THE COMPRESSIBLE LAMINAR BOUNDARY LAYER
ON ROTATING AXISYMMETRIC BODIES, WITH
Some Studies of "SIMILAR" SOLUTIONS

DISSERTATION
Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy
in the Graduate School of The Ohio State University

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1955

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ACKNOWLEDGEMENT

THE AUTHOR WOULD LIKE TO EXPRESS HIS GRATITUDE TO PROFESSOR A. N. TIFFORD FOR HIS GUIDANCE AND COUNSEL DURING THE INVESTIGATIONS REPORTED IN THIS DISSERTATION. HE WOULD ALSO LIKE TO EXPRESS HIS THANKS TO PROFESSOR G. L. VON ESCHEN FOR HIS ENCOURAGEMENT.
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SUMMARY

Part I

All three Sections of Part I deal with the analytical study of the compressible laminar boundary layer on rotating axisymmetric bodies in axial motion when the Prandtl number is one.

In Section 1, a particular integral of the compressible boundary layer equations is obtained. This integral reduces to Busemann's integral when the body is not rotating. It applies, in general, when the surface temperature varies parabolically with the radius of the body. The case of an insulated rotating surface is also covered.

In Section 2, the compressible boundary layer subject to a surface pressure gradient has been correlated with a fictitious incompressible boundary layer when the coefficient of viscosity is proportional to the absolute temperature. The skin friction and the surface rate of heat transfer are evaluated in terms of the fictitious incompressible quantities.

The fictitious incompressible differential system of Section 2 is solved in Section 3 by series development. It is shown that all the unknown functions involved can be expressed in terms of universal functions applicable
to a wide class of problems concerning the compressible laminar boundary layer on rotating axisymmetric bodies.

Part II

Both Sections of Part II deal with "similar" solutions of the compressible boundary layer equations when the Prandtl number is one and the coefficient of viscosity is proportional to the absolute temperature.

In Section 4, a general condition on the local free-stream velocity distribution is established which permits the existence of "similar" solutions for the two-dimensional compressible boundary layer. The surface temperature is required to be constant. The governing differential equations for the "similar" solutions are formulated.

In Section 5, an investigation of "similar" solutions of the compressible boundary layer equations for rotating axisymmetric bodies is made. The analysis shows that the sufficient conditions for the existence of "similar" solutions are, in general, redundant. Only the special case of rotating circular cylinders admits "similar" solutions. The surface frictional and the heat transfer characteristics for a rotating circular cylinder have been determined. The longitudinal skin friction is found to be identical to the
result for the flat plate problem. It is not affected by the circumferential velocity component irrespective of the relative magnitude of the circumferential motion to the longitudinal motion. On the other hand, the longitudinal motion does affect the circumferential skin friction.
PART I

COMPRESSIBLE BOUNDARY LAYER ON ROTATING
AXISYMMETRIC BODIES IN AXIAL MOTION

INTRODUCTION

One of the problems of current interest in the field of aeronautical engineering involves the frictional and heat transfer characteristics at the surface of a rotating body in flight. This is a common problem related to missiles and projectiles designed for high speeds where knowledge of the frictional drag and the surface temperature are required. It is well known that both of these effects are confined within the boundary layer on the surface of the body. An analytical study is therefore made of the compressible laminar boundary layer on rotating axisymmetric bodies.

The theoretical study of the boundary layer on a non-rotating axisymmetric body was made by Mangler. It was shown that the problem of the boundary layer on a non-rotating axisymmetric body can be treated similarly to that of a two-dimensional boundary layer problem, both for
the incompressible\(^{(1)}\) and compressible cases.\(^{(2)}\) However, when the body is rotating, the correspondence between the two-dimensional and the axisymmetric problems breaks down due to the existence of the circumferential velocity component inside the boundary layer.

The literature concerning the theoretical aspects of this problem is meager. The papers available can be classified according to whether they are exact analyses or approximate analyses of the Prandtl boundary layer equations.

The more important investigations among the exact analyses are summarized below. Some initial consideration concerning the boundary layer on rotating cones and cylinders was given by Mangier in his paper on incompressible boundary layer about a body of revolution.\(^{(1)}\) Somewhat later Howarth treated the case of a rotating sphere.\(^{(3)}\) A relatively general analysis of the "constant property" laminar boundary layer on spinning axisymmetric bodies in axial motion has been made by Tifford and Chu.\(^{(4)}\)

Related to the exact solutions of the boundary layer
equations for rotating axisymmetric bodies are exact solutions of the complete Navier-Stokes equations for rotating discs. One such solution is that for axial flow against a rotating disc as given in Refs. (5) and (6). Another is that of flow between rotating discs as given in Ref. (7). It is interesting to note that the exact solution of flow about a rotating disc in axial motion may be applied to the stagnation region of a blunt body in rotation.

The momentum-integral method was employed by Schlichting (8) and Truckenbrodt (9), (10) for approximate analyses of the boundary layer equations. The precision attained from the use of this approximate method depends to some extent on the velocity distribution assumed in the boundary layer. In two-dimensional boundary layer analyses the momentum-integral method has been particularly successful because the choice of the velocity distribution inside the boundary layer was made on the basis of exact solutions and experimental data. In the case of rotating bodies, however, applicable experimental data are lacking. Therefore, the results of the exact analyses should be useful in connection with the choice of velocity distributions for approximate analyses using the momentum-integral method.
When the flight speed or rotating speed, increases to a magnitude comparable with the speed of sound, compressibility effects become significant. The equations of motion and the equation of energy, in general, become interdependent due to the variation of density, viscosity, and conductivity with temperature. The physical ideas underlying the concept of the boundary layer, however, still apply. Therefore the simplifications involved in the compressible boundary layer theory can be carried out analogously as that of the incompressible boundary layer theory.

The problem of the compressible boundary layer about a rotating axisymmetric body was treated by Illingworth.\(11\) He has found that in two cases the compressible boundary layer flow about a rotating axisymmetric body can be correlated with a fictitious two-dimensional boundary layer flow. One such case is the boundary layer flow about a spinning circular cone in a supersonic stream; the other, the general case of axisymmetric slow rotation where the effects of the transverse flow on the longitudinal motion may be neglected.

As a sequel to the general analysis of the "constant property" laminar boundary layer on a rotating axisymmetric
body of Ref. (4), the investigation of the detailed effect therein of the variability of fluid properties is undertaken in this dissertation. Part of this study has been previously reported.\(^{(12)}\) The complete analysis is given in more detail here.
SECTION (1) A PARTICULAR ENERGY INTEGRAL WHEN THE PRANDTL NUMBER IS UNITY

ANALYSIS

The derivation of the equations of compressible boundary layer theory starting from the Navier-Stokes equations of motion and the equation of energy for the steady flow of a compressible fluid with variable viscosity presents no conceptual difference from that of the incompressible case.\(^{(4)}\) The compressible boundary layer equations for a rotating axisymmetric body in axial motion are:

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \tag{1-1}
\]

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial x} \left( \frac{w^2}{r} \right) = \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \tag{1-2}
\]

\[
\rho \left[ u \frac{\partial (rw)}{\partial x} + v \frac{\partial (rw)}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \mu \frac{\partial (rw)}{\partial y} \right] \tag{1-3}
\]

\[
\rho \left( u \frac{\partial i}{\partial x} + v \frac{\partial i}{\partial y} \right) + u \rho \frac{1}{\sigma} \frac{\partial}{\partial x} \left( \mu \frac{\partial i}{\partial y} \right) + \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] \tag{1-4}
\]

\[
\frac{r-1}{\gamma} \rho \frac{1}{p} = \rho \frac{r-1}{\gamma} \rho \frac{1}{i} \tag{1-5}
\]
Here \( x \) is the distance along the surface of the body in a meridian plane, \( y \) is the distance measured normal to the surface, \( r \) is the distance of the surface from the axis of symmetry measured normal to the latter, \( u \) and \( v \) are the velocity components. (See Fig. 1) In obtaining the above equations, the validity of the perfect gas law, a constant Prandtl number are assumed in addition to the usual assumptions used in simplifying the boundary layer equations.

The energy equation in the free stream is

\[
i_1 + \frac{1}{2}u_1^2 = h_1 = \text{Constant} \quad (1-6)
\]

In the analysis of this section, the value of the Prandtl number, \( \sigma \), of the fluid is taken to be unity. We shall then seek a solution of the complete differential system having the enthalpy, \( i \), as a function of \( H \) alone, i.e.,

\[
i = i(H) \quad (1-7)
\]

where

\[
H = \frac{u^2}{2} + \frac{w^2}{2} + \lambda \omega tw \quad , \quad (1-8)
\]

\( \omega \) is the angular velocity of the rotating body, and \( \lambda \) is an arbitrary constant. Combining (1-7), (1-8), (1-2), (1-3), and (1-4), it may be shown that
\[ \frac{\partial i}{\partial \eta} = -1 \]  

(1-9)

is a particular solution of the complete differential system as follows: Multiplying (1-2) by \( u \) and (1-3) by \( w \), we have

\[ \rho \left( u \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} - \frac{uw^2}{r} \frac{\partial r}{\partial x} \right) = u \rho u_1 \frac{\partial u_1}{\partial x} + u \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y}) \]  

(1-10)

\[ \rho \left( uw \frac{\partial w}{\partial x} + vw \frac{\partial w}{\partial y} + \frac{uw^2}{r} \frac{\partial r}{\partial x} \right) = w \frac{\partial}{\partial y} (\mu \frac{\partial w}{\partial y}) \]  

(1-11)

Now, the partial derivatives of \( i \) can be expressed in terms of \( u \) and \( w \) and their partial derivatives as

\[ \frac{\partial i}{\partial x} = i' \left[ \frac{u \partial u}{\partial x} + w \frac{\partial w}{\partial x} + \lambda \omega \frac{\partial (rw)}{\partial x} \right] \]

\[ \frac{\partial i}{\partial y} = i' \left[ \frac{u \partial u}{\partial y} + w \frac{\partial w}{\partial y} + \lambda \omega \frac{\partial (rw)}{\partial y} \right] \]

\[ \frac{\partial}{\partial y} (\mu \frac{\partial i}{\partial y}) = \mu i'' \left[ \frac{u \partial u}{\partial y} + \frac{w \partial w}{\partial y} + \lambda \omega \frac{\partial (rw)}{\partial y} \right]^2 + i' \left\{ \frac{u \partial}{\partial y} (\mu \frac{\partial u}{\partial y}) \right\} \]

\[ + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \frac{w \partial}{\partial y} (\frac{\partial w}{\partial y}) + \mu \frac{\partial}{\partial y} \left[ \mu \frac{\partial (rw)}{\partial y} \right] \]

where the prime denotes total differentiation with respect to \( \eta \).

Substituting the above relations into (1-4) and regrouping terms, we have
Comparing the above equation with (1-3), (1-10), and (1-11), we note that the terms inside the first brace are those of (1-10), the terms inside the second brace are those of (1-3), and are thus cancelled out. What is left now reads as

\[ i' \left\{ \rho \left( u \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} - \frac{uw^2}{r} \frac{\partial r}{\partial x} \right) - u \rho_{11} \frac{\partial u}{\partial x} - u \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y}) \right\} \\
+ i' \left\{ \rho \left( uw \frac{\partial w}{\partial x} + vw \frac{\partial w}{\partial y} + \frac{uw^2}{r} \frac{\partial r}{\partial x} \right) - w \frac{\partial}{\partial y} (\mu \frac{\partial w}{\partial y}) \right\} \\
+ \lambda \omega i' \left\{ \rho \left[ u \frac{\partial (rw)}{\partial x} + v \frac{\partial (rw)}{\partial y} \right] - \frac{\partial}{\partial y} (\mu \frac{\partial w}{\partial y}) \right\} \\
+ (1+i') \left[ u \rho_{11} \frac{\partial u}{\partial x} - \mu \left( \frac{\partial u}{\partial y} \right)^2 - \mu \left( \frac{\partial w}{\partial y} \right)^2 \right] \\
+ \mu i'' \left[ \frac{\partial u}{\partial y} + \frac{\partial w}{\partial y} + \lambda \omega \frac{\partial (rw)}{\partial y} \right]^2 = 0 \]

Comparing the above equation with (1-3), (1-10), and (1-11), we note that the terms inside the first brace are those of (1-10), the terms inside the second brace are those of (1-3), and are thus cancelled out. What is left now reads as

\[ (1+i') \left[ u \rho_{11} \frac{\partial u}{\partial x} - \mu \left( \frac{\partial u}{\partial y} \right)^2 - \mu \left( \frac{\partial w}{\partial y} \right)^2 \right] \]

\[ + \mu i'' \left[ u \frac{\partial u}{\partial y} + w \frac{\partial w}{\partial y} + \lambda \omega \frac{\partial (rw)}{\partial y} \right]^2 = 0 \]

(1-12)

It is seen from (1-12) that \( i' = -1 \) is an integral of the energy equation and satisfies the equation of motion automatically.
It follows that the energy integral reads as

\[ i + \frac{u^2}{2} + \frac{w^2}{2} + \lambda \omega rw = c_1 \]  

(1-13)

where \( c_1 \) is the constant of integration.

For (1-13) to be valid at both edges of the boundary layer, the integration constant, \( c_1 \), must satisfy the relationship,

\[ c_1 = i_1 + \frac{u_1^2}{2} = i_w + \left[ \frac{(1/2) + \lambda}{\omega_r^2} \right] \omega_r^2 \]  

(1-14)
RESULTS AND DISCUSSION

A particular integral of the compressible boundary layer equations for a rotating axisymmetric body in axial motion has been established as (1-13). The only assumption made in obtaining this result from the compressible boundary layer equations is that the Prandtl number is unity. In view of the relation (1-14), it is evident that this particular integral applies, in general, only when the surface enthalpy varies parabolically with r. The arbitrary constant, $\lambda$, may be adjusted, however, so as to allow a variety of parabolic representations of the surface enthalpy distribution to be covered. Additional interesting points worth noting are:

(a) (1-13) reduces to the well-known Busemann integral when the body does not rotate.

(b) The analogy of Ref. 4 between the frictional torque and the surface rate of heat transfer is readily applicable despite the variability of the fluid properties. This can be shown by differentiating (1-13) with respect to $y$ and evaluating all the terms at the wall.

(c) The important case of an insulated rotating surface is covered. It corresponds to the case, $\lambda = -1$. The
surface enthalpy distribution acquired is seen to be

$$i_i = h_1 + (\omega r)^2/2.$$ Conversely, a rotating body having a
surface enthalpy (temperature) corresponding to the free
stream total enthalpy (temperature), $$\lambda = -1/2$$, does experience heat transfer.

(d) In addition to its general character, (1-13) may be
used advantageously in obtaining a correlation between the
compressible boundary layer on a rotating axisymmetric
body having a surface pressure gradient and an associated
fictitious incompressible boundary layer. (See Section 2).
SECTION (2) CORRELATION BETWEEN THE COMPRESSIBLE LAMINAR
BOUNDARY LAYER ON ROTATING AXI-SYMMETRIC BODIES AND A
FICTITIOUS INCOMPRESSIBLE BOUNDARY LAYER

INTRODUCTION

In the analysis of the boundary layer the assumption
of constant fluid properties has been extensively employ-
ed. The resulting simplification manifests itself as a
disengagement of the interdependence of the equations of
motion and the energy equation. Mathematical methods
have been devised to solve the equation under this condi-
tion, particularly in the case of two-dimensional
problems. (14) When the variations of the fluid properties
are to be considered, one logical approach would be to
reduce the compressible boundary layer equations by
suitable transformations into a form somewhat similar to
the two-dimensional incompressible boundary layer equations.
Such a reduction makes it possible to use the conventional
methods of solution for compressible boundary layer
problems.

In the case of the compressible boundary layer about
spinning axisymmetric bodies, it seems natural to use - as Illingworth did\(^{11}\) - two transformations which have proven useful in previous work on nonspinning bodies and rectilinear flow. The first of these is the von Mises transformation which has been used successfully in relating certain cases of two-dimensional compressible flow to corresponding incompressible ones.\(^{15}\) The second is the Mangier transformation,\(^{1}\),\(^{2}\). It relates a given problem of compressible (or incompressible) flow around a non-spinning axisymmetric body to a corresponding problem in compressible (or incompressible) rectilinear flow.

Combining these transformations with the energy integral of Section 1, a correlation is established in this Section between the compressible boundary layer on a rotating axisymmetric body having a surface pressure gradient and an associated fictitious incompressible boundary layer.
ANALYSIS

The compressible boundary layer equations for a rotating axisymmetric body in axial flow were given in Section 1 as (1-1), (1-2), (1-3), (1-4) and (1-5). When the Prandtl number is unity, the energy integral, (1-13), can be used in place of the equation of energy. The complete differential system thus becomes

\[ \frac{\partial (rpu)}{\partial x} + \frac{\partial (rpv)}{\partial y} = 0 \]  \hspace{1cm} (2-1)  
\[ \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \rho_1 u_1 \frac{\partial u_1}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u_1}{\partial y} \right) \]  \hspace{1cm} (2-2)  
\[ \rho \left[ \frac{\partial (rw)}{\partial x} + v \frac{\partial (rw)}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \mu \frac{\partial (rw)}{\partial y} \right] \]  \hspace{1cm} (2-3)  
\[ i + \left( \frac{u^2}{2} \right) + \left( \frac{w^2}{2} \right) + \alpha \omega rw = h_1 = i_1 + \left( \frac{u_1^2}{2} \right) \]  \hspace{1cm} (2-4)  
\[ \rho i = \rho_1 i_1 \]  \hspace{1cm} (2-5)  

It is readily seen that a stream function, \( \psi \), such that

\[ rpu = \frac{\partial \psi}{\partial y} \quad \text{and} \quad rpv = -\frac{\partial \psi}{\partial x}, \]  \hspace{1cm} (2-6)  

satisfies the continuity equation, (2-1).

Next, the von Mises transformation is introduced; that is, the independent variables of the above differential system are changed from \( x \) and \( y \) to \( x \) and \( \psi \). The
derivatives, \( \frac{\partial Z}{\partial x} \) and \( \frac{\partial Z}{\partial y} \), of some function \( Z(x,y) \)
in terms of the new independent variables \( x \) and \( \psi(x,y) \)
are
\[
\begin{align*}
\frac{\partial Z}{\partial x} &= Z_x + \frac{\partial \psi}{\partial x} Z_{\psi} \\
\frac{\partial Z}{\partial y} &= \frac{\partial \psi}{\partial y} Z_{\psi}
\end{align*}
\] (2-7)
where the derivatives with respect to \( x \) and \( \psi \) have been
written in the subscript notation. Combining (2-6) and
(2-7), we obtain
\[
\begin{align*}
\frac{\partial Z}{\partial x} &= Z_x - \rho \nu Z_{\psi} \\
\frac{\partial Z}{\partial y} &= \rho \nu Z_{\psi}
\end{align*}
\] (2-8)
whence
\[
\frac{u}{u} \frac{\partial Z}{\partial x} + \frac{v}{v} \frac{\partial Z}{\partial y} = uZ_x .
\] (2-9)
Substituting these transformations into the boundary
layer equations, (2-2) and (2-3), the following equations
are found.
\[
\begin{align*}
u u_x - \frac{(w^2)}{r}(dr/dx) &= (\rho/\rho)u(du_1/dx) + r^2 u [\rho \mu u]_x \\
(ww)_x &= r^2 [\rho \mu (ww)]_x
\end{align*}
\] (2-10) and (2-11)
Equations (2-10) and (2-11) are considerably
simplified by making the assumption, \(^2\)

\(^2\) This and other assumptions about the viscosity
law have been rather fully discussed in a review paper by Kuerti on the compressible laminar boundary layer theory. Also see Results and Discussion.
\[ \rho u = c \rho L u \]  

(2-12)  

A transformation similar to that of Mangler, namely,  

\[ t = c \int_0^\chi r^2 \rho_0 u_1 u_1 dx \]  

(2-13)  
then yields  

\[ \frac{\partial u}{\partial t} - \frac{w^2}{r} \frac{\partial}{\partial t} + \frac{\partial}{\partial \psi} \left( \frac{u \partial u}{\partial \psi} \right) = \rho \frac{du_1}{dt} + u \frac{\partial}{\partial \psi} \left( \frac{u \partial u_1}{\partial \psi} \right) \]  

(2-14)  
and  

\[ \frac{\partial}{\partial t} \left( r w \right) = u \frac{\partial}{\partial \psi} \left[ u \frac{\partial}{\partial \psi} \left( r w \right) \right] \]  

(2-15)  
The dependent variables are now non-dimensionalized by the following definitions:  

\[ \bar{u} = \frac{u}{u_1}, \quad \bar{w} = \frac{w}{w_1}, \quad \bar{r} = \frac{r}{r_1} = \frac{\rho}{\rho_1}, \]  

(2-16)  
where the last relation follows directly from (2-5).  

Hence, (2-14) and (2-15) may be written as  

\[ \frac{1}{u_1} \frac{du_1}{dt} \left( \bar{u}^2 - 1 \right) + u \frac{\partial u}{\partial \psi} - \frac{\omega^2}{u_1^2} r \frac{dr}{dt} \frac{\partial}{\partial \psi} \left( \bar{u} \frac{\partial u}{\partial \psi} \right) = \bar{u} \frac{\partial}{\partial \psi} \left( \bar{u} \frac{\partial u}{\partial \psi} \right) \]  

(2-17)  

\[ \frac{\partial}{\partial t} \left( \bar{r} \bar{w} \right) = \frac{\partial}{\partial \psi} \left[ \bar{u} \frac{\partial}{\partial \psi} \left( \bar{r} \bar{w} \right) \right] \]  

(2-18)  
From (2-4) we can write,  

\[ \left( \bar{u}^2 - 1 \right) = \left( \bar{u}^2 - 1 \right) \left( 1 + \frac{u_1}{2i_1} \right) + \frac{(\omega r)^2}{2i_1} \bar{w}^2 + \frac{\omega (\omega r)^2}{i_1} \]  

(2-19)
Substituting (2-19) into (2-17) and rearranging terms, we have

\[
\frac{du}{dt} + \frac{1+b}{u_1} \frac{du_1}{dt} \left[ \left( \frac{u^2}{u} - 1 \right) + \left( \frac{\omega^2 r^2}{2h_1} \right) \bar{w}^2 + \left( \frac{\lambda \omega^2 r^2}{h_1} \right) \bar{w} \right] - \frac{\omega^2 u_1^2}{h_1} \frac{dr}{dt} \bar{w}^2 = \frac{\partial}{\partial \psi} \left( \bar{u} \frac{\partial \bar{u}}{\partial \psi} \right)
\]

where \( b = \frac{u_1^2}{2i_1} \).

Since the term \((dr/dt)\) may be expressed as

\[
\frac{dr}{dt} = \frac{dr}{db} \frac{db}{dt} = (1+b)^2 (u_1/h_1) (du_1/dt),
\]

the above equation can thus be written as

\[
\frac{du}{dt} + \frac{1+b}{u_1} \frac{du_1}{dt} \left[ \left( \frac{u^2}{u} - 1 \right) + \left( \frac{\omega^2 r^2}{2h_1} \right) \bar{w}^2 + \left( \frac{\lambda \omega^2 r^2}{h_1} \right) \bar{w} \right] - \frac{\omega^2 u_1^2}{h_1} \frac{dr}{db} \left[ (1+b)^2 \right] \frac{du_1}{dt} = \frac{\partial}{\partial \psi} \left( \bar{u} \frac{\partial \bar{u}}{\partial \psi} \right) \tag{2-20}
\]

and (2-18) retains its original form

\[
\frac{\partial (r^2 \bar{w})}{\partial t} = \frac{\partial}{\partial \psi} \left[ \bar{u} \frac{\partial (r^2 \bar{w})}{\partial \psi} \right] \tag{2-21}
\]

The boundary conditions are

at \( \psi = 0 \), \( \bar{u} = 0 \), \( \bar{w} = 1 \),

at \( \psi \rightarrow \infty \), \( \bar{u} = 1 \), \( \bar{w} = 0 \) \tag{2-22}

We will now try to establish a correspondence between the compressible boundary layer problem represented
by the differential system (2-20), (2-21), and (2-22) for the rotating axisymmetric body having a surface pressure gradient and an associated fictitious incompressible boundary layer problem.

Consider the following system of equations:

\[
\frac{\partial U}{\partial x} + \frac{\partial W}{\partial y} = 0, \tag{2-23}
\]

\[
U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} - \frac{W^2}{R} \frac{\partial R}{\partial x} \left[ 1 + \frac{1}{2} \left( \frac{1}{U_x} \right)^2 \right] \tag{2-24}
\]

\[
+ U_1 \left( \frac{\partial U_1}{\partial x} \right) \left[ \frac{1}{2} \left( \frac{W}{U_x} \right)^2 + \left( \frac{\omega RW}{U_x} \right) - 1 \right] = \nu_s \frac{\partial^2 U}{\partial y^2} \tag{2-25}
\]

Here the capitalized letters denote corresponding quantities in the fictitious incompressible boundary layer problem, and \( \nu_s \) is the uniform kinematic viscosity. Note that when \( W = 0 \), these equations reduce to that of a two-dimensional boundary layer flow of an incompressible fluid about a cylinder. By changing to \( \theta \) and \( \psi \) coordinates, (2-23), (2-24) and (2-25) may be transformed into (2-26) and (2-27)
\[
\frac{\partial \bar{w}}{\partial t} = \frac{1}{U_1} \frac{dU_1}{dt} \left[ \frac{u^2 - 1}{(u^2 - 1 + (\alpha_R / U_1)^2) (\bar{w}^2 + \bar{w})} \right] \quad (2-26)
\]

\[
-\left( \frac{\rho}{\mu} \right)^2 \frac{dR}{dB} \bar{w}^2 \left[ 1 + \frac{1}{2} \left( \frac{U_1}{U_r} \right)^2 \right] \frac{1}{U_1} \frac{dU_1}{dt} \frac{1}{\bar{w}} = \frac{\partial}{\partial \psi} \left( \bar{u} \frac{\partial \bar{w}}{\partial \psi} \right) \quad (2-27)
\]

The following definitions and non-dimensional quantities have been used in obtaining the above differential equations:

\[
L \rho_s U = \frac{\partial \psi}{\partial Y}, \quad L \rho_s V = -\frac{\partial \psi}{\partial X} \quad (2-28)
\]

and

\[
\bar{u} = \frac{U}{U_1}, \quad \bar{w} = \frac{W}{\alpha R}, \quad B = \left( \frac{U_1}{U_r} \right)^2, \quad t = L^2 \rho_s \mu_s \int_0^X U_1 \, dX \quad (2-29)
\]

where \( \rho_s \) is the uniform density; \( \mu_s \), the uniform viscosity; and \( L \), the characteristic length of the incompressible problem.

After comparing (2-26) and (2-27) with (2-20) and (2-21), we choose,

\[
\frac{1}{r} \frac{dr}{dt} = \frac{1}{R} \frac{dR}{dt} \quad (2-30)
\]

and

\[
(1+b) \frac{1}{U_1} \frac{du_1}{dt} = \frac{dU}{dt} \quad (2-31)
\]
The integrals of (2-30) and (2-31) are respectively,

\[ r(t) = c_3 R(t) \quad (2-32) \]

and

\[ \frac{u_1}{i_1}(t) = c_4 U_1(t) \quad (2-33) \]

where \( c_3 \) and \( c_4 \) are integration constants. Furthermore, we choose

\[ \Omega^2 = \omega^2, \quad c_3 = 1, \quad c_4 = \frac{1}{U_r} = \frac{1}{h_1}. \quad (2-34) \]

The last relation gives \( b(t) = B(t) \). It is therefore obvious that with these choices the two sets of equations, (2-20), (2-21) and (2-26), (2-27) are now identical to each other. Besides, the boundary conditions for both sets are the same, as given in (2-22).

The correlation mentioned is thus established. That is, under the conditions given in (2-32), (2-33) and (2-34), the nondimensional functions \( \bar{u} \) and \( \bar{w} \) may be found from the incompressible flow equations (2-23), (2-24) and (2-25) instead of the more complicated equations (2-1), (2-2), (2-3), (2-4), and (2-5).

In order to relate corresponding solutions in the physical and the transformed planes, it is necessary
first to establish explicit relations among $X$, $Y$, $x$, $y$; $R$, $r$; and $U_1$, $u_1$. The variables $X$ and $x$ are related through $t$ by

$$t = C \int_0^X r^2 \mu_1 \rho_1 u_1 dx = L^2 \mu_s \rho_s \int_0^X U_1 dx \quad (2-35)$$

from (2-13) and (2-29) respectively. Therefore,

$$(dX/dx) = Cr^2 \mu_1 \rho_1 u_1 / L^2 \mu_s \rho_s U_1$$

(2-36)

It is convenient here to choose $\mu_s$ and $\rho_s$ as the stagnation quantities of the potential flow outside the compressible boundary layer, and $L$ as $r_m$, the maximum $r$ of the body.

Since the potential field at the outer edge of the compressible boundary layer can be taken as isentropic, it follows

$$\frac{\mu_1 \rho_1}{\mu_s \rho_s} = \left[ 1 + \frac{r-1}{2} M_1(x) \right]^{\frac{r}{r-1}}$$

and

$$\frac{u_1}{U_1} = \left( \frac{h_1}{h} \right)^{1/2} = \left[ 1 + \frac{r-1}{2} M_1^2(x) \right]^{-1/2}$$

Thus (2-36) takes the form

$$X = C \int_0^X \left( \frac{r}{r_m} \right)^2 \left[ 1 + \frac{r-1}{2} M_1^2(x) \right]^{\frac{1-3r}{2(r-1)}} dx$$

(2-37)
Furthermore,

\[ U_1(X) = a_s M_1(x), \quad R(X) = r(x), \quad \Omega = \omega, \quad (2-38) \]

where \( a_s \) is the speed of sound corresponding to the stagnation condition in the free stream.

The relation between the variables \( y \) and \( X, Y \) may be obtained as follows: Since

\[ \psi = \int d\psi = \int \left( \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right) = \int \left( \frac{\partial \psi}{\partial X} dX + \frac{\partial \psi}{\partial Y} dY \right) \]

and \( x \) is a function of \( X \) alone as given by (2-37), we have the equality

\[ (\partial \psi/\partial y)dy = (\partial \psi/\partial Y)dY, \quad \text{or} \quad \rho u dy = \rho_s u_1 dY. \quad (2-39) \]

Now, from the established correlation,

\[ \bar{u} = U/U_1 = u/u_1 \quad \text{and} \quad \bar{w} = w/w_1 = W/W_1 \quad (2-40) \]

we can write (2-39) as

\[ dy = \left( r_m \rho_s U_1 / r \rho u_1 \right) dY = \left( r_m \rho_s U_1 / r \rho u_1 \right) (i/i_1) dY \quad (2-41) \]

From (2-4) and (2-40), we have

\[ i = h_1 \left[ 1 - \left( \frac{u_1}{w_1} / \bar{h}_1 \right) - \left( \frac{w}{w_1} / \bar{h}_1 \right) - \left( \frac{\rho}{\rho_1} \bar{W} / \bar{h}_1 \right) \right] \]
and

\[ i = h_1 \left[ 1 - \left( \frac{u_1^2}{2h_1} \right) - \left( \frac{W}{2h_1} \right) - \left( \frac{\alpha RW}{h_1} \right) \right] \]

Because of (2-33) and (2-34), the above relation becomes

\[ i = h_1 \left[ 1 - \left( \frac{u_1^2}{2h_1} \right) - \left( \frac{W}{2h_1} \right) - \left( \frac{\alpha RW}{h_1} \right) \right] \]

Thus (2-41), after integration, may be written as

\[ y = \left( r_m \rho_s u_1 h_1 / r \rho_p u_1 l_1 l_1 \right) \int_0^Y \left[ 1 - \left( \frac{u_1^2}{2h_1} \right) - \left( \frac{W}{2h_1} \right) - \left( \frac{\alpha RW}{h_1} \right) \right] dY \]

or

\[ y = \left( r_m \rho_s u_1 / r \rho_p u_1 \right) \int_0^Y \left[ 1 - \left( \frac{u_1^2}{2h_1} \right) - \left( \frac{W}{2h_1} \right) - \left( \frac{\alpha RW}{h_1} \right) \right] dY \]

With the known relationship between \( X \) and \( x \) given by (2-37), the above equation is equivalent to

\[ y = A_1 Y + A_2 \int_0^Y U^2 dY + A_3 \int_0^Y W dY + A_4 \int_0^Y W dY \quad (2-42) \]

where \( A_1, A_2, A_3, \) and \( A_4 \) are known functions of \( X \).

The longitudinal skin friction is defined by

\[ c_w(x) = \left( \mu \frac{\partial u}{\partial y} \right)_{y=0} \quad (2-43) \]

Since

\[ \left( \frac{\partial}{\partial y} \right) = (\partial / \partial y) (\partial / \partial y) = r \rho_u (\partial / \partial y) \]
\[
(\partial/\partial Y) = (\partial \psi / \partial Y)(\partial \psi / \partial \eta) = r_m \rho_s U (\partial \psi / \partial \eta)
\]

and

\[
\mu \rho = \mu_1 \rho_1
\]

one obtains as an equivalent form of (2-43) as

\[
\tau_w(x) = (r \mu_1 \rho_1 u_1 / r_m \rho_s U_1) (\partial U / \partial Y)_{Y=0}
\]

This formula may be further reduced when the isentropic relations for the potential field are taken into account,

\[
\tau_w(x) = \mu_s (r / r_m) \left[ 1 + \frac{\hat{r} - 1}{2} M_1(x) \right]^{(1-2 \hat{r})/(\hat{r}-1)} \left( \frac{\partial U}{\partial Y} \right)_{Y=0} (2-44)
\]

The circumferential skin friction is defined by

\[
\tau^*_w(x) = \left( \frac{\mu \partial \omega}{\partial y} \right)_{y=0} (2-45)
\]

This formula can be similarly reduced to

\[
\tau^*_w(x) = \mu_s (r / r_m) \left[ 1 + \frac{\hat{r} - 1}{2} M_1(x) \right]^{(1-3 \hat{r})/2(\hat{r}-1)} \left( \frac{\partial \omega}{\partial Y} \right)_{Y=0} (2-46)
\]

The rate of heat transfer from the surface of the body is given by

\[
q_w = -(k/c_p)(\partial T / \partial y)_{y=0} (2-47)
\]
Since the Prandtl number, \( \sigma \), is assumed to be unity,

\[
q_w = -\left( \frac{\mu \partial i}{\partial y} \right)_{y=0} = -r \mu_1^2 \frac{1}{\rho_1 u_1} (\bar{u} \bar{d} i / \partial \psi)_{\psi=0}.
\]

Substituting

\[
\frac{\partial i}{\partial \psi} = \frac{1}{L} \left[ -u_1 \frac{\partial u_1}{\partial \psi} - (r \omega)^2 \frac{\partial \bar{w}}{\partial \psi} - \lambda (r \omega)^2 \frac{\partial \bar{w}}{\partial \psi} \right]
\]

on account of (2-4), we have

\[
q_w = \frac{r L \rho_1 u_1}{r m \rho_s U_1} \left[ \frac{u_1}{U_1} \frac{\partial U}{\partial \psi} + \frac{\partial W}{\partial \psi} + \lambda (r \omega) \frac{\partial W}{\partial \psi} \right]_{Y=0}
\]

or

\[
q_w = \mu_1 \left( \frac{r}{r m} \right) \left( \frac{\rho_1}{\rho_s} \right) u_1 (r \omega) (1+\lambda) \frac{\partial W}{\partial \psi} \bigg|_{Y=0}.
\]

We obtain finally,

\[
q_w = \frac{\mu_1 (r \omega)}{r m} \left[ 1 + \frac{5-3 \psi}{2 m_1 (x)} \right]^{(1-3 \psi) / 2 (\psi-1)} \left[ \frac{\partial W}{\partial \psi} \right]^{(1+\lambda) (\frac{\partial W}{\partial \psi})} \bigg|_{Y=0}.
\]
RESULTS AND DISCUSSION

The compressible boundary layer with surface pressure gradient on a rotating axisymmetric body has been correlated with a fictitious incompressible boundary layer whose characteristics are described by (2-23), (2-24), and (2-25). The transformations relating the physical variables to the variables in the fictitious plane are given by (2-37), (2-38) and (2-42). The explicit relation between $y$ and $X$, $Y$ can be established only after the solutions of $U(X,Y)$ $W(X,Y)$ are found. In Section 3, we will find that, when these solutions are put into a universal series form, the integrations in (2-42) can be performed once and for all.

The detailed steps of obtaining a solution for the compressible boundary layer on a rotating axisymmetric body are tabulated below:

(a) The problem of the compressible boundary layer on a rotating axisymmetric body is posed with the shape of the body and the surface variation of Mach number known as functions of $x$.

(b) The dependence of $X$ on $x$ is readily obtained from (2-37). The proportionality factor, $C$, in (2-37) is
perhaps best chosen as 

\[
\left(\frac{T_w}{T_1}\right)^{1/2} \frac{(T_1+216)/(T_w+216)}
\]

for air, the value recommended by Chapman and Rebesin. (17)

Knowing the dependence of \( X \) on \( x \), the functions \( U_1(X) \) and \( R(X) \) are obtained from (2-38).

(c) We now proceed to solve the differential system of the fictitious incompressible problem defined by (2-23), (2-24), and (2-25) with the usual boundary conditions.

(d) The solution for the velocity profiles of the compressible boundary layer are obtained from the relations

\[
\frac{U}{U_1}(x,y) = \frac{U}{U_1}(X,Y) \quad \text{and} \quad \frac{W}{\omega_T}(x,y) = \frac{W}{\omega_T}(X,Y).
\]

(e) Finally, the solution for the temperature profile in the boundary layer is obtained from (2-4).

The longitudinal and the circumferential skin frictions are given respectively by (2-44) and (2-46) in terms of the corresponding associated incompressible quantities. The rate of surface heat transfer, on the other hand, is given by (2-48) in terms of the associated incompressible circumferential skin friction. It is interesting to note that in the evaluation of the skin frictions and the surface rate of heat transfer, explicit dependence of \( y \) on \( X \) and \( Y \) is not needed.
The present analysis can be used to study the problem of how much power is required for spinning rockets or projectiles. They normally employ spin for either of two reasons: to decrease dispersion or to attain stability.

Concerning dispersion, it is known that asymmetry of the rocket (chiefly due to fin misalignment) causes the rocket to yaw more to one side than the other, with a consequent accumulation of dispersion. If, however, the rocket is spinning about its longitudinal axis, the asymmetry will be turning and will not always act in the same direction. Thus the accumulation of dispersion in any direction will be much reduced. Normally a low rate of spin will contribute a significant decrease in dispersion. (18)

Projectiles and shells, on the other hand, usually do not have fins, and their motion being stabilized entirely by spin. Therefore, a faster rate of spin is frequently used. The utilization of fast spin gives at the same time a gyroscopic effect which tends to minimize the dispersion of the rocket due to external disturbances.

Another question that can be answered by means of this analysis is how does the spin effect the location of the separation point of the laminar boundary layer.
A qualitative answer can be readily given: Since spin imparts a centrifugal acceleration of fluid elements near the surface of the body, it produces a favorable contribution to the pressure gradient upstream of the maximum section of the body and an adverse contribution to the pressure gradient downstream of this section for the following reasons:

(a) The spinning of the body will bring about a greater increase in velocity magnitude and thus lower pressure at the maximum section of the body.

and (b) The inclination of the body contour always causes a component of the centrifugal force in the direction towards the maximum section of the body.

Therefore the effect of increasing spin is always to move the position of separation nearer to the maximum section of the body. The numerical results following the analysis given in this section should give some quantitative measure as to the change of the location of laminar separation due to the spin of the body. The quantitative answer is of practical value because it is frequently desirable to know the extent of laminar boundary layer on a body in flight.
SECTION (3) REDUCTION OF DIFFERENTIAL SYSTEM INTO A UNIVERSAL FORM SUITABLE FOR NUMERICAL SOLUTION

ANALYSIS

With the correlation established in Section 2, the problem of the compressible laminar boundary layer about a rotating axisymmetric body is reduced to the finding of the solution of the differential system defined by (2-23), (2-24), and (2-25). These differential equations are:

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \]  \hspace{1cm} (3-1)

\[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{W^2}{R} \frac{\partial R}{\partial x} \left[ 1 + \frac{1}{2} \frac{(U/U_r)^2}{(U/U_r)^2} \right] \]

\[ U \frac{1}{U_1} \frac{\partial U}{\partial x} \left[ \frac{1}{2} \frac{W^2}{U_r^2} + \frac{\lambda}{U_r} \frac{\partial W}{\partial y} \right] = \nu_s \frac{\partial^2 U}{\partial y^2} \]  \hspace{1cm} (3-2)

\[ U \frac{\partial (RW)}{\partial x} + V \frac{\partial (RW)}{\partial y} = \nu_s \frac{\partial^2 U}{\partial y^2} \]  \hspace{1cm} (3-3)

with the boundary conditions

at \( Y = 0 \), \quad U = 0, \quad W = \Omega R, \quad (3-4)

at \( Y \to \infty \), \quad U = U_1, \quad W = 0.
The above differential system is amenable to series solutions. We consider, in the following, cases where \( U_1 \) (which is the velocity distribution at the outer edge of the boundary layer) and \( R \) are each represented by a general power series in \( X \). Two different general representations will be treated:

Case (I) \( U_1 = U_r \sum_{j=0}^{n} a_j X^j \) and \( R = \sum_{j=1}^{n} b_j X^j \)\(^{35} \) (3-5)

Case (II) \( U_1 = U_r \sum_{j=1}^{n} a_j X^j \) and \( R = \sum_{j=2}^{n} b_j X^j \)\(^{36} \) (3-6)

Each of these cases leads to a system of total differential equations which can be solved successively. The method, in general, expresses the functions \( U \) and \( W \) at any point as appropriate power series in \( X \) whose coefficients are functions of a new transformed variable \( \eta \). These coefficients can be put into convenient forms such that a system of universal functions (functions independent of the coefficients \( a_j, b_j, \lambda, \) and \( \Omega \)) may be defined. If these universal functions are solved from their governing differential equations, any problem is then reduced to simple arithmetic and the use of tables.
For case (I), the following forms of expansion for $\bar{U}$ and $W$ are first assumed

$$\bar{U} = \sqrt{\frac{X}{Y}} \sum_{j=0}^{n} a_j X^j F_j(\eta) \quad (3-7)$$

$$W = R \sum_{j=0}^{n} X^j G_j(\eta) \quad (3-8)$$

where the function $\bar{U}$, satisfies the relation

$$\frac{U}{U_r} = \frac{\partial \bar{U}}{\partial Y} \quad \text{and} \quad \frac{V}{U_r} = - \frac{\partial \bar{U}}{\partial X} \quad (3-9)$$

and

$$\eta = \sqrt{a_o U_r} \frac{Y/2}{\sqrt{X} S} . \quad (3-10)$$

We then substitute above into (3-1), (3-2), and (3-3) after these equations have been transformed from $X, Y$, to $X, \eta$ coordinates. Collecting terms with the common factor, $X^n$, a series of ordinary differential equations for $F_j$ and $G_j$ in terms of $\eta$ is obtained. Inspection of the differential system shows that it is possible, by expressing $F_j$ and $G_j$ into several functions in a proper way, to transform the equations and boundary conditions into forms which are independent of the coefficients $a_j$, and $b_j$, $\lambda$, and $\Omega$. Thus the solutions of these equations may be applied to all problems having $U_1$ and $R$ represented by (3-5).
It is found that the functions $f_j$ and $G_j$ can be expressed in terms of universal functions $f^j$ and $g^j$ as follows:

\[ F_0 = f_0 \quad (3-11) \]

\[ F_1 = 8 f_1 \quad (3-12) \]

\[ F_2 = 8^2 \left\{ f_{20} + \frac{a_1}{a_0 a_2} f_{21} + \frac{1+(1/2)a_0^2}{a_0 a_2} \Omega^2 f_{22} \right\} \quad (3-13) \]

\[ F_3 = 8^3 \left\{ f_{30} + \frac{a_1 a_2}{a_0 a_3} f_{31} + \frac{a_1}{a_0 a_3} f_{32} + \frac{1}{a_0 a_3} \Omega^2 f_{33} \right\} \quad (3-14) \]

\[ + \Omega^2 \left[ 1+(1/2)a_0^2 \right] \left( \frac{a_1 b_1}{a_0 a_2} f_{34} + \frac{a_2}{a_0 a_3} f_{35} + \frac{b_1 b_2}{a_0 a_3} f_{36} \right) \]

\[ G_0 = g_0 \quad (3-15) \]

\[ G_1 = \frac{a_1}{a_0} g_{11} + \frac{b_2}{b_1} g_{12} \quad (3-16) \]

\[ G_2 = \frac{b_3}{b_1} g_{21} + \frac{b_2}{b_1} g_{22} + \frac{a_1 b_2}{a_0 b_1} g_{23} + \frac{a_2}{a_0} g_{24} + \frac{a_1}{a_0} g_{25} \]

\[ + \Omega^2 \left[ 1+(1/2)a_0^2 \right] \frac{b_1}{a_0} g_{26} \quad (3-17) \]
And the governing differential systems for functions $f_j$ and $g_j$ are:

\[ f''_0 + f'_0 f'' = 0 \]

\[ f''_1 + f'_0 f'_1 - 2f'_0 f'_1 + 3f''_0 f'_1 = -1 \]

\[ L_2 f_{20} \equiv (f''_{20} + f'_0 f''_{20} - 4f'_0 f'_{20} + 5f''_0 f''_{20}) = -\frac{1}{4} \]

\[ L_2 f_{21} = -\frac{1}{8} + 2f'_1 f'' - 3f'_1 f'' \]

\[ L_2 f_{22} = -\frac{1}{8} g_0 \]

\[ L_3 f_{30} \equiv (f'''_{30} + f'_0 f'''_{30} - 6f'_{30} f'_{30} + 7f''_{30} f''_{30}) = -\frac{3}{64} \]

\[ L_3 f_{31} = 6f''_{120} - 3f'_{120} f''_{120} - 5f''_{120} f''_{120} - \frac{3}{64} \]

\[ L_3 f_{32} = 6f''_{121} - 5f''_{121} - 3f''_{121} \]

\[ L_3 f_{33} = \frac{1}{128} g_0^2 + \frac{\lambda}{64} g_0 \]

\[ L_3 f_{34} = 6f''_{122} - 3f''_{122} - 5f''_{122} \]

\[ L_3 f_{35} = -\frac{1}{32} g_0 g_{11} \]

\[ L_3 f_{36} = -\frac{3}{64} g_0^2 - \frac{1}{32} g_0 g_{12} \]

\[ g''_0 + f'_0 g'_0 - 4f'_0 g_0 = 0 \]

\[ L_{11} g_{11} \equiv (g''_{11} + f'_0 g''_{11} - 6f'_0 g'_{11}) = 32 f'_{1} g_0 - 24f'_0 g''_0 \]

\[ L_{11} g_{12} = 32f'_0 g_0 \]
\[ L^*_2 g_{21} = (g_{21}'' + f_0 g_{21}' - 8 f_0' g_{21}) = 8 f_0' g_0 \]

\[ L^*_2 g_{22} = 4 f_0' g_{12} \]

\[ L^*_2 g_{23} = 4 f_0' g_{11} + 48 f_1' g_{12} - 72 f_1' g_{12} - 144 f_1' g_0 \]

\[ L^*_2 g_{24} = 256 f_2' g_{10} - 320 f_2' g_0 \]

\[ L^*_2 g_{25} = 48 f_1' g_{10} - 72 f_1' g_{11} + 256 f_2' g_0 - 320 f_2' g_0 \]

\[ L^*_2 g_{26} = 256 f_2' g_{20} - 320 f_2' g_0 \]

where the prime denotes total differentiation with respect to \( \eta \). The symbols \( L_2, L_3, L_1, \) and \( L^*_2 \) are linear differential operators as defined in these equations.

The boundary conditions on \( f_j \) and \( g_j \) are:

\[ f_0(0) = f_0'(0) = 0 , \quad f_0'(\infty) = 2 , \]

\[ f_1(0) = f_1'(0) = 0 , \quad f_1'(\infty) = \frac{1}{4} , \]

\[ f_{ij}(0) = f_{ij}'(0) = 0 , \quad f_{ij}'(\infty) = 0 \quad \text{except} \quad f_{20}'(\infty) = \frac{1}{32} \quad \text{and} \quad f_{30}'(\infty) = \frac{1}{256} , \]

\[ g_0(0) = 1 , \quad g_0'(\infty) = 0 , \]

\[ g_{ij}(0) = 0 , \quad g_{ij}'(\infty) = 0 , \]
The above differential system covers four terms expansion for \( \Psi \) and three terms expansion for \( W \). Universal functions correspond to more terms in series expansion of \( \Psi \) and \( W \) can be similarly obtained. Note here the differential equations for \( f_j \) are rather similar to those treated in Refs. 19 and 20 and the differential equations for \( g_j \) are to those treated in Ref. 21. In fact, the differential equations for \( f_0, f_1, f_2, l, f_{32} \) are identical with the corresponding ones given in Ref. 19 and their numerical solutions are thus available.

For case (II), the series representation of \( R \) starts with square term in \( X \). The following form for \( \Psi \) and \( W \) are assumed.

\[
\Psi = \int g_s/a_1 u_r \sum_{j=1}^{n} a_j x^j \bar{f}_j(\eta) \quad (3-18)
\]

\[
W = \Omega R \sum_{j=0}^{n} x^j g_j(\eta) \quad (3-19)
\]

where \( \Psi \) satisfies

\[
\frac{\partial \Psi}{u_r} = \frac{\partial^2 \Psi}{\partial y} , \quad \frac{V}{u_r} = -\frac{\partial \Psi}{\partial x} \quad (3-20)
\]

and \( \eta = \sqrt{u_r a_1/v_s} y \) .

(3-21)
Using the same mathematical manipulation as that in case (I), it is found that the functions $\bar{F}_j$ and $\bar{G}_j$ can be expressed in terms of universal functions $\bar{f}_j$ and $\bar{g}_j$ as follows:

\[
\bar{F}_1 = \bar{f}_1 , \quad (3-22)
\]
\[
\bar{F}_2 = \bar{f}_2 , \quad (3-23)
\]
\[
\bar{F}_3 = \bar{f}_{31} + \frac{a_2^2}{a_1 a_3} \bar{f}_{32} + \frac{b_2}{a_1 a_3} \Omega^2 \bar{f}_{33} , \quad (3-24)
\]
\[
\bar{F}_4 = \left\{ \bar{f}_{41} + \frac{a_2 a_3}{a_1 a_4} \bar{f}_{42} + \frac{a_3}{a_1 a_4} \bar{f}_{43} + \Omega^2 \left( \frac{a_2 b_2}{a_1 a_4} \bar{f}_{44} + \frac{b_2 a_3}{a_1 a_4} \bar{f}_{45} \right) \right\} , \quad (3-25)
\]
\[
\bar{G}_0 = \bar{g}_0 , \quad (3-26)
\]
\[
\bar{G}_1 = \left\{ \frac{b_3}{b_2} \bar{g}_{11} + \frac{a_2}{a_1} \bar{g}_{12} \right\} , \quad (3-27)
\]
\[
\bar{G}_2 = \left\{ \frac{b_3^2}{b_2} \bar{g}_{21} + \frac{a_2 b_3}{a_1 b_2} \bar{g}_{22} + \frac{b_4}{b_2} \bar{g}_{23} + \frac{a_2}{a_1} \bar{g}_{24} \right\}
\]
\[ + \frac{a_3}{a_1} \gamma_{25} + \frac{b_2}{a_2} \gamma_{26} \]  

(3-28)

And the governing differential systems for functions \( f_j \) and \( g_j \) are:

\[ \bar{T}_1 + T_1 T_1'' - (T_1)'^2 = -1 \]

\[ \bar{T}_2 + T_2 T_2'' - 3T_1 T_1'' + 2T_1 T_1'' = -3 \]

\[ L_3 \bar{T}_1 \equiv (\bar{T}_1'' + T_1 T_1'' - 4T_1 T_1' + 3T_1'' T_1') = -4 \]

\[ L_3 \bar{T}_2 = -2 + 2(T_2)'^2 - 2T_2 T_2'' \]

\[ L_3 \bar{T}_3 = -2g_0 \]

\[ L_4 \bar{T}_1 \equiv (\bar{T}_1''' + T_1 T_1''' - 5T_1 T_1'' + 4T_1'' T_1') = -5 \]

\[ L_4 \bar{T}_2 = 5T_2 T_2'' - 5 - 2T_1 T_2'' - 3T_1'' T_2' \]

\[ L_4 \bar{T}_3 = 5T_2 T_2'' - 2T_1 T_2'' - 3T_2'' T_2' \]

\[ L_4 \bar{T}_3 = 5T_2 T_2'' - 2T_2 T_2'' - 3T_2'' T_2' - 4g_0 g_1 \]

\[ L_4 \bar{T}_4 = -4g_0 g_{11} - 5g_0 \]

\[ g_0 + T_1 g_0 - 4T_1 g_0 = 0 \]

\[ L_{11} \bar{g}_{11} \equiv (g_{11}'' + T_1 g_{11}' - 5T_1 g_{11}) = 2T_1 g_0 \]
\[ \begin{align*}
L^*_1 e_{12} &= 4 \bar{f}'_{2 \bar{g}} e_0 - 2 \bar{f}_{2 \bar{g}} e_0 \\
L^*_2 e_{21} &= (\bar{e}'_{21} + \bar{f}'_{1 \bar{e}} e_{21} - 6 \bar{f}_{1 \bar{e}} e_{21}) = 2 \bar{f}'_{1 \bar{e}} e_{11} - 2 \bar{f}_{1 \bar{e}} e_{0} \\
L^*_2 e_{22} &= \bar{f}'_{1 \bar{e}} e_{12} + 5 \bar{f}'_{2 \bar{e}} e_{11} - 2 \bar{f}_{2 \bar{e}} e_{0} - 2 \bar{f}_{2 \bar{e}} e_{11} \\
L^*_2 e_{23} &= 2 \bar{f}'_{2 \bar{e}} e_{0} \\
L^*_2 e_{24} &= 5 \bar{f}'_{2 \bar{e}} e_{12} - 2 \bar{f}_{2 \bar{e}} e_{12} - 4 \bar{e}_{0} \bar{f}'_{32} - 3 \bar{e}_{0} \bar{f}_{32} \\
L^*_2 e_{25} &= 4 \bar{e}_{0} \bar{f}'_{31} - 3 \bar{e}_{0} \bar{f}_{31} \\
L^*_2 e_{26} &= 4 \bar{e}_{0} \bar{f}'_{33} - 3 \bar{e}_{0} \bar{f}_{33}
\end{align*} \]

where the prime denotes differentiation with respect to \( \bar{e}_{1} \). The symbols \( L_3 \), \( L_4 \), \( L^*_1 \), and \( L^*_2 \) are linear differential operators as defined.

The corresponding boundary conditions are

\[ \begin{align*}
\bar{f}_1(0) &= \bar{f}'_1(0) = 0 , \quad \bar{f}_1(\infty) = 1 , \\
\bar{f}_2(0) &= \bar{f}'_2(0) = 0 , \quad \bar{f}_2(\infty) = 1 , \\
\bar{f}_{ij}(0) &= \bar{f}'_{ij}(0) = 0 , \quad \bar{f}_{ij}(\infty) = 0 \text{ except } \bar{f}'_{31}(\infty) = 1 , \\
\bar{g}_0(0) &= 1 , \quad \bar{g}_{ij}(\infty) = 0 , \\
\bar{g}_{ij}(0) &= 0 , \quad \bar{g}_{ij}(\infty) = 0 ,
\end{align*} \]
The above differential system covers four terms expansion for $\Psi$ and three terms expansion for $W$. Universal functions correspond to more terms in series expansion of $\Psi$ and $W$ can be similarly obtained. Note here the differential equations for $f_1, f_2, f_3^1, f_3^2, f_4^1, f_4^2,$ and $f_4^3$ are identical with the corresponding ones given in Ref. 20 and their numerical solutions are thus available.
RESULTS AND DISCUSSION

Case (I):

\[
U = \frac{1}{2} \sum_{j=0}^{n} a_j x^j f_j'(\eta) = a_0 f_0' + 2 a_1 x f_1' + \frac{a_2}{2} x^2 \left\{ f_{20}' + \frac{a_1}{a_0 a_2} f_{21}' + \frac{[1+(1/2)a_0^2]b_1^2}{a_0^2} \Omega_f^{12} \right\} + \frac{3a_3}{2} x^3 \left\{ f_{30}' \right\} \\
+ \frac{a_1 a_2}{a_0 a_3} f_3' + \frac{a_1}{a_0 a_3} f_3' + \frac{a_2}{a_0 a_3} f_3' + \frac{a_1 b_1}{a_0 a_3} f_3' + \frac{a_2 b_2}{a_0 a_3} f_3' + \left( \frac{a_1 b_1}{a_0 a_3} \cdot f_3' + \frac{a_1 b_1}{a_0 a_3} f_3' + \frac{b_1 b_2}{a_0 a_3} f_3' \right) + \cdots \\
W = \Omega R \sum_{j=0}^{n} X^j G_j(\eta) = \Omega R \left\{ g_0 + X(\frac{a_1}{a_0} g_{11} + \frac{b_2}{b_1} g_{12}) \right. \\
+ X^2 \left( \frac{a_1 b_1}{a_0 b_1} g_{21} + \frac{b_2}{b_1} g_{22} + \frac{a_1 b_2}{a_0 b_1} g_{23} + \frac{a_2}{a_0} g_{24} + \frac{a_1}{a_0} g_{25} + \frac{a_2}{a_0} g_{26} \right) + \cdots \right\}
\]
\[
(0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} + (0) \mathcal{E}^T (0) \frac{v^T r}{\mathcal{E}^T} (0) \mathcal{E}^T (0) \frac{v^T r}{\mathcal{E}^T} = \left\{ \begin{array}{l}
\ldots + (0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} + (0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} \ni \mathcal{U} = \left( x \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} \right) \\
\ldots + \left[ (0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} + (0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} \right] + \left[ (0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} + (0) \mathcal{E}^T \frac{v^T r}{\mathcal{E}^T} \right]
\end{array} \right.
\]

Case (II)
\[ + \frac{b_2^2}{a_1^3} \Omega \frac{f_{33}''(0)}{\Omega} + a_4^4 \left[ \frac{f_{41}''(0)}{a_1^4} + \frac{a_2^2 a_3}{a_1^4} \frac{f_{42}''(0)}{\Omega} \right] + \frac{a_2^3}{a_1^4} \frac{f_{43}''(0)}{\Omega} + \Omega^2 \frac{a_2 b_3}{a_1^4} \frac{f_{44}''(0)}{\Omega} + \Omega^2 \frac{b_2 b_3}{a_1^4} \frac{f_{45}''(0)}{\Omega} + \cdots \]

\[
\frac{\partial W}{\partial Y} = (a_1 U_r / \gamma_s)^{1/2} \Omega R \sum_{j=0}^{n} x^{j} \frac{g_j'(0)}{\Omega} = (a_1 U_r / \gamma_s)^{1/2} \Omega R \left\{ \begin{array}{c}
\frac{b_2^2}{a_1^3} \frac{g_{11}''(0)}{\Omega} + \frac{a_2^2}{a_1} \frac{g_{12}''(0)}{\Omega} + x^2 \left[ \frac{b_2^2}{a_1^3} \frac{g_{22}'''(0)}{\Omega} + \frac{b_4}{a_1} \frac{g_{23}'''(0)}{\Omega} + \frac{a_2^2}{a_1^3} \frac{g_{24}'''(0)}{\Omega} + \frac{a_2^3}{a_1} \frac{g_{25}'''(0)}{\Omega} \right] + \Omega^2 \frac{b_2^2}{a_1^3} \frac{g_{26}'''(0)}{\Omega} + \cdots \end{array} \right\}
\]

All the unknown universal functions are governed by
linear total differential equations having two-point boundary conditions. They can be solved in succession without resorting to trial and error by either the method given in Ref. (22) or by the method used in Ref. (17). The former one is preferable because of its simplicity and more general applicability.

Due to the fact that successive universal functions depend upon some of the preceding ones, digital computations must be made if accurate numerical solutions are to be obtained.

The numerical integrations can be carried out by using desk computers. However, this would require excessive time and effort because of the large number of equations involved. These equations can be integrated more efficiently by using digit computing machines. On account of the unavailability of the machines, the unknown universal functions are yet to be evaluated.
"SIMILAR" SOLUTIONS OF THE COMPRESSIBLE
BOUNDARY LAYER EQUATIONS

INTRODUCTION

A classic problem of viscous fluid is that of flow along a flat plate. The exact solution of the boundary layer equations for this problem was obtained by Blasius.\(^{(23)}\) He used a "similarity" transformation of variables such that a system of partial differential equations was reduced to a total differential equation. Here by "similarity" is meant that the velocity profiles inside the boundary layer are functions of a single variable of the form, \(y/\sqrt{x}\), where \(x, y\) are distances measured along and perpendicular to the surface respectively. (See Fig. 2)

The similarity approach of Blasius has been successfully used for obtaining solutions of various thermal and velocity boundary layer problems. One of these is the case of a general power-law local free-stream velocity
distribution. The momentum equations were first solved by Falkner and Skan. The energy equation was later treated independently by Eckert and Drewitz and Tifford. The similarity concept was employed throughout.

A related question arises, quite naturally, as to what special forms of the variation of local free-stream velocity with $x$ allow solutions with "similar" velocity and temperature distributions inside the boundary layer.

For an incompressible fluid, it has been shown by Goldstein, Oldroyd, and more generally by Thwaites and Tifford that various families of "similar" velocity profiles exist for certain classes of local free-stream representation.

For a compressible fluid having a constant local free-stream velocity, exact solutions of the boundary layer equations have been given by Busemann and Karman and Tsien for the special case in which the Prandtl number is unity. More extensive numerical solutions have been obtained by Brainerd and Emmons for a series of Prandtl numbers. In all these cases, conditions depend only on a single variable proportional
to, \( y/x^{1/2} \), as in the incompressible cases. Consequently the solutions merely involved the integration of total differential equations.

"Similar" solutions of the compressible boundary layer equations for the case of variable local free-stream were first investigated by Illingworth \(^{33}\) who concluded that such solutions exist only for constant local free-stream. The same conclusion was arrived at by Howarth. \(^{34}\) Somewhat later, Stewartson \(^{35}\) and Illingworth \(^{36}\) independently demonstrated that when the Prandtl number is unity, a compressible boundary layer problem may be correlated to an associated incompressible one. Thus the "similar" solutions of the incompressible boundary layer problems readily lead to solutions of certain compressible problems which are also called the "similar" solutions by Stewartson. The question as to what forms of the variation of local free-stream velocity admit "similar" solutions of the compressible boundary layer problems was reexamined by Li and Nagamatsu. \(^{37}\) Families of "similar" solutions for certain classes of local free-stream representation in terms of a transformed variable were obtained which are analogous in form to that of the incompressible cases given by Tifford in Ref. 30.
In the second part of this dissertation "similar" solutions of the compressible boundary layer equations for two-dimensional as well as rotating axisymmetric bodies are sought. For the two-dimensional problem, Crocco's transformations (38) are used, rather than those of Stewartson employed by Li and Nagamatsu, in the expectation of establishing a more general condition for the local free-stream velocity distribution yielding "similar" solutions. The development is presented in Section 4.

In Section 5, the investigation of "similar" solutions of the compressible boundary layer on rotating axisymmetric bodies is presented. In this case it is found advantageous to utilize an approach analogous to Stewartson's employed by Li and Nagamatsu. (39)
SECTION 4 "SIMILAR" SOLUTIONS OF THE TWO-DIMENSIONAL COMPRESSIBLE BOUNDARY LAYER EQUATIONS

ANALYSIS

The two dimensional compressible boundary layer equations are well known. They are

\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \]  

(4-1)

\[ \rho (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y}) \]  

(4-2)

\[ \frac{\partial p}{\partial y} = 0 \quad \text{or} \quad p = p_1 \]  

(4-3)

\[ c_p \rho (u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) - u \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} (k \frac{\partial T}{\partial y}) + \mu (\frac{\partial u}{\partial y})^2 \]  

(4-4)

The usual boundary conditions are

at \( y = 0 \), \( u = v = 0 \), \( T = T_w(x) \),  

(4-5)

at \( y \to \infty \), \( u = u_1(x) \), \( T = T_1(x) \).

For adiabatic free stream, we have

\[ (c_p T_1) + \left( \frac{u_1^2}{2} \right) = \text{constant} = h_1 \]  

(4-6)

Furthermore, on account of physical considerations,
\( v = 0 \) and \( \partial u / \partial y = 0 \) at \( y \to \infty \), it follows directly from (4-2) that

\[
\rho_{\text{lu}} \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial x} \tag{4-7}
\]

Crocco's transformation, \((38)\) using \( x \) and \( u \) as independent variables will be introduced into the above differential system. The derivatives \( \partial Z / \partial x \) and \( \partial Z / \partial y \) of \( Z(x,y) \) in terms of the new variables \( x \) and \( u(x,y) \) are

\[
\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial Z}{\partial u} \quad , \quad \frac{\partial Z}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial Z}{\partial u} \tag{4-8}
\]

where the differentiation with respect to the new variables have been expressed by the subscript notation. The variable \( y \) can now be considered as a function of \( x \) and \( u \), i.e.,

\[
y = y(x,u) \tag{4-9}
\]

Differentiating (4-9) with respect to \( x \), we have

\[
\frac{\partial y}{\partial x} = y_x + y_u \frac{\partial u}{\partial x} .
\]

This equation is equivalent to
\[
\frac{\partial u}{\partial x} = -\frac{y_x}{y_u}
\]

(4-10)

because \((\partial y/\partial x)\) is identically zero.

Differentiating (4-9) with respect to \(y\), we have

\[
\frac{\partial y}{\partial y} = y_x \frac{\partial x}{\partial y} + y_u \frac{\partial u}{\partial y}
\]

This equation is equivalent to

\[
\frac{\partial u}{\partial y} = \frac{1}{y_u}
\]

(4-11)

because \((\partial y/\partial y) = 1\) and \((\partial x/\partial y) = 0\).

Therefore

\[
\frac{\partial z}{\partial x} = z_x - (y_x/y_u)z_u \quad \text{and} \quad \frac{\partial z}{\partial y} = (1/y_u)z_u
\]

(4-12)

With the above transformation, (4-1) and (4-2) become respectively

\[
uy_u \rho_x - y_x (\rho u)_u + (\rho v)_u = 0
\]

(4-13)

\[
-\rho (uy_x - v) = -y_u \frac{dp}{dx} + (\mu/y_u)_u
\]

(4-14)

In addition, assuming constant value of specific heat of the fluid at constant pressure and introducing enthalpy, \(i\), (4-4) becomes,
\[ \rho \left[ u_i \frac{\partial u}{\partial x} - \left( \frac{y}{y_u} \right) u_i u + (v/y_u) i_u \right] - u \frac{dp}{dx} \]

\[ = \left( \frac{1}{y_u} \right) \frac{k}{c_p} i_u/y_u + \frac{\mu}{y_u}^2 \]

(4-15)

where \((dp/dx)\) is used instead of \((\partial p/\partial x)\) on account of (4-3).

Differentiating (4-14) with respect to \(u\) and eliminating \(v\) in combination with (4-13), we have

\[-u(\rho y_u) u = - (y_u) u \frac{dp}{dx} + \left( \frac{\mu}{y_u} \right) u u\]

(4-16)

Eliminating \(v\) from (4-15) in combination with (4-14), we have after simplification,

\[ \rho u i_x - (i_u + u) \frac{dp}{dx} + \frac{1}{y_u} \left( \frac{\mu}{y_u} \right) u u = \frac{1}{y_u} \left[ \left( \frac{k}{c_p} i_u / y_u \right) u \right] + \frac{\mu}{y_u}^2 \]

(4-17)

Now the number of differential equations defining the two-dimensional compressible boundary layer reduces from three to two. Also, the structure of these equations suggests the introduction of the viscous stress, \(\tau\), instead of \(y\) as one of the dependent variables, where the definition of the viscous stress is

\[ \tau = \mu (\partial u / \partial y) = \frac{\mu}{y_u} \]

(4-18)
Hence (4-16) becomes

\[ u(\rho u/\tau)_x + \tau_{uu} - \frac{dp}{dx} (\mu/\tau)_u = 0 \]  \hfill (4-19)

and (4-17) becomes

\[ \rho u i_x - (i_u + u)i_u + \frac{1}{\mu} \tau_i u - \frac{k}{c_p} \mu i_u \tau - \frac{\tau^2}{u} = 0 \]

Under the assumption of constant Prandtl number, \( \sigma \), the above equation takes the form

\[ (1-\sigma)\tau u i_u + (i_{uu} + \sigma)\tau^2 - \sigma \rho u i_x + \sigma \mu (i_u + u) \frac{dp}{dx} = 0 \]  \hfill (4-20)

The corresponding boundary conditions are

at \( u = u_1(x) \), \( \tau = 0 \), \( i = i_1(x) \), \hfill (4-21)

at \( u = 0 \), \( \tau = \tau_w(x) \), \( i = i_w(x) \).

The above two differential equations, (4-19) and (4-20), are those given by Crocco.\(^{(38)}\) In combination with the boundary conditions specified in (4-21), they constitute a differential system which governs the two-dimensional compressible boundary layer having pressure gradient in the free stream. This form of the boundary layer equations is of practical interest in that it expresses the principal physical quantities \( \tau \) and
i as the unknown variables.

In order to seek "similar" solutions, it is found advantageous of introducing further transformations such that the boundary conditions at the outer edge of the boundary layer are independent of $x$. This is accomplished by changing the independent variables from $x$, $u$ to $x$, $\bar{u}$ and the dependent variables from $\tau$ and $i$ to $\tau$ and $h$ where

$$\bar{u} = \frac{u}{u_1} \quad \text{and} \quad h = \left(1 + \frac{1}{2} \frac{u^2}{u_1^2}\right)/\frac{h}{h_1} = \frac{h}{h_1}.$$  (4-22)

The derivatives $Z_x$ and $Z_u$ of a function $Z(x,u)$ in terms of the new variables $x$ and $\bar{u}$ are

$$Z_x = \frac{\partial Z}{\partial x} - \frac{du_1}{dx} \frac{\bar{u}}{u_1} \frac{\partial Z}{\partial \bar{u}} \quad \text{and} \quad Z_u = \frac{1}{u_1} \frac{\partial Z}{\partial u}$$  (4-23)

Substituting (4-22) and (4-23) into (4-19) and (4-20), we have

$$u_1 \bar{u} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \tau}\right) - \frac{du_1}{dx} \frac{\bar{u}}{u_1} \frac{\partial}{\partial \bar{u}} \left(\frac{\partial u}{\partial \tau}\right) + \frac{1}{u_1} \frac{\partial^2 \tau}{\partial \bar{u}^2} - \frac{1}{u_1} \frac{du}{dx} \frac{\partial}{\partial \bar{u}} \left(\frac{\partial \bar{u}}{\partial \tau}\right) = 0$$  (4-24)

and

$$\frac{(1-\sigma)}{2} \frac{1}{u_1} \frac{\partial}{\partial \bar{u}} \left(\frac{h_1 \frac{\partial h}{\partial \bar{u}}}{u_1 \frac{\partial \bar{u}}{\partial u}} - u_1 \bar{u}\right) + \left(\frac{h_1 \frac{\partial h}{\partial \bar{u}}}{u_1 \frac{\partial \bar{u}}{\partial u}} + \sigma - 1\right) \bar{\zeta}$$
The boundary conditions become

\begin{align*}
\text{at } & \bar{u} = 1, \quad \tau = 0, \quad \bar{h} = 1, \\
\text{at } & \bar{u} = 0, \quad \tau \frac{\partial \tau}{\partial \bar{u}} = \mu \bar{u} \frac{dp}{dx}, \quad \bar{h} = \bar{h}_w
\end{align*}

(4-26)

where \( \tau (\partial \tau / \partial \bar{u}) = \mu \bar{u} \) (dp/dx) at the wall follows from the equation of motion in the form of (4-14).

This equation may be written as

\[-\rho (u_y - v) = -\frac{\mu}{\tau} \frac{dp}{dx} + \tau_u.\]

Since both \( u \) and \( v \) vanish at the wall, we obtain from the above relation

\[\tau \frac{\partial \tau}{\partial \bar{u}} = \mu \bar{u} \frac{dp}{dx}, \quad \text{at } \bar{u} = 0.\]

Examining (4-24) and (4-25), we see that these two equations will be greatly simplified by adopting the following assumptions:

(i) That the coefficient of viscosity varies as the absolute temperature, i.e.,

\[\mu = \mu_0 \frac{T}{T_0}.\]
(ii) That the Prandtl number is unity.

Following the first assumption and the equation of state for a perfect gas, we have

\[ \rho \mu = \rho_0 \mu_0 \frac{p_1}{p_0} \]  \hspace{1cm} (4-27)

Furthermore

\[ \rho \frac{d\rho}{dx} = \rho_1 \frac{d\rho_1}{dx} \]  \hspace{1cm} (4-28)

And

\[ \frac{d\rho}{dx} = - \rho_1 \frac{d\rho_1}{dx} \]  \hspace{1cm} (4-29)

follows directly from (4-2).

Combining (4-28) with (4-22), we have

\[ \rho_1 \rho = (h_1/\rho_1) \bar{h} - (u_1^2/2 \rho_1) \bar{u}^2 \]  \hspace{1cm} (4-30)

Under these two assumptions, (i) and (ii), and using the relations (4-27), (4-28), (4-29), and (4-30), the two governing differential equations (4-24) and (4-25) can be reduced to the following form:

\[ \frac{1}{\rho \mu n u_1 u_1} \tau \frac{d^2 \tau}{d\bar{u}^2} - \frac{u_1 \bar{u}}{\bar{u} h_1 \tau} \frac{d\tau}{dx} + \frac{d\bar{h}}{d\bar{u}} - \frac{2v-1}{v-1} \frac{u_1}{\bar{u}} \bar{u}^2 \]

\[ -(h-\bar{u}^2) \frac{1}{\tau} \frac{d\tau}{d\bar{u}} = 0 \]  \hspace{1cm} (4-31)
and
\[
\frac{1}{\rho \mu h_1 u_1} \cdot \chi^2 \frac{\partial^2 \tilde{h}}{\partial u^2} - \frac{1}{h_1 u_1} \cdot \frac{u}{\partial x} \cdot \frac{\partial \tilde{h}}{\partial u} - (\bar{n} - \bar{n}^2) \frac{\partial \tilde{h}}{\partial u} = 0 \tag{4-32}
\]

where the prime denotes total differentiation.

Further step in the search for "similar" profiles is by assuming the shearing stress, \( \tau \), to be of the form
\[
\tau = \chi(x) \tilde{\tau}(\bar{u}, x). \tag{4-33}
\]

Hence, (4-31) and (4-32) may be written respectively as
\[
\frac{\chi^2 \tilde{\tau}}{\rho \mu h_1 u_1} \cdot \frac{\partial^2 \chi}{\partial u^2} - \frac{1}{h_1 u_1} \cdot \frac{u}{\partial x} \cdot \frac{\chi'}{\chi} - \frac{1}{\chi} \cdot \frac{\partial \tilde{\tau}}{\partial x} + \frac{\partial \tilde{h}}{\partial u} - (\bar{n} - \bar{n}^2) \frac{1}{\chi} \cdot \frac{\partial \tilde{\tau}}{\partial u} = 0 \tag{4-34}
\]

and
\[
\frac{\chi^2 \tilde{\tau}}{\rho \mu h_1 u_1} \cdot \frac{\partial^2 \chi}{\partial u^2} - \frac{1}{h_1 u_1} \cdot \frac{u}{\partial x} \cdot \frac{\partial \tilde{h}}{\partial u} - (\bar{n} - \bar{n}^2) \frac{\partial \tilde{h}}{\partial u} \cdot \frac{\partial \tilde{h}}{\partial u} = 0 \tag{4-35}
\]

A convenient choice of the function \( \chi \) corresponds to
\[
\chi = (\rho \mu h_1 u_1) \cdot \frac{1}{\sqrt{1}}. \]
It may be reduced to a simpler form by using (4-27) and the equation of state

\[ \chi = (c_\mu \rho_1 u_1^2)^{1/2} \]  

(4-36)

With this choice, the first term in each of (4-34) and (4-35) is now a function of \( u \) alone. Closer examination of (4-34), (4-35), and the boundary conditions given in (4-26) indicates that the existence of "similar" solutions is assured provided that \( h_w \) is constant and

\[ \left( \frac{i_1 u_1}{u_1} \right) \frac{\chi'}{\chi} + \frac{2 \gamma - 1}{\gamma - 1} u_1^2 = \alpha \]  

(4-37)

where \( \alpha \) is an arbitrary constant.

From (4-36), we obtain by differentiation

\[ \frac{\chi'}{\chi} = \left( \frac{u_1'}{u_1} + \frac{1}{2} \frac{\rho_1'}{\rho_1} + \frac{1}{2} \frac{u_1''}{u_1} \right) \]

Substituting the above relation into (4-37), we have

\[ \frac{i_1 u_1}{u_1} \left[ \frac{u_1'}{u_1} + \frac{1}{2} \left( \frac{\rho_1'}{\rho_1} + \frac{u_1''}{u_1} \right) \right] + \frac{2 \gamma - 1}{\gamma - 1} u_1^2 = \alpha \]  

(4-38)

Since the local free stream is assumed isentropic, then any two of the three functions \( i_1(x) \), \( \rho_1(x) \), and
u_1(x) can be expressed in terms of the other one by the isentropic relationships, for example:

\[
\frac{T_1}{T_s} = \frac{1}{h_1} = 1 - \frac{r-1}{2} \left( \frac{u_1}{a_s} \right)^2 \tag{4-39}
\]

\[
\frac{\rho_1}{\rho_s} = \left[ 1 - \frac{r-1}{2} \left( \frac{u_1}{a_s} \right)^2 \right]^\frac{1}{r-1} \tag{4-40}
\]

By differentiating (4-40), the following relation can be obtained.

\[
\frac{\rho_1'}{\rho_1} = -\left[ 1 - \frac{r-1}{2} \left( \frac{u_1}{a_s} \right)^2 \right]^{-1} \frac{u_1 u_1'}{a_s^2} \tag{4-41}
\]

Substituting (4-39), (4-40), and (4-41) into (4-38) and simplifying, we have

\[
1 + \frac{u_1 u_1''}{2(u_1')^2} + \left[ 1 - \frac{r-1}{2} \left( \frac{u_1}{a_s} \right)^2 \right]^{-1} \left[ \frac{(4r-3)u_1^2}{2a_s^2} - \frac{\alpha(r-1)}{a_s^2} \right] = 0 \tag{4-42}
\]

The above relation of u_1 represents a sufficient condition for the existence of "similar" profiles. That is, when u_1 satisfies the differential equation (4-42), the two-dimensional compressible boundary layer equations as given by (4-34) and (4-35) admit solutions of the
dependent variables \( \bar{h} \) and \( \bar{\tau} \) as functions of \( \bar{u} \) alone provided that \( \bar{h}_w \) is constant. Under these conditions, (4-34) and (4-35) reduce to

\[
\bar{\tau}^2 \bar{\tau}'' + \bar{\tau} \bar{h}' - (\bar{h} - \bar{u}^2) \bar{\tau}' - \alpha \bar{u} \bar{\tau} = 0 \tag{4-43}
\]

and

\[
\bar{\tau}^2 \bar{h}'' - (\bar{h} - \bar{u}^2) \bar{h}' = 0 \tag{4-44}
\]

where the prime denotes differentiation with respect to \( \bar{u} \).

The corresponding boundary conditions are

at \( \bar{u} = 1 \), \( \bar{\tau} = 0 \), \( \bar{h} = 1 \),

at \( \bar{u} = 0 \), \( \bar{\tau} \bar{\tau}' = -\mu_w/\mu_s \), \( \bar{h} = \bar{h}_w \). \tag{4-45}

Note that \( \bar{h}_w \) = constant corresponds to the case of isothermal surface, therefore the quantity \( \mu_w/\mu_s \) is also a constant.

We will now return to the condition of \( u_1 \), (4-37), under which the "similar" solutions of \( \bar{h} \) and \( \bar{\tau} \) are obtainable.

Introducing

\[
u_\ast = u_1/a_s \quad \text{and} \quad \beta = \alpha (\nu^2 - 1)/a_s^2 \tag{4-46}
\]

we can rewrite (4-42) as
\[
\frac{(u^2 + u u^2)}{u^2 u^2} = -\left(\frac{1}{u^2} + \left[1 - \frac{\gamma - 1}{2} \frac{u}{u^2}\right]^{-1} (4\gamma - 3 - \frac{2\beta}{\gamma})\right) \frac{1}{u^2 u^2}.
\]

or

\[
\frac{(u^2 u^2)^{1/2}}{u^2 u^2} = -\left(\frac{1 - 2\beta}{2u^2} + \left[1 - \frac{\gamma - 1}{2} \frac{u}{u^2}\right]^{-1} \left[\frac{4\gamma - 3 - (\gamma - 1)\beta}{2}\right]\right) \frac{1}{(u^2)^{1/2}}.
\]

Integrating the above equation, we have

\[
\ln(u^2 u^2) = -\frac{1 - 2\beta}{2} \ln(u^2) + \frac{4\gamma - 3 - (\gamma - 1)\beta}{(\gamma - 1)} \ln(1 - \frac{\gamma - 1}{2} \frac{u}{u^2}) + \text{constant}.
\]

That is,

\[
\ln(u^2 u^2) = \frac{(\gamma - 1)u^2}{2(1 - 2\beta)^{1/2}} \left[\frac{(4\gamma - 3 - (\gamma - 1)\beta)}{(\gamma - 1)}\right] (u^2)^{1/2} = c_5 \quad (4-47)
\]

This equation is of the form

\[
\int \frac{\gamma - 1}{(1 - \gamma)^{\gamma - 1}} \frac{\gamma - 1}{\gamma - 1} d\gamma = c_6 \quad (4-48)
\]

where \(\gamma = (\gamma - 1)u^2/2\).

A corresponding result was independently obtained by Li and Nagamatsu \(39\) by using Stewartson's approach. And it was pointed out that the integration of \(4-48\) yields

\[
\int_0^\gamma (1 - \gamma)^{\gamma - 1} \frac{\gamma - 1}{\gamma - 1} d\gamma = c_6 x + c_7 \quad (4-49)
\]
where $c_6$ and $c_7$ are the integration constants. The integral on the left hand side of (4-49) is an incomplete Beta integral which is related to the hypergeometric function

$$\int_0^1 f^{\bar{m}-1}(1-f)^{\bar{n}-1} df = \frac{1}{\bar{m}} f^{\bar{m}} \theta\{-\bar{m}; 1-\bar{n}; \bar{m}+1; f\}$$

where $\theta$ is the hypergeometric function.

The series representation of the hypergeometric function was given as

$$\theta\{-\bar{m}; 1-\bar{n}; \bar{m}+1; f\} = \sum_{j=0}^{\infty} \frac{(\bar{m})_j (1-\bar{n})_j}{(\bar{m}+1)_j (j)!} f^j$$

convergent for $|f| < 1$ and $\bar{n} > 0$.

where $(\bar{m})_j = \Gamma(\bar{m}+j)/\Gamma(\bar{m})$, $j = 0, 1, 2, 3, \ldots \ldots \ldots \ldots \ldots$

Substituting above relations into (4-49), we have the condition on $u_1$ sufficient for the existence of "similar" solutions as

$$\bar{m}^{-1} f^{\bar{m}} \sum_{j=0}^{\infty} \frac{(\bar{m})_j (1-\bar{n})_j}{(\bar{m}+1)_j (j)!} f^j = c_6 x^\frac{(f-1)^{\bar{m}+1}}{2} + c_7 \quad (4-50)$$

The expression of $u_\ast$ as a function of $x$ may thus be obtained by inversion.

Two special cases which have simple expressions
of \( u_*(x) \) are noteworthy.

Case I.

\[ \bar{m} = 1, \quad \beta = 1/2, \quad \bar{n} = \frac{-5\gamma + 3}{2(\gamma - 1)}, \]

Equation (4-49) is reduced to

\[ \int (1-\gamma)^{\bar{n}-1} d\gamma = c_6 x + c_7. \]

Thus

\[ \int = 1 - (c_8 x + c_9) \]

and

\[ u_* = \sqrt{2(\gamma - 1)} \left\{ \left[ \frac{-5\gamma + 3}{2(\gamma - 1)} \right]^{1/2} \left[ 1 - (c_8 x + c_9) \right] \right\}^{1/2} \]

When \( c_9 = 1 \), (4-51) corresponds to the form of \( u_* \) given by Stewartson.

Case II.

\[ \bar{n} = 1, \quad \beta = \frac{4\gamma - 3}{\gamma - 1}, \quad \bar{m} = \frac{-5\gamma + 3}{2(\gamma - 1)}, \]

Equation (4-49) is reduced to

\[ \int \int (m-1) d\gamma = c_6 x + c_7. \]

Thus

\[ \int = \left\{ c_{10} x + c_{11} \right\} \]

and
It is interesting to note that if various transformations of coordinate, \( z = z(x) \), analogous to that of Stewartson's were introduced, an infinite number of simple solutions \( u_\ast = u_\ast(z) \) of (4-47) may be obtained. For instance, (4-47) can be written as

\[
\frac{1}{2} \frac{1 - 2\beta}{2} \frac{x}{a_{12}} + \frac{\beta}{a_{13}} \frac{4r - 3 - \beta + \beta}{r - 1} \frac{dx}{dx}.
\]

This leads to a solution of the form

\[
u_\ast^2 = \left( c_{12} z + c_{13} \right)^{\frac{2}{3 - 2\beta}} \]

provided \( z \) is defined by

\[
z = \int_0^x \left( \frac{a_1}{a_s} \right) \frac{4r - 3 - \beta + \beta}{r - 1} \frac{dx}{dx}.
\]

Similarly, (4-47) can be written as

\[
u_\ast^{-1} \frac{d(u_\ast)}{dx} = c_5 \left( \frac{a_1}{a_s} \right)^{\frac{2r - 3 - \beta}{r - 1}} \left\{ \left( \frac{2}{r - 1} \right) \left[ 1 - \left( \frac{a_1}{a_s} \right)^{\frac{2\beta - 3}{2}} \right] \right\} \frac{dx}{dx}.
\]

This leads to a solution of the form

\[
u_\ast^2 = \text{Exp} \left[ c_{14} z + c_{15} \right].
\]

\[(4-52)\]

\[(4-53)\]

\[(4-54)\]

\[(4-55)\]
provided $z$ is defined by

$$z = \int_0^x \left( \frac{a_1}{a_s} \right)^{\frac{4\gamma - 3 - \gamma \beta + \beta}{\gamma - 1}} \left\{ 1 - \left( \frac{a_1}{a_s} \right)^2 \right\}^{\frac{2\beta - 3}{2}} \, dx \quad (4-56)$$

e tc.

Therefore the various forms of $u_\gamma(z)$ are somewhat artificial in that they are all included in the general relation given by (4-50). However, Stewartson's $z$ transformation was used in his analysis for reducing the compressible boundary layer equations into a form analogous to the incompressible boundary layer equations.
RESULTS AND DISCUSSION

A general condition on \( u_1(x) \) which permits the existence of "similar" solutions of \( \tau \) and \( h \) for two-dimensional compressible boundary layers has been established for the case of isothermal surfaces when the Prandtl number is unity. The general condition is given implicitly by (4-50). The explicit function \( u_1(x) \) can be obtained by inversion.

Two special cases of (4-50) that have simple expressions of \( u_1(x) \) are given by (4-51) and (4-52). The former one contains the form of \( u_1 \) given by Stewartson.

The "similar" solutions for the two-dimensional compressible boundary layer are defined by the differential system consisted of (4-43), (4-44) and (4-45). For this nonlinear differential system, no standard method of integration will yield solutions in a closed form. Numerical methods of integration will have to be used.

A corresponding system of equations of different form was established by Li and Nagamatsu (39) who obtained solution curves by means of analog computers. Their differential system is simpler for setting up on the
computer in that the independent variable is not explicitly involved. However, the range of integration of the independent variable extends from zero to infinity. In using the analog computers to obtain solutions of a two-point boundary value problem of which one boundary point is located at infinity, one has to choose a finite value of the independent variable at which the boundary conditions may be approximately satisfied. This suggests possibly the desirability of a differential system having a finite range of integration as given by the present analysis.

In addition, the solution curves obtained by means of analog computers of a nonlinear differential system usually do not have uniform accuracy throughout the entire range of integration. In certain region, the results tend to be highly sensitive to the chosen initial values due to the nonlinear characteristics of the equations. The present analysis and the analysis given by Li and Nagamatsu are therefore complementary rather than alternative.
SECTION (5) "SIMILAR" SOLUTIONS OF THE COMPRESSIBLE BOUNDARY LAYER EQUATIONS FOR ROTATING AXISYMMETRIC BODIES

INTRODUCTION

The analysis of the compressible boundary layer on rotating axisymmetric bodies given in Part I leads to solutions involving two independent variables. In this Section we will seek "similar" boundary layer solutions for these bodies, that is, solutions which depend upon only one transformed variable.

Two assumptions introduced in the previous Section are employed to simplify the compressible boundary layer equations of the analysis presented in this Section. They are:

(i) that the Prandtl number is unity;
and (ii) that the coefficient of viscosity varies as the absolute temperature.

No restriction as to the relative magnitude of the rotating motion and the axial motion is introduced.
ANALYSIS

Following the presentation of the first Section, the compressible boundary layer on rotating axisymmetric bodies in axial motion may be written as

\[
\frac{1}{r} \frac{\partial \psi}{\partial y} \frac{\partial u}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial u}{\partial y} - \rho \frac{\partial \psi}{\partial x} \frac{\partial u}{\partial x} = \rho \frac{u_1}{x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \tag{5-1}
\]

\[
\frac{1}{r} \frac{\partial \psi}{\partial y} \frac{\partial (\rho \psi)}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial (\rho \psi)}{\partial y} = \frac{\partial}{\partial y} \left[ \mu \frac{\partial (\rho \psi)}{\partial y} \right] \tag{5-2}
\]

\[
\frac{1}{r} \frac{\partial \psi}{\partial y} \frac{\partial (\rho \psi)}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial (\rho \psi)}{\partial y} - u \rho \frac{u_1}{x} \frac{\partial (\rho \psi)}{\partial x} = \frac{1}{\sigma} \frac{\partial}{\partial y} \left[ \mu \frac{\partial (\rho \psi)}{\partial y} \right]
\]

\[
+ \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial (\rho \psi)}{\partial y} \right)^2 \right] \tag{5-3}
\]

where the stream function, \( \psi \), is defined by

\[
\frac{\rho u}{\partial y} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\rho v}{\partial x} = -\frac{\partial \psi}{\partial x}. \tag{5-4}
\]

When the Prandtl number is unity, the energy equation (5-3) may be reduced to

\[
\frac{1}{r} \frac{\partial \psi}{\partial y} \frac{\partial S}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial S}{\partial y} = \frac{\partial}{\partial y} \left[ \mu \frac{\partial S}{\partial y} \right] \tag{5-5}
\]

by introducing a new dependent variable, \( S \), defined as.
\[ S = \frac{1}{h_1} \left( 1 + \frac{u'^2}{2} + \frac{w'^2}{2} + \lambda w^2 \right) - 1 \quad (5-6) \]

The line of thought which leads to the development of the energy integral of the first Section leads also to the choice of the new variable, \( S \). In fact, it is obvious that when \( S \) identically equals zero, \((5-6)\) reduces to the particular solution given by \((1-13)\).

The possibility of obtaining a reduction of \((5-1)\), \((5-2)\), and \((5-5)\) from partial to total differential equations will now be examined. In view of the boundary conditions,

\[ u = u_1(x), \quad w = 0, \quad S = 0 \text{ at } y = \infty \]

and

\[ u = 0, \quad w = r\omega, \quad S = S_w(x) \text{ at } y = 0, \]

we introduce the following substitutions without loss of generality.

\[ \psi = N_1(x)K(\xi), \quad u = u_1J(\xi) \]

\[ w = r\omega Q(\xi), \quad S = S_wI(\xi) \quad (5-8) \]

where \( \xi \), a function of \( x \) and \( y \), is the similarity variable whose form is yet to be determined.
Combining the definition of the stream function given by (5-4) with that in (5-8), we have

\[ rpu = N_1 K' \frac{\partial \xi}{\partial y} \]

where the prime denotes total differentiation.

Thus

\[ \frac{1}{\rho} \frac{\partial \xi}{\partial y} = ru_1 J / N_1 K' \]  \hspace{1cm} (5-9)

The above equation, (5-9), will be used to determine the similarity variable, \( \xi \). It is seen that the definition, \( rpu = \frac{\partial \psi}{\partial y} \), is automatically satisfied irrespective of the forms of \( N_1(x) \) and \( K'(\xi) \). Therefore, the functions \( N_1(x) \) and \( K'(\xi) \) may be adjusted so as to facilitate the determination of "similar" solutions.

The assumption that the coefficient of viscosity be proportional to the absolute temperature will now be introduced, i.e.

\[ \mu = c \mu_0 i / i_o \]

Since the pressure can be taken as constant across the boundary layer, we have

\[ \rho i = \rho i_1 \]

The above two relations yield,
\[ \rho \mu = c \mu \rho_{i1}^i/\rho_0 \] (5-10)

Transforming the independent variables from \( x, y \) to \( x, \mathfrak{T} \) and using the substitutions in (5-8), (5-9) and (5-10), the first equation of motion, (5-1), can be reduced to the following form

\[ u'_{1}N^{'1}K'J-u_{1}N_{1}KJ' - \frac{\omega^2 r'}{(N'_{1}/u_{1}J)}Q^2 - (u'_{1}N_{1}K'/J)(i/1) \]

\[ - C(\mu \rho_{i1}^i u_{1}^2 r'/i_{0}N_{1})(J'K')' = 0 \] (5-11)

From (5-6), we have

\[ i = h_{1}(S_{w}I + 1) - (u_{1}J^2/2) - \frac{(\omega r)^2}{2} Q^2 - \lambda (\omega r)^2 Q \]

Substituting the above relation into (5-11) and multiplying the entire equation by \((J/u_{1}N_{1}K')\), we obtain

\[ (u'_{1}/u_{1})J^2 - (N'_{1}/N_{1})(KJ'/K') - (\omega^2 r'/u_{1}^2)Q^2 \]

\[ - (u'_{1}/u_{1}i_{1})[h_{1}(S_{w}I + 1) - (u_{1}J^2/2) - \frac{(\omega r)^2}{2} Q^2 - \lambda (\omega r)^2 Q] \]

\[ - C(\mu \rho_{i1}^i u_{1}^2 r'/i_{0}N_{1})(J'K')(J'K')' = 0 \]

After rearranging terms,

\[ (N'_{1}/N_{1})(KJ'/K') + (\omega^2 r'/u_{1}^2)Q^2 \]

\[ + (u'_{1}/u_{1}i_{1})[h_{1}(S_{w}I + 1) - \frac{(\omega r)^2}{2} Q^2 - \lambda (\omega r)^2 Q] \]
Similarly, (5-2) becomes
\[ (N_1'/N_1)KQ' - (2r/r)K'Q + C'r/0N_1^2/JJ'/K' = 0 \] (5-13)
and (5-3) becomes
\[ (N_1'/N_1)K'I' - (S'/S_r)K'I_C + C'r/0N_1^2/(JJ'/K') = 0 \] (5-14)

The corresponding boundary conditions for this system of differential equations, (5-12), (5-13), and (5-14), are
\[
\begin{align*}
\text{at } \zeta = 0 : & \quad K = K' = 0, \quad Q = 1, \quad I = 1, \\
\text{at } \zeta \to \infty : & \quad K = 1, \quad Q = 0, \quad I = 0.
\end{align*}
\] (5-15)

The existence of "similar solutions of the above differential system requires that each of the equations (5-12), (5-13), and (5-14) be reducible to a form independent of \( x \). Examining these equations more closely, we note that the sufficient conditions for the existence of "similar" solutions for rotating axisymmetric bodies are, in general, redundant. This redundancy can only be avoided for the special case of rotating cylinders.
In this case, \( r = \text{constant} \) and \( u_1 = \text{constant} \). Therefore (5-12), (5-13), and (5-14) reduce respectively to

\[
\left( \frac{N'_1}{N_1} \right) KJ' + C\left( \mu_0 \rho_0 u_0 r^2 / N'_1 \right) (JJ' / K')' = 0 \quad (5-16)
\]

\[
\left( \frac{N'_1}{N_1} \right) KQ' + C\left( \mu_0 \rho_0 u_0 r^2 / N'_1 \right) (JI' / K')' = 0 \quad (5-17)
\]

\[
\left( \frac{N'_1}{N_1} \right) KL' - \left( S'_w / S_w \right) K' I + C\left( \mu_0 \rho_0 u_0 r^2 / N'_1 \right) (JI' / K')' = 0 \quad (5-18)
\]

where the identities, \( i_1 = i_0 \), \( \rho_1 = \rho_0 \), and \( u_1 = u_0 \) have been used.

The sufficient conditions that (5-16), (5-17), and (5-18) will admit "similar" solutions are obviously

\[
\left( \frac{N'_1}{N_1} \right) / C\mu_0 \rho_0 u_0 r^2 \right) = c_{16} \quad \text{and} \quad S'_w / S_w = mN'_1 / N_1 \quad (5-19)
\]

where \( c_{11} \) and \( m \) are nondimensional arbitrary constants.

Integrating the above two conditions in (5-19), we have

\[
\frac{N'_1}{r}(2C\mu_0 \rho_0 u_0^2) = (c_{16} + c_{17})^{1/2} \quad (5-20)
\]

where \( c_{17} \) may be chosen other than zero, see Results and Discussion of this Section.
\[ c_{16} = \frac{1}{2}, \quad c_{17} = 0, \quad \text{and} \quad k' = J. \]

Thus
\[ N_1 = \left( \frac{C_u \rho_o u}{x} \right)^{1/2} r \quad \text{(5-22)} \]

and
\[ S_w = c_{19} x^{m/2}. \quad \text{(5-23)} \]

From (5-6), we see that (5-23) imposes a limitation on the surface enthalpy distribution. That is, this analysis covers only those cases whose surface enthalpy distribution has the form of
\[ \left( \frac{i_w}{h_1} \right) = c_{19} x^{m/2} + c_{20}. \quad \text{(5-24)} \]

where \( c_{20} \) is an arbitrary constant, related to the arbitrary constant \( \lambda \) by \( c_{20} = 1 - \left[ (\omega r)^2 (1+2\lambda)/2h_1 \right] \).

It follows from the above choice and (5-9) that the similarity variable, \( \xi \), is now defined by
\[ \xi = \left( \frac{C_u \rho_o x}{u_o} \right)^{-1/2} u_o^{1/2} \int_0^y \rho dy. \quad \text{(5-25)} \]

Under these circumstances, (5-16), (5-17), and (5-18) become respectively
\[ K^" + 2K^' = 0 \quad \text{(5-26)} \]
The boundary conditions remain unchanged as given in (5-15), i.e.,

at $\xi = 0$; $K = K' = 0$, $Q = 1$, $I = 1$,

at $\xi \to \infty$; $K = 1$, $Q = 0$, $I = 0$. (5-15)

Equation (5-26) with its boundary conditions is identically the Blasius equation whose solution is available. (23) Equation (5-27) can be solved by direct integration in combination with the boundary conditions given in (5-15). The solution of $Q$, expressed in terms of the Blasius function, reads

$$Q = 1 - \left\{ \int_0^\xi \left[ \exp \int_0^\xi - \frac{K}{2} d\xi \right] d\xi / \int_0^\infty \left[ \exp \int_0^\xi - \frac{K}{2} d\xi \right] d\xi \right\}$$ (5-29)

No analytic solution of $I$ expressible in closed form is obtained from (5-28). For any particular value of $m$, however, this equation can be solved by numerical methods.

A special case, when $m = 0$, is noteworthy. It corresponds to the case of isothermal surfaces as can be seen from (5-6). Moreover, when $m = 0$, (5-28) reduces to
a form identical to (5-27) and their corresponding boundary conditions are also identical. Therefore, for isothermal rotating cylinders, we have

\[ I = Q \quad (5-30) \]

Note that the arbitrary constant, \( \lambda \), can be chosen as zero for this case without loss of generality.

After establishing the above solutions, the frictional and the heat transfer characteristics of a rotating cylinder can be evaluated as follows:

The longitudinal skin friction is defined by

\[ \tau_w = \left( \mu \frac{du}{dy} \right)_{y=0} \quad (5-31) \]

On account of (5-8) and (5-25)

\[ \tau_w = \left[ \mu \rho u_o^2 J'/\left( \rho \rho o u_o x \right)^{1/2} \right]_{y=0} \]

Since \( J = K' \), the above relation yields

\[ \tau_w = C^{-1/2} \left( \frac{\mu w \rho w}{\mu o \rho o} \right) \left( \frac{\rho u o x}{\rho o} \right)^{-1/2} \left[ \rho o u o K''(0) \right] \quad (5-32) \]

Defining \( c_f = 2 \tau_w / \rho o u o^2 \) and \( \text{Re}_x = (\rho o u o x / \mu o) \) and substituting \( \mu w \rho w = \mu o \rho o \), the above equation may be rewritten as
\[ c_f(Re_x)^{1/2} = 2C \frac{1}{K''(0)} \]  
(5-33)

where \( K''(0) = 0.332 \). (See Ref. 14)

The circumferential skin friction is defined by

\[ \tau_w^* = \left( \frac{\mu dw}{dy} \right)_{y=0} \]  
(5-34)

On account of (5-8) and (5-25), we have,

\[ \tau_w^* = C^{1/2} \left( \frac{\rho w}{u_o} \right) \frac{r_w}{u_o} \left( \frac{\rho \rho_x}{\mu_o} \right)^{-1/2} \left[ \rho_o u_o^2 Q'(0) \right] \]  
(5-35)

Similar to the derivation of the longitudinal skin friction, the above equation may be written as

\[ c_f^*(Re_x)^{1/2} = 2C \frac{1}{2} \left( \frac{r_w}{u_o} \right) Q'(0) \]  
(5-36)

where \( c_f^* \) is defined by \( 2\tau_w^*/\rho_o u_o^2 \).

The quantity, \( Q'(0) \), can be directly evaluated after differentiating (5-29).

\[ Q'(0) = \left\{ \int_0^\infty \left[ \text{Exp} \int_0^3 \frac{K}{2} \right] \, d\xi \right\}^{-1} \]  
(5-37)

Its numerical value was given as \( Q'(0) = -0.664 \). (See Ref. 14)

The rate of surface heat transfer is defined by

\[ q_w = -\left( k \frac{dT}{dy} \right)_{y=0} \]  
(5-38)
In terms of enthalpy, it reads

\[ q_w = -\left[ \frac{(k/c_p)(\partial i/\partial y)}{y=0} \right] = -\left[ \frac{\mu/\sigma}{y=0} \right] \]

When the Prandtl number is unity, we have

\[ q_w = -(\mu \partial i/\partial y) \quad y=0 \quad (5-39) \]

It follows from (5-6) that

\[ \frac{\partial i}{\partial y} = h_1 \frac{\partial s}{\partial y} - u \frac{\partial u}{\partial y} - (w+\omega r) \frac{\partial w}{\partial y} \]

By (5-6) and (5-9), we obtain

\[ \left( \frac{\partial i}{\partial y} \right)_{y=0} = \left[ h_1 s_w I'(0) - (\omega r)^2 (1+\lambda) Q'(0) \right] \left( \frac{\rho u_o J}{N_1 K'} \right)_{y=0} \]

Since \( J=K' \) and \( N_1 \) is given by (5-22), the above equation reduces to

\[ \left( \frac{\partial i}{\partial y} \right)_{y=0} = \left[ h_1 s_w I'(0) - (\omega r)^2 (1+\lambda) Q'(0) \right] \rho u_o \left( \frac{1}{2} \right) \left( \frac{C\mu}{\rho_o \rho_{ox}} \right)^{1/2} \quad (5-40) \]

Again from (5-6),

\[ s_w = \frac{1}{h_1} \left[ i_w + \frac{1}{2} (1+\lambda) \omega r^2 \right] - 1 \quad (5-41) \]
Substituting (5-40) and (5-41) into (5-39), we have

\[ q_w = \left\{ (i_w - c_{20} h_1) I'(0) - \left[ h_1 (1-c_{20}) + \frac{(\omega r)^2}{2} \right] Q'(0) \right\} \left( \frac{C_\rho / \mu u_0}{x} \right)^{1/2} \]  

(5-42)

where the relations, \( \lambda = \left\{ \left[ h_1 (1-c_{20}) / (\omega r)^2 \right] - (1/2) \right\} \) and \( \rho_w \mu = C_\rho / \mu u_0 \), have been employed in obtaining the above equation.

Alternatively, if (5-23) were used instead of (5-41), we would obtain,

\[ q_w = \left\{ h_1 c_1 g x m/2 I'(0) - \left[ h_1 (1-c_{20}) + (\omega r)^2 / 2 \right] Q'(0) \right\} \left( \frac{C_\rho / \mu u_0}{x} \right)^{1/2} \]  

(5-43)

The nondimensional local Nusselt number can be defined as

\[ Nu_x = \frac{q_w x}{k_w (T_w - T_0)} \]  

(5-44)

The above equation may be rewritten as

\[ Nu_x = \frac{c_p q_w x}{k_w (i_w - i_0)} \]  

(5-45)

when \( c_p \) is assumed constant.

Substituting (5-43) into (5-45), we have
\[
\frac{\text{Nu}_x}{(\text{Re}_x)^{1/2}} = \frac{i_0 \left[ h_1 (1 - c_{20}) + (\omega_r^2 / 2) \right] Q'(0) - h_1 c_19 x^{m/2} I'(0)}{i_w \left[ c^{1/2} (i_w - i_o) \right]}
\]

(5-46)

where the conditions, \( \sigma = 1 \) and \( \rho_o i_o = \rho_w i_w \) have been used.

For the special case of isothermal surfaces, \((m = 0)\), it has been shown in (5-30) that \( I = Q \) and \( \lambda \) can be conveniently chosen to vanish. Moreover, it follows from (5-24) that

\[
c_{19} = \left[ (i_w / h_1) - c_{20} \right].
\]

Consequently, (5-43) and (5-46) are reduced respectively to

\[
q_w = \left[ h_1 + (\omega_r^2 / 2) - i_w \right] \left( \frac{C \rho_o \mu u_o}{x} \right)^{1/2} Q'(0)
\]

(5-47)

and

\[
\text{Nu}_x (\text{Re}_x)^{-1/2} = C^{-1/2} \frac{i_0}{i_w} \left[ \frac{h_1 + (\omega_r^2 / 2) - i_w}{(i_w - i_o)} \right] Q'(0).
\]

(5-48)

Since \( h_1 = h_o = \left[ i_o + (u_o / 2) \right] \), the final results read as
\[ q_w = \left[ 1_0 + \left( \frac{u_0^2}{2} \right) + \left( \frac{\omega r^2}{2} \right) - \frac{1}{x} \right] \left( \frac{C_{\rho_0} \mu_0 u_0}{x} \right)^{1/2} Q'(0). \] (5-49)

and

\[ \text{Nu}_{\infty}(\text{Re}_{\infty}) = C \left( \frac{1_0}{1_w} \right) \left[ \frac{\left( \frac{u_0^2 + \omega r^2}{2} \right)}{(1_w - 1_0)} - 1 \right] Q'(0) \] (5-50)

where \( Q'(0) = -0.644. \)
RESULTS AND DISCUSSION

An investigation of "similar" solutions of the compressible boundary layer equations for rotating axisymmetric bodies in axial motion has been made. It is found that only the special case of rotating cylinders admits "similar" solutions.

The longitudinal surface frictional characteristics of a rotating cylinder in axial motion are given by both (5-32) and (5-33). These equations are identically the results of the flat plate analysis of Chapman and Rubesin. Thus the circumferential velocity component causes no effect on the longitudinal skin friction irrespective of the relative magnitude of the circumferential motion to the longitudinal motion.

The circumferential surface frictional characteristics are given by both (5-35) and (5-36). It is seen that the longitudinal motion does affect the magnitude of the circumferential skin friction.

The surface heat transfer characteristics are given by both (5-42) and (5-46) for nonisothermal surfaces whose surface enthalpy distribution is represented by (5-24). In the particular case of isothermal surfaces, the surface
heat transfer characteristics are given more simply by (5-49) and (5-50). Examining (5-49), we note that:

(a) When the surface is insulated,

\[ i_w = i_0 + \left( \frac{u_0^2}{2} + \frac{\omega^2 R^2}{2} \right) . \]

(b) When \( i_w < i_0 + \left( \frac{u_0^2}{2} + \frac{\omega^2 R^2}{2} \right) \), heat will be transferred from the fluid to the surface of the cylinder.

(c) When \( i_w > i_0 + \left( \frac{u_0^2}{2} + \frac{\omega^2 R^2}{2} \right) \), heat will be transferred from the cylinder to the fluid.

In obtaining the "similar" solutions, a judicious choice has been made, that is,

\[ c_{16} = \frac{1}{2}, \quad c_{17} = 0, \quad \text{and} \quad k' = J. \]

In the above, \( c_{16} = \frac{1}{2} \) and \( k' = J \) are chosen for the purpose of reducing the longitudinal equation of motion into a form analogous to the Blasius equation. On the other hand, \( c_{17} = 0 \) is chosen merely for simplification. In fact, the constant, \( c_{17} \), may be left as an arbitrary constant in the development. The analysis then covers those cases whose surface enthalpy distribution has the form of

\[ \left( \frac{i_w}{h_1} \right) = c_{19} \left( \frac{x}{2} + c_{17} \right)^{m/2} + c_{20} \]  \hspace{1cm} (5-51)

instead of that given by (5-24).
Although (5-51) is more general than (5-24) it can be shown that the inclusion of $c_{17}$ corresponds effectively only to a translation of the origin of the coordinate axes in the x-direction. In other words, the solution of the cases whose enthalpy distribution is given by (5-51) can be obtained from the simple case corresponding to $c_{17} = 0$ through a translation of axes. Further discussion of the physical significance of a constant similar to $c_{17}$ was given in Ref. 30.
**SYMBOLS AND UNITS**

\( a \)  
speed of sound, ft/sec

\( a_j \)  
coefficients of terms of power series
representation of \( U_1 \)  
\( (j=0,1,2,3, \ldots) \)

\( A_j \)  
functions of \( X \), see (2-42)

\( b \)  
\[ = \frac{u_1^2}{2h_1} \]

\( b_j \)  
coefficients of terms of power series
representation of \( R \)

\( B \)  
\[ = \frac{U_1^2}{2h_1} \]

\( c_j \)  
constants of integration or arbitrary constants

\( c_f \)  
longitudinal skin friction coefficient,
\[ 2\tau/\rho_0 u_0^2 \]

\( c_{f^*} \)  
circumferential skin friction coefficient,
\[ 2\tau^*/\rho_0 u_0^2 \]

\( c_p \)  
specific heat of fluid at constant pressure,
\[ ft^2/sec^2 - ^o_R \]

\( c_v \)  
specific heat of fluid at constant volume,
\[ ft^2/sec^2 - ^o_R \]

\( C \)  
proportionality factor defined by (2-12)

\( f_j \)  
universal functions of \( \eta \), see (3-11) to (3-14)

\( \Phi_j \)  
universal functions of \( \bar{\eta} \), see (3-22) to (3-25)
\( g_j \)  
universal functions of \( \eta \), see (3-15) to (3-17)

\( \bar{g}_j \)  
universal functions of \( \bar{\eta} \), see (3-26) to (3-28)

\( F_j \)  
functions of \( \eta \), see (3-7)

\( \bar{F}_j \)  
functions of \( \bar{\eta} \), see (3-18)

\( G_j \)  
functions of \( \eta \), see (3-8)

\( \bar{G}_j \)  
functions of \( \bar{\eta} \), see (3-19)

\( h \)  
total enthalpy, \( \text{ft}^2/\text{sec}^2 \)

\( \bar{h} \)  
\( = h/h_1 \)

\( H \)  
\( = (u^2/2) + (w^2/2) + \lambda \omega rw \), \( \text{ft}^2/\text{sec}^2 \)

\( i \)  
enthalpy \( = c_p T \), \( \text{ft}^2/\text{sec}^2 \)

\( \bar{i} \)  
\( = i/i_1 \)

\( I \)  
function of \( \Theta \), see (5-8)

\( J \)  
function of \( \Theta \), see (5-8)

\( k \)  
thermal conductivity of fluid, \( \text{ft}-\text{lb}/\text{ft}^2 \text{ sec} \)

\( K \)  
function of \( \Theta \), see (5-8)
L \quad \text{characteristic length, ft}

L_j, L_j^* \quad \text{linear differential operators}

m \quad \text{constant, see (5-19)}

\bar{m} \quad \text{constant, see (4-48)}

M \quad \text{Mach number}

n \quad \text{exponents, see (3-5) and (3-6)}

\bar{n} \quad \text{constant, see (4-48)}

N_x \quad \text{function of } x, \text{ see (5-8)}

Nu_x \quad \text{local Nusselt number, see (5-44)}

p \quad \text{pressure, lb/ft}^2

q \quad \text{heat transfer rate, ft-lb/ft}^2 \text{ sec}

Q \quad \text{function of } \zeta, \text{ see (5-8)}

r \quad \text{distance from the axis of symmetry to the surface, ft}

R \quad \text{distance in the fictitious problem of Sections 2 and 3 corresponding to } r, \text{ ft}

Re_x \quad \text{local Reynolds number } = (\rho_0 u_0 x / \mu_0)
\[ S = \left\{ \left[ i + \left( \frac{u^2}{2} \right) + \left( \frac{w^2}{2} + \lambda \omega \omega \right) \right] / h_1 \right\}^{-1}, \text{ see (5-6)} \]

transformed variable defined by (2-13) and (2-29)

\[ T \]

temperature, °R

\[ \bar{u} \]

velocity component in the x direction, ft/sec

\[ = \frac{u}{u_1} = \frac{U}{U_1} \]

\[ u_* \]

\[ = \frac{u}{a_s} \]

\[ U \]

velocity component in the fictitious problem of Sections 2 and 3 corresponding to u, ft/sec

\[ u_r \]

constant reference velocity \( = (2h_1)^{1/2} \), ft/sec

\[ v \]

velocity component in the y direction, ft/sec

\[ V \]

velocity component in the fictitious problem of Sections 2 and 3 corresponding to v, ft/sec

\[ w \]

spinning velocity component, ft/sec

\[ \bar{w} \]

\[ = \frac{w}{\omega R} = \frac{W}{\omega R} \]

\[ W \]

velocity component in the fictitious problem of Sections 2 and 3 corresponding to w, ft/sec

\[ x \]

distance along the surface, (in the meridian plane) see Fig. 1 for axisymmetric flow, see
Fig. 2 for two-dimensional flow, ft

X distance in the fictitious problem of Sections 2 and 3 corresponding to x, ft

y distance normal to the surface, (normal to x)
see Fig. 1 for axisymmetric flow, see Fig. 2 for two-dimensional flow, ft

Y distance in the fictitious problem of Sections 2 and 3 corresponding to y, ft

z transformed coordinate, $z = z(x)$, see (4-54) and (4-56), ft

Z arbitrary function

$\alpha$ constant, see (4-37)

$\beta$ constant, see (4-46)

$\gamma = c_p/c_v$

$\Gamma$ gamma function

$\mathcal{J} = (\gamma - 1)u_x^2/2$

$\eta = (1/2)\sqrt{a_o U_r/X \nu_s} Y$

$\bar{\eta} = \sqrt{a_1 U_r/\nu_s} Y$
\( \lambda \) arbitrary constant, see (1-8) and (5-6)

\( \mu \) absolute viscosity of fluid, slug/ft sec

\( \nu \) kinematic viscosity of fluid, \( \text{ft}^2/\text{sec} \)

\( \gamma \) similarity variable, function of \( x \) and \( y \), see (5-9)

\( \rho \) density of fluid, slug/ft\(^3\)

Prandtl number, \( c_p \mu/k \)

\( \tau \) longitudinal shearing stress, lb/ft\(^2\)

\( \tau^* \) circumferential shearing stress, lb/ft\(^2\)

function of \( \bar{u} \), see (4-33)

\( \chi \) function of \( x \), see (4-33)

\( \psi \) stream function defined by (2-6) and (2-28)

\( \Phi \) stream function in the fictitious problem of Sections 2 and 3 defined by (3-9)

\( \omega \) angular velocity of the rotating body about its axis of symmetry, rad/sec

\( \Omega \) angular velocity in the fictitious problem of Sections 2 and 3 corresponding to \( \omega \), rad/sec

Unless indicated otherwise, the subscripts have the following significance:

\( l \) local free-stream value

\( i \) dummy subscript, \( i = 0, 1, 2, \ldots \)

\( j \) dummy subscript, \( j = 0, 1, 2, \ldots \)
0 free-stream value
m value at the maximum section of the body
s free-stream stagnation value
w wall surface value
REFERENCES


34. Howarth, L., "Concerning the Effect of Compressibility
on Laminar Boundary Layers and Their Separation,"


Fig. 1. Schematic diagram of axisymmetric coordinate system.
Fig. 2. Schematic diagram of two-dimensional boundary layer coordinates.
I, Sheng To Chu, was born in Shanghai, China, on January 22, 1922. I received my secondary school education in Nanyang Middle School of Shanghai Municipality. My undergraduate training was obtained at The National Central University of China, from which I received the degree of Bachelor of Sciences in 1943. After graduating from the college, I served three years with the China National Aviation Corporation. In the Autumn of 1947, I came to the United States and entered the Graduate School of The Ohio State University. I received the degree of Master of Sciences in 1949. While in residence at The Ohio State University, I held the following positions in the Department of Aeronautical Engineering: Graduate Assistant, 1947-1948, Research Assistant, 1948-1951, Research Associate, 1951-1952, Instructor, 1952-1954.