HELICES IN A FLAT SPACE OF
FOUR DIMENSIONS

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1. FRENET FORMULAE

The main topic of this thesis is the consideration of helices in a flat space of four dimensions. We shall start by deriving the Frenet formulae for a curve in an $N$-dimensional flat space. The method is essentially that of McConnell*. Throughout this paper the metric form is assumed to be positive definite.

Let $x_i$ be rectangular cartesian coordinates in a flat space of $N$-dimensions and positive definite metric form $ds^2 = dx_i dx_i$ where the summation convention is used, i.e., repeated Latin subscripts indicate summation for $i = 1, 2, \ldots, N$. A curve $C$ is defined by the equations $x_i = x_i(s)$ where the parameter $s$ measures the arc length from some point.

The unit tangent vector, defined at any point on the curve, will be denoted by $\lambda_{(0)i}$. Our object is to define $(N-1)$ new unit vectors $\lambda_{(\alpha)i}$, ($\alpha = 1, 2, \ldots, N-1$) which are orthogonal to $\lambda_{(0)i}$ and to each other, also to define $(N-1)$ curvatures $\kappa_{\alpha}$, ($\alpha = 1, 2, \ldots, N-1$) associated with these vectors.

Now let us suppose that we have in some way obtained these vectors up to $\lambda_{(N-1)i}$. Upon differentiation we obtain $D\lambda_{(N-1)i}$, the first derived vector of $\lambda_{(N-1)i}$.

where D stands for \( d/ds \). The set of vectors \( D \lambda_{N,i} \), \( \lambda_{N-1,i} \), \( \ldots \), \( \lambda_{0,i} \), define in general a linear \((\alpha + 2)\) space, \( S_{\alpha+1} \). But it may happen that \( D\lambda_{N,i} \) lies in the linear space \( S_{\alpha+1} \) defined by \( \lambda_{N,i} \), \( \ldots \), \( \lambda_{0,i} \). We shall not consider this singular case. Let us define \( \lambda_{w,i} \) as the unit vector in this \( S_{\alpha+1} \) space which is normal to \( \lambda_{w,i} \), \( \lambda_{(\alpha-1)i} \), \( \ldots \), \( \lambda_{0,i} \), and makes an acute angle with \( D\lambda_{N,i} \). This process determines the required set of vectors, their mutual orthogonality being assured by the process. When the set of vectors \( \lambda_{(0)i} \), \( \ldots \), \( \lambda_{(N-1)i} \) has been obtained, it is impossible to continue the process further, since every vector must be linearly dependent on these \( N \) vectors. To obtain the formulae connecting the vectors, we have

\[
(1.01) \quad k_{\alpha i} \lambda_{\alpha-1,i} - D\lambda_{\alpha-1,i} = \sum_{\beta=0}^{\alpha} A_{\alpha \beta} \lambda_{\beta,i} + \sum_{\beta=0}^{\alpha} A_{\alpha \beta} \lambda_{\beta,i} + \cdots + \sum_{\beta=0}^{\alpha} A_{\alpha \beta} \lambda_{\beta,i}
\]

where \( k_{\alpha}, k_{\beta}, \ldots, k_{N} \) are positive and defined to be the curvatures. We define \( k_{0} = 0 \), so that for \( \alpha = N-1 \), \( \lambda_{(N-1)i} \) does not actually occur. The resulting equation represents the fact that \( D\lambda_{(N-1)i} \) must be a linear function of \( \lambda_{(0)i}, \ldots, \lambda_{(N-1)i} \).

To find the coefficients \( A \), we multiply across by \( \lambda_{\beta,i} (\beta = 0, 1, \ldots, \alpha - 1) \) and obtain

\[
(1.02) \quad \delta = D\lambda_{\alpha,i} \lambda_{\beta,i} + A_{\alpha \beta} \lambda_{\beta,i} \lambda_{\beta,i} + \cdots + A_{\alpha \beta} \lambda_{\beta,i} \lambda_{\beta,i} + \cdots + A_{\alpha \beta} \lambda_{\beta,i} \lambda_{\beta,i}
\]

It is seen from the orthogonality of the vectors that the terms on the right hand side of (1.02) vanish,
except $D\lambda_{\alpha i} \gamma_{\beta i}$ and $A_{\alpha i} \lambda_{\beta i} \gamma_{\alpha i}$. Thus (1.02) can be written

$$0 = D\lambda_{\alpha i} \gamma_{\beta i} + A_{\alpha i} \lambda_{\beta i}. \tag{1.03}$$

From the orthogonality and unit character of the vectors we have

$$\lambda_{\alpha i} \lambda_{\beta i} = 0, \text{ since } \alpha \neq \beta. \tag{1.04}$$

Differentiation gives

$$D\lambda_{\alpha i} \gamma_{\beta i} = -D\lambda_{\beta i} \lambda_{\alpha i} \gamma_{\alpha i}. \tag{1.05}$$

When the value $(-D\lambda_{\beta i} \lambda_{\alpha i})$ is substituted in (1.03) for $D\lambda_{\alpha i} \gamma_{\beta i}$ we have

$$A_{\alpha \beta} = D\lambda_{\beta i} \lambda_{\alpha i} \gamma_{\alpha i}. \tag{1.06}$$

Replacing $D\lambda_{\beta i}$ by its value from (1.01) we have

$$A_{\alpha \beta} = \lambda_{\alpha i} \left( k_{\alpha} \lambda_{\alpha i} \gamma_{\alpha i} - A_{\beta \mu} \lambda_{\beta i} \gamma_{\beta i} - A_{\beta \mu} \lambda_{\beta i} - \cdots - A_{\beta \mu} \lambda_{\mu i} \gamma_{\mu i} \right). \tag{1.07}$$

Thus

$$A_{\alpha \alpha} = A_{\alpha \alpha} = 0 \tag{1.08}$$

$$A_{\alpha \beta} = k_{\beta} \quad (\alpha = 1, \ldots, N-1)$$

$$A_{\alpha \beta} = 0 \quad (\beta = 0, 1, \ldots, \alpha = 2).$$

Substituting the values of (1.08) in (1.01) and solving for $D\lambda_{\alpha i}$ we obtain

$$D\lambda_{\alpha i} = k_{\alpha} \lambda_{\alpha i} \gamma_{\alpha i} = k_{\alpha} \lambda_{\alpha i} \lambda_{\alpha i} \gamma_{\alpha i}, \quad k_{0} = k_{N} = 0, \tag{1.09}$$

$$(\alpha = 0, 1, \ldots, N-1)$$

which is the generalized form of the Frenet formulae.
2. INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF HELICES

A helix in a flat space of four dimensions is defined to be a curve having three constant curvatures, none of which is zero. By definition, each curvature is positive.

The Frenet formulae (1.09) read, for the case of four dimensions

\[
\begin{align*}
D \lambda_{ui} &= \kappa_i \lambda_{ui} \\
D \lambda_{ui} &= \kappa_i \lambda_{ui} = \kappa_i \lambda_{ui} \\
D \lambda_{ui} &= \kappa_3 \lambda_{ui} = \kappa_i \lambda_{ui} \\
D \lambda_{ui} &= \kappa_2 \lambda_{ui} = \kappa_2 \lambda_{ui}.
\end{align*}
\]

For a helix \( \kappa_i, \) \( \kappa_i, \) and \( \kappa_3 \) are (positive) constants.

By differentiation and substitution \( \lambda_{ui}, \lambda_{ui}, \) and \( \lambda_{ui} \) are eliminated from (2.01) so that finally

\[
(2.02) \quad D^4 \lambda_{ui} + K D^2 \lambda_{ui} + \kappa_i^2 \kappa_3 \lambda_{ui} = 0
\]

where \( K \) is defined by

\[
(2.03) \quad K = \kappa_i^2 + \kappa_i^2 + \kappa_3^2.
\]

The solution of the differential equation (2.02) is obtained by substituting \( \lambda_{ui} = e^{pa}; \) this gives the characteristic equation

\[
(2.04) \quad p^4 + K p^2 + \kappa_i^2 \kappa_3^2 = 0
\]

with the four roots

\[
(2.05) \quad p = \pm i M, \pm i N
\]
where M and N are defined by

\[(2.06) \quad M^2 = \frac{1}{2}(K - \sqrt{K^2 - 4 \kappa_1^2 \kappa_3^2}) \quad , M > 0\]
\[N^2 = \frac{1}{2}(K + \sqrt{K^2 - 4 \kappa_1^2 \kappa_3^2}) \quad , N > 0.\]

Here and throughout all radicals indicate the positive square root.

It is noted by (2.06) that for all values of \( \kappa_1 \), \( \kappa_2 \), and \( \kappa_3 \), \( M \) is less than \( N \).

The complete solution of (2.02) is

\[(2.07) \quad \lambda_{ij} = C_i \cos M_\sigma + D_i \sin M_\sigma + E_i \cos M_\sigma + F_i \sin M_\sigma\]

where \( C_i \), \( D_i \), \( E_i \), and \( F_i \) are constant vectors. By integration we obtain

\[(2.08) \quad x_i = \left(\frac{C_i}{M}\right) \sin M_\sigma + \left(\frac{D_i}{M}\right) \cos M_\sigma\]

\[+ \left(\frac{E_i}{N}\right) \sin N_\sigma + \left(\frac{F_i}{N}\right) \cos N_\sigma + G_i\]

with \( G_i \) a constant vector.
3. RELATIONS BETWEEN THE
CONSTANT VECTORS

We shall now show that the vectors \( C_i, D_i, E_i, \)
and \( F_i \) are not completely arbitrary. Since \( \lambda_{\text{co}} \) is a
unit vector we obtain

\[
(3.01) \quad l = (C_i \cos \theta - D_i \sin \theta + E_i \cos \varphi - F_i \sin \varphi)
\]

\[
(C_i \cos \theta - D_i \sin \theta + E_i \cos \varphi - F_i \sin \varphi) = 0
\]

Expanding and simplifying we obtain

\[
(3.02) \quad H_x + H_i \cos 2\theta + H_z \sin 2\theta = H_i \cos (M+N)\theta + H_z \sin (M+N)\theta
\]

\[
+ H_i \cos (M-N)\theta + H_z \sin (M-N)\theta = 0
\]

where

\[
(3.03) \quad H_x = \frac{1}{2}(C^2 + D^2 + E^2 + F^2 - 2) \]

\[
H_i = \frac{1}{2}(C^2 - D^2) \]

\[
H_z = C_i \cdot D_i \]

\[
H^2 = \frac{1}{2}(E^2 - F^2) \]

\[
H_i = E_i \cdot F_i \]

\[
H_x = C_i \cdot E_i = D_i \cdot F_i \]

\[
H_z = -(C_i \cdot F_i + D_i \cdot E_i) \]

\[
H_i = C_i \cdot E_i + D_i \cdot F_i \]

\[
H_z = C_i \cdot F_i - D_i \cdot E_i \]

and \( C^2 = C_i \cdot C_i, D^2 = D_i \cdot D_i, E_i \cdot E_i, F_i \cdot F_i \). Equation

\((3.02)\) is, of course, an identity for all values of the
arc length \( s \).

In order to evaluate the \( H_i \)'s we shall consider
separately the cases where \( M/N \) is rational and where \( M/N \) is irrational. The type of argument used for the second case may also be used for the first, but, since the rational case is slightly simpler, a separate argument will be given for it.

**CASE I:**

If \( M/N \) is rational then

\[
(3.04) \quad M/N = m/n
\]

where \( m \) and \( n \) are integers. We shall define \( q \) as

\[
(3.05) \quad q = M/m = N/n
\]

so that

\[
(3.06) \quad Mq = mqs, \quad Nq = nqs.
\]

If we put \( \Theta = q\Theta \), we can write \((3.02)\) in the form

\[
(3.07) \quad H_o + H_1 \cos 2m\Theta + H_2 \sin 2m\Theta + H_3 \cos 2n\Theta + H_4 \sin 2n\Theta
\]

\[
+ H_5 \cos (m+n)\Theta + H_6 \sin (m+n)\Theta
\]

\[
+ H_7 \cos (n-m)\Theta - H_8 \sin (n-m)\Theta = 0.
\]

We have already seen in Section 2 that \( M \) is less than \( N \) and hence \( m \) is less than \( n \). It follows that \( 2m, 2n, (m+n) \), and \( (n-m) \) are unequal positive integers, except for one singular case, \( n = 3m \). This will be treated separately later. We note the following well known orthogonality relations, \( \alpha \) and \( \beta \) being unequal integers:

\[
(3.08) \quad \int_0^{\frac{\pi}{2}} \sin \alpha \phi \sin \beta \phi \, d\phi = 0, \quad \int_0^{\frac{\pi}{2}} \cos \alpha \phi \cos \beta \phi \, d\phi = 0,
\]

\[
\int_0^{\frac{\pi}{2}} \cos \alpha \phi \sin \beta \phi \, d\phi = 0.
\]

Returning to \((3.07)\), we multiply across by \( d\Theta \)
and integrate between the limits 0 and 2\(\pi\), and obtain

\[
H_0 = 0.
\]

Multiplying (3.07) across by \(\cos 2m\theta\) d\(\theta\) and integrating over the range 0 to 2\(\pi\), we find by using (3.08)

\[
H_i = 0.
\]

Continuing this process of multiplying (3.07) across by the coefficient of the particular \(H\) being sought and integrating over the range 0 to 2\(\pi\), we obtain

\[
H_0 = H_1 = H_2 = H_3 = H_4 = H_5 = H_6 = H_7 = H_8 = 0.
\]

The consequences of the vanishing of the \(H\)'s will be discussed after we have dealt with the irrational case.

**CASE II:**

For the case of \(M/N\) irrational we shall let \(p\) be any positive number and define

\[
\theta = s/p.
\]

Then by (3.02) we have

\[
H_0 + H_1 \cos 2M\theta + H_2 \sin 2M\theta + H_3 \cos 2N\theta + H_4 \sin 2N\theta + H_5 \cos (M + N)p\theta + H_6 \sin (M + N)p\theta + H_7 \cos (M - N)p\theta + H_8 \sin (M - N)p\theta = 0.
\]

We multiply the above equation across by d\(\theta\) and integrate over the range 0 to 2\(\pi\). Letting \(p \to \infty\) we find

\[
H_0 = 0.
\]

Multiplying (3.13) across by \(\cos 2M\theta\) and expressing all terms as the sum and difference of sines and cosines, we obtain
\begin{equation}
(3.15) \quad H_1 (1 + \cos 4M \phi) + H_6 \sin 4M \phi \\
+ H_3 \left[ \cos 2(M+N)p\phi - \cos 2(M-N)p\phi \right] \\
+ H_6 \left[ \sin 2(M+N)p\phi - \sin 2(M-N)p\phi \right] \\
+ H_9 \left[ \cos (3M+N)p\phi + \cos (M-N)p\phi \right] \\
+ H_6 \left[ \sin (3M+N)p\phi + \sin (M-N)p\phi \right] \\
+ H_7 \left[ \cos (3M-N)p\phi + \cos (M+N)p\phi \right] \\
+ H_8 \left[ \sin (3M-N)p\phi - \sin (M+N)p\phi \right] = 0.
\end{equation}

The above is multiplied through by d\phi and integrated over the range 0 to 2\pi. Letting \( p \to \infty \) we obtain

\begin{equation}
(3.16) \quad H_1 = 0.
\end{equation}

We continue this process of multiplying (3.13) across by the coefficient of the particular \( H \) being sought; then after expressing the result as the sum and difference of sines and cosines we multiply across by d\phi and integrate over the range 0 to 2\pi. Letting \( p \to \infty \) we obtain

\begin{equation}
(3.17) \quad H_0 = H_1 = H_2 = H_3 = H_4 = H_5 = H_6 = H_7 = H_8 = 0.
\end{equation}

It is noted that this result is the same as (3.11) where the ratio \( M/N \) was rational, and not equal to 1/3. By continuity, (3.17) must hold even when \( M/N = 1/3 \).

We return to (3.03) and investigate the magnitude and orthogonality relationships among the vectors \( C_i \), \( D_i \), \( E_i \), and \( F_i \). Replacing the \( H \)'s by their values in (3.03) we obtain
\[(3.18)\] \[C^2 = D^2, \quad C_i D_i = 0, \quad D_i E_i = 0, \quad E_i F_i = 0.\]
\[G^2 + E^2 = 1, \quad C_i E_i = 0, \quad D_i F_i = 0,\]
\[D^2 + F^2 = 1, \quad C_i F_i = 0,\]

Thus the vectors \(C_i, D_i, E_i,\) and \(F_i\) are mutually perpendicular. They are not unit vectors; between their magnitudes we have the relations

\[(3.19)\]
\[C = D,\]
\[E = F,\]
\[G^2 + E^2 = 1.\]

Since the helix is determined in form by the curvatures \(\kappa_1, \kappa_2,\) and \(\kappa_3,\) it is clear that \(C, D, E,\) and \(F\) are functions of the curvatures. These functions will be found in Section 5.

To represent the helix in simplified form, we choose the coordinate axes along the vectors \(C_i, D_i,\) \(E_i,\) and \(F_i\) so that

\[(3.20)\]
\[C_i = 0, \quad D_i = C, \quad E_i = 0, \quad F_i = 0,\]
\[C_i = C, \quad D_i = 0, \quad E_i = 0, \quad F_i = 0,\]
\[C_i = 0, \quad D_i = 0, \quad E_i = 0, \quad F_i = E,\]
\[C_i = 0, \quad D_i = 0, \quad E_i = E, \quad F_i = 0.\]

For this choice of coordinate axes the components of the tangent vector \(\lambda(o);\) become by \((2.07)\)
\[ (3.21) \]
\[ \lambda_{01} = -C \sin Ns, \]
\[ \lambda_{01} = C \cos Ns, \]
\[ \lambda_{01} = -E \sin Ns, \]
\[ \lambda_{01} = E \cos Ns. \]

Integration of the above yields the equations of the four dimensional helix in parametric form as
\[ (3.22) \]
\[ x_1 = (C/M) \cos Ms + g_1, \]
\[ x_2 = (C/M) \sin Ms + g_2, \]
\[ x_3 = (E/N) \cos Ns + g_3, \]
\[ x_4 = (E/N) \sin Ns + g_4. \]

The constants of integration may be removed by a proper translation of the axes.

There is an interesting distinction between the case where \( M/N \) is rational, and the case where \( M/N \) is irrational. If \( M/N \) is rational the helix is a closed curve. If \( M/N \) is irrational the helix is not closed.

If the constants \( g_1, g_2, g_3, \) and \( g_4 \) in (3.22) are taken to the left hand side of each equation, and then the equations are squared and added, we observe the following interesting result; namely,
\[ (3.23) \]
\[ (x_1 - g_1)^2 + (x_2 - g_2)^2 + (x_3 - g_3)^2 + (x_4 - g_4)^2 = (C/M)^2 + (E/N)^2. \]

This equation shows that the helix lies on a hyper-sphere with center at \((g_1, g_2, g_3, g_4)\), and with radius \[ \left[ (C/M)^2 + (E/N)^2 \right]^{1/2}. \]
4. DETERMINATION OF THE CURVE FROM THE
PITCH RATIO M/N AND TWO OF THE CURVATURES

We have developed the equations of the four dimen-
sional helix for M/N rational and M/N irrational. We
shall refer hereafter to the ratio M/N as the pitch
ratio, $A$, where from (2.06) we have

\[(4.01) \quad A = \frac{M/N}{\left[\sqrt{K^2 - 4\kappa_1^2 \kappa_3^2}\right]^2}.\]

Another expression for the pitch ratio is found by
multiplying the numerator and denominator on the right
hand side of (4.01) by the numerator and simplifying.
The result is

\[(4.02) \quad A = \frac{K - (K^2 - 4\kappa_1^2 \kappa_3^2)^{1/2}}{2 \kappa_1 \kappa_3}.\]

From the above equation and (2.03) we obtain by simple
algebra

\[(4.03) \quad \kappa_1^2 + \kappa_2^2 + \kappa_3^2 - (1/A + A) \kappa_1 \kappa_3 = 0.\]

This equation can be written in factored form

\[(4.04) \quad (\kappa_1 - A \kappa_3)(1/A \kappa_3 - \kappa_1) = \kappa_2^2\]

from which we deduce the inequalities

\[(4.05) \quad A < \frac{\kappa_1}{\kappa_3} < 1/A.\]

Since (4.03) contains four parameters, it follows
that given any three of them we can determine the
fourth. Equation (4.03) is quadratic in $\kappa_1$, $\kappa_2$, and
$\kappa_3$. Thus, it is possible for roots of this equation

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to be imaginary when solved for \( \kappa_1, \kappa_2, \) or \( \kappa_3 \). If this situation occurs, we shall consider the helix to be non-existent since we are concerned with the real domain only.

**DETERMINATION OF THE HELIX, FROM \( A, \kappa_1, \kappa_3 \).**

It is clear that the pitch ratio, \( A \), lies in the open interval zero to one. It is easily seen that the pitch ratio covers this open interval, in the sense that a helix exists when \( A \) is given any arbitrary value in the range \( 0 < A < 1 \). For, given any \( A \) in this interval, we can choose positive values of \( \kappa_1 \) and \( \kappa_3 \) to satisfy (4.05). The suitable value of \( \kappa_1 \) is given by (4.04), this value being necessarily real, since we have already satisfied (4.05). The helix with \( \kappa_1, \kappa_3 \), satisfying these stated conditions has the required pitch ratio \( A \).

Thus given \( \kappa_1, \kappa_3 \), and \( A \), arbitrary, except for (4.05) and \( 0 < A < 1 \), there exists a helix with these values, \( \kappa_1 \) being given by (4.04).

**DETERMINATION OF THE HELIX, FROM \( A, \kappa_1, \kappa_2 \), OR \( A, \kappa_1, \kappa_3 \).**

If we solve (4.03) for \( \kappa_1 \) or \( \kappa_3 \) we obtain

\[
(4.05) \quad \kappa_1 = \frac{1}{3}(\kappa_3^2(A + 1/A)^2 - \left[ \kappa_3^2(A + 1/A)^2 - 4(\kappa_3^2 + \kappa_2^2) \right]^{1/2})
\]

where \( \alpha = 1 \) when \( \beta = 3 \), and \( \alpha = 3 \) when \( \beta = 1 \). The equation may also be written
\( (4.07) \quad \kappa_4 = \frac{1}{2} \left( \kappa_3 (A + 1/A) \pm \left( \frac{3}{2} (1/A - A) \right)^{1/2} \right) \)

We have already seen that \( A \) must be chosen to satisfy \( 0 < A < 1 \). A necessary and sufficient condition that \( \kappa_4 \) be real is that

\( (4.08) \quad \kappa_4 \leq \frac{1}{2} (1/A - A) \kappa_3 \)

When this condition is satisfied, it is obvious from (4.06) that two positive values of \( \kappa_4 \) are given by (4.06). We can show the inequality (4.08), graphically by plotting on \( \kappa_3, \kappa_4 \) coordinate axes the straight line

\( (4.09) \quad \kappa_4 = \frac{1}{2} (1/A - A) \kappa_3 \)

\[ \tan \theta = \frac{1}{2} (1/A - A) \]

The shaded area in the above figure is the possible region for \( \kappa_4, \kappa_3 \) when \( A \) is given, subject to \( 0 < A < 1 \).

Considering the slope of the line as \( A \) approaches unity, we observe

\( (4.10) \quad \lim_{A \to 1} \frac{1}{2} (1/A - A) \to 0 \)

Therefore, the line tends to coincide with the \( \kappa_3 \) axis.
The angle of the shaded wedge in Figure 1 tends to zero as $A$ tends to unity.

Considering the slope of the line as $A$ approaches zero we observe

\begin{equation}
\lim_{A \to 0} \frac{1}{2}(l/A - A) \to \infty.
\end{equation}

Therefore, the line tends to coincide with the $\kappa_2$ axis.

Since the limits are never actually attained ($A$ can never become zero or unity), without degeneracy of the helix, the line $\kappa_2 = \frac{1}{2}(l/A - A)\kappa_\phi$ can never coincide with either the $\kappa_\phi$ axis or the $\kappa_2$ axis.
5. EVALUATION OF THE CONSTANTS C AND E

IN TERMS OF THE CURVATURES

In Section 2 of this thesis the general equations of the helix in a flat space of four dimensions have been developed. For convenience we shall rewrite the equations. (3.22) below:

\[(5.01)\]
\[
\begin{align*}
    x_1 &= (C/M) \cos Ms + g_1, \\
    x_2 &= (C/M) \sin Ms + g_2, \\
    x_3 &= (E/N) \cosNs + g_3, \\
    x_4 &= (E/N) \sinNs + g_4.
\end{align*}
\]

The above equations were the result of solving a fourth order differential equation, which in turn had been developed from the Frenet formulae. We have already, in (3.19), established the relation

\[(5.02)\]
\[C^2 + E^2 = 1\]

between the constants C and E. By substitution from (5.01) in the Frenet formulae we can complete the evaluation of the constants C and E in terms of the curvatures \(\kappa_1\), \(\kappa_2\), and \(\kappa_3\).

Differentiation of (5.01) twice yields

\[(5.03)\]
\[
\begin{align*}
    D\lambda_{\theta_1} &= -MC \cos Ms, \\
    D\lambda_{\theta_2} &= -MC \sin Ms, \\
    D\lambda_{\theta_3} &= -NE \cos Ns, \\
    D\lambda_{\theta_4} &= -NE \sin Ns.
\end{align*}
\]

Using the relation \(D\lambda_{\theta_1} D\lambda_{\theta_1} = \kappa_1^2\) from the Frenet
formulae, we have

\( M^2 C^2 + N^2 E^2 = \kappa_i^2 \).

With (5.02) this gives

\[
C^2 = \frac{N^2 - \kappa_i^2}{N^2 - M^2},
\]

\[
E^2 = \frac{M^2 - \kappa_i^2}{M^2 - N^2}.
\]

Using (2.06) for \( M^2 \) and \( N^2 \) we obtain

\[
C^2 = \frac{-\kappa_i^2 + \kappa_1^2 + \kappa_3^2}{2(K^2 - 4 \kappa_i^2 \kappa_3^2)} + \frac{1}{2}, \quad C > 0,
\]

\[
E^2 = \frac{\kappa_i^2 - \kappa_1^2 - \kappa_3^2}{2(K^2 - 4 \kappa_i^2 \kappa_3^2)} + \frac{1}{2}, \quad E > 0.
\]

We now have the constants \( C \) and \( E \) as well as \( M \) and \( N \) expressed in terms of the curvatures. With this result in mind we note that the radius of the hyper-sphere (3.23) is determined by the values of the curvatures \( \kappa_i, \kappa_1, \kappa_3 \).
6. REPRESENTATION OF A HELIX BY MOVING
POINTS ON TWO REPRESENTATIVE CIRCLES

Before discussing the representation of helices by moving points, let us collect essential formulae and definitions thus far developed. From Section 2 we have

\( K = \kappa_1^2 + \kappa_2^2 + \kappa_3^2 \) \hspace{1cm} (6.01)

and

\( M^2 = \frac{1}{2}(K - (K^2 - 4 \kappa_1^2 \kappa_3^2)^{\frac{1}{2}}), \quad M > 0, \) \hspace{1cm} (6.02)

\( N^2 = \frac{1}{2}(K + (K^2 - 4 \kappa_1^2 \kappa_3^2)^{\frac{1}{2}}), \quad N > 0. \)

From Section 3 we have

\( C = D, \)

\( E = F, \)

\( C^2 + E^2 = 1. \) \hspace{1cm} (6.03)

From Section 5 we have

\( C^2 = \frac{-\kappa_1^2 + \kappa_2^2 + \kappa_3^2}{2(K^2 - 4 \kappa_1^2 \kappa_3^2)^{\frac{1}{2}}} + \frac{1}{2}, \quad C > 0, \)

\( E^2 = \frac{\kappa_1^2 - \kappa_2^2 - \kappa_3^2}{2(K^2 - 4 \kappa_1^2 \kappa_3^2)^{\frac{1}{2}}} + \frac{1}{2}, \quad E > 0. \) \hspace{1cm} (6.04)

The equations of the helix, (5.01), rewritten with the appropriate translation of the coordinate axes to remove the constants of integration \( g_1, g_2, g_3, \) and \( g_4 \) read

18
\[(6.05)\]
\[x_1 = \frac{C}{M} \cos Ms,\]
\[x_2 = \frac{C}{M} \sin Ms,\]
\[x_3 = \frac{E}{N} \cos Ns,\]
\[x_4 = \frac{E}{N} \sin Ns.\]

If we square and add the first two equations of (6.05), then square and add the second two equations of (6.05), we obtain
\[(6.06)\]
\[x_1^2 + x_2^2 = \left(\frac{C}{M}\right)^2,\]
\[x_3^2 + x_4^2 = \left(\frac{E}{N}\right)^2.\]

It is noted that the projection of the curve on the \((x_1,x_2)\) plane is a circle having radius \(\frac{C}{M}\), and the projection of the curve on the \((x_3,x_4)\) plane is a circle having radius \(\frac{E}{N}\). Our representations of the curve will be made in terms of these two circles. The circle in the \((x_1,x_2)\) plane will be referred to as the \(R_{12}\) circle, and the circle in the \((x_3,x_4)\) plane will be referred to as the \(R_{34}\) circle. We shall also use the symbols \(R_{12}\) and \(R_{34}\) to denote the radii of these circles. Thus,
\[(6.07)\]
\[R_{12} = \frac{C}{M},\]
\[R_{34} = \frac{E}{N}.\]

Let us consider two points moving on the circumferences of \(R_{12}\) and \(R_{34}\), according to the equations (6.05). These points start together for \(s = 0\) at \((\frac{C}{M},0)\) for \(R_{12}\), and \((\frac{E}{N},0)\) for \(R_{34}\). The values of the angular
velocities for R_{12} and R_{34} with respect to \sigma are M and N respectively. It is evident that a helix is completely represented (to within a translation) by the two representative moving points as just described. The points (P and Q) are shown in Figure 2 which is drawn for the particular case of \eta_1: \eta_2: \eta_3.

The following shows a summary of the results of the various cases considered below. The radii as well as the angular velocities on the representative circles are given, and in addition the pitch ratio. In some cases the numbers of the equations have been inserted since some of the equations are too complicated to appear in the table. The table is read lengthwise, the type of helix considered being stated in the left hand column.
<table>
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<td>$K_2 \to 0$, $K_1 = K_3$</td>
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<td>$(\sqrt{2} K_1)^{-1}$</td>
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<tr>
<td>$K_2 \to 0$, $K_1 &lt; K_3$</td>
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<td>0</td>
<td>$K_1$</td>
</tr>
<tr>
<td>$K_3 \to 0$</td>
<td>$\infty$</td>
<td>$K_1 \left { \frac{K_1^2 + K_3^2}{2} \right }^{-1}$</td>
<td>0</td>
</tr>
</tbody>
</table>
We shall now establish the results shown in the table.

**CASE I:** \( \kappa_1 = \kappa_2 = \kappa_3 \).

Let us consider the case where all three curvatures are equal, i.e., \( \kappa_1 = \kappa_2 = \kappa_3 \). Under these conditions we find

\[
M^2 = \kappa_i \frac{(3 - \sqrt{5})}{2}, \quad N^2 = \kappa_i \frac{(3 + \sqrt{5})}{2}
\]

from (6.02), and using (6.04) we obtain

\[
C^2 = \frac{(5 + \sqrt{5})}{10}, \quad E^2 = \frac{(5 - \sqrt{5})}{10}.
\]

From (6.07) the radii of the representative circles are found to be

\[
R_{12} = \kappa_i^{-1} \left[ \frac{5}{2} + \frac{\sqrt{5}}{5} \right]^{\kappa_i} \quad , \\
R_{34} = \kappa_i^{-1} \left[ \frac{5}{2} - \frac{\sqrt{5}}{5} \right]^{\kappa_i} .
\]

We note that for all values of \( \kappa_i \), \( R_{12} > R_{34} \). From (6.08) the pitch ratio is

\[
A = M/N = \frac{3}{5}(3 - \sqrt{5}).
\]
Figure 2

Representative circles for the case $\kappa_1, \kappa_2, \Omega_3 = 1$. Points $P$ and $Q$ show rates of angular velocities.
CASE II: \( R_{12} = R_{34} \)

Let us now investigate helices for which \( R_{12} = R_{34} \).

For such a helix we have

(6.12) \[ \frac{C}{M} = E/N. \]

But, \( C^2 + E^2 = 1 \) from (6.03), and hence

(6.13) \[ (N^2 + M^2)C^2 = M^2. \]

We now substitute the values of \( M^2, N^2, \) and \( C^2 \) from (6.02) and (6.04). After simplification (6.13) becomes

(6.14) \[ (\kappa_2^2 + \kappa_3^2)^2 + \kappa_1^2(\kappa_2^2 - \kappa_3^2) = 0 \]

which is the condition to be satisfied for \( R_{12} \) and \( R_{34} \) to be equal. From the above equation \( \kappa_1^2 \) is found to be

(6.15) \[ \kappa_1^2 = (\kappa_2^2 + \kappa_3^2)^2(\kappa_3^2 - \kappa_3^2)^{-1}. \]

Substituting (6.15) in (6.02) and (6.04) we can express \( M, N, C, \) and \( E \) as functions of \( \kappa_2 \) and \( \kappa_3 \) as follows

(6.16) \[ M^2 = \kappa_3(\kappa_2^2 + \kappa_3^2)(\kappa_2 + \kappa_3)^{-1}, \]

\[ N^2 = \kappa_3(\kappa_2^2 + \kappa_3^2)(\kappa_3 - \kappa_3)^{-1}, \]

and

(6.17) \[ C^2 = (\kappa_3 - \kappa_2)/2\kappa_3, \]

\[ E^2 = (\kappa_3 + \kappa_2)/2\kappa_3. \]

Calculating the ratios \( C/M \) and \( E/N \) we find

(6.18) \[ R_{12} = R_{34} = \frac{C}{M} = E/N = \left[\frac{\kappa_3^2 - \kappa_3^2}{2\kappa_3^2(\kappa_1^2 + \kappa_2^2)}\right]^{\kappa_3}. \]

From (6.16) the pitch ratio is found to be

(6.19) \[ A = \left[\frac{(\kappa_3 - \kappa_2)(\kappa_3^2 + \kappa_3^2)^{-1}}{\kappa_3}\right]. \]
It is noted that \( \kappa_3 \) must be greater than \( \kappa_1 \) in order that \( C \) may be real.

Equation (6.14) or (6.15) gives a relation between \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) which must be satisfied if \( R_{12} \) and \( R_{34} \) are to be equal. It is interesting to give a geometrical interpretation to this relationship. We might take a three space with \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) as rectangular cartesian coordinates, so that a point in this space would represent a definite helix. But we get a simpler picture if we introduce auxiliary variables \( x, y, \) and \( z \) as follows.

In (6.15) let us substitute the following

\[
\begin{align*}
\kappa_1^2 &= y + z, \\
\kappa_2^2 &= (x/\sqrt{2}) + y - z, \\
\kappa_3^2 &= (x/\sqrt{2}) - y + z.
\end{align*}
\]

After simplification we have the result

\[
x^2 + y^2 - z^2 = 0,
\]

which is the equation of a cone. By solving (6.20) we find the coordinates \((x,y,z)\) are expressed in terms of the curvatures as follows

\[
\begin{align*}
x &= (\kappa_1^2 + \kappa_3^2)/\sqrt{2}, \\
y &= \frac{1}{2}(2\kappa_1^2 + \kappa_2^2 - \kappa_3^2), \\
z &= \frac{1}{2}(2\kappa_1^2 - \kappa_2^2 + \kappa_3^2).
\end{align*}
\]

Since we restrict our attention to the non-degenerate case in which \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) are all positive, we have from (6.20)

25
(6.23) \[ \kappa_2^2 z, \quad y + z > 0, \]
\[ \kappa_1^2 = x/\sqrt{2} + y - z > 0, \]
\[ \kappa_3^2 = x/\sqrt{2} - y + z > 0. \]

In terms of the quantities \(x, y,\) and \(z\) we find \(M\)
and \(N\) have the general form
\[ (6.24) \quad M = \left[ \frac{1}{2}(\sqrt{2}x + y + z - \sqrt{2x^2 + 5y^2 + 2yz - 3z^2}) \right]^{\frac{1}{2}}, \]
\[ N = \left[ \frac{1}{2}(\sqrt{2}x + y + z + \sqrt{2x^2 + 5y^2 + 2yz - 3z^2}) \right]^{\frac{1}{2}}. \]
By means of (6.04), \(C\) and \(E\) in terms of \(x, y,\) and \(z\)
have the general form
\[ (6.25) \quad C = \left[ \frac{(\sqrt{2}x - y - z + \sqrt{2x^2 + 5y^2 + 2yz - 3z^2})}{2 \sqrt{2x^2 + 5y^2 + 2yz - 3z^2}} \right]^\kappa, \]
\[ E = \left[ \frac{(-\sqrt{2}x + y + z + \sqrt{2x^2 + 5y^2 + 2yz - 3z^2})}{2 \sqrt{2x^2 + 5y^2 + 2yz - 3z^2}} \right]^\kappa. \]
Investigating the ratios \(C/M\) and \(E/N\) we find by using
(6.21) to simplify
\[ (6.26) \quad C/M = (\sqrt{2}x + y + z)^{-\frac{3}{2}} = (\kappa_2^2 + \kappa_1^2 + \kappa_3^2)^{-\frac{1}{2}}, \]
\[ E/N = (\sqrt{2}x + y + z)^{-\frac{3}{2}} = (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)^{-\frac{1}{2}}. \]
These ratios are of course equal, each being the common
value of the radii, \(R_{12} = R_{34} \cdot \)

Figure 3 shows a section of the cone,
\(x^2 + y^2 - z^2 = 0,\) for \(z = \text{constant}.\) The lines indicated by
(6.23) are shown in dark ink. The possible region
satisfying the inequalities (6.23) is the region shaded
in red and blue. The red portion of the circle is the
region for $R_{12} > R_{34}$ as is easily shown by taking any point within the red portion of the circle and investigating the ratios $C/M$, $E/N$ by means of (6.24) and (6.25). For the remaining part of the region $R_{12} < R_{34}$, except on the circumference of the circle where $R_{12} = R_{34}$. 
CASE III: \[ \kappa_0 \to 0. \]

We shall consider certain limiting cases. If we put \( \kappa_0 = 0 \), the helix is a straight line. We shall suppose \( \kappa_0 \) to be small, but not zero, with \( \kappa_1 \) and \( \kappa_3 \) remaining fixed and finite. We find as \( \kappa_0 \to 0 \), from (6.02) and (6.04) that, \( M^2, N^2, C^2 \), and \( E^2 \) have limits

\[ (6.27) \]
\[ M^2 \to 0, N^2 \to \kappa_1^2 + \kappa_3^2, \]
\[ C^2 \to 1, E^2 \to 0. \]

Hence we have

\[ (6.28) \]
\[ R_{12} \to \infty, R_{34} \to 0. \]

Thus, if \( \kappa_0 \) is small with \( \kappa_1 \) and \( \kappa_3 \) finite, the radius of \( R_{12} \) becomes very large, while the radius of \( R_{34} \) becomes very small.

An investigation of the pitch ratio, \( M/N \), shows it to be very small for \( \kappa_0 \) small with \( \kappa_1 \) and \( \kappa_3 \) finite.
**CASE IV:** \( \kappa_1 \to 0 \).

Let us take \( \kappa_1 \) small with \( \kappa_1 \) and \( \kappa_3 \) finite. In the limit we have from (6.02)

\[
(6.29) \quad M^2 \to \frac{1}{3}(\kappa_1^2 + \kappa_3^2 - \sqrt{(\kappa_1^2 + \kappa_3^2)^2 - 4\kappa_1^2\kappa_3^2}).
\]

Since the positive value of the radical is to be taken, we must separate the cases into

\[
(6.30) \quad \begin{align*}
\kappa_1 &> \kappa_3 \\
\kappa_1 &= \kappa_3 \\
\kappa_1 &< \kappa_3
\end{align*}
\]

Accordingly we have, as \( \kappa_1 \to 0 \),

\[
(6.31) \quad \begin{align*}
M &\to \kappa_3, \quad \text{if } \kappa_1 > \kappa_3, \\
M &\to \kappa_1, \quad \text{if } \kappa_1 = \kappa_3, \\
M &\to \kappa_1, \quad \text{if } \kappa_1 < \kappa_3.
\end{align*}
\]

For \( N \) we have

\[
(6.32) \quad N^2 \to \frac{1}{3}(\kappa_1^2 + \kappa_3^2 + \sqrt{(\kappa_1^2 + \kappa_3^2)^2 - 4\kappa_1^2\kappa_3^2})
\]

and so

\[
(6.33) \quad \begin{align*}
N &\to \kappa_1, \quad \text{if } \kappa_1 > \kappa_3, \\
N &\to \kappa_1, \quad \text{if } \kappa_1 = \kappa_3, \\
N &\to \kappa_3, \quad \text{if } \kappa_1 < \kappa_3.
\end{align*}
\]

In a like manner for \( C^2 \) we have from (6.04), if \( \kappa_1 \neq \kappa_3 \),

\[
(6.34) \quad C^2 \to \frac{-\kappa_1^2 + \kappa_3^2}{2 \left[(\kappa_1^2 + \kappa_3^2)^2 - 4\kappa_1^2\kappa_3^2\right]^{1/4}} + \frac{1}{3}
\]

and hence, as \( \kappa_1 \to 0 \),

30
\[(6.35) \quad C \to 0, \text{ if } \kappa_1 > \kappa_3, \]
\[C \to 1, \text{ if } \kappa_1 < \kappa_3. \]

Similarly
\[(6.36) \quad E^2 \to \frac{\kappa_3^2 - \kappa_2^2}{2\left[(\kappa_1^2 + \kappa_3^2)^2 - 4 \kappa_1^2 \kappa_2^2 \right]^{\frac{1}{2}}} + \frac{1}{2}, \]

and hence,
\[(6.37) \quad E \to 1, \text{ if } \kappa_1 > \kappa_3, \]
\[E \to 0, \text{ if } \kappa_1 < \kappa_3. \]

If \(\kappa_1 = \kappa_3\), we have from (6.04)
\[(6.38) \quad \begin{align*}
C^2 & \to \frac{\kappa_3^2}{2(\kappa_3^4 + 4 \kappa_1^2 \kappa_2^2)^{\frac{1}{2}}} + \frac{1}{2}, \\
E^2 & \to \frac{-\kappa_3^2}{2(\kappa_3^4 + 4 \kappa_1^2 \kappa_2^2)^{\frac{1}{2}}} + \frac{1}{2}.
\end{align*} \]

and so, in this case as \(\kappa_3 \to 0,\)
\[(6.39) \quad C \to 1/\sqrt{2}, \quad E \to 1/\sqrt{2}. \]

Collecting the results from (6.31) on, the limits of \(M, N, C,\) and \(E\) as \(\kappa_1 \to 0,\) with \(\kappa_1 > \kappa_3\) are
\[(6.40) \quad \begin{align*}
M & \to \kappa_3, \\
N & \to \kappa_3, \\
C & \to 0, \\
E & \to 1.
\end{align*} \]

For the radii of the representative circles we have
\[(6.41) \quad R_{12} \to 0, \quad R_{34} \to 1/\kappa_1, \]

and for the pitch ratio
\[(6.42) \quad A = M/N \to \kappa_3/\kappa_1. \]
The limits of $M$, $N$, $C$, and $E$ from (6.31) and following for $\kappa \to 0$, with $\kappa_4 = \kappa_2$ are

\begin{align*}
M & \to \kappa_1 , \\
N & \to \kappa_2 , \\
C & \to 1/\sqrt{2} , \\
E & \to 1/\sqrt{2} .
\end{align*}

For the radii of the representative circles we have

\begin{align*}
R_{12} & \to (\sqrt{2} \kappa_1 )^{-1} , \\
R_{34} & \to (\sqrt{2} \kappa_2 )^{-1} .
\end{align*}

It is noted that if $\kappa_2 \to 0$, with $\kappa_4 = \kappa_2$, the representative circles tend to have equal radii. The angular velocities of the points on the circles are equal in the limit as is shown by $M$ and $N$ in (6.43), and so for the ratio $M/N = A$ we find

\begin{equation}
A \to 1 .
\end{equation}

Collecting the results from (6.31) on, the limits of $M$, $N$, $C$, and $E$ as $\kappa \to 0$, with $\kappa_4 = \kappa_2$ are

\begin{align*}
M & \to \kappa_1 , \\
N & \to \kappa_2 , \\
C & \to 1 , \\
E & \to 0 .
\end{align*}

For the radii of the representative circles we have

\begin{align*}
R_{12} & \to 1/\kappa_1 , \\
R_{34} & \to 0 .
\end{align*}

The pitch ratio $M/N = A$, is found to have the limit
(6.48) \[ A \rightarrow \frac{K_1}{K_3}. \]
CASE V: \( \kappa_3 \to 0 \).

Let us consider the case of \( \kappa_3 \) small with \( \kappa' \) and \( \kappa_2 \) finite.

From (6.02) we have

\[
\begin{align*}
M & \to 0, \\
N & \to \left( \kappa_i^2 + \kappa_2' \right)^{\frac{1}{2}},
\end{align*}
\]

and from (6.04) we obtain

\[
\begin{align*}
C & \to \kappa_2 (\kappa_i^2 + \kappa_2' - \frac{1}{2}), \\
E & \to \kappa_i (\kappa_i^2 + \kappa_2' - \frac{1}{2}).
\end{align*}
\]

For the radii of the representative circles we have

\[
\begin{align*}
R_{12} & \to \infty, \\
R_{34} & \to \kappa_i (\kappa_i^2 + \kappa_2' - 1).
\end{align*}
\]

Thus, if we let \( \kappa_3 \) become small while keeping \( \kappa' \) and \( \kappa_2 \) finite we find \( R_{12} \) approaches infinity, and \( R_{34} \) approaches the constant \( \kappa_i (\kappa_i^2 + \kappa_2' - 1) \).

Examination of the pitch ratio, \( M/N = \lambda \), shows it to be approaching zero as \( \kappa_3 \to 0 \), with \( \kappa_i \) and \( \kappa_2 \) finite. The angular velocity of a point on \( R_{12} \) approaches zero shown by the limit of \( M \) in (6.49). The angular velocity of a point on \( R_{34} \) approaches the constant value, \( \kappa_i (\kappa_i^2 + \kappa_2' - \frac{1}{2}) \), as shown by the limit of \( N \) in (6.49).