Propagation and Excitation of Electromagnetic Modes for Travelling-wave MRI Applications

THESIS

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By

Yi Chen

Graduate Program in Electrical and Computer Science

The Ohio State University

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Master's Examination Committee:

Roberto G. Rojas, Advisor

Patrick Roblin
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Abstract

Due to deficiencies of the conventional coil Magnetic Resonance Imaging (MRI) under ultra-high DC magnetic field to improve the imaging quality, a Travelling-Wave MRI (TWMRI) is discussed here. The main purpose of this research is to investigate the propagation characteristics of electromagnetic waves within the imaging area and the connection with the excitation source of the travelling wave. The travelling wave inside the imaging area has been modeled as a waveguide excitation problem with and without the human body inside. The research discussed here can be divided into near field analysis of the source (antenna) inside a circular waveguide that is empty (free-space) and partially filled with a phantom to account for the presence of a human body. Both analytical solution and numerically simulated results with various excitation sources are shown. The analytical solutions are implemented with MATLAB [Matrix Laboratory] and the numerical simulation is done with ANSYS HFSS [High Frequency Structural Simulator]. As expected, the results show that the excitation of the propagating waveguide modes is closely related to the near fields of the source. With these results, a proper source to improve the excitation of the desired waveguide modes can be designed. The desired modes have to be determined based on the image processing schemes use in this application. This thesis does not consider image processing.
Dedication

This document is dedicated to my family.
Acknowledgments

I would like to express my sincere appreciation to Prof. Roberto Rojas for providing me great help in accomplishing this thesis and Prof. Patrick Roblin for being my committee member. I also thank Prof. Ronald Reano for the license of HFSS in accomplishing this research.
Vita

August 2009......................................................Xiangming High School

June 2013 ....................................................B.S. Information Engineering, East China University of Science and Technology

December 2013 ..................................................B.S. Electrical and Computer Engineering, University of Missouri - Columbia

2014 to present ...............................................M.S. Electrical and Computer Engineering, The Ohio State University

Fields of Study

Major Field: Electrical and Computer Engineering
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Chapter 1: Introduction

Principle of Magnetic Resonant

Nuclear magnetic resonance (NMR) and magnetic resonance Imaging (MRI) are widely used in cancer detection, anatomy and many other fields. The principle of MRI is based on a characteristic of particles that have motion or precession behavior [1]. Atoms or molecules have electrons that rotate around the proton. The magnetic moment in a molecule or atom is due to electron rotation around the nucleus as well as the spin of (quantum mechanical effect) electrons and the nucleus. The orbital rotation of electrons can generate in some atoms/molecules a net magnetic momentum perpendicular to the orbital plane as depicted in Figure 1(a). The net momentum of each atom/molecule is in random directions and cancel each other as shown in Figure 1(b) [1]. However, when a strong static magnetic field ($B_0$) is present, these momenta will precess around the direction of the magnetic field as illustrated in Figure 1(c).

![Figure 1](image)

*Figure 1 (a) Particle’s net magnetic dipole moment; (b) Zero net magnetic dipole moment in absence of DC magnetic field; (c) Particles in the presence of a DC Magnetic Field $B_0$; (d) Particle’s Procession in presence of applied DC magnetic field.*
**Magnetic Resonant Frequency**

When the DC magnetic field is initially applied, a torque will be generated resulting in the precession of the particles around the applied magnetic field direction as depicted in Figure 1(d). The torque can be calculated as the time rate of change of angular momentum; namely,

\[
-\frac{1}{\gamma} \frac{d\vec{m}}{dt} = \vec{T} = \mu_0 \vec{m} \times \vec{H}_0
\]  

(1.1)

where the \( \vec{m} \) is the net magnetic dipole momentum of particles (hydrogen in this case), \( \gamma \) is the gyro magnetic ratio of hydrogen and \( \vec{H}_0 \) is the static magnetic field. Using separation of variables and solving the differential equation, \( \omega_0 = \mu_0 \gamma H_0 \) which is called the Larmor, or procession frequency [3] can be derived. When another time-varying magnetic field \( (B_1) \) perpendicular to the DC magnetic field, commonly referred as the RF signal, is applied and the magnetic field changes with the Larmor frequency, the momentum of the particles will also change at the same rate as the torque’s rate of change. This phenomenon is called magnetic resonance.

**Conventional Magnetic Resonant Imaging**

When the particles flip back to align with the static magnetic field, the energy of all particles will change and photons will be released, which is the principle of sensing in magnetic resonance imaging. In medical magnetic resonant imaging (MRI), the particle detected is hydrogen and the Larmor frequency is 42.58 MHz/T. In conventional MRI, the static magnetic field \( (B_0) \) is about 1.5 T, which corresponds to a Larmor frequency of
63.87 MHz. At this frequency, the RF magnetic field \( B_1 \) is typically generated by a coil with time-varying (AC) current. This coil is frequently referred to as a bird cage as depicted in Figure 2. This coil usually surrounds the object to be scanned. With the AC current in the coil, a standing magnetic wave is generated and magnetic resonant happens close to the coil.

![Figure 2 Bird Cage Coil for Standing-wave MRI](image)

**Travelling-wave Magnetic Resonant Imaging**

As depicted in Equation 1.1, the Larmor frequency is a strong function of the \( B_0 \) field. Meanwhile, the strength of the static magnetic field is proportional to the number of particles that can be aligned by the field and the number of aligned particles directly affect the quality of the image. With the goal of generating high quality imaging, an ultra-high field (7 Tesla) was proposed in [2]. The Larmor frequency of hydrogen particles under this condition increases to 298 MHz. The first limitation of traditional MRI is that it is nearly impossible to generate a homogeneous standing wave inside the bird cage coil at this Larmor frequency, resulting in a distorted image. Keeping these concerns in mind, the travelling wave method was introduced in [2]. These authors show that a homogeneous travelling electromagnetic wave propagating in the MRI bore can be excited by a properly designed antenna (source).
There are several advantages in using travelling wave MRI. First, the ultra-high field improves the image quality. Secondly, the area of the image can be enlarged from a bird cage to the whole MRI bore. In addition, bird cage-free imaging makes the MRI more comfortable for the patient.

**Purpose of Research**

The main focus of this thesis is to investigate the properties of the propagating electromagnetic wave inside the bore. One objective is to develop an analytical approach to analyze the excitation of electromagnetic modes within the MRI bore in order to design an optimized excitation source (antenna). The approach proposed here is based on analytical as well as numerical techniques. The structure of this thesis is divided into an analytical method that involves the study of an empty waveguide with a patch antenna, empty waveguide analysis with a bowtie dipole antenna. Furthermore, two types of phantom models in the waveguide to study the propagating of EM modes. A simpler model of a phantom is used for an analytical study while a more realistic model is used and studied numerically. This study concludes with a comparison and discussion of the results obtained with two types of sources (antennas) as well as brief discussion of future work.
Chapter 2: Strategy and Method

Research Analysis Strategy

The purpose of this research is to develop an analytical approach to analyze the propagation and excitation of electromagnetic modes within the MRI bore. With this analytical approach, an optimized excitation source (antenna) can be designed to excite one or more predetermined electromagnetic mode(s). The study starts with an investigation of the electromagnetic modes of the empty bore and the modes excited by two types of sources. The modes and the strength of their respective excitations by the two sources can then be compared. The first excitation source considered here is the same patch antenna used in the first TWMRI experiment by Brunner [1]. The second proposed excitation source is a wide band bowtie-dipole antenna. The key steps in the investigation of the mode excitation by the two sources are illustrated in Figure 3.

![Flow Chart of Analysis Procedure](image)

*Figure 3 Flow Chart of Analysis Procedure*

Keeping in mind that the walls of the MRI are metallic, the whole travelling wave analysis inside the bore can be modeled as a simple cylindrical waveguide problem where
the cylinder walls are perfect electric conductors (PEC). These two type of sources (antennas) are first analyzed in free space and then introduced in an empty metal waveguide for the study of their near fields which expanded in terms of a set of electromagnetic waveguide modes. The free space and near field analysis are carried out by means of a FEM method tool, ANSYS HFSS. After the numerical data of the near field has been acquired, an orthogonally property of the electromagnetic waveguide modes can be applied to extract the excitation coefficients of various modes. Next, a phantom is introduced in the MRI bore to study the electromagnetic wave propagation performance when a phantom is present. Two types of phantom are introduced. The first phantom is a simple circular cylindrical phantom. The EM modes for this structure can be obtained analytical if it is modeled as a partially dielectric filled waveguide problem. The second phantom is more realistic and it is closer to a human body. The differences in the excitation coefficients for different modes and the electromagnetic wave performance of both sources (antennas) is also discussed.

**Free Space Performance**

The operating frequency of the two sources (antenna) in free-space is first designed roughly for the Larmor frequency in the TWMRI application. However, in the TWMRI application, the antenna is used as a near field electromagnetic wave excitation source while a normal antenna design is based on the far field performance. So it is understood that the antennas initially designed for free-space operation may have to be tuned for impedance matching purposes. Thus, when each of the sources is introduced inside the
MRI bore, impedance tuning is performed by slightly changing the dimensions so the resonance is close to the Larmor frequency.

**Waveguide Analysis**

As previously mentioned, the near field analysis of the sources is first performed for an empty cylindrical waveguide problem which is simulated in ANSYS HFSS as depicted in Figure 4. At the end of the bore, a radiation boundary condition is used and a perfect match layer is used at the backend of the antenna model to eliminate reflected waves. The analytical waveguide analysis is done in two parts. The first step is to extract the excitation coefficients of the waveguide modes due to the near field antenna fields. The excitation coefficient of each mode can be calculated based on the orthogonal properties of the waveguide modes once the aperture electric fields are known. The second part is focused on the simulation of the propagation modes in an empty waveguide and waveguide with phantom present. A cylindrical phantom is introduced first for solving the eigenvalues and eigenfunctions in closed form of the waveguide electromagnetic waves. The problem is modeled as a partially dielectric filled cylindrical waveguide problem as illustrated in Figure 5. At the middle cross section of the bore shown in Figure 4 and Figure 5, the same orthogonal property can be applied to determine the excitation coefficients of the modes in the partially filled. By comparing the strength of the electric field for each mode in the near field region and the middle cross section, the electromagnetic propagation performance for each mode can be determined.
The verification of the propagation and evanescent modes obtained analytically has also been done in ANSYS HFSS by using a wave port to excite a certain electromagnetic mode with the same excitation coefficient. The model used for this verification is shown in Figure 6 and the wave port is applied right at the position where the near field is extracted. With the same excitation coefficient applied on the wave port, the same polarization and strength of the electrical field should be expected.

With the analytical solution of the excitation coefficients for the evanescent and propagation modes in the near field of the source, the waveguide performance of each source can be studied. In other words, the performance of the electromagnetic wave propagation can be determined based on the excitation coefficient of one or several electromagnetic waveguide mode(s).
Although the analytical solution for the bore with a circular phantom plays an important role for the initial understanding of the modes excited within the bore, the human body is not a perfect circular cylindrical structure as modeled in the analytical solution. The analytical eigenfunctions analysis becomes much more complicated if the phantom is modified to make it more realistic. A realistic geometry of the phantom as illustrated in Figure 7 is modeled and the result is completely based on the numerical simulation.

![Figure 7 Phantom with Human Body Geometry](image)

With these improved result, a more detailed and accurate field and power distribution at various cross sections within the MRI bore can be obtained.
Chapter 3: Empty Waveguide Analysis

Patch antenna modeling

A circular patch antenna with circular polarization, similar to the antenna discussed by Brunner [2], is considered first. The patch antenna is built with copper on a plexiglass substrate with a ground plane on the back as depicted in Figure 8. There is an air gap in the substrate for resonant frequency tuning purposes. Two feed points with $90^\circ$ rotation around the center point of the circular patch and fed with a $90^\circ$ phase shift between them are used to generate circular polarization.

Figure 8 (a) Top view of Patch antenna; (b) Cross section of patch antenna

The antenna dimensions are shown in Table 1 and the resonant frequency of the antenna is around 290MHz as illustrated in Figure 9.
The gain patterns of the theta and phi components in different planes are depicted in Figure 10.

Two coaxial cables used to feed the patch antenna are also simulated in HFSS to more accurately model the feeding structure. The geometry of the coaxial cable is calculated by means of Equation (3.1) to have a 50Ω characteristic impedance.

\[
Z_0 = \frac{\sqrt{\mu}}{\sqrt{\varepsilon}} \frac{\ln b/a}{2\pi}
\]  

(3.1)
Figure 10 (a) Total Gain; (b) Gain Theta; (c) Gain Phi; (d) Gain Z

As mentioned before, the antenna is designed for circular polarization so that the input signals at the two feeding points should be 90 degrees out of phase. In order to simulate the antenna structure more accurately and verify the result of the resonant frequency, the 50Ω referred S-parameter data is exported from HFSS and feed to a 90 degree hybrid coupler in Advance Design System (ADS) as shown in Figure 11.
Empty cylindrical waveguide

As previously mentioned, the radiation fields in backward direction (opposite direction to waveguide) needs to be absorbed to avoid spurious reflections. This is accomplished by introducing a PML layer illustrated in Figure 12.

When the antenna is placed at one end of the cylindrical bore as Figure 12 shows, the resonant frequency of the antenna increases from 290 MHz to 317 MHz. In this circumstance, the antenna is no longer used as an ordinary antenna but a waveguide source for exciting the electromagnetic wave. In an empty or homogenous waveguide with metallic walls, the electromagnetic wave can be divided into transverse electric (TE) or transverse magnetic (TM) mode. Referred from Pozar [3], the propagation constant
and cutoff frequency of each mode of a homogenous waveguide is a strong function of the geometry of the waveguide as depicted in Equation (3.2) and (3.3).

\[ \begin{align*}
  & \text{TE}_{m,n} \text{ mode:} & \quad (f_c)_{m,n} = \frac{\chi'_{m,n}}{2\pi a \sqrt{\mu \varepsilon}} \\
  & \text{TM}_{m,n} \text{ mode:} & \quad (f_c)_{m,n} = \frac{\chi_{m,n}}{2\pi a \sqrt{\mu \varepsilon}}
\end{align*} \tag{3.2} \tag{3.3} \]

where the \( \chi_{m,n} \) and \( \chi'_{m,n} \) are the roots of the Bessel functions and the derivative of the Bessel functions, respectively. From this equation and given the radius of the cylindrical bore is 29 cm, the dominant mode which has the lowest cutoff frequency is the TE\(_{11} \) mode and its cutoff frequency is 303 MHz which is right above the Larmor frequency.

When a phantom is introduced, the cutoff frequencies of the modes will change. An eigenmodes expansion is a proper approach to analyze the different modes that the antenna excites.

**Eigenmodes expansions**

As mentioned previously, the cutoff frequencies of the empty waveguide can be determined by the size of the waveguide and the only propagating mode in the empty waveguide is the TE\(_{11} \) mode and the other modes are actually evanescent modes. In other word, for the given dimensions of the bore, only the TE\(_{11} \) mode is required and the source should excite mostly this mode and to a lesser extend other modes in the homogenous waveguide case. In order to determine whether the patch antenna has a decent performance TE\(_{11} \) mode, the orthogonality properties of the solution of the waveguide
modes are used. From Balanis [4], the eigenfunctions of a cylindrical homogenous waveguide are:

**TE<sub>m,n</sub> mode:**

\[
E_{\rho, mn}^{TE} = \frac{A_{mn}m}{\epsilon \rho} J_m(\beta_{\rho, mn \rho})[C \sin(m\phi) - D \cos(m\phi)]e^{-j\beta_z z} \quad (3.4)
\]

\[
E_{\phi, mn}^{TE} = \frac{A_{mn}m \beta_{\rho, mn \rho}}{\epsilon} J'_m(\beta_{\rho, mn \rho})[C \cos(m\phi) + D \sin(m\phi)]e^{-j\beta_z z} \quad (3.5)
\]

**TM<sub>m,n</sub> mode:**

\[
E_{\rho, mn}^{TM} = -\frac{B_{mn}m \beta_{z, mn \rho}}{\omega \mu \epsilon} J_m(\beta_{\rho, mn \rho})[C \cos(m\phi) + D \sin(m\phi)]e^{-j\beta_z z} \quad (3.6)
\]

\[
E_{\phi, mn}^{TM} = \frac{B_{mn}m \beta_{z, mn \rho}}{\omega \mu \epsilon} J'_m(\beta_{\rho, mn \rho})[C \sin(m\phi) - D \cos(m\phi)]e^{-j\beta_z z} \quad (3.7)
\]

\[
E_{z, mn}^{TM} = -j\frac{B_{mn}m^2 \beta_{\rho, mn \rho}}{\omega \mu \epsilon} J_m(\beta_{\rho, mn \rho})[C \cos(m\phi) + D \sin(m\phi)]e^{-j\beta_z z} \quad (3.8)
\]

where \( C \) and \( D \) are the excitation coefficient that need to be determined and the \( \beta_{\rho}, \beta_{\phi} \) and \( \beta_{\phi} \) are the propagation constants of each direction and satisfied the well-known dispersion equation:

\[
\beta_{\rho}^2 + \beta_{\phi}^2 + \beta_{\phi}^2 = \omega^2 \mu^2 \epsilon^2 \quad (3.9)
\]

Since heavy use of the orthogonality properties is made here, a detailed of the proof of these properties can be found in Appendix A. Equation (3.10) shows the orthogonal between different polarizations.

\[
\iint (\vec{e}_{mn} + \vec{e}_{z, mn}) \cdot (\vec{e}_{lk} + \vec{e}_{z, lk}) \, ds
\]

\[
= \int_0^{2\pi} \int_0^a (\vec{e}_{mn} \cdot \vec{e}_{lk} + \vec{e}_{z, mn} \cdot \vec{e}_{z, lk}) \, ds \quad (3.10)
\]
where the $\vec{e}_{mn}$ is the transverse field and $\vec{e}_{z,mn}$ is the longitudinal field [4]. Note that for the TE$_{mn}$ mode, the $\vec{e}_{z,mn}$ term is actually zero. In the near field cross section, the aperture E-field can be expanded in terms of the eigenfunctions with different coefficient as

Equation (3.11) shows.

$$\vec{E}_{aperture} = \sum_{i=1}^{n} \sum_{i=0}^{m} C_{mn} \cdot \vec{E}_{mn}$$  \hspace{1cm} (3.11)

where the $\vec{E}_{aperture}$ in this case is the electric field obtained from the HFSS simulation, and the $C_{mn}$ term is the excitation coefficient for the specified (m,n) mode. The excitation coefficient can be calculated by Equation (3.12).

$$\vec{E}_{aperture} \cdot \vec{E}_{lk} = \sum_{i=1}^{n} \sum_{i=0}^{m} C_{mn} \cdot \vec{E}_{mn} \cdot \vec{E}_{lk} = C_{lk} \cdot \vec{E}_{lk} \cdot \vec{E}_{lk}$$

$$\text{Take integration} \quad \Rightarrow \quad C_{lk} = \frac{\iint \vec{E}_{aperture} \cdot \vec{E}_{lk} \, ds}{\iint \vec{E}_{lk} \cdot \vec{E}_{lk} \, ds} \quad (3.12)$$

After the excitation coefficients have been calculated based on the orthogonality property of eigenfunctions, the total electric field can be obtained by comparing summing the different modes with the calculated excitation coefficients. This total field can be compared with the original exported aperture electric field to verify the eigenfunction expansion.

In this research, the original aperture data was exported from HFSS with data points meshed in a 580mm×580mm aperture with 1mm spacing in Cartesian coordinates. The eigenfunctions, excitation coefficients and plots are all calculated by MATLAB. The
comparison of the aperture electric field shows some similarity of each component as illustrated in Figure 13 and 14. The details of each mode can be found in Appendix B and the details of the MATLAB codes in Appendix C.

![Figure 13 Linear Polarization Electric Field Verification (Patch Antenna)](image1)

![Figure 14 Linear Polarization Electric Field Verification (Patch Antenna)](image2)

It is easy to see that the original electric field is not exactly the same as the regenerated field. The error is about 22% and it is calculated based on Equation (3.13).
\[
\text{Error} = \frac{\iint |\vec{E}_{\text{aperture}} - \vec{E}_{\text{regenerate}}|^2 \, ds}{\iint |\vec{E}_{\text{aperture}}|^2 \, ds} \cdot 100\% \quad (3.13)
\]

There are two main reasons which introduce large errors. The first is the numerical interpolation and surface integration calculation built-in function in MATLAB. The other reason is the insufficient number of eigenmodes used in the. In order to achieve enough accuracy and better interpolation, the spacing between each point is equal to 1mm. With the 580mm×580mm cross section, there are over 300,000 numerical data points need to be calculated per integration. The regenerate field plots contains 50 eigenmodes in total (m is from 0 to 4 and n is from 1 to 5 with both TE and TM modes). Although there are still 22% error exists, the error is decreased by increasing the number of eigenmodes. In addition, the objective to regenerate the total electric field is to verify the excitation coefficient calculation. The excitation coefficients for the first several dominant modes are the first thing needs to be concerned. So the error here does not have a significant affection on the final result.

In Appendix B, it is shown that for both, the circular polarization and linear polarization excitations, several other modes have the same or higher magnitude than the dominant TE$_{11}$ mode. These results will be further compared with the phantom inserted in the waveguide.
Chapter 4: Bowtie Dipole Source

With the analysis of a patch antenna, the near field analysis shows reasonable results in terms of the excitation coefficient of different modes. To access the performance of this patch antenna used in the initial TWMRI studies [2] it is necessary to consider other sources. In this study a bowtie dipole antenna has been investigated and modeled following the same procedure used in the previous patch antenna. Since these antennas are used as excitation sources for MRI applications, the size of the antenna is limited by the cylindrical bore which is 29cm in radius. In order to fit the antenna in the cylindrical tube, a slot bowtie dipole antenna is considered to minimize its size [5]. The slot bowtie dipole antenna shown in Figure 15 has been modeled on a copper sheet mounted on a 25mm thick plexiglass with the size specified in Table 2. As the previous antenna, the bowtie antenna is modeled and simulated in ANSYS HFSS.

Table 2 Slot Bowtie Dipole Geometry

<table>
<thead>
<tr>
<th>L</th>
<th>W</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>240 mm</td>
<td>180 mm</td>
<td>10 mm</td>
</tr>
</tbody>
</table>

Figure 15 Slot Bowtie Dipole
Unlike an ordinary dipole antenna which operates at a series resonant frequency, the first resonant frequency of the slot bowtie dipole is a parallel resonant at 280MHz as depicted in Figure 16.

![Z-Parameter of bowtie dipole](image1.png) ![S-Parameter of bowtie dipole](image2.png)

Figure 16 Z-Parameter of bowtie dipole
Figure 17 S-Parameter of bowtie dipole

Referring to [5] two metal tuning stubs are used in order to match the input impedance at the resonant frequency. With the contribution of the two stubs, the input impedance is matched to 50 Ohms as depicted in Figure 17 where the S\textsubscript{11} parameter is shown versus frequency. The resonant frequency for this antenna is also around 280MHz which is close to the resonant frequency of the patch antenna considered in the previous chapter. When the antenna is introduced in the cylindrical bore, it will couple with the metal bore and the resonant frequency will increased as in the patch antenna case. So the 280 MHz frequency leaves a margin to meet the Larmor frequency requirement for the MRI application and meet the cutoff frequency requirement of the waveguide.

The radiation gain pattern in free-space is illustrated in Figure 18. Since the bowtie antenna is designed as a slot type antenna, the radiation pattern and resonant behavior are reversed to ordinary bowtie dipole antenna built in metal. This is a consequence of the
Duality Theorem. For example, the $\theta$ component radiation pattern looks similar to $\phi$ component of the ordinary bowtie dipole antenna and vice versa. The same backward radiation as in the patch antenna exists in the bowtie dipole antenna and the PML boundary condition is still necessary to perform the near field analysis where the antenna is introduced to the bore. Note that unlike the previous patch antenna, this antenna is linearly polarized.

![Figure 18](a) Total Gain in $xz$ plane; (b) Total Gain in $yz$ plane

**Near field analysis**

To perform the near field analysis, the same procedure used in the previous patch antenna is followed here; namely, using the orthogonality properties of the eigenfunctions. The generated eigenfunctions including their respective excitation coefficients are shown in Figure 19 and 20. Due to the limitations previously mentioned for the patch antenna analysis, the total error of the generated field has been verified to be lower than 30% when a total of 25 eigenmodes are used. The circular polarization case is investigated by using the superposition of the fields generated by two dipole antennas rotated 90° with
respect to each other and fed by two sources that are out of phase by 90° as well. Since the antenna is a slot-type radiator, the actual horizontal linear polarization is when the dipole is on the y axis and the same applies to the vertical linear polarization; namely, vertical polarization is obtained when the axis of the dipole is along the x-direction.

*Figure 19  Linear Polarization Electric Field Verification (Bowtie Dipole Antenna)*

*Figure 20  Circular Polarization Electric Field Comparison (Bowtie Dipole Antenna)*
The error in generating the total near field by summing a set of eigenmodes has been verified to be below 25% when the same number of eigenmodes are used (50) as in the patch antenna case. One thing can be observed is the difference of the magnitudes of the total electric fields appear to be larger than the differences of the magnitudes of the fields of each mode; however, the total error is still lower than 25%.

Additional observations are discussed in Appendix B. It turns out that the TE\(_{11}\) mode is not stronger than the TM\(_{11}\), TM\(_{12}\) and TM\(_{13}\) modes. But the magnitude of the TE\(_{11}\) mode is higher than in the case of the patch antenna. These results will be fully explained in Chapter 6 when the phantom is inserted in the bore.
Chapter 5: Waveguide with Phantom Present

Circular cylindrical phantom

It has been shown that in an empty waveguide, the only mode that propagates is the TE$_{11}$ mode. The near field analysis of the empty waveguide shows that it is possible extract different modes from the aperture field based on the eigenfunction expansion. However, when a phantom is introduced in the cylindrical bore, the analysis becomes more complicated and difficult to have a closed form solution. In order to have a brief idea of the propagation modes from an analytical solution, the phantom is assumed to be a cylinder of circular cross section. This shape was chosen because the waveguide can be then modeled as a partially cylindrical dielectric filled waveguide. The cross section is modeled as Figure 21 and the parameter values are illustrated in Table 3.

Table 3 Phantom field parameter

<table>
<thead>
<tr>
<th>$\varepsilon_{II}$</th>
<th>$\varepsilon_{I}$</th>
<th>$\mu_{II}$</th>
<th>$\mu_{I}$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>56.7</td>
<td>1</td>
<td>1</td>
<td>140 mm</td>
<td>290 mm</td>
</tr>
</tbody>
</table>

Figure 21 Cross section of cylindrical phantom
Although the eigenfunctions of this problem becomes more complicated than the homogeneous case, it is still possible to solve for them analytically. From Harrington [6], the eigenfunctions still satisfy the Helmholtz equation and the form of the electric and magnetic vector potentials are given in Equations (5.1) to (5.4):

\[ F_z^I (\rho, \phi, z) = A_{mn} B_{m}^{F,I} (\beta_{\rho,II} \rho) \left\{ \frac{\sin (m\phi)}{\cos (m\phi)} \right\} e^{-j\beta_z z} \] (5.1)

\[ F_z^II (\rho, \phi, z) = B_{mn} B_{m}^{F,II} (\beta_{\rho,II} \rho) \left\{ \frac{\sin (m\phi)}{\cos (m\phi)} \right\} e^{-j\beta_z z} \] (5.2)

\[ A_z^I (\rho, \phi, z) = C_{mn} B_{m}^{A,I} (\beta_{\rho,II} \rho) \left\{ \frac{\sin (m\phi)}{\cos (m\phi)} \right\} e^{-j\beta_z z} \] (5.3)

\[ A_z^II (\rho, \phi, z) = D_{mn} B_{m}^{A,II} (\beta_{\rho,II} \rho) \left\{ \frac{\sin (m\phi)}{\cos (m\phi)} \right\} e^{-j\beta_z z} \] (5.4)

where the \( B_{m}^{F or A, I or II} (\beta_{\rho,II} \rho) \) represents a combinations of Bessel functions and the propagation constant \( \beta_\rho \) can still be determined from the boundary conditions. Recall that the empty or homogenous waveguide only contain Bessel functions of the first kind because the fields are finite in the center of the waveguide. However, for the partially filled waveguide case, the electric field in the II region does not include the center of the waveguide, the eigenfunctions contain both, the first and second kind Bessel functions, in region II. As a result, the roots of the eigenfunction solutions are more complicated as shown in Table 4.

\[
\begin{array}{|c|c|c|}
\hline
\textbf{TM modes} & \textbf{Empty waveguide} & \textbf{Partially Filled Waveguide} \\
\hline
J_m (\beta_{\rho,II} b) = 0 & J_m (\beta_{\rho,II} b) + Y_m (\beta_{\rho,II} b) = 0 \\
\hline
\textbf{TE Modes} & J'_m (\beta_{\rho,II} b) = 0 & J'_m (\beta_{\rho,II} b) + Y'_m (\beta_{\rho,II} b) = 0 \\
\hline
\end{array}
\]

*Table 4 Change of Bessel Function*
The cutoff frequencies can then be calculated from the new boundary condition based on Equation (5.5) and (5.6).

\[ (f_c)_{m,n} = \frac{x_{m,n}}{2\pi a \sqrt{\mu \varepsilon}} \]  \hspace{1cm} (5.5)  
\[ (f_c)_{m,n} = \frac{x_{m,n}}{2\pi a \sqrt{\mu \varepsilon}} \]  \hspace{1cm} (5.6)

where the \( u_{m,n}' \) and \( u_{m,n} \) are the solution of \( J_m(x_{m,n}) + Y_m(x_{m,n}) = 0 \) and \( J'_m(x_{m,n}') + Y'_m(x_{m,n}') = 0 \), respectively. The changes in the cutoff frequencies are listed in Table 9.

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( \Psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Psi )</td>
<td>( \chi )</td>
</tr>
</tbody>
</table>

Table 5 Change of Roots of Eigenfunctions

<table>
<thead>
<tr>
<th>( m )</th>
<th>( TM_{mn} ) (Empty)</th>
<th>( TM_{mn} ) (Phantom)</th>
<th>( TE_{mn} ) (Empty)</th>
<th>( TE_{mn} ) (Phantom)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2.41</td>
<td>3.83</td>
<td>5.14</td>
<td>0.23</td>
</tr>
<tr>
<td>2</td>
<td>5.52</td>
<td>7.02</td>
<td>8.42</td>
<td>3.18</td>
</tr>
<tr>
<td>3</td>
<td>8.65</td>
<td>10.2</td>
<td>11.6</td>
<td>6.33</td>
</tr>
<tr>
<td>4</td>
<td>11.8</td>
<td>13.32</td>
<td>14.8</td>
<td>9.44</td>
</tr>
</tbody>
</table>

It is obvious that the TM01 and TM11 modes become the first two dominant modes with cutoff frequencies at 37.9MHz and 219MHz, respectively, instead of \( TE_{11} \) mode with cutoff frequency at 348MHz. The verification of this eigenfunction analysis can be conducted with the HFSS simulation. The simulation is done by using a single wave port to excite a certain mode with the same near field excitation coefficient. Figure 22 depicts the model used for this verification.
In the middle cross section, the total aperture field can also be decomposed into different modes. Meanwhile, if a wave port is used, the resulting excited field of a given mode can also be achieved. The comparison for the first three dominant modes at the middle of the bore are shown in Figure 23.

*Figure 22 Verification setup of cylindrical phantom*

*Figure 23 Verification of Different Modes*
As the comparisons in the previous near field analysis, the error in generating these fields has been calculated with Equation (3.13) in Chapter 3 and all the error values turned out to be are less than 18%. The error is introduced by the numerical integration function in MATLAB.

The result of the cylindrical phantom analysis indicates that the TM$_{01}$, TM$_{11}$ and TE$_{11}$ modes are all strongly excited by the sources instead of only the TE$_{11}$ mode in the empty waveguide case. With this result, a further investigation of a more realistic phantom will be done to generate more realistic field excitations within the bore. These results will be generated for the patch and bowtie antennas which will be used to determine which of these antennas is more desirable for TWMRI applications.

**Phantom geometry to model human body**

With the introduction of a more realistic phantom geometry (see Figure 24), the analytical solution of the fields excited in the bore becomes impossible. The human body phantom is modeled with a combination of different cylindrical structures as illustrated in Figure 24. The dielectric is assumed to have a relative dielectric constant of 56.7 and a conductivity of 0.94 S/m. The cross sections where the fields and power levels will be calculated are also shown in Figure 24.
A full comparison of the fields excited by the patch and bowtie antennas for the above structure will be discussed in Chapter 6.
Chapter 6: Comparison and Results

Excitation coefficients in empty waveguide

The calculation of excitation coefficients in empty waveguide for three waveguide modes with the same input power are shown Table 6. They are used to represent the strength of each mode and the phase of the excitation coefficient is ignored. It can be seen that the bowtie dipole antenna has a better performance in exciting the TE_{11} mode for the empty waveguide. The patch antenna has a better performance in exciting both TM_{01} and TM_{11} modes in the empty waveguide.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Polarization Type</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>TM_{01} Mode</td>
<td>Patch (Circular Polarization)</td>
<td>6.65×10^{-9}</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Patch (Linear Polarization)</td>
<td>9.4×10^{-9}</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bowtie (Circular Polarization)</td>
<td>6.18×10^{-9}</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bowtie (Linear Polarization)</td>
<td>5.88×10^{-9}</td>
<td>0</td>
</tr>
<tr>
<td>TM_{11} Mode</td>
<td>Patch (Circular Polarization)</td>
<td>1.29×10^{-8}</td>
<td>1.24×10^{-8}</td>
</tr>
<tr>
<td></td>
<td>Patch (Linear Polarization)</td>
<td>1.33×10^{-8}</td>
<td>1.31×10^{-8}</td>
</tr>
<tr>
<td></td>
<td>Bowtie (Circular Polarization)</td>
<td>1.86×10^{-9}</td>
<td>1.88×10^{-8}</td>
</tr>
<tr>
<td></td>
<td>Bowtie (Linear Polarization)</td>
<td>5.33×10^{-11}</td>
<td>2.7×10^{-8}</td>
</tr>
<tr>
<td>TE_{11} Mode</td>
<td>Patch (Circular Polarization)</td>
<td>1.28×10^{-10}</td>
<td>1.32×10^{-10}</td>
</tr>
<tr>
<td></td>
<td>Patch (Linear Polarization)</td>
<td>1.34×10^{-10}</td>
<td>1.35×10^{-10}</td>
</tr>
<tr>
<td></td>
<td>Bowtie (Circular Polarization)</td>
<td>1.75×10^{-10}</td>
<td>1.74×10^{-10}</td>
</tr>
<tr>
<td></td>
<td>Bowtie (Linear Polarization)</td>
<td>2.47×10^{-10}</td>
<td>2.39×10^{-13}</td>
</tr>
</tbody>
</table>
Empty waveguide comparison

The result comparison in empty waveguide is shown in Figure 25. Although the real MRI situation is phantom presented, the empty waveguide is still a good reference for the verification of the expected result based on TE_{11} mode strength comparison. The difference orientations of the fields is due to the feeding method of the antennas. From the color bar scale of the field magnitude evaluated in the middle cross section of the MRI bore, both plots show the TE_{11} mode is the only propagating mode inside the empty bore. The strength of the electric field excited by the bowtie dipole antenna is stronger than the field excited by the patch antenna, which is consistent with the excitation coefficients shown in Table 6.

![Figure 25 Empty waveguide result comparison](image)

Waveguide with phantom present

As verified in the partially filled waveguide analysis, when a phantom is present in the waveguide, the TM_{01} mode will become the new dominant mode and the TM_{11} mode will also propagate in the waveguide in the presence of the circular cylindrical phantom. So when a phantom is present in the waveguide, it is expected that the patch antenna will
have a better performance as the results in Table 6 indicate. The results of partially filled waveguide, where the more realistic phantom model shown in Figure 20 is used, has been numerically obtained because no analytical solution exists for this complicated structure. A straightforward comparison has been done regarding the power passing through each cross section at different locations as depicted in Figure 26.

![Figure 26](image)

*Figure 26 (a) Total power in cross section; (b) Power in phantom; (c) Total power in cross section with reversed phantom; (d) Power in phantom with reversed phantom.*
Right hand circular polarization, horizontal linear polarization and vertical polarization has been considered for this configuration. Furthermore, the fields were calculated for two cases; namely, when the phantom’s head and feet are close to source. The vertical polarization and horizontal polarization designation is with respect to the phantom’s orientation. Horizontal polarization refers to the case where the electric field is parallel to the plane containing the shoulders and torso of the phantom. Vertical polarization is when the electric field is perpendicular to the plane containing the shoulders and torso of the phantom. Since the phantom is not rotationally symmetric, the horizontal and vertical linear polarization does affect the result. The solid lines in Figure 26 show the power of the fields excited by the patch antenna at different locations; while the dash lines show the power of the fields excited by the bowtie dipole antenna. As expected, Figure 26 (a) and (c) shows that the patch antenna has a better performance than the bowtie dipole antenna when the phantom is inside the bore since it has a better near field performance regarding the excitation of the TM_{01} and TM_{11} modes. For each source, the vertical linear polarization has the strongest excitation implying the phantom has the smallest interaction with the electromagnetic wave. Another observation that can be made is that the “reversed phantom case” (phantom’s feet closest to source) has much higher power at each cross section along the length of the MRI bore.

For MRI applications, it is desirable to have fields that strongly interact with the phantom. Keeping this mind, the power within the phantom is shown in Figure 26 (b) and (d). As the result shows, the horizontal polarization interacts more strongly with the
phantom, therefore, making it more desirable than the vertical polarization. Comparing both sources, it can be observed that the patch antenna has better performance in each case than the bowtie dipole antenna. The details of the electric field distribution are shown in Appendix C.
Chapter 7: Conclusion and Further Work

From the analytical and numerically simulated results obtained here, several observations can be made regarding the electromagnetic field behavior inside the waveguide (MRI bore). First, the TE_{11} mode is the only propagation mode in the given empty cylindrical bore at the frequency of interest. Second, if a phantom is present, the dominant mode changes from the TE_{11} mode to the TM_{01} mode; the latter mode has better wave propagation characteristic in the waveguide, thus helping with the imaging that needs to be done. Finally, although the analytical result does not show the impact of the polarization because it considers an empty waveguide or a waveguide with a circularly symmetric phantom, the wave polarization is important when a more realistic phantom is used to model the human body. Based on the results of Chapter 6, horizontally polarized fields have the best performance because they have the strongest interaction with the phantom. The numerically simulated results with the phantom inside the bore shows consistency with the near field analysis of the antenna; namely, the expansion of the near fields in terms of a set of waveguide modes. This implies that a proper excitation source for the TWMRI application can be designed based on this near field analysis.

There are several aspects of this research that can be expanded. First of all, a physical experiment (imaging) can be done to assess the accuracy of the analysis performed here.
that is mainly focused on electromagnetic aspects. Secondly, there is a need to design an optimized probe (antenna) to excite the desired mode(s) based on the needs of the imaging algorithms. It will also be important to implement and test the performance of this probe. Finally, the appropriate power levels inside the human body have to be determined keeping in mind safety and the quality of the TWMRI-based imaging.
References


[5]. Eldek, Abdelnasser et al. Wideband Bow-tie Slot Antenna with Tuning Stubs. IEEE, 2004

Appendix A: Proof of Orthogonality

E-Field Eigenfunctions in cylindrical waveguide:

**TE\(_{mn}\) mode:**

\[ E_{\rho,\text{TE}_{mn}} = \frac{A_{mn}}{\varepsilon \rho} J_m(\beta_{\rho,\text{mn}} \rho) [C \sin(m\phi) - D \cos(m\phi)] e^{-j\beta_z z} \tag{A.1} \]
\[ E_{\phi,\text{TE}_{mn}} = \frac{A_{mn} \beta_{\rho,\text{mn}}}{\varepsilon} J'_m(\beta_{\rho,\text{mn}} \rho) [C \cos(m\phi) + D \sin(m\phi)] e^{-j\beta_z z} \tag{A.2} \]

**TM\(_{mn}\) mode:**

\[ E_{\rho,\text{TM}_{mn}} = -\frac{B_{mn} \beta_{\rho,\text{zn}}}{\omega \mu \varepsilon \rho} J_m(\beta_{\rho,\text{mn}} \rho) [C \cos(m\phi) + D \sin(m\phi)] e^{-j\beta_z z} \tag{A.3} \]
\[ E_{\phi,\text{TM}_{mn}} = \frac{B_{mn} m \beta_{\rho,\text{zn}}}{\omega \mu \varepsilon \rho} J_m(\beta_{\rho,\text{mn}} \rho) [C \sin(m\phi) - D \cos(m\phi)] e^{-j\beta_z z} \tag{A.4} \]
\[ E_{z,\text{TM}_{mn}} = -j \frac{B_{mn} \beta_{\rho,\text{zn}}^2}{\omega \mu \varepsilon \rho} J_m(\beta_{\rho,\text{mn}} \rho) [C \cos(m\phi) + D \sin(m\phi)] e^{-j\beta_z z} \tag{A.5} \]

**Orthogonality of Trigonometry Function:**

\[ \int_{0}^{2\pi} \sin(m\phi) \sin(l\phi) \ d\phi = \begin{cases} 0, & m \neq l \\ \pi, & m = l \end{cases} \tag{A.6} \]
\[ \int_{0}^{2\pi} \sin(m\phi) \cos(l\phi) \ d\phi = \begin{cases} 0, & m \neq l \\ 0, & m = l \end{cases} \tag{A.7} \]
\[ \int_{0}^{2\pi} \cos(m\phi) \cos(l\phi) \ d\phi = \begin{cases} 0, & m \neq l \\ \pi, & m = l \end{cases} \tag{A.8} \]

**Bessel Function Recurrence Relations:**

\[ \frac{2\alpha}{x} Z_{\alpha}(x) = Z_{\alpha-1}(x) + Z_{\alpha+1}(x) \tag{A.9} \]
\[
2 \frac{dZ_\alpha(x)}{dx} = Z_{\alpha-1}(x) - Z_{\alpha+1}(x)
\] (A.10)

**General Orthogonality of Bessel Function:**

\[
\int_0^1 \rho \cdot J_m(\alpha \rho) \cdot J_m(\beta \rho) \, d\rho = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2} J_{m+1}^2(\alpha), & \alpha = \beta \end{cases}
\] (A.11)

**Proof of Orthogonality of TE\(_{mn}\) and TE\(_{lk}\) mode:**

\[
\iiint (\epsilon_{mn}^{TE} + \epsilon_{z,mn}^{TE}) \cdot (\epsilon_{lk}^{TE} + \epsilon_{z,lk}^{TE}) \, ds = \int_0^{2\pi} \int_0^a (E_{\rho, mn}^{TE} E_{\rho, lk}^{TE} + E_{\phi, mn}^{TE} E_{\phi, lk}^{TE}) \rho \, d\rho \, d\phi
\] (A.12)

Where \(\epsilon_{mn}^{TE}\) and \(\epsilon_{lk}^{TE}\) are the notations for transvers field. Plug in the Equation (A.1) and Equation (A.2) into Equation (A.12) and separate the integral:

\[
\int_0^{2\pi} \int_0^a E_{\rho, mn}^{TE} E_{\rho, lk}^{TE} \rho \, d\rho \, d\phi
\] (A.13)

\[
\int_0^{2\pi} \int_0^a E_{\phi, mn}^{TE} E_{\phi, lk}^{TE} \rho \, d\rho \, d\phi
\] (A.14)

*From Equation (A.6) to (A.8), the integral on \(\phi\) only has value when \(m = l\), and the integral can be simplified as:*
\[ \int_0^{2\pi} \left[ C_2 \sin(m\phi) - D_2 \cos(m\phi) \right]^2 \, d\phi = \int_0^{2\pi} \left[ C_2^2 \sin^2(m\phi) - 2CD \sin(m\phi) \cos(m\phi) + D_2^2 \cos^2(m\phi) \right] \, d\phi = C_2^2\pi + D_2^2\pi \] (A.15)

\[ \int_0^{2\pi} \left[ C_2 \cos(m\phi) + D_2 \sin(m\phi) \right]^2 \, d\phi = \int_0^{2\pi} \left[ C_2^2 \cos^2(m\phi) + 2CD \sin(m\phi) \cos(m\phi) + D_2^2 \sin^2(m\phi) \right] \, d\phi = C_2^2\pi + D_2^2\pi \] (A.16)

With this orthogonality and the recurrence relations of Bessel function (Equation (A.9) and Equation (A.10)), the integral on \( \rho \) can also be simplified as:

\[ \int_0^a \frac{A_{mn}m}{\varepsilon \rho} J_m(\beta_{p,mn}\rho) \cdot \frac{A_{lk}l}{\varepsilon \rho} J_l(\beta_{p,lk}\rho) \cdot \rho \, d\rho = \int_0^a \frac{A_{mn}\beta_{p,mn}A_{mk}\beta_{p,mk}}{4\varepsilon^2} \left[ J_{m-1}(\beta_{p,mn}\rho) + J_{m+1}(\beta_{p,mn}\rho) \right] \rho \, d\rho \] (A.17)

\[ \int_0^a \frac{A_{mn}\beta_{p,mn}}{\varepsilon} J_m'(\beta_{p,mn}\rho) \cdot \frac{A_{lk}\beta_{p,lk}}{\varepsilon} J_l'(\beta_{p,lk}\rho) \rho \, d\rho = \int_0^a \frac{A_{mn}\beta_{p,mn}A_{mk}\beta_{p,mk}}{4\varepsilon^2} \left[ J_{m-1}(\beta_{p,mn}\rho) - J_{m+1}(\beta_{p,mn}\rho) \right] \rho \, d\rho \] (A.18)

It is easy to see that the coefficient of Equation (A.17) and (A.18) are the same. It is also obvious that \( J_{m-1}(\beta_{p,mn}\rho) \cdot J_{m-1}(\beta_{p,mk}\rho) \) and \( J_{m+1}(\beta_{p,mn}\rho) \cdot J_{m+1}(\beta_{p,mk}\rho) \) components follows the orthogonality of Bessel function and vanish when \( n \neq k \). Also, Equation (A.15) ~ (A.18) show that the coefficient of Equation (A.12) and (A.14) are the same so that the cross product in Equation (A.17) and (A.18) will cancel each other because of the negative sign. So the orthogonality of Equation (A.12) can be proved.

**Proof of Orthogonality of TM_{mn} and TM_{lk} mode:**
\[
\iint (\vec{e}_{mn}^{TM} + \vec{e}_{z,mn}^{TM}) \cdot (\vec{e}_{lk}^{TM} + \vec{e}_{z,lk}^{TM}) \, ds
= \iint \vec{e}_{mn}^{TM} \cdot \vec{e}_{lk}^{TM} + \vec{e}_{z,mn}^{TM} \cdot \vec{e}_{z,lk}^{TM} \, ds
= \int_0^{2\pi} \int_0^{a} (E_{\rho,mn}^{TM} \cdot E_{\rho,lk}^{TM} + E_{\phi,mn}^{TM} \cdot E_{\phi,lk}^{TM}) \rho \, d\rho \, d\phi
+ \int_0^{2\pi} \int_0^{a} (E_{z,mn}^{TM} \cdot E_{z,lk}^{TM}) \rho \, d\rho \, d\phi
\]

(A.19)

where \(\vec{e}_{mn}^{TM}\) and \(\vec{e}_{lk}^{TM}\) are the notations for transverse field. Plug in the Equation A.1 and (A.2) into Equation (A.12) and separate the transverse part in the integral:

\[
\int_0^{2\pi} \int_0^{a} E_{\rho,mn}^{TM} \cdot E_{\rho,lk}^{TM} \cdot \rho \, d\rho \, d\phi
= \int_0^{2\pi} \left[ C \cos(m\phi) + D \sin(m\phi) \right]
\cdot \left[ C \cos(l\phi) + D \sin(l\phi) \right] d\phi
\cdot \int_0^a \frac{B_{mn} \beta_{mn} B_{z,mn}}{\omega \mu \epsilon} J_m(\beta_{\rho,mn} \rho) 
\cdot \frac{B_{lk} \beta_{lk} B_{z,lk}}{\omega \mu \epsilon} J_l(\beta_{\rho,lk} \rho) \rho \, d\rho
\]

\[
\int_0^{2\pi} \int_0^{a} E_{\phi,mn}^{TM} \cdot E_{\phi,lk}^{TM} \cdot \rho \, d\rho \, d\phi
= \int_0^{2\pi} \left[ C \sin(m\phi) - D \cos(m\phi) \right]
\cdot \left[ C \sin(l\phi) - D \cos(l\phi) \right] d\phi
\cdot \int_0^a \frac{B_{mn} m \beta_{z,mn}}{\omega \mu \epsilon \rho} J_m(\beta_{\rho,mn} \rho) 
\cdot \frac{B_{lk} l \beta_{z,lk}}{\omega \mu \epsilon \rho} J_l(\beta_{\rho,lk} \rho) \rho \, d\rho
\]

(A.20)

(A.21)

The same procedure from Equation (A.17) to (A.18) can be repeated in Equation (A.20) and (A.21) and the orthogonality of the transverse part can be proved. The axial part in Equation (A.19) can be expanded as Equation (A.22)
\[
\int_0^{2\pi} \int_0^a E_{z,mn}^{TM} \cdot E_{z,lk}^{TM} \cdot \rho \, d\rho \, d\phi \\
= - \int_0^{2\pi} [C \cos(m\phi) + D \sin(m\phi)] \\
\cdot [C \cos(l\phi) + D \sin(l\phi)] \, d\phi \\
\cdot \int_0^a \frac{B_{mn} \beta_{pmn}^2}{\omega\mu\varepsilon} J_m(\beta_{pmn}\rho) \cdot \frac{B_{lk} \beta_{lk}^2}{\omega\mu\varepsilon} J_l(\beta_{lk}\rho) \rho \, d\rho
\] (A.22)

It is easy to see that both integrals in Equation (A.22) satisfy the orthogonality condition and the axial component is orthogonal.

**Proof of orthogonal property of TE\(_{mn}\) and TM\(_{lk}\) mode:**

\[
\iint (e_{mn}^{TE} + e_{z,mn}^{TE}) \cdot (e_{lk}^{TM} + e_{z,lk}^{TM}) \, ds \\
= \int_0^{2\pi} \int_0^a (E_{\rho,mn}^{TE} \cdot E_{\rho,lk}^{TM} + E_{\phi,mn}^{TE} \cdot E_{\phi,lk}^{TM}) \cdot \rho \, d\rho \, d\phi
\] (A.23)

The Equation A.23 can be expanded as:

\[
\int_0^{2\pi} \int_0^a (E_{\rho,mn}^{TE} \cdot E_{\rho,lk}^{TM}) \cdot \rho \, d\rho \, d\phi \\
= - \int_0^a \frac{A_{mn}}{\varepsilon \rho} J'_m(\beta_{pmn}\rho) \\
\cdot \frac{B_{lk} \beta_{lk}^2}{\omega\mu\varepsilon} J'_l(\beta_{lk}\rho) \rho \, d\rho \\
\cdot \int_0^{2\pi} [C \sin(m\phi) - D \cos(m\phi)] \\
\cdot [C \cos(l\phi) + D \sin(l\phi)] \, d\phi
\] (A.24)

\[
\int_0^{2\pi} \int_0^a (E_{\phi,mn}^{TE} \cdot E_{\phi,lk}^{TM}) \cdot \rho \, d\rho \, d\phi \\
= \int_0^a \frac{A_{mn} \beta_{pmn}}{\varepsilon} J'_m(\beta_{pmn}\rho) \cdot \frac{B_{lk} \beta_{lk}^2}{\omega\mu\varepsilon} J'_l(\beta_{lk}\rho) \rho \, d\rho \\
\cdot \int_0^{2\pi} [C \cos(m\phi) + D \sin(m\phi)] \\
\cdot [C \sin(l\phi) - D \cos(l\phi)] \, d\phi
\] (A.25)

The integral on \(\phi\) in Equation (A.24) and (A.25) can be expanded as:
\[ \int_{0}^{2\pi} [C \sin(m\phi) - D \cos(m\phi)] \cdot [C \cos(l\phi) + D \sin(l\phi)] d\phi \]
\[ = \int_{0}^{2\pi} [C^2 \sin(m\phi) \cos(l\phi)] d\phi \]
\[ + CD \sin(m\phi) \sin(l\phi) \]
\[ - CD \cos(m\phi) \cos(l\phi) - D^2 \cos(m\phi) \sin(l\phi)] d\phi \]

\[ \int_{0}^{2\pi} [C \cos(m\phi) + D \sin(m\phi)] \cdot [C \sin(l\phi) - D \cos(l\phi)] d\phi \]
\[ = \int_{0}^{2\pi} [C^2 \cos(m\phi) \sin(l\phi)] d\phi \]
\[ + CD \sin(m\phi) \sin(l\phi) \]
\[ - CD \cos(m\phi) \cos(l\phi) - D^2 \sin(m\phi) \cos(l\phi)] d\phi \]

In Equation (A.26) and (A.27), the square parts will always be zero and the cross products will be zero if \( m \neq l \). However, if \( m = l \), these cross products will cancel each other. So the product of TE and TM mode will always be zero.
Appendix B: Details of Eigenmodes in Empty Waveguide

Patch Antenna (Linear Polarization): $TE_{mn}$ Modes
Patch Antenna (Linear Polarization): $TM_{mn}$ Modes
Patch Antenna (Circular Polarization): $TE_{mn}$ Modes
Patch Antenna (Circular Polarization): $TM_{mn}$ Modes
Slot Bowtie Dipole Antenna (Linear Polarization): $TE_{mn}$ Modes
Slot Bowtie Dipole Antenna (Linear Polarization): $TM_{mn}$ Modes
Slot Bowtie Dipole Antenna (Circular Polarization): $TE_{mn}$ Modes
Slot Bowtie Dipole Antenna (Circular Polarization): $TM_{mn}$ Modes
Appendix C: Details of Electrical-Field in different cross section

*Right Hand Circular Polarization*

<table>
<thead>
<tr>
<th>Head close to source</th>
<th>Feet close to source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patch</td>
<td>Bowtie Dipole</td>
</tr>
</tbody>
</table>

![Graphs showing electrical-field distribution for different configurations and polarizations.](image-url)
Horizontal Linear Polarization

Head close to source
Patch
Bowtie Dipole

Feet close to source
Patch
Bowtie Dipole
Vertical Linear Polarization

Head close to source
- Patch
- Bowtie Dipole

Feet close to source
- Patch
- Bowtie Dipole
Appendix D: MATLAB Codes

.FETCHING DATA IN HFSS AND CONVERT TO MATRIX

E_Real = importdata('E_Real.csv');
E_xr = E_Real(:,2); E_yr = E_Real(:,3); E_zr = E_Real(:,4);
E_Imag = importdata('E_Imag.csv');
E_xi = E_Imag(:,2); E_yi = E_Imag(:,3); E_zi = E_Imag(:,4);
E_x_compl = E_xr + 1i .* E_xi;
E_y_compl = E_yr + 1i .* E_yi;
E_z_compl = E_zr + 1i .* E_zi;

CONVERT EXTRACTED DATA INTO CORRECT MESH FORMAT

size = 581;
E_x_original = ones(size);
E_y_original = ones(size);
E_z_original = ones(size);
for i = 1:size;
    for j = 1:size;
        E_x_original(i,j) = E_x_compl(1-i+size*j);
        E_y_original(i,j) = E_y_compl(1-i+size*j);
        E_z_original(i,j) = E_z_compl(1-i+size*j);
    end;
end;

Creat mesh for X, Y based on -29cm, 29cm and 1 mm spacing
Meshing the waveguide for Calculation

[X,Y] = meshgrid(-0.29 : 0.001 : 0.29, 0.29 : -0.001 : -0.29);
[PHI,RHO]=cart2pol(X,Y);

Converted Aperture field into cylindrical System
E_rho_aperture = E_x_original .* cos(PHI) + E_y_original .* sin(PHI);
E_phi_aperture = E_x_original .* sin(PHI) + E_y_original .* cos(PHI);
E_z_aperture = E_z_original;
E_mag_aperture = sqrt(abs(E_rho_aperture).^2 + abs(E_phi_aperture).^2 + abs(E_z_aperture).^2);

Plotting Aperture Fields for reference

figure('Units', 'pixels', 'Position', [100 200 600 480]);
pcolor(X,Y,E_mag_aperture);
title_Original = title ('E_Total [HFSS]');
set (title_Original, 'FontSize', 20);
caxis([0, 300]); shading flat; colorbar('FontSize',20);
figure('Units', 'pixels', 'Position', [100 200 600 480]);
pcolor(X,Y,E_rho_aperture);
title_Original = title ('E_Rho [HFSS]');
set (title_Original, 'FontSize', 20);
shading flat; caxis([0, 300]); colorbar('FontSize',20);
figure('Units', 'pixels', 'Position', [100 200 600 480])
pcolor(X,Y,abs(E_phi_aperture));
title_Original = title ('E_P_h_i [HFSS]');
set (title_Original, 'fontsize', 20);
shading flat; caxis([0, 40]); colorbar('FontSize',20);
figure('Units', 'pixels', 'Position', [100 200 600 480])
pcolor(X,Y,abs(E_z_aperture));
title_Original = title ('E_Z [HFSS]');
set (title_Original, 'fontsize', 20);
shading flat; caxis([0, 300]); colorbar('FontSize',20);

%%%%%%%%%%%%%%%%%%%% Convert the Nan Data outside the boundary for integration purpose
for i = 1:size;
    for j = 1:size;
        if (~isnan(E_x_compl(1-i+size*j)))
            E_x_original(i,j) = E_x_compl(1-i+size*j);
        else
            E_x_original(i,j) = 0;
        end
        if (~isnan(E_y_compl(1-i+size*j)))
            E_y_original(i,j) = E_y_compl(1-i+size*j);
        else
            E_y_original(i,j) = 0;
        end
        if (~isnan(E_z_compl(1-i+size*j)))
            E_z_original(i,j) = E_z_compl(1-i+size*j);
        else
            E_z_original(i,j) = 0;
        end
    end
end

E_rho_aperture = E_x_original .* cos(PHI) + E_y_original .* sin(PHI);
E_phi_aperture = -E_x_original .* sin(PHI) + E_y_original .* cos(PHI);
E_z_aperture = E_z_original;

%%%%%%%%%%%%%%%%    Calculation of Excitation Coefficient %%%%%%%%%%%%%%%%
%%%%%%
%%%%%%
for m = 0:4
    for n = 1:5
        Beta = omega/c;
        Beta_rho_TE = dbessel_root(n,m+1)/a; %Solution of bessel function;
        if (Beta >= Beta_rho_TE)
            Betaz_TE = sqrt(Beta^2 - Beta_rho_TE^2); %Propagation constant in Z Direction
        else
            Betaz_TE = -1i * sqrt(Beta_rho_TE^2 - Beta^2);
        end
        TE_C = ones(6); TE_D = ones(6); TM_C = ones(6); TM_D = ones(6);
    end
end
if (Beta >= Beta_rho_TM)
    Betaz_TM = sqrt(Beta^2 - Beta_rho_TM^2);
else
    Betaz_TM = -1i * sqrt(Beta_rho_TM^2 - Beta^2);
end

% Eigen Function of TEmn mode: Ez is zero in this case;
E_cos_rho_TE_D = m./(epsilon.*RHO).*besselj(m, Beta_rho_TE.*RHO).*cos(m.*PHI);
E_sin_rho_TE_C = m./(epsilon.*RHO).*besselj(m, Beta_rho_TE.*RHO).*sin(m.*PHI);
E_cos_phi_TE_C = Beta_rho_TE./epsilon.*dbesselj(m,Beta_rho_TE.*RHO).*cos(m.*PHI);
E_sin_phi_TE_D = Beta_rho_TE./epsilon.*dbesselj(m,Beta_rho_TE.*RHO).*sin(m.*PHI);

% Eigen Function of TMmn mode;
E_cos_rho_TM_C = - Beta_rho_TM .* Betaz_TM ./ (omega .* mu .* epsilon) .* d0; (m, Beta_rho_TM .* RHO) .* cos(m .*PHI);
E_sin_rho_TM_D = - Beta_rho_TM .* Betaz_TM ./ (omega .* mu .* epsilon) .* dbesselj(m, Beta_rho_TE.*RHO) .* sin(m .*PHI);
E_cos_phi_TM_D = - m .* Betaz_TM ./ (omega .* mu .* epsilon .* RHO) .* cos(m.*PHI);
E_sin_phi_TM_C = - m .* Betaz_TM ./ (omega .* mu .* epsilon .* RHO) .* sin(m.*PHI);
E_cos_z_TM_C = - 1i .* Beta_rho_TE^2 ./ (omega .* mu .* epsilon) .* dbesselj(m, Beta_rho_TE.*RHO) .* cos(m.*PHI);
E_sin_z_TM_D = - 1i .* Beta_rho_TE^2 ./ (omega .* mu .* epsilon) .* dbesselj(m, Beta_rho_TE.*RHO) .* sin(m.*PHI);

 fprintf(' Applying the boundary condition

for i = 1:size;
    for j = 1:size;
        if (RHO(i,j) >= a)
            E_cos_rho_TE_D(i,j) = NaN;
            E_sin_rho_TE_C(i,j) = NaN;
            E_cos_phi_TE_C(i,j) = NaN;
            E_sin_phi_TE_D(i,j) = NaN;
            E_cos_rho_TM_C(i,j) = NaN;
            E_sin_rho_TM_D(i,j) = NaN;
            E_cos_phi_TM_D(i,j) = NaN;
            E_sin_phi_TM_C(i,j) = NaN;
            E_cos_z_TM_C(i,j) = NaN;
            E_sin_z_TM_D(i,j) = NaN;
        end
    end
end

 fprintf(' Calculation based on the derived equation

% TEmn mode:

Numerator_C_TE = @(phi,rho)TE_Numerator_C(phi, rho, Beta_rho_TE,...
E_phi_aperture, E_rho_aperture, m, epsilon);
Int_Numerator_C_TE = quad2d(Numerator_C_TE, 0, 2*pi, 0.001, 0.289);

Numerator_D_TE = @(phi,rho)TE_Numerator_D(phi, rho, Beta_rho_TE,...
E_phi_aperture, E_rho_aperture, m, epsilon);
Int_Numerator_D_TE = quad2d(Numerator_D_TE, 0, 2*pi, 0.001, 0.289);

Denominator_C_TE = @(phi,rho)TE_Denominator_C(phi, rho, Beta_rho_TE, m,
epsilon );
Int_Denominator_C_TE = quad2d(Denominator_C_TE, 0, 2*pi, 0, 0.29);

Denominator_D_TE = @(phi,rho)TE_Denominator_D(phi, rho, Beta_rho_TE, m,
epsilon );
Int_Denominator_D_TE = quad2d(Denominator_D_TE, 0, 2*pi, 0, 0.29);

% TMmn mode:

Numerator_C_TM = @(phi,rho)TM_Numerator_C{ phi, rho, Beta_rho_TM, ...
\( \text{E}_\phi\text{_aperture, E}_r\text{ho}_\text{aperture, E}_z\text{_aperture, m, \epsilon, \omega, \mu, Betaz}_\text{TM}; \)

\[
\text{Int\_Numerator}_C\text{_TM} = \text{quad2d}(\text{Numerator}_C\text{_TM, 0}, 2\pi, 0.001, 0.289);
\]

\[
\text{Numerator}_D\text{_TM} = @(\text{phi, rho})\text{TM\_Numerator}_D(\text{ phi, rho, Beta}_\text{rho}\_\text{TM}, \ldots \text{ E}_\phi\text{_aperture, E}_r\text{ho}_\text{aperture, E}_z\text{_aperture, m, \epsilon, \omega, \mu, Betaz}_\text{TM});
\]

\[
\text{Int\_Numerator}_D\text{_TM} = \text{quad2d}(\text{Denominator}_C\text{_TM, 0}, 2\pi, 0.001, 0.289);
\]

\[
\text{Denominator}_C\text{_TM} = @(\text{phi, rho})\text{TM\_Denominator}_C(\text{ phi, rho, Beta}_\text{rho}\_\text{TM}, \ldots \text{ E}_\phi\text{_aperture, E}_r\text{ho}_\text{aperture, E}_z\text{_aperture, m, \epsilon, \omega, \mu, Betaz}_\text{TM});
\]

\[
\text{Int\_Denominator}_C\text{_TM} = \text{quad2d}(\text{Denominator}_C\text{_TM, 0}, 2\pi, 0, 0.29);
\]

\[
\text{Denominator}_D\text{_TM} = @(\text{phi, rho})\text{TM\_Denominator}_D(\text{ phi, rho, Beta}_\text{rho}\_\text{TM}, \ldots \text{ E}_\phi\text{_aperture, E}_r\text{ho}_\text{aperture, E}_z\text{_aperture, m, \epsilon, \omega, \mu, Betaz}_\text{TM});
\]

\[
\text{Int\_Denominator}_D\text{_TM} = \text{quad2d}(\text{Denominator}_D\text{_TM, 0}, 2\pi, 0, 0.29);
\]

\[
%\text{TEmn mode:}
\]

\[
\text{Coefficient}_C\text{\_TE} = \text{Int\_Numerator}_C\text{\_TE} / \text{Int\_Denominator}_C\text{\_TE};
\]

\[
\text{Coefficient}_D\text{\_TE} = \text{Int\_Numerator}_D\text{\_TE} / \text{Int\_Denominator}_D\text{\_TE};
\]

\[
\text{if} \ (m == 0) \ 
\text{Coefficient}_D\text{\_TE} = 0; 
\text{end};
\]

\[
\text{E}_\text{rho}\_\text{TE} = \text{E}_\text{sin}\_\text{rho}\_\text{TE}\_C \ast \text{Coefficient}_C\text{\_TE} + \text{E}_\text{cos}\_\text{rho}\_\text{TE}\_D \ast \text{Coefficient}_D\text{\_TE};
\]

\[
\text{E}_\text{phi}\_\text{TE} = \text{E}_\text{cos}\_\text{phi}\_\text{TE}\_C \ast \text{Coefficient}_C\text{\_TE} + \text{E}_\text{sin}\_\text{phi}\_\text{TE}\_D \ast \text{Coefficient}_D\text{\_TE};
\]

\[
\text{TE}\_C(n,m+1) = \text{Coefficient}_C\text{\_TE};\text{ TE}\_D(n,m+1) = \text{Coefficient}_D\text{\_TE};
\]

\[
\text{EField\_Calculated\_rho}\_\text{TE} = \text{EField\_Calculated\_rho}\_\text{TE} + \text{E}_\text{rho}\_\text{TE};
\]

\[
\text{EField\_Calculated\_phi}\_\text{TE} = \text{EField\_Calculated\_phi}\_\text{TE} + \text{E}_\text{phi}\_\text{TE};
\]

\[
%\text{TMmn mode:}
\]

\[
\text{Coefficient}_C\text{\_TM} = \text{Int\_Numerator}_C\text{\_TM} / \text{Int\_Denominator}_C\text{\_TM};
\]

\[
\text{Coefficient}_D\text{\_TM} = \text{Int\_Numerator}_D\text{\_TM} / \text{Int\_Denominator}_D\text{\_TM};
\]

\[
\text{if} \ (m == 0) \ 
\text{Coefficient}_D\text{\_TM} = 0; 
\text{end};
\]

\[
\text{E}_\text{rho}\_\text{TM} = \text{E}_\text{cos}\_\text{rho}\_\text{TM}\_C \ast \text{Coefficient}_C\text{\_TM} + \text{E}_\text{sin}\_\text{rho}\_\text{TM}\_D \ast \text{Coefficient}_D\text{\_TM};
\]

\[
\text{E}_\text{phi}\_\text{TM} = \text{E}_\text{sin}\_\text{phi}\_\text{TM}\_C \ast \text{Coefficient}_C\text{\_TM} + \text{E}_\text{cos}\_\text{phi}\_\text{TM}\_D \ast \text{Coefficient}_D\text{\_TM};
\]

\[
\text{E}_z\_\text{TM} = \text{E}_\text{cos}\_\text{z}\_\text{TM}\_C \ast \text{Coefficient}_C\text{\_TM} + \text{E}_\text{sin}\_\text{z}\_\text{TM}\_D \ast \text{Coefficient}_D\text{\_TM};
\]

\[
\text{TM}\_C(n,m+1) = \text{Coefficient}_C\text{\_TM};\text{ TM}\_D(n,m+1) = \text{Coefficient}_D\text{\_TM};
\]

\[
\text{EField\_Calculated\_rho}\_\text{TM} = \text{EField\_Calculated\_rho}\_\text{TM} + \text{E}_\text{rho}\_\text{TM};
\]

\[
\text{EField\_Calculated\_phi}\_\text{TM} = \text{EField\_Calculated\_phi}\_\text{TM} + \text{E}_\text{phi}\_\text{TM};
\]

\[
\text{EField\_Calculated\_z}\_\text{TM} = \text{EField\_Calculated\_z}\_\text{TM} + \text{E}_z\_\text{TM};
\]

\[
%\text{Plot Degenerate Mode}
\]

\[
\text{figure(}'Units', 'pixels', 'Position', [100 100 600 480])
\text{pcolor}(X,Y,\text{abs}(\text{sqrt}(\text{abs}(E}_\text{rho}\_\text{TE}^2 + \text{abs}(E}_\text{phi}\_\text{TE}^2)));\text{Eigen\_m}\_\text{TE} = \text{int2str}(m);\text{Eigen\_n}\_\text{TE} = \text{int2str}(n);\text{title}
\]

\[
\text{molecule - strcat('TE', ', ', ', ', ', ', Eigen\_m}\_\text{TE});\text{title\_TE} = \text{title} (\text{title\_TE});\text{set}
\]

\[
\text{title\_TE, 'fontsize', 20)};\text{shading flat; colorbar('FontSize',20);}
\]

\[
\text{figure(}'Units', 'pixels', 'Position', [100 100 600 480])
\text{pcolor}(X,Y,\text{abs}(\text{sqrt}(\text{abs}(E}_\text{rho}\_\text{TM}^2 + \text{abs}(E}_\text{phi}\_\text{TM}^2)));\text{Eigen\_m}\_\text{TM} = \text{int2str}(m);\text{Eigen\_n}\_\text{TM} = \text{int2str}(n);\text{title}
\]

\[
\text{molecule - strcat('TM', ', ', ', ', ', ', Eigen\_m}\_\text{TM});\text{title\_TM} = \text{title} (\text{title\_TM});\]
set (title_TM, 'fontsize', 20); shading flat; colorbar('FontSize',20);
end

%Total Field:
EField_Calculated_rho = EField_Calculated_rho_TE + EField_Calculated_rho_TM;
EField_Calculated_phi = EField_Calculated_phi_TE + EField_Calculated_phi_TM;
EField_Calculated_z = EField_Calculated_z_TM;
EField_Total = sqrt(abs(EField_Calculated_rho).^2 + abs(EField_Calculated_phi).^2 +
abs(EField_Calculated_z).^2);

figure('Units', 'pixels', 'Position', [100 300 600 480]);
hold on;

%Calculate the total field values

title_Calculated = title ('E_Total [Eigen Function]');
set (title_Calculated, 'fontsize', 20);
caxis([0, 300]);
shading flat; colorbar('FontSize',20);

figure('Units', 'pixels', 'Position', [100 300 600 480])
hold on;

%caxis([0, 300]);

%Calculate the error values

%%%%%%%%%%%%%%%%% Error Calculation

%%%%%%%%%%%%%%%%% For singularity, Nan Value needs to be converted first 

Err_rho = E_rho_aperture - EField_Calculated_rho;
Err_phi = E_phi_aperture - EField_Calculated_phi;
Err_z = E_z_aperture - EField_Calculated_z;
Err_total = abs(Err_rho).^2 + abs(Err_phi).^2 + abs(Err_z).^2;

Error_Fun_rho = @(phi,rho)Error_Integration( phi, rho, abs(Err_rho) );
Int_Err_rho = quad2d(Error_Fun_rho, 0, 2*pi, 0, 0.29);
Error_Fun_phi = @(phi,rho)Error_Integration( phi, rho, abs(Err_phi) );
int_Err_phi = quad2d(Error_Fun_phi, 0, 2*pi, 0, 0.29);
Error_Fun_z = @(phi,rho)Error_Integration( phi, rho, abs(Err_z) );
int_Err_z = quad2d(Error_Fun_z, 0, 2*pi, 0, 0.29);
int_Err_total = Int_Err_rho + Int_Err_phi + Int_Err_z;

Error_calculated_rho = @(phi,rho)Error_Integration( phi, rho, abs(EField_Calculated_rho) );
int_calculated_rho = quad2d(Error_calculated_rho, 0, 2*pi, 0, 0.29);
Error_calculated_phi = @(phi,rho)Error_Integration( phi, rho, abs(EField_Calculated_phi) );
int_calculated_phi = quad2d(Error_calculated_phi, 0, 2*pi, 0, 0.29);
Error_calculated_z = @(phi,rho)Error_Integration( phi, rho, abs(EField_Calculated_z) );
int_calculated_z = quad2d(Error_calculated_z, 0, 2*pi, 0, 0.29);
int_calculated_total = int_calculated_rho + int_calculated_phi + int_calculated_z;

Err_Nor_rho = Int_Err_rho ./ int_calculated_rho;
Err_Nor_phi = Int_Err_phi ./ int_calculated_phi;
Err_Nor_z = Int_Err_z ./ int_calculated_z;
Err_Nor_total = Int_Err_total ./ int_calculated_total;

Error_aperture_rho = @(phi,rho)Error_Integration( phi, rho, abs(E_rho_aperture) );
Int_aperture_rho = quad2d(Error_aperture_rho, 0, 2*pi, 0, 0.29);
Error_aperture_phi = @(phi,rho)Error_Integration( phi, rho, abs(E_phi_aperture) );
Int_aperture_phi = quad2d(Error_aperture_phi, 0, 2*pi, 0, 0.29);
Error_aperture_z = @(phi,rho)Error_Integration( phi, rho, abs(E_z_aperture) );
Int_aperture_z = quad2d(Error_aperture_z, 0, 2*pi, 0, 0.29);
Int_aperture_total = Int_aperture_rho + Int_aperture_phi + Int_aperture_z;

Err_Nor_rho_ap = Int_Err_rho ./ Int_aperture_rho;
Err_Nor_phi_ap = Int_Err_phi ./ Int_aperture_phi;
Err_Nor_z_ap = Int_Err_z ./ Int_aperture_z;
Err_Nor_total_ap = Int_Err_total ./ Int_aperture_total;