $\bar{\theta}_o \approx 26$ over three assemblies (with dual clutch damper configuration).

<table>
<thead>
<tr>
<th>Measurement set</th>
<th>Peak-to-peak $\bar{\theta}(t)$ ($\mu \pm \sigma$) for period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>ASM1</td>
<td>2230 ± 91</td>
</tr>
<tr>
<td>ASM2</td>
<td>2040 ± 113</td>
</tr>
<tr>
<td>ASM3</td>
<td>2070 ± 39</td>
</tr>
<tr>
<td>Overall</td>
<td>2110 ± 123</td>
</tr>
</tbody>
</table>

Table 2.3 Analysis of measurements in terms of the mean ($\mu$) and standard deviation ($\sigma$) of response regime transition times and relative displacement between initial and final operating points for $\bar{\theta}_o \approx 26$ over three assemblies (with dual clutch damper configuration).

<table>
<thead>
<tr>
<th>Measurement set</th>
<th>Response regime transition times ($\mu \pm \sigma$)</th>
<th>Relative displacement between initial and final operating points ($\mu$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{T}(di \to si)$</td>
<td>$\bar{T}(si \to ni)$</td>
</tr>
<tr>
<td>ASM1</td>
<td>1.61 ± 0.98</td>
<td>13.8 ± 0.06</td>
</tr>
<tr>
<td>ASM2</td>
<td>0.76 ± 0.01</td>
<td>11.1 ± 0.18</td>
</tr>
<tr>
<td>ASM3</td>
<td>0.78 ± 0.01</td>
<td>10.3 ± 0.43</td>
</tr>
<tr>
<td>Overall</td>
<td>1.05 ± 0.66</td>
<td>11.7 ± 1.58</td>
</tr>
</tbody>
</table>
The responses to other loading conditions are also measured; Fig. 2.7 shows the typical measured responses for $\bar{\theta}_o \approx 21$ and $\bar{\theta}_o \approx 33$. These responses are qualitatively similar to the previous loading condition ($\bar{\theta}_o \approx 26$, Fig. 2.5). All responses have the desired initial and final operating points and distinct non-linear regimes (di, si, and ni). The regime transition times and peak-to-peak $\bar{\theta}(\bar{t})$ are very similar; however there is a significant difference for the di regime. As the amplitude of $T_o$ increases, the peak-to-peak value of $\bar{\theta}(\bar{t})$ in the di regime also increases and peak values of $\bar{\theta}(\bar{t})$ occur earlier for all regimes.
Figure 2.7  Comparison of normalized measured motions for $\bar{\theta}_o \approx 21$ and $\bar{\theta}_o \approx 33$, and demonstration of non-linear response regimes (with dual clutch damper configuration). Key: 

- measured motion for $\bar{\theta}_o \approx 21$; (---) – measured motion for $\bar{\theta}_o \approx 33$; ( | ) – response regime transition for $\bar{\theta}_o \approx 21$; ( ) – response regime transition for $\bar{\theta}_o \approx 33$; and (− · −) – stage transitions.
2.6 Development of a simple non-linear dynamic model

The experiment is first simplified by adjusting the position of the shaft so that it contacts only one of the two clutch dampers. This overcomes complexities due to asynchronous contact and misalignment between the clutch dampers and the shaft. The step-like response of the single clutch damper configuration is measured using the previously described procedures. To verify that the measured motions are representative of the designed system, comparisons are made to the predictions from a simplified SDOF non-linear model. The experiment is described by the following equation of motion corresponding to Fig. 2.2, where it is assumed that all dissipation sources are lumped into a single viscous element $c$, and the dry friction is ignored for the sake of simplicity

$$J \ddot{\theta}(t) + c \dot{\theta}(t) + \Gamma(\theta, \dot{\theta}) = T(t).$$ (2.15)

The non-linear function $\Gamma(\theta, \dot{\theta})$ is the torque transmitted to $J$ through the spline clearance in series with the clutch damper and is the sum of elastic and hysteretic torques

$$\Gamma(\theta, \dot{\theta}) = \Phi(\theta) + \Psi(\theta, \dot{\theta}).$$ (2.16)

The elastic torque $\Phi(\theta)$ is defined as the following where $\Xi(\Theta) = 0.5(\text{sgn}(\Theta) + 1)$

$$\Phi(\theta) = \sum_{j=1}^{N} \left\{ \left[ \Phi_{j-1} + k_{j} (\theta - \Theta_{j-1}) \right] \left[ \Xi(\theta - \Theta_{j-1}) - \Xi(\theta - \Theta_{j}) \right] + \cdots \right\},$$ (2.17)

$$k_{j+} = \frac{\Phi_{j+} - \Phi_{(j-1)+}}{\Theta_{j+} - \Theta_{(j-1)+}}, \quad k_{j-} = \frac{\Phi_{(j-1)-} - \Phi_{j-}}{\Theta_{(j-1)-} - \Theta_{j-}}.$$ (2.18a-b)

The dissipative torque $\Psi(\theta, \dot{\theta})$ is written as the following where $h(\theta)$ is a piecewise linear function
\[ \Psi(\theta, \dot{\theta}) = h(\theta) \tanh(\eta \dot{\theta}), \quad (2.19a-b) \]

\[ h(\theta) = \sum_{j=1}^{N} \left[ h_{j-1} + \Delta h_j \left( \theta - \Theta_{j-1} \right) \right] \left[ \Xi(\theta - \Theta_j) - \Xi(\theta - \Theta_{j-1}) \right] + \ldots \], \quad (2.20) \]

\[ \Delta h_j = \left( \frac{h_{j-1} - h_{j}}{\Theta_{j-1} - \Theta_j} \right), \quad \Delta h_j = \left( \frac{h_{j-1} - h_j}{\Theta_{j-1} - \Theta_j} \right). \quad (2.21a-b) \]

The inertia parameters \( J_A \) and \( J_B \) are estimated by using a commercial solid modeling code [2.30]. The geometry of the torsion arm and shaft are measured to the nearest millimeter; material density is assumed to be uniform and estimated by measuring the mass of each component to the nearest 0.5 kg. Viscous damping \( (c = 2 \zeta (J_{\text{fl}})^{0.5}) \) is estimated by assuming a damping ratio of \( \zeta = 0.015 \), which is typical for the torsional surge mode of a vehicle driveline [2.18]. The clearance transition \( \Theta_{\text{cl}} \) is measured using a digital level [2.25] to the nearest 0.05°. All clutch damper parameters are estimated from measured quasi-static performance curves.

The non-linear model is numerically integrated using MATLAB [2.31]. The discontinuous function \( \Xi(\theta) \) is approximated by the following smoothened function where \( \gamma \) is an empirical regularizing factor [2.33]

\[ \Xi(\theta) \approx \hat{\Xi}(\theta) = 0.5 \left( \tanh(\gamma \theta) + 1 \right). \quad (2.22) \]

The maximum allowable time step for integration and the uniform output time step are chosen as \( \tau_r \). Three Runge-Kutta methods, of which one is intended for numerically stiff systems [2.31], are evaluated; however no significant difference among the results is found.
A comparison of predicted and measured motions is given in Fig. 2.8 and Table 2.4. The simulated motions are characterized by the same response regimes as the measured signals. There is indeed very good agreement between the simple model and measurement for the di regime and most of the si regime. Nevertheless, the predicted motions decay less quickly and the transition from si to ni occurs much later. This suggests that the assumed dissipation model is insufficient and that further work would be needed to better quantify dissipation sources and refine $h(\theta)$. 
Figure 2.8 Comparison of measured and predicted motions for $\bar{\theta}_o = 23$ (with single clutch damper configuration). Key: (▬▬) – measured motion; (▬▬) – predicted motion using SDOF model, ( | ) – regime transition for the measured motion, ( | ) – regime transition for the simulated motion, (−⋅−) – stage and clearance transitions.
Table 2.4 Comparison of measured and predicted motions in terms of the mean (μ) and standard deviation (σ) of: (a) peak-to-peak $\bar{\theta}(T)$ and (b) response regime transition times for $\bar{\theta}_o \approx 23$ (with single clutch damper configuration).

(a) Peak-to-peak $\bar{\theta}(T)$ (μ ± σ) for period

<table>
<thead>
<tr>
<th>Motion</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured (over 12 sets)</td>
<td>802 ± 35</td>
<td>660 ± 19</td>
<td>541 ± 18</td>
<td>413 ± 9</td>
<td>342 ± 14</td>
<td>243 ± 15</td>
</tr>
<tr>
<td>Predicted</td>
<td>880</td>
<td>592</td>
<td>445</td>
<td>321</td>
<td>256</td>
<td>208</td>
</tr>
</tbody>
</table>

(b) Response regime transition times (μ ± σ)

<table>
<thead>
<tr>
<th>Motion</th>
<th>$T(di \rightarrow si)$</th>
<th>$T(si \rightarrow ni)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured (over 12 sets)</td>
<td>2.94 ± 0.59</td>
<td>11.53 ± 0.50</td>
</tr>
<tr>
<td>Predicted</td>
<td>2.98</td>
<td>15.16</td>
</tr>
</tbody>
</table>

2.7 Conclusion

The chief contribution of this article is the successful development of a new, refined vibratory (non-rotating) experiment that demonstrates the step response of a highly non-linear torsional system. This experiment differs from prior work [2.2, 17-18] in that it includes multi-staged torsional stiffness and friction elements as provided by production clutch dampers. The experiment is conceptually designed using a SDOF torsional system model with discontinuous non-linear stiffness $k(\theta)$ and frictional hysteresis $h(\theta)$ which are assumed from quasi-static performance curves. To ensure relevance to vehicle powertrain
subsystems and accuracy of measurement, a set of specific design objectives (such as the sizing for natural period, system actuation, instrumentation, and signal processing) are achieved. The angular motions are calculated from direct measurements of translational velocity (laser vibrometer) and translational acceleration (translational accelerometer). Typical measurements show that the step response is characterized by three pronounced non-linear regimes (di, si, and ni) which correspond to the multi-staged features of the clutch damper. Each measurement was quantified by two metrics: peak-to-peak $\bar{\theta}(\bar{r})$ and response regime transition times $(\bar{r}(\text{di} \rightarrow \text{si})$ and $\bar{r}(\text{si} \rightarrow \text{ni}))$. The metrics for 12 measurements collected over three assemblies show that the results are repeatable. Calculation of $\bar{\theta}(\bar{r})$ is also verified by observing that all values of $\bar{\theta}_o - \bar{\theta}_f$ are within the tolerance of the directly measured values.

The predictions of a SDOF non-linear model exhibit the same qualitative nature as the measurements from the new experiment as both demonstrate the same non-linear response regimes, though there is only a partial agreement for the metrics. Therefore, this new experiment should provide time-domain benchmark data for the validation of higher dimensional non-linear models, while permitting an efficient process to evaluate the design of practical multi-staged torsional stiffness and friction devices. Even though a SDOF non-linear model is sufficient for experiment design and some predictions, further work is needed to develop better models as well as to extract dissipative parameters.
References for Chapter 2


List of symbols for Chapter 2

- $c$: torsional viscous damping
- $d$: effective diameter (pneumatic cylinder)
- $E$: modulus of elasticity
- $f$: frequency
- $g$: gravitational acceleration
- $h$: Coulomb hysteresis amplitude
- $J$: torsional inertia
- $k$: torsional stiffness
- $l$: translational throw (pneumatic cylinder)
- $L$: length
- $p$: gauge pressure
- $t$: time
- $T$: external torque
- $u, \dot{u}$: measured translational displacement and velocity by laser vibrometer
- $\ddot{v}$: measured translational acceleration by accelerometer $V$
- $\ddot{w}$: measured translational acceleration by accelerometer $W$
- $x, y, z$: translational coordinates
- $\beta$: wave number
- $\Gamma$: torque transmission
- $\gamma$: displacement regularizing factor
$\zeta$ damping ratio

$\eta$ velocity regularizing factor

$\theta, \dot{\theta}, \ddot{\theta}$ angular displacement, velocity, and acceleration

$\Theta$ angular stage transition

$\mu$ arithmetic mean

$\Xi$ ideal unit step function

$\rho$ material density

$\sigma$ standard deviation

$\tau$ oscillatory period

$\Phi$ elastic torque

$\Psi$ dissipative torque

$\omega$ angular frequency

**Subscripts**

1,2,... modal index

0, I, II,... stage index

$A$ torsion arm

$B$ clutch shaft

$D$ clutch damper

$e$ external

$f$ final

max maximum
\( n \) natural
\( o \) initial
\( r \) modal index \((r = 1, 2, \ldots)\)
\( s \) sampling
\( u \) laser point
\( v \) translational accelerometer \( V \)
\( w \) translational accelerometer \( W \)
\( x, y, z \) translational coordinate index
\( + \) drive side \((\theta > 0)\)
\( - \) coast side \((\theta < 0)\)

**Abbreviations**

SDOF single degree of freedom
ASMI2,3 experiment assembly 1, 2, and 3
ni no-impact response regime
si single-sided impact response regime
di double-sided impact response regime
Chapter 3: Development of a scientific torsional system experiment containing controlled single or dual-clearance non-linearities: Examination of step-responses

3.1 Introduction

Most torsional systems in vehicles and machinery contain one or more discontinuous non-linear features by design or otherwise [3.1-9]. Such non-linear elements include clearances (e.g. backlashes between gears), multi-staged torsional springs, pre-load and stopper features (e.g. torque transmission devices), and multi-staged dry friction components. Of this group, the clearance or gap element is the most fundamental as it is required to assemble components without interference while providing space for lubrication. Presence of such clearances in multi-degree of freedom systems induce conditions for vibro-impact phenomena, depending on the value of mean and alternating loads. Gear rattle [3.4, 9-10] and vehicle driveline clunk [3.4-8] are common physical manifestations of such systems as evident from many noise and vibration issues in the ground vehicle industry. While several researchers have developed non-linear simulation models, they often include many simplifications and/or incorporate assumed or empirical parameters. There is clearly a need for a scientific experiment that could yield physical insight, accurate parameters, and benchmark time domain data for the validation of non-linear simulation models. Therefore, it is the chief goal of this article and as such a new
laboratory experiment with controlled clearance element(s), while being distinctly different from the prior experimental studies [3.4-8], will be proposed.

3.2 Problem formulation

Courderc et al. [3.4] proposed a system-level rotating experiment to better understand gear rattle during engine run-up. Their experiment accommodates several production components, such as a clutch damper and vehicle transmission, while containing several clearances spread across multiple locations. Likewise, a different type of experiment must be considered to study vehicle driveline clunk, typically induced by a sudden change in the mean operating point of the driveline (often known as a tip-in or tip-out event) [3.4-8]. The simplest way to implement such excitation in a laboratory setting is to apply a step-down torque under vibratory conditions, which is essentially free vibration about a static equilibrium. Few system-level experiments that employ the step-response method have been proposed in the literature [3.5-7]. Common features across all test rigs [3.5-7] include the following: i) Production vehicle driveline components are often utilized; ii) The systems are made positive-definite to achieve vibratory conditions (i.e. at least one torsional component is fixed to ground); iii) Two or more clearance elements are present; and iv) A step-down torque is applied via a variable mass-drop from a torsion arm. Although these experiments [3.5-7] provide much needed system-level insight, it is also necessary to study non-linear dynamics at the component-level. For example, consider the experiment recently proposed by Krak et al. [3.8], which has been developed to provide parameter estimation for a clutch damper under dynamic conditions. This particular
experiment [3.8] accommodates one or two production clutch dampers and is excited by a step-like torque like the clunk experiments [3.5-7]; however, the external torque is supplied by a pneumatic actuator rather than a mass-drop due to the relatively high torque capacity of the device.

These aforementioned large-scale experiments [3.4-8] have the following intrinsic benefits: i) Interactions between multiple non-linear features are maintained; ii) System and component-level dynamics can be studied under realistic boundary conditions; iii) Real-world type excitation is more easily achieved; and iv) Parameter estimation has greater fidelity [3.11]. However, there are also several inherent disadvantages, such as increased cost (e.g. labor, hardware, instrumentation, and laboratory space), a higher degree of complexity, and, most important, a lack of controllability. The latter two are largely due to the use of production components, which can contain multiple (known and unknown) non-linear features while being subject to variations in assembly and manufacturing. Therefore, the goal of this paper is to address this critical need by proposing a more controlled and scientific version of prior large-scale experiments [3.5-8]. Accordingly, the following specific objectives are defined: 1) Develop a new controlled laboratory (“bench-top”) experiment that exhibits the step-response of a torsional system that contains one or two clearance non-linearities located at different locations (to allow single and dual-clearance cases); 2) Verify the feasibility of the proposed experiment by comparing typical measurements with predictions from simple, low-dimensional non-linear models and 3) Demonstrate the utility of the proposed experiment (denoted X1)
through comparative studies between a single-clearance configuration and to a large-scale laboratory experiment (denoted X2 in this paper) [3.8].

### 3.3 Conceptual design considerations

First, it is assumed that clearance non-linearities (with torsional stiffness \( k(\theta) \), where \( \theta \) is the relative angular displacement) can be described by a general piecewise linear form as illustrated in Fig. 3.1. Here, \( \Phi(\theta) \) is the elastic torque transmitted through the clearance, subscripts I and II denote stages (where stage I is the clearance), \( k_j \) is the torsional stiffness (linear) of stage \( j \), and \( \Theta_j \) is the angular transition from stage \( j \) to \( j+1 \). For the sake of simplicity, the clearance is assumed to be symmetric about \( \theta = 0 \) and static stiffness values are known (and valid under dynamic conditions) while ignoring contact mechanics. Accordingly, \( k(\theta) \) and \( \Phi(\theta) \) are defined by the following expressions:

\[
k(\theta) = k_i + (k_{n} - k_i) \Xi (|\theta| - \Theta_i), \tag{3.1}
\]

\[
\Phi(\theta) = k_i \theta + \text{sgn}(\theta)(k_{n} - k_i)(|\theta| - \Theta_i) \Xi (|\theta| - \Theta_i). \tag{3.2}
\]

Next, the proposed experiment (in its dual, single, and no-clearance configurations) is conceptually described by the non-linear models illustrated in Fig. 3.2. Here, \( J \) is the torsional inertia, \( k \) is the torsional stiffness (linear), \( h \) is the Coulomb friction amplitude, \( T \) is the external torque, subscripts \( \{A, B, C\} \) are coordinate and element indices, and \( \{\theta, \dot{\theta}, \ddot{\theta}\} \) are the angular displacement, velocity, and acceleration.
Figure 3.1 Illustration of an angular symmetric clearance element $k(\theta)$. Here, elastic torque transmission is given by $\Phi(\theta)$, $\theta$ is the relative angular displacement, $\Theta_1$ is the angular transition between stage I (clearance with torsional stiffness $k_1 = 0$) and II, $k_\Pi$ is the torsional stiffness of stage II, $(\theta_o, \Phi_o)$ is the initial operating point, and $(\theta_f, \Phi_f)$ is the final operating point.

respectively (see the end of this chapter for a full list of symbols). The dual-clearance configuration (denoted X1-2 and shown in Fig. 3.2a) is a three degree of freedom (3DOF) positive-definite system that contains two clearance non-linearities ($k_{AB}(\theta_{AB})$ and $k_C(\theta_C)$) and has the following set of governing equations:

\begin{align}
J_A \ddot{\theta}_A + h_A \tanh(\eta \dot{\theta}_A) + h_{AB} \tanh(\eta \dot{\theta}_{AB}) + \Phi_{AB}(\theta_{AB}) &= T_A(t), \\
J_B \ddot{\theta}_B - h_{AB} \tanh(\eta \dot{\theta}_{AB}) + h_{BC} \tanh(\eta \dot{\theta}_{BC}) - \Phi_{AB}(\theta_{AB}) + k_{BC}\theta_{BC} &= 0, \\
J_C \ddot{\theta}_C - h_{BC} \tanh(\eta \dot{\theta}_{BC}) + h_C \tanh(\eta \dot{\theta}_C) - k_{BC}\theta_{BC} + \Phi_C(\theta_C) &= 0.
\end{align}

(3.3a-c)
Here, $\Phi_{AB}(\theta_{AB})$ and $\Phi_c(\theta_c)$ are the elastic torque transmissions (as defined by Eq. (3.2)) through $k_{AB}(\theta_{AB})$ and $k_c(\theta_c)$, respectively. Clearance element $k_{AB}(\theta_{AB})$ (key and keyway) is located between torsional inertias $J_A$ (torsion arm and shaft) and $J_B$ (jaw coupling hub), and $k_c(\theta_c)$ (key and keyway) is located between $J_C$ (jaw coupling hub) and ground. It is assumed that $k_{AB}(\theta_{AB})$ and $k_c(\theta_c)$ are identical, and thus share the same values of $\theta_j$ and $k_j$. A torsional spring (linear) $k_{BC}$ (achieved via coil springs between the coupling jaws) is utilized to couple $J_B$ to $J_C$. Similar to prior work [3.8] concerning clutch dampers, it is assumed that all dissipative elements (which include: $h_A$ (shaft and bearing interface), $h_{AB}$ (shaft and coupling hub interface), $h_{BC}$ (coil springs), and $h_C$ (shaft and coupling hub interface)) can be simply described by a Coulomb friction model (hyperbolic tangent approximation where $\eta$ is a regularizing factor [3.12]). Lastly, system excitation is provided by an external torque $T_A(t)$ that is applied to inertia $J_A$ ($T_B = T_C = 0$); $T_A(t)$ is a step-down at time $t = 0$ from initial $T_{Ao}$ to final $T_{Af}$ torque values and is defined by the following:

$$T_A(t) = T_{Ao} \left[1 - \Xi(t) \right] + T_{Af} \Xi(t).$$  \hspace{1cm} (3.4)

The single-clearance configuration (denoted X1-1a and shown in Fig. 3.2b) is a two degree of freedom (2DOF) positive-definite system that contains one clearance element ($k_{AB}(\theta_{AB})$).

Here, $k_c(\theta_c) \rightarrow \infty$ such that $J_C$ is fixed to ground. There is an alternate version (say X1-1b) in which $k_c(\theta_c)$ is the sole clearance element ($J_C$ is free to vibrate but $k_{AB}(\theta_{AB}) \rightarrow \infty$).
However, version X1-1a is favored over X1-1b for two reasons: i) Element $k_{AB}(\theta_{AB})$ is closer to the excitation $T_A(t)$, thus it influences the step-response more that $k_{BC}(\theta_c)$; and ii) The layout of X1-1a is analogous to a prior large-scale experiment [3.8] such that $k_{AB}(\theta_{AB})$ is similar to a spline clearance (or a very compliant spring) and $k_{BC}$ may be considered a linearized clutch damper. The single-clearance configuration (X1-1a) is a simplified case of X1-2; hence, X1-1a has the following set of governing equations, which are derived from Eq. (3.3):

\[
\begin{align*}
J_A \ddot{\theta}_A + h_A \tanh(\eta \dot{\theta}_A) + h_{AB} \tanh(\eta \dot{\theta}_{AB}) + \Phi_{AB}(\theta_{AB}) &= T_A(t), \\
J_B \ddot{\theta}_B - h_{AB} \tanh(\eta \dot{\theta}_{AB}) + h_{BC} \tanh(\eta \dot{\theta}_{BC}) - \Phi_{AB}(\theta_{AB}) + k_{BC} \theta_B &= 0.
\end{align*}
\] (3.5a-b)

Lastly, the no-clearance configuration (denoted X1-0 and shown in Fig. 3.2c) is a single degree of freedom (1DOF) positive-definite, linear time-invariant system as both clearance elements are removed ($k_{AB}(\theta_{AB}) \to \infty$ and $k_{BC}(\theta_c) \to \infty$) so that $J_A$ is rigidly attached to $J_B(\theta_A = \theta_B)$ and $J_C$ is fixed to ground. Similar to X1-1a, configuration X1-0 is a simplification of X1-2; accordingly, its (X1-0) governing equation is the following:

\[
(J_A + J_B) \ddot{\theta}_A + h_A \tanh(\eta \dot{\theta}_A) + h_{BC} \tanh(\eta \dot{\theta}_{BC}) + k_{BC} \theta_A = T_A(t).
\] (3.6)

This configuration (X1-0) serves several important roles as it provides much needed no-impact benchmark data for comparison while representing a special constant-contact case (where $|\theta_{AB}| > \Theta_{\text{AB-1}}$) of the single-clearance configuration (X1-1a). Crucial system parameters (including $J_A$ and $h_A$) can be estimated from its step-response.
Figure 3.2 Conceptual illustration of the proposed experiment X1 with clearances as defined in Fig. 3.1: (a) dual-clearance configuration (X1-2); (b) single-clearance configuration (X1-1a); and (c) no-clearance configuration (X1-0). Here, \( \{\theta, \dot{\theta}\} \) is the angular displacement and velocity (respectively), \( J \) is the torsional inertia, \( k \) is the torsional stiffness, \( h \) is the Coulomb friction, and \( T \) is the external torque. Subscripts \( \{A, B, C\} \) are element and coordinate indices. Key: 1 – torsion arm, shaft, and disks; 2 – key and keyway (clearance); 3 – coupling hub; 4 – torsional spring (linear); and 5 – shaft and bearing interface.
The following experimental design objectives are identified to ensure the relevance and applicability to the literature [3.4-8]:

1) The experiment must qualify as “bench-top,” meaning its footprint must fit on a typical table-top found in a laboratory setting (say approximately 1 m by 0.5 m);
2) For sizing purposes, the first torsional mode of all configurations must have a natural frequency \((\omega_{n,1})\) between 5 and 15 Hz (with a corresponding natural period \(\tau_{n,1}\) between 0.07 and 0.2 s), similar to a vehicle driveline and prior experiments [3.4-8];
3) The first flexural mode of the torsion arm within the plane of rotation must have a frequency greater than \(10\omega_{n,1}\) to avoid interference with the dynamics of interest;
4) Coulomb friction must be minimized so that non-linear behavior is controlled by the clearance elements;
5) To ensure vibro-impact phenomena, the initial \(((\theta_{AB,o}, \Phi_{AB,o}) \text{ and } (\theta_{Co}, \Phi_{Co}))\) and final \(((\theta_{AB,f}, \Phi_{AB,f}) \text{ and } (\theta_{C,o}, \Phi_{C,o}))\) operating points at the clearances must lie on separate stages (see Fig. 3.1); and
6) The experiment must be instrumented to measure time domain angular acceleration signals, which is consistent with prior work [3.4-8].

3.4 Physical design, instrumentation, and measurement considerations

The proposed experiment is physically illustrated (for the sake of clarity) by the simplified solid model shown in Fig. 3.3. First, a steel baseplate that has a footprint of
approximately 50 cm² and can be bolted or clamped onto a laboratory work surface is selected. Next, a sleeve bearing is chosen and bolted to the baseplate; a corresponding keyed shaft (of length $L_{\text{shaft}}$, radius $R_{\text{shaft}}$, and material density $\rho_{\text{shaft}}$) is supported by the bearing. A torsion arm (of length $L_{\text{arm},x}$, height $L_{\text{arm},y}$, thickness $L_{\text{arm},z}$, and material density $\rho_{\text{arm}}$) is then rigidly attached to the shaft. Steel disks (with radius $R_{\text{disk}}$, thickness $L_{\text{disk},z}$, and material density $\rho_{\text{disk}}$) are then adhered to bottom-side of the torsion arm at both of its ends; a symmetrical layout is chosen so that the final operating point has a near zero torque ($T_{Af} \approx 0$).

Figure 3.3 Solid model of the proposed experiment X1 as conceptually described in Fig. 3.2. Key: 1 – torsion arm, shaft, and disks ($J_d$); 2 – bearing; 3 – coupling hub ($J_p$); 4 – coil spring sets between coupling jaws; 5 – coupling hub ($J_c$); 6 – electromagnet mass drop; and 7 – base plate.
These components compose $J_A$ as defined by $J_A = J_{\text{shaft}} + J_{\text{arm}} + 2J_{\text{disk}}$ where $J_{\text{shaft}}$, $J_{\text{arm}}$, and $J_{\text{disk}}$ are the torsional inertias of the shaft, torsion arm, and a single disk, respectively, and are approximated by the following [3.13]:

$$\begin{align*}
J_{\text{shaft}} &\approx 0.5\pi\rho_{\text{shaft}} R_{\text{shaft}} L_{\text{shaft}}^4, \\
J_{\text{arm}} &\approx \rho_{\text{arm}} L_{\text{arm}_x} L_{\text{arm}_y} L_{\text{arm}_z} (L_{\text{arm}_x}^2 + L_{\text{arm}_y}^2)/12, \\
J_{\text{disk}} &\approx \pi\rho_{\text{disk}} L_{\text{disk}_x} R_{\text{disk}}^2 \left[ \left( 3R_{\text{disk}}^2 + L_{\text{disk}_z}^2 \right)/12 + \left( L_{\text{arm}_x} - R_{\text{disk}} \right)^2 \right].
\end{align*}$$

Next, a jaw coupling set, consisting of an elastomer spider and two identical hubs with keyways and set screws (see [3.14] for an example), is chosen. One of the hubs ($J_B$) mates to the keyed shaft on the end opposite to the torsion arm; the other hub ($J_C$) mates to a separate short keyed shaft that is fixed to the baseplate. The key and keyway interfaces between the hubs and shafts provide clearance elements $k_{AB}(\Theta_{AB})$ and $k_C(\Theta_C)$. The clearance gaps ($\Theta_{AB}$ and $\Theta_C$) are controlled by the key widths. If the set screw is tightened at either location, the hub becomes rigidly attached to the shaft (i.e. $k_{AB}(\Theta_{AB}) \to \infty$ and $k_C(\Theta_C) \to \infty$). The elastomer spider between the hubs is removed and replaced by six coil springs, each with translational stiffness $k_{\text{spring}}$ (measured under static loading). Accordingly, torsional stiffness $k_{BC}$ is defined by the expression

$$k_{BC} = 6k_{\text{spring}} R_{\text{spring}}^2$$

where

$R_{\text{spring}}$ is the radial distance to the centerline of the coil springs.

It is assumed that the first torsional mode of all experiment configurations can be approximated by the natural frequency of configuration X1-0, as defined by:
\[
\omega_{n,1} = \left[ k_{BC} / (J_A + J_B) \right]^{0.5}.
\]  

(3.10)

Because the coil springs and coupling hubs are purchased components, it is easier to control \( \omega_{n,1} \) through \( J_A \). For the sake of convenience, torsion arm length \( L_{arm,x} \) is selected so that \( 5 \text{ Hz} \leq \omega_{n,1} \leq 15 \text{ Hz} \). However, \( L_{arm,x} \) must also be chosen so that the frequency of the first flexural mode of the torsion arm within the plane of rotation (say \( \omega_{arm,1} \)) is greater than \( 10\omega_{n,1} \). This is verified by approximating the torsion arm as a cantilevered beam of length \( L'_{arm,x} = 0.5L_{arm,x} - 2R_{disk} \) with a point mass \( m' = \pi R_{disk}^2 L_{disk,z} + 2\rho_{arm} R_{disk} L_{arm,y} L_{arm,z} \) at its end [3.15]. The characteristic equation of this beam is defined by the following where subscript \( r \) is the modal index, \( \beta_r \) is the \( r \)th wave number \( (\beta_r = (\omega_{arm,r}^2 \gamma_{arm} E_{arm}^{-1} I_{arm}^{-1})^{0.25}) \), \( \gamma_{arm} \) is the mass per unit length of the torsion arm \( (\gamma_{arm} = \rho_{arm} L_{arm,y} L_{arm,z}) \), \( E_{arm} \) is the Young’s modulus of the arm material, and \( I_{arm} \) is the area moment of inertia of the arm’s cross section \( (I_{arm} = L_{arm,y}^3 L_{arm,z} / 12) \) [3.13, 15]:

\[
\beta^3_r \left\{ \frac{2\gamma_{arm} \left[ 1 + \cos(\beta_r L'_{arm,x}) \cosh(\beta_r L'_{arm,x}) \right] + \cdots}{2m' \beta_r \left[ \cos(\beta_r L'_{arm,x}) \sinh(\beta_r L'_{arm,x}) - \cosh(\beta_r L'_{arm,x}) \sin(\beta_r L'_{arm,x}) \right]} \right\} = 0. \tag{3.11}
\]

Next, Eq. (3.11) is solved (numerically) for the smallest positive value of \( \beta_r \), yielding \( \beta_1 \) and subsequently \( \omega_{arm,1} = (\beta_1^4 E_{arm} I_{arm}^{-1} \gamma_{arm}^{-1})^{0.5} \).

After the experiment is assembled, the torsional inertias are estimated using solid modeling software [3.16] (where \( \rho \) is estimated from the measured mass and volume of the respective physical component) and the flexural mode of the torsion arm is verified.
using commercial finite element code [3.17]. Additionally, \( k_{BC} \) is estimated by applying known torque values (with finite increments of \( \Delta T_A \)) to the torsion arm (in configuration X1-0) and measuring the resulting relative displacement \( \Delta \theta_A \) with a digital level to the nearest 0.1° [3.18] (see Appendix B for a list of hardware and instrumentation). The system natural frequency \( \omega_{n,1} \) is then checked by substituting the estimated values of \( J_A, J_B, \) and \( k_{BC} \) into Eq. (3.10).

Next, it is necessary to reduce the effect of dissipative elements (e.g. dry friction) and thus, the experiment is assembled so that shaft, bearing, and coupling hubs are well aligned (this reduces the normal forces at radial interfaces). Additionally, Teflon sheeting is adhered to sliding interfaces (where possible), such as between the coupling hubs.

Finally, an electromagnet mass drop is chosen to provide the external torque \( T_A(t) \), similar to prior work [3.5-7]. For time \( t < 0 \), the electromagnet (with additional mass) is attached to one of the steel disks on the torsion arm, as shown in Fig. 3.3. At \( t = 0 \), a switch disrupts the electrical current and the electromagnet (with additional mass) falls away from the disk. The amount of additional mass is determined by the torque required to fully displace spring \( k_{BC} \) (say by \( \theta_{BC,max} \), thus \( T_{Ao} = k_{BC} \theta_{BC,max} \)). This ensures that the initial operating point at each clearance is within stage II (i.e. contact), and maximizes the initial energy of the system.

To measure the angular motions, a translational accelerometer [3.19] (see Appendix B for specifications) is attached to each torsional inertia at a radial distance \( R_j \), as shown in Fig. 3.4. The measured translational acceleration signals \( \ddot{q}_j \) are recorded by a
commercial data acquisition system [3.20-21] and computer with associated software [3.22]. Measurement is triggered by a threshold value of $\dot{q}_A$, which is judiciously chosen so that the step consistently occurs near $t = 0$ for all configurations. Sampling frequencies of 12.8, 25.6, and 51.2 kHz are initially considered, and it is found that peak to peak values of $\ddot{\theta}_A(t)$ (for the single (X1-1a) and dual-clearance (X1-2) configurations) increase with sampling frequency.

**Figure 3.4** Instrumentation schematic for experiment X1. Here, $\ddot{\theta}$ is the calculated angular acceleration, $\ddot{q}$ is the measured translational acceleration, $J$ is the torsional inertia, and $R$ is the radial distance from the axis of rotation to the translational accelerometers [3.19]. Key: 1 – torsion arm, shaft, and disks; 2 – coupling hub; 3 – coupling hub; 4 – translational accelerometer; 5 – data acquisition device (DAQ) [3.20-21]; and 6 – computer (CPU) [3.22].
Although this is a significant observation, a sampling frequency of 12.8 kHz is chosen because the truly impulsive nature of the response is not a primary concern of this study. Next, it is assumed that measured signals $\dot{q}_j$ are tangential to the corresponding angular motions, thus angular accelerations $\ddot{\theta}_j$ are calculated using the following equation:

$$
\ddot{\theta}_j = \dot{q}_j R_j^{-1}, \quad j = A, B, C.
$$

(3.12)

Typical measured angular accelerations (normalized) for the no (X1-0), single (X1-1a), and dual-clearance (X1-2) configurations are shown in Figs. 3.5-7, respectively; the typical external torque $T_A(t)$ (all configurations) is shown in Fig. 3.5b.

**Figure 3.5** Measured (normalized) angular acceleration (a) $\ddot{\theta}_A(t)$ for the no-clearance configuration of Fig. 2c (X1-0), given (b) step-like excitation $\overline{T}_A(t)$.  

61
Here, time is scaled by $\tau_{n-1} = 2\pi / \omega_{n-1}$, angular acceleration by $\theta_{AB\text{ max}} \tau_{n-1}^{-2}$, and torque by $k_{BC} \theta_{BC\text{ max}}$. Several important observations are made. First, the measured motion of the no-clearance configuration (X1-0) appears to be almost linear and its oscillatory period $\overline{\tau}_{\text{osc}}$ (defined by the elapsed time between extrema and then normalized) is nearly constant. Additionally, the nature of the amplitude decay is neither distinctly linear nor exponential.

In contrast, the measured motions of the single (X1-1a) and dual-clearance (X1-2) configurations clearly exhibit rich non-linear behavior. In both X1-1a and X1-2, the double-sided impact regime (denoted di) is observed across the entire response (with amplitudes about 20 times greater than in X1-0).

Figure 3.6 Measured angular accelerations for the single-clearance configuration of Fig. 3.2b (X1-1), given step-like excitation $\overline{T}_s(\overline{\tau})$ of Fig. 3.5b: (a) $\overline{\theta}_A(\overline{\tau})$; and (b) $\overline{\theta}_B(\overline{\tau})$. 

62
Prior work [3.8] has shown that these impacts occur when there is a sudden change from a relatively low (or zero) to high stiffness level, such as from $k_{j_{-1}}$ to $k_{j_{+1}}$ at $\pm \Theta_{j_{-1}}$. It is also noted that $\overline{\tau}_{\text{osc}}$ increases (nearly exponentially) by about 40% from the first to last oscillatory period (softening effect). Furthermore, the response durations of the clearance configurations (X1-1a and X1-2) are much shorter than that of the no-clearance configuration (X1-0).

**Figure 3.7** Measured angular accelerations for the dual-clearance configuration of Fig. 3.2a (X1-2), given step-like excitation $\overline{F}_{a}(\overline{t})$ of Fig. 3.5b: (a) $\overline{\theta}_{a}(\overline{t})$; (b) $\overline{\theta}_{b}(\overline{t})$; and (c) $\overline{\theta}_{c}(\overline{t})$. 63
This suggests that there are additional dissipative elements in X1-1a and X1-2, such as the sliding interfaces \( h_{4b} \) and \( h_c \), and possibly impact damping (though this issue is beyond the scope of the paper). In the next section, the proposed experiment will be verified by comparing measurements to predictions from the non-linear models shown in Fig. 3.2.

### 3.5 Verification of the experiment via comparison with non-linear models

Since all linear and non-linear model parameters must be estimated prior to the simulation, necessary parameters, methods, and their procedural sequence are listed in Table 3.1. First, \( k_{BC} \) is estimated (as stated in section 3.4) by applying known static torque values (with relative amplitudes \( \Delta T_A \)) to the \( J_A \) element (in the no-clearance configuration X1-0) and measuring the resulting change in angular displacement \( \Delta \theta_A \) with a digital level [3.18]; accordingly, \( k_{BC} = \Delta T_A / \Delta \theta_A \). Next, a sub-experiment (denoted X1-P1 and conceptually illustrated in Fig. 3.8a) is proposed to estimate torsional inertia \( J_B \) and Coulomb friction \( h_{BC} \). It is built by removing \( J_A \) from the single-clearance configuration (X1-1a) and has the following governing equation where \( T_B(t) \) is an impulsive external torque applied to \( J_B \) via an impulse hammer:

\[
J_B \ddot{\theta}_B + h_{BC} \tanh(\eta \dot{\theta}_B) + k_{BC} \theta_B = T_B(t).
\] (3.13)

The angular acceleration \( \ddot{\theta}_B(t) \) is calculated from the measured translational acceleration \( \ddot{q}_B(t) \), as shown in Fig. 3.4 and given by Eq. (3.12). The value of \( J_B \) (which includes the translational accelerometer) is then extracted from \( J_B = k_{BC} (\tau_{osc,\mu} / 2\pi)^2 \) where \( \tau_{osc,\mu} \)
Table 3.1 Summary of the necessary system parameters (including the sequence, method of estimation, and corresponding figure) for non-linear models of the proposed experiment (X1).

<table>
<thead>
<tr>
<th>Sequence of estimation</th>
<th>Estimated parameter(s)</th>
<th>Method and corresponding figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>$k_{BC}$</td>
<td>No-clearance configuration X1-0 (static loading) – Fig. 3.2c</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>$J_B$, $J_C$, and $h_{BC}$</td>
<td>Sub-experiment X1-P1 – Fig. 3.8a</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>$J_A$ and $h_A$</td>
<td>No-clearance configuration X1-0 (step-response) – Fig. 3.2c</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt;</td>
<td>$h_{AB}$ and $h_C$</td>
<td>Sub-experiment X1-P2 – Fig. 3.8b</td>
</tr>
<tr>
<td>5&lt;sup&gt;th&lt;/sup&gt;</td>
<td>$\Theta_{AB-1}$, $\Theta_{C-1}$, $k_{AB-II}$, and $k_{C-II}$</td>
<td>Sub-experiment X1-P3 – Fig. 3.8c</td>
</tr>
</tbody>
</table>

is the average oscillatory period of the measured response $\ddot{\theta}_B(t)$. Due to similarity, it is assumed that $J_C = J_B$. Since the response amplitude decay is nearly linear, $h_{BC}$ is estimated through the well-known logarithmic decrement method [3.15].

Next, the step-response (shown in Fig. 3.5a) of the no-clearance configuration (X1-0) is utilized to estimate $J_A$ (including a translational accelerometer) and $h_A$. Like the sub-experiment X1-P1, $J_A$ is estimated from $J_A = k_{BC} \left(\tau_{osc,II} / 2\pi\right)^2 - J_B$ where $\tau_{osc,II}$ is the average oscillatory period of the step-response (Fig. 3.5a). It is assumed that the response
amplitude decay is nearly linear and that the total Coulomb friction amplitude \( h_A + h_{BC} \) can be estimated as before \([3.15]\); as \( h_{BC} \) is already known, \( h_A \) is easily calculated.

A second sub-experiment (denoted X1-P2 and shown in Fig. 3.8b) is proposed to estimate Coulomb friction \( h_{AB} \). It is a modified version of the single-clearance experiment (X1-1a), where \( k_{AB}(\theta_{AB}) = 0 \) and \( J_B \) is fixed to the ground, and thus given by the following:

\[
J_A \ddot{\theta}_A + (h_A + h_{AB}) \tanh\left( \eta \dot{\theta}_A \right) = T_A(t).
\]  

(3.14)

The external torque \( T_A(t) \) is applied to the \( J_A \) element via a translational spring scale at the end torsion arm. Here, \( T_A(t) \) is slowly increased from zero amplitude (at \( t = 0 \)) until sliding (at constant velocity) occurs, and then it is slowly decreased until sticking occurs (say at \( T_A(t) = T_{stick} \)). It is assumed that \( h_{AB} \approx T_{stick} - h_A \).
Figure 3.8 Sub-experiments of X1 utilized for parameter estimation: (a) sub-experiment X1-P1 (estimation \( J_B \) and \( h_{BC} \)); (b) sub-experiment X1-P2 (estimation \( h_{AB} \)); and (c) sub-experiment X1-P3 (estimation of \( \Theta_{AB_1} \) and \( k_{AB_2} \)). Here, \( \{ \theta, \dot{\theta} \} \) is the angular displacement and velocity (respectively), \( J \) is the torsional inertia, \( k \) is the torsional stiffness, \( h \) is the Coulomb friction, and \( T \) is the external torque. Subscripts \( \{ A,B,C \} \) are element and coordinate indices. Key: 1 – torsion arm, shaft, and disks; 2 – key and keyway (clearance); 3 – coupling hub; 4 – torsional spring (linear); and 5 – shaft and bearing interface.
A third sub-experiment (denoted X1-P3 shown in Fig. 3.8c) is proposed for the estimation of $\Theta_{AB_{-1}}$ and $k_{AB_{-II}}$. It is constructed by removing all components except $J_C$, $k_C(\theta_C)$, and $h_C$ from the dual-clearance configuration (X1-2) and it is given by the following:

$$J_C \dot{\theta}_C + h_C \tanh(\eta \dot{\theta}_C) + \Phi_C(\theta_C) = T_C(t).$$

(3.15)

Here, $T_C(t)$ is applied to the $J_C$ element via a torsion arm with variable mass at its end (the subsequent measurements occur under static conditions, thus the additional torsional inertia of the arm and mass is inconsequential). The angular displacement $\theta_C$ is calculated to the nearest 0.01° using $\theta_C = q_C R_{\text{probe}}^{-1}$ where $q_C$ is the translational displacement measured by a probe [3.23] located at radial distance $R_{\text{probe}}$ (where it is assumed that $q_C$ is tangential to $\theta_C$). First, $J_C$ is moved between the two contact points of the clearance $k_C(\theta_C)$ and the relative angular displacement (say $\theta_{gap}$) is measured. It has been assumed that the clearance elements are symmetric, thus $\Theta_{C_{-1}} = 0.5 \theta_{gap}$. Torsional inertia $J_C$ is then positioned at either $\pm \Theta_{C_{-1}}$ and known static torque values (with relative amplitudes $\Delta T_C$) are applied to $J_C$ via a torsion arm and variable masses. The resulting change in angular displacement $\Delta \theta_C$ is measured and stiffness $k_{C_{-II}}$ is calculated using

$$k_{C_{-II}} = \frac{\Delta T_C}{\Delta \theta_C}.$$ Nevertheless, measurements reveal that $k_{C_{-II}}$ is very stiff compared to $k_{BC}$ (say by a factor of nearly 200); thus, the value of $k_{C_{-II}}$ is approximated as $k_{C_{-II}} \approx 200 k_{BC}$. Recall from section 3.3, it is assumed that $k_{AB_{-II}} = k_{C_{-II}}$ and $\Theta_{AB_{-1}} = \Theta_{C_{-1}}$. 

68
After the parameter estimation process is complete, the step-responses of each configuration are predicted by numerically integrating the corresponding governing equations (Eq. (3.3) for X1-2, Eq. (3.5) for X1-1a, and Eq (3.6) for X1-0). Several variable order and time-step Runge-Kutta algorithms [3.24] are used and the maximum allowable time-step and resolution of the resulting time array are set equal to the sampling period of the measurements (about 78.1 μs). Since there is negligible difference between the integration algorithms, the non-linear simulation is considered successful. First, a comparison between the measurement and prediction for the no-clearance configuration (X1-0) is shown in Fig. 3.9. The motions have nearly equivalent oscillatory periods (in terms of duration and quantity) along with peak to peak acceleration amplitudes, which are listed in Table 3.2. This agreement provides fidelity to the parameter estimation process as well as the instrumentation and signal processing method employed.
Figure 3.9 Measured and predicted angular acceleration $\ddot{\theta}_A(\bar{T})$ (normalized) step-response of the no-clearance configuration (X1-0). Key: (▬▬) – measurement; and (▬▬) – prediction. Here, ($j^{th}$) marks are the peak to peak values of their oscillatory periods.

Table 3.2 Measured and predicted peak to peak values for oscillatory periods in $\ddot{\theta}_A(\bar{T})$ of the no-clearance configuration (X1-0) given step excitation.

<table>
<thead>
<tr>
<th>Period Number</th>
<th>2nd</th>
<th>4th</th>
<th>6th</th>
<th>8th</th>
<th>10th</th>
<th>12th</th>
<th>14th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measurement</td>
<td>74</td>
<td>63</td>
<td>50</td>
<td>38</td>
<td>26</td>
<td>17</td>
<td>8.3</td>
</tr>
<tr>
<td>Prediction</td>
<td>72</td>
<td>62</td>
<td>52</td>
<td>41</td>
<td>31</td>
<td>21</td>
<td>10</td>
</tr>
</tbody>
</table>
Next, measured and predicted motions of the single-clearance configuration (X1-1a) are compared in Fig. 3.10. Additionally, peak to peak values and times of occurrence for selected impacts (which are labeled in Fig. 3.10) are listed in Tables 3.3 and 3.4, respectively. Both predicted and measured motions exhibit the double-sided impact regime and have very similar duration (about 18 impacts and 9 or 10 oscillatory periods). However, the predicted motions seem to have a lead that increases with time as clearly observed in the times of occurrence in Table 3.4. Although the reduced-order non-linear model fails to predict the precise peak to peak values of impacts in $\overline{\theta_a(\bar{t})}$, it is relatively accurate with respect to the impacts in $\overline{\theta_b(\bar{t})}$ (say within a factor of 2) and long time-scale motions (which are nearly sinusoidal with an oscillatory period of about $\overline{\tau_{n,1}}$ in both $\overline{\theta_a(\bar{t})}$ and $\overline{\theta_b(\bar{t})}$).

Finally, a comparison between the measurement and prediction for the dual-clearance configuration (X1-2) is shown in Fig. 3.11 and their peak to peak values as well as times of occurrence for impacts (as labeled in Fig. 3.11) are shown in Tables 3.5 and 3.6, respectively. Most comments regarding the predictions of X1-1a also apply to X1-2 except for two notable differences. First, the prediction (X1-2) exhibits one more oscillatory period (and about two more impacts) than the measurement for some responses, and secondly, the leading effect of the prediction is much greater in X1-2 than seen previously in X1-1a. Overall, the predicted and measured motions (for all configurations) exhibit similar quantitative (in some aspects) and qualitative behavior, which verifies the
feasibility of the experiment. To improve the accuracy of the predictions, refined higher-dimensional non-linear models must be considered.

\[\ddot{\theta}_A(\bar{t}) \text{ and } \ddot{\theta}_B(\bar{t})\] (normalized) step-response of the single-clearance configuration (X1-1a). Key: (▬▬) – measurement; and (▬▬) – prediction. Here, \((j^{th})\) impacts are marked.

**Figure 3.10** Measured and predicted angular accelerations \(\ddot{\theta}_A(\bar{t})\) and \(\ddot{\theta}_B(\bar{t})\) (normalized) step-response of the single-clearance configuration (X1-1a). Key: (▬▬) – measurement; and (▬▬) – prediction. Here, \((j^{th})\) impacts are marked.
Table 3.3 Measured and predicted peak to peak values for selected impacts: (a) in $\bar{\theta}_a(T)$; and (b) in $\bar{\theta}_b(T)$ of the single-clearance configuration (X1-1a) given step excitation.

<table>
<thead>
<tr>
<th>Impact Number</th>
<th>Measurement</th>
<th>Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2230</td>
<td>78</td>
</tr>
<tr>
<td>2nd</td>
<td>1420</td>
<td>80</td>
</tr>
<tr>
<td>9th</td>
<td>1450</td>
<td>66</td>
</tr>
<tr>
<td>10th</td>
<td>920</td>
<td>65</td>
</tr>
<tr>
<td>17th</td>
<td>470</td>
<td>33</td>
</tr>
<tr>
<td>18th</td>
<td>160</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 3.4 Measured and predicted times of occurrence selected impacts: (a) in $\bar{\theta}_a(T)$; and (b) in $\bar{\theta}_b(T)$ of the single-clearance configuration (X1-1a) given step excitation.

<table>
<thead>
<tr>
<th>Impact Number</th>
<th>Measurement</th>
<th>Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>0.31</td>
<td>0.31</td>
</tr>
<tr>
<td>2nd</td>
<td>0.83</td>
<td>0.82</td>
</tr>
<tr>
<td>9th</td>
<td>4.47</td>
<td>4.39</td>
</tr>
<tr>
<td>10th</td>
<td>5.01</td>
<td>4.91</td>
</tr>
<tr>
<td>17th</td>
<td>8.88</td>
<td>8.59</td>
</tr>
<tr>
<td>18th</td>
<td>9.56</td>
<td>9.19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Impact Number</th>
<th>Measurement</th>
<th>Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>0.32</td>
<td>0.31</td>
</tr>
<tr>
<td>2nd</td>
<td>0.83</td>
<td>0.82</td>
</tr>
<tr>
<td>9th</td>
<td>4.48</td>
<td>4.39</td>
</tr>
<tr>
<td>10th</td>
<td>5.01</td>
<td>4.91</td>
</tr>
<tr>
<td>17th</td>
<td>8.88</td>
<td>8.59</td>
</tr>
<tr>
<td>18th</td>
<td>9.55</td>
<td>9.19</td>
</tr>
</tbody>
</table>
Figure 3.11 Measured and predicted angular accelerations (normalized) for step-response of the dual-clearance configuration (X1-2): (a) $\ddot{\theta}_A(t)$; (b) $\ddot{\theta}_B(t)$; and (c) $\ddot{\theta}_C(t)$. Key: (▬) – measurement; and (▬) – prediction. Here, ($j^{th}$) impacts are marked.
Table 3.5 Measured and predicted peak to peak values for selected impacts: (a) in $\overline{\theta}_A(T)$; (b) in $\overline{\theta}_B(T)$; and (c) in $\overline{\theta}_C(T)$ of the dual-clearance configuration (X1-2) given step excitation.

<table>
<thead>
<tr>
<th></th>
<th>Peak to peak value (normalized) for impact number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>(a) $\overline{\theta}_A(T)$</td>
<td></td>
</tr>
<tr>
<td>Measurement</td>
<td>1660</td>
</tr>
<tr>
<td>Prediction</td>
<td>80</td>
</tr>
<tr>
<td>(b) $\overline{\theta}_B(T)$</td>
<td></td>
</tr>
<tr>
<td>Measurement</td>
<td>1540</td>
</tr>
<tr>
<td>Prediction</td>
<td>1170</td>
</tr>
<tr>
<td>(c) $\overline{\theta}_C(T)$</td>
<td></td>
</tr>
<tr>
<td>Measurement</td>
<td>2500</td>
</tr>
<tr>
<td>Prediction</td>
<td>1220</td>
</tr>
</tbody>
</table>

Table 3.6 Measured and predicted times of occurrence for selected impacts: (a) in $\overline{\theta}_A(T)$; (b) in $\overline{\theta}_B(T)$; and (c) in $\overline{\theta}_C(T)$ of the dual-clearance configuration (X1-2) given step excitation.

<table>
<thead>
<tr>
<th></th>
<th>Time of occurrence (normalized) for impact number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>(a) $\overline{\theta}_A(T)$</td>
<td></td>
</tr>
<tr>
<td>Measurement</td>
<td>0.33</td>
</tr>
<tr>
<td>Prediction</td>
<td>0.31</td>
</tr>
<tr>
<td>(b) $\overline{\theta}_B(T)$</td>
<td></td>
</tr>
<tr>
<td>Measurement</td>
<td>0.32</td>
</tr>
<tr>
<td>Prediction</td>
<td>0.31</td>
</tr>
<tr>
<td>(c) $\overline{\theta}_C(T)$</td>
<td></td>
</tr>
<tr>
<td>Measurement</td>
<td>0.34</td>
</tr>
<tr>
<td>Prediction</td>
<td>0.31</td>
</tr>
</tbody>
</table>
3.6 Relevance and applicability to a large-scale experiment

As stated in section 3.3, the single-clearance configuration (X1-1a) is analogous to the large-scale experiment recently proposed by Krak et al. [3.8] (denoted X2). The prior experiment [3.8] can be simply described by a 1DOF non-linear model with torsional inertia $J$, multi-staged torsional stiffness element $k(\theta)$, multi-staged dry friction element $h(\theta)$, step-like external torque $T(t)$, and angular displacement, velocity, and acceleration \{\theta, \dot{\theta}, \ddot{\theta}\}, respectively (see [3.8] for a schematic and full discussion). In particular, the stiffness $k(\theta)$ and dry friction $h(\theta)$ elements are provided by a production vehicle clutch damper. The elastic torque transmission $\Phi(\theta)$ for two example cases (denoted C1 [3.8] and C2) is illustrated in Fig. 3.12. Both C1 and C2 contain four-stages; stages II, III, and IV are a pre-load feature, main operating stage, and stopper, respectively. Stage I has a very compliant spring ($k_1 \approx 0$) in C1 and a spline backlash ($k_i = 0$) in C2. The abrupt change in stiffness from stages I to II (in C1 and C2) is similar to the clearance non-linearity contained in the proposed experiment (X1-1a).

The measured angular accelerations $\ddot{\theta}(\bar{T})$ for X2-C1 [3.8] and X2-C2, given a step-like external torque $T(t)$ excitation (with initial and final operating points as labeled in Fig. 3.12), are shown in Figs. 3.13a and 3.14a, respectively. Here, results are normalized where the time is scaled by $\tau_{n_{III}}$ (natural period of stage III) and the angular acceleration is scaled by $\Theta_{IV} / \tau_{n_{III}}^2$. The step-response of both X2-C1 and X2-C2 exhibit double (di), single (si), and no-impact (ni, though not shown for X2-C1 in Fig. 3.12).
Figure 3.12 Characteristics of two multi-staged clutch dampers (denoted C1 and C2) that are utilized in a prior large-scale experiment (denoted X2) [3.8]: (a) C1 (where stage I is a very compliant spring); and (b) C2 (where stage I is a spline clearance). Here, $\Phi(\theta)$ is the elastic torque transmission; see Fig. 3.1 for other symbols.
3.13a) regimes [3.8-9]. The time of transition between these regimes are denoted by $t_{di}$ (from di to si regimes) and $t_{si}$ (from si to ni regimes). These regimes correspond to travel across stage transitions $\Theta_1$ and $-\Theta_1$ as discussed in [3.8]. The responses can be further characterized by the time-varying oscillatory period $\tau_{osc}^{(j)}$, which is defined here as the elapsed (normalized) time between the $j^{th}$ and $(j+1)^{th}$ impact at $\Theta_1$ with a corresponding time of occurrence $\tau_{osc}^{(j)}$ (median time between the $j^{th}$ and $(j+1)^{th}$ impact at $\Theta_1$). Points $(\tau_{osc}^{(j)}, \tau_{osc}^{(j)})$ are labeled in Figs. 3.13-14a and shown in Figs. 3.13-14b (for the di and si regimes only). Observe that points $(\tau_{osc}^{(j)}, \tau_{osc}^{(j)})$ exhibit asymptotic trends within each regime; for instance, the double-sided impact regime (di) shows a softening effect ($\tau_{osc}^{(j)} \to \infty$) while the single-sided impact regime (si) has a hardening effect ($\tau_{osc}^{(j)} \to 0$).

This behavior is also exhibited by the single-clearance configuration of the proposed experiment (X1-1a); points $(\tau_{osc}^{(j)}, \tau_{osc}^{(j)})$ from motion $\bar{\theta}_y(t)$ are labeled in Fig. 3.15a and shown in Fig. 3.15b. Here, only the hardening effect is apparent because the entire response is in the double-sided impact regime (di) only. This qualitative agreement clearly demonstrates the relevance and applicability of the proposed scientific experiment.
Figure 3.13 Measured step-response of clutch damper C1 (X2-C2) from the prior experiment X2 [3.8] with: (a) angular acceleration $\ddot{\theta}(\bar{t})$; and (b) impulse period $\bar{\tau}_{osc}^{(j)}$ (for double (di) and single-sided (si) impact regimes only). Here, $\bar{t}_{di}$ is the time of transition from double to single-sided impact regimes and $\bar{t}_{si}$ is the time of transition from the single to no-impact regime. Key: (▬▬) $\ddot{\theta}(\bar{t})$ and (▬ ▬) $\bar{\tau}_{osc}^{(j)}$. 

79
Figure 3.14 Measured step-response of clutch damper C2 (X2-C2) from the prior experiment [3.8] with: (a) angular acceleration $\bar{\theta}(\bar{T})$; and (b) impulse period $\bar{\tau}_{osc}^{(j)}$ (for double (di) and single-sided (si) impact regimes only). Here, $\bar{t}_{di}$ is the time of transition from double to single-sided impact regimes and $\bar{t}_{si}$ is the time of transition from the single to no-impact regime. Key: (▬▬) – $\bar{\theta}(\bar{T})$ and (▬…”–) – $\bar{\tau}_{osc}^{(j)}$. 
Figure 3.15 Measured step-response of the single-clearance configuration (X1-1a) of the proposed experiment: (a) angular acceleration \( \ddot{\theta}_B(T) \); and (b) impulse period \( \tau^{(j)}_{\text{osc}} \). Key: (▬▬) – \( \ddot{\theta}_B(T) \) and (▬×▬) – \( \tau^{(j)}_{\text{osc}} \).

3.7 Conclusion

The chief contribution of this article is a new scientific experiment that exhibits the step-response of a torsional system that contains one or two controlled clearance nonlinearities (single or dual-clearance configurations). Although it is similar in several aspects to large-scale experiments proposed in prior work [3.4-8], this new experiment is...
quite unique since it contains only well-known and controlled non-linear features while intentionally designed to be scaled (“bench-top”). Typical measurements (for both the single and dual-clearance configurations) exhibit richly non-linear behavior, including the double-sided impact regime [3.8-9] and a time-varying oscillatory period. Reasonable agreement between the measurements and predictions from simplified, reduced-order non-linear models verify that the experiment (in various configurations) operates as intended. However, it is also obvious that more refined higher-dimensional non-linear models are necessary to better describe the physics of the system. A qualitative comparison to a prior large-scale experiment [3.8] demonstrates the relevance and applicability to the literature [3.4-8]. Further, a redesign of the proposed experiment could enhance its utility. For example, introducing asymmetry to the torsion arm would migrate the final operating point \( (\theta_{AB,f}, \Phi_{AB,f}) \) towards an angular transition \( \pm\theta_{AB,1} \), thus increasing the likelihood of the single-sided impact regime (si). Finally, the proposed scientific experiment should yield critical time domain benchmark data that is collected under laboratory conditions with well-known and controlled features. This dataset is essential for the validation of non-linear simulation models, parameter estimation [3.11], as well as providing insight into the role of one or multiple clearance non-linearities within a torsional system.

References for Chapter 3


List of symbols for Chapter 3

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>$h$</td>
<td>Coulomb friction amplitude</td>
</tr>
<tr>
<td>$I$</td>
<td>area moment of inertia</td>
</tr>
<tr>
<td>$J$</td>
<td>torsional inertia</td>
</tr>
<tr>
<td>$k$</td>
<td>torsional stiffness</td>
</tr>
<tr>
<td>$L$</td>
<td>translational distance (length)</td>
</tr>
<tr>
<td>$\dot{q}$</td>
<td>translational acceleration</td>
</tr>
<tr>
<td>$R$</td>
<td>radius</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T$</td>
<td>external torque</td>
</tr>
<tr>
<td>$\beta$</td>
<td>wave number</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>mass per unit length</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>finite change</td>
</tr>
<tr>
<td>$\eta$</td>
<td>regularizing factor for Coulomb friction</td>
</tr>
<tr>
<td>$\theta, \dot{\theta}, \ddot{\theta}$</td>
<td>angular displacement, velocity, and acceleration</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>stage transition (angular)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>material density</td>
</tr>
<tr>
<td>$\tau$</td>
<td>oscillatory period</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>elastic torque transmission</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>ideal unit step function</td>
</tr>
</tbody>
</table>
\( \omega \) circular frequency

**Subscripts**

\( A, B, C \) absolute coordinate and element indices

\( AB, BC \) relative coordinate and element indices

arm torsion arm

di double-sided impact

disk disk

\( f \) final point

imp impulse

I,II,... stage index

gap gap or clearance

max maximum

\( n \) natural

\( ni \) no-impact

\( o \) initial point

osc oscillatory

probe translational displacement probe

\( r \) modal index

shaft shaft

\( si \) single-sided impact

stick sticking condition (dry friction)
x,y,z  spatial (length, width, thickness)

µ  average

Abbreviations

1,2,3DOF  one, two, and three degree(s) of freedom

X1-0  proposed experiment, no-clearance configuration

X1-1a,b  proposed experiment, single-clearance configuration(s)

X1-2  proposed experiment, dual-clearance configuration

X1-P1,2,3  proposed sub-experiments for parameter estimation

X2  prior large-scale experiment (see Ref. [3.8])
Chapter 4: Asymptotic trends in time-varying oscillatory period for a dual-staged torsional system

4.1 Introduction

Most torque-transmission devices contain discontinuous non-linearities, such as clearances, multi-staged springs, pre-load elements, and stoppers [4.1-11]; prime examples of this family are vehicle powertrains and drivelines, which are composed of gears, spline shafts, synchronizers, and clutch dampers. This inherent set of non-linear features gives rise to undesirable vibro-impact phenomena, often known in the ground vehicle industry in terms of gear rattle [4.4, 9] and vehicle driveline clunk [4.5-8] problems. Numerical and experimental studies concerning free vibration or step-response [4.5-8] clearly show regime-dependent (say double-sided or single-sided impact [4.9]) and time-varying oscillatory periods, as well as distinct asymptotic trends [4.4-11] (hardening, softening, or linear). In an effort to better understand previously observed results and assess such non-linear phenomena in measured or numerical motion signatures [4.4-11], this paper will propose a new analysis tool. The chief goal is to estimate the trends in time-varying oscillatory periods of a non-linear torsional oscillator with a dual-staged spring and associated dry friction element. The method will be computationally verified and experimentally validated.
4.2 Analytical formulation of the problem

Consider a simple and yet representative time-invariant, non-linear torsional system described by a single degree of freedom (SDOF) model of Fig. 4.1a. The corresponding governing equation of motion is the following:

\[ J\ddot{\theta} + h \tanh(\eta \dot{\theta}) + \Phi(\theta) = T(t). \]  

Here, \( \theta \) is the angular displacement, \( J \) is torsional inertia, \( h \) is the Coulomb friction amplitude, \( \eta \) is an empirical regularizing factor \([4.16]\), \( \Phi(\theta) \) is the torque transmitted through the stiffness element \( k(\theta) \), and \( T(t) \) is the external torque (see the end of this chapter for a full list of symbols). For the sake of simplicity, \( k(\theta) \) is a symmetric dual-staged torsional spring as defined by the following equation where \( \Xi \) is the unit step function, subscripts I and II denote stages, \( k_j \) is the stage dependent torsional stiffness value, and \( \Theta_j \) is an angular stage transition:

\[ k(\theta) = k_i + (k_{ii} - k_i) \Xi(|\theta| - \Theta_i). \]  

Accordingly, \( \Phi(\theta) \) is illustrated in Fig. 4.1b and described by the following piecewise linear equation where \( \text{sgn} \) is the triple-valued sign function:

\[ \Phi(\theta) = k_i \theta + \text{sgn}(\theta)(k_{ii} - k_i)(|\theta| - \Theta_i) \Xi(|\theta| - \Theta_i). \]  

The dynamic response within each stage only may be addressed by a linear sub-system with the corresponding natural periods:

\[ \tau_{nj} = 2\pi \left(\frac{J}{k_j}\right)^{0.5}, \; j = I, II. \]
Figure 4.1 Conceptual illustration of the non-linear torsional system: (a) single-degree of freedom model; (b) elastic torque $\Phi(\theta)$; and (c) examples of near backlash ($\alpha = 0.1$) and pre-load ($\alpha = 10$) non-linearities examined in this article. Here, $\theta$ is the angular displacement, $J$ is the torsional inertia, $h$ is a Coulomb friction element, $k(\theta)$ is a non-linear torsional stiffness element, and $T(t)$ is the external torque.
The stiffness of the first stage is scaled such that $k_1 = \alpha k_{ll}$ where $\alpha \geq 0$. For the case of backlash non-linearity with $\alpha = 0$, stage I is a clearance element and $\tau_{nl} \rightarrow \infty$. However, when $\alpha \rightarrow \infty$, stage I is a pre-load feature and $\tau_{nl} \rightarrow 0$; the limited case is $\alpha = 1$, where $k(\theta)$ is simply a linear spring. External torque $T(t)$ is a step applied at $t = 0$ and is such that the initial and final operating points $((\theta_o, T_o) \text{ and } (\theta_f, T_f))$, respectively) lie on different stages, as shown in Fig. 4.1b. This ensures a comprehensive non-linear response with three distinct regimes and asymptotic trends of the time-varying oscillatory period, as seen in prior work [4.4-8].

Specific objectives are as follows: 1) Propose the instantaneous effective stiffness concept and utilize it to estimate the asymptotic trends of the time-varying oscillatory period; 2) Propose the necessary digital signal processing (DSP) parameters for related time domain analyses; and 3) Validate the proposed method by estimating the asymptotic trends in the time-varying oscillatory period of two recent laboratory experiments. The scope of this paper is limited to the time domain analysis of the step-response of time-invariant torsional systems. Additionally, it is assumed that damping can be approximated as Coulomb friction and therefore the oscillatory period may be approximated by the natural period of the first mode. This article extends the stochastic linearization techniques [4.12-13] proposed by Wallaschek [4.14] and Rook and Singh [4.15].
4.3 Time-varying oscillatory period

The typical step-responses of the example case (Fig. 4.1) is predicted for two illustrative cases, a nearly backlash non-linearity ($\alpha = 0.1$) and nearly pre-load non-linearity ($\alpha = 10$). The values of $k_\parallel$ and $\Theta_1$ are estimated from a production vehicle clutch damper [4.8]. Torsional inertia $J$ is sized so that $\tau_{\text{all}} = 0.1$ s, which is within the range of the first torsional surge mode of a vehicle driveline [4.4-9]. The amplitudes of $h$, $(\theta_o, T_o)$, and $(\theta_j, T_j)$ are judiciously selected for each case so that the responses have similar durations though each is sufficiently long and exhibits the desired regimes. Angular motions are predicted through numerical integration1 [4.17] of Eq. (4.1); several variable-step Runge-Kutta algorithms intended for low to high mathematically stiff systems are utilized [4.17], but negligible differences between the results are found. Additionally, the maximum allowable time step for integration and the time resolution of resulting signals are set equal to approximately 78.1 $\mu$s, which corresponds to a sampling frequency of 12.8 kHz. The predicted angular motions are shown in Figs. 4.2 and 4.3; time and angular displacement are normalized by $\tau_{\text{all}}$ and $\Theta_1$, respectively. Responses of Figs. 4.2 and 4.3 exhibit three distinct regimes, similar to prior work [4.8]: i) the double-sided impact regime (di) is characterized by $\theta$ crossing both $-\Theta_1$ and $\Theta_1$; ii) the single-sided impact regime (si) occurs when $\theta$ crosses $-\Theta_1$ only; and iii) the no-impact regime (ni) is defined when $\theta$ does not cross any $\Theta_j$. 
The responses are further characterized by oscillatory period \( \tau^{(j)} \) with corresponding time of occurrence \( t^{(j)} \). Values \( \tau^{(j)} \) and \( t^{(j)} \) are defined as the elapsed and median times, respectively, between the \( j \)th and \( (j + 2) \)th extrema (peak or valley) of a signal, as shown in Figs. 4.2 and 4.3. A continuous time domain signal \( \tau(t) \) is then

![Diagram](image)

**Figure 4.2** Predicted angular motions (normalized) for a backlash type non-linear system \((\alpha = 0.1)\). Here, \( \dot{\theta} \) is the angular displacement, \( \ddot{\theta} \) is the angular velocity, \( \dddot{\theta} \) is the angular acceleration, \( \tau^{(j)} \) is oscillatory period \( j \), \( t_{di} \) is the time of transition from the double-sided to single-sided impact regimes, \( t_{si} \) is the time of transition from the single-sided to no-impact regimes, and \( \Theta_1 \) is the angular transition from stage I to II. Key: (---) – angular motion; (–––) – \( \Theta_1 \); and (|) – \{\( t_{di}, t_{si} \)\}.  

93
Figure 4.3 Predicted angular motions (normalized) for a pre-load type non-linear system \((\alpha = 10)\).

Here, \(\bar{\theta}\) is the angular displacement, \(\bar{\theta}\) is the angular velocity, \(\bar{\theta}\) is the angular acceleration, \(\tau^{(j)}\) is oscillatory period \(j\), \(\tau_{t_{ii}}\) is the time of transition from the double-sided to single-sided impact regimes, \(\tau_{t_{ii}}\) is the time of transition from the single-sided to no-impact regimes, and \(\Theta_{t_i}\) is the angular transition from stage I to II. Key: (▬▬) – angular motion; (▬▬) – \(\bar{\theta}\); and (|) – \(\{t_{t_{ii}}, \tau_{t_{ii}}\}\).

calculated from a smoothened spline fit [4.17] of the discrete points \((\tau^{(j)}, \tau^{(j)})\) where \(j = 1, 2, \ldots, P\) and \(P\) is the total number of oscillatory periods; calculated \(\tau(t)\) is shown in Fig. 4.4 for \(\theta(t), \dot{\theta}(t), \text{ and } \ddot{\theta}(t)\). It is observed that \(\tau(t)\) calculations are similar across all
angular motions and exhibit regime-dependent asymptotic trends, summarized in Table 4.1. These trends can be described as hardening \((\tau(t) \to 0)\), softening \((\tau(t) \to \infty)\), or linear \((\tau(t) \approx \tau_y)\) in nature. Next, a concept of instantaneous effective stiffness is proposed and later utilized to estimate \(\tau(t)\).

**Figure 4.4** Calculated oscillatory period \(\bar{T}(\bar{I})\) (normalized) for a torsional system with: (a) backlash type non-linearity \((\alpha = 0.1)\); and (b) pre-load type non-linearity \((\alpha = 10)\). Here, \(\bar{\tau}_{\text{all}}\) is the natural period of stage II, \(\bar{t}_{\text{di}}\) is the time of transition from double-sided to single-sided impact regime, and \(\bar{t}_{\text{si}}\) is the time of transition from the single-sided to no-impact regime. Key: (-----) – calculated from \(\bar{\theta}(\bar{I})\); (-----) – calculated from \(\bar{\theta}(\bar{I})\); (-----) – calculated from \(\bar{\theta}(\bar{I})\); (-----) – \(\bar{\tau}_{\text{all}}\); and (-----) – \(\{\bar{t}_{\text{di}}, \bar{t}_{\text{si}}\}\).
Table 4.1  Regime-dependent asymptotic trends of $\bar{t}(T)$ for $\alpha = 0.1$ and $\alpha = 10$.

<table>
<thead>
<tr>
<th>Regime</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>double-sided impact, di</td>
<td>softening</td>
<td>hardening</td>
</tr>
<tr>
<td>single-sided impact, si</td>
<td>hardening</td>
<td>softening</td>
</tr>
<tr>
<td>no-impact, ni</td>
<td>linear, $\bar{t}(T) \approx \bar{t}_{\text{lin}}$</td>
<td>linear, $\bar{t}(T) \approx \bar{t}_{\text{lin}}$</td>
</tr>
</tbody>
</table>

4.4 Concept of instantaneous effective stiffness

Assume that over a small interval of time, with period $\tau_w$ centered at time $t = t'$, the following approximation [4.13-15] can be made $\{\Phi(\theta)\}_{t=\tau} \approx \{\hat{\Phi}(\theta)\}_{t=\tau'}$; here, $\hat{\Phi}(\theta)_{t=\tau'}$ is a linear time-invariant approximation of $\Phi(\theta)_{t=\tau'}$. Like prior work [4.14-15], $\hat{\Phi}(\theta)_{t=\tau'}$ is defined by the following where $\hat{k}_m_{t=\tau'}$ and $\hat{k}_a_{t=\tau'}$ are mean and alternating stiffness components, respectively, and $\langle \theta, t' \rangle_t$ is the instantaneous expected value (windowed time average) of $\theta(t)$ over period $\tau_w$ centered at time $t = t'$:

$$\{\hat{\Phi}(t)\}_{t=\tau'} = \{\hat{k}_m(\theta, t) \cdot \hat{k}_a[\theta(t) - \langle \theta, t' \rangle_t]\}_{t=\tau'}.$$  \hfill (4.5)

The proposed instantaneous expected value operator is defined by the following equation where $w_{\text{shp}}(t-t')$ is a windowing function:

$$\langle \theta, t' \rangle_t = \frac{\int_t^\prime \theta(t) w_{\text{shp}}(t-t') dt}{\int_t^\prime w_{\text{shp}}(t-t') dt}. \hfill (4.6)$$

96
The windowing function $w_{shp}(t-t')$ has the following general form, chosen such that the instantaneous expected value operator can truly be localized in time:

$$w_{shp}(t-t') =\begin{cases} 
0 & t < t' - 0.5\tau_w \\
> 0 & t' - 0.5\tau_w \leq t \leq t' + 0.5\tau_w \\
0 & t > t' + 0.5\tau_w
\end{cases}$$

(4.7)

To estimate $\hat{k}_m|_{t=t'}$ and $\hat{k}_a|_{t=t'}$, the following error is defined [4.14-15]:

$$e|_{t=t'} = \{\Phi(\theta) - \dot{\Phi}(\theta)\} \big|_{t=t'}.$$  (4.8)

Next, $\langle (e|_{t=t'})^2, t' \rangle_t$ is minimized with respect to $\hat{k}_m|_{t=t'}$ and $\hat{k}_a|_{t=t'}$ as follows:

$$\frac{\partial \langle (e|_{t=t'})^2, t' \rangle_t}{\partial \hat{k}_m|_{t=t'}} = 0, \quad \frac{\partial \langle (e|_{t=t'})^2, t' \rangle_t}{\partial \hat{k}_a|_{t=t'}} = 0.$$  (4.9)

Expanding Eq. (4.9), $\hat{k}_m|_{t=t'}$ and $\hat{k}_a|_{t=t'}$ are defined by

$$\hat{k}_m|_{t=t'} = \frac{\langle \Phi, t' \rangle_t \langle \theta, t' \rangle_t^2}{\langle \theta, t' \rangle_t^2},$$  (4.10)

$$\hat{k}_a|_{t=t'} = \frac{\langle \Phi \theta, t' \rangle_t - \langle \Phi, t' \rangle_t \langle \theta, t' \rangle_t}{\langle \theta^2, t' \rangle_t - \langle \theta, t' \rangle_t^2}.$$  (4.11)

Following prior work [4.14-15], instantaneous effective stiffness $\hat{k}|_{t=t'}$ is assumed to be equivalent to $\hat{k}_a|_{t=t'}$:

$$\hat{k}|_{t=t'} = \hat{k}_a|_{t=t'}.$$  (4.12)

However, $\hat{k}|_{t=t'}$ becomes undefined under static conditions; therefore, $t' \in [t_o + 0.5\tau_w, t_f - 0.5\tau_w]$ where $t_o$ and $t_f$ are the initial and final times of the dynamic conditions.
response, respectively. Furthermore, $t'$ and $\tau_w$ are restricted by sampling parameters and the duration of the dynamic response. Finally, a continuous time domain signal $\hat{k}(t)$ is calculated from a smoothened spline fit\(^\text{17}\) of the discrete signal $\hat{k}_{t=t'}$.

For the sake of illustration, $\tilde{k}(\tau)$ (normalized, $\tilde{k}_{\text{II}} = 1$) is estimated over several $w_{\text{shp}}$ and a range $\tau_w$ for a near backlash type non-linearity ($\alpha = 0.1$). The normalized window length $\tilde{\tau}_w$ is limited to \{0.5, 1, 2\} and windowing function $w_{\text{shp}}$ is simply defined by the following (illustrated in Fig. 4.5): i) box-car $w_{\text{box}}$; ii) triangular $w_{\text{tri}}$; and iii) right-skewed saw-tooth $w_{\text{saw}}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.5.png}
\caption{Uniform windowing functions $w_{\text{shp}}$ used for estimations: (a) box-car, $w_{\text{box}}$; (b) triangular, $w_{\text{tri}}$; and (c) saw-tooth, $w_{\text{saw}}$. Here, $t'$ is an arbitrary time and $\tau_w$ is the window length.}
\end{figure}
Figure 4.6  Instantaneous effective stiffness \( \tilde{k}(\tilde{T}) \) (normalized) for a backlash type non-linear system \( (\alpha = 0.1) \) with windowing function \( w_{\text{box}} \) and uniform window lengths of: (a) \( \tau_w = 0.5 \); (b) \( \tau_w = 1 \); and (c) \( \tau_w = 2 \). Here, \( \tilde{T}^{(1)} \) and \( \tilde{T}^{(P)} \) are the domain limits of \( \tilde{T} \), \( \tilde{t}_i \) and \( \tilde{t}_s \) are regime transition times, and \( \tilde{k}_I \) and \( \tilde{k}_II \) are the torsional stiffness values for stages I and II, respectively.

Key: (▬▬) – \( \tilde{k}(\tilde{T}) \); (—young) – \( \tilde{T}^{(1)}, \tilde{T}^{(P)}, \tilde{t}_i, \tilde{t}_s \); and (——) – \( \{\tilde{k}_I, \tilde{k}_II\} \).

These three window shapes are chosen so that comparisons can be made between uniform and non-uniform weighting, and symmetric and asymmetric windows. The effect of window length \( \tau_w \) is shown in Fig. 4.6 where \( w_{\text{box}} \) is utilized. It is clear that a smaller
value of $\tau_w$ produces a local estimation of $\tilde{k}(\tau)$; likewise, a larger value produces a global or smoothened estimation. Additionally, it is observed that the range of $\tilde{k}$ is bounded by physical stiffness elements $\bar{k}_I$ and $\bar{k}_II$. The effect of window shape is shown in Fig. 4.7 where $\tau_w = 1$. As before, $\tilde{k}(\tau)$ is bounded by $\bar{k}_I$ and $\bar{k}_II$ for all window shapes; however, each produces a unique result. Windows shapes which have a uniform weighting ($w_{box}$) give a smoothened estimation of $\tilde{k}$ compared to the windows that do not ($w_{tri}$ and $w_{saw}$). It is also observed that estimations from asymmetric windows ($w_{saw}$) exhibit a time-shift effect when compared to the symmetric windows ($w_{box}$ and $w_{tri}$).

It is important to note that $\hat{k}(t)$ is a mathematical concept (and not a physical spring element). Additionally, $\hat{k}|_{t=t'}$ is significantly different from $\langle k(\theta), t' \rangle$, which is the instantaneous expected value of $k(\theta)$ at $t = t'$ over $\tau_w$. This is simply illustrated by the comparison of $\tilde{k}(\tau)$ to $\tilde{k}_\mu(\tau)$ (a continuous time domain signal of $\langle k(\theta), t' \rangle$) for $\tau_w = 1$ and $w_{box}$, as shown in Fig. 4.8; the greatest absolute difference of $\tilde{k}_\mu(\tau)$ with respect to $\tilde{k}(\tau)$ roughly 35%.
Figure 4.7  Instantaneous effective stiffness $\tilde{k}(\tilde{t})$ (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with uniform window length $\bar{w}_u = 1$ and the following windowing functions: (a) $w_{box}$; (b) $w_{tri}$; and (c) $w_{saw}$. Here, $\bar{T}^{(1)}$ and $\bar{T}^{(p)}$ are the domain limits of $\bar{t}(\tilde{t})$, $\bar{t}_d$ and $\bar{t}_s$ are regime transition times, and $k_1$ and $k_2$ are the torsional stiffness values for stages I and II, respectively. Key: (▬▬) $\tilde{k}(\tilde{t})$; ( | ) $\{\bar{T}^{(1)}, \bar{T}^{(p)}, \bar{t}_d, \bar{t}_s\}$; and (– – –) $\{k_1, k_2\}$. 
Figure 4.8 Instantaneous effective $\tilde{k}(\bar{\tau})$ and mean $\bar{k}_\mu(\bar{\tau})$ stiffness values (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with uniform window length $\tau_w = 1$, and windowing function $w_{\text{box}}$. Here, $\bar{T}^{(i)}$ and $\bar{T}^{(p)}$ are the domain limits of $\bar{\tau}(\bar{T}), \bar{t}_{\text{di}}$ and $\bar{t}_{\text{si}}$ are regime transition times, and $\bar{k}_{\text{II}}$ is the torsional stiffness of stage II. Key: (▬▬) – $\tilde{k}(\bar{T})$; (▬▬) – $\bar{k}_\mu(\bar{T})$; ( | ) – $\{\bar{T}^{(i)}, \bar{T}^{(p)}, \bar{t}_{\text{di}}, \bar{t}_{\text{si}}\}$; and (– – –) – $\bar{k}_{\text{II}}$.

4.5 Equivalent linear system with instantaneous stiffness

At time $t = t'$ the SDOF non-linear system of Fig. 4.1 may be approximated by an undamped time-invariant linear system (conceptually illustrated in Fig. 4.9) consisting of torsional inertia $J$ and torsional stiffness $\hat{k}\mid_{t=t'}$. The corresponding natural period (linear system) is

$$\hat{\epsilon}_n\mid_{t=t'} = 2\pi \left(\frac{J}{\hat{k}\mid_{t=t'}}\right)^{0.5}. \quad (4.13)$$
A continuous time domain signal $\hat{\tau}_n(t)$ is calculated from a smoothened spline fit [4.17] of $\hat{\tau}_n\vert_{\text{est}}$; it is assumed that $\hat{\tau}(t) = \hat{\tau}_n(t)$ where $\hat{\tau}(t)$ is the estimated oscillatory period of the non-linear system. The normalized signal $\overline{\tau}(\overline{r})$ is estimated for $\overline{r}_w \in [0.1, 6]$ across all $w_{\text{shp}}$; $\overline{\tau}(\overline{r})$ for a near backlash type non-linearity ($\alpha = 0.1$) and $\overline{r}_w = 2$ is shown in Fig. 4.10. For the sake of comparison to $\overline{\tau}(\overline{r})$ (which is calculated from $\overline{\theta}(\overline{r})$), the domain of $\overline{\tau}(\overline{r})$ is limited to $\overline{r} \in [\overline{r}^{(1)}, \overline{r}^{(P)}]$. The quantitative agreements between calculated $\overline{\tau}(\overline{r})$ and estimated $\overline{\tau}(\overline{r})$ in the double-sided impact regime (di), single-sided impact regime (si), and the overall response (ov) are defined by metrics $\Pi_{\text{rr}}$ where $\text{rr} = \{\text{di}, \text{si}, \text{ov}\}$:

$$
\Pi_{\text{di}} = \exp\left[-\int_{\overline{r}^{(1)}}^{\overline{r}^{(P)}} \overline{\tau}(\overline{r}) - \overline{\tau}(\overline{r}) \, d\overline{r}\right],$$

$$
\Pi_{\text{si}} = \exp\left[-\int_{\overline{r}^{(1)}}^{\overline{r}^{(P)}} \overline{\tau}(\overline{r}) - \overline{\tau}(\overline{r}) \, d\overline{r}\right],$$

$$
\Pi_{\text{ov}} = \exp\left[-\int_{\overline{r}^{(1)}}^{\overline{r}^{(P)}} \overline{\tau}(\overline{r}) - \overline{\tau}(\overline{r}) \, d\overline{r}\right].$$

If $\Pi_{\text{rr}} = 1$, then $\overline{\tau}(\overline{r}) = \overline{\tau}(\overline{r})$ within the corresponding regime (deemed “best agreement”); otherwise, as $\Pi_{\text{rr}} \to 0$, then $|\overline{\tau}(\overline{r}) - \overline{\tau}(\overline{r})| \to \infty$ (possibly the “worst agreement”). Metrics $\Pi_{\text{rr}}$ for all $w_{\text{shp}}$ and $\overline{r}_w$ are shown in Fig. 4.11 (near backlash type non-linear, $\alpha = 0.1$) where $R[\overline{r}_{\text{rr}}]$ is the range of $\overline{\tau}(\overline{r})$ in the corresponding response regime.

It is easily noted that all window shapes exhibit similar trends; there is increasing accuracy from $\overline{r}_w = 0.1$ to 1, followed by a leveling-off for roughly $1 \leq \overline{r}_w \leq 4$, and
decreasing accuracy for $\bar{r}_w > 4$. However, the box-car window $w_i$ has a slight loss of accuracy within $1 \leq \bar{r}_w \leq 2$. The maximum values of $\Pi_{rr}$ and corresponding $\bar{r}_w$, say $\Pi_{\text{rr, best}}$ and $\bar{r}_{w, \text{rr, best}}$, are listed in Tables 4.2 and 4.3, respectively. It is obvious that no single pairing of $w_{\text{shp}}$ and $\bar{r}_w$ (within this limited set) achieves best accuracy for all response regimes and given range of $\alpha$ values, and that $\bar{r}_{w, \text{rr, best}}$ has a wide range across response regimes. Nonetheless, one important observation is made: Best $\Pi_{rr}$ is achieved when $R[\bar{r}_{rr}] \leq \bar{r}_w \leq 3R[\bar{r}_{rr}]$. However, a uniform $\bar{r}_w$ might not satisfy this condition across all response regimes. Therefore, it stands to reason that a windowing function ($w_{\text{shp, adp}}$) that utilizes an adaptive window length ($\tau_w(t')$) could improve the accuracy.

![Figure 4.9](image.png)

**Figure 4.9** Effective undamped linear time-invariant system with instantaneous effective stiffness $\hat{k}_{\text{eff,}}$ at time $t = t'$. Here, $\theta$ is the angular displacement and $J$ is the torsional inertia.
Table 4.2 Maximum values of estimation metrics $\Pi_{rr,\text{best}}$ using uniform window $w_{\text{shp}}$.

<table>
<thead>
<tr>
<th>Window shape (shp)</th>
<th>$\Pi_{di,\text{best}}$</th>
<th>$\Pi_{si,\text{best}}$</th>
<th>$\Pi_{ov,\text{best}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\alpha = 0.1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>0.91</td>
<td>0.86</td>
<td>0.77</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>0.91</td>
<td>0.89</td>
<td>0.81</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>0.92</td>
<td>0.82</td>
<td>0.71</td>
</tr>
<tr>
<td>(b) $\alpha = 10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>0.84</td>
<td>0.90</td>
<td>0.74</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>0.83</td>
<td>0.91</td>
<td>0.75</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>0.83</td>
<td>0.92</td>
<td>0.76</td>
</tr>
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</table>

Table 4.3 Normalized window length $\bar{w}_{rr,\text{best}}$ necessary for $\Pi_{rr,\text{best}}$ using uniform window $w_{\text{shp}}$.

<table>
<thead>
<tr>
<th>Window shape (shp)</th>
<th>$\bar{w}_{ri,\text{best}}$</th>
<th>$\bar{w}_{si,\text{best}}$</th>
<th>$\bar{w}_{ov,\text{best}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\alpha = 0.1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>1.3</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>2.0</td>
<td>4.1</td>
<td>2.2</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>5.4</td>
<td>1.7</td>
<td>2.9</td>
</tr>
<tr>
<td>(b) $\alpha = 10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>1.7</td>
<td>3.1</td>
<td>1.7</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>2.4</td>
<td>1.9</td>
<td>2.0</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>6.0</td>
<td>3.5</td>
<td>2.7</td>
</tr>
</tbody>
</table>
Figure 4.10 Calculated $\tau(T)$ and estimated $\tilde{\tau}(T)$ oscillatory periods (normalized) for a backlash type non-linear system ($\alpha = 0.1$) with uniform window length $T_w = 2$ and the following windowing functions: (a) $w_{\text{box}}$; (b) $w_{\text{tri}}$; and (c) $w_{\text{saw}}$. Here, $\tau^{(1)}$ and $\tau^{(p)}$ are the domain limits of $\tau(T)$, and $\bar{T}_{si}$ and $\bar{T}_{di}$ are regime transition times. Key: (▬▬) $\tau(T)$; (▬▬) $\tilde{\tau}(T)$; (|) $\{\tau^{(1)}, \tau^{(p)}, \bar{T}_{si}, \bar{T}_{di}\}$; and (––––) $\bar{T}_{\text{all}}$.

Therefore, the following new adaptive method is proposed: 1) An initial window length $T_{wo}$ is judiciously chosen; 2) Estimation of $\hat{k}_{\mid_{t=t'}}$ is first estimated at time $t' = 0.5T_{wo}$ using the uniform windowing function defined by Eq. (4.7) where $\tau_w = T_{wo}$; 3) For $t' > 0.5T_{wo}$,
\( \hat{k}_{\tau_{t'}} \) is estimated using the following adaptive windowing function where \( t' \) is the time immediately prior to \( t' \), \( \beta \) is an arbitrary window length factor \((\beta > 0)\), and \( \tau_{n_{\tau_{t'}}} \) is the natural period of the time-invariant linear system at \( t = t' \):

\[
\hat{k}_{\tau_{t'}} = \text{estimated using the following adaptive windowing function where } t' \text{ is the time immediately prior to } t', \ \beta \text{ is an arbitrary window length factor } (\beta > 0), \ \text{and } \tau_{n_{\tau_{t'}}} \text{ is the natural period of the time-invariant linear system at } t = t'.
\]

Figure 4.11 Regime-dependent metrics \( \Pi_{\tau} \) for a backlash type non-linear system \((\alpha = 0.1)\) with uniform window \( w_{\text{shp,adp}} \). Here, \( \bar{w}_w \) is the uniform window length (normalized), \( d_i \) is the double-sided impact regime, \( s_i \) is the single-sided impact regime, \( o_v \) is the entire response, and \( R[\bar{\tau}_\tau] \) is the range of \( \tau(\tau) \) (observed) in regime \( \tau \). Key: \(-\) box; \(-\) tri; \(-\) saw; and \( | \) - \( R[\bar{\tau}_\tau] \).
4) Estimated oscillatory period \( \hat{\tau}_\beta \) (non-linear system) is then calculated from \( \hat{k}_{\beta} \) and the related continuous time domain signal \( \hat{\tau}(t) \) is calculated [4.17].

For example, \( \hat{\tau}(\tau) \) is estimated using \( w_{\text{shp}_{-\text{adp}}} \) over \( 0.1 \leq \beta \leq 6 \) and the corresponding metrics \( \Pi_\beta \) are shown in Fig. 4.12 for a near backlash type non-linearity \( (\alpha = 0.1) \); here, \( \tau_{\alpha \beta} = 2\tau^{(1)} \). Accuracy increases from \( \beta = 0.1 \) to 1, then reaches its maximum within \( 1 \leq \beta \leq 4 \), and decreases for \( \beta > 4 \). The \( \beta \) values for \( \Pi_{\text{rr}_{-\text{best}}} \) (say \( \beta_{\text{rr}_{-\text{best}}} \)) are given in Table 4.4. Similar to \( \tau_{\alpha \beta_{-\text{rr}_{-\text{best}}} \beta} \) for the uniform window, \( \beta_{\text{rr}_{-\text{best}}} \) varies greatly across the response regimes; however, \( 1 \leq \beta_{\alpha \beta_{-\text{ov}_{-\text{best}}}} \leq 3 \) (roughly), which is consistent with the prior observation concerning \( \tau_{\alpha \beta} \). While \( \beta_{\alpha \beta_{-\text{ov}_{-\text{best}}}} \) is similar across \( \alpha \) for \( w_{\text{box}_{-\text{adp}}} \) and \( w_{\text{tri}_{-\text{adp}}} \), it has significant variation for \( w_{\text{saw}_{-\text{adp}}} \). A comparison of \( \Pi_{\alpha \beta_{-\text{ov}_{-\text{best}}}} \) for the uniform and adaptive windows, (\( \Pi_{\alpha \beta_{-\text{ov}_{-\text{best}}}}(w_{\text{shp}}) \) and \( \Pi_{\alpha \beta_{-\text{ov}_{-\text{best}}}}(w_{\text{shp}_{-\text{adp}}}) \), respectively) is shown in Table 4.5; \( w_{\text{shp}_{-\text{adp}}} \) improves overall accuracy for most but not all cases of \( \alpha \) and window shape. Given these limited observations, it is recommended that \( w_{\text{tri}_{-\text{adp}}} \) with \( \beta \approx 2 \) be utilized to achieve best possible accuracy in \( \hat{\tau}(t) \) estimations. Although it is not addressed here, proper selection of \( \tau_{\alpha \beta} \) is also significant. Furthermore, \( \tau_{\alpha \beta}(t') \), and
thus $\beta$, are restricted by a choice of sampling parameters and the length of the dynamic response, similar to $\iota'$ and $\tau_w$.

Figure 4.12  Regime-dependent metrics $\Pi_{\text{rr}}$ for a backlash type non-linear system ($\alpha = 0.1$) with adaptive window $w_{\text{shp_adp}}$. Here, $\beta$ is the window length factor, $d_i$ is the double-sided impact regime, $s_i$ is the single-sided impact regime, $o_v$ is the entire response, and $R[\tau_{\text{rr}}]$ is the range of $\bar{T}(\overline{t})$ (observed) in regime $\text{rr}$. Key: (▬▬) – $w_{\text{box_adp}}$; (▬▬▬) – $w_{\text{tri_adp}}$; (*****) – $w_{\text{saw_adp}}$; and (|) – $R[\tau_{\text{rr}}]$. 

109
In summary, the following observations are made regarding the window process:

1) Amplitude of $\hat{k}(t)$ is bounded by the stiffness values of the corresponding real spring for all windowing parameters; 2) Smaller window lengths produce a more temporally-localized value, unlike larger lengths which give a more global value; 3) Uniformly and non-uniformly weighted window shapes (e.g. box-car $w_{\text{box}}$ and triangular window $w_{\text{tri}}$ shapes, respectively) of the same length can produce significantly different amplitudes; and 4) Asymmetric window shapes (e.g. saw-tooth $w_{\text{saw}}$) can produce a leading or lagging effect with respect to symmetric window shapes (e.g. box-car and triangular) of similar length.

### Table 4.4 Window length coefficient $\beta_{\text{rr\_best}}$ necessary for $\Pi_{\text{rr\_best}}$ using adaptive window $w_{\text{shp\_adp}}$.

<table>
<thead>
<tr>
<th>Window shape (shp)</th>
<th>$\beta_{\text{di_best}}$</th>
<th>$\beta_{\text{si_best}}$</th>
<th>$\beta_{\text{ov_best}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\alpha = 0.1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>1.0</td>
<td>3.3</td>
<td>1.0</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>1.7</td>
<td>3.7</td>
<td>2.1</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>4.3</td>
<td>1.4</td>
<td>2.3</td>
</tr>
<tr>
<td>(b) $\alpha = 10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>1.1</td>
<td>2.4</td>
<td>1.1</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>2.8</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>4.6</td>
<td>3.5</td>
<td>3.5</td>
</tr>
</tbody>
</table>
Table 4.5 Summary of $\Pi_{ov_{best}}$ for uniform and adaptive windows, $\Pi_{ov_{best}}(w_{shp})$ and $\Pi_{ov_{best}}(w_{shp_{adp}})$, respectively.

<table>
<thead>
<tr>
<th>Window shape (shp)</th>
<th>$\Pi_{ov_{best}}(w_{shp})$</th>
<th>$\Pi_{ov_{best}}(w_{shp_{adp}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\alpha = 0.1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>0.778</td>
<td>0.779</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>0.809</td>
<td>0.810</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>0.711</td>
<td>0.714</td>
</tr>
<tr>
<td>(b) $\alpha = 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-car (box)</td>
<td>0.736</td>
<td>0.742</td>
</tr>
<tr>
<td>Triangular (tri)</td>
<td>0.750</td>
<td>0.749</td>
</tr>
<tr>
<td>Saw-tooth (saw)</td>
<td>0.755</td>
<td>0.747</td>
</tr>
</tbody>
</table>

4.6 Validation of estimation method using measurements from experiments

To validate the proposed method, two laboratory experiments (X1 and X2), which can be approximated by the SDOF non-linear model illustrated in Fig. 4.1, are considered next [4.8, 18]. Experiment X1 is a controlled benchtop setup containing a single torsional clearance [4.18], and X2 is a large-scale system containing a production vehicle clutch damper [4.8] (with four stages). The elastic function $\phi(\theta)$ for each experiment is shown in Fig. 4.13. External torque $T(t)$ is applied in a step-like manner with initial and final operating points lying on separate stages, as shown in Fig. 4.13. The primary operating stages of X1 and X2 are II and III, respectively. Thus, the experiments are sized such that $\tau_{aII} \approx 0.1s$ in X1 and $\tau_{aIII} \approx 0.1s$; furthermore, normalized time for each system is scaled by these values. The time-varying oscillatory period $\tau(t)$ is calculated.
from measurements, as described in section 4.3, and shown in Figs. 4.14 and 4.15; angular motions [4.8, 18] are omitted for the sake of brevity. For experiment X1, only the double-sided impact regime (di) with a softening trend occurs; however,

Figure 4.13 Illustration of multi-staged elastic torque curves $\Phi(\theta)$, and initial $(\theta_o, T_o)$ and final $(\theta_f, T_f)$ operating points for two experiments used to validate estimation methods: (a) experiment X1 [3.18]; and (b) experiment X2 [3.8]. Here, $\theta$ is the angular displacement, $T$ is external torque, $\Theta_j$ are angular stage transitions, and subscript {I, II,...} denote stages.
experiment X2 demonstrates all possible regimes and trends (although regime transition times are not labeled). This difference can be attributed to the following factors: a. In X1, the final operating point \((\theta_f, \Phi_f) \approx (0, 0)\), whereas in X2, \((\theta_f, \Phi_f) \approx (\Theta_f, > 0)\), which increases the likelihood that both double (di) and single-sided (si) impact regimes occur; and b. Energy dissipation has a large contribution to the response of X1 due to the presence of a backlash non-linearity \(k_i = 0\), unlike X2 for which \(k_i\) is a very soft spring.

Following the suggestions from section 4.5, \(\check{\tau}(\tau)\) signals for both X1 and X2 are estimated using \(\hat{w}_{\text{tri, adp}}\) with \(\beta = 2\); an initial guess of \(\check{\tau}_\text{wo} = 1\) is selected. Quantitative

![Figure 4.14](image)

**Figure 4.14** Calculated \(\check{\tau}(\check{T})\) and estimated \(\hat{\check{\tau}}(\check{T})\) oscillatory periods (normalized) for experiment X1 [3.18] using adaptive windowing function \(\hat{w}_{\text{tri, adp}}\) with window length factor \(\beta = 2\) and initial window length \(\check{\tau}_\text{wo} = 1\). Here, \(\check{T}^{(i)}\) and \(\check{T}^{(p)}\) are the domain limits of \(\check{\tau}(\check{T})\). Key: (---) \(-\check{\tau}(\check{T})\); (--) \(-\hat{\check{\tau}}(\check{T})\); (-----) \(\check{T}^{(i)}\) \(-\check{T}^{(p)}\); and (---) \(-\check{\tau}_\text{all} \).
Figure 4.15 Calculated $\tau(\bar{t})$ and estimated $\hat{\tau}(\bar{t})$ oscillatory periods (normalized) for experiment X2 [3.8] using adaptive windowing function $w_{\text{tri, adp}}$ with window length factor $\beta = 2$ and initial window length $\tau_{\text{inp}} = 1$. Here, $\bar{t}^{(1)}$ and $\bar{t}^{(P)}$ are the domain limits of $\tau(\bar{t})$. Key: (▬▬) – $\tau(\bar{t})$; (▬▬) – $\hat{\tau}(\bar{t})$; ( | ) – $\{\bar{t}^{(1)}, \bar{t}^{(P)}\}$; and (– – –) – $\tau_{\text{inp}}$.

Comparisons between $\tau(\bar{t})$ (observed) and $\hat{\tau}(\bar{t})$ (estimated) are shown in Figs. 4.14 and 4.15 for experiments X1 and X2, respectively. In particular, the signal $\hat{\tau}(\bar{t})$ correctly exhibits the hardening and softening nature observed in each experiment though it lacks complete agreement with $\tau(\bar{t})$. Several factors could contribute to this difference. First, the suggested windowing parameters and algorithm (developed from simplified example cases) may not be optimal for these real experiments. Furthermore, the corresponding non-linear models of these experiments may require additional degrees of freedom and the corresponding features would need a higher level of characterization. Finally, if significant
viscous damping is present with a relatively low stiffness, then the oscillatory period cannot be accurately estimated by an effective natural period. Instead, an effective damped natural period would need to be considered.

4.7 Conclusion

The main contribution of the paper is the development of a new analysis tool that estimates the trends in time-varying oscillatory periods in the step-response of a torsional system containing a multi-staged spring. Development of the method begins with the formulation of a new concept of instantaneous effective stiffness. Although it is strictly mathematical in nature (and not a physical element), its amplitude is limited by the stiffness values of the corresponding real spring. The related computations employ windowing functions and thus a range of window shapes and lengths are investigated. Next, the instantaneous effective stiffness is used to approximate the real non-linear system (at some instant during its step-response) as an undamped time-invariant linear oscillator. Here, it is assumed that the damping in the real system is relatively low such that its oscillatory period at that time can be approximated by the natural period of the linear oscillator. The method is then applied to numerical example cases and a set of windowing parameters, which include window shape and an adaptive length algorithm, are suggested from the results. Finally, the proposed method is validated by correctly estimating the asymptotic trends (such as hardening or softening) observed in the time-varying oscillatory periods of two recently proposed experiments by the authors [4.8, 17]. The method of this article should serve as an important diagnostic tool for the system identification of an unknown
device. Assuming that angular displacement and torque can be measured for a step-
response, the proposed method should provide value insight for the characterization of non-
linear features (say using a SDOF approximation), as well as amplitude dependence [4.19].
In cases of relatively high viscous damping, the method could be applied to only the first
few cycles of oscillations. Overall, the proposed method should improve the efficiency
and accuracy of the model building process for mechanical devices with clearance non-
linearities [4.2, 19].

References for Chapter 4


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L. Verdillon, S. Gairmard, Vehicle driveline dynamic behavior: Experimentation

transmissions – measurement, modelling, and simulation, *Proc. IMechE Part K:
Journal of Multi-body dynamics*, 213 (1999) 53-60 (doi:
10.1243/1464419991544054).

torsional impact of automotive drivelines, *Proc. IMechE. Part D: Journal of

automatic transmission system with multiple clearances: Formulation, simulation,


<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$e$</td>
<td>error</td>
</tr>
<tr>
<td>$h$</td>
<td>Coulomb friction amplitude</td>
</tr>
<tr>
<td>$J$</td>
<td>torsional inertia</td>
</tr>
<tr>
<td>$k$</td>
<td>torsional stiffness</td>
</tr>
<tr>
<td>$N$</td>
<td>total number of stages</td>
</tr>
<tr>
<td>$P$</td>
<td>total number of oscillatory periods</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T$</td>
<td>external torque</td>
</tr>
<tr>
<td>$w$</td>
<td>windowing function</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>stiffness ratio</td>
</tr>
<tr>
<td>$\beta$</td>
<td>window length coefficient</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>total torque transmission</td>
</tr>
<tr>
<td>$\eta$</td>
<td>regularizing factor for Coulomb friction</td>
</tr>
<tr>
<td>$\theta, \dot{\theta}, \ddot{\theta}$</td>
<td>angular displacement, velocity, and acceleration</td>
</tr>
<tr>
<td>$\phi$</td>
<td>stage transition (angular)</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>estimation metric</td>
</tr>
<tr>
<td>$\tau$</td>
<td>oscillatory period</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>elastic torque</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>ideal unit step function</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>dissipative torque</td>
</tr>
</tbody>
</table>
Subscripts

\begin{itemize}
  \item \textit{a} \quad \text{alternating coefficient}
  \item \textit{adp} \quad \text{adaptive window}
  \item \textit{best} \quad \text{“best”}
  \item \textit{box} \quad \text{box-car}
  \item \textit{di} \quad \text{double-sided impact}
  \item \textit{f} \quad \text{final point}
  \item \textit{I,II,...} \quad \text{stage index}
  \item \textit{m} \quad \text{mean coefficient}
  \item \textit{n} \quad \text{natural}
  \item \textit{ni} \quad \text{no-impact}
  \item \textit{o} \quad \text{initial point}
  \item \textit{ov} \quad \text{overall response}
  \item \textit{t} \quad \text{instantaneous (time dependent)}
  \item \textit{saw} \quad \text{saw tooth}
  \item \textit{shp} \quad \text{shape}
  \item \textit{si} \quad \text{single-sided impact}
  \item \textit{tri} \quad \text{triangular}
  \item \textit{w} \quad \text{windowing}
\end{itemize}

Superscripts

\begin{itemize}
  \item \textit{(i)} \quad \text{oscillatory period number} (i=1,2,...P)
\end{itemize}
\(^\wedge\) approximated

Operators

\(\langle \rangle_i\) instantaneous expected value

Abbreviations

SDOF single degree of freedom

DSP digital signal processing
Chapter 5: Correlation between quasi-static and dynamic experiments for a practical torsional device with multiple discontinuous non-linearities

5.1 Introduction

There is an extensive body of literature on the estimation of system parameters [5.1-9]; see Ref. [5.1] for a thorough review of prior work as well as a list of over 300 papers. For the purpose of tractability, many researchers assume that non-linear features or functions are well characterized (or known) before attempting parameter extraction [5.1]. Alternatively, the “black box” modeling method is used when the underlying physics of a device is truly unknown though ambiguous results may be found [5.1]. Overall, many of the estimation methods rely on steady-state excitation and assume smooth (differentiable) and/or weak non-linearities [5.1]. Prior work [5.2-8] has often utilized measurements from well-controlled, laboratory or scientific experiments. These experiments [5.2-8] are intentionally designed to isolate a singular feature of interest, say a spring or dissipative element, and accommodate the necessary instrumentation. Although such methods and experiments are useful for understanding particular non-linear features, their extension to practical devices [5.9-21] is severely limited by a lack of controllability found in many applications (say due to variability in manufacturing, assembly, or operating environment). This problem is particularly acute when practical components contain multiple (and yet unknown) discontinuous non-linearities.
Most real-life components, including non-linear devices used for vibration isolation [5.10], are subjected to a wide range of mean operating points and dynamic excitation due to intentional product functions. Even if these devices could be disassembled into separate non-linear features for individual study, in situ interaction between the built-in features would be lost, and the estimation process would become incredibly complex. It is thus desirable to estimate parameters at the component-level. Nevertheless, this approach has its own unique challenges, such as laboratory space for large-scale experiments, selection of actuators than can provide in situ loading, and location and selection of instrumentation. Accordingly, the primary goal of this article is to propose a time domain method for estimating stiffness and damping properties of a non-linear torsional isolation device (vehicle clutch damper) that is illustrated in Fig. 5.1a (and further described in section 5.2). It has multiple discontinuous non-linearities (as shown in Fig. 5.1b) and is often subject to dynamic, transient loading over a large range of torques and angular displacements. The methods proposed in this article should benefit noise and vibration engineers and developers of non-linear simulation codes while contributing to the scientific literature on non-smooth dynamic systems.

5.2 Example case and literature review

A typical vehicle clutch damper [5.11-21] primarily serves to transmit a mean torque while attenuating torque pulsations between an engine and transmission [5.11]. It is intentionally designed to contain multi-staged torsional springs, pre-load features, clearances, and multi-staged dry friction elements [5.11-21]. Furthermore, clutch
Figure 5.1 Illustration of a multi-staged vehicle clutch damper: (a) schematic with parts labeled; (b) photograph of a typical production component; and (c) measured (normalized) quasi-static performact curve (arrows indicate path direction). Here, $\bar{\tau}$ is the normalized torque transmitted through the device, $\bar{\theta}$ is the normalized relative angular displacement ($\theta = \theta_{o-hub} - \theta_{i-hub}$), and $\bar{\dot{\theta}}$ is the normalized relative angular velocity. Key for (a): A – flywheel and pressure plate interface; B – rivet; C – multi-staged coil spring; D – outer hub (angular displacement $\theta_{o-hub}$); E – inner hub (angular displacement $\theta_{i-hub}$); and F – multi-staged Coulomb friction element.
dampers are often asymmetric; the drive side (positive relative angular displacement) of the device differs from its coast side (negative relative angular displacement) [5.11-12]. The typical torque range usually spans from near zero (under idling conditions) to as high as 1000-2000 Nm (under driving conditions); to accommodate this, the ratio of torsional stiffness between adjacent stages may be as high as 100 [5.20-21]. It is common in high-load ground vehicles for two devices to be installed in a parallel configuration [5.21]; this introduces further complexity, such as indexing error and non-identical features. In practice, a clutch damper can be critical to controlling vibro-impact and impulsive loading phenomena in transmissions [5.14-15, 19-20] and resonance growth during engine start-up [5.16]. Design and dynamic analysis of such a device could be facilitated by suitable non-linear simulation tools.

A literature review of clutch dampers [5.11-21] shows that characterization is usually conducted at the component-level on a commercial test rig [5.22-23] under quasi-static loading conditions only. An example performance curve of a production clutch camper is shown in Fig. 5.1b where \( \Gamma \) is the torque transmission and \( \theta \) is the relative angular displacement. It is described by a multi-staged torsional spring in parallel with a multi-staged Coulomb friction element; elastic and dissipative parameters are then estimated from physical domain representations of torque transmission. Prior analysts [5.12, 14-21] have assumed these static parameters when simulating dynamic responses of both the device itself and powertrain systems. Further, many models [5.12, 14-16, 21] include a parallel linear torsional viscous damper and assume damping values (or from a modal experiment) to improve agreement with measurement. This damping scheme relies
on frequency domain methods and assumes that the operating range of the device lies within a single stage. Thus, it is questionable to apply it to a transient non-linear response in which the mean operating point varies significantly, spanning multiple stages rapidly (e.g. vehicle clunk phenomenon [5.14, 19-20, 24]). In addition, this parameter is sometimes extracted from a system-level measurement, such as provided by Biermann et al. [5.19] and Menday et al. [5.20]. While such experiments are valuable to understanding the role of clutch dampers within the context of a powertrain, they do not provide any physical insight at the component-level. To overcome this hurdle, Krak et al. [5.21] proposed a step-response type experiment which is a simplified version of prior experiments [5.19-20, 24]. Overall, there is clearly a void in the literature concerning the parameter estimation for vehicle clutch dampers (and similar devices) under dynamic, transient loading. The method proposed in this article will attempt to address this particular need; thus, both quasi-static and dynamic experiments are utilized and compared. Issues related to the mean shaft speed (such as the drag torque) are beyond the scope of this article.

5.3 Analytical formulation of problem

Consider a torsional device that contains multiple discontinuous non-linear features, including multi-staged stiffness and dissipative elements. For practical reasons, manufacturers characterize such devices under quasi-static loading conditions only (relatively low angular velocity, denoted by subscript $Q$ and conceptually illustrated in Fig. 5.2a). The governing equation of the device is the following where $\theta$ is the relative
angular displacement, \( T_Q(t) \) is external torque, \( J_\varepsilon \) is the associated torsional inertia \( (J_\varepsilon \approx 0) \), and \( \Gamma_Q(\theta, \dot{\theta}) \) is the torque transmitted through the device

\[
J_\varepsilon \ddot{\theta} + \Gamma_Q(\theta, \dot{\theta}) = T_Q(t).
\]  

(5.1)

See the end of this chapter for a complete list of symbols.

Figure 5.2 Conceptual illustration of torque transmission through a symmetric, two-stage vehicle clutch damper under quasi-static loading conditions. Here, \( \Gamma_Q(\theta, \dot{\theta}) \) is transmitted torque, \( \theta \) is angular displacement, and \( \dot{\theta} \) is angular velocity. Stages are denoted by subscripts I and II, torsional stiffness is denoted by \( k \), and angular stage transitions are denoted by \( \Theta \). Key: (----) – transmitted torque \( \Gamma_Q(\theta, \dot{\theta}) \); (-----) – elastic torque component \( \Phi_Q(\theta) \); and (---) – angular stage transition \( \Theta \).
The torque transmitted through the device can be described by the following function

\[ \Gamma_Q(\theta, \dot{\theta}) = \Psi_Q(\theta, \dot{\theta}) + \Phi_Q(\theta). \]  

(5.2)

For the sake of illustration, a symmetric, dual-staged example case is illustrated in Fig. 5.3; stages are denoted by subscript Roman numerals I and II, torsional stiffness is denoted by \( k \), and angular stage transitions are denoted by \( \Theta \). The ratio of \( k_1 \) to \( k_{II} \) is critical for describing \( \Phi_Q(\theta) \); for instance, consider the following cases: a backlash is defined by \( k_1 \to 0 \) (or \( k_1 << k_{II} \)); a pre-load feature requires \( k_1 \to \infty \) (or \( k_1 >> k_{II} \)); a stopper element is given by \( k_{II} \to \infty \) (or \( k_{II} >> k_1 \)); and the linear spring element only occurs when \( k_1 = k_{II} \).

The path-dependent nature of \( \Gamma_Q(\theta, \dot{\theta}) \) is due to \( \Psi_Q(\theta, \dot{\theta}) \), which is assumed to be described by Coulomb friction elements only.

Such devices usually operate under dynamic and high speed loading conditions (relatively high angular velocity, denoted by subscript \( D \) conceptually illustrated in Fig. 5.2b). Torque transmission may now be described by the function

\[ \Gamma_D(\theta, \dot{\theta}) = \Psi_D(\theta, \dot{\theta}) + \Phi_D(\theta, \dot{\theta}). \]  

(5.3)

Unlike the quasi-static conditions, it is assumed that \( \Phi_D(\theta, \dot{\theta}) \) may depend on both \( \theta \) and \( \dot{\theta} \), and \( \Psi_D(\theta, \dot{\theta}) \) may be described by viscous damping and/or Coulomb friction elements.

The governing equation of the device is described below where \( T_D(t) \) is external torque and \( J \) is the associated torsional inertia \((J \neq 0)\):
Figure 5.3 Quasi-static and dynamic experiments for a vehicle clutch damper: (a) conceptual illustration for quasi-static loading (denoted X-Q); (b) conceptual and (c) physical illustrations for dynamic loading (denoted X-D). Here, $\theta$ is angular displacement, $\dot{\theta}$ is angular velocity, $\Phi$ is the elastic torque transmission, $\Psi$ is the dissipative torque transmission, $\Gamma$ is the total torque transmission ($\Gamma = \Psi + \Phi$), $T$ is external torque, and $J$ is torsional inertia. Key for (c): A – flywheel (ground); B – clutch assembly (houses one clutch damper); C – shaft; D – bearing; E – torsion arm; F – pneumatic actuators; G – structural supports (ground); and H – bed plate (ground).
\[ J \ddot{\theta} + \Gamma_D(\theta, \dot{\theta}) = T_D(t). \]  

(5.4)

As stated in section 5.2, it is often assumed that \( \Gamma_D(\theta, \dot{\theta}) \approx \Gamma_Q(\theta, \dot{\theta}) \) when predicting dynamic responses. Therefore, the chief objective of this article is to essentially address the following question: How does \( \Gamma_D(\theta, \dot{\theta}) \) relate to \( \Gamma_Q(\theta, \dot{\theta}) \)? More specific questions (as sub-objectives) are posed as follows: a. Does \( \Phi_D(\theta, \dot{\theta}) \approx \Phi_Q(\theta) \) per assumptions made in prior work [5.12-21]? b. Does \( \Psi_Q(\theta, \dot{\theta}) \) fully account for dissipation under dynamic loading? c. What dissipative mechanism (viscous damping or Coulomb friction) best describes \( \Psi_D(\theta, \dot{\theta}) \)? d. Given a dynamic response in which the mean operating point rapidly crosses multiple stages, can \( \Psi_D(\theta, \dot{\theta}) \) be described in a global manner (as assumed in prior work [5.12, 14-16, 21]), or must it be described locally (i.e. stage dependent)? Answers to these questions require experimental and analytical investigation of a multi-staged device, but the scope of analysis is limited to time domain and single degree of freedom non-linear models. For the sake of simplification, all multi-staged features shall be piecewise linear. Time domain methods of characterizing elastic and dissipative parameters under step responses form the basis of this article.

5.4 Characterization under quasi-static loading

Characterization under quasi-static loading is typically carried out using a commercial test machine [5.22-23] (denoted X-Q), as conceptually illustrated in Fig. 5.2a and described by Eq. (5.1-2). The device is fixed to ground at one end and a low-rate,
ramp-like external torque $T_{\varphi}(t)$ is slowly applied at the other end. The test machine is instrumented to measure $\theta(t)$ and $\Gamma_{\varphi}(t)$ only; therefore, $\theta(t)$ is considered to be the (motion) excitation source, though it is assumed from Eq. (5.1) that $T_{\varphi}(t) \approx \Gamma_{\varphi}(t)$.

Angular velocity $\dot{\theta}(t)$ is estimated in this paper by applying a forward differencing method to $\theta(t)$, then smoothened using a moving average method [5.25]. For example, consider a four-staged asymmetric clutch damper (single-disk configuration) with a shaft spline clearance (denoted by subscript 0), drive side ($\theta > 0$, denoted by subscript +) and coast side ($\theta < 0$, denoted by subscript -). Typical (normalized) measurements are given in Fig. 5.4; here, time is normalized by $\tau_s \approx 0.1 \text{s}$ (approximately the first natural period of a vehicle powertrain torsional surge mode [5.24]), angular displacement is normalized by $\theta_s = \max(|\Theta_j|)$, and torque is normalized by $\Gamma_s = \max(|\Gamma_{\varphi}(\theta, \dot{\theta})|)$. Measured $\Gamma_{\varphi}(\theta, \dot{\theta})$ clearly exhibits path dependence; path P1 begins at $\theta = 0$ and ends at $\theta = \Theta_{IV}$ with $\dot{\theta} > 0$; path P2 begins at $\theta = \Theta_{IV}$ and ends at $\theta = \Theta_{I}$ with $\dot{\theta} < 0$; and path P3 begins at $\theta = \Theta_{IV}$ and ends at $\theta = 0$ with $\dot{\theta} > 0$. Paths P1 and P3 are referred to as the upper paths and P2 as the lower path.

It is assumed that $\Phi_{\varphi}(\theta)$ is piecewise linear in nature and can be described by the following equation where $\Phi_{\varphi_{\pm}}$ is the median point between the upper and lower paths at $\theta = \Theta_{j_{\pm}}$, $k_{j_{\pm}}$ is the torsional stiffness of each stage, $\Xi(\theta)$ is the unit step function, and $N$ is the total number of stages:
\[
\Phi_Q(\theta) = \sum_{j=1}^{N} \left\{ \Phi_Q(\theta_{(j-1)+}) + k_j (\theta - \theta_{(j-1)+}) \right\} \left[ \Xi (\theta - \theta_{(j-1)+}) - \Xi (\theta - \theta_{+}) \right] + \ldots
\]

\[
k_j = \frac{\Phi_Q(\theta_+ - \Phi_Q(\theta_{(j-1)+})}{\theta_{+} - \theta_{(j-1)+}}, \quad k_j = \frac{\Phi_Q(\theta_{(j-1)+}) - \Phi_Q(\theta_{+})}{\theta_{(j-1)+} - \theta_{+}} \quad \text{for} \quad j = \text{I, II...N}.
\]
Since stage \( j = 0 \) is a clearance, \( \Phi_{\theta}(\theta) = 0 \) for \( \theta \in [\Theta_0, \Theta_{0^+}] \) and \( k_{0^\pm} = 0 \). Dissipative torque \( \Psi_{\theta}(\theta, \dot{\theta}) \) is described by a multi-staged Coulomb friction element of the following form where \( h_{\theta}(\theta) \) is the friction amplitude and \( \eta \) is a regularizing factor [5.16]

\[
\Psi_{\theta}(\theta, \dot{\theta}) = h_{\theta}(\theta) \tanh\left(\eta \dot{\theta}\right). \tag{5.7}
\]

The friction amplitude \( h_{\theta}(\theta) \) is assumed to be piecewise linear and described by the following where \( h_{\theta_{j^\pm}} \) is one half of the difference between the upper and lower paths at \( \theta = \Theta_{j^\pm} \) and \( \Delta h_{\theta_{j^\pm}} \) is the change in friction amplitude within stage \( j^\pm \)

\[
h_{\theta}(\theta) = \sum_{j=1}^{N} \left[ h_{\theta_{j}} + \Delta h_{\theta_{j}} \left( \theta - \Theta_{j-1}^j \right) \right] \sum_{j=1}^{N} \left[ \Xi\left( \theta - \Theta_{j-1}^j \right) - \Xi\left( \theta - \Theta_{j}^j \right) \right] \], \tag{5.8}
\]

\[
\Delta h_{\theta_{j}} = \frac{h_{\theta_{j}} - h_{\theta_{j-1}^j}}{\Theta_{j}^j - \Theta_{j-1}^j}, \Delta h_{\theta_{j}} = \frac{h_{\theta_{j}} - h_{\theta_{j-1}^j}}{\Theta_{j}^j - \Theta_{j-1}^j} \text{ for } j = 1, \ldots, N. \tag{5.9}
\]

Similar to \( \Phi_{\theta}(\theta) \), \( \Psi_{\theta}(\theta, \dot{\theta}) = 0 \) for \( \theta \in [\Theta_0, \Theta_{0^+}] \) and \( h_{0^\pm} = 0 \). For illustrative purposes, physical domain representations of \( \Phi_{\theta}(\theta) \) and \( \Psi_{\theta}(\theta, \dot{\theta}) \) are given in Fig. 5.5. To validate the static characterization, measured and predicted \( \Gamma_{\theta}(t) \) and \( \dot{\Gamma}_{\theta}(t) \) are compared in Fig. 5.6; note that \( \dot{\Gamma}_{\theta}(t) \) is estimated here through numerical differentiation. There is close agreement between the measured and predicted signals as indicated by the mean and maximum (absolute) deviations which are: 0.008 and 0.05, respectively, for \( \Gamma_{\theta}(\overline{t}) \); 0.005 and 0.05, respectively, for \( \dot{\Gamma}_{\theta}(\overline{t}) \).
Figure 5.5 Predicted elastic and dissipative torques in quasi-static experiment X-Q. Here, $\Phi_\theta$ is the normalized elastic torque and $\Psi_\theta$ is the normalized dissipative torque. Key: 

- path P1; (---) – path P2; (••••) – path P3; (−−−−) – angular stage transitions $\Theta_j$; \{0, I,...IV\} 

- stage indices; (+) – drive side; and (-) – coast side.

Figure 5.6 Predicted and measured (normalized) torque transmission $\Gamma_\theta$ for quasi-static experiment X-Q. Here, $\dot{\Gamma}_\theta$ is the time derivative of $\Gamma_\theta$. Key: (---) – measurement; (---) – prediction; (−−−−) – angular stage transitions $\Theta_j$; \{0, I,...IV\} – stage indices; (+) – drive side; and (-) – coast side.
5.5 Characterization under dynamic loading

For characterization under dynamic loading, consider the laboratory experiment proposed recently by Krak et al. [5.21] (denoted X-D), which is conceptually and physically illustrated in Fig. 5.2b-c and described by Eqs. (5.3-4). Here, the device is fixed to ground at one end and has some associated torsional inertia $J$ attached at the other end. A step-like torque $T_p(t)$ is applied to $J$ such that the initial and final operating points of the device ($\theta_o, \Gamma_{Do}$) and ($\theta_f, \Gamma_{Df}$) lie on separate stages. The experiment is instrumented to measure the translational velocity of a point on $J$ only; then $\theta(t)$, $\dot{\theta}(t)$, and $\ddot{\theta}(t)$ are estimated using system geometry and numerical methods. See Ref. [5.21] for further details regarding the experiment design, instrumentation, and signal processing. For example, again consider the four-staged asymmetric clutch damper as introduced in section 5.1. Measured responses are given in Fig. 5.7; the normalization follows the same scheme given in section 5.4. The responses exhibit richly non-linear behavior with three distinct response regimes: double-sided impact (di), single-sided impact (si), and no-impact (ni) [5.21]. These regimes are characterized by significant peak values of $\ddot{\theta}(t)$ which occur at abrupt changes in torsional stiffness, such is found at $\theta = \Theta_{1 \pm}$. Point ($\theta_o, \Gamma_{Do}$) lies on stage III+ and point ($\theta_f, \Gamma_{Df}$) lies on stage II+; the path between these points crosses several stages (III- to III+). Note that $\Theta_{0 \pm}$ differs between X-D and X-Q experiments due to a variation in assembly.
Figure 5.7 Typical measured motions from dynamic experiment X-D. Here, $\bar{\theta}$ is the normalized angular displacement, $\bar{\dot{\theta}}$ is the normalized angular velocity, and $\bar{\ddot{\theta}}$ the is normalized angular acceleration. Key: (----) – measured motion; (---·---) – angular stage transitions $\Theta_{j\alpha}$; (|) – response regime transition; (di, si, ni) – double-sided, single-sided, and no-impact regimes; \{0, I, II\} – stage indices; (+) – drive side; and (-) – coast side.
5.6 Estimation of elastic parameters

It is problematic to directly estimate $\Gamma_D(\theta, \dot{\theta})$ by substituting measured (estimated) signals $\theta(t), \dot{\theta}(t),$ and $\ddot{\theta}(t)$ into Eq. (5.4) due to the impulsive and often noisy nature of the measurements. Furthermore, it is difficult to decompose $\Gamma_D(\theta, \dot{\theta})$ into its elastic and dissipative components without a priori knowledge of $\Psi_D(\theta, \dot{\theta})$ and $\Phi_D(\theta, \dot{\theta})$. Therefore, alternative estimation methods must be pursued. For instance, Wallaschek [5.9] and Rook et al. [5.26] proposed a stochastic linearization method for estimating effective stiffness parameters of a discontinuous feature under harmonic excitation (stationary process). A similar method is employed here to estimate $\Phi_D(\theta, \dot{\theta})$ though it must be modified to accommodate a non-stationary process, evident from the measured response of X-D. Assume that over a small time window of period $\tau_g$ centered at time $t$ the following approximation can be made $\Gamma_D(t) \approx \hat{\Gamma}_D(t)$. Here, $\hat{\Gamma}_D(t)$ is the time history of a linear function as defined below where $\hat{\gamma}_d(t)$ is the time history of a mean coefficient, $\hat{\gamma}_u(t)$ is the time history of an alternating coefficient, and $\langle \theta, t \rangle$ is an instantaneous expected value operator (windowed time average):

$$\hat{\Gamma}_D(t) = \hat{\gamma}_d(t) \langle \theta, t \rangle + \hat{\gamma}_u(t) \left[ \theta(t) - \langle \theta, t \rangle \right].$$  

The instantaneous expected value operator $\langle \theta, t \rangle$ is defined by the following where $g(t' - t)$ is a sliding rectangular window of period $\tau_g$, $t'$ is a dummy time variable, $t_o$ is the initial time of the measured response, and $t_f$ is the final time of the measured response.
\[
\langle \theta, t \rangle_i = \frac{\int_{t_i}^{t_f} \theta(t') g(t' - t) dt'}{\int_{t_i}^{t_f} g(t' - t) dt'},
\]
(5.11)
\[
g(t' - t) = \Xi(t' - t + 0.5\tau_g) - \Xi(t' - t - 0.5\tau_g).
\]
(5.12)

The time histories \(\hat{\gamma}_{DM}(t)\) and \(\hat{\gamma}_{DA}(t)\) are treated as time-invariant at time \(t\) and thus the following property is assumed:

\[
\langle \hat{\gamma}_{DM}(t), t \rangle_i = \hat{\gamma}_{DM}(t), \quad \langle \hat{\gamma}_{DA}(t), t \rangle_i = \hat{\gamma}_{DA}(t).
\]
(5.13)

Next, \(\hat{\Gamma}_D(t)\) is substituted into Eq. (5.4) to define the following error signal \(e(t)\) where

\[
A(t) = T_D(t) - J\hat{\theta}(t)
\]
\[
e(t) = A(t) - \hat{\Gamma}_D(t).
\]
(5.14)

Then, \(\langle e^2(t), t \rangle_i\) is minimized with respect to \(\hat{\gamma}_{DM}(t)\) and \(\hat{\gamma}_{DA}(t)\) at every time \(t\) as follows

\[
\frac{\partial \langle e^2(t), t \rangle_i}{\partial \hat{\gamma}_{DM}(t)} = 0, \quad \frac{\partial \langle e^2(t), t \rangle_i}{\partial \hat{\gamma}_{DA}(t)} = 0,
\]
(5.15)
\[
\hat{\gamma}_{DM}(t) = \frac{\langle A, t \rangle_i \langle \theta, t \rangle_i}{\langle \theta, t \rangle_i^2},
\]
(5.16)
\[
\hat{\gamma}_{DA}(t) = \frac{\langle A\theta, t \rangle_i - \langle A, t \rangle_i \langle \theta, t \rangle_i}{\langle \theta^2, t \rangle_i - \langle \theta, t \rangle_i^2}.
\]
(5.17)

Calculation of \(\hat{\Gamma}_D(t)\) is highly dependent on the duration of \(\tau_g\); the approximation becomes local as \(\tau_g \to 0\) and global as \(\tau_g \to \infty\). To demonstrate this, \(\hat{\Gamma}_D(t)\) is calculated for \(\tau_g = 0.05\tau_s\) and \(\tau_g = 0.4\tau_s\), and displayed in Fig. 5.8; it is obvious that \(\hat{\Gamma}_D(t)\) is
smoothened as $\tau_g$ increases. Therefore, for this analysis, $\tau_g = 0.05\tau_s$ is selected while recognizing that the duration of $\tau_g$ must lie between the sampling period (roughly $0.007\tau_s$) and total length of the measured response (about $20\tau_s$).

For the sake of comparison, $\Gamma_\theta(t)$ is calculated using the measured $\theta(t)$ and $\dot{\theta}(t)$ from X-D; the time domain representation is shown in Fig. 5.9. Observe a close agreement between the two as the mean and maximum absolute differences between

![Graph showing the effect of window length on the estimated dynamic torque transmission $\hat{\Gamma}_D(\tau)$ for dynamic experiment X-D in time domain. Key: (---) $\tau_g = 0.05\tau_s$; and (—) $\tau_g = 0.4\tau_s$.](image)

**Figure. 5.8** Effect of window length on the estimated dynamic torque transmission $\hat{\Gamma}_D(\tau)$ for dynamic experiment X-D in time domain. Key: (---) $\tau_g = 0.05\tau_s$; and (—) $\tau_g = 0.4\tau_s$. 
and $\hat{\Gamma}_D(\bar{t})$ are 0.02 and 0.10 respectively. The physical domain representation over the first oscillatory period $\tau^{(1)}$ is displayed in Fig. 5.10; $\hat{\Gamma}_D(\theta, \dot{\theta})$ exhibits both stage and path dependence similar to $\Gamma_q(\theta, \dot{\theta})$. However, the stage transitions are heavily smoothened during high angular velocity. This is attributed to: i) a limited sampling period, which is inherent in any measured signal; ii) the finite window length used by the approximation; and iii) the fact the linearization approximates the sum of elastic and dissipative torques. Despite this, it is possible to extract some physical insight about $\Phi_D(\theta, \dot{\theta})$.

![Graph showing comparison of torques](image)

**Figure 5.9** Comparison of quasi-static $\Gamma_q(\bar{t})$ and estimated dynamic $\hat{\Gamma}_D(\bar{t})$ torques for dynamic experiment X-D in time domain. Key: (-----) $\Gamma_q(t)$; (----) $\hat{\Gamma}_D(t)$; and (□) limits of period $\tau^{(1)}$. 

139
Figure 5.10  Comparison of quasi-static $\Gamma_q(\bar{\theta}, \dot{\bar{\theta}})$ and estimated dynamic $\tilde{\Gamma}_D(\bar{\theta}, \dot{\bar{\theta}})$ torque for experiment X-D in physical domain over the oscillatory period $\tau^{(1)}$. Key: (▬▬) $\Gamma_q(\theta, \dot{\theta} > 0)$; (▬▬) $\Gamma_q(\theta, \dot{\theta} < 0)$; (▬▬) $\tilde{\Gamma}_D(\theta, \dot{\theta} > 0)$; (▬▬) $\tilde{\Gamma}_D(\theta, \dot{\theta} < 0)$; (−∙−) angular stage transitions $\Theta_{j\pm}$; {0, I, II} – stage indices; (+) – drive side; and (-) – coast side.

As stated in section 5.3, it is assumed that all dissipation ($\Psi_q(\theta, \dot{\theta})$ and $\Psi_D(\theta, \dot{\theta})$) can be described by viscous damping and/or Coulomb friction, and therefore $\Gamma_q(\theta, 0) = \Phi_q(\theta)$ and $\Gamma_D(\theta, 0) = \Phi_D(\theta, 0)$. Accordingly, $\Phi_D(\theta, 0) \approx \tilde{\Gamma}_D(\theta, 0)$, which allows $\Phi_D(\theta, \dot{\theta})$ to be compared to $\Phi_q(\theta)$ at $\dot{\theta} = 0$, as shown in Fig. 5.11. The first observation is that $\Phi_D(\theta, 0)$ and $\Phi_q(\theta)$ exhibit stage transitions and amplitudes. The mean and maximum absolute differences between $\Phi_D(\bar{\theta}, 0)$ and $\Phi_q(\bar{\theta})$ are roughly 0.02.
and 0.05 respectively. Second, $\Phi_D(\theta,0)$ is not strictly piecewise linear, which is most clearly seen at stage II+ (pre-load feature); nevertheless, the absolute difference between $\Phi_D(\theta)$ and $\Phi_D(\theta,0)$ in stage II+ is at most 0.03. The close agreement between $\Phi_D(\theta)$ and $\Phi_D(\theta,0)$ supports the assumption made in prior work [5.12-21] that $\Phi_D(\theta,\dot{\theta}) \approx \Phi_D(\theta)$; this will be further verified by numerical simulation presented in section 5.8.

**Figure 5.11** Comparison of quasi-static elastic $\Phi_D(\theta)$ and dynamic $\Phi_D(\theta,0)$ elastic torques for experiment X-D in physical domain at $\dot{\theta} = 0$. Key: \[\text{(\textcolor{red}{solid line})} - \Phi_D(\theta); \text{(\textcolor{green}{dashed line})} - \Phi_D(\theta,0); \text{(\textcolor{blue}{dotted line})} - \text{angular stage transitions } \Theta_{\pm}; \{0, I, II\} - \text{stage indices; (+) - drive side; and (-) - coast side.}\]
5.7 Estimation of dissipative parameters

Seven dissipation formulations, which are conceptually illustrated in Fig. 5.12 and summarized in Table 5.1, are considered for $\Psi_D(\theta, \dot{\theta})$. It is assumed that one or more of the proposed formulations can effectively describe all dissipative elements within the device. Formulation D0 simply assumes that $\Psi_D(\theta, \dot{\theta}) = \Psi_D(\theta, \dot{\theta})$, which is consistent with literature [5.12-21]. Formulations D1.1-2 assume global damping, where D1.1 is a single-staged torsional viscous damper $c_{D1}$, and D1.2 is a single-staged Coulomb element $h_{D1}$. To facilitate the estimation of parameters, the dissipated energy $E_{D1}$ is defined in terms of both the energy balance principle and dissipation formulations, as given by the following expressions where $t_o$ and $t_f$ are the initial and final times of the measured response (X-D), $W$ is the external work, $\Delta U$ is the change in potential energy, and $\Delta V$ is the change in kinetic energy:

$$E_{D1} = W - \Delta U - \Delta V,$$  \hspace{1cm} (5.18)

$$E_{D1} = \int_{t_o}^{t_f} c_{D1} \dot{\theta}^2 dt, \quad E_{D1} = \int_{t_o}^{t_f} h_{D1} \tanh(\eta \dot{\theta}) \dot{\theta} dt,$$  \hspace{1cm} (5.19)

$$W = \int_{t_o}^{t_f} T_D(t) \dot{\theta} dt,$$  \hspace{1cm} (5.20)

$$\Delta U = \int_{t_o}^{t_f} \Phi_D(\theta, \dot{\theta}) \dot{\theta} dt,$$  \hspace{1cm} (5.21)

$$\Delta V = 0.5 J \left[ \dot{\theta}^2(t_f) - \dot{\theta}^2(t_o) \right].$$  \hspace{1cm} (5.22)

In Eq. (19), dissipated energy $E_{D1}$ is defined by a time-integral of the dissipative power associated with each formulation (D1.1 and D1.2). This is equivalent to the path-
integral of the dissipative torque; however, the time-integral is chosen for the sake of convenience (here and for other dissipative formulations). Next, dissipative parameters $c_{D_1}$ and $h_{D_1}$ are defined by the following:

\[
c_{D_1} = E_{D_1} \left[ \int_{t_0}^{t_f} \dot{\theta}^2 \, dt \right]^{-1}, \quad (5.23)
\]

\[
h_{D_1} = E_{D_1} \left[ \int_{t_0}^{t_f} \dot{\theta} \tanh(\eta \dot{\theta}) \, dt \right]^{-1}. \quad (5.24)
\]

### Table 5.1
Summary of proposed formulations for dissipative torque $\Psi_D(\theta, \dot{\theta})$ corresponding to Fig. 5.11.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Psi_D(\theta, \dot{\theta})$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>D0</td>
<td>$h_Q(\theta) \tanh(\eta \dot{\theta})$</td>
<td>Local (stage dependent) damping</td>
</tr>
<tr>
<td>D1.1</td>
<td>$c_{D_1} \dot{\theta}$</td>
<td>Global damping</td>
</tr>
<tr>
<td>D1.2</td>
<td>$h_{D_1} \tanh(\eta \dot{\theta})$</td>
<td></td>
</tr>
<tr>
<td>D2.1</td>
<td>$h_Q(\theta) \tanh(\eta \dot{\theta}) + c_{D_2} \dot{\theta}$</td>
<td>Global and local (stage dependent) damping</td>
</tr>
<tr>
<td>D2.2</td>
<td>$[h_Q(\theta) + h_{D_2}] \tanh(\eta \dot{\theta})$</td>
<td></td>
</tr>
<tr>
<td>D3.1</td>
<td>$c_{D_3}(\theta) \dot{\theta}$</td>
<td>Local (stage dependent) damping</td>
</tr>
<tr>
<td>D3.2</td>
<td>$h_{D_3}(\theta) \tanh(\eta \dot{\theta})$</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.12 Conceptual illustration of formulations D0-D3 for the dissipative torque \( \Psi_d(\theta, \dot{\theta}) \).

Here, \( \theta \) is the angular displacement, \( c \) denotes torsional viscous damping, and \( h \) denotes Coulomb friction amplitude.
In contrast to D1.1-2, formulations D2.1-2 assume that $\Psi_\theta(\theta, \dot{\theta})$ is valid under dynamic loading, though insufficient to completely describe the dissipation. It is assumed here that the residual dissipated energy can be attributed to a global damping element, such as a single-staged torsional viscous damper $c_{d_2}$ for formulation D2.1 (consistent with assumptions from prior work [5.12, 14-16, 21]) or a single-staged Coulomb element $h_{d_2}$ for formulation D2.2. Thus, D2.1-2 is considered a combination of local and global damping elements. To estimate $c_{d_2}$ and $h_{d_2}$, an energy balance is again applied to the measured responses of X-D as defined below where $W$, $\Delta U$, and $\Delta V$ are given by Eqs. (5.20-22) and $E_{D_2}$ is the dissipated energy:

$$E_{D_2} = W - \Delta U - \Delta V,$$

$$E_{D_2} = \int_{t_c}^{t_f} \left[ c_{d_2} \dot{\theta} + \Psi_\theta(\theta, \dot{\theta}) \right] \dot{\theta} dt, \quad E_{D_2} = \int_{t_c}^{t_f} \left[ h_{d_2} \tanh(\eta \dot{\theta}) + \Psi_\theta(\theta, \dot{\theta}) \right] \dot{\theta} dt. \quad (5.26)$$

Dissipative parameters $c_{d_2}$ and $h_{d_2}$ are then defined by the following

$$c_{d_2} = \left[ E_{D_2} - \int_{t_c}^{t_f} \Psi_\theta(\theta, \dot{\theta}) \dot{\theta} dt \right] \left[ \int_{t_c}^{t_f} \dot{\theta}^2 dt \right]^{-1}, \quad (5.27)$$

$$h_{d_2} = \left[ E_{D_2} - \int_{t_c}^{t_f} \Psi_\theta(\theta, \dot{\theta}) \dot{\theta} dt \right] \left[ \int_{t_c}^{t_f} \tanh(\eta \dot{\theta}) \dot{\theta} dt \right]^{-1}. \quad (5.28)$$

Formulations D3.1-2 assume that damping must be described locally, where D3.1 utilizes a multi-staged torsional viscous damper $c_{d_3}(\theta)$, and D3.2 employs a multi-staged Coulomb element $h_{d_3}(\theta)$. Functions $c_{d_3}(\theta)$ and $h_{d_3}(\theta)$ are assumed to be piecewise linear as given below:
Parameter estimation first requires the definition of dissipative power $C_D^3(t)$ that is:

$$C_D^3(\theta) = \sum_{j=1}^{N} \left\{ \frac{c_{D3,j+1} + \ldots + c_{D3,N}}{\theta_{j+1} - \Theta} \left[ \Xi\left(\theta - \Theta_{j+1}\right) - \Xi\left(\theta - \Theta_j\right) \right] + \ldots \right\}, \quad (5.29)$$

$$\Delta C_{D3,j} = \frac{c_{D3,j+1} - c_{D3,j}}{\theta_{j+1} - \Theta}, \quad \Delta C_{D3,j} = \frac{c_{D3,j-1} - c_{D3,j}}{\theta_{j} - \Theta}, \quad (5.30)$$

$$h_{D3}(\theta) = \sum_{j=1}^{N} \left\{ \frac{h_{D3,j+1} + \ldots + h_{D3,N}}{\Delta \theta} \left[ \Xi\left(\theta - \Theta_{j+1}\right) - \Xi\left(\theta - \Theta_j\right) \right] + \ldots \right\}, \quad (5.31)$$

$$\Delta h_{D3,j} = \frac{h_{D3,j+1} - h_{D3,j}}{\theta_{j+1} - \Theta}, \quad \Delta h_{D3,j} = \frac{h_{D3,j-1} - h_{D3,j}}{\theta_{j} - \Theta}, \quad (5.32)$$

The energy dissipated in each stage $E_{D3,j+1}$ is calculated using the following:

$$E_{D3,j+1} = \int_{t_0}^{t_f} \left[ \Xi\left(\theta - \Theta_{j+1}\right) - \Xi\left(\theta - \Theta_j\right) \right] dt, \quad (5.34)$$

$$E_{D3,j-1} = \int_{t_0}^{t_f} \left[ \Xi\left(\theta - \Theta_{j-1}\right) - \Xi\left(\theta - \Theta_{j-1}\right) \right] dt. \quad (5.35)$$

The energy dissipated in each stage is also defined as follows:

$$E_{D3,j+1} = \int_{t_0}^{t_f} \left[ c_{D3,j+1} + \Delta C_{D3,j+1}\left(\theta - \Theta_{j+1}\right) \right] \frac{d}{dt} \left[ \Xi\left(\theta - \Theta_{j+1}\right) - \Xi\left(\theta - \Theta_j\right) \right] dt, \quad (5.36)$$

$$E_{D3,j-1} = \int_{t_0}^{t_f} \left[ c_{D3,j-1} + \Delta C_{D3,j-1}\left(\theta - \Theta_{j-1}\right) \right] \frac{d}{dt} \left[ \Xi\left(\theta - \Theta_{j-1}\right) - \Xi\left(\theta - \Theta_j\right) \right] dt, \quad (5.37)$$
Parameters $c_{D3j\pm}$ and $h_{D3j\pm}$ are then calculated by combining Eqs. (5.34-39) as given below where it is assumed that $c_{D30\pm} = h_{D30\pm} = 0$, and intermediate parameters $\alpha_{cj\pm}$, $\beta_{cj\pm}$, $\alpha_{hj\pm}$, and $\beta_{hj\pm}$ are defined in Table 5.2.

\[
E_{D3j^+} = \int_{t_0}^{t_f} \left[ \frac{h_{D3(j-1)^+} + \ldots}{\Delta h_{D3j^+} (\theta - \Theta_{(j-1)^+})} \right] \tanh(\eta \hat{\theta}) \frac{\Xi(\theta - \Theta_{(j-1)^+}) - \ldots}{\Xi(\theta - \Theta_{j^+})} \; dt ,
\]

\[
E_{D3j^-} = \int_{t_0}^{t_f} \left[ \frac{h_{D3(j-1)^-} + \ldots}{\Delta h_{D3j^-} (\theta - \Theta_{(j-1)^-})} \right] \tanh(\eta \hat{\theta}) \frac{\Xi(\theta - \Theta_{(j-1)^-}) - \ldots}{\Xi(\theta - \Theta_{j^-})} \; dt .
\]

(5.38)  

(5.39)
Table 5.2 Intermediate parameters used in the estimation of $c_{D3jk}$ and $h_{D3jk}$ for dissipative formulation D3.1-2.

<table>
<thead>
<tr>
<th>Intermediate parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{ij+}$</td>
<td>$\int_{t_j}^{t_i} \dot{\theta}^2 \left[ \Xi \left( \theta - \Theta_{(j-1)^+} \right) - \Xi \left( \theta - \Theta_{j+} \right) \right] dt$</td>
</tr>
<tr>
<td>$\alpha_{ij-}$</td>
<td>$\int_{t_j}^{t_i} \dot{\theta}^2 \left[ \Xi \left( \theta - \Theta_{j-} \right) - \Xi \left( \theta - \Theta_{(j-1)^-} \right) \right] dt$</td>
</tr>
<tr>
<td>$\beta_{ij+}$</td>
<td>$\int_{t_j}^{t_i} \dot{\theta}^2 \left( \frac{\theta - \Theta_{(j-1)^+}}{\Theta_{j+} - \Theta_{(j-1)^-}} \right)^2 \Xi \left( \theta - \Theta_{(j-1)^+} \right) \Xi \left( \theta - \Theta_{j+} \right) dt$</td>
</tr>
<tr>
<td>$\beta_{ij-}$</td>
<td>$\int_{t_j}^{t_i} \dot{\theta}^2 \left( \frac{\theta - \Theta_{j-}}{\Theta_{(j-1)^-} - \Theta_{j-}} \right)^2 \Xi \left( \theta - \Theta_{j-} \right) \Xi \left( \theta - \Theta_{(j-1)^-} \right) dt$</td>
</tr>
<tr>
<td>$\alpha_{ij+}$</td>
<td>$\int_{t_j}^{t_i} \tanh (\eta \dot{\theta}) \dot{\theta} \left[ \Xi \left( \theta - \Theta_{(j-1)^+} \right) - \Xi \left( \theta - \Theta_{j+} \right) \right] dt$</td>
</tr>
<tr>
<td>$\alpha_{ij-}$</td>
<td>$\int_{t_j}^{t_i} \tanh (\eta \dot{\theta}) \dot{\theta} \left[ \Xi \left( \theta - \Theta_{j-} \right) - \Xi \left( \theta - \Theta_{(j-1)^-} \right) \right] dt$</td>
</tr>
<tr>
<td>$\beta_{ij+}$</td>
<td>$\int_{t_j}^{t_i} \tanh (\eta \dot{\theta}) \dot{\theta} \left( \frac{\theta - \Theta_{(j-1)^+}}{\Theta_{j+} - \Theta_{(j-1)^-}} \right)^2 \Xi \left( \theta - \Theta_{(j-1)^+} \right) \Xi \left( \theta - \Theta_{j+} \right) dt$</td>
</tr>
<tr>
<td>$\beta_{ij-}$</td>
<td>$\int_{t_j}^{t_i} \tanh (\eta \dot{\theta}) \dot{\theta} \left( \frac{\theta - \Theta_{j-}}{\Theta_{(j-1)^-} - \Theta_{j-}} \right)^2 \Xi \left( \theta - \Theta_{j-} \right) \Xi \left( \theta - \Theta_{(j-1)^-} \right) dt$</td>
</tr>
</tbody>
</table>
5.8. Comparative assessment of dissipative formulations

To validate the proposed formulations, time domain simulations of the dynamic experiment (X-D) are conducted using a commercial numerical solver [5.21, 25] since it is commonly used in the vehicle industry. Comparisons between measurements and predictions from formulations D0 and D3.2 are shown in Fig. 5.13 and 5.14 respectively; predictions from formulations D1.1-2, D2.1-2, and D3.1 are displayed in Appendix C. Additionally, comparisons between measurements and predictions from formulation D0, D2.1, and D3.2 are shown in Fig. 5.15. Upon initial observation, it is evident that \( \Psi_D(\theta, \dot{\theta}) \) (formulation D0) alone is not sufficient to describe dissipation under dynamic loading and that predictions from D1-3 are superior to D0. Geometric norms of prediction error are defined over each regime (di, si, and ni) of the measured responses as given below where \( \theta(t) \) is the measured signal, \( \theta(t, \text{Da.b}) \) is predicted from formulation Da.b (a = \{1, 2, 3\}, b = \{1, 2\}), and \( \theta(t, \text{D0}) \) is the prediction from formulation D0

\[
R1(\theta) = \left( \frac{\int_{t_o}^{t_o+\tau_a} \left[ \theta(t) - \theta(t, \text{Da.b}) \right]^2 dt}{\int_{t_o}^{t_o+\tau_a} \left[ \theta(t) - \theta(t, \text{D0}) \right]^2 dt} \right)^{0.5}, \quad (5.44)
\]

\[
R2(\theta) = \left( \frac{\int_{t_o+\tau_a}^{t_o+\tau_a+\tau_d} \left[ \theta(t) - \theta(t, \text{Da.b}) \right]^2 dt}{\int_{t_o+\tau_a}^{t_o+\tau_a+\tau_d} \left[ \theta(t) - \theta(t, \text{D0}) \right]^2 dt} \right)^{0.5}, \quad (5.45)
\]

\[
R3(\theta) = \left( \frac{\int_{t_o+\tau_a+\tau_d}^{t_f} \left[ \theta(t) - \theta(t, \text{Da.b}) \right]^2 dt}{\int_{t_o+\tau_a+\tau_d}^{t_f} \left[ \theta(t) - \theta(t, \text{D0}) \right]^2 dt} \right)^{0.5}. \quad (5.46)
\]
For additional metrics, the absolute differences between the normalized measured and predicted response regime periods are defined as:

\[ R_4 = \left| \bar{\tau}_{\text{di}} - \bar{\tau}_{\text{di}} \,(\text{Da.b}) \right|, \quad (5.47) \]

\[ R_5 = \left| \bar{\tau}_{\text{si}} - \bar{\tau}_{\text{si}} \,(\text{Da.b}) \right|, \quad (5.48) \]

\[ R_6 = \left| \bar{\tau}_{\text{ni}} - \bar{\tau}_{\text{ni}} \,(\text{Da.b}) \right|. \quad (5.49) \]

The ideal value for each metric is 0 (zero) and the worst case is associated with a very high positive value; thus, “best” metric values are those nearest to 0 (as summarized in Table 5.3). It is evident (from the metrics, Figs. 5.13-14, and Appendix C) that predictions from dissipative formulations D2.1, D2.2, and D3.2 (see Fig. 5.12 for illustrations) have the closest agreement to the measurements. These predictions are very similar to the measurement (and each other) in the double (di) and single-sided impact (si) regimes, where the motion spans several stages and the angular velocity is relatively high. However, when the response is confined to a single stage (no-impact regime (ni)), the angular velocity decreases; now, the prediction from D2.1 (which is the formulation most similar to prior work [5.12-16, 19-21]) is significantly worse than those from D2.2 and D3.2 (current work). This can clearly be seen in Fig. 5.15. The reduced accuracy of D2.1 could be explained by the following: 1) As angular velocity decreases, the effect of viscous damping also decreases, unlike a Coulomb friction element; and 2) The stage-dependent element in D2.1 is estimated under quasi-static loading only. The predictions from D2.2 and D3.2 are too similar to declare that one is more accurate than the other. Nevertheless, these results suggest that over wide ranges of angular displacement (across several stages) and velocity,
the effective dissipative formulation of the device is best described by a pure Coulomb friction element (unlike formulation D2.1 and prior work [5.12-16, 19-21]) that incorporates stage-dependence, such as formulations D2.2 and D3.2. Additionally, both quasi-static and dynamic experiments must be utilized to accurately estimate all parameters (e.g. angular stage transitions $\Theta$, Coulomb friction amplitude $h$, and regularizing factor $\eta$).
Figure 5.13  Comparison between measured and predicted (with formulation D0) motions for experiment X-D. Key: (▬▬) – measured motion; (▬▬) – predicted motion; (− · −) – angular stage transitions $\Theta_{j/2}$; (|) – regime transitions of the measured response; $\{\text{di, si, ni}\}$ – double-sided, single-sided, and no-impact regimes of the measured response; $\{0, I, II\}$ – stage indices; (+) – drive side; and (-) – coast side.
Figure 5.14 Comparison between measured and predicted (with formulation D3.2) motions for experiment X-D. Key: (▬▬) – measured motion; (▬▬) – predicted motion; (−∙−) – angular stage transitions $\Theta_j$; (|) – regime transitions of the measured response; \{di, si, ni\} – double-sided, single-sided, and no-impact regimes of the measured response; \{0, I, II\} – stage indices; (+) – drive side; and (-) – coast side.
Figure 5.15. Comparison between measured and predicted (with formulations D0, D2.1, and D3.2) $\ddot{\theta}(\bar{t})$ for experiment X-D. Key: (-----) – measurement; and (-----) – prediction.
Table 5.3 Summary of the response metrics (R1-R6) for several dissipation formulations (D1-D3): (a-c) normalized geometric norm of error between measurement and prediction; and (d) absolute difference between measured and predicted response regime periods. The ideal value for each metric is 0 (zero) and the worst case is associated with a very high positive value; lowest values ("best") per metric are emboldened.

<table>
<thead>
<tr>
<th></th>
<th>D1.1</th>
<th>D1.2</th>
<th>D2.1</th>
<th>D2.2</th>
<th>D3.1</th>
<th>D3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Measured double-sided impact regime di</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R1(θ)</td>
<td>0.18</td>
<td>0.95</td>
<td>0.28</td>
<td>0.24</td>
<td>1.28</td>
<td>0.38</td>
</tr>
<tr>
<td>R1(\dot{θ})</td>
<td>0.30</td>
<td>0.90</td>
<td>0.36</td>
<td>0.28</td>
<td>0.98</td>
<td>0.38</td>
</tr>
<tr>
<td>R1(\ddot{θ})</td>
<td>0.58</td>
<td>0.92</td>
<td>0.58</td>
<td>0.51</td>
<td>1.06</td>
<td>0.56</td>
</tr>
<tr>
<td>(b) Measured single-sided impact regime si</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R2(θ)</td>
<td>0.32</td>
<td>0.51</td>
<td>0.31</td>
<td>0.31</td>
<td>0.41</td>
<td>0.26</td>
</tr>
<tr>
<td>R2(\dot{θ})</td>
<td>0.45</td>
<td>0.67</td>
<td>0.45</td>
<td>0.46</td>
<td>0.50</td>
<td>0.40</td>
</tr>
<tr>
<td>R2(\ddot{θ})</td>
<td>0.64</td>
<td>0.82</td>
<td>0.64</td>
<td>0.64</td>
<td>0.62</td>
<td>0.60</td>
</tr>
<tr>
<td>(c) Measured no-impact regime ni</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R3(θ)</td>
<td>0.19</td>
<td>0.04</td>
<td>0.10</td>
<td>0.04</td>
<td>0.20</td>
<td>0.04</td>
</tr>
<tr>
<td>R3(\dot{θ})</td>
<td>0.41</td>
<td>0.04</td>
<td>0.22</td>
<td>0.05</td>
<td>0.44</td>
<td>0.04</td>
</tr>
<tr>
<td>R3(\ddot{θ})</td>
<td>0.54</td>
<td>0.11</td>
<td>0.33</td>
<td>0.12</td>
<td>0.58</td>
<td>0.12</td>
</tr>
<tr>
<td>(d) Absolute difference between measured and predicted response regime periods</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R4 (di)</td>
<td>0.04</td>
<td>1.85</td>
<td>0.07</td>
<td>0.03</td>
<td>0.27</td>
<td>0.06</td>
</tr>
<tr>
<td>R5 (si)</td>
<td>&gt; 5.84</td>
<td>2.04</td>
<td>2.84</td>
<td>0.26</td>
<td>&gt; 5.81</td>
<td>0.27</td>
</tr>
<tr>
<td>R6 (ni)</td>
<td>N/A</td>
<td>0.19</td>
<td>2.91</td>
<td>0.29</td>
<td>N/A</td>
<td>0.21</td>
</tr>
</tbody>
</table>
5.9. Conclusion

The main contribution of this article is the development of a new time domain parameter estimation method for a practical torsional device (such as a vehicle clutch damper) with multiple discontinuous non-linearities that subject to dynamic, transient loading. Under current engineering practice [5.11-23], vehicle clutch dampers are usually characterized from a quasi-static experiment only and linear viscous damping parameters are assumed. The proposed method compares characterization of the device under quasi-static and dynamic loading conditions, which addresses a critical need in both industrial practice and scientific literature. The proposed method compares the characterization of a vehicle clutch damper under quasi-static and dynamic loading conditions, which addresses a critical need in both industrial practice and scientific literature. Following the suggestions of Kerschen et al. [5.1], the proposed method utilizes measurements collected from a component-level dynamic experiment, which offers the following intrinsic advantages: 1) the interaction between multiple non-linear features is maintained; 2) experimentation conditions could be more similar to the operating environment; and 3) fewer measurements from laboratory experiments are required.

An instantaneous stochastic linearization technique is first proposed to estimate and verify the underlying elastic parameters. Dissipative parameters are estimated using the energy balance principle and a model-based approach (with seven alternate formulation) through it is limited to single or multi-staged torsional viscous dampers and Coulomb friction elements. Answers to the questions posed (under objectives in section 5.3) are summarized as follows: a. Elastic parameters under quasi-static and dynamic loading are
very similar, which is consistent with the assumptions made in literature [5.12-21]. b. The multi-staged Coulomb friction estimated under quasi-static loading is insufficient to account for the dissipation under dynamic loading; however, the addition of a parallel single-staged torsional viscous damper or Coulomb friction element improves predictions. c. Dissipation under dynamic loading is better described by the Coulomb friction rather than torsional viscous damping. d. When the mean operating point of a dynamic response crosses one or more stages, damping must be described in a local or stage dependent manner. Given the successful application demonstrated in this study, the proposed method could be extended to the characterization and modeling of similar physical systems with discontinuous non-linearities. Finally, this method should be valuable tool for improving the design and dynamic analysis of a family non-linear isolators and dampers.

References for Chapter 5


List of symbols for Chapter 5

0, I, II,... stage index

c torsional viscous damping

$E, \dot{E}$ dissipative energy and power

$h$ Coulomb hysteresis amplitude

$J$ torsional inertia

$k$ torsional stiffness

$N$ total number of stages

$t$ time

$T$ external torque

$U$ potential energy

$V$ kinetic energy

$W$ external work

$\alpha$ coefficient for parameter estimation

$\beta$ coefficient for parameter estimation

$\gamma$ coefficient for torque transmission approximation

$\Gamma$ torque transmission

$\Delta$ finite change

$\eta$ angular velocity regularizing factor (Coulomb friction)

$\theta, \dot{\theta}, \ddot{\theta}$ angular displacement, velocity, and acceleration

$\Theta$ angular stage transition
\( \Xi \) ideal step function

\( \tau \) period (time)

\( \Phi \) elastic torque transmission

\( \Psi \) dissipative torque

+ drive side

- coast side

**Subscripts**

0, I, II,... stage index

\( a \) alternating

\( D \) dynamic loading conditions

\( d_i \) double-sided impact regime

\( f \) final

\( i \)-hub clutch damper inner hub

\( j \) stage index

\( m \) mean

\( n \) natural

\( n_i \) no-impact regime

\( o \) initial

\( o \)-hub clutch damper outer hub

\( Q \) quasi-static loading conditions

\( s \) scaling factor

\( s_i \) single-sided impact regimes
\( t \)  
instantaneous (time dependent)

\( \varepsilon \)  
small value approximated by 0

+  
drive side

-  
coast side

**Superscripts**

(i)  
oscillatory period index

−  
normalized

'  
dummy variable

^  
approximation

**Operators**

\( \langle \rangle_t \)  
instantaneous expected value

**Abbreviations**

di  
double-sided impact regimes

si  
single-sided impact regimes

ni  
no-impact regime

P1,2,3  
paths under quasi-static loading

X-D  
experiment under dynamic loading

X-Q  
experiment under quasi-static loading

D0,Da,b  
proposed dissipative formulations for dynamic loading (\( \{a, b\} = \{1, 2, 3\} \))

R1,2,..6  
predicted response metrics
Chapter 6: Conclusion

6.1 Summary

This dissertation examines the step-response of discontinuous non-linear torsional devices, such as ground vehicle drivelines and their components [6.1-6]. Typical non-linearities of such systems include clearances, multi-staged torsional springs, stopper elements, pre-load features, and multi-staged dry friction elements [6.1-4, 7].

A new laboratory experiment that demonstrates the step-response of a component which contains both known and unknown discontinuous non-linear features is proposed (in Chapter 2). For design purposes, the experiment is simply described by a single degree of freedom (1DOF) positive-definite non-linear model. A production vehicle clutch damper [6.1] is chosen to provide multi-staged stiffness and dry friction elements, and a torsion arm with shaft provide the torsional inertia. Like prior work [6.2-4], the elastic and dissipative parameters are estimated from a separate quasi-static experiment [6.8], and the torsion arm is then sized so that the natural frequency (within the main stage of the clutch damper) is between 5 and 15 Hz (about the first torsional mode of a vehicle driveline) [6.2-6]. Due to the high torque capacity of the device, excitation is provided by pneumatic cylinders with a mechanical quick-release rather than an electromagnet mass drop [6.3-6]. The angular motions are measured by a laser vibrometer to overcome the difficulties associated with accelerometers (such as “ringing”) and to more accurately estimate the
angular displacement. The measurements clearly exhibit a rich non-linear behavior including double and single-single vibro-impacts with time-varying oscillatory periods. The performance of the experiment is verified by comparing the measurements to predictions from a minimal order non-linear model. Although both motions share the same qualitative behavior, the longer endurance of the predicted response suggests that the quasi-static dissipative parameters are insufficient for dynamic cases.

Next, a new laboratory experiment that demonstrates the step-response of a component that contains only known and well-controlled discontinuous non-linear features is proposed (in Chapter 3). It is intentionally designed to be a scaled (“benchtop”) version of prior experiments [6.2-6] (and Chapter 2) and to have no-clearance, single-clearance, and dual-clearance configurations, each described by positive-definite non-linear models (1, 2, and 3DOF, respectively). Here, the clearances are provided by key and keyway interfaces between shafts and jaw coupling hubs. Similar to prior experiments [6.2-6] (and Chapter 2), the torsional inertias and stiffness elements are sized so that the first natural frequency of the system is between 5 and 15 Hz. In order that the response be dominated by the clearance non-linearities, dry friction is minimized. A step-down external torque is then applied via an electromagnet mass drop from a torsion arm [6.3-6], and the resulting motions are measured by translational accelerometers. The no clearance configuration has a near linear response, but the single and dual-clearance configurations exhibit double-sided vibro-impacts and time-varying oscillatory periods with a softening trend. The feasibility of the experiment is demonstrated by comparing the measurements to predictions from the non-linear models (all configurations). There is good qualitative and
quantitative (in some aspects) agreement between the motions; however, the prediction accuracy decreases as the number of clearances increase. It is evident that more refined higher dimensional non-linear models are required to better describe the physics of the system. Finally, the applicability of the new experiment is demonstrated by performing a comparative study between the single-clearance configuration and the experiment proposed in Chapter 2.

Existing stochastic linearization techniques [6.9-10] are extended to estimate the trends of the time-varying oscillatory periods in the step-response of a non-linear torsional device (in Chapter 4). First, 1DOF computational example cases that contain near backlash and pre-load non-linearities (in parallel with a dry friction element) are introduced, and the regime-dependent trends of the time-varying oscillatory periods are discussed (hardening or softening). Next, stochastic linearization techniques developed by Wallaschek [6.9] and Rook and Singh [6.10] are utilized to propose a new concept of instantaneous effective stiffness. The calculations require relative angular displacement and torque signals (across a non-linear spring) as well as an instantaneous expected value operator with windowing functions. Although this new concept is purely mathematical, its amplitude is bounded by the stiffness values of the non-linear spring. The instantaneous effective stiffness concept is then employed to define a linear time-invariant 1DOF system at some instant during the step-response. The oscillatory period (non-linear response) near this time is approximated by the natural period of the corresponding linear system. The proposed estimation method is applied to the examples cases (near backlash and pre-load non-linearities) and digital signal processing parameters (such as uniform/adaptive window sizes and shapes) for time
domain analysis are investigated. Finally, the utility and applicability of the proposed method is demonstrated by estimating the trends of the time-varying oscillatory periods exhibited by the experiments proposed in Chapters 2 and 3.

A time domain method to estimate elastic and dissipative parameters of a component which contains known and unknown discontinuous non-linear features (from Chapter 2) is formulated (in Chapter 5). Elastic parameters are estimated through stochastic linearization techniques [6.9-10] and a model-based approach that utilizes the energy balance principle is utilized to estimate the dissipative parameters. In the process of developing the estimation method, several hypothetical questions concerning the nature of elastic and dissipative parameters are addressed. The findings are summarized as follows: i) Elastic parameters estimated from a quasi-static experiment are nearly equivalent to those estimated under dynamic conditions, as assumed in prior work [6.2-6]); ii) Dissipative parameters estimated from a quasi-static experiments are insufficient to accurately predict the energy loss under dynamic conditions; iii) The dominant dissipative mechanism under dynamic conditions is dry friction; and iv) When the mean operating point of the device rapidly crosses multiple stages, it better to define the dissipative parameters locally (stage dependent) rather than globally, as done in prior work [6.2-6].

6.2 Contributions

This dissertation provides four distinct and yet inter-related contributions towards the state of the art for discontinuous non-linear torsional systems. The first two contributions address the design, construction, and verification of original laboratory
experiments that demonstrate step-responses; the first one investigates a practical component that contains both known and unknown discontinuous non-linear features (in Chapter 2), and the second is a more scientific experiment, as it contains only known and well-controlled discontinuous non-linear features (in Chapter 3). Both new experiments yield much-needed physical insight at the component-level, which prior experimental driveline system work [6.2-6] fails to provide. In summary, both experiments overcome the general void in the field of non-linear dynamics where very few experimental investigations have been carried out [6.11]. The benchmark datasets should be equally valuable to the literature and engineering practice as they may be utilized to validate non-linear simulation models and estimate/select parameter values (say for dynamic or quasi-static conditions).

The next two contributions address refined parameter or system identification methods [6.12]. The first of these (in Chapter 4) develops a time domain method to estimate the trends in the time-varying oscillatory periods for the step-response of a non-linear torsional system; this is an extension of the prior work [6.9-10]. This estimation method should serve as an important diagnostic tool for system identification of unknown and/or non-linear devices, thus improving the accuracy and efficiency of the dynamic model building process. Furthermore, the utility of the method is demonstrated through the application to new laboratory experiments from Chapters 2 and 3. The last contribution (in Chapter 5) formulates a refined time domain method to estimate elastic and dissipative parameters of a component which contains both known and unknown discontinuous non-linear features. Prior methods [6.2-6, 13] have either utilized the linear system theory or
assumed that all non-linear features are smooth (differentiable). Therefore, the proposed method is specifically developed for systems that contain discontinuous and stage-dependent non-linear features. The method is applied to the benchmark data provided by the new component experiment (in Chapter 2), and the results address several questions that were not resolved by the literature [6.2-5]. First, elastic parameters estimated under quasi-static and dynamic conditions are nearly equivalent, as previously assumed [6.2-5]. However, a quasi-static estimation of dissipative parameters inaccurately predicts the energy loss under a dynamic loading. Furthermore, it is necessary to describe dissipative mechanisms locally (stage-dependent) rather than globally (single-staged, say through empirical modal damping values in most publications [6.2-7]), especially when the mean operating point rapidly crosses several stiffness transitions. This estimation method is not limited to the devices discussed in this dissertation, but can be extended to the characterization and modeling of similar non-smooth physical systems.

6.3 Future work

The work reported in this dissertation leads to several promising research topics. First, both proposed experiments can be modified to accommodate other excitation types, such as various step, impulsive, harmonic, or periodic torques. These benchmark datasets would facilitate further studies in parameter estimation [6.2-7] (including the determination of impact damping coefficients [6.10]) as well as the evaluation of damping parameters under gear rattle [6.2] and driveline clunk phenomena [6.2-7]. Perhaps the most fertile topic is the interaction between multiple clearance non-linearities, as observed in the dual-
clearance configuration of the proposed experiment (Chapter 3) and prior work [6.2-7, 10].

These types of driveline systems may be viewed as non-linear sub-systems coupled by linear stiffness elements. The new scientific experiment should be modified to explore the following aspects of this problem: i) The role of distance between the clearance elements; ii) The effect of the stiffness value at the coupling; and iii) The significance of the excitation source location in relationship to the clearances. Finally, the proposed estimation method for the time-varying oscillatory periods is developed only for lightly damped systems, with a limited set of digital signal processing parameters. Thus, it is natural to extend this work to moderately and heavily damped systems. This would require the formulation of a new concept of instantaneous effective viscous damping [6.9] as well as further investigations into window shapes/sizes and adaptive algorithms.

References for Chapter 6


Appendix A: List of instrumentation (Chapter 2)
Polytec PSV-400 scanning laser vibrometer [2.22] and PSV-400-M4 data acquisition system [2.23]:

- Measurement range: ± 10 m/s
- Minimum sampling period: 1 μs
- Additional channels: 4

PCB translational accelerometer model 355B02 [2.25]:

- Sensitivity: 10 mV/g
- Measurement range: ± 500 g (peak)
- Frequency range for ± 5% accuracy: 1 to 10000 Hz
- Frequency range for ± 10% accuracy: 0.6 to 12000 Hz
- Frequency range for ± 3 dB accuracy: 0.3 to 17000 Hz
- Resonant frequency: ≥ 35 kHz
- Broadband amplitude resolution (1 to 10000 Hz): 0.0005 g RMS

PCB handheld shaker model 394C06 [2.24]:

- Operating frequency: 159.2 ± 1.6 Hz
- Acceleration amplitude: 9.81 ± 0.01 m/s² RMS
- Velocity amplitude: 9.81 mm/s RMS
- Displacement amplitude: 9.81 μm RMS

Craftsman Digital Torpedo Level [2.26]:

- Measurement range: 0 to 360°
- Accuracy of digital display: ± 0.1°
- Accuracy of vials: ± 0.029°
Appendix B: List of instrumentation (Chapter 3)
Craftsman Digital Torpedo Level [3.17]:
- Measurement range: 0 to 360°
- Accuracy of digital display: ± 0.1°
- Accuracy of vials: ± 0.029°

PCB translational accelerometer model 355B02 [3.18]:
- Sensitivity: 10 mV/g
- Measurement range: ± 500 g (peak)
- Frequency range for ± 5% accuracy: 1 to 10000 Hz
- Frequency range for ± 10% accuracy: 0.6 to 12000 Hz
- Frequency range for ± 3 dB accuracy: 0.3 to 17000 Hz
- Resonant frequency: ≥ 35 kHz
- Broadband amplitude resolution (1 to 10000 Hz): 0.0005 g RMS

National Instruments analog input module model NI 9234 [3.19]:
- Sampling frequency: 51.2 kHz per channel
- Voltage range: ±5 V
- Digital resolution: 24-bit
- Dynamic range: 102 dB
- Anti-aliasing filters

OMEGA translational displacement transducer Model LD500-2.5 [3.22]:
- Stroke: ±2.5 mm
- Spring rate: 13 g/mm
- Voltage range: ±10 V
- Sensitivity: 73.31 mV/V/mm
Appendix C: Comparative results for other dissipation formulations (Chapter 5)
Figure C.1 Comparison between measured and predicted (with formulation D1.1) motions for dynamic experiment X-D. Here, $\bar{\theta}$ is the normalized angular displacement, $\bar{\dot{\theta}}$ is the normalized angular velocity, and $\bar{\ddot{\theta}}$ is the normalized angular acceleration. Key: (▬▬) – measured motion; (▬▬) – predicted motion; (▬▬) – angular stage transitions $\bar{\Theta}_{j\mu}$; (|) – regime transitions of the measured response; {di, si, ni} – double-sided, single-sided, and no-impact regimes of the measured response; {0, I, II} – stage indices; (+) – drive side; and (-) – coast side.
Figure C.2 Comparison between measured and predicted (with formulation D1.2) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption.
Figure C.3  Comparison between measured and predicted (with formulation D2.1) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption.
Figure C.4 Comparison between measured and predicted (with formulation D2.2) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption.
Figure C.5 Comparison between measured and predicted (with formulation D3.1) motions for dynamic experiment X-D. Key is described in the Fig. C.1 caption.
Bibliography


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