A SELF-EXCITED FREQUENCY MULTIPLIER

FOR THE

MILLIMETER WAVE-LENGTH RANGE

DISSERTATION

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1. INTRODUCTION

1.1 The Importance of Millimeter Wave-length Research. There has been a gradually increasing interest in applications of millimeter waves in recent years as the lower frequency bands have become crowded by radio, television, radar, and many other systems. It has also been found that some of the physical properties of matter could be investigated very effectively by means of millimeter wave-length spectroscopy, and this is now a rapidly growing field of research. Although the use of wave lengths shorter than ten millimeters is not yet common in military applications, it seems likely that systems will be devised which take advantage of the small size of directional antennas and other special characteristics of the millimeter region.

It is therefore not surprising that very intensive research programs have been pursued at many industrial and university laboratories throughout the world to advance millimeter wave-length technology. The experimental problems, however, in the study of millimeter equipment are difficult ones, and consequently there is strong incentive to look for new techniques and new devices. This has led to the development of a wide variety of experimental sources of power and also of circuit components, although very few are yet available commercially.

The tube which is the subject of this report is a somewhat different application of the familiar technique of frequency multiplication. It is essentially a retarding-field oscillator in which a second

cavity, tuned to a harmonic, has been added. Under the normal operating conditions of the retarding-field oscillator, this tube will therefore simultaneously deliver power at a fundamental frequency and also at a harmonic frequency. At anode currents too low to sustain fundamental oscillations, a signal of fundamental frequency may be injected, and harmonic output can be obtained just as with a conventional frequency multiplier.

The remainder of this introductory section will be devoted to a summary of the present state of the art of generating millimeter signals. The main purpose is to present the characteristics of existing oscillators in such a way as to permit comparison with the retarding-field harmonic generator.

1.2 Methods of Generating Millimeter Waves. An obvious approach to the problem of generating millimeter waves is to scale tubes which were satisfactory at lower frequencies. This has led to the development of millimeter wave-length reflex klystrons, magnetrons, traveling wave tubes, and retarding-field oscillators. Another method which has been very widely used is to employ the non-linear characteristics of a silicon crystal or vacuum tube to generate harmonics. Some progress has also been made in developing oscillators using new physical principles. These include the Cerenkov oscillator and the Doppler-effect tubes. However, since these latter tubes do not ordinarily produce coherent oscillations, they will not be further discussed in this report.

Magnetrons in the millimeter range have been used almost exclusively as transmitting tubes because of their capabilities as high power
pulsed oscillators. The most remarkable of these tubes at present are the ones which have been developed at Columbia University\textsuperscript{2} to operate at wave-lengths as short as 3.3 millimeters with peak output power of the order of ten kilowatts. Reflex klystrons and retarding-field oscillators, on the other hand, are useful for local operator service in receivers, for bench oscillators, and for other applications requiring low-power continuous operation. They have the advantage of being tunable both mechanically and electrically. Both have been operated in experimental versions to wave lengths as short as five millimeters\textsuperscript{3}.

Backward-wave oscillators have been developed also primarily for low power cw operation and are characterized by very wide electronic tuning range. They differ from the previous types of tubes in that they employ a distributed traveling-wave structure rather than resonant cavities. Such structures are generally believed to be usable at shorter wave-lengths than cavities; and, in fact, useful amounts of power have been obtained at wave lengths somewhat below three millimeters.

Among the earliest harmonic generators were vacuum triodes, but they have not so far been useful in the millimeter range. The first microwave frequency-multiplier tubes were two-cavity klystrons in which the low frequency signal was introduced into the buncher cavity, and harmonic power was extracted from a small cavity placed at the end of the drift space\textsuperscript{4}. In most of these tubes the tenth harmonic was employed. An outstanding advantage of this harmonic generator is the fre-

quency stability which may be realized if a chain of frequency-stabilized multipliers are driven by a quartz crystal oscillator.

More recently, magnetrons have been used as sources of harmonics at Columbia University\(^5\). The primary application was in molecular spectroscopy. These magnetrons were operated at K-band with half microsecond pulses and a pulse repetition rate of a thousand cycles per second. The fundamental pulse power output was at least 20 kilowatts and produced harmonics which were usable as far as the tenth. The peak power output of the tenth harmonic was of the order of 100 microwatts. The variation of power output with the order of the harmonic is shown in Fig. 1. They have also operated their 3.3 mm magnetrons as harmonic generators, and, with 6 kilowatts peak power, have produced 900 milliwatts at 1.1 mm. Although these harmonics have very small average power, they still produce quite usable signal-to-noise ratios in systems with proper band width.

Harmonic generators, particularly silicon crystals, have probably been the most commonly used sources of millimeter wave-length power. Such harmonic sources, however, have been used primarily for laboratory purposes and in most cases have delivered very small power. This method was used at Bell Laboratories in the early 40's to produce 6 mm output for test purposes. It has also been developed to a very high degree in various laboratories for use in microwave spectroscopy. Probably the leader in this field has been Duke University, where an announcement\(^6\) has recently been made of the precision measurement of the frequency of

\(^5\)Klein, Loubser, Nethercot and Townes, Rev. Sci. Inst. 23, 78 (1953)

\(^6\)Burras and Gordy, Phys. Rev. 93, 897 (1954)
Fig. 1. Typical harmonic output from K-band magnetrons at Columbia Radiation Laboratory.
a molecular transition at a wave length of .771 mm. In their work they have used the Raytheon 2K33 reflex klystron as a K-band source and have generated the harmonics in specially mounted silicon crystals. The power level decreases with the order of the harmonics, but the output is still usable as far as the 16th harmonic - the one used in the experiment mentioned above. The output power of the 2K33 is approximately 30 milliwatts, and the relative powers obtainable in the harmonics vary tremendously from crystal to crystal and also depend rather critically on circuit arrangements and tuning. In Fig. 2 is shown typical variation of power output with harmonic number for such a system. Again in this system, the relatively low frequency 2K33 was preferred because it facilitated frequency stabilization.

1.3 Capabilities and Limitations of Various Oscillators as Sources of Millimeterwave Power. Several basic difficulties are encountered with any type of vacuum tube oscillator when one attempts to scale the oscillator to higher frequencies. Many of the recent developments of new types of oscillators have come about in attempts to minimize one or more of these difficulties.

The first problem is that of mechanical construction of small tube parts. It becomes increasingly more difficult to maintain the relative tolerances in the dimensions of tube parts as the parts become smaller, and it is even more difficult to maintain close assembly tolerances. To give an order of magnitude of the sizes of parts involved, a resonant cavity for a reflex klystron or a retarding-field oscillator for the 6 mm wave-length range might have a diameter of about 100 mils and a
Fig. 2. Harmonic output obtained with crystal multipliers at Johns Hopkins University.
height of 25 mils. Although the construction problems in scaling such
a cavity even to 3 mm are serious, special techniques such as hobbing
greatly facilitate the production of small cavities. Another way to
avoid some of the difficulties in the construction of cavities is to use
some type of distributed structure, as in the backward-wave oscillator.
These distributed structures are, however, somewhat complex in themselves,
and the construction problems have by no means been entirely avoided.

A second and perhaps more serious difficulty is that of providing
a suitable electron beam. The electron gun becomes smaller as the tube
is scaled, and it becomes more and more difficult to obtain the neces­sary electron emission. This difficulty is accentuated by the fact that
as the tube is scaled to small size, circuit losses increase and conse­quently the amount of current necessary to sustain oscillations increases.

A third problem is that of dissipating the power of the electron
beam in smaller and smaller tube parts. When the linear dimensions are
reduced by a factor of two, the surface areas and hence the heat radia­tion are reduced by a factor of four. Heat conduction is reduced by a
factor of two at the same time. Since the input power is at least as
high in the scaled model, it is apparent that the temperatures of the
parts will be much higher than in the larger version and that deforma­tion or melting of the tube details imposes a limit on the ultimate
oscillator frequency. One way of avoiding the heat dissipation limita­tion, at least to some extent, is to use pulsed operation. This has
been done in millimeter wave magnetrons. Pulsed operation is not suit­able, however, for some applications; and in any case it does not help
very much in reflex type tubes because breakdown voltage limitations in close-spaced electron guns do not permit high pulse voltages.

These and other considerations have been used by Elliot\textsuperscript{7} and Carter\textsuperscript{8} to predict ultimate frequency limits for the reflex klystron and the retarding-field oscillator respectively in the neighborhood of 3 mm. The frequency limit for low-field magnetrons has not been clearly established. Experimental magnetrons have been operated at wave-lengths shorter than 3 mm and it appears likely that some further scaling will be possible. The ultimate wave-length limitation on backward-wave oscillators is even more vague. Experimental tubes have operated in the neighborhood of 3 mm and considerable effort is being devoted to scaling them still further.

The chief advantage of the reflex type tube is probably the simplicity of the tube and of its associated equipment. The reflex klystron has moderate mechanical tunability and rather limited electronic tunability; while the retarding-field oscillator has a very wide mechanical tuning range and comparable electronic tuning. The principal advantage of the magnetron is its high peak power, but it suffers from the disadvantages of requiring a magnetic field, being ordinarily untuned, having poor frequency stability, and in addition having a broad bandwidth because of its pulsed operation. The advantages of the backward-wave oscillator are wide electronic tuning range and high ultimate frequency. The disadvantages of the backward-wave oscillator lie mainly in its complexity and the requirement of a special magnetic field.

\textsuperscript{7} Elliot, R. S., \textit{J. Appl. Phys.} 23, 812 (1952)

\textsuperscript{8} Carter, C. J., "Limitations on the Maximum Frequency of the Retarding-field Oscillator", WADC Technical Report, The Ohio State University (1958)
1.4 Possible Capabilities and Limitations of Frequency Multipliers.

The same considerations that applied to other vacuum tube oscillators also apply to vacuum tube multipliers, but in some respects the multipliers have an advantage. The construction of a frequency-multiplier tube requires only that the harmonic cavity and harmonic coupling system be small in size and have close tolerances. The input cavity and gun parts may be large. This is more of an advantage than is apparent at first sight because with hobbing methods, small cavities can be made fairly easily.

A very important advantage of the frequency-multiplier type of operation is that the starting current limitation of an oscillator is no longer present. It is therefore possible to avoid the electron emission density limitation encountered in millimeter oscillators. At the same time, the possibility of avoiding the large starting current of millimeter oscillators results in a reduction in power which must be dissipated. It may also be possible, in the harmonic generator, to dissipate the beam power on the relatively large size parts associated with the fundamental frequency operation rather than on the delicate high-frequency parts.

From the above considerations it can be seen that the difficulties that limit the frequency of operation of an oscillator apply mainly to the fundamental frequency of a frequency multiplier and not to the harmonic frequency. It might therefore be expected that a tube could be designed for a fundamental frequency as high as is feasible for an oscillator of a given type, and that several times this frequency could be obtained by means of harmonics. Additional advantages of the
frequency multiplier include the fact that it is adaptable to frequency-stabilized systems, and that frequency measurements can be made at the fundamental frequency rather than at the harmonic frequency, where measuring devices are difficult to construct.

The possible advantages of frequency multipliers lead to a consideration of various methods of harmonic production. One such method is to apply a high-frequency signal to a non-linear impedance, as is done in the case of the crystal multiplier. It would be desirable to use some non-linear characteristic of an electron beam rather than a crystal, however, because of the low power level at which crystals burn out. The other method is to employ a higher harmonic component in the beam current of an oscillator to drive a harmonic circuit.

Ways in which the retarding-field oscillator may be used to deliver harmonic power will be discussed in more detail in the next section.

2.0 SELF-EXCITED FREQUENCY MULTIPLIERS EMPLOYING THE RETARDING-FIELD OSCILLATOR

2.1 Advantages and Difficulties in the Use of a Single-cavity System. Probably the most obvious method for obtaining harmonic output from the retarding-field oscillator is to make use of the non-sinusoidal character of the induced current. This would require that the harmonic load impedance, as transformed to the interaction gap, have a reasonably high value. This situation does not ordinarily exist in a reentrant cavity since the frequencies of higher-mode resonances are not multiples of the fundamental frequency. However, if two separate output circuits are used for the fundamental and the harmonic, it may be possible by
means of external tuners to bring a higher mode resonance into harmonic relationship with the fundamental.

Assuming that such a circuit could be constructed, the next important question is whether the harmonic content of the induced current is sufficient to provide useful harmonic output. This requires a detailed calculation of the instantaneous induced current and its Fourier analysis (see Section 4.10); but the essential results are summarized in Fig. 3, which shows the harmonic content of the induced current, expressed as percent of the fundamental, for various values of signal level. The parameter representing signal level is K, which is defined as the ratio of the peak ac gap voltage to the dc gap voltage. In self-excited operation, it is very difficult to attain a signal level greater than $K = 0.25$, which corresponds roughly to 3% overall efficiency of the basic oscillator. Under this condition the second harmonic is about 15% and the higher harmonics are below 5% of the fundamental current.

An attempt was made by C. J. Carter to detect second harmonic output with the tube shown in Fig. 4. The fundamental frequency was about 2500 mc. Harmonic output, if any, was too weak to be detected.

In view of the negative results of this experiment and the low harmonic content of the induced current, no further work was done with this type of tube.

A closely related method has more recently been used in France by Bernier and Leboutet. They obtained harmonic output from a high power reflex klystron having a fundamental wave-length of about 4 cm.

Bernier and Leboutet, Private Communications.
Fig. 3. Harmonic content of the total induced current.
Fig. 4. The single-cavity harmonic generator.
2.2 The Two-cavity Method for Utilizing the Electron Bunches at the End of their Transits. A detailed theoretical study of the retarding-field oscillator suggests another approach. The induced current has the rather small harmonic content mentioned in the last section because of a smoothing effect resulting from the fact that electrons in the gap at any time are moving with a wide continuous range of velocities. Nevertheless, the electron convection current at the anode is strongly bunched and hence contains high harmonics of considerable magnitude. The harmonic content of this current, as calculated in Section 4.2, is indicated in Fig. 5.

In order to utilize this strongly-bunched current without encountering the smoothing effect of the rest of the beam, it is necessary to place a second interaction gap and resonant cavity in the region where the electrons are collected. An arrangement for doing this is shown in Fig. 6.

The space available for a second cavity is very limited unless marked changes are made in the electron gun design. The space is sufficient, however, if the fourth or higher harmonic is used. The radial decoupling line and second cavity serve two purposes. The first is to provide the proper degree of coupling between the harmonic cavity and the second is to avoid the necessity of introducing a large wave guide between the main cavity and the electron gun.

The details of the design of the self-excited frequency multiplier must necessarily be determined empirically since the geometry is not susceptible to mathematical analysis. In general, the fundamental section is made to conform as nearly as possible to designs of the basic
Fig. 5. Harmonic content of the anode conduction current at optimum bunching for each harmonic and at $K = 0.29$. 
Fig. 6. Self-excited frequency multiplier.
oscillator which give highest efficiency. In this way, as high a degree of bunching as possible is obtained. The way in which the external load in the fundamental wave-guide is transformed to the interaction gap is adjusted by a stub tuner to increase further the degree of bunching. The second interaction gap must be sufficiently wide to accept substantially all of the returning electron beam but at the same time must be narrow enough to provide a good beam-coupling coefficient. In the simple design shown, these two requirements are not compatible and a compromise must be made on an experimental basis. Suitable radial grid wires projecting part way across a wide gap might be expected to improve the beam-coupling coefficient. They have not been tried, however, because a basic design objective has been to keep the parts simple enough to permit scaling to as high a frequency as possible.

3.0 THEORY OF THE PLANAR RETARDING-FIELD OSCILLATOR.

As was pointed out earlier, the self-excited frequency multiplier is essentially a modification of the retarding-field oscillator, and consequently the operating characteristics of the two devices are closely related. It is not possible, in fact, to predict the performance of the multiplier without the use of a fairly complete and detailed theory of the retarding-field oscillator. For that reason Section 3.0 and much of Section 4.0 are devoted to theoretical discussions of various aspects of retarding-field oscillator operation.

The geometry of practical retarding-field oscillators is too complex to permit a simple analysis, but it has been found that the results calculated for an idealized oscillator agree qualitatively with the experi-
mental performance of actual oscillators. The assumptions made for these idealized tubes are discussed in more detail in the later sections, but the main assumption is that of a uniform field between the anode and repeller. This greatly simplifies the mathematical problem.

There is some arbitrariness in the application of such a planar theory to experimental tubes because both the geometry of the interaction gap and also space charge make the field non-uniform. One approach is to make the gap width, d, arbitrary in the equations for the planar tube. The usefulness of the theory would then depend upon how consistently it describes actual tubes when an appropriate "effective" gap spacing is used. As an illustration, the following are typical experimental values for the 2 - 4 cm retarding-field oscillator:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam potential</td>
<td>800 v.</td>
</tr>
<tr>
<td>Beam current</td>
<td>52 ma.</td>
</tr>
<tr>
<td>Extinction tuning range, 1st mode</td>
<td>77 mc.</td>
</tr>
<tr>
<td>Starting current, 1st mode</td>
<td>12 ma.</td>
</tr>
<tr>
<td>Starting current, 2nd mode</td>
<td>6 ma.</td>
</tr>
</tbody>
</table>

Although it was not measured in this particular test, the extinction tuning range is usually about twice as great in the second mode as in the first. The values calculated from the planar theory with d=0.012 inch are:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extinction tuning range, 1st mode</td>
<td>77 mc.</td>
</tr>
<tr>
<td>Extinction tuning range, 2nd mode</td>
<td>160 mc.</td>
</tr>
<tr>
<td>Starting current, 1st mode</td>
<td>14.8 ma.</td>
</tr>
<tr>
<td>Starting current, 2nd mode</td>
<td>7.2 ma</td>
</tr>
</tbody>
</table>

These calculations agree fairly well with the experimental data, particularly in the relative values between modes. Of more importance, however, is the fact that the theory indicates qualitatively the way in
which the tube performance depends upon beam current, local conditions and other operating conditions. Other examples of correlation between the planar theory and tube operation will be discussed as they arise in later sections.

In Section 3.1 a brief description of the operating mechanism of the retarding-field oscillator is presented as an introduction to the small-signal theory of Section 3.2 and the large signal theory of Sections 3.3 and 3.4.

3.1 Qualitative Description of the Operation of the Retarding-field Oscillator. The retarding-field oscillator is in some respects similar to the reflex klystron. It differs from the reflex klystron, however, in that the repeller is a part of the resonant circuit as shown in Fig. 7. The second grid of the reflex klystron has been eliminated, and consequently the ac and dc fields are no longer separated. The repeller is maintained at a fixed negative dc potential of sufficient magnitude to prevent electrons from striking it, at least in the absence of oscillations. When oscillations are present in the resonant cavity, an ac potential between repeller and anode is superimposed on the dc potential. The electrons enter this combined field with a velocity modulation, bunching, and delivery of ac power to the circuit are all carried out in the same interaction space, rather than in separate regions as in the reflex klystron. One can still visualize the velocity modulation as taking place during the initial part of the electron transit, followed by bunching while the electrons move slowly near the place where they reverse direction, and finally by delivery of ac power as the electrons are collected. However, since this picture does not define clearly the regions in which these operations take place,
Fig. 7. The retarding-field oscillator with repeller output coupling.
it cannot lead to quantitative results.

A very extensive discussion of the general operation of the retarding-field oscillator, including its tuning characteristics, was given by J. J. Ebers\textsuperscript{10}. In this paper he compared and contrasted the essential mechanism in the operation of the RFO with that of the Barkhausen-Kurz oscillator and the reflex klystron. He also obtained, analytically, an expression for the small-signal electronic efficiency. From this he deduced an equation for the starting current in terms of the operating conditions of the tube. These results were then extended to large signal operation by graphical means. The methods used by Ebers did not, however, lend themselves to the computation of electronic tuning or output loading effects since only the real part of the electronic admittance was obtained. The present work presents a somewhat more general theory of the retarding-field oscillator, in that reactive effects are included.

A somewhat similar retarding-field tube was developed in Germany about 1937, but was coaxial in geometry and operated at a much lower frequency. As a first approximation to the theory of this tube, Kleinsteuber and others studied the planar retarding-field mechanism and presented a general analysis of it. Reference will be made to these sources for some of the theoretical results used in later sections.

3.2 Small-Signal Theory of the Retarding-Field Oscillator. The actual geometry employed in practical retarding-field oscillators will be idealized here in order to clarify the fundamental electronic mechanism. It will be assumed that the electrodes are parallel planes, and fringing fields will be neglected. Both the ac and the dc electric
fields between the electrodes will be assumed uniform, which implies the neglect of space charge. The anode is maintained positive at a constant potential $V_0$ with respect to the cathode (see Fig. 8). Since thermal

Fig. 8. The planar Retarding-field Oscillator

velocity spread is neglected, this causes all electrons to enter the interaction space between the anode and repeller with the same velocity. Between anode and repeller there is a combined ac and dc field represented by

$$E = \frac{1}{\sigma} \left( V_i + V_m \cos \omega t \right)$$  \hspace{1cm} (1)

The $\text{ac}$ part of the field is produced by means of a resonant circuit or cavity connected between the anode and repeller, and the $\text{dc}$ part by the repeller bias, $-V_1$. The positive direction for displacement, velocity, force, and electric field will be from anode to repeller (upward in Fig. 8). $V_1$ is sufficiently large to return all the electrons to the anode except for large $V_m$.

Probably the simplest method for deriving an expression for the electronic admittance is to compute the instantaneous current induced in
the circuit connected to the interaction gap by the modulated electron beam. This current, if considered to be induced in the upper electrode, may be written by means of Ramo's equation \(^{11}\)

\[
\frac{di}{dt} = E, u_E dq
\]  

(2)

where \(dq\) is the charge inducing the current. \(E_1\) is a fictitious field which would be present at the location of \(dq\) if the upper electrode were at unit potential with respect to the lower, and \(u_E\) is the component of the velocity of the charge in the direction of \(E_1\). For the geometry of Fig. 8, \(E = \frac{1}{d}\), and \(u_E\) is simply the velocity of the electrons which entered the gap at \(t = t_1\). \(I_0\) is the dc beam current.

Thus

\[
\frac{di}{dt} = \frac{I_0}{d} u(t, t_1) dt
\]  

(3)

In order to integrate \(u(t, t_1)\) to obtain the effect of all the electrons in the gap it is necessary to derive an equation for \(u\) and to establish the limits on the integration. To do this requires that the equation of motion of the electrons be solved for the displacement and velocity. The equation of motion is

\[
\frac{du}{dt} = -\frac{e}{m} V_1 (1 + K \cos \Theta)
\]  

(4)

subject to the conditions: \(u = u_0\) and \(x = 0\) for \(t = t_1\). \(K\) is defined as \(V_m/V_1\) and is thus a measure of signal level. It is convenient for numerical computations to define some new dimensionless variables as

\(^{11}\)Ramo, S., Proc. I.R.E. 27, 584 (1939)
The reciprocal of the width of the interaction gap occurs very frequently in later formulas and will be denoted by

$$M \equiv \frac{u_o}{\omega d}$$

(6)

It is also convenient to lump some of the constants of Eqs. (3) and (4) together as

$$\frac{eV_t}{\omega ma u_o} \equiv \frac{2}{\beta_o}$$

(7)

The constant $\beta_o$ defined in this way will be shown later to be the $dc$ transit angle for the electrons. In terms of the quantities thus defined, Eq. (4) becomes

$$\frac{dV}{d\theta} = -\frac{2}{\beta_o} (1 + K \cos \theta); \quad V = 1 \quad \text{for} \quad \theta = \theta_i$$

(8)

This may be integrated at once, and

$$V = 1 - \frac{2}{\beta_o} \left[ \theta - \theta_i + K (\sin \theta - \sin \theta_i) \right]$$

(9)

Since $V = \frac{d\xi}{d\theta}$, a second integration yields

$$\xi = \theta - \theta_i - \frac{2}{\beta_o} \left\{ \left( \frac{\theta - \theta_i}{2} \right)^2 + K \left[ \cos \theta - \cos \theta_i - (\theta - \theta_i) \sin \theta_i \right] \right\}$$

(10)

Eqs. (9) and (10) specify the velocity and position of an electron at a time $t = \frac{\theta}{\omega}$, assuming that it entered the gap at $t_1 = \frac{\theta_1}{\omega}$.

The transit angle for an electron leaving at the phase angle $\theta$ is given by setting $\xi = 0$ in Eq. (10). Using the notation $\beta = \theta_2 - \theta_1$.
for the transit angle, where \( \theta_2 \) is the exit angle, we may write the relation as

\[
0 = \beta - \frac{2}{\beta_o} \left( \frac{\beta^2}{2} + K \left[ \cos (\theta_2 - \beta) - \cos \theta_2 - \beta \sin (\theta_2 - \beta) \right] \right)
\] (11)

This equation must be solved for \( \beta \) as a function of \( \theta_2, \beta_o, \) and \( K. \) Although it is obviously not possible to write an explicit solution in general, it is possible for the small-signal case. The procedure is the common one of assuming the transit angle to consist of a dc part plus a small varying part. Thus

\[
\beta = \beta_o + \delta
\] (12)

The maximum value of \( \delta \) will be small compared with \( \beta_o \) for \( K \ll 1, \) and Eq. (11) may be written approximately as

\[
0 = -\delta - \frac{2K}{\beta_o} \left[ \cos (\theta_2 - \beta_o) - \cos \theta_2 - \beta_o \sin (\theta_2 - \beta_o) \right]
\] (13)

and

\[
\beta = \beta_o + \frac{2K}{\beta_o} \left[ \cos (\theta_2 - \beta_o) - \cos \theta_2 - \beta_o \sin (\theta_2 - \beta_o) \right]
\] (14)

The most serious errors in this approximation come from dropping \( \delta \) in the arguments of the sine and cosine functions. For example, although the error in the algebraic terms may not be serious for \( K = 0.1, \) it would correspond to an error of about 12° in the trigonometric terms and could be troublesome.

Returning now to Eq. (3), we may introduce the dimensionless variables and write
\[ i(\theta) = I_o M \int_{\theta-\beta}^{\theta} \mathcal{V}(\theta, \theta_i) \, d\theta_i \]  

(15)

With the limits chosen in this way, the contributions of all the electrons present in the interaction gap at the time corresponding to \( \theta \) have been included.

When the dimensionless velocity from Eq. (9) is substituted into Eq. (15), the instantaneous current becomes

\[ i(\theta) = I_o M \int_{\theta-\beta}^{\theta} \{1 - \frac{2}{\beta_0} [\theta - \theta_i + k (\sin \theta - \sin \theta_i)]\} \, d\theta_i \]  

(16)

Integration gives

\[ i(\theta) = I_o M \frac{1}{\beta_0} \left\{ \rho_0 e^{-\rho^2} - 2k \left[ \rho \sin \theta + \cos \theta - \cos (\theta - \rho) \right] \right\} \]  

(17)

Use of the approximate expression for \( \beta \) from Eq. (14) leads to

\[ i(\theta) = \frac{I_o M}{\rho_0} \left\{ -2k \left[ \cos \theta - \cos (\theta - \rho_0) + \rho_0 \sin (\theta - \rho) \right] - 2k \rho_0 \sin \theta + \cos \theta - \cos (\theta - \rho_0) \right\} \]  

(18)

This equation may be simplified by combination of terms and the use of appropriate trigonometric identities to yield

\[ i(\theta) = -\sqrt{\nu} I_o \frac{M^2 \rho_0}{2 \nu_o} \left[ 2 \rho_0 \cos \frac{\rho_0}{2} - 4 \sin \frac{\rho_0}{2} \right] \sin (\theta - \frac{\rho_0}{2}) \]  

(19)

or, in complex notation,

\[ i(\theta) = \Re \left[ I_o e^{i \theta} \right] = \Re \left[ \sqrt{\nu} I_o \frac{M^2 \rho_0}{2 \nu_o} \left( 2 \rho_0 \cos \frac{\rho_0}{2} - 4 \sin \frac{\rho_0}{2} \right) e^{i(\theta - \frac{\rho_0}{2})} \right] \]  

(20)
By definition,

$$Y_e = -\frac{I_e}{V_m}$$

(21)

and therefore

$$Y_e = -G_0 \frac{M^2 \beta_o}{2} \left( 2 \beta_o \cos \frac{\beta_o}{2} - 4 \sin \frac{\beta_o}{2} \right) e^{j \frac{\beta_o}{2}}$$

(22)

where $G_0 = \frac{I_0}{V_o}$.

The components of $Y_e$ may be readily obtained from Eq. (22) in the following form

$$G_e = -\frac{1}{2} G_0 M^2 \left( \beta_o \sin \beta_o - 2 + 2 \cos \beta_o \right)$$

(24)

and

$$B_e = -\frac{1}{2} G_0 M^2 \left( \beta_o \cos \beta_o + \beta_o - 2 \sin \beta_o \right)$$

(25)

The variations of $G_e$ and $B_e$ with $\beta_o$ are shown in Fig. 9, and a polar plot of $Y_e$ is sketched in Fig. 10. This polar plot may be used in the same way as the familiar admittance spiral for reflex klystrons although its appearance is quite different. The representation of the load admittance on the same plot may be clarified by a brief consideration of the equivalent circuit of the oscillator as referred to the interaction gap. The subscript c refers to quantities related to the resonant cavity and the coupling system between the cavity and the load. The subscript L refers to those introduced by the final load. All of the elements to the right of a-b in Fig. 11 may be combined to give an equivalent $C$, $L$, and $G$. 

28
Fig. 9. Electronic conductance and susceptance in units of $\frac{1}{2} G_0 w^2$ plotted as functions of transit angle.
Fig. 10. Polar plot of the electronic admittance of the planar retarding-field oscillator. Plotted in units of $\frac{1}{2} G_0 M^2$. 
These produce an admittance

\[ Y = G + j \left( \omega C - \frac{1}{\omega L} \right) = G + j \omega C \left( 1 - \frac{1}{\omega^2 LC} \right) \]  \hspace{1cm} (26)

If we denote the resonant angular frequency by \( \omega_0 \) and an angular frequency near resonance by \( \omega = \omega_0 + \Delta \omega \),

\[ Y = G + j \left( \frac{\omega - \omega_0}{\omega} (\omega + \omega_0) \right) = G + j \omega \frac{\Delta \omega (2 \omega_0 + \Delta \omega)}{\omega_0 + \Delta \omega} \]  \hspace{1cm} (27)

or

\[ Y \approx G + j 2 \omega \Delta \omega \]  \hspace{1cm} (28)

Near resonance \( G \) is practically constant, and the locus of the total admittance to the right of \( a-b \) is a vertical straight line. Since under oscillating conditions, the load admittance must be equal and opposite to the electronic admittance, the locus of load admittance has been drawn on the left of the origin as the line \( a-b \) in Fig. 10. The conditions for minimum starting current will occur when this line is tangent to the admittance spiral. This implies that \( G_0 \) be negative and have an extreme value, or
Carrying out the differentiation of $G_e$ as given in Eq. (24) leads to

\[ \frac{d G_e}{d \beta_o} = 0 \]  

Equation (29)

or

\[ \sin \beta_o + \beta_o \cos \beta_o - 2 \sin \beta_o = 0 \]  

Equation (30)

The roots of Eq. (31) satisfying the condition $G_e < 0$ are the following:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.725</td>
</tr>
<tr>
<td>2</td>
<td>14.07</td>
</tr>
<tr>
<td>3</td>
<td>20.37</td>
</tr>
<tr>
<td>4</td>
<td>26.67</td>
</tr>
<tr>
<td>5</td>
<td>32.96</td>
</tr>
</tbody>
</table>

Here $n$ is called the mode number and represents the nearest number of whole cycles that the electrons remain in the interaction space. It may be noted that

\[ \beta_o \simeq \left( n + \frac{1}{4} \right) \pi, \quad n = 1, 2, \ldots \]  

Equation (32)

The values of $\beta_o$ tabulated above do not agree exactly with those obtained by Ebers because of an error in his function $f(\beta_o)$. A corrected form of $f(\beta_o)$ will now be derived in connection with the calculation of the small signal efficiency.
The ac power delivered by the electron stream is divided between the cavity and circuit losses and the useful load. It may be expressed as

\[ P = \frac{1}{2} V_m^2 (G_L + G_c), \]  

(33)

where \( G_L \) is the load conductance and \( G_c \) is the circuit conductance, both referred to the interaction gap. The input power to the beam, assuming \( K \) to be very small is

\[ P_{in} = V_o I_{os} \]  

(34)

\( I_{os} \) being the oscillator starting current. The electronic conversion efficiency is then

\[ \eta_e = \frac{V_m^2 (G_L + G_c)}{2 V_o I_{os}} = \frac{K^2 V_o^2 (G_L + G_c)}{2 V_o I_{os}} \]  

(35)

Eq. (7) can be written also in a form which does not involve \( u_0 \).

\[ \frac{e V_i^2}{\omega^2 d^2 m} = \frac{8 V_o}{\beta_o^2} \]  

(36)

If Eq. (36) is used to eliminate the repeller voltage in Eq. (35), and the sum of \( G_L \) and \( G_c \) is replaced by \(-G_o\) from Eq. (24), the electronic efficiency becomes

\[ \eta_e = 4 K^2 \frac{1}{\beta_o^2} \left[ \beta_o \sin \beta_o - 2 + 2 \cos \beta_o \right] \]  

(37)

\[ \approx \frac{4 K^2}{\beta_o} \sin \beta_o, \text{ for large } \beta_o \]  

(38)
This equation is true only for very small values of $K$ and only for $\beta_0$ very near an optimum transit angle. Comparison of Eq. (37) with Ebers' equation for electronic efficiency shows that the correct form for his $f(\beta_0)$ is

$$f(\beta_0) = \frac{1}{\beta_0^2} \left[ \beta_0 \sin \beta_0 - 2 \cos \beta_0 \right].$$

(39)

This result may also be verified by Ebers' method, although the labor required is considerable.

An equation for the starting current of the oscillator may be obtained immediately from the requirement that $G_L + G_c = -G_e$. The starting current is thus

$$I_{os} = \frac{(G_L + G_c) 2 V_0}{M^2 \left( \beta_0 \sin \beta_0 - 2 \cos \beta_0 \right)}.$$  \hspace{1cm} (40)

with the numerical values of the constants inserted, the results in terms of gap spacing and frequency are:

$$I_{os} = 9.60 \times 10^{-13} \omega^2 d^2 (G_L + G_c), \quad \text{1st mode}$$

$$I_{os} = 4.67 \times 10^{-13} \omega^2 d^2 (G_L + G_c), \quad \text{2nd mode}$$

$$I_{os} = 3.08 \times 10^{-13} \omega^2 d^2 (G_L + G_c), \quad \text{3rd mode}$$

(41)

Since the factor $\omega^2 d^2$ is constant with linear scaling, the variation of starting current with scaling factor will depend upon how the load is coupled to the cavity. If a constant load is heavily coupled, the starting current would vary nearly as the square root of the frequency. The variation of starting current with tuning was discussed by Ebers\textsuperscript{12}.

The electronic tuning range between extinction points may be investi-
gated by means of this small signal theory. Referring again to line a-b on Fig. 10, the difference in $B$ between the two intersection points on the spiral is related to the tuning range according to equation

$$\Delta f = \frac{\Delta B}{4\pi C}$$

where $C$ is the capacitance of the interaction gap and may be determined approximately by field plotting methods. The tuning range may be expressed also in a way which does not involve the capacitance explicitly. The loaded $Q$ is defined by

$$Q_L = \frac{\omega C}{G_L + G_c}$$

and hence

$$\Delta f = \frac{\Delta B}{2 Q_L (G_L + G_c)} f$$

Consideration of Eqs. (43) and (44) and the admittance spiral suggests various ways in which the electronic tuning range might be increased. For example, the loaded $Q$ might be increased by a decrease of $(G_L + G_c)$ without a change in $C$ or $I_0$. This might be done by decreasing circuit losses or decreasing the coupling to the load or both. According to Eq. (41) this would decrease the starting current and hence would move line a-b of Fig. 10 to the right. It is apparent that the improvement in $\Delta f$ would be very slight except for beam currents close to the starting current. This is shown more directly in Fig. 12.
Fig. 12. Variation of electronic tuning range with starting current.
where the tuning range relative to that of a lossless oscillator is plotted against the ratio of starting current to beam current. Since it is relatively difficult to make marked increases in $Q_L$ and since oscillators are commonly operated with $I_o$ several times $I_{os}$, this method of increasing the range is not very promising. For example, if $I_o = 2.5 I_{os}$ and the loaded $Q$ were doubled, the electronic tuning range would be increased by about 8%.

Another possibility is to decrease $Q_L$ by decreasing the gap capacitance, while keeping the total conductance and the beam current constant. In this case, it is apparent from Eq. 41 that the starting current would not be changed. Consequently the electronic tuning range would be increased as $\frac{1}{C}$. There are physical limitations on how far the decrease in $C$ can be carried, particularly in tubes which depend upon variation of capacitance for mechanical tuning.

If both the capacitance and the total conductance were decreased while the loaded $Q$ remained constant, the fractional increases in $\Delta f$ discussed above would be multiplied. Even in this way, the improvement practically attainable might be rather small.

Another, and much more effective way, to increase the electronic tuning range is to increase the beam current. This follows from the fact that the electronic admittance, according to Eq. 22, is proportional to the beam conductance. The theoretical variation of $\Delta f$ with current, as plotted in Fig. 13, has also been verified experimentally.
Fig. 13. Variation of electronic tuning range with beam current.
3.3 Large-signal Theory of the Retarding-field Oscillator. The foregoing discussion was concerned mainly with the starting conditions for the operation of the retarding-field oscillator, and consequently many important aspects of the tube performance were not considered. Among the important large-signal characteristics of the oscillator are the variation of efficiency with signal level, the variation of output power and oscillation frequency with changes in loading, the variation of output power and frequency with repeller voltage, and changes in current wave forms with signal level. All of these characteristics are important in predicting and interpreting the behavior of the tube as a frequency multiplier.

A rather large amount of work has been done on the large-signal theory of the retarding-field oscillator, the principal investigators being Kleinsteuber\textsuperscript{13}, Ebers\textsuperscript{14} and Neubauer\textsuperscript{15}. A summary of the methods used in their investigations and of the results they obtained will be presented in order to facilitate the discussion of the harmonic generator.

Kleinsteuber used the induced-current approach in his calculation of electronic admittance. His starting point was essentially Eq. 15 of Section 3.2. The use of this equation is complicated by the fact that the limits on the integral are implicit multiple-valued functions of the integration variable. Two methods of solution were used by Kleinsteuber. One was a series expansion in powers of K, and the other was an exact numerical evaluation of the integral after changes in

\textsuperscript{13}Kleinsteuber, W., Hochfreq. u Elektroak, 57, 1 (1941)
\textsuperscript{15}Neubauer, Doctoral Dissertation, The Ohio State University 1954.
variables.

The power-series solution involves very cumbersome coefficients and only two terms were retained. The first term leads to an electronic admittance identical with that obtained in the previous section. The second term gives an approximate large-signal correction, and calculated values of electronic admittance for various values of dc transit angle and for values of $K$ as high as 1.4 were presented. Questions of the convergence of the series and of the accuracy of the results obtained with two terms were not settled. The optimum dc transit angle was calculated for small-signal conditions and the apparently erroneous value of $\beta_0 = 7.56$ was obtained. The electronic efficiency of the oscillator for this transit angle was calculated and is shown in Fig. 14.

This series method was justified by Kleinsteuber mainly on the basis of agreement with the results of numerical and graphical evaluations of the induced current in his second method. These numerical results are difficult to obtain, however, because the transformations necessary to simplify the limits make the integrals very complex. His exact calculations were limited mainly to a calculation of electronic efficiency for $\beta_0 = 7.56$. He indicated in a general way how the reactive component might be calculated numerically, but did not include either details or results.

The large-signal calculations made by both Ebers and Neubauer were based on a calculation of the average energy given to the ac field by the electrons in the beam. The procedure is to subtract the kinetic energy of each electron at the end of its transit from the energy it had
Fig. 14. Electronic conversion efficiency.
When it entered the interaction space. If this difference is divided by the initial kinetic energy and the result averaged over all the electrons during one cycle of operation, the following simple relation results

\[ \eta = 1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{u^2}{u_s^2} \, d\theta, \]  

(45)

In carrying out the calculations indicated in this equation, one must compute first the exit velocity, \( u \), as a function of both the entrance angle and the level of operation of the tube. This must be done numerically except in the limiting case of small signals.

Although this method has the great advantage of simplicity, it requires great care if accurate results are to be obtained. The main source of difficulty lies in the fact that the velocities of the electrons are changing rapidly as they leave the gap, and hence a small error in the calculated transit angle may make a relatively large error in the velocity and a still larger one in the square of the velocity.

The calculation of electronic conversion efficiency was carried out by Ebers for a dc transit angle of 7.60 and for values of \( K \) up to about 1.0. Neubauer used a \( \beta_0 \) of 7.725 but did not obtain results for \( K \) greater than 0.5. Reactive effects were not calculated in either case, and operation in higher modes was not treated in detail.

The numerical results obtained in these three papers appear to be in disagreement. The discrepancies in the optimum transit angle are only about two percent, but those in efficiency are in some instances about twenty percent.
In view of this disagreement and the relative lack of information on reactive effects, it appears to be desirable to make at least a limited number of independent calculations in order to coordinate and supplement the previous work.

In the following calculations the electronic admittance is obtained by considering the power given by the electron beam to the associated circuit. However, instead of using the change in kinetic energy as Ebers did, the energy exchange is obtained by integrating the force on an electron throughout its trajectory. This is, of course, only a formal difference, but it permits an easy extension to include reactive effects. This method has been commonly used for other oscillators,\textsuperscript{16} and was used by J. L. Moll\textsuperscript{17} in calculating the small-signal admittance of the retarding-field oscillator.

The way in which the conductance and susceptance expressions are derived can be seen from a brief consideration of phasor power relations in ordinary circuit analysis. The complex power, in phasor notation is given by

\[ P = I \overline{V} \]  

(46)

where \( \overline{V} \) is the complex conjugate of the voltage drop across the circuit elements in question. Also

\[ I = Y \overline{V}, \]  

(47)

\textsuperscript{16}Mueller and Rostas, Helv. Phys. Acta. 18, 435 (1940)

\textsuperscript{17}Moll, J. L., Unpublished Notes.
and therefore

\[ P = Y V \overline{V} = Y V^2 \]  \hspace{1cm} (48)

Writing out the components of \( P \) and \( Y \), one obtains

\[ P + jQ = (G + jB)V^2 \]  \hspace{1cm} (49)

Thus the electronic conductance and susceptance can be calculated from the real and reactive powers delivered to the electron beam by the field in the interaction gap. The notation and basic assumptions are the same as in Section 3.2. The first step in the derivation of an expression for the real power is to integrate the force on an electron with respect to distance along its trajectory in the interaction gap. The resulting equation for the work done on the electron is

\[ W = \int_{t_1}^{t_2} f \, d\kappa = \int_{t_1}^{t_2} f \, u \, dt \]  \hspace{1cm} (50)

The expressions for the force and the velocity are substituted from Eqs. (1) and (9) to give

\[ W = -\int_{\Theta_i}^{\Theta_f} \left( \frac{eV}{\omega d} \right) \left( 1 + K \cos \Theta \right) u_o \left( 1 - \frac{2}{\beta_0} \Theta + \frac{2}{\beta_0^2} \Theta \right) e^{2k \sin \Theta} + \frac{2k}{\beta} \sin \Theta \right) d\Theta \]  \hspace{1cm} (51)

The integral of the constant part of the force is zero since the electron returns to its starting point at \( \Theta = \Theta_2 \). The integral may therefore be simplified to

\[ W = -\int_{\Theta_i}^{\Theta_f} \left( \frac{eV}{\omega d} u_o k \right) \cos \Theta \left( 1 - \frac{2}{\beta_0} \Theta + \frac{2}{\beta_0^2} \Theta \right) e^{2k \sin \Theta} + \frac{2k}{\beta} \sin \Theta \right) d\Theta \]  \hspace{1cm} (52)
If \( n \) electrons flow into the gap per second, the average power delivered to the beam is

\[
P_e = \frac{1}{2\pi} \int_{0}^{2\pi} n W d\theta,
\]

The rate at which charge enters the gap in this case is constant and equal to the magnitude of the dc beam current.

\[
ne = I_0
\]

The equation for the power input to the beam is therefore

\[
P_e = -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{I_0 V_k U_e}{\omega} \cos\theta \left( 1 - \frac{2}{\beta_0} \theta_e + \frac{2}{\beta_0} \theta_e^2 - \frac{2k}{\beta_0} \sin\theta_e + \frac{2k}{\beta_0} \sin\theta_e^2 \right) d\theta d\theta_e
\]

The reactive power may be obtained by a similar calculation except that a fictitious quadrature component of force is used with the actual velocity. Thus

\[
Q_e = \frac{1}{2\pi} \int_{0}^{2\pi} \int \phi^* u d\phi d\theta
\]

Here also the constant term in the force expression integrates to zero and may be omitted.

The phasor power may therefore be written as

\[
P_e - jQ_e = -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{I_0 V_k U_e}{\omega} e^{j\theta} \left( 1 - \frac{2}{\beta_0} \theta_e + \frac{2}{\beta_0} \theta_e^2 - \frac{2k}{\beta_0} \sin\theta_e + \frac{2k}{\beta_0} \sin\theta_e^2 \right) d\theta d\theta_e
\]

The constant in the integral may be expressed in terms of the beam voltage and the dc transit angle in the following way.
An alternative expression for the phasor power is

\[
\mathcal{P}_e \cdot Q_e = \frac{2kV_0I_o}{\rho_o \pi} \int_{\frac{2\pi}{\rho_o}} j^\theta \left( \frac{2}{\rho_o} \theta - \frac{2k}{\rho_o} \sin \theta \right) d\theta \tag{59}
\]

The first integration may be readily performed with the result:

\[
\mathcal{P}_e \cdot Q_e = \frac{2kV_0I_o}{\rho_o \pi} \int_{\frac{2\pi}{\rho_o}} \left\{ j^\theta \left[ j - j^2 \frac{2}{\rho_o} \theta + \frac{2k}{\rho_o} \sin \theta \right] + e^{j\theta} \left[ \frac{k}{2\rho_o} \sin \theta - j \frac{k}{\rho_o} \sin \theta \right] \right\} d\theta \tag{60}
\]

In this integral the terms in \( e^{j\theta_1} \) and \( e^{j2\theta_1} \) integrate to zero and may be omitted. Also the expression may be simplified by use of

\[
\Theta_2 - \Theta_1 = \rho_o + \delta \tag{61}
\]

With these changes,

\[
\mathcal{P}_e \cdot Q_e = \frac{2kV_0I_o}{\rho_o \pi} \int_{\frac{2\pi}{\rho_o}} \left\{ j^\theta \left[ j - j^2 \frac{2}{\rho_o} \theta + \frac{2k}{\rho_o} \sin \theta \right] + e^{j\theta} \left[ \frac{k}{2\rho_o} \sin \theta - j \frac{k}{\rho_o} \sin \theta \right] \right\} d\theta \tag{62}
\]
This equation is exact, but the integral is difficult to evaluate because \( \Theta_2 \) and \( \delta \) are not expressible in a simple way in terms of \( \Theta_1 \).

If the signal level is assumed to be very small so that \( \delta \) can be expressed by Eq. (13), then \( P_e \) and \( Q_e \) reduce to the equations

\[
\begin{align*}
P_e &= -\frac{4K^2V_0I_0}{\rho^2_0} \left[ 2\cos^2 \theta_0 - 2 + \rho^2_0 \sin^2 \theta_0 \right] \tag{63}
\end{align*}
\]

and

\[
\begin{align*}
Q_e &= -\frac{4K^2V_0I_0}{\rho^2_0} \left[ 2\sin^2 \rho_0 + \rho_0^2 + \rho_0 \cos \rho_0 \right] \tag{64}
\end{align*}
\]

These give \( G_e \) and \( B_e \) in the same form as in Section 3.2, and constitute a partial check on the accuracy of Eq. (62).

It is of course, possible to evaluate the integral in Eq. (62) numerically, and this computation will now be outlined. The integral may be split into real and imaginary parts such that

\[
\begin{align*}
P_e &= -\frac{2K^2V_0I_0}{\rho_0 \pi} \int_0^{\pi} \left\{ -\frac{2K}{\rho_0} \sin^2 \Theta_1 - \frac{2}{\rho_0} \cos \Theta_2 - \sin \Theta_2 - \frac{2\delta}{\rho_0} \sin \Theta_2 \\
&\quad + \frac{2K}{\rho_0} \sin \Theta_1 \sin \Theta_2 + \frac{K}{2\rho_0} \cos 2 \Theta_2 \right\} d \Theta_1 \tag{65}
\end{align*}
\]

and

\[
\begin{align*}
Q_e &= \frac{2K^2V_0I_0}{\rho_0 \pi} \int_0^{\pi} \left\{ \cos \Theta_2 + \frac{2\delta}{\rho_0} \cos \Theta_2 - \frac{2K}{\rho_0} \sin \Theta_1 \cos \Theta_2 \\
&\quad - \frac{2\delta}{\rho_0} \sin \Theta_2 + \frac{K}{2\rho_0} \sin 2 \Theta_2 - \frac{K\delta}{\rho_0} \right\} d \Theta_1 \tag{66}
\end{align*}
\]
The integrands are next plotted point by point as functions of \( \theta_1 \), making use of a table of \( \beta \) as a function \( \theta_1 \). The integrals are then evaluated by means of a polar planimeter.

The results of these calculations are plotted as efficiency rather than power to permit easier comparison with other oscillators. In Fig. 15 the calculated efficiency is plotted as a function of signal level for \( \beta_0 = 7.725 \). The small-signal efficiency curve is plotted also for comparison purposes.

The components of the electronic admittance are related to the phasor power by

\[
G_e + jB_e = \frac{2}{\sqrt{m}} (P_e + jQ_e)
\]

This equation may also be written in terms of \( K \) the beam input power, and \( M \) as defined in Eq. (6). Thus,

\[
G_e + jB_e = \frac{2G_o}{V_o I_o} \left( \frac{M \beta_o}{4K} \right)^2 (P_e + jQ_e)
\]

or

\[
G_e + jB_e = \frac{1}{2} G_o M^2 \frac{\beta_o^2}{4K^2 V_o I_o} (P_e + jQ_e)
\]

In this form it may be readily compared with the small signal results. The calculated theoretical variation of \( G_e \) and \( B_e \) with \( K \) are shown in Fig. 16. It is of interest to note that the variation of \( B_e \) with signal level is very slight.

The variation of of electronic admittance with signal level may also be presented in polar form, as in Fig. 17. Only the first mode is shown. A load admittance locus is shown as a vertical straight line.
Fig. 15. Efficiency as a function of signal level. The small-signal curve is shown dashed.
Fig. 16. Components of the electronic admittance for $\sigma_o = 7.725$ and increasing $K$. In units of $\frac{1}{2} G_0 M^2$. 
Fig. 17. First-mode electronic admittance of the planar retarding-field oscillator. Expressed in units of $\frac{1}{2} G_0 m^2$. 
through the $K = 1.0$ point. Ideally, if a tube having this load were
suddenly turned on, the signal level would rise until an equilibrium
condition was reached at $K = 1.0$. This would correspond to an elec-
tronic conversion efficiency of 18.5%, according to Fig. 15.

Although the electronic admittance calculated in this way is precise,
it is not convenient for some purposes because it is not expressed in
analytical form. It is therefore desirable to supplement these calcu-
lations with an approximate analytical formulation, which follows in the
next section.

3.4 Approximate Large-signal Theory for Higher-mode Operation.
The well-known first-order theory of the reflex klystron\textsuperscript{18} is an
approximation valid for long transit angles only. The operation it pre-
dicts for the first mode is seriously in error. Nevertheless, this
first-order theory is very useful as a qualitative guide to klystron
behavior and has been very extensively used.

A similar theory of the large-signal behavior of the retarding-
field oscillator is possible. The integrals for the real and reactive
power given in Eqs. (65) and (66) can be evaluated quite readily under
the assumption that

$$\beta_o >> 1$$  \hspace{1cm} (71)

These integrals then become

$$P_e = - \frac{2K}{\beta_o} \int_{0}^{2\pi} \sin \theta \, d\theta,$$

and
\[- Q_e = - \frac{2K V_0 I_e}{\beta_o \pi} \int_0^{2\pi} (\cos \theta - k) d\theta, \quad (73)\]

Under the same assumption
\[\theta_2 = \theta + \beta = \theta + \beta_o + 2K \sin \theta, \quad (74)\]

Substitution of Eq. (74) into Eq. (73) permits the equations to be written as
\[P_e = - \frac{2K V_0 I_o}{\beta_o \pi} \int_0^{2\pi} \sin (\theta + \beta_o + 2K \sin \theta) d\theta, \quad (75)\]
and
\[- Q_e = - \frac{2K V_0 I_o}{\beta_o \pi} \int_0^{2\pi} \left[-k + \cos (\theta + \beta_o + 2K \sin \theta)\right] d\theta, \quad (76)\]

These two integrals can be reduced to standard forms by means of trigonometric identities, with the following results:
\[P_e = - \frac{2K V_0 I_o}{\beta_o \pi} \left[2\pi \sin \beta_o J_1(2K)\right], \quad (77)\]
and
\[Q_e = - \frac{2K V_0 I_o}{\beta_o \pi} \left[2\pi K + 2\pi \cos \beta_o J_1(2K)\right], \quad (78)\]

After simplification and regrouping of the constants, the equations may be written in a form which permits easy comparison with the small-signal results of Eqs. (63) and (64).
\[P_e = - \frac{4K^2 V_0 I_e}{\beta_o^2} \left[\frac{J_1(2K)}{K} \beta_o \sin \beta_o\right], \quad (79)\]
and

\[ Q_e = - \frac{4k^2V_e I_0}{\beta_o^2} \left[ \beta_o + \frac{J_i(2k)}{K} \beta_o \cos \beta_o \right] \]  

(80)

For small signals these expressions reduce to the high mode case of Section 3.2 since

\[ \lim_{K \to 0} \frac{J_i(2k)}{K} = 1 \]  

(81)

The electronic admittance as a function of both \( \beta_o \) and \( K \) is shown in Fig. 18 for the first mode and in Fig. 19 for the second modes. Just as in the corresponding theory of the reflex klystron, there are marked departures from exact results such as those of Fig. 17. Neverthe less, the qualitative agreement is good enough to permit the approximate theory to be used for descriptive purposes.

For example, the mode shape for a typical load admittance may be calculated from Fig. 18. It is plotted as a function of repeller voltage in Fig. 20 since this is the way it is commonly observed experimentally. An alternative presentation is shown in Fig. 21, where the abscissa is relative frequency. From this curve it can be seen that the half-power band width is about 57% of the extinction band width.

The difference in shape between the curves of Figs. 20 and 21 is, of course, the result of a non-linear relationship between repeller voltage and oscillator frequency. This relationship, which is of great importance in frequency modulation applications, is displayed in Fig. 22. The half-power tuning range is indicated by the limits a and b. Within this half-power range, the curve differs only slightly from the tuning characteristic of the reflex klystron.
Fig. 18. First mode electronic admittance spirals.
Fig. 19. Electronic admittance spirals for the second mode.
Fig. 20. Repeller mode shape calculated for typical operating conditions.
Fig. 21. Power output as a function of frequency for electronic tuning. Arbitrary frequency scale.
Fig. 22. Oscillator frequency as a function of repeller voltage.
It is of interest to compare the first order theories of the retarding-field oscillator and the reflex klystron. This comparison is facilitated if it is noted that the signal level parameter $K$, after a little rearrangement of Eq. (7) is given by

$$2K = \frac{\beta_0 V_m M}{2 V_o}$$

(82)

Thus $2K$ is identical with the bunching parameter $X$ used by Pierce and Shepherd. Also the quantity $M$, which was previously defined as the reciprocal of the dimensionless gap width, may be formally identified with the beam-coupling coefficient of klystron theory. In this case, however, it has not been found possible to show that the two quantities have the same physical significance.

The electronic conductance of the retarding-field oscillator, except for differences in notation, is almost identical in form with that of the reflex klystron. In the same notation the two equations are:

for the retarding-field oscillator,

$$G_e = -G_o M^2 \frac{J_1(2K)}{2K} \beta_0 \sin \beta_o$$

(83)

and for the reflex klystron,

$$G_e = +G_o M^2 \frac{J_1(X)}{X} \beta_o \sin \beta_o$$

(84)

The only differences are the difference in optimum dc transit angle resulting from the difference in sign and the difference in the significance of $M$. If these differences are accounted for, that part of the very extensive theory of the reflex klystron which is based on the electronic conductance may be applied also to the retarding-field oscillator.
Perhaps it should be pointed out that in the case of the reflex klystron the value of $M$ for an ideal gap is 1.0, and in practical tubes the figure is often about 0.7. On the other hand, for the retarding-field oscillator, from Eq. (82),

$$M = \frac{4K V_0}{\beta_0 V_m} = \frac{4V_m V_0}{\beta_0 V_1 V_m} = \frac{4}{\beta_0} \frac{V_0}{V_1}$$  \hspace{1cm} (85)$$

Since the dc gap voltage must be larger than the beam voltage in order for the electrons to return to the anode, the highest possible value of $M$ is $\beta_0/4$. This is approximately 0.52, and the values encountered in practice are about 0.4.

The electronic susceptance of the retarding-field oscillator is different in form from that of the klystron, as shown in the following two equations:

for the retarding-field oscillator,

$$B_e = - G_0 M^2 \left( \frac{\beta_0^2}{2} + \frac{J_1(2K)}{2K} \beta_0 \cos \beta_0 \right)$$  \hspace{1cm} (86)$$

and for the reflex klystron,

$$B_e = + G_0 M^2 \frac{J_1(X)}{X} \beta_0 \cos \beta_0$$  \hspace{1cm} (87)$$

In addition to the differences noted for the conductance, the susceptance of the retarding-field oscillator contains a linear term in $\beta_0$. It is this term which causes the electronic admittance spirals to lie below the conductance axis. It is also responsible for the lack of symmetry in the repeller mode shape (Figure 20). It is apparent that, in general, the part of reflex klystron theory which deals with electronic tuning and other effects depending on the susceptance will not
apply to the retarding-field oscillator.

As an illustration of the similarity of the two types of oscillators, the electronic efficiencies of both according to the first-order theory are shown in Fig. 23. The efficiencies lie on the same curve, but corresponding modes occur at different transit angles.

4.0 THEORETICAL CHARACTERISTICS OF SELF-EXCITED FREQUENCY MULTIPLIERS

4.1 The discussion in Section 3 was concerned with the operation of the planar retarding-field oscillator at its fundamental frequency. It is now necessary to extend this theory to include harmonic effects in order to predict the operation of frequency multipliers.

Two methods were described in Section 2 by means of which it should be possible to obtain harmonic output from the retarding-field oscillator. The first method utilizes the harmonic components of the induced current in the interaction gap, while the second method depends upon the harmonic content of the convection current at the anode. More detailed calculations of the potentialities of these two methods are presented in the following subsections.

4.2 Calculation of the Instantaneous Induced Current in the Interaction Gap of a Planar Retarding-field Oscillator. In the calculation of the instantaneous-induced current, the assumptions made at the beginning of Section 2 will apply. However, since harmonic production is a large-signal effect, the approximate results of the small-signal theory must be replaced by more exact graphical or numerical calculations. Although the numerical method has been found to be the more satisfactory, the graphical procedure provides a clearer insight into the physical
Fig. 23. Optimum efficiencies according to the long transit angle approximation.
- Reflex Klystron
- x Retarding Field Oscillator
phenomena involved and will therefore be discussed first.

In the graphical calculation, the differential relation of Eq. (3) is replaced by the difference equation

$$\Delta i = \frac{I_0}{d} \ u(t, t_i) \Delta t,$$

(88)

The entering electrons are thus grouped into packets having equal charges given by

$$\Delta q = I_0 \Delta t,$$

(89)

The total induced current is then obtained by making a summation over all the electrons in the interaction gap at the time $t$.

$$i(t) = \sum \frac{I_0}{d} \ u(t, t_i)$$

(90)

Curves of $u(t, t_1)$ from Eq. (9) are plotted for uniform intervals in $t_1$ as shown in Fig. 24. The plotting of such a set of curves may be completed very readily with a single template of the form $y = \theta + K \sin \theta$ since all the curves are of the same shape. It is necessary, of course, that each velocity curve be terminated at a point corresponding to the exit angle for the particular electron packet. This exit angle is simply $\theta_1 + \beta(\theta_1)$. The total induced current at a particular time $t$ is obtained by adding all the velocity ordinates at the value of the time and multiplying by the constant $I_0/d$. If a desk calculator is used, the addition may be carried out for every quarter of a radian throughout a cycle in a moderate amount of time.

This graphical method has been tried for a number of combinations of parameters and experience shows that useful results may be obtained.
Fig. 24. Electron velocities in units of $\frac{2u_o}{\beta_o}$ as a function of $\Theta$. The lower dotted curve is the locus of exit velocities.
if sufficient care is exercised in the plotting. The chief difficulty stems from the fact that the use of a single template may result in systematic errors.

A better method appears to be a numerical calculation of the current following the integration of Eq. (16). Fortunately the integrand is one which is readily integrated, and for sufficiently small values of $K$ the formal result in Eq. (17) is valid. As $K$ is increased, however, a point is reached beyond which $\beta$ is a multiple-valued function of $\theta$. Typical plots of $\beta$ vs. $\theta$ are shown in Fig. 25. The critical value of $K$ below which $\beta$ is single valued is about 0.52. For $K$ greater than about 2.18, there are ranges of $\theta$ for which $\beta$ has five values, and for still larger signal levels the curve becomes even more complex.

These highly complex cases could be studied by a straightforward extension of the present method but are actually of very little practical importance since none of the oscillators which have been constructed operates at values of $K$ greater than unity. In fact, typical values of $K$ are in the neighborhood of 0.25. In view of these facts, is it sufficient to develop formulas for the single-valued case and for the one in which $K$ may have three values.

In the range of $0.52 < K < 2.18$, the integral of Eq. (16) must be replaced by two integrals as follows:

$$i(\theta) = \frac{I_e}{\omega} \frac{u_a}{d} \left[ \int_{\beta_1}^{\theta} \psi(\theta, \theta_1) d\theta_1 + \int_{\theta_1}^{\psi - \beta_2} \phi(\theta, \theta_1) d\theta_1 \right]$$
Fig. 25. A plot of transit angle versus exit angle.
The definitions of $\beta_1$, $\beta_2$, and $\beta_3$ may be understood with the help of Fig. 25. The electrons in the interaction gap at $\theta$ consist of two groups: those which entered during the interval extending back from $\theta$ to $\theta - \beta_1$ and those which entered in the interval between $\theta - \beta_2$ and $\theta - \beta_3$.

The first integral when the expression for $\sqrt{ }$ from Eq. (16) is used becomes

$$
\int_0^{\theta - \beta_1} d\theta = \frac{1}{\beta_0} \left\{ \frac{\beta_0}{2} \left( \beta_0 - \beta_2^2 - 2K [\beta_1 \sin \theta + \cos \theta - \cos(\theta - \beta_1)] \right) \right\}
$$

Similarly the second integral is

$$
\int_0^{\theta - \beta_2} d\theta = \left[ \theta - \frac{2}{\beta_0} \theta + \frac{2}{\beta_0} \right]^{\theta - \beta_2} \left[ \theta (\theta - \beta_2) - \theta (\theta - \beta_3) \right]^{\theta - \beta_3} + \frac{(\theta - \beta_3)^2}{2} - \frac{(\theta - \beta_2)^2}{2} + K (\theta - \beta_3) \sin \theta - K (\theta - \beta_2) \sin \theta + k [\cos(\theta - \beta_2) - \cos(\theta - \beta_3)]
$$

$$
= \frac{1}{\beta_0} \left\{ \left( \beta_0 - \beta_2 \right) \left( \beta_0 - \beta_3 \right) - 2K \left[ \beta_3 - \beta_1 \right] \sin \theta + \cos(\theta - \beta_1) - \cos(\theta - \beta_3) \right\}
$$

The sum of the two integrals may then be combined into the single expression

$$
i(\theta) = \frac{1}{\omega} \frac{\mu_0}{\beta_0} \left\{ \left( \beta_0 - \beta_2 + \beta_3 \right) - \left( \beta_1^2 + \beta_2^2 + \beta_3^2 - 2K \beta_1 \beta_2 \beta_3 \sin \theta - \cos(\theta - \beta_2) + \cos(\theta - \beta_3) - \cos(\theta - \beta_1) \right) \right\}
$$
As $K$ decreases toward 0.52, $\beta_2 \to \beta_3$, and the above equation becomes identical with Eq. (16).

The numerical, point-by-point, evaluation of $i(\theta)$ can be carried out in a straightforward way using a table or graph of $\beta$ as a function of $\theta$. In addition to the graph in Fig. 25, a short table of values of $\beta$, together with some comments on its computation, will be found in Appendix I.

The total induced current for several values of $K$, as calculated by this method are shown in Figs. 26, 27, and 28. It is apparent that the wave form of the induced current departs markedly from a pure sinusoid as $K$ increases.

In order to obtain a quantitative idea of the harmonic content of these wave forms, a Fourier analysis was made for the lower harmonics. A plot of the variation of harmonic amplitude with $K$ was shown in Fig. 3. The feasibility of using these harmonics is discussed in the next section.

4.3 Theoretical Harmonic Power Output from the Single-cavity Type of Harmonic Generator. The harmonic analysis of the induced current presented in the previous pages may be used to obtain an approximate idea of the harmonic power that might be realized from a single-cavity system. An exact analysis is greatly complicated by the fact that, in order to obtain harmonic power output, it is necessary to produce a fairly high gap impedance at the harmonic frequency. This results in the introduction of a harmonic voltage across the gap and thus modifies the conditions assumed in the original analysis.
Fig. 26. Instantaneous induced current for $K = 0.1$ and 0.2.
Fig. 27. Instantaneous induced current for $K = 0.3$, $0.4$, and $0.5$. 
Fig. 28. Instantaneous induced current for $K = 0.6, 0.7, 0.8$ and 1.0.
As a first order approximation the non-linear effects of the harmonic gap voltage will be neglected. That is, it will be assumed that the fundamental operation is not disturbed by the presence of the relatively small harmonic voltage. Under this assumption the harmonic current will consist of two parts, that produced by the interaction of the beam with the fundamental gap voltage and that produced by the harmonic gap voltage. According to Fig. 3 the first component of the second harmonic current would be about 15% of the fundamental at a typical signal level of $K = 0.25$. The other harmonics are much lower. In order to estimate the effects of the interaction of the beam with the harmonic voltages, it is necessary to consider the effective harmonic transit angles involved compared with the transit angle ranges for negative beam conductance. These are approximately as follows:

<table>
<thead>
<tr>
<th>Harmonic Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$ in cycles</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{3}{4}$</td>
<td>5</td>
</tr>
<tr>
<td>$\beta_0$ range for $G_e &lt; 0$</td>
<td>$1 - \frac{1}{2}$</td>
<td>$2 - \frac{1}{2}$</td>
<td>$3 - \frac{1}{2}$</td>
<td>$5 - \frac{1}{2}$</td>
</tr>
</tbody>
</table>

From this it is apparent that for the third harmonic the beam exerts a loading effect, and for the second and fourth harmonics the main effect is susceptive.

Since the beam does not have a negative conductance for these harmonics, there should be no further amplification of the harmonic power produced by the fundamental voltage. An upper limit to the harmonic power could therefore be estimated by multiplying the square of the harmonic current by the harmonic shunt resistance. For the second harmonic the output might be expected to be about 20 db below the
fundamental output.

For still higher harmonics, the beam might have a negative conductance in some cases. The higher harmonics in the beam current are so small, however, that they would probably be masked by debunching and other secondary effects.

From the foregoing it might be expected that, if a second harmonic impedance comparable to the fundamental impedance could be reflected into the interaction gap, a small but detectable harmonic output could be obtained.

4.4 Harmonic Content of the Convection Current at the Anode. The induced current described in the Section 4.2 resulted from the motions of all the electrons present in the interaction gap. Since some of these electrons were moving away from the anode and some were moving toward it, it is not surprising that their effects tend to cancel to some degree. Even under conditions such that the electrons returning to the anode were strongly bunched (K = 0.6), the harmonic content of the induced current was not very great.

A more promising source of harmonic power would be the bunched beam as it is collected at the anode, provided that the smoothing effect of the rest of the beam could be avoided. This is the mechanism involved in the self-excited frequency multiplier described in Section 2.2. The bunched electron beam as it passes through the second interaction gap at the anode must therefore be analyzed in order for an estimate of the performance of this device to be made.

To a first approximation, the second interaction gap may be assumed
to be very short and therefore to have a beam-coupling coefficient of unity. Under this condition the induced current will be the same as the electron convection current at the anode plane.

The method for calculating the instantaneous electron convection current at the anode is similar to that for the reflex klystron. The basic principle employed in this calculation is that of conservation of charge. This principle may be stated mathematically in a way which relates the current entering the interaction gap, \( I_o \), to the current which leaves the gap, that is, \( i(0,\theta_2) \) in the notation of Section 3.

If a quantity of charge entering during the time \( \Delta t_1 \) is followed through its motion in the interaction gap, it will be found to return to the anode during the time interval \( \Delta t_2 \). The statement that all the charge which enters the interaction gap must later leave it is written as

\[
I_o \Delta t_1 = i(0,\theta_e) \Delta t_2 \tag{97}
\]

or more exactly as

\[
i(0,\theta_e) = I_o \frac{dt_1}{dt_2} \tag{98}
\]

As pointed out by Pierce and Shepherd,\(^{19}\) this relation holds only if \( t_1 \) is a single-valued function of \( t_2 \). In the more general case

\[
i(0,\theta_e) = I_o \sum \left| \frac{dt_1}{dt_2} \right| \tag{99}
\]

where the values of \( dt_1/dt_2 \) to be added are calculated at each of the entrance times which correspond to \( t_2 \).

\(^{19}\) Ibid p. 642
The numerical use of this equation is facilitated if, as in the previous calculations, transit angles and exit angles are introduced. Thus

$$\frac{dt_1}{dt_2} = \frac{d\theta_1}{d\theta_2} = \frac{d}{d\theta_2} (\theta_1 - \beta) = 1 - \frac{d\beta}{d\theta_2}$$  \hspace{1cm} (100)

The working equation is therefore

$$i(\theta_1, \theta_2) = I_o \sum \left| 1 - \frac{d\beta}{d\theta_2} \right|$$  \hspace{1cm} (101)

The exact evaluation of the instantaneous current from this equation is complicated by the fact that $\beta$ cannot be expressed in a simple way as a function of $\theta_2$. It is, of course, possible to obtain the current graphically from a plot of $\beta$ vs. $\theta_2$ such as Fig. 25. Examination of this plot gives a clear picture of the general appearance of the current waveform. For example, the current will have an infinite peak when the curve has an infinite slope; and, for values of $K$ greater than about 0.52, the current will have two infinite current peaks.

A more satisfactory method of calculating the current, at least from the standpoint of accuracy, is to differentiate the expression for the transit angle, Eq. 18) implicitly. The result is

$$0 = \beta' - \frac{2}{\rho_o} \left\{ \beta \beta' + k \left[ - \sin(\theta_2 - \beta)(1 - \beta') + \sin \theta_2 ight. ight. \\
\left. \left. \left. - \beta' \sin(\theta_2 - \beta) - \beta \cos(\theta_2 - \beta) (1 - \beta') \right] \right\}$$  \hspace{1cm} (102)

Separation of terms in $\beta'$ yields

$$0 = \beta' \left[ 1 - \frac{2\beta}{\rho_o} - \frac{2K}{\rho_o} \cos(\theta_2 - \beta) \right] - \frac{2K}{\rho_o} \left[ \sin \theta_2 - \sin(\theta_2 - \beta) - \beta \cos(\theta_2 - \beta) \right]$$  \hspace{1cm} (103)
The expression of terms for $\beta'$ from this equation permits the instantaneous current to be written as

$$i(\alpha, \theta_2) = I_0 \sum \left| 1 - \frac{2K \left[ \sin \theta_2 - \sin(\theta_2 - \beta) - \beta \cos(\theta_2 - \beta) \right]}{\beta_0 \left[ 1 - \frac{2\beta}{\beta_0} \left[ 1 + K \cos(\theta_2 - \beta) \right] \right]} \right|$$

(104)

Point-by-point calculation of the current with this equation is rather easy to carry out with a desk calculator after a graph of $\beta'$ vs. $\theta_2$ has been plotted. For low values of $K$ the summation reduces to a single term; as, in fact, it does at higher values of $K$ over most of a cycle of $\theta_2$.

The convection current wave forms obtained in this way are shown in Figs. 29 and 30. The second of these figures shows the wave forms for low-signal levels plotted with an expanded scale. The harmonic content of these wave forms can be determined by the usual numerical methods, but it is more convenient to use a harmonic analyzer. The analysis obtained with the Corati instrument is shown in Table I.

**TABLE I**

Graphical Fourier Analysis of the Convection Current at the Anode

<table>
<thead>
<tr>
<th>Harmonic number</th>
<th>$K = 0.1$</th>
<th>$K = 0.2$</th>
<th>$K = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>19.2</td>
<td>39.6</td>
<td>57.6</td>
</tr>
<tr>
<td>3</td>
<td>3.3</td>
<td>17.5</td>
<td>32.7</td>
</tr>
<tr>
<td>4</td>
<td>5.5</td>
<td>7.5</td>
<td>21.9</td>
</tr>
<tr>
<td>5</td>
<td>8.8</td>
<td>4.2</td>
<td>14.9</td>
</tr>
</tbody>
</table>
Fig. 29. Convection current at the anode for $K=0.1$, $0.2$, and $0.3$. 
Fig. 30. Convection current at the anode for $K = 0.5, 0.6,$ and 1.0.
Although the analysis presented above is quite general and is limited in accuracy only by the care used in the calculations, it is nevertheless sometimes useful to have approximate analytical expressions for the harmonic amplitudes. Under suitable simplifying assumptions, such expressions can be derived quite readily by the method employed by Webster for the reflex klystron.

In such an analysis the instantaneous current is assumed to be of the form

\[ i(\theta_1, \theta_2) = a_0 + a_1 \cos(\theta_2 + \phi) + a_2 \cos 2(\theta_2 + \phi) + \ldots \]

\[ + b_1 \sin(\theta_2 + \phi) + b_2 \sin 2(\theta_2 + \phi) + \ldots \]  \hspace{1cm} (105)

where the coefficients are determined by the usual formulas,

\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} i(\theta_1, \theta_2) \cos n(\theta_2 + \phi) d\theta_2 \]  \hspace{1cm} (106)

and

\[ b_n = \frac{1}{\pi} \int_{0}^{2\pi} i(\theta_1, \theta_2) \sin n(\theta_2 + \phi) d\theta_2 \]  \hspace{1cm} (107)

The next step is to change variables from \( \theta_2 \) to \( \theta_1 \) by means of the continuity of charge relation

\[ I_c \ d\theta_1 = i(\theta_1, \theta_2) \ d\theta_2 \]  \hspace{1cm} (108)

The equation for \( a_n \) then becomes

\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} I_c \cos n(\theta_1 + \phi) d\theta_1 \]  \hspace{1cm} (109)

\[^{20}\text{Webster, D. L., J. Appl. Phys. 10, 501 (1939)}\]
Also $\beta$ can be written as $\beta_0 + \delta$; and, if $\delta$ is defined as $-\beta_0$, the result is

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} I_o \cos n(\theta, + \delta) d\theta,$$

(110)

To this point the result is general, but since no exact expression for $\delta$ in terms of $\theta_1$ is available it is now necessary to make an approximation. For example, the assumption of a very long transit angle permits $\delta$ to be written approximately as

$$\delta = 2K \sin \theta_1$$

(111)

Then the coefficients become

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} I_o \cos n(\theta_1 + 2K \sin \theta_1) d\theta_1 = 2 I_o J_n (-2nK)$$

(112)

and

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} I_o \sin n(\theta_1 + 2K \sin \theta_1) d\theta_1 = 0$$

(113)

The series for $i(\theta_2)$ is

$$i(\theta_2) = I_o + 2 I_o \left[ -J(2K) \cos (\theta_2 - \beta_0) + J(4K) \cos 2(\theta_2 - \beta_0) + \ldots \right]$$

(114)

This equation differs from that of the reflex klystron only in the alternation of the signs of the terms and the values of $\beta_0$ for best oscillations. It is valid for all values of $K$ provided that $\beta_0$ is vary large compared to unity.

Another approximation may be obtained by assuming small-signal conditions. The derivation proceeds as before except that

$$\delta = \frac{2K}{\beta_0} \left[ \cos (\theta_1 + \beta_0) - \cos \theta_1 + \beta_0 \sin \theta_1 \right]$$

(115)
The direct substitution of this expression into the equation for the coefficients results in integrals too complicated for easy evaluation, but a simpler equation can be obtained if it is noted that \( \delta \) is a sinusoidal function of \( \theta_1 \). Eq. (115) can then be written as

\[
\delta = \frac{2K}{\beta_0} A \sin (\theta + \alpha)
\]  

(116)

where the \( A \) and \( \alpha \) are to be evaluated numerically for a particular \( \beta_0 \).

The integral for \( a_n \) then becomes

\[
a_n = \frac{I_o}{2\pi} \int_0^{2\pi} \cos \left[ n\theta + \frac{2nKA}{\beta_0} \sin (\theta + \alpha) \right] d\theta
\]

(117)

This integral may be reduced to a standard form by a change of variables

\( \theta + \alpha = \Theta \)

Then

\[
a_n = \frac{I_o}{2\pi} \int_0^{2\pi} \cos \left[ n\Theta + \frac{2nKA}{\beta_0} \sin \Theta \right] d\Theta
\]

(118)

\[
= \frac{I_o}{2\pi} \int_0^{2\pi} \cos n\alpha \cos \left[ n\Theta + \frac{2nKA}{\beta_0} \sin \Theta \right] d\Theta
\]

\[
+ \frac{I_o}{2\pi} \int_0^{2\pi} \sin n\alpha \sin \left[ n\Theta + \frac{2nKA}{\beta_0} \sin \Theta \right] d\Theta
\]

(119)

\[
= 2I_o \cos n\alpha J_n \left( \frac{2nKA}{\beta_0} \right)
\]

(120)

As before, \( b_n = 0 \) and \( a_0 = I_o \).

The resulting series in this case is valid for small values of \( K \) and any value of \( \beta_0 \). It is apparent that it approaches the previous
series as \( \beta_0 \) increases, and in fact differs only slightly from it even in the first mode.

There are various ways in which the results of the approximate harmonic analysis may be presented graphically. In Fig. 31 the variation of harmonic amplitude with \( K \) is plotted for several values of the harmonic number. An interesting characteristic is that there is an optimum value of \( K \) for which the ratio of a given harmonic amplitude to the fundamental is a maximum. The variation of this optimum \( K \) with harmonic number is shown in Fig. 32. It is especially interesting that the optimum \( K \) is as low as 0.6 for the higher harmonics. This is quite low compared with the theoretical signal level for maximum fundamental efficiency (\( K=1.0 \)).

The amplitude of the harmonics are quite high under optimum signal conditions, as shown in Fig. 33, and they decrease very slowly with increasing harmonic number. This, of course, is a highly optimistic picture; and, in the next subsection, some of the practical limitations on the use of these harmonics will be considered.

4.5 Factors Influencing the Theoretical Harmonic Power Output of the Self-excited Frequency Multiplier. Harmonic power output proportional to the squares of the current components in Fig. 33 could be obtained from the self-excited frequency multiplier if the following conditions were met:
Fig. 31. Amplitudes of the harmonic components of the anode convection current in % of the fundamental.
Fig. 32. Optimum value of signal level for the first ten harmonics.
Fig. 33. Amplitudes in percent for the first ten harmonics at the optimum signal level for each harmonic.
a. Fundamental operation at \( K = 0.6 \).
b. No debunching in the beam.
c. All the beam passing into the second interaction gap.
d. Beam-coupling coefficient of the second gap equal to unity.
e. Harmonic impedance of the second gap comparable to the fundamental impedance of the first gap.
f. Harmonic output-coupling system as efficient as the fundamental coupling system.

Unfortunately it is not possible in practice to comply perfectly with any of these conditions. The highest signal level attained in this Laboratory has been about \( K = 0.3 \). As Fig. 31 shows, the harmonic amplitudes are much smaller at this level than at the optimum level. For the fourth harmonic, for example, the decrease is about ten db.

Questions concerning debunching, beam transmission and beam coupling cannot be answered readily because of the complex geometry of the tube. An enlarged sketch of the interaction region of a typical self-excited frequency multiplier is shown in Fig. 34. The dotted curve is intended to show in a very approximate way a possible electron path. Because of the high space-charge density in the beam and the fact that both the ac and dc fields have radial components, the calculation of accurate trajectories does not appear feasible. In view of these complexities, an estimate of the power loss from these effects is of very questionable value and will not be attempted.
Fig. 34. The interaction region of the self-excited frequency multiplier.
In contrast, the impedance of the second gap and the efficiency of the harmonic coupling system can be calculated by the self-consistent field method of R. C. Ward. He found that in a typical case the combination of the inner cavity, radial line and outer cavity would have the following characteristics:

\[ \lambda_0 = 6.4 \text{ mm} \]
\[ R_{sh} = 35,000 \text{ ohms} \]
\[ Q_0 = 1025 \]

The degree of coupling to the output wave guide can be adjusted over a considerable range. As a comparison, the fundamental cavity operating at 25.6 mm would have roughly the same unloaded \( Q \) and perhaps twice the shunt resistance. These figures for the harmonic and fundamental cavities are intended merely to indicate orders of magnitude since variations in fabrication of the tube parts as well as design changes may result in large variations in circuit characteristics.

The preceding discussion indicates descriptively where power losses may occur, but the magnitudes of these losses must be determined by measurement. For comparison purposes, however, it is useful to estimate an upper limit to the harmonic output. This may be done for various experimentally observed signal levels (as measured by the fundamental output power) under the assumptions of complete utilization of the returning beam, unity beam coupling, and the same characteristics for the harmonic cavity as for the fundamental cavity.

The results of such a calculation are shown in Figure 35. The four typical sets of fundamental operating conditions shown below were assumed:

<table>
<thead>
<tr>
<th>$V_o$</th>
<th>Output Power</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>800 v</td>
<td>1 w</td>
<td>0.24</td>
</tr>
<tr>
<td>800 v</td>
<td>1.5 w</td>
<td>0.29</td>
</tr>
<tr>
<td>800 v</td>
<td>2 w</td>
<td>0.34</td>
</tr>
<tr>
<td>1200 v</td>
<td>5 w</td>
<td>0.32</td>
</tr>
</tbody>
</table>

The harmonic power is expressed in db above one milliwatt for frequencies from the fundamental to the 10th harmonic. As might be expected, the power output increases rapidly with fundamental power for the 1w, 1.5 and 2w curves since these curves represent increases in $K$. For example, for the fifth harmonic, the powers are 5 mw, 32 mw and 125 mw. The increase from 2 w to 5 w operation at constant efficiency produces the much less marked relative increase in harmonic output to 320 mw.

5.0 EXPERIMENTAL RESULTS OBTAINED WITH SELF-EXCITED FREQUENCY MULTIPLIERS

5.1 Introduction

5.1.1 Objectives. The general objective of the experimental part of this research is to determine whether the two-cavity type of self-excited frequency multiplier is potentially a useful source of low-power cw output in the millimeter wave length range. Such a broad objective has many aspects. For example, the new device may be compared with existing sources in regard to output frequency, output power, mechanical tuning range, electronic tuning range, input power requirements, ease of construction, frequency stability, size and weight, mechanical
Fig. 35. Upper limit of harmonic power output in db above 1 mw. for typical fundamental operating conditions.
ruggedness, and many other characteristics. Probably none of these characteristics can be evaluated accurately until the tube has gone through a prolonged period of development; but, nevertheless, fairly reliable estimates can be based upon the preliminary experimental results described in the following pages.

5.1.2 General Design Considerations. The self-excited frequency multiplier is essentially a modification of the retarding-field oscillator. Since the retarding-field oscillator has already reached a rather high stage of development, the first design problem is to introduce a harmonic cavity and coupling system without causing serious deterioration of the fundamental operation; and the second problem is to make gradual changes in the geometry of the tube in order to optimize the harmonic operation.

The space available for the harmonic cavity is very small because the electron gun must be close to the interaction gap. This is not too serious a problem, however, if a cavity of the radial line type is used and if it is designed for the fourth or higher harmonic. The construction and characteristics of the harmonic cavities and coupling systems will be described in more detail in the discussion of specific tubes.

The second problem is one in which theoretical considerations offer only general guidance; the detailed changes to be made in the tube geometry must necessarily be determined experimentally. The changes should be in the direction of improving the efficiency of the output-coupling system, increasing the fraction of the beam current entering the second interaction gap, and increasing the modulation coefficient of the second
Again the changes which have been made and the effects which they have produced will be discussed in more detail for specific tubes.

5.1.3 Measurements at the Harmonic Frequencies. Measurements of the output of the frequency multipliers can be made, in principle, just as they would be in any other wave guide circuit. Certain difficulties arise, however, at wave lengths shorter than six millimeters because of increased wall losses and also because of the small sizes of the parts. For example, the attenuation of silver wave guide becomes gradually more serious at shorter wave lengths (1 db per foot in RG 99/U wave guide at 5 mm), and in some of these experiments it has been found desirable to use a dielectric wave guide to connect the tube to the test equipment. Also it is usually not possible to measure wave lengths as accurately at the harmonic wave lengths as at the fundamental because surface losses reduce the Q's of the wave meters. In this case, however, it is sufficient to be able to determine which harmonic is being produced, and accurate values can then be calculated from the fundamental wave meter readings. Tuners are also more difficult to use at these shorter wave lengths because they tend to be relatively lossy. Signal detectors and power meters present probably the most serious difficulties at these wave lengths. The crystal detectors which have been used in this work are difficult to tune, have a sensitivity which varies rapidly with frequency, and of course are very easily damaged. The bolometer power meter is the only type which has been available so far. It has the disadvantage that the bolometer element has a very short life, and replacements are both expensive and difficult to obtain. It has also been
found that power readings with this type of meter may be about 1.1 db low at full-scale.22

Measuring equipment for wave lengths below six millimeters is now becoming commercially available and possibly some of these experimental difficulties will be alleviated in the near future.

5.2 The X-band to 6.4 mm Frequency Multiplier. The first model of the self-excited frequency multiplier had the form shown in Fig. 6. It can be seen that the essential parts of the 2 - 4 cm retarding-field oscillator were changed only slightly. The fundamental power was coupled out in the usual way through the repeller coaxial line. The only change in the fundamental power was a reduction in height from 0.125 to 0.105 inch. This change was made to permit the use of available retarding-field oscillator cavity blocks. The electron gun was the same as used in the standard 2 - 4 cm oscillators. It provides an electron beam 0.020 inch diameter with a perveance of about 2.5 x 10^-6 amp-volt^-3/2.

The harmonic cavity and coupling system were located in the lower end-wall of the fundamental resonant cavity, as shown in Fig. 6. This wall, which consisted just of a disc containing the anode aperture in the retarding-field oscillator, was replaced by two discs containing the harmonic system between them. The harmonic coupling system was very similar to that used in the Western Electric 1464 reflex oscillator and related tubes in that an inner critical resonant cavity was coupled to an outer cavity by means of a quarter wave length radial line. In the case of the harmonic generator, however, it served a different

22 Bell Telephone Laboratory's Thirteenth Interim Report on Millimeter Wave Research, October 1954.
purpose. It was intended to permit the junction to the harmonic output wave guide to be located at a radius greater than the fundamental cavity radius. This was necessary because the inside height of the wave guide was 0.094 inch and the space available between the fundamental cavity and the electron gun region was less than 0.060 inch.

The method of making the connection between the harmonic system and the wave guide is also shown in Fig. 6. Since all the tests were made on continuously-pumped demountable tubes, it was necessary to make the blocks containing the cavities in such a way that they could be readily interchanged between tests. The block was therefore designed with a milled slot in the side so that when it was fitted into place in the tube mounting the end of the harmonic wave guide would fit tightly over an opening in the outer harmonic cavity. This junction was probably not as good as could be obtained if a choke flange were placed on the end of the wave guide, but it was considered adequate for preliminary testing.

Several forms of harmonic interaction gap were tried with this model of the tube. Some of them are drawn to a larger scale in Fig. 36.

The first one tried, shown in Fig. 36, was based on the "flat anode" retarding-field oscillator. (The "flat anode" tube was one in which there was no conical projection of the anode into the main interaction gap, and it was characterized by good efficiency but somewhat higher starting current). The distance from the edge of the entering beam to the inner edge of the harmonic gap was 0.002 inch and the average width of the gap was 0.008 inch. In the first operation of this tube, fourth harmonic operation was obtained initially over the range from 6.13 to 6.62 mm,
Fig. 36. First interaction-gap geometry.

Fig. 37. Second interaction-gap geometry.
in good agreement with the design wave length. The repeller was then inadvertently allowed to touch the top plate and presumably decrease the width of the second harmonic gap slightly. The fourth harmonic tuning range then became about 6.5 mm to 7.3 mm and fifth harmonic output was observed from 6.42 mm to 6.60 mm. The repeller voltage was swept sinusoidally and the repeller modes were observed on an oscilloscope. The harmonic output gave about a two-inch deflection on a Dumont type 322 oscilloscope at full gain using a 1N53 crystal with about 5 or 6 db attenuation in the wave guide. The operation of the fundamental oscillator was quite poor in comparison with usual results and required more than twice the normal starting current.

The tube was also operated as an externally-excited harmonic generator. It was driven by means of a Varian type X-13 oscillator with the beam current of the frequency multiplier slightly below the starting value. Harmonic output was readily obtained although at lower power than under self-excited conditions.

There was some evidence from microscopic examination of the tubes just described that part of the returning electrons were being collected at a radius greater than that of the second gap. It was therefore decided to increase the average radius of the gap to determine whether these electrons represented any substantial part of the beam current. The dimensions chosen are shown in Fig. 37.

The results obtained with this gap were, in general, not as satisfactory as with the previous one. Third and fourth harmonic output was obtained in the range 6.9 - 8 mm, but the level was low. It was noted
that the strongest output was obtained in the fourth harmonic rather than in the third. This occurred when the repeller spacing was very small and may have been the result of a favorable focusing effect. Again with this arrangement, the starting current was high and the fundamental operation was rather poor.

In an effort to reduce the starting current, the geometry was changed to that shown in Fig. 38. The conical nose projected into the gap about 0.003 inch and had a sharp edge. This had the desired effect on the starting current, and the fundamental power output was also greatly increased. The harmonic output was very weak, however. Examination after disassembly showed that the current was almost entirely collected on the top plate, and there was very little evidence of bombardment of the nose. It was also found that the gun spacing was too great. A second attempt with the correct gun spacing gave essentially the same results.

From these results it was concluded that very little of the returning beam was entering the second gap. In order to avoid the delay of having new parts made the top plate was depressed by driving the repeller against it. This resulted in decreasing the second gap width to about 0.006 inch and approximately doubled the distance to which the nose projected above the top plate as shown in Fig. 39. The harmonic power output was immediately increased, and both fourth and fifth harmonics could be obtained. The harmonic tuning range was approximately 5.9 to 7.8 mm with maximum power occurring at about 6.2 mm. The power output was not continuous over this tuning range, but instead consisted of a series of peaks spaced about 0.1 mm apart in wavelength. Insuffi-
Fig. 38. Third interaction-gap geometry.

Fig. 39. Fourth interaction-gap geometry.
cient data were obtained to permit a complete power curve to be plotted, but the relative heights of the first few peaks in the fourth harmonic curve are shown in Fig. 40. The shapes of these power peaks were not measured in detail. A crude attempt at harmonic-power measurement, however, was made by connecting a K-band bolometer power meter to the small 6 mm wave guide without a transition section or a tuner. The power was estimated to be about 1.8 mw at 6.2 mm with an 800 volt 52 ma beam. At a beam voltage of 1000 volts, the power increased to about 4.5 mw.

From a comparison of oscilloscope deflections, it was estimated that this output of 4.5 mw was probably 6 to 10 db below the maximum that had been obtained with the first model.

In a subsequent test, very similar results were obtained, except that some degree of overlapping of the harmonic tuning ranges was noted. This is shown in the tuning curve of Fig. 41.

The variation of harmonic-power output with beam voltage at 6.2 mm was also measured with the mismatched bolometer meter. It was found that the best repeller centering adjustment for low voltage operation did not give the best high voltage operation. Power output curves as a function of beam voltage for two centering conditions are shown in Fig. 42. It is noteworthy that harmonic power output was still detectable with the bolometer at 480 volts and 24 ma, which was only slightly above starting conditions for the fundamental oscillator.

At the conclusion of these tests, the top plate was again depressed by means of the repeller until the anode projected about 0.013 inch.
Fig. 40. An example of power variations over a small portion of the tuning range of a frequency multiplier.
Fig. 41. An example of the tuning ranges of a self-excited frequency multiplier.
Fig. 42. Harmonic power output as a function of beam voltage.
This decreased the harmonic-gap width and increased the harmonic wave length for best operation to about 7.5 mm with a fundamental wave length of 3 cm. At this wave length harmonic power was detectable at an anode potential of 350 volts and a beam current of 14 ma. The power output at 800 volts was only about 0.6 mw, however.

The qualitative conclusions arrived at from tests of the type described in this section were: that the anode should project well above the top cavity plate for good fundamental operation, that the radial position of the harmonic interaction gap was not very critical, and that the width of the harmonic interaction gap should be moderately wide. To put such conclusions on a quantitative basis will require a very considerable development effort because of the large number of man-hours represented by each new model.

As a supplement to this type of work, it was decided to investigate the scaling characteristics of the frequency multiplier by going to the next higher frequency range in which test equipment was available. This work is described in the next section.

5.3 The 2 cm to 5 mm Frequency Multiplier. The choice of a new frequency range was dictated mainly by the availability of test equipment. The best combination appeared to be to design the fundamental parts around RG-91/U wave guide and to use RG-98/U wave guide for the harmonics. The wave length range for the fundamental was thus about 1.7 - 2.4 cm, and that of the harmonics was 4 - 6 mm.

The scaling was not exact, mainly because a standard electron gun of the required size was not available. Instead, the cavity parts were
scaled by approximately three-fourths, and the same gun was used. In the case of the repeller, the diameter of the hole in the end was left unchanged in order to match the anode, but the outside diameter was reduced in the same scale as the rest of the cavity. The width of the harmonic interaction gap was made about 0.004 inch and nose of the anode projected about 0.008 inch above the top plate. The essential parts are shown in Fig. 43.

The best fundamental operation occurred at a wavelength of 1.95 cm, where oscillations started at a beam potential of 540 volts. The power output at 800 volts was 0.3 watts and at 1000 volts was 1.9 watts. The fundamental output was tunable over the range 1.38 - 2.70 cm with power almost constant at 0.5 watt from 2.1 to 2.7 cm for 800 volt operation. The starting current was markedly higher than for the 2 - 4 cm oscillator.

Fourth harmonic output was observed over the range 4.84 - 5.02 mm and also in the neighborhood of 6.3 mm. A rough power measurement was made using a bolometer power meter without tuning. It indicated that the maximum power occurred at 4.86 mm and that the magnitude was about 0.2 mw. More detailed measurements in this range were delayed until replacement bolometers could be obtained.

5.4 A Frequency Multiplier Employing Higher Harmonics. Another approach to the problem of reaching higher frequencies is to employ higher harmonics rather than to scale the entire system. In order to obtain experimental insight into the possibilities of the method it was decided to construct tubes to generate the fifth through the ninth harmonics.
Fig. 43. 2cm to 5cm frequency multiplier.
Two alternatives were considered. The first was to use an X-band fundamental oscillator with a series of very small harmonic cavities and coupling systems. The second was to use a single harmonic cavity and drive it with a fundamental oscillator tuning over a range from one-fifth to one-ninth the harmonic frequency.

The first system was considered to have the advantage that the fundamental cavity and the electron motions would not change. On the other hand considerable difficulties might be encountered in attempting to make a series of harmonic cavities giving consistent behavior. For example, the nose diameter would be fixed, and the gap loading would therefore increase as the overall dimensions decreased. Cavity losses might also be expected to change markedly with increasing frequency. The coupling to the output wave guide would change unless accurately scaled wave guides were specially made. In addition to these difficulties the most serious problem would be the lack of test equipment in the region below 5 mm.

In comparison, the second system would permit the harmonic circuit and test equipment to operate at a fixed frequency. This would transfer the problems of data interpretation to the fundamental frequency where relatively reliable and diversified equipment is available.

For these reasons it was decided to construct a new mount employing RG-50/U wave guide in the fundamental circuit and having a fixed-frequency 6.4mm harmonic cavity. A fundamental tuning range of approximately 3 - 6 cm could then be used. Data on harmonics from the third through the ninth would then be obtainable with this tube and the previous 2 - 4 cm model.
Another change in construction was also investigated. It was found that the cavity block and the plates containing the harmonic system could be pressed together rather than brazed without serious deterioration of the tube operation. This had the advantage that minor changes in anode shape, for example, could be made without requiring that a complete new set of parts be made.

In preliminary tests on this model of the frequency multiplier, it was found that the fundamental oscillation level was rather low, and consequently the harmonic output was poor. The fifth, sixth and seventh harmonics were readily observable, however. The best operation was obtained at a fundamental wave length of 6.67 mm. The harmonic power was about 80 microwatts.

The difficulties in these tests were apparently mainly the result of an incorrect repeller design in the fundamental oscillator. The geometry that was used differed only slightly from that of the 2 - 4 cm oscillator, and probably there was not space available for the longer trajectories present necessarily in the 4 - 6 cm range. Harmonic measurements were therefore postponed until an improved empirical design of the fundamental oscillator could be developed.

6.0 SUMMARY AND CONCLUSIONS

6.1 Theoretical Background. An understanding of the operation of the self-excited frequency multiplier must be based upon a fairly complete study of the theory of the retarding-field oscillator. The first part of this theory is presented in Section 2.0, where an expression for the small-signal electronic conductance is derived. This permits the
behavior of the tube near starting conditions to be discussed. In the following section a first-order theory of the large-signal operation of the retarding-field oscillator is outlined. Certain similarities of this theory to the first-order theory of the reflex klystron permit much of the experience accumulated with reflex klystrons to be applied to the retarding-field oscillator. Precise calculations of the large-signal electronic admittance and conversion efficiency at the optimum dc transit angle are also included as a verification of the qualitative results of the first-order theory.

These calculations at the fundamental frequency of the oscillator are followed in Section 4.0 by studies of the origins and possible applications of harmonic components of current in the bunched-electron beam. It is shown that the induced current in the interaction gap is relatively weak in harmonics, while the convection current at the anode may have large harmonic components.

The results of this analysis are then applied to predict the power output of the two-cavity self-excited frequency multiplier. In order to establish an approximate upper limit in power output, it is assumed that all the beam is utilized in the second interaction gap and that this gap has perfect coupling to the beam. This limit is a function of the level of the fundamental oscillation and may be expressed in terms of fundamental efficiency. Under the above assumptions, the upper limit for fourth harmonic output at 2% fundamental efficiency is found to be about 22 db below the fundamental power output. The harmonic power increases rapidly with fundamental signal level, and at 4% fundamental efficiency the fourth harmonic is only 13 db below the fundamental.
6.2 Observed Results. All of the models of the self-excited frequency multiplier which have been tested so far have been based upon retarding-field oscillators tuning over the ranges 2.1 - 4.0 cm or 1.8 - 3.0 cm. Both of these oscillators used the same electron gun, the one-fifth size Heil gun, which produces a beam 0.020 inch in diameter. The observed harmonic wave lengths have been in the range from 8.0 to 4.8 mm. All harmonics from the third through the seventh have been studied, but the most extensive work has been done with the fourth harmonic. The best power output obtained with the fourth harmonic, while not accurately measured, was probably less than ten db below the value calculated as the limit for the idealized frequency multiplier. Under some conditions, harmonic output has been detected at beam currents only slightly greater than necessary to start oscillations.

6.3 Comparison of the Self-excited Frequency Multiplier with Other Sources of Millimeter Wave Length cw Power. A comparison of the frequency multiplier with the retarding-field oscillator is of particular interest. In principle, a retarding-field oscillator scaled to the output frequency of the frequency multiplier would be superior to it in power. This conclusion is not valid, however, at extremely high frequencies because eventually frequencies are reached beyond which further scaling is not feasible. In practice, scaling difficulties begin to arise at relatively low frequencies. The best retarding-field oscillator which has been designed so far covers a range of about 2 - 4 cm with a power between one and two watts at 800 volts. The emission density in the electron gun is reasonable, and the tube is capable of long life.
When this tube is scaled directly to the 1 - 2 cm range, the output power is about as would be expected from scaling theory, but certain difficulties arise. The emission density in the electron gun is excessive and tube life is very short. The power handling capacity of the anode is reduced because of its smaller size and the input power must be reduced. The starting current is increased because of increased losses. Since all of the parts are smaller and the assembly tolerances are closer, new techniques of fabrication must be used.

In attempts to scale the basic oscillator to still shorter wavelengths, the difficulties mentioned above rapidly become more serious.

The frequency multiplier is not subject to these limitations to the same degree. In general, the size of the gun is determined by the fundamental oscillator, as is the power dissipation. It was thus possible to use the same gun and the same beam power in the multiplier for 4.8 mm output as is normally used in the 2 - 4 cm oscillator. The starting current was also the same since it was determined by the fundamental oscillator. Most of the parts involved are the same as in the prototype fundamental oscillator; the only small parts being those connected with the harmonic cavity and coupling system. In the design chosen, these parts are not unreasonably complicated and the machining tolerances are the same as for the fundamental cavity.

From this it appears reasonable to conclude that for any frequency at which a retarding-field oscillator can be made to operate well, a frequency multiplier can be constructed to provide output at a frequency three or four times as high.
The reflex klystron could also be modified to obtain harmonic output. This is possible since it is common practice to collect the returning beam outside the first aperture in an effort to avoid hysteresis. A sketch of an arrangement for introducing the second cavity is shown in Fig. 44. It would be important to arrange the second gap in such a way that it could exert little effect on the electrons during their upward motion. The operation of such a frequency multiplier has not been studied theoretically, but in view of the similarities of the reflex klystron and retarding-field oscillator it might be expected to operate successfully.

In comparing the retarding-field type of frequency multiplier with the reflex klystron, two types of millimeter range klystrons must be considered. One is the gridded type developed by E. D. Reid of Bell Telephone Laboratories and the other is the Raytheon ungridded type. The Reid tube has an output of about 20 mw at 5.4 mm and requires a beam potential of 600 volts. The Raytheon tubes operate at 7.5 mm with about 3 mw output at a beam potential of about 2500 volts. Both types are capable of some degree of further scaling, but the difficulties involved increase rapidly. A possible advantage of the frequency multiplier over the gridded klystron is in ease of construction since most of the parts are large and no grids are involved. In operating characteristics, the frequency multiplier appears to have advantages over the Raytheon tubes in output power, tunability, and operating voltages. On the other hand, the model of the multiplier which was tested in the 5 mm range had somewhat smaller output than the highly-developed gridded klystron.
Fig. 4. Sketch of a possible self-excited frequency multiplier based on the reflex klystron.
In contrast, crystal multipliers have the advantage of not requiring a cavity, but they have very low output power and are susceptible to burn out. It is also necessary to use wave guide filters or some type of grating to separate the various harmonics produced by the crystal. Since the retarding-field multiplier uses a cavity the problem of separating frequencies does not occur, but probably the ultimate frequencies attainable are considerably lower than with a crystal. The multiplier tube shares with the crystal the advantage of being capable of frequency stabilization.

6.4 Comparison of the Experimental Performance with Theory.
Although the details of the behavior of the self-excited frequency multiplier cannot be calculated because of the complexity of the geometry, it is of interest to note that the theory predicts the correct order of magnitude. For example, in Section 5.3 the fourth harmonic output at 4.86 mm was 0.2 mw. This occurred with a fundamental power of 1.9 watts. The experimental conversion loss was therefore 39.8 db.

The value of $K$ corresponding to this fundamental power is 0.21 and the conversion loss for an ideal frequency multiplier would be given by $J_4^2 (8 \times 0.21)/J_1^2(2 \times 0.21)$ which is 0.0115 or -19.4 db. In order to make the comparison more realistic, it is necessary to estimate the fraction of the returning beam entering the second gap and also the modulation coefficient of the second gap. The fact that the position of the second gap was found experimentally to be not very critical indicates that the fraction of the current entering the gap is approximately equal to the projected area of the gap divided by the area of
the returning beam. This relative area is about 0.22. This results in a further loss of 13.4 db, giving a total of -32.8 db. The difference of 7 db between this result and the experimental result could be explained by a beam coupling coefficient of the second gap about half as large as that of the first.

The agreement thus appears to be about as good as could be expected on the basis of present knowledge of the geometrical characteristics of the interaction gaps.

6.5 Possibilities for Further Research on the Self-excited Frequency Multiplier. As a long-range study, the most promising work would probably be in the direction of increasing the operating signal level of the basic retarding-field oscillator. Small improvements here would result in relatively large increases in harmonic output. The basic problem is to determine why the efficiency saturation effect occurs at a level considerably below that predicted by theory. Some of the phenomena involved may be repeller bombardment by the beam, transit angle dispersion in the beam, and debunching in the beam.

Work on the frequency multiplier itself should probably be in the direction of studying the beam focusing in order to make better use of the returning electrons. It is also desirable to investigate the effects of changes in the geometry of the second interaction gap on its beam coupling coefficient.
APPENDIX I.

Calculation of Transit Angles for the Planar Retarding-Field Oscillator. Tables of ac transit angles are necessary in many of the numerical calculations for the planar retarding-field oscillator. Of a number of methods which have been used the one described here has been found to be simplest in practice.

The transit angle equation is

\[ 0 = \beta - \frac{2}{p_0} \left( \frac{E^2}{2} + K \left[ \cos(\theta - \beta) - \cos \theta - \beta \sin (\theta - \beta) \right] \right) \] \hspace{1cm} (1-a)

Since this equation is transcendental in \( \beta \), a graphical or numerical solution is indicated. However, rather than to solve the equation as it stands, it is easier to solve explicitly for \( K \) as

\[ K = \frac{p_0 \beta - \beta^2}{2 \left[ \cos \theta_1 - \cos (\theta_1 + \beta) - \beta \sin \theta_1 \right]} \] \hspace{1cm} (2-a)

For each \( \theta_1 \) of interest, \( K \) is calculated for several values of \( \beta \). From the results of this calculation \( K \) may be plotted as a function of \( \beta \). The desired values of \( \beta \) as a function of \( K \) are then determined by interpolation. Fortunately the curves are almost linear, and the interpolation can be made quite accurately from a small number of calculated points.

The results of such a calculation for \( p_0 = 7.725 \) are presented in Table II, and typical curves are plotted in Fig. 45.

As an aid in such a plot it may be noted that the curves for all values of \( K \) cross the \( p_0 \) line at the same points. These points may be found by setting the denominator of Eq. (2-a) equal to zero, and are
Fig. 45. Transit angles for the planar retarding-field oscillator with $\beta_0 = 7.725$. 

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Table II
Transit Angles for the Planar Retarding-field Oscillator

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given by $\theta_1 = 0.100, 3.24, 6.38, \text{ etc.}$ The maxima and minima are also
easily obtained by differentiating Eq. (1-a) with respect to $\theta_1$. The
loci of maximum and minimum points are then given by

$$\tan \theta_1 = \frac{\sin \beta - \beta}{1 - \cos \beta}.$$

These loci are indicated by dashed lines on the plots.
BIBLIOGRAPHY


19. Ibid. p. 642.


28. Allering, Dä llenbach, u Kleinsteuber, "Der Resotank, ein neuer Generator für Mikrowellen", Hochfrequenz und Elektroak, Vol. 51 p. 96 (1938)

I, Marlin Oakes Thurston, was born in Denver, Colorado, September 20, 1918. I received my secondary school education in the public schools of Littleton, Colorado. My undergraduate training was obtained at the University of Colorado, where I received the degree Bachelor of Arts in 1940. I was enrolled in the Graduate School of the University of Colorado until 1942, at which time I entered military service. At the conclusion of this service I returned to the University of Colorado and was granted the degree Master of Science in 1946. During the next five years I served on the faculty of the United States Air Force Institute of Technology, where I attained the position of Associate Professor of Electrical Engineering. In January 1952, I received an appointment as Research Associate at the Ohio State University, where I specialized in the Department of Electrical Engineering. I held this position for three and one-half years while completing the requirements for the degree Doctor of Philosophy.