ON CERTAIN BOREL SETS ASSOCIATED WITH
A CONTINUOUS PLANE TRANSFORMATION

A Thesis Presented for the
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by
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INTRODUCTION

A point set $E$ in the linear interval $(a,b)$ is said to be Lebesgue measurable if the sum of its Lebesgue exterior measure and that of its complement in $(a,b)$ is exactly $b-a$, the length of the interval in which they are contained. In the first chapter we show that each of three other definitions of measurability is equivalent to this one.

Borel sets are then defined in terms of closed additive classes of sets, and there follows a discussion of the implications concerning their structure.

Next the definitions of exterior measure, measurability, and Borel sets are extended to plane sets of points. This is done in such a way that the results obtained are comparable to those already found for linear sets.

The second chapter is concerned with bounded continuous transformations $T: z = t(w)$, which map points from a given unit square in the complex $w$-plane into points of the complex $z$-plane. Several point sets in the $w$-plane which are defined in terms of maximal model continua of points in the $z$-plane (that is, components of the inverse sets) prove to be of some interest. These are the sets $E(T)$, the totality of points belonging to some essential maximal model continuum under the transformation $T$; $E(T)$, the totality of points each of which is itself an essential maximal model continuum; and $N(T)$, a certain subset of $E(T)$.

These three sets are shown to be Borel sets, a result which is
in itself interesting enough to furnish the motivation for this study. Moreover, further application in the theory of continuous transformations has been found for these sets [1] and [10] (A number in square brackets pertains to the reference given that number in the bibliography at the end of this paper.).
Chapter 1
Measure: Borel Sets

1.1 Let us consider a given set \( E \) in the linear interval \((a,b)\), and a covering \( \Gamma \) of \( E \) consisting of a denumerable or finite number of open intervals \( \{i\} \). We define the Lebesgue exterior measure of \( E \) to be the g.l.b. \( \sum_{i \in \Gamma} |i| \) for all such \( \Gamma \). The symbol \( m_\varnothing E \) will be used to denote the exterior measure of \( E \).

The following facts result from this definition:

1. The exterior measure of an interval is its length.
2. If \( E_1 \supseteq E_2 \), then \( m_\varnothing E_1 \geq m_\varnothing E_2 \).
3. For any set \( E \), \( m_\varnothing E + m_\varnothing CE \geq b - a \), where \( CE = (a,b) - E \). The set \( CE \) will be referred to as the complement of \( E \).
4. \( m_\varnothing (E_1 + \cdots + E_n) \leq m_\varnothing E_1 + \cdots + m_\varnothing E_n \). The sign of equality holds if there is a positive distance between every pair of sets in the class under consideration. [2].
5. \( m_\varnothing (E_1 - E_2) \geq m_\varnothing E_1 - m_\varnothing E_2 \) for any sets \( E_1 \) and \( E_2 \).

1.2 Lemma: For any set \( E \subseteq (a,b) \) and any \( \varepsilon > 0 \), there exists an open set \( \mathcal{O} \supseteq E \) such that \( m_\varnothing \mathcal{O} - m_\varnothing E < \varepsilon \).

Proof: By the definition of exterior measure, there exists a covering of \( E \) consisting of at most a denumerable number of open intervals such that \( \sum_{n=1}^{\infty} |i_n| < m_\varnothing E + \varepsilon \). But \( \sum_{n=1}^{\infty} i_n \) is an open set, call it \( \mathcal{O} \). Then \( m_\varnothing \mathcal{O} \leq \sum_{n=1}^{\infty} |i_n| < m_\varnothing E + \varepsilon \).

1.3 We will now turn our attention to the class of sets \( E \) for which it is true that \( m_\varnothing E + m_\varnothing CE = b - a \). Such sets are said to be
Lebesgue measurable, and we define the measure of $E$, denoted by $mE$, to be the same as its exterior measure. There exist sets $E'$ for which $mE'+mCE'>b-a$. This is of fundamental importance in the theory of measure, since it shows that not all point sets are Lebesgue measurable. In fact, the actual construction of these non-measurable sets indicates that there is no measure function $\varphi(E)$ which associates with each set $E \subseteq (a,b)$ a real number $\varphi(E) \geq 0$ satisfying the following conditions:

1. $\varphi([a,b]) = b-a$.
2. $\varphi(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \varphi(E_i)$, if $E_j \cap E_k = \emptyset$ for $j \neq k$.
3. $\varphi(E_1) = \varphi(E_2)$, if $E_1$ and $E_2$ are congruent in the ordinary geometrical sense [3, p. 374; 7, p. 90].

Since $C(CE) = E$, it follows that the complement of a measurable set is itself measurable.

There are a number of other definitions for measurability which are intuitively more appealing and in some cases more workable. By a series of theorems in sections 1.5 to 1.9, we will show the following four definitions to be equivalent: A set $E$ is measurable if:

1. $mE + mCE = b-a$;
2. for every $\varepsilon > 0$ there is a closed set $F \subseteq E$ such that $mE - mF < \varepsilon$;
3. for every set $Q \subseteq (a,b)$, $mQ = mE(Q \cdot E) + mE(Q \cdot CE)$;
4. for every $\varepsilon > 0$ there exists a finite number of intervals such that $E = \sum_{k=1}^{n} i_k + E_1(\varepsilon) - E_2(\varepsilon)$, where $mE_1 < \varepsilon$ and $mE_2 < \varepsilon$.

The equivalence will be shown by proving that (1) implies (2),
(2) implies (3), (3) implies (4), and finally that (4) implies (1).
Thus the condition stated in any one of these definitions implies
that in any other, and consequently.

Before setting out to prove these theorems, we must know for
once and for all that open sets and closed sets satisfy the condi-
tion $m_o E + m_o CE = b - a$, that is, they are measurable in accordance
with our original definition. The terms open and closed will be
taken to mean open and closed relative to the interval $(a, b)$.

1.4 Lemma: If $E = \sum_{k=1}^{\infty} I_k$, non-overlapping open intervals, then
$m_o E = \sum_{k=1}^{\infty} |I_k|$, and $E$ is measurable.

Proof: $b - a \geq \sum_{k=1}^{\infty} |I_k|$, hence the series of positive terms is
bounded and convergent. Remove from $(a, b)$ the first $n$ intervals.
The remaining segments and points contain CE. Now we have for each $n$:

$m_o CE \leq (b - a) - \sum_{k=1}^{\infty} |I_k|$. Therefore, in the limit, $(b - a) - m_o CE \geq \sum_{k=1}^{\infty} |I_k|$.

We also have $m_o E = \sum_{k=1}^{\infty} |I_k|$ [cf. (4), section 1.1]. But on the other
hand it is true that $m_o E \geq (b - a) - m_o CE$ [cf. (3), section 1.1].

Combining all these into one statement, we have

$(b - a) - m_o CE \geq \sum_{k=1}^{\infty} |I_k| \geq m_o E \geq (b - a) - m_o CE$.

It is evident that the equality sign must hold throughout, and

$m_o E = \sum_{k=1}^{\infty} |I_k| = (b - a) - m_o CE$.

Theorem 1: Any open set is measurable (i.e., $m_o O + m_o CO = b - a$).

Proof: Any open set can be expressed as a denumerable sum of
non-overlapping intervals. The latter is a measurable set, by the
preceding lemma.

Theorem 2: Any closed set is measurable.
Proof: The complement of a closed set is open, therefore measurable. Hence the complement of the open set, which is the given closed set, is also measurable.

1.5 Theorem: If \( m_\sigma E + m_\sigma CE = b - a \), then for every \( \varepsilon > 0 \) there exists a closed set \( F \) such that \( F \subseteq E \) and \( m_\sigma (E - F) < \varepsilon \).

Proof: From the lemma, section 1.2, we have an open set \( \Omega \) such that \( \Omega \supseteq CE \), and
\[
m_\sigma \Omega - m_\sigma CE < \varepsilon. \tag{1}
\]
Since \( \Omega \) is open, it is measurable. So is its complement \( F \), a closed set. Then \( m_\sigma \Omega = (b - a) - m_\sigma F \). Also, by hypothesis, we have \( m_\sigma CE = (b - a) - m_\sigma E \). Substituting these two values in (1), we obtain the desired inequality, namely
\[
m_\sigma E - m_\sigma F < \varepsilon.
\]
Also, \( F \subseteq E \), for \( CF = \Omega \supseteq CE \).

1.6 Lemma: If \( E_2 \supseteq E_1 \), and \( E_1 \) is measurable, then
\[
m_\sigma (E_2 - E_1) = m_\sigma E_2 - m_\sigma E_1.
\]
Proof: \( m_\sigma E_1 + m_\sigma CE_1 = b - a \). By the theorem, section 1.5, this implies that for a given \( \varepsilon > 0 \), there exists a closed set \( F \subseteq E_1 \) such that \( m_\sigma (E_1 - F) < \varepsilon \). Also, there is an open set \( \Omega \supseteq E_2 \) such that \( m_\sigma \Omega - m_\sigma E_2 < \varepsilon \). Using (2), section 1.1, and theorem 1 of section 1.4 (since \( \Omega - F = \Omega \supseteq CF \), which is an open set), we have
\[
m_\sigma (E_2 - E_1) = m_\sigma (\Omega - F) = (b - a) - m_\sigma [C(\Omega - F)].
\]
Replace \( C(\Omega - F) \) by \( \Omega + F \), and apply (4), section 1.1, to find the relation \( m_\sigma [C(\Omega - F)] = m_\sigma \Omega + m_\sigma F \) (this follows from the fact that...
F and \( C \) are disjoint closed sets, and therefore have a positive
distance between them. Now we see that
\[ m_\varepsilon(b - a) = m_\varepsilon(C) - m_\varepsilon F. \]
Apply theorem 1 of section 1.4:
\[ m_\varepsilon(E_2 - E_1) \leq m_\varepsilon C - m_\varepsilon F. \]
Upon substitution for \( m_\varepsilon C \) and \( m_\varepsilon F \), this becomes
\[ m_\varepsilon(E_2 - E_1) \leq m_\varepsilon E_2 - m_\varepsilon E_1 + 2\varepsilon. \]
But this is true for all \( \varepsilon > 0 \); hence in the limit, as \( \varepsilon \) tends to 0,
\[ m_\varepsilon(E_2 - E_1) \leq m_\varepsilon E_2 - m_\varepsilon E_1. \]
On the other hand, for any two sets \( E_1 \) and \( E_2 \), it is true that
\[ m_\varepsilon(E_2 - E_1) \geq m_\varepsilon E_2 - m_\varepsilon E_1. \]
These complementary inequalities give us the desired result, namely
\[ m_\varepsilon(E_2 - E_1) = m_\varepsilon E_2 - m_\varepsilon E_1. \]

1.7 Theorem: If for every \( \varepsilon > 0 \) there is a closed set \( F \subseteq E \)
such that \( m_\varepsilon E - m_\varepsilon F < \varepsilon \), then \( m_\varepsilon(Q) = m_\varepsilon(Q \cdot E) + m_\varepsilon(Q \cdot C \setminus E) \) for any set
\( Q \subseteq (a, b) \).

Proof: We have an open set \( \Theta \supseteq E \) such that \( m_\varepsilon \Theta - m_\varepsilon E < \varepsilon \), for
a given \( \varepsilon > 0 \). By hypothesis we also have a closed set \( F \subseteq E \) such
that \( m_\varepsilon E - m_\varepsilon F < \varepsilon \). Then \( m_\varepsilon \Theta - m_\varepsilon F < 2\varepsilon \). By the lemma, section 1.6,
m_\varepsilon(\Theta - F) = m_\varepsilon \Theta - m_\varepsilon F < 2\varepsilon.

Now we consider any set \( Q \subseteq (a, b) \). \( Q = Q \cdot E + Q \cdot C \setminus E \). Therefore
\[ m_\varepsilon Q \leq m_\varepsilon(Q \cdot E) + m_\varepsilon(Q \cdot C \setminus E) \leq m_\varepsilon(Q \cdot F) + m_\varepsilon[(Q \cdot E - F)] + m_\varepsilon[Q(\Theta - F)] + m_\varepsilon(Q \cdot C \Theta). \]
This last follows from (4), section 1.1. Apply (2), section 1.1:
\[ m_\varepsilon Q \leq m_\varepsilon(Q \cdot F) + m_\varepsilon(Q \cdot C \Theta) + 2m_\varepsilon[(Q \cdot E - F)] \leq m_\varepsilon(Q \cdot F) + m_\varepsilon(Q \cdot C \Theta) + 4\varepsilon. \]
But \( F \) and \( C \Theta \) are disjoint closed sets. therefore there is a positive
distance between them. Hence there is also a positive distance between the two sets \( Q \cap F \) and \( Q \cap G \). It follows that

\[
m_\varepsilon(Q \cap F) + m_\varepsilon(Q \cap G) = m_\varepsilon(Q(F + G)) \leq m_\varepsilon Q \quad \text{[cf. (2) and (4), section 1.1].}
\]

Applying this result, we have

\[
m_\varepsilon Q \leq m_\varepsilon(Q \cap E) + m_\varepsilon(Q \cap CE) < m_\varepsilon Q + 4 \varepsilon.
\]

But the choice of \( \varepsilon \) was arbitrary, hence

\[
m_\varepsilon Q \leq m_\varepsilon(Q \cap E) + m_\varepsilon(Q \cap CE) \leq m_\varepsilon Q.
\]

This establishes the desired equality.

1.8 Theorem: If for every set \( Q \subseteq (a,b) \) it is true that

\[
m_\varepsilon Q = m_\varepsilon(Q \cap E) + m_\varepsilon(Q \cap CE),
\]

then \( E = \sum_{k=1}^\infty i_k + E_1(\varepsilon) - E_2(\varepsilon) \), where \( n \) is finite, \( m_\varepsilon E_1 < \varepsilon \), and \( m_\varepsilon E_2 < \varepsilon \), for any given \( \varepsilon > 0 \).

Proof: Give \( \varepsilon > 0 \). From the definition of exterior measure, we know that we can cover \( E \) with at most a denumerable number of open intervals such that \( \sum_{k=1}^\infty |i_k| < m_\varepsilon E + \varepsilon \). The series \( \sum_{k=1}^\infty |i_k| \) is bounded and therefore convergent. Hence for some \( n \), \( \sum_{k=1}^n |i_k| < \varepsilon \).

Let \( E_1 = E - \sum_{k=1}^n i_k \) and \( E_2 = \sum_{k=1}^n i_k - E \). Then \( E = \sum_{k=1}^n i_k + E_1 - E_2 \).

We must still establish that the exterior measures of \( E_1 \) and \( E_2 \) are less than \( \varepsilon \). For the set \( E_1 \) we have

\[
m_\varepsilon E_1 = m_\varepsilon(E - \sum_{k=1}^n i_k) \leq m_\varepsilon(\sum_{k=1}^\infty i_k - \sum_{k=1}^n i_k) \leq m_\varepsilon \sum_{k=1}^\infty i_k \leq \sum_{k=1}^\infty |i_k| < \varepsilon.
\]

This follows from (2) and (4), section 1.1. To obtain information on the set \( E_2 \), we let the set \( Q \) of the hypothesis be \( \sum_{k=1}^\infty i_k \). Then

\[
m_\varepsilon \sum_{k=1}^\infty i_k = m_\varepsilon E + m_\varepsilon(\sum_{k=1}^\infty i_k - E).
\]

Rearranging terms in this equation, we have

\[
m_\varepsilon(\sum_{k=1}^\infty i_k - E) = m_\varepsilon \sum_{k=1}^\infty i_k - m_\varepsilon E.
\]

But \( m_\varepsilon \sum_{k=1}^\infty i_k - m_\varepsilon E \leq \sum_{k=1}^\infty |i_k| - m_\varepsilon E < \varepsilon \). The exterior measure of \( E_2 \) is given
by the following:

\[ m_0 E_2 = m_0 \left( \sum_{k \in E} i_k - E \right) \leq m_0 \left( \sum_{k \in E} i_k - E \right) < \epsilon. \]

1.9 Theorem: If for every \( \epsilon > 0 \), there exists a finite number of intervals such that \( E = \sum_{k \in E_1} i_k + E_1(\epsilon) - E_2(\epsilon) \), where \( m_0 E_1 < \epsilon \) and \( m_0 E_2 < \epsilon \) (the sets \( E_1 \) and \( E_2 \) are taken such that \( E_1 \cdot \sum_{k \in E} i_k = 0 \) and \( E_2 \subseteq \sum_{k \in E} i_k \)), then \( m_0 E + m_0 CE = b - a \).

Proof: \( CE = C \sum_{k \in E} i_k - E_1 + E_2. \)

\[ m_0 CE \leq m_0 C \sum_{k \in E} i_k + m_0 E_2 - m_0 C \sum_{k \in E} i_k + \epsilon. \]

\[ m_0 E \leq m_0 \sum_{k \in E} i_k + m_0 E_1 < m_0 \sum_{k \in E} i_k + \epsilon. \]

\[ m_0 E + m_0 CE < m_0 \sum_{k \in E} i_k + m_0 C \sum_{k \in E} i_k + 2 \epsilon. \]

\( \sum_{k \in E} i_k \) can be written as \( \sum_{k \in E} i'_k \), where the intervals \( i'_k \) are non-overlapping, and \( n' \leq n \). Then we have

\[ m_0 \sum_{k \in E} i'_k + m_0 C \sum_{k \in E} i'_k = b - a, \] whence

\[ m_0 \sum_{k \in E} i'_k + m_0 C \sum_{k \in E} i'_k = b - a. \] Thus

\[ m_0 E + m_0 CE < b - a + 2 \epsilon. \]

But this holds for all \( \epsilon > 0 \), hence as \( \epsilon \to 0 \),

\[ m_0 E + m_0 CE \leq b - a. \]

For any set \( E \) it is true that

\[ m_0 E + m_0 CE \geq b - a. \]

These complementary inequalities establish the theorem.

1.10 Let us now consider a class of sets in the interval \((a, b)\) having the following properties:

1. If a certain set is included in the class, then its complement is also included.
(2) If each member of a denumerable subclass is included, then the sum (union) of these sets is also included.

We observe that this class is closed under any finite number or denumerable number of additions, complementations, and combinations of the two. Such a class is called a closed additive class of sets [4, p. 598].

The product and difference of point sets can be expressed as a combination of complementation and addition:

\[ E_1 \cdot E_2 \cdot E_3 \cdots = C\left(CE_1 + CE_2 + CE_3 + \cdots \right). \]
\[ E - (E_1 + E_2 + E_3 + \cdots) = C\left(CE + E_1 + E_2 + E_3 + \cdots \right). \]

Thus a closed additive class of sets is also closed under the operations of taking products and differences, up to a denumerable number of times.

Consider any closed additive class of sets in \((a, b)\) which contains all the open intervals in \((a, b)\). If this class is minimal, it is called the Borel class, and the sets in it are Borel sets (sometimes referred to as Borel measurable sets).

Application of the definition for Lebesgue measurability (or, more conveniently, the equivalent definitions stated in section 1.3) shows that sums, products, differences, and complements of measurable sets (finite or denumerable in number) are again measurable sets [5, pp. 177-178; 6, pp. 70-77]. This leads to the conclusion that all Borel sets are measurable, since each one can be formed from open intervals by certain combinations of the four operations mentioned.

It has been shown, on the other hand, that not all measurable
sets are Borel sets, i.e., the Borel class of sets is a proper subclass of Lebesgue measurable sets, which in turn form a proper subclass of all sets in the interval \((a, b)\) [cf. section 1.3]. There is a close relationship between Borel sets and measurable sets, however. In fact, every measurable set can be expressed as the sum of a Borel set and a set of measure zero [5, p. 179]. If this sum is not a Borel set, then it is evident that the set of measure zero is likewise not a Borel set. But it can be expressed, for example, as the sum of itself and the null set, which is a Borel set formed by the product of two non-overlapping open intervals.

Borel sets play a very important role in the theory of measure, and are often very convenient to use in dealing with point sets because of their relatively simple and definite structure. Once we establish that a particular set is a Borel set, we know that it is measurable; furthermore, we are able to tell much concerning its structure.

1.11 In the second chapter of this paper, we will be dealing with points sets in a plane rather than on a line. We must, therefore, find a suitable extension of the concept of exterior measure so that the results for plane sets will be consistent with those obtained for the line, i.e., the same definitions for and theorems concerning measurability will hold, and to the classes of Lebesgue measurable sets and Borel sets in the line will correspond analogous classes in the plane.

It will be seen that the replacement of the interval \((a, b)\) by
a closed rectangle, and of open intervals by open rectangles (with
length-sum of intervals being replaced by area-sum of rectangles)
will produce exactly the desired results [5, p. 175]. In the case
of two dimensional space, the measure of common geometrical figures
such as the square or circle will be associated with their areas
rather than lengths, as for the line. The measure of a one-dimen-
sional set in a two-dimensional space will be zero. For example
the measure of a square is its area, whether we consider the open
square or the closed one, and the measure of its boundary is zero.
It is interesting to note that a non-measurable set on the line
becomes a measurable set of measure zero in the plane.

The class of Borel sets in the plane is again the minimal
closed additive class of sets in the given closed rectangle
which contain all the open rectangles. And again this class is
closed under the additional operations of taking products and
differences, as well as sums and complements.
Chapter 2

Plane Transformations

2.1 In the second chapter we will be concerned with bounded continuous transformations \( T \) of the form \( x = x(u, v) \) and \( y = y(u, v) \), where \( x(u, v) \) and \( y(u, v) \) are defined, bounded, and continuous on the unit square \( S: 0 \leq u \leq 1, 0 \leq v \leq 1 \). For convenience, we shall use the complex notation \( w = u + iv \), \( z = x + iy \). Then the transformation \( T \) may be written in the form \( z = t(w) \), where \( w \in S \). The function \( t(w) \) is defined, bounded, and continuous on \( S \). The transform \( T(A) \) of a set \( A \) in the \( w \)-plane will be considered as defined only on \( A \cdot S \); and the inverse \( T'(A) \) of a set \( A \) in the \( z \)-plane will consist only of those points \( w \in S \) each of which has an image in \( A \) (the symbol representing a set in the \( z \)-plane will be underscored).

2.2 In the sequel, the following concepts, terms, and quantities are used so often that it seems advisable to make the definitions at this time, for once and for all:

(1) A simply connected Jordan region is one bounded by a simple closed Jordan curve. A multiply connected Jordan region is one bounded by a finite number (greater than one) of non-intersecting simple closed Jordan curves. These two terms will be combined under the single heading of Jordan region, and \( R \) will be the generic notation for a member of this class. The positive direction of the boundary curves is taken to be such that the region \( R \) lies to the left as we traverse any one of the curves in its positive direction. Denote the boundary curves by \( C_i \ (i = 1, \ldots, n) \) and let \( \sum_{i=1}^{n} C_i = B_R \)
called the boundary of $R$. Then the set $R^0$ is defined as $R - B_x$.

(2) A domain, or open domain, is a connected open set.

(3) Let $s(A)$ be the generic notation for a neighborhood of a point set $A$, where by neighborhood is meant an open connected set containing $A$. In discussing neighborhoods of a point, there will be no loss in generality if we use only spherical neighborhoods; the symbol $s(w, \alpha)$ will denote a spherical neighborhood of $w$, i.e., the open circular disk with $w$ as center and the real number $\alpha > 0$ as radius.

(4) If we are given two points $w_1$ and $w_2$, we will define the distance between them, $|w_1 - w_2|$, to be the ordinary Euclidean distance $\sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$; this is the modulus of the complex number $w_1 - w_2$.

(5) Given two continuous transformations $T_1: z = t_1(w)$ and $T_2: z = t_2(w)$ defined on $S$, we define $||T_1, T_2/A|| = \text{l.u.b. of } |t_1(w) - t_2(w)|$ for $w \in A$, any set in $S$. This quantity will be called the distance of $T_1$ and $T_2$ on the set $A$ [2, p. 202].

(6) The symbol $\langle a, b, c \rangle$ denotes the greatest one or ones of the three real numbers $a$, $b$, and $c$. Similarly, $\langle a, b, c \rangle$ is the least one or ones of the three.

(7) The diameter of a point set $A$ is defined to be the l.u.b. $|w_1 - w_2|$ for $w_1, w_2 \in A$.

2.3 Lemma: Under a continuous transformation, the inverse of an open set is itself an open set.

Proof: Given the transformation $T: z = t(w)$, continuous on a
Jordan region \( R \). Consider any open set \( \mathcal{O} \) in \( T(R) \). Let \( z_0 \) be a point of \( \mathcal{O} \), and \( w_0 \in R \) be a point of \( T'(z_0) \). Assert that there exists a \( \delta(w_0) > 0 \) such that if \( |w - w_0| < \delta \), then \( w \in T'(\mathcal{O}) \), hence \( T^{-1}(\mathcal{O}) \) is open.

In the proof of this assertion, we first observe that since \( \mathcal{O} \) is open, there exists an \( \varepsilon > 0 \) such that if \( |z - z_0| < \varepsilon \), the point \( z \) is in \( \mathcal{O} \). Since the transformation is continuous, there exists a \( \delta(\varepsilon, w_0) > 0 \) such that for \( |w - w_0| < \delta \), \( |t(w) - t(w_0)| < \varepsilon \); hence \( t(w) \in \mathcal{O} \), and \( w \in T'(\mathcal{O}) \).

2.4 Given a continuous transformation \( T : z = t(w) \neq 0 \) defined on a simple closed oriented curve \( C \subset \mathbb{C} \); if \( C \) be divided suitably into a finite number of simple arcs \( C_i \) (\( i = 1, \ldots, n \)), there exists a continuous argument of \( t(w) \) over each of the arcs \([8, p. 23]\).

Let the endpoints of \( C_i \) be \( w_i \) and \( w_{i+1} \), where \( w_{n+1} = w_1 \). Then we define \( V_{C_i} \arg [t(w)] = \arg [t(w_{i+1})] - \arg [t(w_i)] \). This function is called the variation of the argument of \( t(w) \) over \( C_i \). We also define \( V_C \arg [t(w)] = \sum_{i=1}^{n} V_{C_i} \arg [t(w)] \).

Similarly, for \( t(w) = z_0 \), \( w \in C_i \),

\[ V_{C_i} \arg [t(w) - z_0] = \arg [t(w_{i+1}) - z_0] - \arg [t(w_i) - z_0] \], and

\[ V_C \arg [t(w) - z_0] = \sum_{i=1}^{n} V_{C_i} \arg [t(w) - z_0] \].

Since \( C \), the image - in fact, the continuous image - of \( C \), is a closed directed curve, then \( V_C \arg [t(w) - z_0] = 2n \pi \), where \( n \) is some integer or zero, and is referred to as the topological index of \( z_0 \) with respect to the curve \( C \) \([9, p. 195]\).
2.5 Let $R$ be a Jordan region in $S$, and $C$ one of its boundary curves. As a point $w$ describes $C$ once in a positive sense with respect to $R$, $t(w)$ describes a directed closed continuous curve in the image plane, call it $\mathcal{C}$. For a point $z_0$ in the image plane, we define the number $\mu(z_0, T, C)$ as follows:

$$
\mu(z_0, T, C) = \begin{cases} 0 & \text{if } z_0 \in \mathcal{C}, \\
\text{topological index of } z_0 \text{ with respect to } \mathcal{C} & \text{if } z_0 \notin \mathcal{C}.
\end{cases}
$$

Let $B_R$ be the image of $B_R$ under $T$. then we will define

$$
\mu(z_0, T, B_R) = \begin{cases} 0 & \text{if } z_0 \in B_R, \\
\sum_{c \in B_R} \mu(z_0, T, C) & \text{if } z_0 \notin B_R.
\end{cases}
$$

2.6 If $\mu(z_0, T, B_R) \neq 0$, we say $R$ is an indicator for the point $z_0$ under the transformation $T$, or more simply, an indicator region $(z_0, T)$. If $T$: $z = t(w)$ is continuous on a Jordan region $R$ which is an indicator region $(z_0, T)$, then there exists a point $w_0 \in R^0$ such that $t(w_0) = z_0$ [8, p. 50].

2.7 Theorem of Rouche [8, p. 45]: If $T_1$: $z = t_1(w)$ and $T_2$: $z = t_2(w)$ are both defined and continuous on a Jordan curve $C$ such that $|t_1(w)| < |t_2(w)|$, then $V \arg [t_1(w) + t_2(w)] = V \arg [t_2(w)]$.

Definition: $\varepsilon(z_0, T, B_R)$ = minimum of $|t(w) - z_0|$ for $w \in B_R$.

It is evident that $\varepsilon = 0$ if and only if $z_0 \in T(B_R)$.

Corollary 1 to the theorem of Rouche: Given $\varepsilon(z_0, T, B_R) > 0$.

If $z^*$ be any point of the $z$-plane such that $|z^* - z_0| < \varepsilon$, then

$$
\mu(z^*, T, B_R) = \mu(z_0, T, B_R).
$$
Proof: \( v_{B_k} \text{arg} \left[ t(w) - z^* \right] = v_{B_k} \text{arg} \left[ \{t(w) - z_0\} + \{z_0 - z^*\} \right] \).

|z_0 - z^*| \leq \mathcal{C} \leq |t(w) - z_0|, by hypothesis. Therefore

\( v_{B_k} \text{arg} \left[ t(w) - z^* \right] = v_{B_k} \text{arg} \left[ t(w) - z_0 \right], \) by the theorem of Rouche.

Hence \( \mu(z^*, T, B_k) = \mu(z_0, T, B_k). \)

Corollary 2: Given \( \varepsilon(z_0, T, B) > 0 \). If \( T^*: z = t^*(w) \) be any continuous transformation on \( R \) such that \( \|T^*, T/B_R\| < \mathcal{C} \), then

\( \mu(z_0, T^*, B_k) = \mu(z_0, T, B_k). \)

Proof: \( v_{B_k} \text{arg} \left[ t^*(w) - z_0 \right] = v_{B_k} \text{arg} \left[ \{t^*(w) - t(w)\} + \{t(w) - z_0\} \right] \).

But \( |t^*(w) - t(w)| \leq \|T^*, T/B_R\| < \mathcal{C} \leq |t(w) - z_0| \). Therefore

\( v_{B_k} \text{arg} \left[ t^*(w) - z_0 \right] = v_{B_k} \text{arg} \left[ t(w) - z_0 \right], \) by the theorem of Rouche.

Hence \( \mu(z_0, T^*, B_k) = \mu(z_0, T, B_k). \)

2.8 The symbol \( n(z_0, T, A) \) denotes the number of distinct points \( w \) in the set \( A \cdot T'(z_0) \), where \( A \) is any set contained in \( S \). Note that it is possible for this number to be \( +\infty \) [1, p. 262].

Given \( k \), a positive integer or zero, we define the kernel of order \( k \), denoted by \( K(k, T, R) \), to be the set of points \( z \in T(R) \) for each of which there exists a number \( \eta(k, z, T, R) > 0 \) such that \( n(z, T^*, R) \leq k \) for every continuous transformation \( T^* \) satisfying the relation \( \|T, T^*/R\| < \eta \) [cf. (5), section 2.2]. That is to say, \( T^*(z) \) consists of at least \( k \) distinct points in \( R \) for \( T^* \) within a distance \( \eta \) of \( T \) [1, p. 262].

Theorem: Every \( K(k, T, R) \) of finite order is an open set.

Proof: If \( K \) is empty, we have the null set, which is open.

If \( K \) is not empty, take a point \( z_0 \in K \). Consider any other point \( z_1 \) such that \( |z_0 - z_1| < \eta_0 = \eta(k, z_0, T, R) \).
Assert that $z_1 \in \mathbb{K}$, therefore $\mathbb{K}$ is open. To prove this, we let $\eta_i = \eta_0 - |z_0 - z_1|$. Then $0 < \eta_i < \eta_0$. Let $T_*$ be any continuous transformation on $R$ satisfying the relation $||T_*, R|| < \eta_i$. Now we define the transformation $T^*: t^*(w) = t_*(w) + \lfloor z_0 - z_1 \rfloor$. $T^*$ is continuous on $R$, since $T_*$ is continuous and $z_0 - z_1$ is constant. $||T^*, R|| \leq ||T^*, T_|| + ||T_*, T|| < |z_0 - z_1| + \eta_i = \eta_0$. Hence $n(z_0, T^*, R) \equiv k$ (follows from the definition of $\mathbb{K}$). But each point of $T_*^*(z_0)$ maps into $z_1$ under $T_*$, and each point of $T_*^*(z_1)$ maps into $z_0$ under $T^*$; hence $T_*^{-1}(z_1) = T_*(z_0)$, and $n(z_1, T_*, R) = n(z_0, T^*, R) \geq k$. This shows that $\eta_i$ is a suitable $\gamma (k, z_1, T, R)$, and $z_1 \in \mathbb{K}$.

2.9 Given a point $z$ in the $z$-plane, let us consider a component of its inverse $T^{-1}(z)$. Such a component, being a maximal connected subset of $T^{-1}(z)$, is closed, compact, and therefore a continuum. We call it a maximal model continuum of $z$ under $T$, and use the symbol $\sigma(z, T)$. The totality of all $\sigma(z, T)$ for $z \in T(A)$ includes the set $A$ itself. Furthermore, every point in $T^{-1}(A)$, where $A$ is a set in $T(S)$, belongs to some maximal model continuum.

A $\sigma(z, T)$ is called essential if every open set containing it also contains a Jordan region $R$ which is an indicator region $(z, T)$ and which contains the $\sigma(z, T)$ necessarily in its interior - for assuming that a point $w$ belongs to both $B_R$ and $\sigma(z, T)$ leads to the contradictions that $A[t(w), T, B_R] = 0$, since $t(w) \in T(B_R)$.

2.16 We will now define three point sets in the $w$-plane which
are contained in the unit square $S$ and depend on the (bounded) continuous transformation $T$.

The set $E(T)$ consists of all those points $w \in S$ which are points of some essential maximal model continuum, under the transformation $T$, of any point $z \in T(S)$. Every point of $S$ is a member of some maximal model continuum, but not necessarily of an essential maximal model continuum, hence $E(T)$ need not comprise the whole square $S$. On the other hand, we have no way of determining how many essential maximal model continua will exist, in general, for some transformation. For example, there might be non-denumerably many for one transformation and none at all for another.

The set $C(T)$ consists of all those points $w \in S$ each of which is itself an essential maximal model continuum of its image point $t(w)$ under $T$.

The set $N(T)$ consists of those points $w \in C(T)$ for each of which there exists a neighborhood $s(w)$ such that $s(w) - w$ contains no point of any essential maximal model continuum of $t(w)$, under $T$.

Clearly we have the relation $E(T) \supseteq C(T) \supseteq N(T)$, where the equality signs in general do not hold.

In the remainder of this paper, we shall establish the remarkable fact that all three of these sets are Borel sets. To this end we shall define several other point sets which will be shown to be Borel sets themselves, and then develop point set identities among these and our three given sets, $E(T)$, $C(T)$, and $N(T)$.

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2.11 Given any positive integer \( n \), and any integer \( m \), both different from zero, but not necessarily distinct, denote by \( E_{m,n} \) the set of all points \( w \in S \) for each of which there exists a Jordan region \( R \subset S \) satisfying the following three conditions:

1. \( w \in R^0 \).
2. \( R \subset S(w,1/n) \).
3. \( \mu_[t(w),T,B_R] = m \).

We wish to establish that each set \( E_{m,n} \) is an open set. This will be the case if for each point \( w \in E_{m,n} \) we can find a number \( \eta(w) > 0 \) such that any point within a distance \( \eta \) of \( w \) is also a member of \( E_{m,n} \).

Consider any point \( w_0 \in E_{m,n} \). Then there is a Jordan region \( R \subset S \) such that \( w_0 \in R^0 \), \( R \subset S(w_0,1/n) \), and \( \mu_[t(w_0),T,B_R] = m \). Let the generic notation for a point of \( B_R \) be \( w_0 \). Then \( \min |w_0 - w| = \eta > 0 \), and \( \max |w_0 - w| = \gamma < 1/n \).

Recalling that \( \epsilon[t(w_0),T,B_R] = \min |t(w) - t(w_0)| \), we have the fact that \( \mu [z,T,B_R] = \mu [t(w_0),T,B_R] \) for any point \( z \) such that \( |t(w_0) - z| < \epsilon \), by corollary 1, section 2.7. Because \( T \) is continuous, there exists a \( \delta(w_0,\epsilon) > 0 \) such that \( |t(w_0) - t(w_1)| < \epsilon \) for \( |w_0 - w_1| < \delta \). Hence, for such a point \( w_1 \), it is true that \( \mu [t(w_1),T,B_R] = \mu [t(w_0),T,B_R] = m \).

Choose \( \gamma = \delta, \eta, 1/n - \delta \) \[ cf. (6), section 2.2 \]. Clearly \( \gamma > 0 \), and for any point \( w \) such that \( |w - w_0| < \gamma \), we have

1. \( w \in R^0 \) (the same Jordan region as for \( w_0 \)).
2. \( R \subset S(w,1/n) \).
3. \( \mu[t(w),T,B_R] = m \).
Thus \( w \in E_{m,n} \), and \( E_{m,n} \) is an open set.

2.12 Having established the fact that each set \( E_{m,n} \) is an open set, we observe that it is also a Borel set. It follows that \( \prod_{n=1}^{\infty} (\sum_{m=0}^{\infty} E_{m,n}) \) is a Borel set, too; for here we have denumerably many products of denumerable summations of Borel sets, which by definition comprise a closed additive class. To show that \( C(T) \) is a Borel set, we assert that

\[
C(T) = \prod_{n=1}^{\infty} (\sum_{m=0}^{\infty} E_{m,n}).
\]

Proof: Take a point \( w_0 \in C(T) \). Then \( w_0 \) is an essential maximal model continuum of \( t(w_0) \) under \( T \). Hence every open set containing \( w_0 \) contains an indicator region \( R \) which contains \( w_0 \) in its interior; \( \mu [t(w_0), T, B_R] > 0 \). If, in particular, the open sets are taken to be \( s(w_0, 1/n) \), where \( n = 1, 2, \ldots \), then it is apparent that \( w_0 \) is contained on one of the sets \( E_{m,n} \) for each \( n \), hence also in \( \sum_{m=0}^{\infty} E_{m,n} \) for each \( n \). Therefore \( w_0 \in E_{m,n} \), and

\[
C(T) \subseteq \prod_{n=1}^{\infty} (\sum_{m=0}^{\infty} E_{m,n}).
\]

Take a point \( w_0 \in \prod_{n=1}^{\infty} (\sum_{m=0}^{\infty} E_{m,n}) \). For each value of \( n \), \( w_0 \in \sum_{m=0}^{\infty} E_{m,n} \), and is therefore a member of one of the sets \( E_{m,n} \). By definition of the set \( E_{m,n} \), there exists a Jordan region \( R \) such that \( w_0 \in R^0 \), \( R \subseteq s(w_0, 1/n) \), and \( \mu [t(w_0), T, B_R] = m + 0 \) (i.e., \( R \) is an indicator region of \( t(w_0) \) under \( T \)). The point \( w_0 \) belongs to some \( v [t(w_0), T] \).

Assume this maximal model continuum has other points. Then it has a certain diameter \( \eta > 0 \) [cf. (7), section 2.2]. For \( n > 2/\eta \), \( s(w_0, 1/n) \) clearly contains no Jordan region whose boundary \( B_k \) does not contain a point in common with \( v [t(w_0), T] \). Then we have \( \mu [t(w_0), T, B_k] = 0 \).
which is a contradiction. Thus the point $w_0$ is itself a maximal model continuum of $t(w_0)$ under $T$. Since $s(w_0,1/n)$ contains an indicator region of $t(w_0)$ for all $n$, we infer that every open set containing $w_0$ also contains an indicator region of $t(w_0)$ under $T$. The point $w_0$ is therefore an essential maximal model continuum, and a member of the set $\mathcal{E}(T)$. This gives us the relation

$$\bigcap_{n=1}^{\infty} \left( \bigcup_{m=0}^{\infty} E_{m,n} \right) \subseteq \mathcal{E}(T).$$

Combining it with the complementary, we have

$$\mathcal{E}(T) = \bigcap_{n=1}^{\infty} \left( \bigcup_{m=0}^{\infty} E_{m,n} \right).$$

2.13 Let $D_p$ denote the subdivision of the unit square $S$ into $p_j^2$ congruent squares $q$, where $p_j$ is the $j$th positive prime. Consider any point $w_0$ in the interior of $S$. There exists an integer $j_0(w_0)$ such that for every $j > j_0$, the point $w_0$ is contained in the interior of some square $q$ of each $D_p$. To arrive at a proof of this statement, it will be sufficient to show that the point $w_0$ cannot lie on a vertical line of division for more than one subdivision $D_p$.

Assume that for $j_1 \neq j_2$, $w_0$ lies on a vertical line of division in both $D_{p_1}$ and $D_{p_2}$. Then its distance from the vertical axis is $\frac{\alpha}{P_j} = \frac{\beta}{P_j}$, where $\alpha$ and $\beta$ are integers such that $0 < \alpha < p_j$, and $0 < \beta < p_j$. It follows that $\frac{p_j \cdot \beta}{P_j}$ must be an integer, which is impossible, since $p_j$ divides neither $p_j$ nor $\beta$, and being a prime, cannot then divide their product.

2.14 In the $z$-plane, we now define the function $K(z,T,R)$.
as follows:

\[ \kappa(z, T, R) = \begin{cases} 
  k & \text{for } z \in [K(k, T, R) - K(k+1, T, R)], \\
  +\infty & \text{for } z \in K(\infty, T, R). 
\end{cases} \]

Theorem: \( \kappa(z, T, R) \) is equal to the number of distinct essential maximal model continua of \( z \) under \( T \) in \( R \).

Though this theorem is fundamental, its proof is beyond the scope of this paper. The details of the proof will be found in the reference [1, pp. 271-276].

2.15 We now define a point set \( U \subseteq S \), which will be of use in showing that \( N(T) \) is a Borel set. Let \( U \) be the set of points \( w \in S \) for each of which there is a neighborhood \( s(w, \omega) \) such that \([s(w, \omega) - w]\) contains no essential maximal model continuum for \( t(w) \) under \( T \) (a bounded continuous transformation). This does not imply, however, that \([s(w, \omega) - w]\) contains no point of an essential maximal model continuum.

To show that \( U \) is a Borel set, we require the point set identity

\[ U = \sum_{j=1}^{\infty} \sum_{q \in K_j} \{ q^* - T^{-1}[K(j, T, q)] \}; \]

for we have shown that the set \( K(k, T, R) \) is open for finite \( k \) (section 2.8), and that the inverse of an open set is open (section 2.3). The difference of two open sets (which are necessarily Borel sets) is a Borel set, and the denumerable summation of Borel sets is still a Borel set.

To prove that we have in fact an identity, first take any point \( w_0 \in U \). Then there is a neighborhood \( s(w_0, \omega) \) such that \([s(w_0, \omega) - w_0]\) contains no essential maximal model continuum for
$t(w_0)$ under $T$. There exists an integer $j_0(w_0)$ such that $w_0 \in q^0$ for some square $q$ of each $D_{pj}$ for which $j > j_0$. Also, for $pj > 2/\alpha$, any square $q$ that contains $w_0$ is contained in $s(w_0, \alpha)$. Hence, for $pj > \frac{p_1}{2/\alpha}$, some $q$ of each $D_{pj}$ contains $w_0$ in its interior, and is in turn contained in $s(w_0, \alpha)$. Choose one of the squares $q$ fulfilling these conditions, call it $q_0$. We observe that there is at most one essential maximal model continuum of $t(w_0)$ under $T$ contained in $q_0$ (there would be exactly one if and only if the point $w_0$ were a member of an essential maximal model continuum contained in $q_0$).

Then, according to the theorem stated in section 2.14, the point $t(w_0) \subseteq K(2, T, q_0)$, leading to the conclusion that the point $w_0 \subseteq T^{-1}[K(2, T, q_0)]$. Therefore $w_0 \in \{q_0^0 - T^{-1}[K(2, T, q_0)]\}$, and

$$U \subseteq \sum_{j_0} \sum_{q \in D_{pj}} \{q_0^0 - T^{-1}[K(2, T, q)]\}.$$

Now take a point $w_0 \in \sum_{j_0} \sum_{q \in D_{pj}} \{q_0^0 - T^{-1}[K(2, T, q)]\}$. Then there is some square $q_0$ such that $w_0 \in q_0^0$ and $w_0 \subseteq T^{-1}[K(2, T, q_0)]$. According to the theorem stated in section 2.14, $t(w_0)$ has at most one essential maximal model continuum contained in $q_0$, under the transformation $T$. If it has none, $w_0$ is obviously a member of $U$.

Therefore let us assume there is one essential maximal model continuum of $t(w_0)$ in $q_0$. If $w_0$ is a point of this set, then

$$[s(w_0, \alpha) - w_0]$$

contains no essential maximal model continuum of $t(w_0)$, where $0 < \alpha < \min|w - w_0|$, for $w \in B_q$. If $w_0$ is not a point of this set, there is a positive distance $\nu$ between them (since both are closed sets). Then

$$[s(w_0, \alpha) - w_0]$$

contains no essential maximal model continuum of $t(w_0)$ under $T$, where $0 < \alpha < \nu, \min|w - w_0|$, for
w \in B_q$. Hence the point $w_0 \in U$, and
\[
U \supseteq \sum_{j'} \sum_{q \in B_j} \{q^0 - T' \cdot [K(2, T, q)]\}.
\]
From this and the complementary inequality already established, we have the desired result:
\[
U = \sum_{j'} \sum_{q \in B_j} \{q^0 - T' \cdot [K(2, T, q)]\}.
\]

2.16 In this section will be demonstrated the set identity
\[ N(T) = U \cdot C(T). \]
Since both sets on the right hand side are Borel sets, it follows that their product, $N(T)$, is also a Borel set.

Take a point $w_0 \in U \cdot C(T)$. Then $w_0 \in U$ and $w_0 \in C(T)$. The point $w_0$ is itself an essential maximal model continuum of $t(w_0)$ under $T$. By definition of an essential maximal model continuum, in every open set containing $w_0$, hence in every $s(w_0, \alpha)$, there is a Jordan region $R$ such that $w_0 \in R^0$ and $\mu(w_0, T, R) = 0$. Since $w_0 \in U$, one of these neighborhoods, say $s(w_0, \alpha)$, has the property that $[s(w_0, \alpha)] - w_0$ contains no essential maximal model continuum for $t(w_0)$ under $T$. The particular Jordan region $R_1$ associated with and contained in $s(w_0, \alpha)$ can therefore contain no point of an essential maximal model continuum of $t(w_0)$ other than $w_0$ itself (for if it contained another, it would follow that $\mu(t(w_0), T, R_1) = 0$, a contradiction). Take the number $\alpha_2$ such that $0 < \alpha_2 < \min |w - w_0|$, for $w \in B_R$. Then $s(w_0, \alpha_2) \subseteq R_1$, and $[s(w_0, \alpha_2) - w_0]$ contains no point of an essential maximal model continuum of $t(w_0)$ under $T$.

We now have $w_0 \in N(T)$, and
\[ U \cdot C(T) \subseteq N(T). \]
Take a point \( w_0 \in N(T) \). Then \( w_0 \in E(T) \), by the definition of \( N(T) \). Also there is a neighborhood \( s(w_0, \infty) \) such that \( [s(w_0, \infty) - w_0] \) contains no point of any essential maximal model continuum of \( t(w_0) \) under \( T \). Therefore \( w_0 \in U \) (cf. definition of \( U \), section 2.15). Hence
\[ N(T) \subseteq U \cdot E(T). \]

This combined with the complementary inequality previously obtained, yields the result
\[ N(T) = U \cdot E(T). \]

2.17 To show that the set \( E(T) \) is also a Borel set, we shall make use of the set \( A_n \) (\( n \) any positive integer), which is defined to consist of all those points \( w \in S \) for each of which there exists an indicator region \( R \) of \( t(w) \) under \( T \) such that \( w \in R^0 \) and \( T(R) \subseteq s[t(w), 1/n] \).

First we must know that each set \( A_n \) is open. This will be evident if we can exhibit for each point \( w \in A_n \) a number \( \eta(w) > 0 \) such that \( w^* \in A_n \) if \( |w^* - w| < \eta \). Let us consider such a point \( w \in A_n \). There exists an indicator region \( R \) of \( t(w) \) under \( T \) (i.e., \( \mu[t(w), T, B_R] > 0 \)) such that \( w \in R^0 \) and \( T(R) \subseteq s[t(w), 1/n] \). Since \( T \) is continuous, \( |t(w^*) - t(w)| < \varepsilon \) if \( |w^* - w| < \delta(\in, w) \). Recalling that \( \varepsilon = \min |t(w) - t(w_R)| \) for \( w_R \in B_R \), let the number \( \varepsilon \) be given as \( \varepsilon, 1/n - \max |t(w) - t(w)| \). Note: that \( \varepsilon > 0 \). This choice determines some \( \delta \). Let \( \eta = \frac{\delta}{\min |w - w_R|} \). It follows that if \( |w^* - w| < \eta \), \( w^* \in R^0 \) (the same Jordan region), \( T(R) \subseteq s[t(w^*), 1/n] \), and \( \mu[t(w^*), T, B_R] = \mu[t(w), T, B_R] \neq 0 \). (cf.
corollary 1, section 2.7). Hence \( w^* \subseteq A_n \), and \( A_n \) is an open set.

2.18 Since each set \( A_n \) is open, it follows that the set 
\( A_n \) is a Borel set. We now set out to prove that 
\[
\mathcal{E}(T) = \bigcap_{n=1}^{\infty} A_n.
\]

Take a point \( w_0 \in \mathcal{E}(T) \). Then \( w_0 \) is a member of some essential maximal model continuum of \( t(w_0) \) under \( T \), call the set \( \mathcal{V}_0 \).
\[
T(\mathcal{V}_0) = t(w_0) = z_0.
\]
Every open set containing \( \mathcal{V}_0 \) contains an indicator region \( R \) such that \( R^0 \supset \mathcal{V}_0 \). Let \( w_0 \) be the generic notation for a point of \( \mathcal{V}_0 \), and \( w_R \) a point of \( R \). Now the transformation \( T \) is uniformly continuous. Therefore \( |t(w) - z_0| < \varepsilon \) for 
\[
|w - w_0| < \delta(\varepsilon).
\]
Let \( \varepsilon = 1/n \), \( n=1,2,\cdots \). For each \( \delta(1/n) \) we can construct an open set \( \mathcal{O}_n = \sum s(w_0, S) \supset \mathcal{V}_0 \), each of whose points is within a distance \( \delta \) of some point \( w_0 \). Hence for every point of \( R_n \), \( |t(w_0) - z_0| < 1/n \), \( n=1,2,\cdots \). Thus we have \( w_0 \in A_n \) for each \( n \), and
\[
\mathcal{E} \subseteq \bigcap_{n=1}^{\infty} A_n.
\]

Take a point \( w_o \in \bigcap_{n=1}^{\infty} A_n \). For each \( n \), there is an indicator region \( R_n \) of \( t(w_0) \) under \( T \) such that \( w_o \in R_n^0 \) and 
\[
T(R_n) \subseteq s[t(w_0), 1/n].
\]
The point \( w_o \) is a member of some \( c[t(w_0), T] \), call it \( \mathcal{V}_0 \). Then \( T(\mathcal{V}_0) = t(w_o) = z_0 \).

Assert that \( \mathcal{V}_0 \) is an essential maximal model continuum of \( z_0 \) under \( T \), hence \( w_o \in \mathcal{E}(T) \). Note that \( R_n^0 \supset \mathcal{V}_0 \) for each \( n \), else 
\[
\mu(z_0, T, B_{R_n}) = 0,
\]
a contradiction.

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In the proof of this assertion, we consider the set
\[ T^{-1}[s(z_0, 1/n)], \]
which contains \( T^{-1}(z_0) \). By the lemma, section 2.3, \( T^{-1}[s(z_0, 1/n)] \) is an open set. Let \( \Gamma_n \) be the component of this set which contains \( w_0 \). \( \Gamma_n \) is an open set, and contains \( R_n \) for each \( n \). Also, the \( \Gamma_n \)'s form a decreasing sequence, that is
\[ \Gamma_n \supset \Gamma_{n+1}, \quad n=1,2,\ldots. \]

Clearly \( \bigcap_{n=1}^{\infty} \Gamma_n \supset V_0 \). Since \( T(\Gamma_n) = s(z_0, 1/n) \), and \( z_0 \) is the only point contained in all these disks, the continuity of \( T \) gives us the fact that \( T(\bigcap_{n=1}^{\infty} \Gamma_n) = z_0 \). Furthermore, \( \bigcap_{n=1}^{\infty} \Gamma_n \) is connected [11].

But \( V_0 \) is the maximal connected set in \( T^{-1}(z_0) \) containing \( w_0 \);
therefore \( \bigcap_{n=1}^{\infty} \Gamma_n \subset V_0 \). This gives us the equality \( V_0 = \bigcap_{n=1}^{\infty} \Gamma_n \).

Now let us consider any open set \( \mathcal{O} \subset V_0 \). Since there is a positive distance between \( V_0 \) and \( C \mathcal{O} \), it is evident that for \( n \) sufficiently large, \( \Gamma_n \subset \mathcal{O} \). But \( R_n \subset \Gamma_n \), and \( R_n \) is an indicator region \( (z_0, T) \). Therefore \( V_0 \subset \mathcal{O} \) is an essential maximal model continuum of \( z_0 \) under \( T \). We now have
\[ \bigcap_{n=1}^{\infty} A_n \subset E(T). \]

This and the complementary inequality already obtained give
\[ E(T) = \bigcap_{n=1}^{\infty} A_n. \]

2.19 The result in section 2.18 completes the proof that \( E(T) \), \( \mathcal{C}(T) \), and \( N(T) \) are Borel sets. This fact alone is remarkable enough to warrant investigation of the three sets. But the very property of being Borel sets makes them useful in the further development of the theory of continuous plane transformations. For results along this line, see [1] and [10].

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