CLASSIFICATION OF SOLITON GRAPHS ON TOTALLY POSITIVE GRASSMANNIAN

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

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2015

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It has been known that certain class of nonlinear wave equations admits stable solitary wave solutions that are regular, non decaying and localized along distinct lines in the $xy$-plane. The most well-known example of such equations is the KP equation, and it provides an excellent model for shallow water waves. These solutions are known as the line-soliton solutions, and they form complex interaction patterns of line-solitons resembling web-like structures. Here the patterns generated by those soliton solutions will be called the soliton graphs.

In this thesis, we consider mainly the soliton solutions of the KP equation. It is well known that a soliton solution $u_A(x, y, t)$ of the KP equation can be constructed from a point $A$ of the real Grassmannian $Gr(N, M)$, and has been proven that the regularity of soliton solution $u_A(x, y, t)$ is equivalent to the total non-negativity of $A$, that is, $A$ is an element of totally nonnegative Grassmannian, denoted by $Gr(N, M)_{\geq 0}$.

The main purpose of the thesis is to classify the soliton graphs using geometric combinatorics. For this purpose, we consider the soliton solutions of the KP hierarchy, which consists of the symmetries of the KP equation parameterized by a sequence of compatible time variables $t = (t_3, t_4, \cdots)$. Each soliton graph can be expressed as a point configuration $\mathcal{A}$, where each element of $\mathcal{A}$ represents a set of soliton parameters such as the wavenumber and the propagation direction. Then we consider a subdivision of the point configuration $\mathcal{A}$, which we call soliton subdivisions, and we relate the soliton subdivisions with polyhedral fan structure in the multi-time $t$. 
space. Here is a summary of the main results. We develop an explicit algorithm to construct the soliton subdivisions by lifting the configuration $\mathcal{A}$ with specific weights for these points in $\mathcal{A}$. We use the Gale transform to identify the polyhedral cones of $Gr(1, M)_{>0}$ case to the secondary polytopes. Then we extend the Gale transform for the $Gr(N, M)_{>0}$ case, and give a realizability checking theorem about soliton subdivisions, which induces a parameterization of a polyhedral cone in multiple time $t$ space by a soliton subdivision. In the end, we present the detailed analysis for the case of $Gr(3, 6)$. 
Dedicated to my family and friends
I would like to thank my advisor, Professor Yuji Kodama, for introducing me to the area of Integrable systems, and helping me overcome the difficulties of learning and research. I would like to thank Professor Lauren Williams and Professor Yuji Kodama for providing the materials for their work and bringing up the questions for this thesis.

I would like to thank Professor Yuji Kodama, Professor Dan Burghelea, Professor Mike Davis, Professor Matthew Kahle, Professor Thomas Kerler, Professor Jean-François Lafont, Professor Crichton Ogle for their courses.

I would like to thank Professor Mike Davis, Professor David Anderson, Professor Vladimir Kogan for proofreading this thesis.

I would like to thank Professor Yuji Kodama, and my classmates Xiaohui Wang, Tinghao Hsu, Shenghui Liu for discussion about the questions I meet during the thesis preparation.

Finally, I would like to thank my parents for bringing me up and continuously supporting my education.
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CHAPTER 1
INTRODUCTION

The KP equation is a two-dimensional nonlinear dispersive wave equation given by

$$\frac{\partial}{\partial x} \left( -4 \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0,$$  

(1.0.1)

where $u = u(x, y, t)$ represents the wave amplitude at the point $(x, y)$ in the $xy$-plane at the time $t$. The equation was proposed by Kadomtsev and Peviashvili in 1970 to study the transversal stability of the soliton solutions of the Korteweg-de Vries (KdV) equation [9]. The KP equation can also be used to describe shallow water waves, and in particular, the equation provides an excellent model for the resonant interaction of those waves (see [11] for recent progress).

One of the main breakthroughs in the KP theory was given by Sato [18] in 1980, who realized that solutions of the KP equation can be expressed as an orbit on an infinite-dimensional Grassmannian. The present thesis deals with a real, finite-dimensional version of the Sato theory. In particular, we consider the so-called line-soliton solutions which can be constructed from a point of the real Grassmannian denoted by $Gr(N, M)$, with the point expressed as an $N \times M$ rectangle matrix $A$, [18, 19, 6, 7]. See chapter 2 for the precise definition of Grassmannian $Gr(N, M)$, and totally non-negative (totally positive) Grassmannian $Gr(N, M)_{\geq 0}$ ($Gr(N, M)_{> 0}$). We denote by $u_A(x, y, t)$ the solution associated to $A \in Gr(N, M)$,
and it can be constructed under the transformation \( u_A(x, y, t) = \frac{2}{\partial x^2} \ln \tau_A(x, y, t) \), where the \( \tau_A(x, y, t) \) function can be obtained as follows.

\[
\tau_A(x, y, t) = \sum_{1 \leq i_1 < \cdots < i_N \leq M} \Delta_{i_1 \cdots i_N}(A) E_{i_1, \ldots, i_N}(x, y, t),
\]

where \( \Delta_{i_1 \cdots i_N}(A) \) is the \( N \times N \) minor of the matrix \( A \) with the column indices given by \( \{i_1 < \cdots < i_N\} \), and \( E_{i_1, \ldots, i_N}(x, y, t) \) is the Wronskian determinant of the exponential functions \( \{E_i(x, y, t) = \exp(\kappa_i x + \kappa_i^2 y + \kappa_i^3 t) : i = 1, \ldots, M\} \), i.e.

\[
E_{i_1, \ldots, i_N} := \begin{vmatrix}
E_{i_1} & E_{i_1}^{(1)} & \cdots & E_{i_1}^{(N-1)} \\
E_{i_2} & E_{i_2}^{(1)} & \cdots & E_{i_2}^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
E_{i_N} & E_{i_N}^{(1)} & \cdots & E_{i_N}^{(N-1)}
\end{vmatrix}
\]

with \( E_{i_1}^{(j)} \) the \( j \)-th derivative of \( E_i(x, y, t) \) with respect to \( x \). The function \( E_{i_1, \ldots, i_N} \) can be explicitly given by

\[
E_{i_1, \ldots, i_N}(x, y, t) = K_{i_1 \cdots i_N} \prod_{s=1}^{N} E_{i_s}(x, y, t),
\]

where \( K_{i_1 \cdots i_N} = \prod_{1 \leq s < t \leq N} (\kappa_{i_t} - \kappa_{i_s}) \) with the order \( \kappa_1 < \kappa_2 < \cdots < \kappa_M \). It has been proven that the solution is regular if and only if all the minors \( \Delta_{i_1 \cdots i_N}(A) \) are nonnegative, that is, the matrix \( A \) should be an element of the totally nonnegative Grassmannian, denoted by \( Gr(N, M)_{\geq 0} \). If all the minors are positive, we call it the totally positive Grassmannian denoted by \( Gr(N, M)_{> 0} \).

In this thesis, we are interested in classifying the two dimensional patterns generated by the solution \( u_A(x, y, t) \) for a fixed time \( t \), which can be illustrated by the contour plot of the solution in the \( xy \)-plane. We call the pattern the soliton graph (we will give a precise definition of the soliton graph in Chapter 2). For example, when \( N = 2 \) and \( M = 5 \), the soliton graphs for \( Gr(2, 5)_{> 0} \) is shown in Figure 1.1.
In each region of the soliton graph shown in the example, one exponential $E_{i_1 \cdots i_N}(x, y, t)$ in the $\tau$-function is dominant over other exponentials, so that the solution becomes trivial, i.e. $u_A(x, y, t) = \frac{2}{\partial x^2} \ln \tau_A(x, y, t) \approx \frac{2}{\partial x^2} \ln E_{i_1 \cdots i_N}(x, y, t) = 0$ in this region. Thus each region of soliton graphs corresponds to a dominant exponential $E_{i_1 \cdots i_N}(x, y, t)$ indexed by $i_1 \cdots i_N$.

Figure 1.1: An example of soliton graph for $Gr(2, 5)$ at three different times. A pair of numbers in each region represents the dominant exponential indexed by those numbers, e.g. 12 means that in this region $E_{12}$ dominates over other exponentials.

Let us provide a background of work on the problem about classifying the regular line-soliton solutions. Kodama and his collaborators have worked on classifying the regular line-soliton solutions [2, 10, 1, 3, 4]. They found the asymptotic pattern of the line-solitons at $y \to \pm \infty$ of the solution, which has a tight connection with totally non-negative Grassmannian structure. In the field of combinatorics, Postnikov [16] studied the totally non-negative part of the Grassmannian $Gr(N, M)_{\geq 0}$ from a combinatorial point of view. He constructed a decomposition of $Gr(N, M)_{\geq 0}$ into
positroid cells, and introduced several remarkable families of combinatorial objects, including decorated permutations, J-diagrams, plabic graphs, and Grassmann necklaces. All these structures have been found out to have a connection with the soliton solutions. In [12], Kodama and Williams exploited the connection between the theory of total positivity for the Grassmannian and the structure of regular soliton solutions to the KP equation. They completely characterized the soliton graphs for the time asymptotic $t \to \pm \infty$, using an algorithm based on the J-diagram. They also provided a complete classification of the soliton graphs coming from the totally positive Grassmannian $Gr(2, M) > 0$ by using triangulations of an $M$-gon.

In this thesis, we consider the soliton graphs for the KP hierarchy, which consists of the symmetries of KP equation (see e.g. [14]). Each symmetry has a new compatible time parameter, and the KP hierarchy is defined on the multi-time space $t = (t_3, t_4, t_5, \cdots)$.

Then the exponential terms in soliton solution $u_A(x, y, t)$ of KP hierarchy can be updated by $E_i(x, y, t) = \exp(p_i x + q_i y + \omega_i(t))$, and each weight $\omega_i(t)$ can be obtained by time $t = (t_3, t_4, \ldots)$ variables. Thus we will study all soliton graphs in the multi-time $t$-space of the KP hierarchy. Figure 1.2 shows a classification of all $Gr(2, 5) > 0$ soliton subdivisions in $(t_3, t_4)$-space.

With the weight set $\omega = (\omega_1, \cdots, \omega_M)$ for each $t$, we define a set of point configurations $A^\omega = \{(p_I, q_I, \omega_I)\}$ with the index set $I = \{i_1, \ldots, i_N\}$, where the parameters are given by $p_I = \sum_{i \in I} p_i, q_I = \sum_{i \in I} q_i, \omega_I = \sum_{i \in I} \omega_i$. We then construct a regular subdivision of $A$, and obtain a soliton graph as a dual graph of the subdivision. In this thesis, we will mainly discuss the case for the totally positive Grassmannian, that is, we assume $\Delta_I(A) > 0$ for all set $I = \{i_1, \cdots, i_N\}$. The main goal of this thesis is
to classify the soliton graphs of the KP hierarchy by constructing the regular subdivisions of the point configuration $\mathcal{A}_A^\omega$ for $A \in Gr(N, M)_{>0}$, which is denoted by $\mathcal{A}_{N,M}^\omega$.

Each subdivision of the point configuration $\mathcal{A}_{N,M}$ is a collection of polygons (also called polygon tiles), satisfying

- the union of all these polygons covers the convex hull of $\mathcal{A}_{N,M}$,
- all the vertices of each polygon are from $\mathcal{A}_{N,M}$,
- each pair of polygons intersect along a common face (possibly empty).

If all the polygons are triangles, we call it a triangulation of $\mathcal{A}_{N,M}$.

Let us first consider the point configuration for $Gr(1, M)_{>0}$.

\[ \mathcal{A}_{1,M} = \{(p_i, q_i) \in \mathbb{R}^2 : i = 1, \ldots, M\} \]

Figure 1.2: Soliton graphs for $Gr(2, 5)_{>0}$ of a KP hierarchy with $(t_3, t_4)$ variable.
with the $M$ points cyclically sitting in $\mathbb{R}^2$, called the $pq$-plane. Since the points $(p_i, q_i)$ are on a parabola, i.e. $q_i = p_i^2$, the convex hull $\text{convex}(A) = P_0$ is an $M$-gon. Then we add a weight $\omega_i$ on each point $(p_i, q_i)$, and define a point configuration on $\mathbb{R}^3$, 

$$A_{1,M}^{\omega} = \{(p_i, q_i, \omega_i) \in \mathbb{R}^2 \times \mathbb{R} : i = 1, \ldots, M\}$$

and define a three dimensional convex polytope $\text{convex}(A^\omega) = P^\omega$. We project the faces of $P^\omega$ down into the $pq$-plane, and obtain a subdivision $Q_{1,M}(\omega)$ of $P_0$.

In Figure 1.3, we illustrate a triangulation $Q_{1,5}(\omega)$, and the dual of $Q_{1,5}(\omega)$ is the soliton graph for the corresponding the weight $\omega$. (A precise definition of the duality will be given in Chapter 3.) Notice that the classification of the soliton graphs for $Gr(1, M) > 0$ is equivalent to the triangulations of $M$-gon.

![Figure 1.3: Duality between the triangulation of a pentagon and KP soliton graph for $Gr(1, 5) > 0$. The left panel shows the point configuration $A_{1,5}^{\omega} = \{(\kappa_i, \kappa_i^2, \omega_i) : i = 1, \ldots, 5\}$ and the right panel shows the corresponding soliton graph which is dual to $A_{1,5}^{\omega}$.](image)

In the general $Gr(N, M) > 0$ case, we start with

$$A_{N,M}^{\omega} = \{(p_I, q_I, \omega_I) \in \mathbb{R}^3 : I = \{i_1, \ldots, i_N\}, \text{ with } 1 \leq i_1 < \cdots < i_N \leq M\}.$$
Then one can show that the convex hull $P_{N,M}^0 = \text{conv}(A_{N,M}^0)$ gives an $M$-gon. By projecting higher faces oriented top of convex hull $P_{N,M}^{\omega} = \text{conv}(A_{N,M}^{\omega})$ into $\mathbb{R}^2$, we have a subdivision $Q_{N,M}(\omega)$ of the $M$-gon $P_{N,M}^0$.

Figure 1.4 demonstrates the duality structure between the subdivisions of $A_{2,5}^{\omega}$ and the KP soliton graphs for $Gr(2,5)_{>0}$. In the example, we choose different three choices of the weight $(\omega_1, \omega_2, \cdots, \omega_5)$. Figure 1.5 illustrates an example of inductive construction algorithm (see Algorithm 4.6.1 in chapter 4) of the soliton subdivisions for given weight $\omega = (\omega_1, \cdots, \omega_M)$.

Figure 1.4: Soliton subdivisions and the KP soliton graphs of $Gr(2,5)_{>0}$. The left panel shows the case with the weight $\omega = (0, 0, 0, 0, 0)$; The middle panel shows the case with $\omega = (24, 2, 0, 0, 8)$; The right panel shows the case with $(8, 1, 0, -1, -8)$.

The structure of the thesis is organized as follows.

In chapter 2, we provide background information for Hirota bilinear equations, and $\tau$ functions. We present soliton solutions for three special reductions of Hirota bilinear
equations, namely, the KP equation, 2D Toda equations and the Davey-Stewartson equations. We define the soliton graph based on a piecewise linear function, which gives the dominance relation among the exponentials in $\tau$ function. Then we explain a result that describes the unbounded region of a soliton graph in [3, 4].

In chapter 3, we introduce a duality map from a plane configuration to a point configuration, and define soliton subdivisions of the point configuration, which is dual to the soliton graphs. Then we state the main problems for the thesis, which is to classify the soliton subdivisions using several combinatorial tools.

In chapter 4, we introduce a quadrilateral checking formula for the triangulation of a 4-gon, and we develop an algorithm to construct the soliton subdivision of $Gr(1, M)_{>0}$ case. Then we give an induction map which induces a soliton subdivision
for $Gr(N + 1, M) > 0$ from that for $Gr(N, M) > 0$. The main work in this chapter is to use the quadrilateral checking formula and the induction map to provide an algorithm to construct the soliton subdivision for the general case of $Gr(N, M) > 0$.

In chapter 5, we introduce the KP hierarchy with the multi-time variables $t$. We give a classification of soliton subdivision of $Gr(1, M) > 0$ case by constructing the Gale transform in the multi-time space. Then we prove the main theorem, which parametrizes soliton subdivision with polyhedral cone in the multi-time space. Using the theorem, we inductively give the polyhedral structure in the multi-time space whose polyhedral cone corresponds to each soliton subdivision. In the end, we illustrate a complete classification of the soliton graphs for $Gr(3, 6) > 0$, based on the results of this chapters.
CHAPTER 2
SOLITON SOLUTIONS AND SOLITON GRAPHS

In this chapter, we introduce soliton solutions and soliton graphs. We provide background information for Hirota bilinear equations, and \( \tau \) functions (see for example, [5], [20]). Then we introduce the soliton solutions of the KP equation, 2D Toda equations and the Davey-Stewartson equations. In the end, we define the soliton graph based on a piecewise linear function and explain a result which classifies the unbounded region of a soliton graph in [3, 4].

2.1 Notations

In this thesis, we denote \([M]\) as the \( M \) digit set \( \{1, 2, \cdots, M\} \), and \( \binom{[M]}{N} \) as the collectoin of all the \( N \) digit subsets of \([M]\). For an element \( I = \{i_1, \cdots, i_N\} \) in \( \binom{[M]}{N} \), we may denote it as \( I = i_1 \cdots i_N \). For an \( N + 1 \) digit subset \( I \cup \{i_{N+1}\} \in \binom{[M]}{N+1} \), we may denote it as \( I \cup i_{N+1} \), or \( I i_{N+1} \) if there is no confusion.

The real Grassmannian \( Gr(N, M) \) is the space of all \( N \)-dimensional subspaces of \( \mathbb{R}^M \). An element of \( Gr(N, M) \) can be viewed as a full-rank \( N \times M \) matrix modulo left multiplication by nonsingular \( N \times N \) matrices. In other words, two \( N \times M \) matrices represent the same point in \( Gr(N, M) \) if and only if they can be obtained from each other by row operations. Let \( \binom{[M]}{N} \) be the set of all \( N \)-element subsets of \([M]\).

For a \( N \) digits set \( I \in \binom{[M]}{N} \), let \( \triangle_I(A) \) denote the maximal minor of a \( N \times M \)
matrix $A$ located in the column set $I$, where $A$ represents an element in $Gr(N,M)$. The map $A \mapsto (\Delta_I(A))$, where $I$ ranges over $\binom{[M]}{N}$, induces the Plücker embedding $Gr(N,M) \hookrightarrow \mathbb{RP}^{\binom{M}{N} - 1}$.

**Definition 2.1.1.** Given an element $A \in Gr(N,M)$, we define the associated matroid $M(A)$ as the collection of all the $N$-subsets $I \subset [M]$ such that $\Delta_I(A) \neq 0$.

**Definition 2.1.2.** The totally non-negative Grassmannian $(Gr(N,M))_{\geq 0}$ (respectively, totally positive Grassmannian $(Gr(N,M))_{> 0}$) is the subset of $(Gr(N,M))$ that can be represented by $N \times M$ matrices $A$ with all $\Delta_I(A)$ non-negative (respectively, positive).

### 2.2 Hirota bilinear equations

Consider the following set of bilinear equations for the $\tau$-functions $\tau^{(s)}(x_k; k \in \mathbb{Z})$ with $s \in \mathbb{R}$, [20]

\[
\int \tau^{(s')}(x' - [\lambda^{-1}], \bar{x}') \tau^{(s)}(x + [\lambda^{-1}], \bar{x}) \lambda^{s'-s} e^{\theta(x'-x, \lambda)} d\lambda = \int \tau^{(s'+1)}(x', \bar{x}' - [\lambda]) \tau^{(s-1)}(t, \bar{x} + [\lambda]) \lambda^{s'-s} e^{\theta(\bar{x}'-\bar{x}, \lambda^{-1})} d\lambda
\]

where $x = (x_k : k \in \mathbb{Z}_{>0}), \bar{x} = (x_j : j \in \mathbb{Z}_{<0})$, and

\[
\theta(x, \lambda) = \sum_{n=1}^{\infty} \lambda^n x_n, \\
x - [\lambda^{-1}] = \left( x_1 - \frac{1}{\lambda}, x_2 - \frac{1}{2\lambda^2}, \ldots \right), \text{ etc.}
\]

In the equations we have $\lambda \in \mathbb{C}$, and the integral is taken for any circular bounded loops in $\mathbb{C}$ not passing the origin.

Setting $s' = s$ and $x \rightarrow x + a, \bar{x} \rightarrow \bar{x} - a$, and expanding the identity w. r. t. $a$, one can get the following system of equations,
\[ [D_1(-4D_3 + D_1^3) + 3D_2^2] \tau^{(s)} \circ \tau^{(s)} = 0, \]  
(2.2.1)  
\[ (D_2 + D_1^2) \tau^{(s)} \circ \tau^{(s+1)} = 0, \]  
(2.2.2)  
\[ (D_{-2} - D_{-1}^2) \tau^{(s)} \circ \tau^{(s+1)} = 0 \]  
(2.2.3)  
\[ D_1D_{-1} \tau^{(s)} \circ \tau^{(s)} = 2[(\tau^{(s)})^2 - \tau^{(s+1)}\tau^{(s-1)}], \]  
(2.2.4)

where the Hirota derivatives \( D_k \) for \( k \in \mathbb{Z} \setminus \{0\} \) are

\[ D_k^n f \circ g = \left( \frac{\partial}{\partial x_k} - \frac{\partial}{\partial y_k} \right)^n f(x)g(y) \bigg|_{y=x}, \]

The first equation (2.2.1) is the KP bilinear equation, and the second one (2.2.2) is referred to as the Bäcklund transformations of the KP equation between two \( \tau \)-functions \( \tau = \tau^{(s)} \) and \( \tau' = \tau^{(s+1)} \). The third one (2.2.3) is an extension of the Bäcklund transformation for the negative flows. The forth equation (2.2.4) is two-dimensional Toda bilinear equations for \( s \in \mathbb{Z} \). We then note that the forth equation together with the second and third equations provides the bilinear form of the Davey-Stewartson equation (see the next few sections for the details).

In this thesis, we study real and regular line-soliton solutions of those equations in terms of real Grassmannian \( Gr(N, M) \), the set of \( N \)-dimensional subspaces in \( \mathbb{R}^M \). The main goal of this chapter is to show how one can generate soliton graph of the equations associated to those bilinear equations.

### 2.3 The \( \tau \)-functions for soliton solutions

It is well-known that for each \( s \), the \( \tau^{(s)} \)-functions in the Wronskian of some functions \( \{ f_i^{(s)}(x) : i = 1, \ldots, N \} \) satisfy the Hirota bilinear equations for \( x := (x_k; k \in \mathbb{Z}) \),

\[ \tau^{(s)}(x) = \text{Wr}(f_1^{(s)}(x), \ldots, f_N^{(s)}(x)), \]  
(2.3.1)
Here the derivatives in the Wronskian are with respect to the $x_1$-variable, and each $f^{(s)}_i$ satisfies
\[
\frac{\partial f^{(s)}_i}{\partial x_k} = f^{(s+k)}_i \quad \text{for} \quad k \in \mathbb{Z} \setminus \{0\}.
\] (2.3.2)
That is, we have (see [7] for the proof):

**Proposition 2.3.1 ([7]).** *The set of those $\tau$-functions satisfies the Hirota bilinear equations.*

For the equations, we consider the following exponential functions which are simple solutions of those equations,
\[
E_j^{(s)}(x) = p_j^s e^{\theta_j(x)} \quad \text{with} \quad \theta_j(x) := \sum_{k \in \mathbb{Z} \setminus \{0\}} p_j^k x_k
\] (2.3.3)
with constant parameters $p_j \in \mathbb{C}$. We then consider $f^{(s)}_i$ as a finite sum of those exponential functions $\{E_j^{(s)}: j = 1, \ldots, M\}$, i.e.
\[
f^{(s)}_i(x) = \sum_{j=1}^M a_{ij} E_j^{(s)}(x),
\]
The $N \times M$ matrix $A = (a_{ij})$ can be considered as a point of Grassmannian $Gr(N, M)$.
We assume the matrix $A$ to be *irreducible*, i.e. the reduced row echelon form of $A$ contains no zero column(s) and has no row(s) with only pivot as the nonzero element.

Then using Cauchy-Binet formula to expand the determinant, the $\tau^{(s)}$-function can be expressed as
\[
\tau^{(s)}(x) = \sum_{I \in \binom{[M]}{N}} \Delta_I(A) E_I^{(s)}(x), \quad \text{with} \quad E_I^{(s)} = \text{Wr}(E_{i_1}^{(s)}, \ldots, E_{i_N}^{(s)}),
\]
for \( I = \{i_1, \ldots, i_N\} \). Here \( \Delta_I(A) \) is the \( N \times N \) minor of the matrix \( A \) with the columns having the indices in \( I \), and the Wronskian \( E_I^{(s)} \) is

\[
E_I^{(s)} = \begin{vmatrix}
E_{i_1}^{(s)} & E_{i_1}^{(s+1)} & \cdots & E_{i_1}^{(s+N-1)} \\
E_{i_2}^{(s)} & E_{i_2}^{(s+1)} & \cdots & E_{i_2}^{(s+N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
E_{i_N}^{(s)} & E_{i_N}^{(s+1)} & \cdots & E_{i_N}^{(s+N-1)}
\end{vmatrix} = \left( \prod_{k=1}^{N} p_{i_k}^s e^{\theta_{i_k}} \right) \begin{vmatrix}
1 & p_{i_1} & \cdots & p_{i_1}^{N-1} \\
1 & p_{i_2} & \cdots & p_{i_2}^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & p_{i_N} & \cdots & p_{i_N}^{N-1}
\end{vmatrix}
\]

\[
= \left( \prod_{k=1}^{N} p_{i_k}^s e^{\theta_{i_k}} \right) \prod_{j>k} (p_{i_j} - p_{i_k}) =: P_I^{(s)} e^{\Theta_I(x)} ,
\]

where \( P_I \) and \( \Theta_I(x) \) for the ordered set \( I = \{i_1 < \cdots < i_N\} \) are defined by

\[
P_I^{(s)} = \left( \prod_{k=1}^{N} p_{i_k}^s \right) \prod_{j>k} (p_{i_j} - p_{i_k}), \quad \text{and} \quad \Theta_I(x) = \sum_{k=1}^{N} \theta_{i_k}(x) .
\]

Notice that the parameter \( s \) appears only in \( P^{(s)} \) and the \( \tau \)-function has a simple form,

\[
\tau^{(s)}(x) = \sum_{I \in \binom{[M]}{N}} \Delta_I(A) P^{(s)} e^{\Theta_I(x)} . \tag{2.3.4}
\]

We construct real and regular solutions of those two-dimensional soliton equations mentioned in the previous section by choosing appropriate parameters \( p_j \)'s in the exponential functions. The matrix \( A \) should be chosen as a point of totally non-negative Grassmannian \( Gr(N, M)_{\geq 0} \). This implies that all the minors \( \Delta_I(A) \) for \( A \in Gr(N, M)_{\geq 0} \) are non-negative.

Recall the definition of the matroid associated to the matrix \( A \) from \( Gr(N, M)_{\geq 0} \), i.e.

\[
\mathcal{M}(A) := \left\{ I \in \binom{[M]}{N} : \Delta_I(A) \neq 0 \right\} .
\]

Then, we may write the \( \tau \)-function in the form,

\[
\tau^{(s)}(x) = \sum_{I \in \mathcal{M}(A)} \exp \left( \Theta_I(x) + \ln(\Delta_I(A) P^{(s)}) \right) .
\]

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We consider a large scale, i.e. with a small parameter \(0 < \epsilon \ll 1\),

\[ x_k \rightarrow \frac{x_k}{\epsilon} \quad \text{for all} \quad k \in \mathbb{Z} \setminus \{0\}. \]

Then we define a piecewise linear function \(f_{\mathcal{M}(A)}(x)\) by the limit,

\[
f_{\mathcal{M}(A)}(x) := \lim_{\epsilon \to 0} \epsilon \ln \left[ \tau^{(s)} \left( \frac{x}{\epsilon} \right) \right] = \lim_{\epsilon \to 0} \epsilon \ln \left[ \sum_{I \in \mathcal{M}(A)} \exp \left( \frac{1}{\epsilon} \Theta_I(x) \right) \right] \quad (2.3.5)
\]

\[= \max \{ \Theta_I(x) : I \in \mathcal{M}(A) \}.\]

We will use this piecewise linear function \(f_{\mathcal{M}(A)}(x)\) to define the soliton graphs which represent the patterns of the soliton solutions.

### 2.4 The KP solitons

The KP equation for \(u = u(x, y, t)\) is given by

\[
(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \quad (2.4.1)
\]

where \((x, y, t)\) are identified as

\[x = x_1, \quad y = x_2, \quad t = x_3,\]

in the first equation (2.2.1) of the bilinear equations. The solution \(u\) is expressed by

\[u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \ln \tau^{(s)}(x, y, t), \quad \text{with} \quad s = 0.\]

In this case, the exponential functions \(E_j^{(0)}(x, y, t)\) are given by

\[E_j^{(0)}(x, y, t) = e^{\theta_j(x, y, t)} \quad \text{with} \quad \theta_j(x, y, t) := \kappa_j x + \kappa_j^2 y + \kappa_j^3 t.\]

That is, we choose

\[p_j = \kappa_j \in \mathbb{R}, \quad \text{and assume the order} \quad \kappa_1 < \kappa_2 < \cdots < \kappa_M.\]
Figure 2.1: A polygon inscribed in a parabola $\zeta = \xi^2$ for the KP soliton.

Note that each parameter $(p_j, q_j)$ is a point on the parabola, $q = p^2$.

Choosing the $\kappa$-parameters, we get a convex $M$-gon inscribed in the parabola, $q = p^2$, with vertices $v_j = (\kappa_j, \kappa_j^2)$, $j = 1, \ldots, M$. See Figure 2.1.

Then the $\tau$-function is given by

$$
\tau(0)(x, y, t) = \sum_{I \in \binom{[M]}{N}} \Delta_I(A) K_I e^{\Theta_I(x, y, t)} \quad \text{with} \quad K_I = \prod_{j > k} (\kappa_{i_j} - \kappa_{i_k}).
$$

Therefore, the choice $A \in Gr(N, M)_{\geq 0}$ implies the regularity of the solution $u(x, y, t)$ for all $(x, y, t) \in \mathbb{R}^3$ (the converse is also true).

### 2.5 Two-dimensional Toda lattice solitons

The two-dimensional Toda lattice has the form,

$$
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \ln(1 + V_n) = V_{n+1} - 2V_n + V_{n-1}.
$$

The bilinear equation of the 2-dim Toda lattice is given by (2.2.4). The variable $V_n$ is expressed in terms of the $\tau^{(n)}$-function,

$$
V_n(x, t) = -\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \ln \tau^{(n)}(x, t).
$$
with the variables

\[ x = x_1 - x_{-1}, \quad t = x_1 + x_{-1}, \quad n = s \in \mathbb{Z}. \]

Then the exponential function is given by

\[
E_j^{(n)}(x,t) = p_j^n \exp \left[ \frac{1}{2} \left( p_j - \frac{1}{p_j} \right) x + \frac{1}{2} \left( p_j + \frac{1}{p_j} \right) t \right]
= (\pm 1)^n \exp (\pm x \sinh \phi_j \pm t \cosh \phi_j + n \phi_j) =: (\pm 1)^n e^{\theta_j^{\pm}(x,t,n)}.
\]

where we have used \( p_j = \pm \exp \phi_j \in \mathbb{R} \). Note that writing the exponent in the form

\[ \tilde{p}_j x + \tilde{q}_j y + \omega_j n, \]

the parameters \((\tilde{p}_j, \tilde{q}_j)\) are lying on the hyperbola, \( \tilde{q}^2 - \tilde{p}^2 = 1 \), and the variable \( \phi_j \) represents the area of the region bounded by the hyperbola and the lines \( q_j x = p_j y \) and \( x = 0 \) (the \( \phi_j \) with negative \( x \) should be considered as \(-\)the corresponding area)).

We consider the hyperbola \((p, q) = (\pm \sinh \phi, \pm \cosh \phi)\). Note that the sign gives a specific branch of the curve, i.e \(+(-)\) corresponds to the upper (lower) branch.

---

**Figure 2.2:** A hyperbola for the two-dimensional Toda lattice solitons. Recall that \( \phi_j \) measures the area surrounded by the \( y \)-axis, the line \( q_j p = q_j p \) and the hyperbola (note \( \phi_1 < \phi_2 < 0 < \phi_3 < \phi_4 < \phi_5 \) in this example).
Then we find the Wronskian \( E_I^{(n)} = \text{Wr}(E_{i_1}^{(n)}, \ldots, E_{i_N}^{(n)}) = P_I^{(n)} e^{\theta_I} \) with \( \theta_I = \sum_k \theta_{i_k}^\pm \) and

\[
P_I^{(n)} = \left( \prod_{k=1}^{N} p_{i_k}^n \right) \prod_{j>k} (p_{i_j} - p_{i_k}) = (\pm 1)^n N \prod_{k=1}^{N} e^{n \phi_{i_k}} 2^{N(N-1)/2} \prod_{j>k} e^{(\phi_{i_j} + \phi_{i_k})/2} \sinh \frac{1}{2} (\phi_{i_j} - \phi_{i_k})
\]

\[
= (\pm 1)^n N^2 2^{N(N-1)/2} e^{(n+N-1)/2} \sum_k \phi_{i_k} \text{Sh}_I(\phi) \quad \text{with} \quad \text{Sh}_I(\phi) := \prod_{j>k} \sinh \frac{1}{2} (\phi_{i_j} - \phi_{i_k}).
\]

The positivity of \( \text{Sh}_I(\phi) \) is obtained, if we order the \( \phi_j \) as

\[
\phi_1 < \phi_2 < \cdots < \phi_M.
\]

Note that the signs \( (\pm 1)^n \) do not give any contribution to the solution. Then the \( \tau \)-functions for the two-dimensional Toda lattice are given by

\[
\tau^{(n)}(x, t) = (\pm 1)^n N^2 2^{N(N-1)/2} \sum_{I \in \binom{[M]}{N}} e^{(n+N-1)/2} \Phi_I \Delta_I(A) \text{Sh}_I(\phi) e^{\Theta_I^+(x,t,n)},
\]

where \( \Phi_I = \sum_k \phi_{i_k} \) for the index set \( I = \{i_1, \ldots, i_N\} \). The common factor in the \( \tau \)-function does not contribute the solution (\( \tau \)-function may be considered as a homogeneous coordinate on the Grassmannian).

### 2.6 The Davey-Stewartson solitons

Here we consider the Davey-Stewartson equation of type II and defocusing case, which has the form,

\[
\begin{align*}
\dot{q}_t + \frac{1}{2}(q_{xx} - q_{yy}) + 2qQ + 4|q|^2q &= 0, \\
Q_{xx} + Q_{yy} &= -4(|q|^2)_{xx}
\end{align*}
\]

where the variables \((x, y, t)\) are expressed by \((x_{-k}, x_k = x_{-k})\) for \( k = 1, 2, \ldots \), i.e.

\[
x = i(x_1 - x_{-1}), \quad y = x_1 + x_{-1}, \quad t = -ix_2 = ix_{-2}.
\]
Also we assume $\tau^{(s+1)} = [\tau^{(s-1)}]^s$. Then the complex function $q$ and the real function $Q$ are given by

$$q = \frac{\tau^{(s+1)}}{\tau^{(s)}} e^{4it}, \quad \text{and} \quad Q = \frac{\partial^2}{\partial x^2} \ln \tau^{(s)}.$$

We take the exponential function,

$$E_j^{(s)}(x, y, t) = p_j^s \exp \left[ \frac{1}{2i} \left( p_j - \frac{1}{p_j} \right) x + \frac{1}{2} \left( p_j + \frac{1}{p_j} \right) y + \frac{1}{2i} \left( p_j^2 - \frac{1}{p_j^2} \right) t \right]$$

$$= \exp (-x \sin \psi_j + y \cos \psi_j - t \sin(2\psi_j) - i s \psi_j) =: e^{\theta_j(x,y,t) - i s \psi_j},$$

where we have chosen $p_j = e^{-i\psi_j} \in \mathbb{C}$ with $\psi_j \in \mathbb{R}$. Notice that $E_j^{(s)}$ is a complex factor $e^{-i s \psi_j}$.

Here the parameter $(p_i, q_i)$ is given by a unit circle $(p, q) = (-\sin \psi, \cos \psi)$ where $\psi$ is measured in the counterclockwise direction with the positive $y$-axis as $\psi = 0$.

![Diagram](image_url)

Figure 2.3: Parameters for the Davey-Stewartson soliton.
With those $E_j^{(s)}$, we calculate the Wronskian $E_I^{(s)} = \text{Wr}(E_{i_1}^{(s)}, \ldots, E_{i_N}^{(s)}) = P_1^{(s)} e^{\Theta_I}$

where $P_I^{(s)}$ is given by

$$P_I^{(s)} = \left( \prod_{k=1}^N p_{ik}^{s} \right) \prod_{j>k} (p_{ij} - p_{ik}) = \left( \prod_{k=1}^N e^{-is\psi_{ik}} \right) (-2i)^{\frac{N(N-1)}{2}} \prod_{j>k} e^{-\frac{i}{2}(\psi_{ij} + \psi_{ik})} \sin \frac{1}{2}(\psi_{ij} - \psi_{ik})$$

Thus, if we choose $s = -\frac{N-1}{2}$, then $E_I^{(s)}$ becomes

$$E_I^{(\frac{1-N}{2})}(x, y, t) = (-2i)^{\frac{N(N-1)}{2}} S_I(\psi) e^{\Theta_I(x, y, t)}$$

Then the $\tau$-function is given by

$$\tau^{(\frac{1-N}{2})}(x, y, t) = (-2i)^{\frac{N(N-1)}{2}} \sum_{I \in \binom{[M]}{N}} \Delta_I(A) S_I(\psi) e^{\Theta_I(x, y, t)}$$

Note again that the factor $(-2i)^{\frac{N(N-1)}{2}}$ does not give any contribution to the solution.

If we order the parameters

$$0 \leq \psi_1 < \psi_2 < \cdots < \psi_M < 2\pi,$$

then each $S_I^{(s)}$ is a positive function. Then choosing $A \in Gr(N, M)_{\geq0}$, the $\tau$-function has sign-definite, and the solutions $q(x, y, t)$ and $Q(x, y, t)$ are regular.

### 2.7 Soliton graphs

We consider several explicit choices of independent variables $(x_k; k \in \mathbb{Z})$ so that the solutions give real and regular solitons for the equations including the KP, 2D Toda lattice and the Davey-Stewartson equations. In particular, we are interested in the patterns in real two-dimensional space $\mathbb{R}^2$ generated by the soliton solutions and their dynamics with some time variable. We may denote those variables by $(x, y, t)$ (for the 2-D Toda case, just consider $t_0 = n_0$ for a fixed $n_0$).

Let us consider the KP soliton solution of $Gr(1, 2)_{>0}$ as an example.
Example 2.7.1. We compute the KP soliton solution $u_A(x, y, t)$ associated to the $1 \times 2$ matrix $A = (1 \ a)$ with $a > 0$, considered as an element of $Gr(1, 2)_{>0}$. Write $E_1 = e^{\theta_1}$ and $aE_2 = e^{\theta_2 + \ln a} = e^{\tilde{\theta}_2}$. Then the $\tau$-function $\tau_{A}$ and the soliton solution $u_A$ are given by

$$\tau_A(x, y, t) = e^{\theta_1} + e^{\tilde{\theta}_2} = 2e^{\frac{1}{2}(\theta_1 + \tilde{\theta}_2)} \cosh \frac{1}{2}(\theta_1 - \tilde{\theta}_2), \text{ and}$$

$$u_A(x, y, t) = \frac{1}{2}(\kappa_1 - \kappa_2)^2 \sech^2 \frac{1}{2}(\theta_1 - \tilde{\theta}_2).$$

This is one line-soliton solution, and the peak of the solution (wave crest) is given by the equation $\theta_1 = \tilde{\theta}_2$. When $x \to -\infty$, $\tau_{A}(x, y, t) \approx e^{\theta_1}$, then the solution $u_A(x, y, t) \approx 0$ at this region. The Figure 2.4 marks the dominant exponential $E_1$ in $x \to -\infty$ region.

For each fixed $t$, $\theta_1 = \tilde{\theta}_2$ gives a line which divides the $xy$-plane into two regions. The exponential $E_1$ dominates in the region including $x \ll 0$, and $E_2$ dominates the other region where $x \gg 0$. We label each region by its dominant exponential. Figure 2.4 depicts $u_A(x, y, t)$, where $t = (0, \ldots, 0)$, $a = 1$, and $(\kappa_1, \kappa_2) = (-1, 2)$. 

Figure 2.4: One line-soliton solution $u_A(x, y, t)$ where $A = (1, 1) \in Gr(1, 2)_{>0}$, $E_i$ represents the dominant exponential in each region.
In general, we have
\[ \theta_i(x,y,t) = p_i x + q_i y + \omega_i(t), \quad \Theta_I(x,y,t) = \sum_{i \in I} \theta_i(x,y,t). \]

Recall \( \mathcal{M}(A) = \{ I \in \binom{[M]}{N} : \Delta_I(A) \geq 0 \} \) for \( A \in \text{Gr}(N,M) \geq 0 \). Under a large scales of \((x,y,t)\), we have defined a piecewise linear function \( f_{\mathcal{M}(A)}(x,y,t) \),

\[ f_{\mathcal{M}(A)}(x,y,t) = \max \{ \Theta_I(x,y,t) : I \in \mathcal{M}(A) \}, \]

Note that \( f_{\mathcal{M}(A)}(x,y,t_0) \) represents a dominant plane \( z = \Theta_I(x,y,t_0) \) at each point \((x,y)\) for fixed \( t_0 \). Then the graph \( C_{t_0} (\mathcal{M}(A)) \) shows the pattern to identify the dominant plane at each point in \( \mathbb{R}^2 \). We may define soliton graph to describe the pattern for the dominant relation among \( \Theta_I(x,y,t) \).

**Definition 2.7.1.** The soliton graph \( C_{t_0} (\mathcal{M}(A)) \) for fixed time \( t = t_0 \) is defined by

\[ C_{t_0} (\mathcal{M}(A)) := \{ \text{the locus in } \mathbb{R}^2 \text{ where } f_{\mathcal{M}(A)}(x,y,t = t_0) \text{ is not linear} \}. \quad (2.7.1) \]

Each region of the complement of the soliton graph \( C_{t_0} \) is a domain of linearity for \( f_{\mathcal{M}(A)}(x,y) \) at time \( t_0 \), and hence each region is naturally associated to a dominant \( \Theta_I(x,y) \), \( I \in \mathcal{M}(A) \) from the \( \tau \)-function. We label this region by \( I \).

A line-soliton is a finite or unbounded line segment in a soliton graph which represents a balance between two dominant \( \Theta_I \)'s. Lemma 2.7.1 provides the method to calculate the equation for a line-soliton.

**Lemma 2.7.1.** [4, Proposition 5] Consider a line-soliton in a soliton graph. The index sets of the dominant \( \Theta_I \)'s in adjacent regions of the soliton graph in the xy-plane are of the form \( \{i,m_2,\ldots,m_N\} \) and \( \{j,m_2,\ldots,m_N\} \).

According to Lemma 2.7.1, those two \( \Theta_I \)'s have \( N - 1 \) common digits, and we call the line separating them a line-soliton of type \([i,j]\), or simply an \([i,j]\)-soliton. Locally we have the line-soliton is along the location given by \( \Theta_{i,m_2,\ldots,m_N} = \Theta_{i,m_2,\ldots,m_N} \), or

\[ \theta_i = \theta_j. \]
Remark. For a soliton graph $C_{t_0}(\mathcal{M}(A))$, we may also color a trivalent vertex black if the index sets $\{I, J, K\}$ of the three neighbor regions of the vertex satisfy $I = K_0 \setminus \{i\}, J = K_0 \setminus \{j\}$ and $L = K_0 \setminus \{l\}$ for some common $(N + 1)$-index set $K_0$; we may color it white if $\{I, J, K\}$ satisfy $I = I_0 \cup \{i\}, J = I_0 \cup \{j\}$ and $L = I_0 \cup \{l\}$ for some common $(N - 1)$-index set $I_0$.

See Figure 2.5 for an example of a KP soliton graph.

Figure 2.5: An example of the soliton graph for $Gr(3,6)_{>0}$. Each number set $ijk$ represents the dominate exponential with the index set $I = \{i, j, k\}$.

Although we have not labeled all regions or all edges, the remaining labels can be determined using Lemma 2.7.1.
2.8 Unbounded line solitons of a soliton graph

Chakravarty and Kodama [3] have a classification theorem for unbounded line solitons of KP soliton graphs of $Gr(N,M)$.

**Theorem 2.8.1.** Let $A \in Gr(N,M)_{\geq 0}$ (totally nonnegative Grassmannian). Let $\{i_1, \ldots, i_N\}$ be the pivot set of $A$, and $\{j_1, \ldots, j_{M-N}\}$ be the non-pivot set. Also assume that $\{k_1 < \cdots < k_M\}$ is generic.

Then the soliton solution $u$ has the following structure:

- for $y \gg 0$, $\exists N$ solitons of $[i_n, p_n]$-types for some $p_n > i_n$,
- for $y \ll 0$, $\exists (M-N)$ solitons of $[q_m, j_m]$-types for some $q_m < j_m$.

Moreover, the pairing map

$$
\begin{cases}
\pi(i_n) = p_n, & n = 1, \ldots, N \\
\pi(j_m) = q_m, & m = 1, \ldots, M - N
\end{cases}
$$

is a bijection, i.e. a derangement of the symmetric group $S_M$.

This theorem gives an explicit description of the unbounded regions of soliton graphs. In the next chapter, we will introduce several combinatorics tools to describe the geometric pattern in the bounded region of soliton graphs.
Figure 2.6: Unbounded line-solitons for $Gr(N, M)_{\geq 0}$. 

\[ [i_k, p_k] \quad y = \infty \quad [i_1, p_1] \]

Interaction Region

\[(i_1, i_2, \ldots, i_N) \quad x = -\infty \quad (p_2, p_3, \ldots, p_N) \quad x = \infty \]

\[ y = -\infty \quad [q_m, j_m] \]
CHAPTER 3
SOLITON SUBDIVISIONS

In this chapter, we introduce soliton subdivisions, and study the duality relation between soliton graphs and soliton subdivisions.

3.1 Point configuration \( \mathcal{A}_{N,M}^\omega \) and Duality

Recall the piece-wise linear function defined in previous sections,

\[
f_{\mathcal{M}(A)}(x, y) = \max \{ \Theta_I(x, y) = p_I x + q_I y + \omega_I : I \in \mathcal{M}(A) \},
\]

where \( p_I = \sum_{i \in I} p_i, q_I = \sum_{i \in I} q_i, \omega_I = \sum_{i \in I} \omega_i. \) We will study this plane arrangement by using its duality structure. In this chapter, we assume \( \{(p_i, q_i), i \in [M]\} \) sit counter-clockwise on a conic curve in \( \mathbb{R}^2 \). For KP solitons, we know they are on a parabola.

We define a map, called duality map, which associates a plane in \( \mathbb{R}^3 \) to a point in \( \mathbb{R}^3 \) such that

\[
\mu : (p, q, \omega) \rightarrow \{(x, y, z) : z = px + qy + \omega\}
\]

The vector \( \langle p, q, -1 \rangle \) is the normal vector of the plane, and the vector \( \langle p, q \rangle \) gives the increasing direction of the plane, i.e. \( \nabla z = \langle p, q \rangle \). Then for two planes, \( z = \theta_1(x, y) = p_1 x + q_1 y \) and \( z = \theta_2(x, y) = p_2 x + q_2 y \), the vector \( v_{[1,2]} = \langle p_1 - p_2, q_1 - q_2 \rangle \) gives the increasing direction in the difference \( \theta_1 - \theta_2 \). Notice that the intersection line of those planes is given by \( (p_1 - p_2)x + (q_1 - q_2)y = 0 \) which is perpendicular to this vector. Then the vector \( v_{[1,2]} \) gives the dominant relation between those planes.
Figure 3.1: The vector \( \langle p, q, -1 \rangle \) is the normal vector of the plane \( z = px + qy + \omega \) and the vector \( \langle p, q \rangle \) gives the increasing direction of \( z \).

In the case of \( \omega = (\omega_1, \ldots, \omega_M) = 0 \), we have \( M \) planes, \( z = \theta_k(x, y) = p_k x + q_k y \) for \( k = 1, \ldots, M \). We use the following example to demonstrate the duality structure.

**Example 3.1.1.** We consider the example when \( M = 3 \) as shown in figures 3.2, 3.3, 3.4. The three planes corresponding to the three points in the \( pq \)-plane divide the \( xy \) plane into six regions, each of which is labeled by the order relation among the planes.

In general, a triangle is called *white*, if the index sets \( \{I, J, L\} \) of the triangle satisfy \( I = I_0 \cup \{i\}, J = I_0 \cup \{j\}, L = I_0 \cup \{l\} \) for some common \((N - 1)\)-index set \( I_0 \); a triangle is called *black* if the index sets \( \{I, J, L\} \) of the triangle satisfy \( I = K_0 \setminus \{i\}, J = K_0 \setminus \{j\}, L = K_0 \setminus \{l\} \) for some common \((N + 1)\)-index set \( K_0 \).

We consider the generic choice of the parameters \((p_i, q_i)\) in the sense that the vectors \( v_{[i,j]} := \langle p_i - p_j, q_i - q_j \rangle \) for \( 1 \leq i < j \leq M \) are all distinct. Then there are \( \binom{M}{2} \) lines associated to \( \theta_i = \theta_j \) (giving the corresponding transposition), and we let \([i, j]\) denote this line. Those lines divide the \( xy \)-plane in \( 2 \times \binom{M}{2} = M(M - 1) \) regions (polyhedral fans, see 5.1.3 for the definition), each of which can be parametrized by the dominant relation among those planes. That is, each region is labeled by an
ordered index set $i_1i_2 \cdots i_M$, and this implies $	heta_{i_1}(x, y) > \theta_{i_2}(x, y) > \cdots > \theta_{i_M}(x, y)$ for any points $(x, y)$ in this region (see Figure 3.2). One should also note that the adjacent regions of the line $[i, j]$ have the following labels,

$$**\cdots**i j**\cdots** \quad \text{and} \quad **\cdots**j i**\cdots**$$

and if $i > j$, then the vector $v_{[i,j]}$ has the direction from the region labeled by $**\cdots**$ $j i \cdots$ to the one by $**\cdots**i j \cdots$. Here the *’s indicate the same ordered indices.

One can also assign a unique direction to each line $[i, j]$, so that the right side of the line along the direction corresponds to the dominant index $i$, i.e. $\theta_i > \theta_j$ at the right side of the line. Those directions play the same role as $y \gg 0$ (incoming to the origin) and $y \ll 0$ (out going from the origin) for the KP case.

Now we define a point configuration associated with $\mathcal{M}(A)$ which is dual to $f_{\mathcal{M}(A)}(x, y)$ under the duality map.
Figure 3.3: Each section corresponds to the dominance of one $\theta$ for $Gr(1, 3)$. The triangle is marked white, because the index sets $\{I, J, L\}$ of the triangle satisfy $I = I_0 \cup \{i\}, J = I_0 \cup \{j\}, L = I_0 \cup \{l\}$ for some common $(N-1)$-index set $I_0$, with $N = 1$ here.

$$A(\mathcal{M}(A)) := \{(v_I, \omega_I) \in \mathbb{R}^2 \times \mathbb{R} : I \in \mathcal{M}(A)\},$$

where $v_I = (p_I, q_I), p_I = \sum_{j=1}^{N} p_{ij}, q_I = \sum_{j=1}^{N} q_{ij}$ and $\omega_I = \sum_{j=1}^{N} \omega_{ij}$. We may denote $(p_I, q_I, \omega_I)$ as $\hat{v}_I$.

### 3.2 Soliton subdivisions

In this section, we study the point configuration defined in the previous section, and define the soliton subdivision.

We define irreducibility of the $N \times M$ matrix $A$ representing a Grassmannian point, if it satisfies

(a) for any $i \in [M]$, there exists $I \in \mathcal{M}(A)$ s.t. $i \in I$,

(b) $\bigcap_{I \in \mathcal{M}(A)} I = \emptyset$ (i.e. no common index).
Figure 3.4: Each section represents the dominance of $\theta_i + \theta_j$ for $Gr(2, 3)$. The triangle is marked black, because the index sets $\{I, J, L\}$ of the triangle satisfy $I = K_0 \setminus \{i\}, J = K_0 \setminus \{j\}, L = K_0 \setminus \{l\}$ for some common $(N + 1)$-index set $K_0$, with $N = 2$ here.

We define the genericity of the vertices $v_i$ such that

$$v_{[i,j]} := v_i - v_j \text{ is not parallel to any } v_{[k,l]} \text{ if } (i, j) \neq (k, l).$$

For the KP solitons, the genericity is equivalent to say that $\kappa_i + \kappa_j$ are different for any pair of the $\kappa$-parameters.

Under those conditions of irreducibility of the matrix $A$ and genericity of the parameters $v_i$, we have

**Proposition 3.2.1.** The convex hull of $A(M(A))$

$$\text{conv} \left\{ v_I = (p_I, q_I) \in \mathbb{R}^2 : I \in M(A) \right\} \text{ is an } M\text{-gon.}$$

**Proof.** This is a corollary of Theorem 2.8.1, under the duality map. □

In particular, if $M(A) = \binom{[M]}{N}$ (i.e. $A$ is in $Gr(N, M)_{>0}$), then the convex hull of those points forms an $M$-gon whose vertices are

$$\{ v_{I_j} : I_j = \{j, j+1, \ldots, j+N-1\}, \ j = 1, \ldots, M \text{ (cyclic order)} \}$$
Example 3.2.1. For $N = 2$ and $M = 6$, we have a hexagon with the vertices $v_{i,j} = v_i + v_j$, we have

$$\{v_{1,2}, v_{2,3}, v_{3,4}, v_{4,5}, v_{5,6}, v_{6,1}\}.$$ 

Note here that all other 9 points $v_{i,j}$ with $|i - j| > 1$ (cyclic sense) are inner points of the hexagon.

Now we consider the convex hull of the point set $A^\omega(\mathcal{M}(A))$ in $\mathbb{R}^3$, which is the polytope associated to $A^\omega(\mathcal{M}(A))$

$$P^\omega(\mathcal{M}(A)) := \text{conv} \{A^\omega(\mathcal{M}(A))\} \subset \mathbb{R}^3.$$ 

Note that $P(\mathcal{M}(A)) := P^0(\mathcal{M}(A))$ is the convex $M$-gon given in the Proposition 3.2.1 above.

We define the faces of $P^\omega(\mathcal{M}(A))$ as below.

Definition 3.2.1. A nonempty set $S \subset P^\omega(\mathcal{M}(A))$ is an upper $m$-face of $P^\omega(\mathcal{M}(A))$ if there exists a plane $\mathcal{P} := \{z = ax + by + c\}$ such that

(a) $S = P^\omega(\mathcal{M}(A)) \cap \mathcal{P}$ with dim $S = m$ and

(b) any point in the region $z > ax + by + c$ has no intersection with $P^\omega(\mathcal{M}(A))$.

Similarly, we define a nonempty set $S \subset P^\omega(\mathcal{M}(A))$ is a lower $m$-face of $P^\omega(\mathcal{M}(A))$, if there exists a plane $\mathcal{P} := \{z = ax + by + c\}$ with property (a), and the region $z < ax + by + c$ has no intersection with $P^\omega(\mathcal{M}(A))$.

Definition 3.2.2. A subdivision of a point configuration $\mathcal{A}$ is a collection of polygons, satisfying

- the union of all these polygons covers the convex hull of $\mathcal{A}$,

- all the vertices of each polygon are from $\mathcal{A}$,
each pair of polygons meets along a common face (possibly empty).

In our case, we are studying a point configuration $A_{N,M}$ in $\mathbb{R}^2$, a subdivision $Q$ of $A_{N,M}$ is a collection of polygons $\{\sigma_1, \cdots, \sigma_m\}$ with $\sigma_i \subset \binom{[M]}{N}$. Each polygon is given by $\text{conv}\{v_I : I \in \sigma_i\}$.

We first consider the point configuration for $Gr(1, M)$

$$A_{1,M} = \{v_i = (p_i, q_i) \in \mathbb{R}^2 : i = 1, \cdots, M\}$$

with the $M$ points cyclically sitting in $\mathbb{R}^2$. The convex hull $\text{conv}(A_{1,M}) = P_{1,M}^0$ is the $M$-gon with all vertices in $A_{1,M}$.

We add the weight $\omega = (\omega_1, \cdots, \omega_M)$ on each point $(p_i, q_i)$, and obtain the point configuration

$$A_{1,M}^{\omega} = \{(p_i, q_i, \omega_i) \in \mathbb{R}^2 \times \mathbb{R} : i = 1, \cdots, M\},$$

and $\text{conv}(A_{1,M}^{\omega}) = P_{1,M}^{\omega}$ is a 3-dimension polytope. We project the faces of $P_{1,M}^{\omega}$ which can be seen from top into $\mathbb{R}^2$, and obtain a subdivision $Q(A_{1,M}, \omega)$ of $P_{1,M}^0$, which actually is a triangulation of the $M$-gon $P_{1,M}^0$.

**Definition 3.2.3.** Generally, the weight $\omega_I$ lifted to $v_I$ induces a subdivision $Q(M(A), \omega)$ of $A(M(A))$ as follows:

- Obtain all upper faces in $P^{\omega}(M(A))$,
- Project these faces from $\mathbb{R}^3$ down into $\mathbb{R}^2$.

We call it a *soliton subdivision* of $A(M(A))$ for the weight $\omega = (\omega_1, \cdots, \omega_M)$. 32
3.3 Soliton graphs induced by subdivisions

It has been shown in [12], the KP soliton graphs for $t \to \pm \infty$ have only trivalent vertices or X-crossing vertices. Correspondingly, we first describe a local graph corresponding to a triangle or a parallelogram. See Figure 3.5. The trivalent vertex for the Y-shape interaction is given by

$$\theta_i(x, y, t) = \theta_j(x, y, t) = \theta_k(x, y, t),$$

where $\theta_i = p_i x + q_j y + \omega_k t$. The interaction point for X-crossing is given by

$$\theta_i(x, y) = \theta_j(x, y, t) \quad \text{and} \quad \theta_k(x, y, t) = \Theta_l(x, y, t).$$

![Figure 3.5: Two cases for a triangle or a parallelogram duality. For the first case (a), $J_a = I_0 \cup \{i\}, J_b = I_0 \cup \{j\}$ and $J_c = I_0 \cup \{k\}$ with $|I_0| = N - 1$. For the second case (b), $J_a = J_0 \cup \{i, k\}, J_b = J_0 \cup \{j, k\}, J_c = J_0 \cup \{j, l\}$ and $J_d = J_0 \cup \{j, l\}$ with $|J_0| = N - 2$.](image)

Figures 3.6 and 3.7 illustrate some examples of the duality of the soliton graphs.
Generally, it is very difficult to study the soliton graphs and soliton subdivisions with general weight vector $\omega$ for lower dimension cell of $Gr(N, M) \geq 0$, which is the main work we will extend from the current results. In this thesis, we will focus on the totally positive Grassmannian $Gr(N, M) > 0$, and classify the soliton subdivisions with general weight vector $\omega$ for $Gr(N, M) > 0$ in the next few chapters.

Before closing this chapter, we give the algorithm from the generic soliton subdivision for $Gr(N, M) \geq 0$ to the soliton graph for the given weight $\omega = (\omega_1, \cdots, \omega_M)$.

**Algorithm 3.3.1. From the generic soliton subdivision $Q$ for $Gr(N, M) > 0$ to the generic soliton graph:**

1. For each triangle $\{v_{I_1}, v_{I_2}, v_{I_3}\}$ or $\{v_{I_1i_2}, v_{I_2i_3}, v_{I_3i_3}\}$ in the subdivision, we draw a point $p_{i_1i_2i_3}$ in $(x, y)$ plane, where the point is the solution of

$$\theta_{i_1} = \theta_{i_2} = \theta_{i_3}$$
Figure 3.7: Soliton graphs of $\text{Gr}(4,8)$ with unbounded line-solitons marked by a permutation $\pi = (83276541)$. This is a regular 4-line soliton solution. Dimension of the Grassmannian cell $= 11 - 1 - 6 = 4$.

(2) For each inner edge $\{v_{i_1}, v_{i_2}\}$ between two triangles $\triangle$ and $\triangle'$ in the subdivision $Q$, we draw a line between the two points corresponding to the two triangles. This gives the soliton line $[i_1, i_2]$.

(3) For each boundary edge $\{v_{j, j+1, \ldots, j+N-1}, v_{j+1, \ldots, j+N}\}$, we have a triangle $\triangle$ in $Q$ adjacent to the edge. We shoot out an unbounded soliton line $[j, j+N]$ from the point $p$ corresponding to the triangle $\triangle$.

The points and edges constructed from this algorithm then provide the corresponding soliton graph.
CHAPTER 4

CONSTRUCTION OF SOLITON SUBDIVISIONS

In this chapter, we introduce a quadrilateral checking formula and an induction map between soliton subdivisions with different $N$ values. Then we provide an algorithm to construct the soliton subdivision for the general case of $\text{Gr}(N, M)_{>0}$.

4.1 Notations review

First, we review some notations we use in this chapter.

Recall $[M] := \{1, 2, \ldots, M\}$, and $\binom{[M]}{N}$ which is the collection of all the $N$ digit subsets of $[M]$. An $N$ digit subset $I = \{i_1, \ldots, i_N\}$ of $[M]$ can be written as $I = i_1 \cdots i_N$. An $N + 1$ digit subset $I \cup \{i_{N+1}\} \in \binom{[M]}{N + 1}$ can be denoted as $I i_{N+1}$, or $I \cup i_{N+1}$.

In this chapter, we will focus on soliton subdivisions for totally positive Grassmannian case, which means we consider the matrix $A \in \text{Gr}(N, M)_{>0}$ such that $\mathcal{M}(A) = \binom{[M]}{N}$, and we have

$$\mathcal{A}_{N,M} := \left\{ \mathbf{v}_I = (p_I, q_I) : I \in \binom{[M]}{N} \right\},$$

with $\{\mathbf{v}_1, \ldots, \mathbf{v}_M\}$ cyclically sitting on a conic curve in $\mathbb{R}^2$, and $\mathbf{v}_I = \sum_{i \in I} \mathbf{v}_i$. and we have $P_{N,M} = P^0_{N,M} := \text{conv}(\mathcal{A}_{N,M})$ the convex hull of $\mathcal{A}_{N,M}$ is an $M$-gon.

Given a weight vector $\omega = (\omega_1, \ldots, \omega_M)$, we give each point $\mathbf{v}_i$ a weight $\omega_i$, and
denote \( \hat{v}_i := (v_i, \omega_i) \). We then give a weight \( \omega_I = \sum_{k \in I} \omega_k \) to the point \( v_I \), and denote \( \hat{v}_I := (v_I, \omega_I) \). Then we consider the point configuration

\[
\mathcal{A}_{N,M}^\omega := \left\{ \hat{v}_I = (p_I, q_I, \omega_I) : I \in \binom{[M]}{N} \right\},
\]

and the convex hull of the point configuration

\[
P_{N,M}^\omega = \text{conv}(\mathcal{A}_{N,M}^\omega).
\]

The \( m \)-upper face \( S \) of \( P_{N,M}^\omega \), defined in previous chapter, means the existence of a plane \( P : z = ax + by + c \) such that the intersection of \( z \geq ax + by + c \) and \( P_{N,M}^\omega \) is \( S \). We will use the definition in the later sections to identify the upper faces of \( P_{N,M}^\omega \).

We project the upper 2-faces of \( P_{N,M}^\omega \) into \( \mathbb{R}^2 \), and obtain the collection of projected polygons in \( \mathbb{R}^2 \), which is the subdivision \( Q_{N,M}(\omega) \) induced by the weight vector \( \omega = (\omega_1, \ldots, \omega_M) \).

**Remark.** We may call a polygon in a subdivision \( Q \) a polygon tile or a tile of the subdivision \( Q \), since these polygons build up the subdivision.

For a polygon in \( Q_{N,M}(\omega) \), assume its vertices are given by the set \( \{v_{I_1}, \ldots, v_{I_k}\} \).

Each vertex \( v_I \) can be represented by its index set \( I \), and we may denote the polygon \( \text{conv}\{v_{I_1}, \ldots, v_{I_k}\} \) as its vertex set \( \{v_{I_1}, \ldots, v_{I_k}\} \), or its index set \( \{I_1, I_2, \ldots, I_k\} \) for short. For the corresponding face of the polygon in \( P_{N,M}^\omega \), we may denote it as \( \{\hat{v}_{I_1}, \hat{v}_{I_2}, \ldots, \hat{v}_{I_k}\} \) or \( \{I_1, I_2, \ldots, I_k\} \). We may denote the edge \( \text{conv}\{v_{I_i}, v_{I_{i+1}}\} \) of the polygon \( \{I_1, I_2, \ldots, I_k\} \) as \( \{v_{I_i}, v_{I_{i+1}}\} \) or \( \{I_i, I_{i+1}\} \), and denote the edge \( \text{conv}\{\hat{v}_{I_i}, \hat{v}_{I_{i+1}}\} \) as \( \{\hat{v}_{I_i}, \hat{v}_{I_{i+1}}\} \) or \( \{I_i, I_{i+1}\} \).

For a polygon tile in the subdivision \( Q_{N,M}(\omega) \), we mark it as *white* if it has the form \( \{I_{i_1}, I_{i_2}, \ldots, I_{i_k}\} \) for certain \( I \in \binom{[M]}{N-1} \); we mark it as *black* if it has the form \( \{K \setminus \{i_1\}, K \setminus \{i_2\}, \ldots, K \setminus \{i_k\}\} \), for certain \( K \in \binom{[M]}{N+1} \).
4.2 Extreme weight orders

First, we may consider a straightforward situation. Suppose we have for the $M$ vertices $(p_i, q_i), i \in [M]$, the values $p_i - p_j, q_i - q_j$, for any $i \neq j \in [M]$, are of the order 1. Then we assume that the weight vector $\omega = (\omega_1, \cdots, \omega_M)$ satisfies certain extreme order, denoted by $\omega_{i_1} \cong \omega_{i_2} \cong \omega_{i_3} \gg \omega_{i_4} \gg \cdots \gg \omega_{i_M}$ which is defined as below.

**Definition 4.2.1.** We call a weight vector $(\omega_1, \cdots, \omega_M)$ an extreme order, if inductively,

- the vertex $\hat{v}_{i_k}$ is below the planes spanned by any three vertices $\hat{v}_{i_r}, \hat{v}_{i_s}, \hat{v}_{i_t}$, for $1 \leq r < s < t < k$.

Then we can construct the subdivision $Q_{1,M}(\omega)$ for an extreme order weight.

**Proposition 4.2.1.** We can construct a triangulation $T(\omega)$ using the extreme order $\omega_{i_1} \cong \omega_{i_2} \cong \omega_{i_3} \gg \omega_{i_4} \gg \cdots \gg \omega_{i_M}$ as follows.

1. Create a triangle with vertices $\{i_1, i_2, i_3\}$
2. For $4 \leq k \leq n$, add two edges connecting $i_k$ to its clockwise and counterclockwise neighbors from $\{i_1, i_2, \ldots, i_{k-1}\}$.
3. Repeat the step (2) until adding vertex $i_M$.

Then we obtain the triangulation $T(\omega)$ which gives the subdivision $Q_{1,M}(\omega)$ corresponding to the extreme order.

**Proof.** The result is obtained from the definition of extreme order. \(\Box\)

**Example 4.2.1.** We give an example for $Q_{1,11}(\omega)$ for the extreme order

$$\omega_1 \cong \omega_5 \cong \omega_8 \gg \omega_2 \gg \omega_6 \gg \omega_9 \gg \omega_4 \gg \omega_3 \gg \omega_7 \gg \omega_{11} \gg \omega_{10}.$$
Figure 4.1: For the extreme order $\omega_1 \cong \omega_5 \cong \omega_8 \gg \omega_2 \gg \omega_6 \gg \omega_9 \gg \omega_4 \gg \omega_3 \gg \omega_7 \gg \omega_{11} \gg \omega_{10}$. We start from triangle with the index set $\{1, 5, 8\}$ in the left figure, then we add new digit 2, 6, 9. By several more middle adding, we obtain the final triangulation as in the right figure.

4.3 Quadrilateral checking

Consider $A_{1,4}^\omega$, if we have a weight $\omega_1 = 1, \omega_2 = \omega_3 = \omega_4 = 0$, we will have the edge $\{\hat{v}_1, \hat{v}_3\}$ in $P_{1,4}^\omega$. Thus we obtain the triangulation $Q_{1,4}(\omega)$ with diagonal $\{v_1, v_3\}$ or $\{1, 3\}$ for short. Actually, we only need to check a determinant formula to decide the diagonal.

**Definition 4.3.1.** The quadrilateral checking formula associated with four digits $\{i_1 < i_2 < i_3 < i_4\}$, is defined as

$$D_{i_1i_2i_3i_4} = \begin{vmatrix} 1 & p_{i_1} & q_{i_1} & w_{i_1} \\ 1 & p_{i_2} & q_{i_2} & w_{i_2} \\ 1 & p_{i_3} & q_{i_3} & w_{i_3} \\ 1 & p_{i_4} & q_{i_4} & w_{i_4} \end{vmatrix}$$

**Lemma 4.3.1.** Subdivision $Q_{1,4}(\omega)$ has the diagonal $\{i_1, i_3\}$ if the quadrilateral checking formula $D_{i_1i_2i_3i_4} < 0$; $Q_{1,4}(\omega)$ has the diagonal $\{i_2, i_4\}$ if $D_{i_1i_2i_3i_4} > 0$. 

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Proof. Recall \( v_\omega = (p_i, q_i, w_i) \). When \( D_{i_1i_2i_3i_4} > 0 \), we may consider the plane given by the three vertices \( \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_3} \)

\[
\begin{align*}
1 & \quad p_{i_1} & q_{i_1} & w_{i_1} \\
1 & \quad p_{i_2} & q_{i_2} & w_{i_2} \\
1 & \quad p_{i_3} & q_{i_3} & w_{i_3} \\
1 & \quad x & y & z
\end{align*}
= 0.
\]

Thus we know \( \hat{v}_{i_4} \) is below the plane, which implies \( \{i_1, i_2, i_3\} \) is a tile of the subdivision \( Q_{1,4}(\omega) \). Equivalently, \( \{i_1, i_3\} \) is the diagonal for the subdivision.

Similarly, when \( D_{i_1i_2i_3i_4} < 0 \), \( Q_{1,4}(\omega) \) has the diagonal \( \{i_2, i_4\} \).

Remark. We may also consider the normal direction given by \( (\hat{v}_2 - \hat{v}_1) \times (\hat{v}_3 - \hat{v}_1) \), this is the direction that \( \hat{v}_1, \hat{v}_2, \hat{v}_3 \) have the same weight, and this direction includes the normal direction of the plane containing \( \hat{v}_1, \hat{v}_2, \hat{v}_3 \). The quadrilateral formula \( D_{i_1i_2i_3i_4} \) determines the sign of \( [(\hat{v}_2 - \hat{v}_1) \times (\hat{v}_3 - \hat{v}_1)] \cdot (\hat{v}_4 - \hat{v}_1) \), and thus we can know \( \hat{v}_4 \) is above or below the plane containing \( \hat{v}_1, \hat{v}_2, \hat{v}_3 \).

Since the quadrilateral checking formula \( D_{i_1i_2i_3i_4} \) is dealing with the relative position of two diagonals in \( \mathbb{R}^3 \), we may also state that \( \{\hat{v}_{i_1i_3}, \hat{v}_{i_2i_4}\} \) is vertically above \( \{\hat{v}_{i_2i_3}, \hat{v}_{i_1i_4}\} \) when \( D_{i_1i_2i_3i_4} < 0 \). This means, when on top of intersection point of \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) and \( \{\hat{v}_{i_2}, \hat{v}_{i_4}\} \) in \( \mathbb{R}^2 \), \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is vertically above \( \{\hat{v}_{i_2}, \hat{v}_{i_4}\} \). We will state \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is vertically above \( \{\hat{v}_{i_2}, \hat{v}_{i_4}\} \) which meaning a quadrilateral condition in later sections.

The sign of quadrilateral checking formula \( D_{i_1i_2i_3i_4} \) induces the triangulation of the point configuration \( A_{1,4}^\omega \).

4.4 Soliton subdivision for \( Gr(1, M) \)

Now we give the algorithm to construct \( Q_{1,M}(\omega) \) using a uniform adding process.

Algorithm 4.4.1. Soliton Triangulation for \( Gr(1, M) \):
(1) Start from the triangle \( \{1,2,3\} \), we add another vertex \( v_4 \). We mark the original boundary edge \( \{v_1,v_3\} \) as dotted line now. If \( D_{1234} < 0 \), then we mark \( \{v_1,v_3\} \) as solid line, meaning it is a true diagonal for \( Q_{1,4}^{(\omega)} \); else if \( D_{1234} > 0 \), we mark \( \{v_2,v_4\} \) as the true diagonal for \( Q_{1,4}^{(\omega)} \).

(2) Suppose we have a triangulation of the polygon \( \{1,2,\cdots,m\} \). Then add another vertex \( v_{m+1} \).

   (i) We mark the original boundary edge \( \{v_1,v_m\} \) as dotted line now, we have a quadrilateral \( \{1,j,i,m+1\} \) containing the dotted line, for some vertex \( v_j \) between \( v_1 \) and \( v_m \).

   (ii) If \( D_{1,j,i,m+1} < 0 \), we mark \( \{v_1,v_m\} \) as true diagonal again, and finish this step; else if \( D_{1,j,i,m+1} > 0 \), we have a new true diagonal \( \{v_j,v_{m+1}\} \) which breaks the whole polygon \( \{1,2,\cdots,m+1\} \) into two parts, and we mark \( \{v_1,v_j\} \) and \( \{v_j,v_m\} \) as dotted lines.

   (iii) For polygons \( \{1,2,\cdots,j,m+1\} \) and \( \{j,j+1\cdots,m+1\} \), we repeat the process in (ii). Inductively we finish the step (2) in finite steps.

(3) We repeat the step (2), and finally obtain the soliton triangulation after adding \( v_M \).

**Remark.** We may see, it does not matter which ordering of the \( M \) vertices we add in each step. Thus generally, we may choose a suitable order from the raw information of weights. Then we add one more step in the beginning of the algorithm.

- For the generic chosen weights \( \{\omega_I\}_{I \in [M]} \), we order the \( M \) vertices first by the weights \( \omega_{i_1} \geq \omega_{i_2} \geq \omega_{i_3} > \omega_{i_4} > \cdots > \omega_{i_M} \), and use this order as the adding order of the vertices.

We use the triangulations of \( A_{1,5}^\omega \) as an example.
**Example 4.4.1.** We use an example of $A_{1,5}^\omega$ to demonstrate the algorithm below, as shown in Figure 4.2.

We start from the 4-gon $\{1,2,3,4\}$, and use the quadrilateral formula to obtain the triangulation for it. Then we add one more digit $v_5$, and we assume the vertex $v_5$ is between $v_1$ and $v_4$.
Now we prepare some lemmas to prove the algorithm.

To prove an edge \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \), with \( i_1 < i_3 \), is an upper 1-face of \( P_{1,M}^\omega \), we will use the definition to find a plane \( z = ax + by + c \). While for the opposite direction, we also give several lemmas about an edge \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \), with \( i_1 < i_3 \) not being an upper 1-face.

**Lemma 4.4.2.** If there exists one point \( v = (p_0, q_0, z_0) \) in \( P_{1,M}^\omega \) such that \( (p_0, q_0) \) is in \( \{v_{i_1}, v_{i_3}\} \) and \( v \) is above \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) at the pq-location \( (p_0, q_0) \), then the edge \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is not an upper 1-face of \( P_{1,M}^\omega \).

**Proof.** If the edge \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is an upper 1-face of \( P_{1,M}^\omega \), then there exists a plane \( z = ax + by + c \) such that the region \( z > ax + by + c \) does not include any point in \( P_{1,M}^\omega \), which also means all points above \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is not in \( P_{1,M}^\omega \). □

Consequently, we can prove the following lemma.

**Lemma 4.4.3.** The following two conditions are equivalent.

1. \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is not an upper 1-face of \( P_{1,M}^\omega \).
2. There exists \( i_2, i_4 \) such that \( i_1 < i_2 < i_3 < i_4 \), and \( D_{i_1i_2i_3i_4} > 0 \), or there exists \( i_2, i_4 \) such that \( i_2 < i_1 < i_4 < i_3 \) and \( D_{i_2i_1i_4i_3} < 0 \). Or equivalently, there exists \( i_2, i_4 \) such that \( \{\hat{v}_{i_2}, \hat{v}_{i_4}\} \) is vertically above \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \).

**Proof.** If \( \{\hat{v}_{i_1}, \hat{v}_{i_3}\} \) is not an upper 1-face, we know \( \{v_{i_1}, v_{i_3}\} \) is not a boundary edge of \( P_{1,M} \) (otherwise, it is always an upper 1-face). Thus \( \{v_{i_1}, v_{i_3}\} \) breaks the polygon \( P_{1,M} \) into two parts. Consider one side, which is also a polygon \( P_1 \), with \( \{v_{i_1}, v_{i_3}\} \) a boundary edge for it. We focus on \( P_1^\omega \), and we can find an upper 2-face of \( P_1^\omega \) with the index containing \( i_1, i_3, i_m \) for certain \( i_m \). We can tilt the plane spanned by \( \hat{v}_{i_1}, \hat{v}_{i_3}, \hat{v}_{i_m} \) a little such that the new plane \( p \) intersects \( P_1^\omega \) exactly at \( \hat{v}_{i_1}, \hat{v}_{i_3} \). On the other side, there is at least one vertex \( \hat{v}_{i_l} \) above the new plane \( p \). It is not difficult to see \( i_1, i_m, i_3, i_l \) satisfies the condition (2).
On the other direction, in both cases, we can find a point on the other edge \( \hat{v}_{i_2}, \hat{v}_{i_4} \), that is vertically above \( \hat{v}_{i_1}, \hat{v}_{i_3} \), and thus we can not find a plane \( z = ax + by + c \) such that the region of \( z \geq ax + by + c \) intersecting the polytope is exactly \( \{ \hat{v}_{i_1}, \hat{v}_{i_3} \} \). □

The lemma 4.4.3 also implies that if we have \( \{ \hat{v}_{i_1}, \hat{v}_{i_3} \} \), with \( i_1 < i_3 \), is an upper 1-face, then for any \( \{ v_{i_2}, v_{i_4} \} \) intersecting \( \{ v_{i_1}, v_{i_3} \} \) in \( \mathbb{R}^2 \), we have \( \{ \hat{v}_{i_1}, \hat{v}_{i_3} \} \) vertically above \( \{ \hat{v}_{i_2}, \hat{v}_{i_4} \} \).

If \( \{ \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_3} \} \) is an upper 2-face, then we know \( \{ \hat{v}_{i_1}, \hat{v}_{i_2} \}, \{ \hat{v}_{i_2}, \hat{v}_{i_3} \} \) and \( \{ \hat{v}_{i_1}, \hat{v}_{i_3} \} \) are upper 1-faces. We only need to tilt the plane containing \( \hat{v}_{i_1} \hat{v}_{i_2} \hat{v}_{i_3} \) a little, we can obtain a plane that only containing only one edge of the triangle, thus by definition, we have the edge is an upper 1-face. While on the other hand, we may also need to obtain an upper 2-face from its boundary edges.

**Lemma 4.4.4.** If \( \{ \hat{v}_{i_1}, \hat{v}_{i_2} \}, \{ \hat{v}_{i_2}, \hat{v}_{i_3} \} \) and \( \{ \hat{v}_{i_1}, \hat{v}_{i_3} \} \) are upper 1-faces of \( P_{1,M}^\omega \), then \( \{ \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_3} \} \) is an upper 2-face.

*Proof.* If not, we know there is a vertex \( \hat{v}_m \) above the plane spanned by \( \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_3} \). WLOG, we may assume \( i_1 < i_2 < i_3 < i_m \), then we know \( D_{i_1i_2i_3i_m} > 0 \). By lemma 4.4.3, we know it is not an upper 1-face. Contradiction. □

Thus we can see that if we obtain all the upper 1-faces, then the corresponding diagonals induce the subdivision directly.

**Lemma 4.4.5. Separate Line lemma** Assume one diagonal \( \{ v_{i_1}, v_{i_2} \} \) breaks the polygon \( P_{1,M} \) into two polygons \( P_1 \) and \( P_2 \), and the diagonal \( \{ v_{i_1}, v_{i_2} \} \) is in the subdivision \( Q_{1,M}(\omega) \). If \( L \) is a diagonal of the subdivision for \( P_{1}^\omega \), then \( L \) is also in the subdivision \( Q_{1,M}(\omega) \).

*Proof.* Suppose \( L = \{ v_{i_3}, v_{i_4} \} \), and \( L \) is not a diagonal for the subdivision \( Q_{1,M}(\omega) \). Then by lemma 4.4.3, we can find \( i_5 i_6 \) such that \( \{ \hat{v}_{i_5}, \hat{v}_{i_6} \} \) is above \( \{ \hat{v}_{i_3}, \hat{v}_{i_4} \} \), with \( i_5 \)
is in between \( \{i_3, i_4\} \) away from \( \{i_1, i_2\} \), and \( i_6 \) is in the other polygon \( P_2 \). See Figure 4.4.5 for the structure.

![Figure 4.3: For the proof of the lemma 4.4.5.](image)

Since \( \{\hat{v}_{i_1}, \hat{v}_{i_2}\} \) is an upper 1-face of \( P_{1,M}^\omega \), and \( \{v_{i_1}, v_{i_2}\} \) intersects with \( \{v_{i_5}, v_{i_6}\} \) at a point \( v \), we can see \( \{\hat{v}_{i_1}, \hat{v}_{i_2}\} \) is vertically above \( \{\hat{v}_{i_5}, \hat{v}_{i_6}\} \). Let us denote the point on \( \{\hat{v}_{i_1}, \hat{v}_{i_2}\} \) above the intersection point \( v \) as \( \hat{v} \), the connection segment between \( \hat{v}_{i_5} \) and \( \hat{v} \) thus is above \( \{\hat{v}_{i_5}, \hat{v}_{i_6}\} \) in \( P_1 \), and thus \( \{\hat{v}_{i_5}, \hat{v}_1\} \) is above \( \{\hat{v}_{i_3}, \hat{v}_{i_4}\} \) in \( P_1 \), contradiction with \( \{\hat{v}_{i_3}, \hat{v}_{i_4}\} \) being an upper 1-face of \( P_{1,M}^\omega \). \( \Box \)

Now we can give the proof of the algorithm.

**Proof.** [Proof of Algorithm 4.4.1]

Assume we have a soliton subdivision of the polygon \( P_{\{i_1 \ldots i_m\}} := \text{conv}\{v_{i_j} : j = 1, \ldots, m\} \), and we add one more vertex \( v_{im+1} \) to near the original boundary edge \( \{v_{i_1}, v_{i_m}\} \) of \( P_{\{i_1 \ldots i_m\}} \). Assume we have the tile \( \{i_1, i_l, i_m\} \) in the subdivision of \( P_{\{i_1 \ldots i_m\}} \) for some vertex \( v_l \).
If \( \hat{v}_{i_1}, \hat{v}_{i_m} \) is vertically above \( \hat{v}_{i_{m+1}}, \hat{v}_i \), then we have the vertex \( \hat{v}_{i_{m+1}} \) is below the plane containing \( \{ \hat{v}_{i_1}, \hat{v}_{i_m}, \hat{v}_i \} \), thus \( \{ \hat{v}_{i_1}, \hat{v}_{i_m}, \hat{v}_i \} \) is still an upper 2-face of the polytope \( P_{\{i_1, \ldots, i_{m+1}\}} \). If \( \{ \hat{v}_{i_{m+1}}, \hat{v}_i \} \) is vertically above \( \{ \hat{v}_{i_1}, \hat{v}_{i_m} \} \), by lemma 4.4.5, we know \( \{ \hat{v}_{i_{m+1}}, \hat{v}_i \} \) breaks the polygon \( P_{\{i_1, \ldots, i_{m+1}\}} \) into two sub-polygons \( P_1, P_2 \), and \( \hat{v}_{i_{m+1}} \hat{v}_i \) is an upper 1-face of the polytope \( P_{\{i_1, \ldots, i_{m+1}\}} \). Inductively we can consider the subdivision of the two sub-polygons \( P_1, P_2 \). 

\[ \square \]

### 4.5 Induction map \( \mathcal{H} \)

Now we introduce an \textit{induction map} \( \mathcal{H} \) from the triangulation \( Q_{1,M}(\omega) \) to a subdivision \( Q_{2,M} \), where \( Q_{2,M} \) is partially completed to be the soliton triangulation \( Q_{2,M}(\omega) \), and only white cliques (maximal white polygons in a subdivision) should be triangulated.

For the triangulation \( Q_{1,M}(\omega) \) obtained by the algorithm in previous section, we may imagine to oversize the each vertex in the polygon twice as new vertices \( 2v_1, \ldots, 2v_M \), such that the middle point of \( 2v_i \) and \( 2v_j \) is \( v_i + v_j \) in \( \mathbb{R}^2 \). Then we take the mid-point of each edge of the triangulation. One should note that the total number of vertices is \( 2M - 3 \), where \( M - 3 \) gives the number of internal vertices of the subdivision corresponding to the diagonals of the triangulation. Then for each triangle of the triangulation with \( \{i, j, N\} \) index set, one can form a triangle by connecting the mid-points of the triangle, that is, the new triangle has the vertices marked by \( \{ij, jk, ki\} \). Then the obtained subdivision \( Q_{2,M} \) has the following structure:

(a) The boundary vertices are given by \( I = \{i, i + 1\} : i = 1, \ldots, M, \text{ mod}(M) \} \).

(b) If the degree \( d_i \geq 3 \), the vertex \( v_i \) forms \( d_i \)-gon with the vertices \( \{a_k : N = 1, \ldots, d_i\} \) where each vertex \( v_{a_k} \) is connected to \( v_i \).
The $d_i$-gon corresponds to the case of $\text{Gr}(1, d_i)$. We note the relation,

$$\frac{1}{2} \sum_{i=1}^{M} (d_i - 2) = M - 3 = \{\text{the number of the internal vertices}\}$$

We demonstrate the induction map by an example with $M = 11$ in Figure 4.4.

![Figure 4.4: The induction map from a triangulation of 11-gon to a subdivision $Q_{2,11}$. The numbers in ellipse at each polygon in the right figure show the common indices for this polygon. The black triangles are of the type $\{S \setminus \{j\} : j = i_1, i_2, i_3\}$ with $S = \{i_1, i_2, i_3\}$.

We can reconsider the above process as follows. For $M = 3$ case, as shown in Figure 4.5, we have a map $\mathcal{H}$ from $P_{1,3}$ to $P_{2,3}$ by introducing a variable $\epsilon$. In the Figure 4.5 we have the left figure is $P_{13}$, and the right figure is $P_{23}$. With $\epsilon > 0$, we have a blow up of $P_{1,3}$ which is a hexagon whose vertices are $v_i + \epsilon v_{i+1}, v_i + \epsilon v_{i-1}, i = 1, 2, 3$ cyclically, in the middle figure of Figure 4.5. Then we let the parameter $\epsilon$ increase to 1, and we have $P_{1,3}$ evolves to be $P_{2,3}$.

Now we introduce a process called $\epsilon$-blow up of a generic subdivision $Q_{1,M}(\omega)$ and then we define an induction map from $Q_{1,M}(\omega)$ to a subdivision $Q_{2,M}$.
**Definition 4.5.1.** Given a generic subdivision $Q_{1,M}(\omega)$,

- for each white triangle $\{i_1, i_2, i_3\}$, we blow up it as $\{v_{i_1} + \epsilon v_{i_3}, v_{i_1} + \epsilon v_{i_2}, v_{i_2} + \epsilon v_{i_1}, v_{i_2} + \epsilon v_{i_3}, v_{i_3} + \epsilon v_{i_1}, v_{i_3} + \epsilon v_{i_2}\}$; i.e. the white triangle blows up to a hexagon whose corresponding vertices in $\mathbb{R}^3$ are coplanar, and

- for each vertex $v_i$, denote all the vertices connected to it with a diagonal in the subdivision $Q_{1,M}(\omega)$ as $\{i_1, \cdots, i_d\}$, if $d \geq 3$, we blow up the vertex $v_i$ as a polygon $v_i + \epsilon v_{i_1}, \cdots, v_i + \epsilon v_{i_d}$. Note that when $d = 2$, the blow up result gives a new edge.

We define the collection of all the blow up polygons in above process, as the $\epsilon$ blow up of a generic subdivision $Q_{1,M}(\omega)$.

**Remark.** It is not difficult to see the union of all the polygons in the $\epsilon$ blow up of $Q_{1,M}(\omega)$, for small $\epsilon \ll 1$, is an $2M$-gon.

**Notation 4.5.2.** For a vertex $v_i + \epsilon v_j$, we may use $i + \epsilon \cdot j$ for short. Later, when we consider a vertex $v_I + \epsilon v_j$ for some $I \subset [M], j \notin I$, we may use $I + \epsilon \cdot j$ for short.
**Definition 4.5.3.** When $\epsilon = 1$, we can obtain the same subdivision $Q_{2,M}(\omega)$ we have done above. We define the process (which is a well defined map) from the generic subdivision (triangulation) $Q_{1,M}(\omega)$ to the subdivision $Q_{2,M}(\omega)$ as an *induction map* $\mathcal{H}$.

Now we consider the *induction map* $\mathcal{H}$ from a generic subdivision (triangulation) $Q_{N,M}(\omega)$ to a subdivision $Q_{N+1,M} = \mathcal{H}(Q_{N,M}(\omega))$.

**Remark.** The subdivision $\mathcal{H}(Q_{N,M}(\omega))$ is the soliton subdivision $Q_{N+1,M}(\omega)/\sim_w$, meaning, after triangulating the white polygons in $\mathcal{H}(Q_{N,M}(\omega))$, we can obtain $Q_{N+1,M}(\omega)$. (see definition ?? $\sim_w$ and explanation in the next chapter for more details.)

First, we extend the $\epsilon$ blow up for $Q_{N,M}(\omega)$.

**Remark.** The essential idea is that we are trying to blow up the level from $N$ to $N+1$. For a chosen vertex $v_I$, and for each of its neighbor vertices $v_J$, we know there is only one digit different between $I$ and $J$, thus we are blowing up $v_I$ along $v_Jv_I$ direction as a new point $v_I + \epsilon v_J$, and when $\epsilon$ increases to 1, this point will become a vertex in $A_{N+1,M}$.

Recall the notation $I + \epsilon \cdot j$ meaning the vertex $v_I + \epsilon v_j$, we give the following definition of $\epsilon$-*blow up* of a generic subdivision $Q_{N,M}(\omega)$.

**Definition 4.5.4.** We define the $\epsilon$ *blow up* of a generic subdivision $Q_{N,M}(\omega)$ as a collection of blow up polygons in following process.

- For each white triangle $\{Ii_1, Ii_2, Ii_3\}$, for some $I \in \left( \left[ \frac{M}{N-1} \right] \right)$, we blow up it as $\{Ii_1 + \epsilon \cdot i_3, Ii_1 + \epsilon \cdot i_2, Ii_2 + \epsilon \cdot i_1, Ii_2 + \epsilon \cdot i_3, Ii_3 + \epsilon \cdot i_2, Ii_3 + \epsilon \cdot i_1\}$;

- For each black triangle $\{K \setminus \{i_1\}, K \setminus \{i_2\}, K \setminus \{i_3\}\}$, $K \in \left( \left[ \frac{M}{N+1} \right] \right)$, we can also write it as $\{Ii_1i_2, Ii_2i_3, Ii_1i_3\}$, with $I = K \setminus \{i_1, i_2, i_3\}$. We give a shrinking triangle $\{Ii_1i_2 + \epsilon \cdot i_3, Ii_2i_3 + \epsilon \cdot i_1, Ii_1i_3 + \epsilon \cdot i_2\}$. 

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• For each vertex $I$, denote all the vertices connected to it with an edge in the subdivision $Q_{N,M}(\omega)$ as $\{J_1, \cdots, J_d\}$, with each $J_i \setminus I$ has exactly one digit, we blow up the vertex $I$ as a polygon $I + \epsilon \cdot J_1 \setminus I, \cdots, I + \epsilon \cdot J_d \setminus I$. Note when $\{I, J_{i_1}, J_{i_2}\}$ form a black triangle, we have $J_{i_1} \setminus I = J_{i_2} \setminus I$, meaning they give the same blow up points.

• The collection of all the tiles from previous two steps forms the $\epsilon$-blow up of $Q_{N,M}(\omega)$.

Follow up the definition of $\epsilon$-blow up, we may let the $\epsilon$ value increases from a very small value to 1, and in this process, the index sets of the blow up tiles will keep the same. When $\epsilon = 1$, each black tile shrinks to a point, and each white triangle becomes a black tile in $Q_{N+1,M}$. We describe the result in the following proposition.

**Proposition 4.5.1.** We define the induction map $\mathcal{H}$, with $Q_{N+1,M} = \mathcal{H}(Q_{N,M}(\omega))$ as follows,

1. For each white triangle $v_{I_1,l_1}v_{I_1,l_2}v_{I_1,l_3}$ in $Q_{N,M}(\omega), I = i_1, \cdots, i_{N-1}$, we insert a black triangle $v_{I_1,l_1,i_1}v_{I_1,l_2,i_2}v_{I_1,l_3,i_3}$ into $Q_{N+1,M}$.

2. For each vertex $v_{i_1, \cdots, i_N}$ of the subdivision $Q_{N,M}(\omega)$, we check all the black triangles and edges attached to $v_{i_1, \cdots, i_N}$, each triangle or edge has $N + 1$ digits $i_1, \cdots, i_N, i_m$. Thus we obtain a list of $N + 1$ digit sets. if the list have three or more sets, we mark a white polygon whose vertices corresponding to these $N + 1$ digit sets; and we insert this white polygon into $Q_{N+1,M}$.

3. We insert all the sub-faces of the obtained tiles into $Q_{N+1,M}$.

Then $Q_{N+1,M}$ is a subdivision of $A_{N+1,M}$.

**Proof.** We use $\epsilon$ blow up of the generic subdivision $Q_{N,M}(\omega)$, and increases $\epsilon = 1$, then we obtain the subdivision $Q_{N+1,M}$ described in the above proposition. □
4.6 Construction algorithm of soliton subdivision

With the induction map $\mathcal{H}$, we can give the inductive algorithm to construct the subdivision $Q_{N,M}(\omega)$.

**Algorithm 4.6.1. From soliton subdivision $Q_{N,M}(\omega)$ of $Gr(N,M)$ to soliton subdivision $Q_{N+1,M}(\omega)$ of $Gr(N+1,M)$:**

1. Use the induction map $\mathcal{H}$ to obtain a subdivision $Q_{N+1,M}$ of $A_{N+1,M}$.

2. We use Algorithm 4.4.1 to triangulate each white polygon in $Q_{N+1,M}$ with more than three vertices.

3. The collection of all the tiles from above process forms the soliton subdivision $Q_{N+1,M}$.

![Figure 4.6: From a triangulation $Q_{1,M}(\omega)$ to the subdivision $Q_{2,M}(\omega)$ with the same weight vector $\omega$.](image)

The algorithm gives an inductive process to obtain all soliton subdivisions for $N = 1, \cdots, M - 1$. First, from the induction map $\mathcal{H}$, we already know the algorithm
Figure 4.7: From soliton subdivision of 11-gon for $Gr(2, 11)$ to construct the subdivision for $Gr(3, 11)$. Use the white triangles to obtain all black triangles first. Then use the algorithm for $Gr(1, M)$ triangulation for each white pieces.

gives a subdivision of $\mathcal{A}_{N+1, M}$. To prove the algorithm, we only need to prove the obtained subdivision is induced by the weight $\omega$. Or equivalently, all the triangle tiles correspond to certain upper 2 faces in $P^\omega_{N+1, M}$.

For a black triangle tile $\{I_{i_1 i_2}, I_{i_2 i_3}, I_{i_1 i_3}\}$ obtained from $\{I_{i_1}, I_{i_2}, I_{i_3}\}$ in the algorithm, with $I \in \left(\left[\frac{M}{N}\right]\right)$, we know $\{\hat{v}_{I_{i_1}}, \hat{v}_{I_{i_2}}, \hat{v}_{I_{i_3}}\}$ is an upper 2-face of $P^\omega_{N, M}$, all the other vertices $\hat{v}_J$, $J \in \left(\left[\frac{M}{N}\right]\right)$, are below the plane containing $\hat{v}_{I_{i_1}}, \hat{v}_{I_{i_2}}, \hat{v}_{I_{i_3}}$. Thus we can see all the vertices $\hat{v}_K$, $K \in \left(\left[\frac{M}{N}\right]\right)$ are below the plane containing $\hat{v}_{I_{i_1 i_2}}, \hat{v}_{I_{i_2 i_3}}, \hat{v}_{I_{i_1 i_3}}$, meaning $\hat{v}_{I_{i_1 i_2}}, \hat{v}_{I_{i_2 i_3}}, \hat{v}_{I_{i_1 i_3}}$ is an upper 2-face of $P^\omega_{N+1, M}$ Now we only need to prove the triangulated white triangles in the above algorithm correspond to the upper 2-faces of $P^\omega_{N+1, M}$.

For a white triangle tile $\{I_{i_a}, I_{i_b}, I_{i_c}\}$ obtained in the algorithm with $I \in \left(\left[\frac{M}{N}\right]\right)$, we review the process to obtain it as follows. In the algorithm, we collect all the upper 1-faces or black triangles adjacent to the vertex $\hat{v}_I$ in $P^\omega_{N, M}$, and each upper 1-faces or black triangle induces a vertex in $P^\omega_{N+1, M}$. (The black triangle have two edges, which are two upper 1-faces attached to $\hat{v}_I$, and both of them will induce
We denote all the vertices as \( \{I_1, I_2, \ldots, I_m\} \), and the convex hull of these vertices as a polytope \( P_{\omega}^{i_1, i_2, \ldots, i_m} \). The corresponding polygon \( \{I_1, I_2, \ldots, I_m\} \) is a white polygon tile in \( \mathcal{H}(Q_{N,M}(\omega)) = Q_{N+1,M} \), and we triangulate it using Algorithm 4.4.1. We obtain the white triangle tile \( \{I_a, I_b, I_c\} \) from an upper 2-face of the polytope \( P_{\omega}^{i_1, i_2, \ldots, i_m} \).

Now we prove the white triangle tile \( \{I_a, I_b, I_c\} \) is also an upper 2-face of the polytope \( P_{N+1,M}^{i_1, i_2, \ldots, i_m} \). First, we have a plane \( \mathcal{P} : z = ax + by + c \) containing \( \{\hat{v}_{I_a}, \hat{v}_{I_b}, \hat{v}_{I_c}\} \), is above all the vertices \( \{\hat{v}_{I_1}, \hat{v}_{I_2}, \ldots, \hat{v}_{I_m}\} \). Along the normal direction of the plane \( \mathcal{P} \), the weight of \( \hat{v}_{I_a}, \hat{v}_{I_b}, \hat{v}_{I_c} \) are the same and higher than \( \hat{v}_s, s \neq a, b, c \). Consider the plane \( \mathcal{P}' \) with the same normal direction containing \( \hat{v}_I \) in \( P_{N,M}^{\omega} \), and we can see it is above all the upper 1-faces of \( P_{N,M}^{\omega} \) adjacent to vertex \( \hat{v}_I \). By convexity of \( P_{N,M}^{\omega} \), \( \mathcal{P}' \) is above all other vertices \( \hat{v}_J \) for all \( J \in \binom{[M]}{N} \), and consequently, we have \( \mathcal{P} \) containing \( \{\hat{v}_{I_a}, \hat{v}_{I_b}, \hat{v}_{I_c}\} \) is above all the vertices \( \hat{v}_K \) for all \( K \in \binom{[M]}{N+1} \). This finishes the proof of the algorithm.

### 4.7 Properties of soliton subdivisions

In this section, we give several combinatorial properties of soliton subdivisions.

Below we give several definitions related to maximal weakly separated collections and plabic tilings, and quote several main results in [15]. See [15, 16] for more details.

**Definition 4.7.1.** For two \( N \) element subsets \( I \) and \( J \) of \( [M] \), we say \( I \) and \( J \) are weakly separated if \( I \setminus J \) and \( J \setminus I \) are separated by a chord in a cyclical order. We call \( \mathcal{F} \) a weakly separated collection inside \( \binom{[M]}{N} \), if any two elements of \( \mathcal{F} \) are weakly separated. When it is maximal for the inclusion poset, we call it a maximal weakly separated collection.

**Proposition 4.7.1** (Proposition 11.2 in [15]). If we have a maximal weakly separated
collection \( \{ J_i, \ i = 1, \cdots, m \} \subset \binom{[M]}{N} \), identify each \( J_i \) with the vertex \( v_{J_i} \). We define \( W_{N,M} \), a collection of polygons as follows.

- For each \( N + 1 \) digit set \( K \subset [M] \), we find all elements \( \sigma = \{ J_{ik} : J_{ik} \subset K \} \) in \( \{ J_i, \ i = 1, \cdots, m \} \). If the elements in \( \sigma \) is more that 2, we insert a black polygon \( \sigma \) into \( W_{N,M} \).

- For each \( N - 1 \) digit set \( I \subset [M] \), we find all elements \( \sigma = \{ J_{ik} : I \subset J_{ik} \} \) in \( \{ J_i, \ i = 1, \cdots, m \} \). If the elements in \( \sigma \) is more than 2, we insert a white polygon \( \sigma \) into \( W_{N,M} \).

With the collection of all the polygons from the above process, \( W_{N,M} \) forms a subdivision of \( A_{N,M} \), called a plabic tiling associated with the maximal weakly separated collection \( \{ J_i, \ i = 1, \cdots, m \} \subset \binom{[M]}{N} \).

The proposition gives a general process to construct a subdivision from a pure combinatorial object, namely weakly separated collection, a collection of index sets satisfying certain combinatorial property. It is not so difficult to see that a generic soliton subdivision satisfies the rules of the black polygon tiles and white polygon tiles in the construction process of the plabic tiling. Thus we have a generic soliton subdivision, ignoring the triangulation of its black polygon regions and the triangulation of white polygon regions, is a plabic tiling.

**Proposition 4.7.2.** A generic soliton subdivision \( Q_{N,M}(\omega) \) has

1. \( N(M - N) - M + N \) black triangles,
2. \( N(M - N) - N \) white triangles,
3. \( N(M - N) + 1 \) points,
4. All points form a maximal weakly separated collection. (Theorem 1.3 in [15])

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Proof. We use the construction process to give an inductive proof of the first three counting properties; in the proof, we add one more condition

(5) \( Q_{N,M}(\omega) \) has \( N(M - N) + 1 + 2k(M - N) - M - 1 \) edges, that is \( 3k(M - N) - M \) edges.

When \( N = 1 \), the subdivision \( Q_{N,M} \) is a triangulation of \( M \)-gon, thus it satisfies all three conditions.

Now we assume all three conditions hold for \( Gr(N, M) \) case, for some \( N, 1 \leq N \leq M - 2 \). Now we consider \( Gr(N + 1, M) \). Recall that the construction process gives the partial subdivision of \( Q_{N+1,M} \) from the \( Q_{N,M} \) with the same weight, and then make the triangulation for the white remaining regions in \( Q_{N+1,M} \).

The black triangles of \( Q_{N+1,M} \) are exactly from the white triangles of \( Q_{N,M} \), thus the amount is \( N(M - N) - N = (N + 1)(M - N - 1) - M + N + 1 \).

The white triangles are related to the degree of each vertex in \( Q_{N,M} \). Here the degree of a vertex \( I \) means how many vertices in \( Q_{N+1,M} \) will contain the index set \( I \). Thus we know these points sharing index set \( I \) are obtained from edges in \( Q_{N,M} \) attached to the point \( p \) indexed by \( I \). The black triangles will eliminate the degree by 1 from the edge number attached to the point \( p \). Totally each black triangle will eliminate three degree for three points. If the degree of a point \( p \) is \( l \), with \( l \geq 3 \), it will induces a white \( l \)-gon in \( Q_{N+1,M} \). As we can see no point will have degree smaller than 2, (otherwise it only have black triangles around it, which is impossible by definition of black triangle.) Thus a vertex of degree \( l \) will induces \( l - 2 \) white triangles in \( Q_{N+1,M} \). Thus the number of white triangles is twice of edge number minus triple of black triangle number minus twice of vertex number in \( Q_{N,M} \), which is \( 2(3N(M - N) - M) - 3(N(M - N) - M + N) - 2(N(M - N) + 1) \), equal to \((N + 1)(M - N - 1) - N - 1 \).

The point number of \( Q_{N+1,M} \) are the number of edges in \( Q_{N,M} \) minus twice of
black triangles, since the three edges of each black triangle contribute the same point in $Q_{N+1,M}$. Thus we obtain

$$3N(M - N) - M - 2(N(M - N) - M + N) = (N + 1)(M - N + 1) + 1$$

With the above three numbers, we can obtain the number of edges in condition (5). □

We extend the above proposition to state the following conjecture for the subdivision of $\mathcal{A}(\mathcal{M}(A))$ for some other lower dimensional Grassmannian cells in $Gr(N, M)_{\geq 0}$.

**Conjecture 4.7.3 (General case).** If the Grassmannian point $A$ is coming from a cell with dimension $D$, then a generic soliton subdivision $Q(\mathcal{M}(A), \omega)$ consist of

(a) $D - M + N$ number of black triangles,

(b) $D - N$ number of white triangles, and

(c) $D + 1 - p$ number of vertices where $p$ is the number of parallelograms.

We may consider the examples in Figures 3.5 and 3.6 in the previous chapter. We can count the numbers of black, white triangles, and the parallelograms in the figures, and the numbers satisfy the conditions in the conjecture.
CHAPTER 5
POLYHEDRAL FAN IN THE MULTI-TIME SPACE

In this chapter, we discuss a polyhedral structure in the multi-time space for the soliton subdivisions for $Gr(N, M)_{>0}$, and prove the main theorem in this chapter that identifies a polyhedral cone for each generic soliton subdivision.

5.1 The KP hierarchy and the multi-time space

The KP hierarchy can be expressed by the set of infinite equations

$$\left[D_1D_k - 2h_{k+1}(\bar{D})\right] \tau \circ \tau = 0, \quad \text{for} \quad k = 1, 2, \ldots,$$

where $D_k = D_{t_k}$, $\bar{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots)$ and $h_k(x)$ with $x = (x_1, x_2, \ldots)$ is the completely homogeneous symmetric function generated by

$$\exp \left( \sum_{j=1}^{\infty} x_j \lambda^j \right) = \sum_{k=0}^{\infty} h_k(x) \lambda^k.$$

Here $x_1 = x$, $x_2 = y$, $x_l = t_l$ for $l \geq 3$, where $\{t_l, \ l \geq 3\}$ represent the symmetries of the KP equation (see e.g. [14, 5]). Note that the first two equations for $k = 1, k = 2$ are trivial, and the equation for $k = 3$ gives the KP equation,

$$(-4D_1D_3 + D_1^4 + 3D_2^2) \tau \circ \tau = 0.$$

The KP hierarchy has the same soliton solutions as the KP equation, given by $\tau$ function in the Wronskian form

$$\tau_A(x, y, t) = \text{Wr}(f_1, f_2, \ldots, f_k),$$

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where the scalar functions \( \{ f_j : j = 1, \ldots, k \} \) satisfy the linear differential equations

\[
\frac{\partial^m f}{\partial x^m} = \frac{\partial f}{\partial x_m} \quad \text{for all } m \in \mathbb{N}.
\]

We denote \( t = (t_3, t_4, \cdots) \) with \( t_n = x_n \), and consider a finite dimensional solution space of the scalar functions \( f \)'s. Recall the calculation for KP solitons in chapter 2, we have the \( \tau \) function

\[
\tau(x, y, t) = \sum_{I\in\left[\begin{bmatrix} M \\ N \end{bmatrix}\right]} \Delta_I(A) K_I e^{\Theta_I(x,y,t)},
\]

where \( \theta_i = \kappa_i x + \kappa_i^2 y + \kappa_i^3 t_3 + \kappa_i^4 t_4 + \cdots \), and \( \Theta_I = \sum_{i\in I} \theta_i \).

Then we can show that

\[
u_A(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \ln \tau_A(x, y, t),
\]

gives the soliton solution of the KP hierarchy. In this chapter we consider \( A \) from the totally positive Grassmannian, meaning \( \Delta_I(A) > 0 \) for all \( I \in \left[\begin{bmatrix} M \\ N \end{bmatrix}\right] \).

**Remark.** We have \( \theta_i = \sum_{k=0}^{M-1} \kappa_i^k t_k \), with \( t_0 = C, x = t_1, y = t_2 \) for some constant \( C \); the transform from \( \{t_0, t_1, \cdots, t_{M-1}\} \) to \( \theta_1, \cdots, \theta_M \) is the Vandermonde matrix \( V = (\kappa_j^{i-1})_{M \times M} \), which is of full rank. Thus we only need to consider the multi-time up to \( t_{M-1} \), and we denote

\[
t = (t_3, t_4, \cdots, t_{M-1}).
\]

The piecewise linear function \( f_{N,M}(x, y, t) \) for the KP hierarchy is given by

\[
f_{N,M}(x, y, t) = \max_{I \in \left[\begin{bmatrix} M \\ N \end{bmatrix}\right]} \left\{ \Theta_I(x, y, t) = \sum_{i\in I} \Theta_i(x, y, t) \right\}
\]

\[
\Theta_i(x, y, t) = p_i x + q_i y + \omega_i(t) \quad \text{with} \quad q_i = p_i^2 = \kappa_i^2,
\]

where \( \omega_i(t) \) with \( t = (t_3, \ldots, t_{M-1}) \) are given by

\[
\omega_i(t) = \sum_{k=3}^{M-1} \kappa_i^k t_k.
\]
Dual to the plane arrangement, we may consider the point configuration

\[ A_{N,M}^\omega = \left\{ \hat{v}_I \in \mathbb{R}^3 : I \in \begin{pmatrix} [M] \\ N \end{pmatrix} \right\}, \]

with \( \hat{v}_I = (\kappa_I, \kappa_I^{(2)}, \omega_I(t)) \) and \( \kappa_I^{(2)} = \sum_{i \in I} \kappa_i^2. \)

We explain the following notations and definitions we use in the remaining sections.

If \( \{v_j : j \in J\} \) are all the vertices on a tile of \( Q_{1,M}(\omega) \) for some \( J \subset [M] \), we may denote the tile by \( \{j : j \in J\} \). If \( \{\hat{v}_j : j \in J\} \) is a \(|J|\)-face of \( P^\omega \), we may denote the face by \( \{\hat{j} : j \in J\} \). With \( B = \{b_1, \cdots, b_M\} \subset \mathbb{R}^{M-3} \), we denote \( B_J = \{b_j : j \in J\} \). We define the affine-spanned cone \( \text{cone}(B_J) \) of \( B_J \), and the relative interior of the cone \( \text{relint}(\text{cone}(B_J)) \) as below.

**Definition 5.1.1.** A *polyhedral cone* or cone (for short) \( K \subset \mathbb{R}^m \) is a subset of \( \mathbb{R}^m \) such that

1. for \( u_1, u_2 \in K \), \( u_1 + u_2 \in K \);
2. for \( u \in K \) and \( a \geq 0 \), \( a \, u \in K \).

**Definition 5.1.2.** We define \( \text{cone}(R) \) for \( R \subset \mathbb{R}^m \) is a subset of \( \mathbb{R}^m \) containing all elements of the form

- \( a \, u_1 + b \, u_2 \), for any \( a \geq 0, b \geq 0, u_1, u_2 \in R \).

We call \( \text{cone}(R) \) the affine span cone of \( R \).

If the dimension of the linear space spanned by \( \text{cone}(R) \) is \( k \), we define \( \text{relint}(\text{cone}(R)) \) is the collection of all points \( p \) in \( \text{cone}(R) \) satisfying

- there exists a small ball \( B_p \) of dimension \( k \) centered at \( p \) such that \( B_p \subset \text{cone}(R) \).

We call \( \text{relint}(\text{cone}(R)) \) the relative interior of \( \text{cone}(R) \).
Note that a cone in $\mathbb{R}^m$ may have smaller dimensions than $m$, and when we consider the relative interior, we consider the interior in the topology of subspace spanned by $\text{cone}(R)$ in $\mathbb{R}^m$.

**Example 5.1.1.** Consider $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ in $\mathbb{R}^3$. We have $\text{cone}\{e_1, e_2\}$ is the region $\{(x, y, 0) : x \geq 0, y \geq 0\}$, and $\text{relint}(\text{cone}\{e_1, e_2\})$ is the region $\{(x, y, 0) : x > 0, y > 0\}$.

**Definition 5.1.3.** A **polyhedral fan** is a finite collection of strictly convex polyhedral cones, such that the intersection of any two cones of the fan is a face of each of them and each face of each cone is also a cone of the fan.

**Remark.** A complete polyhedral fan means the union of all the cones in the fan fills up the whole space which contains all the cones. In this thesis, when we state polyhedral fan, we actually mean the complete polyhedral fan.

In this chapter, we will focus on the subdivision of the point configuration $\mathcal{A}_{N,M}^{\omega}$ with $\omega_i = \sum_{m=3}^{M-1} \kappa_i^m t_m$. We study will a polyhedral fan structure in $t$ space, with each cone labeled by a generic soliton subdivision.

### 5.2 Gale transform

First we quote some notations and results from [17] about the Gale transform. These notations and results are commonly stated in combinatorics books.

Consider $M$ points $\{\mathbf{v}_1, \cdots, \mathbf{v}_M\}$ in $\mathbb{R}^{d-1}$ whose affine hull has dimension $d-1$, or the rank of the vectors $(1, \mathbf{v}_1), \cdots, (1, \mathbf{v}_M)$ is $d$. In our case, we have these $M$ points $\{(\kappa_i, \kappa_i^2)\}_{i \in [M]}$ in $\mathbb{R}^2$, so let us directly fix $d = 3$ for this section. We put the $M$ column vectors $(1, \mathbf{v}_i)^t$ together as a matrix $A$, 

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\[
A = \begin{pmatrix}
  1 & 1 & \cdots & 1 \\
  \kappa_1 & \kappa_2 & \cdots & \kappa_M \\
  \kappa_1^2 & \kappa_2^2 & \cdots & \kappa_M^2
\end{pmatrix},
\]

and consider the kernel of \( A \)

\[
\ker_{\mathbb{R}}(A) := \{ u \in \mathbb{R}^M : A u = 0 \}.
\]

Let \( u_1, \ldots, u_{M-3} \in \mathbb{R}^M \) be a basis for the vector space \( \ker_{\mathbb{R}}(A) \). We organize these vectors as the columns of an \( M \times (M - 3) \) matrix

\[
B := (u_1 u_2 \cdots u_{M-3}),
\]

which has \( M \) rows.

**Definition 5.2.1.** The \( M \) ordered rows of \( B, B = \{ b_1, \cdots, b_M \} \subset \mathbb{R}^{M-3}, \) is called a *Gale transform* of \( \{ v_1, \cdots, v_M \} \). The associated *Gale diagram* of \( \{ v_1, \cdots, v_M \} \) is the vector configuration \( B \) drawn in \( \mathbb{R}^{M-3} \).

Let \( \omega = (\omega_1, \cdots, \omega_M) \in \mathbb{R}^M \) be a weight vector, and consider the "lifted" point configuration

\[
\mathcal{A}_{1,M}^{\omega} = \{ \hat{v}_m = (v_m, \omega_m) : m = 1, 2, \cdots, M \} \subset \mathbb{R}^3,
\]

and its convex hull \( P_{1,M}^{\omega} \).

Recall the process to obtain the subdivision induced by the weight vector \( \omega = (\omega_1, \cdots, \omega_M) \):

- Obtain all the upper 2-faces of \( P_{1,M}^{\omega} \);
- Project these faces from \( \mathbb{R}^3 \) down into \( \mathbb{R}^{3-1} \).

The collection of all the polygons in \( \mathbb{R}^2 \) from the above process, forms the *regular subdivision* \( Q(\omega) \) of \( \mathcal{A} \) with respect to \( \omega \).
Remark. In $A_{1,M}$ case, regular subdivision is equivalent to soliton subdivision; but in general $A_{N,M}$ case, for $N > 1$, they are not equivalent. Soliton subdivision has a restriction on the choice of $\omega_I = \sum_{i \in I} \omega_i$ for the given weight vector $\omega = (\omega_1, \cdots, \omega_M)$. We say that the soliton subdivision $Q_{N,M}(\omega)$ is realizable, if there exists $Q_{N,M}(\omega)$ for given $\omega$.

The Gale diagram $\mathcal{B}$ is a very powerful tool to read off the tiles in the subdivision $Q(\omega)$, and the face lattice of the polytope $P^\omega = \text{conv}(\{\hat{v}_1, \cdots, \hat{v}_M\})$. The following theorem gives the method to check the regularity of a subdivision using the Gale transform.

**Theorem 5.2.1** ([8]). Let $Q = \{J_1, \cdots, J_m\}, J_i \subset [M]$ for $i = 1, \cdots, m$, be a subdivision of $\mathcal{A}$ and $\mathcal{B}$ be a Gale transform of $\mathcal{A}$. Then $Q$ is regular if and only if

$$\bigcap_{i=1}^m \text{relint}(\text{cone}(\mathcal{B}_{\bar{J}_i})) \neq \emptyset,$$

where $\bar{J}_i = [M] \setminus J_i$.

### 5.3 Polyhedral fan for soliton subdivisions of $A_{1,M}$

In this section, we will study the KP soliton subdivision of $Gr(1, M)$ using the Gale transform introduced in the previous section. Recall that in KP soliton subdivision, the weight vector $\omega = (\omega_1, \cdots, \omega_M)$, is induced by a time $t$ variable with the form $\omega_i = \sum_{m=3}^{M-1} \kappa^m_i t_m$.

First, let us consider a non-generic subdivision of $A_{1,M}$ induced from a special weight vector $\omega = (-1, 0, 0, \cdots, 0)$. With this weight vector, we know $\hat{v}_2, \hat{v}_3, \cdots, \hat{v}_M$ have the same height, while $\hat{v}_1$ is below the plane spanned by $\{\hat{v}_2, \hat{v}_3, \cdots, \hat{v}_M\}$. Thus we have a special non-generic subdivision with only one diagonal $\{v_2, v_M\}$. There exists a time variable $r_1^-$ in multi-time $t$-space $\mathbb{R}^{M-3}$, such that the weight vector
induced by \( r_i^- \) gives a non-generic subdivision. The negative \(-\) sign in the notation \( r_i^- \) means the \(-1\) in the weight vector \((-1,0,0,\cdots,0)\). Similarly, we can define \( r_i^+ \), \( i = 1,\cdots,M \). On the other hand, we define \( r_i^- = -r_i^- \), and we will explain more details about \( r_i^+ \) in next few sections later. We call these \( r_i^+ = \pm r_i \) the \textit{main rays} in \( t \)-space for non-generic subdivisions. See Figure 5.1 for the main rays in \((t_3, t_4)\) space for soliton subdivisions of \( A_{1,5} \).

Before solving \( r_i \), we state a proposition

\textbf{Lemma 5.3.1.} \textit{The weight vector} \( \omega = (\omega_1, \cdots, \omega_M) \) \textit{and the weight vector} \( \omega' = (c + \kappa_1 x_0 + \kappa_1^2 y_0 + \omega_1, \cdots, c + \kappa_M x_0 + \kappa_M^2 y_0 + \omega_M) \), \textit{for any} \( (c, x_0, y_0) \), \textit{give the same subdivision} \( Q_{N,M}(\omega) \).

\textit{Proof.} Consider the weight vector given by \( \omega_{xy} = (c + \kappa_1 x_0 + \kappa_1^2 y_0, \cdots, c + \kappa_M x_0 + \kappa_M^2 y_0) \). We can see all vertices under this weight \( \omega_{xy} \) are co-planer. \( \Box \)

\textbf{Remark.} With a set of weights \((\Omega_1, \cdots, \Omega_M) \in \mathbb{R}^M\), one can identify a unique point \((t_0, x, y, t)\) where \( t_0 + xt_1 + yt_2 + \omega_i(t) = \Omega_i \) for \( i = 1, \cdots, M \). With different choice \((t_0, x, y)\), \( \omega = (\omega_1, \cdots, \omega_M) \) and \( \omega' = (c + \kappa_1 x_0 + \kappa_1^2 y_0 + \omega_1, \cdots, c + \kappa_M x_0 + \kappa_M^2 y_0 + \omega_M) \) induce the same subdivision. Thus for any weight vector \( \omega \), we can find an unique \((t_0, x, y, t)\), such that \( t \) induce the same subdivision as \( \omega \).
To solve the $\mathbf{r}_1^-$, we can find a vector $\hat{\mathbf{t}}_0 = (c, x_0, y_0, t_0)$ such that

$$\hat{\mathbf{t}}_0 \cdot V = (-1, 0, 0, \ldots, 0)^t,$$

where $V = (\kappa_j^i)^{n \times n}$ is the Vandermonde matrix. By Lemma 5.3.1, we can ignore the $(c, x_0, y_0)$ part of $\hat{\mathbf{t}}_0$, and obtain $\mathbf{r}_1^-$. Thus we are trying to solve a matrix $T_{n \times n}$ such that

$$M \cdot V = \text{Id}.$$

to solve all the $\mathbf{r}_i^-$, $i = 1, 2, \cdots, M$.

Now we can relate the above process with Gale transform. We have the point configuration, $\mathcal{A}_{1,M} = \{(\kappa_i, \kappa_i^2) : i \in [M]\}$. The first three rows of the Vandermonde matrix $V = (\kappa_j^i)^{n \times n}$ is the $A$ matrix for the Gale process of $\mathcal{A}_{1,M}$. Here we introduce one Gale transform of $\mathcal{A}_{1,M}$ relative to $\{\mathbf{r}_i, i \in [M]\}$ as follows.

We obtain the inverse of $V$, by directly solving $T \cdot V = \text{Id}$. Set $T = V^{-1} = (b_{ij})_{n \times n}$, and we have

$$\sum_{k=1}^{n} b_{ik} \kappa_j^k = \delta_{ij}.$$

$\sum_{k=1}^{n} b_{ik} \kappa_j^k$ is a degree $n-1$ polynomial of $\kappa$ and we have $n$ points on the polynomial. Thus by Lagrange interpolation, we can solve

$$b_{ij} = \phi(i) (-1)^{n-j} \epsilon_{n-j-1}([n] \setminus \{i\}).$$

where $\phi(i) = 1/\prod_{j \neq i} (\kappa_j - \kappa_i)$.

Now let us write $T = V^{-1}$ as $n$ rows $\tilde{\mathbf{r}}_i$, and ignore the first three coordinates of each row, we obtain $n$ rows $\mathbf{r}_i$, which have length $n-3$. We have $V \cdot V^{-1} = I$, $A$ is the first three rows of $V$, and $B$ is the part of $V^{-1}$ ignoring the first three columns. Thus we have $A \cdot B = \mathbf{0}$, and the $n$ rows of $B$, $\{\mathbf{r}_i, i \in [n]\}$, is one Gale transform of $\mathcal{A}_{1,M}$. 64
Since we are considering the higher faces of the lifted polytope as the soliton subdivisions, and the original coherent subdivisions is using lower faces, we may use \( B = \{ r_i^- = -r_i : i \in [M] \} \) which is another Gale transform of \( A_{1,M} \), and we will focus on this Gale transform.

We already know the meaning of these \( M \) special rays in the multi-time space. By plugging into \( \omega_i = \sum_{m=3}^{M-1} \kappa_i^m t_m, -r_i \) gives a weight vector \( \omega = (0, \cdots, 0, -1, 0, \cdots, 0) \) with \(-1\) at position \( i \) and \( 0 \) at other positions. By Lemma 5.3.1, the weight \( (\omega_i = \sum_{m=3}^{M-1} \kappa_i^m t_m)_{i \in [M]} \) given by \( r_i \) will induce the same subdivision. \( r_i \) means the vertices \( \{ \hat{v}_j, j = 1, \cdots, i - 1, i + 1, \cdots, M \} \) are co-planar in \( \mathbb{R}^3 \). We can consider \( r_i \) from the perspective that all the quadrilateral checking formulas of \( i_1, i_2, i_3, i_4 \) for \( \{ \hat{v}_j, j = 1, \cdots, i - 1, i + 1, \cdots, M \} \) are 0. Recall the quadrilateral checking formula associated with four digits \( \{ i_1 < i_2 < i_3 < i_4 \} \) introduced in the previous chapter,

\[
D_{i_1i_2i_3i_4} = \begin{vmatrix} 1 & p_{i_1} & q_{i_1} & w_{i_1} \\ 1 & p_{i_2} & q_{i_2} & w_{i_2} \\ 1 & p_{i_3} & q_{i_3} & w_{i_3} \\ 1 & p_{i_4} & q_{i_4} & w_{i_4} \end{vmatrix}
\]

The sign of this formula determines the triangulation of a 4-gon marked by \( \{ i_1, i_2, i_3, i_4 \} \).

Now use the coordinates \( p_i = \kappa_i, q_i = \kappa_i^2, \omega_i = \sum_{m=3}^{M-1} \kappa_i^m t_m \), the quadrilateral formula is
where \( h_k(i_1, i_2, i_3, i_4) := h_k(\kappa_{i_1}, \kappa_{i_2}, \kappa_{i_3}, \kappa_{i_4}) \) is the completely homogenous symmetric polynomials defined by

\[
\begin{align*}
\frac{1}{M-1} & \sum_{m=3}^{M-1} h_{m-3}(i_1, i_2, i_3, i_4) t_m,
\end{align*}
\]

We then find that \( r_i = (t_3, \ldots, t_{M-1})^T \) satisfies

\[
\sum_{k=3}^{M-1} h_{k-3}(i_1, i_2, i_3, i_4) t_k = 0.
\]

Now we apply the Theorem 5.2.1 to the example for \( N = 1, M = 5 \) case as below. We denote the secondary fan for the choice of \( B = \{ r_i^+ \} \) as \( \mathcal{F}(A_{1, M}) \).

**Example 5.3.1.** For \( N = 1, M = 5 \), we have the polyhedral fan given in Figure 5.2. In this example, we use the parameters \( \kappa = (-2, -1, 0, 1, 2) \).

**Proposition 5.3.2.** A white tile \( J \) shows up in the subdivision \( Q_{1, M}(\omega(t)) \) if and only if the time variable \( t \) belongs to the relative interior \( \text{relint} (\text{cone}(B_{\bar{J}})) \), where \( \bar{J} = [M] \setminus J \).
Figure 5.2: Left figure shows a process to obtain the cone corresponding to the generic subdivision with diagonals 13 and 35. The three tiles \{123, 135, 345\} give an intersection of cones cone(\{4, 5\}), cone(\{2, 4\}), cone(\{1, 2\}). Range over all \( t \), we obtain the polyhedral fan \( F_{1,5} \) in the right figure, and we can obtain the generic subdivision in each cone by using the subdivisions corresponding to the boundary ray of the cone.

5.4 Polyhedral fan of soliton subdivisions for \( Gr(2, M) \)

Now we consider the subdivision of the point configuration

\[
\mathcal{A}^\omega_{2,M} = \left\{ \hat{v}_I = (p_I, q_I, \omega_I) : I \in \binom{[M]}{2} \right\},
\]

with \( \omega_i = \sum_{m=3}^{M-1} \kappa_i^m t_m \).

**Remark.** Recall the induction map \( \mathcal{H} \) from soliton triangulation \( Q_{1,M}(\omega) \) to a subdivision \( \mathcal{H}(Q_{1,M}(\omega)) \). We can naturally extend the induction map \( \mathcal{H} \) to any non-generic subdivision \( Q_{1,M}(\omega) \), which uses the same process.

After obtaining \( \mathcal{H}(Q_{1,M}(\omega)) \), we again use the weight \( \omega \) to triangulate the white
cliques in $\mathcal{H}(Q_{1,M}(\omega))$ to obtain $Q_{2,M}(\omega)$. We want to ignore the second step temporarily and focus on $\mathcal{H}(Q_{1,M}(\omega))$ first. For this reason we define an equivalence class as follows.

**Definition 5.4.1.** We say two subdivisions $Q$ and $Q'$ of $A_{2,M}$ are **white equivalent**, if $Q$ and $Q'$ are the same after ignoring the triangulation of the white cliques.

**Remark.** Similarly, we can define **black equivalent**. We denote white equivalence relation as $\sim_w$, and black equivalence relation as $\sim_b$; and we denote an equivalence relation under both of the two equivalence relations as $\sim_{wb}$.

With the equivalence relation, we can mark the subdivision in a same equivalence class as $Q_{2,M}(\omega)/\sim_w$. Then we know the induction map $\mathcal{H}$ will map a generic subdivision $Q_{1,M}(\omega)$ into an equivalence class $Q_{2,M}(\omega)/\sim_w$ with the same weight $\omega$. In the $t$-space, each full dimension cone corresponds to a generic subdivision $Q_{1,M}(\omega(t))$, and it exactly corresponds to the equivalence class $Q_{2,M}(\omega(t))/\sim_w$. Thus we have the exact same polyhedral fan $\mathcal{F}(A_{1,M})$ for all the equivalence class $Q_{2,M}(\omega(t))/\sim_w$. Note that for a subdivision $Q_{2,M}(\omega(t))$, a black clique can only be a triangle of the form $\{i_1i_2, i_2i_3, i_1i_3\}$. Thus the equivalence class $Q_{2,M}(\omega(t))/\sim_w$ is the same as the equivalence class $Q_{2,M}(\omega(t))/\sim_{wb}$.

**Remark.** By ignoring the triangulation of the white cliques in $Q_{2,M}(\omega)$, we obtain a subdivision $Q$ of $A_{2,M}$, and we use this $Q$ to represent the equivalence class $Q_{2,M}(\omega)/\sim_w$, meaning the white cliques can have any triangulation depending on different choices of the weight $\omega$. Similarly, when we mention $Q_{2,M}(\omega)/\sim_{wb}$, we mean the subdivision ignoring the triangulation of the white and black cliques in $Q_{2,M}(\omega)$.

We will denote the polyhedral fan for all the equivalence class of $Q_{2,M}(\omega(t))/\sim_{wb}$ as $\mathcal{F}_{2,M}$, which is already given by $\mathcal{F}(A_{1,M})$ as stated in the above explanation.
Figure 5.3: We obtain the polyhedral fan \( F_{2,M} \) from \( F(A_{1,M}) \). Note that the polyhedral fan \( F_{1,M} \) is the entire \( t_3t_4 \)-plane (left figure), and \( F_{2,M} \) consists of five cones as shown in the right figure. Apply the induction map \( H \) to the cone(\( \{2, 4\} \)), we can obtain a subdivision \( Q_{2,M}/\sim_{wb} \) by inner vertices 13, 35 as shown in the right figure.

5.5 Polyhedral fan of soliton subdivisions for \( Gr(N, M) \)

We start from the example for \( M = 5 \) case, from \( Q_{2,5}(\omega)/\sim_{wb} \) to \( Q_{2,5}(\omega) \).

Example 5.5.1. We have a polyhedral fan for the equivalence classes \( Q_{2,5}(\omega)/\sim_{wb} \). We may consider the triangulation for the white cliques in each \( Q_{2,5}(\omega)/\sim_{wb} \), which breaks some cones in \( F_{2,5} \) into smaller cones by some dotted lines which separate two generic subdivisions.

Let \( \sigma \) be the index set for a white triangle in a subdivision \( Q_{N,M} \). For the white triangle \( \sigma \), the vertices are expressed by

\[ K_\sigma \cup \{a\}, \ K_\sigma \cup \{b\}, \ K_\sigma \cup \{c\}, \]

with some \((N - 1)\) index subset \( K_\sigma \). Let \( K^+_\sigma := K_\sigma \) and \( K^-_\sigma = [M] \setminus K_\sigma \setminus \{a, b, c\} \).
Figure 5.4: For each cone in $F_{2,5}$, the subdivision has two choice of triangulations for the white clique. Consider the subdivision with inner vertices 13, 14, we need to triangulate the white clique $\{12, 13, 14, 15\}$, when we have the white triangulation by diagonal 13, 15, We have the white triangle tile $\{13, 14, 15\}$, which corresponds to the cone($\{r_3^+, r_5^-\}$) using Theorem 5.5.2. Note the dotted line corresponds to $r_i^-$. 

Let $\sigma$ be the index set for a black triangle in the subdivision $Q_{N,M}$. For the black triangle $\sigma$, the vertices are expressed by

$$K_\sigma \setminus \{a\}, \ K_\sigma \setminus \{b\}, \ K_\sigma \setminus \{c\},$$

with some $(N + 1)$ index subset $K_\sigma$ including $\{a, b, c\}$. Let $K_\sigma^+ := K_\sigma \setminus \{a, b, c\}$ and $K_\sigma^- = [M] \setminus K_\sigma$. Then we have

**Theorem 5.5.1.** A subdivision $Q_{N,M}(\omega(t))$ contains a triangle tile $\sigma$, if and only if the time variable $t$ belongs to

$$\text{relint}(\text{cone}\{r_\alpha^+, r_\beta^- : \alpha \in K_\sigma^+, \beta \in K_\sigma^-\}).$$

**Proof.** Recall that the time variable $t = r_i^-$ implies a subdivision equivalent with
the weight vector \( \omega = (0, \cdots, 0, -1, 0, \cdots, 0) \) with \(-1\) at position \(i\) and 0 at other positions.

Similarly, the time variable \( t = r_i^+ \) is equivalent with the weight vector \( \omega = (0, \cdots, 0, +1, 0, \cdots, 0) \) with \(+1\) at position \(i\) and 0 at other positions.

We consider the white triangle case in the proof. For one direction, if time variable \( t \) is inside the relative interior \( \text{relint}(\text{cone}\{r_i^+, r_i^- : \alpha \in K_\sigma^+, \beta \in K_\sigma^- \}) \), with the index sets of the three vertices to be \( K_\sigma \cup \{a\}, K_\sigma \cup \{b\}, K_\sigma \cup \{c\} \) for some \( K_\sigma \in \left( \begin{array}{c} [M] \\ N-1 \end{array} \right) \). Thus we have the weight of \( \hat{v}_{K_\sigma \cup \{a\}}, \hat{v}_{K_\sigma \cup \{a\}}, \hat{v}_{K_\sigma \cup \{a\}} \) is \( N - 1 \), which is the highest among all vertices in \( A_{N,M}^\sigma \), and all other vertices in \( A_{N,M}^\sigma \) have weight smaller than \( N - 1 \). Thus we have this triangle tile.

On the other direction, if the \( \hat{v}_{K_\sigma \cup \{a\}}, \hat{v}_{K_\sigma \cup \{a\}}, \hat{v}_{K_\sigma \cup \{a\}} \) is the upper 2 face, we may find some \((c, x_0, y_0)\) such that \( \hat{v}_{K_\sigma \cup \{a\}}, \hat{v}_{K_\sigma \cup \{a\}}, \hat{v}_{K_\sigma \cup \{a\}} \) have the highest weight \( N - 1 \) while all other vertices in \( A_{N,M}^\sigma \) are lower than \( N - 1 \). This means \( t \) is inside the relative interior \( \text{relint}(\text{cone}\{r_i^+, r_i^- : \alpha \in K_\sigma^+, \beta \in K_\sigma^- \}) \). \( \square \)

**Example 5.5.2.** Recall \( r_i^+ = r_i, r_i^- = -r_i, i \in [M] \), the main rays in the multi-time \( t \)-space. We can use the Theorem 5.5.1 to obtain the subdivisions at these special rays as shown in Figures 5.5 and 5.6.

Similar to the definition of regular subdivision, we give a definition for realizable subdivision.

**Definition 5.5.1.** We say that a subdivision \( Q_{N,M} \) of \( A_{N,M} \) is realizable if it is a soliton subdivision \( Q_{N,M}(\omega) \) induced from certain weight vector \( \omega = (\omega_1, \cdots, \omega_M) \).

Using Theorem 5.5.1 we obtain the main theorem of this chapter which we call the realizable-checking theorem.
Theorem 5.5.2. Each generic subdivision having the triangle tiles $\triangle = \{\sigma_1, \ldots, \sigma_m\}$ can be constructed by taking any point from the set

$$\bigcap_{\sigma \in \triangle} \text{relint}(\text{cone}(\{r_\alpha^+, r_\beta^- : \alpha \in K_\sigma^+, \beta \in K_\sigma^+\})).$$

If the set is empty, then the subdivision is not realizable with this choice of the $\kappa$-parameters.

When we focus on the black triangles $\triangle_B$ in a subdivision $Q_{N,M}(\omega)$, and consider $Q_{N,M}(\omega)/\sim_w$, we know $Q_{N,M}(\omega)/\sim_w$ can be constructed from the set

$$\bigcap_{\sigma \in \triangle_B} \text{relint}(\text{cone}(\{r_\alpha^+, r_\beta^- : \alpha \in K_\sigma^+, \beta \in K_\sigma^-\})).$$

5.6 Soliton subdivisions of $A_{3,6}$

In this section, we discuss some details of the soliton subdivisions of $A_{3,6}^\omega$.

We start from the secondary polyhedral fan $F(A_{1,6})$, which has 6 main rays $r_i, i = 1, 2, \cdots, 6$, in $(t_3, t_4, t_5)$ space. We draw these rays in a plane, with each
Figure 5.6: Subdivision $Q_{N,M}$ corresponding to $r_1^+$. 

ray represented by a point. Each line in $\mathbb{R}^2$ plane corresponds to a dimension two fan in $(t_3, t_4, t_5)$ space. Each region bounded by three lines corresponds to a triangular cone attached to the origin in $(t_3, t_4, t_5)$ space.

From the fan $F_2,6$, whose cones correspond to $Q(\omega)_{2,6}/\sim_{wb}$, we may consider the triangulation of the white cliques in each $Q(\omega)_{2,6}/\sim_{wb}$. Then we break each cone into finer cones.

Let us study the Figure 5.10 in details. We draw a very light blue circle in the $N = 3$ polyhedral fan in Figure 5.9, and the circle cut the polyhedral fan into two parts. Then we draw the inner and outer of the circles separately, and obtain the two figures in Figure 5.10.

For the fan $F_3,6$, in the center of each figure in Figure 5.10, we have three lines (actually three dimension 2 fans) intersect at one point (actually one ray), and we call it a double point of $F_3,6$. We use this temporal notation for the example of $F_3,6$ only. For more general case, we do not have a systematic description yet.

We may see that the three lines for a double point do not always intersect at one
Figure 5.7: The polyhedral fan $F(A_{1,6})$ in $\mathbb{R}^3$. We draw the six rays $r_i^-$'s as six points in $\mathbb{R}^2$, the fan is given by refinement of all simplicial cones spanned by 3 rays. The dual polytope is the associahedron of $A_3$ type, which has 14 vertices given by the Catalan number $C_4$.

point. The line (fan) spanned by $r_6^-$ and $r_3^+$ satisfy the quadrilateral formula

$$D_{1,2,4,5} = t_3 + h_1(1, 2, 4, 5)t_4 + h_2(1, 2, 4, 5)t_5 = 0,$$

since it is obtained from a white triangulation in $Gr(2, 6)$ step. Similarly we have the formula for $r_i^- r_1^+$ and $r_2^- r_5^+$. The intersection of these three lines meaning there is a non-zero solution (which is the intersection point) such that

$$\begin{pmatrix}
1 & h_1(1, 2, 4, 5) & h_2(1, 2, 4, 5) \\
1 & h_1(2, 3, 4, 6) & h_2(2, 3, 4, 6) \\
1 & h_1(1, 3, 4, 6) & h_2(1, 3, 4, 6)
\end{pmatrix} \mathbf{t} = 0$$

Then we have

$$\begin{vmatrix}
1 & h_1(1, 2, 4, 5) & h_2(1, 2, 4, 5) \\
1 & h_1(2, 3, 4, 6) & h_2(2, 3, 4, 6) \\
1 & h_1(1, 3, 4, 6) & h_2(1, 3, 4, 6)
\end{vmatrix} = 0,$$
Figure 5.8: We obtain $\mathcal{F}_{2,6}$ for the soliton subdivisions of $\mathcal{A}_{2,6}$ with the same fan as $\mathcal{F}(\mathcal{A}_{1,6})$. The subdivision in the figure is the equivalence class $Q_{2,6}(\omega)/\sim_w$.

Assume $\kappa_1 = -\kappa_6, \kappa_2 = -\kappa_5, \kappa_3 = -\kappa_4$ for simplification, we obtain

$$\kappa_2^2 = \kappa_1 \kappa_3.$$ 

Using Theorem 5.5.2, we know in this case, the subdivisions in Figure 5.11 are not realizable. In addition, we have two directions to resolve the double point here, either $\kappa_2^2 > \kappa_1 \kappa_3$ or $\kappa_2^2 < \kappa_1 \kappa_3$, the subdivision for these two situations are shown in Figure 5.11.

Without resolving the double point in $\mathcal{F}_{3,6}$, the total number of full dimension polyhedral cones in Figure 5.10 is 30, and after resolving the double points, one can get 2 additional regions for each resolving direction. The total number of equivalent soliton subdivisions under $\sim_{wb}$ is then given by 34.

**Remark.** Recall the weak separation introduced in the previous chapter. There are 34 maximal weakly separated collections in $\binom{[6]}{3}$, which are all obtained by soliton subdivisions by choosing different $\kappa$ variables.
Figure 5.9: We break each cone (solid lines in the left figure) in the fan $\mathcal{F}_{2,6}$ into finer cones (dotted lines in the left figure) corresponding to triangulation of the white cliques in the subdivision $Q_{3,6}$. The right figure shows the fan $\mathcal{F}_{3,6}$.

We state a conjecture that describes a more deep relation between weak separation and soliton subdivisions.

**Conjecture 5.6.1.** *All the maximal weakly separated collections can be obtained by soliton subdivisions with different choices of $\kappa$ variables.*
Figure 5.10: Using the induction map, we obtain the subdivision $Q_{3,6}$ from the generic subdivision $Q_{2,6}$. There is a double point in the center of the graph, which means the three lines crossing the point are not always intersecting at only one point, and the topology at this position may be resolved into two conditions by chosen different $\kappa$ parameters.

Figure 5.11: Resolution of the double point at the center of the graph in the figure 5.10. There are two ways to resolve the point, and one obtains two different subdivisions. Notice that they correspond to different choices of the $\kappa$-parameters.
BIBLIOGRAPHY


