MODULAR CURVATURE FOR TORIC
NONCOMMUTATIVE MANIFOLDS

DISSERTATION

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ABSTRACT

In this paper, we extend recent results on the modular geometry on noncommutative two tori to a larger class of noncommutative manifolds: toric noncommutative manifolds. We first develop a pseudo differential calculus which is suitable for spectral geometry on toric noncommutative manifolds. As a main application, we derive a general expression for the modular curvature with respect to a conformal change of metric on toric noncommutative manifolds. By specializing our results to the noncommutative two and four tori, we recovered the modular curvature functions found in the previous works. An important technical aspect of the computation is that it is free of computer assistance.
To my father who is blessing me from the heaven
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TABLE OF CONTENTS

Abstract ................................................................. ii
Dedication ................................................................. ii
Acknowledgments ........................................................ iv
Vita .............................................................................. v

CHAPTER PAGE
1 Introduction ......................................................... 1
2 Deformation along $\mathbb{T}^n$ ..................................... 9
   2.1 Deformation of Fréchet algebras ......................... 9
   2.2 Deformation of operators .................................. 16
3 Deformation of Riemannian geometry ......................... 25
   3.1 Deformation of $C^\infty(M)$ ............................... 26
   3.2 Deformation of tensor fields .............................. 32
   3.3 Phase functions with respect to a given connection. . 39
   3.4 Tensor calculus on $T^*M$ ................................. 42
4 Pseudo differential operators on noncommutative manifolds .... 49
   4.1 Pseudo differential operators on $\mathbb{R}^m$ and Fréchet topologies .... 49
   4.2 Deformations of pseudo differential operators .......... 55
   4.3 Deformation of classical pseudo-differential operators .. 63
5 Widom’s pseudo differential calculus .......................... 65
   5.1 The symbol calculus ........................................... 65

vi
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2</td>
<td>Quantization map Op and the Schwartz Kernels</td>
<td>74</td>
</tr>
<tr>
<td>6</td>
<td>Deformation of the symbol calculus</td>
<td>77</td>
</tr>
<tr>
<td>7</td>
<td>Heat kernel asymptotic</td>
<td>82</td>
</tr>
<tr>
<td>7.1</td>
<td>Pseudo differential operators with parameter</td>
<td>82</td>
</tr>
<tr>
<td>7.2</td>
<td>Perturbation of the scalar Laplacian $\Delta$ via a Weyl factor</td>
<td>84</td>
</tr>
<tr>
<td>7.3</td>
<td>Resolvent approximation</td>
<td>88</td>
</tr>
<tr>
<td>7.4</td>
<td>Heat kernels and its variations</td>
<td>90</td>
</tr>
<tr>
<td>7.5</td>
<td>Zeta functions and conformal indices</td>
<td>96</td>
</tr>
<tr>
<td>8</td>
<td>Modular curvature</td>
<td>99</td>
</tr>
<tr>
<td>8.1</td>
<td>Computation of $b_2$</td>
<td>100</td>
</tr>
<tr>
<td>8.2</td>
<td>Integration on the sphere $S^{m-1}$</td>
<td>105</td>
</tr>
<tr>
<td>8.3</td>
<td>Integration in $\lambda$ and the rearrangement lemma</td>
<td>110</td>
</tr>
<tr>
<td>8.4</td>
<td>Modular curvature on noncommutative two tori</td>
<td>114</td>
</tr>
<tr>
<td>8.5</td>
<td>Modular curvature for even dimensional toric noncommutative manifolds</td>
<td>118</td>
</tr>
<tr>
<td>8.6</td>
<td>Comparison with [15] and [16]</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>127</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

In the noncommutative differential geometry program (cf. for instance, Connes’s book [8]), the geometric data is given in the form of a spectral triple \((\mathcal{A}, \mathcal{H}, D)\), where \(\mathcal{A}\) is a \(^*\)-algebra which serves as the algebra of coordinate functions of the underlying space, and \(D\) is an unbounded self-adjoint operator such that the commutators \([D, a]\) are bounded operators on \(\mathcal{H}\) for all \(a \in \mathcal{A}\). More than the topological structure, the spectral data also reflects the metric and differential structure of the geometric space. The prototypical example comes from spin geometry: \((C^\infty(M), L^2(\mathcal{S}), \mathcal{D})\), where \(M\) is a closed spin manifold with spinor bundle \(\mathcal{S}\) and \(\mathcal{D}\) is the Dirac operator.

In Riemannian geometry, local geometric invariants, such as the scalar curvature function can be recovered from the asymptotic expansion of Schwartz kernel function of the heat operator \(e^{-t\Delta}\):

\[
K(x, x, t) \sim \sum_{j \geq 0} V_j(x) t^{(j-d)/2}, \text{ where } d \text{ is the dimension of the manifold.}
\]

Equivalently, one turns to the asymptotic expansion of the heat kernel trace

\[
\text{Tr}(f e^{-tD^2}) \sim \sum_{j \geq 0} V_j(f, D^2) t^{(j-d)/2}, \quad f \in \mathcal{A}, \quad (1.0.1)
\]
which makes perfect sense in the operator theoretic setting. In the spirit of Connes and Moscovici’s work [12], for a metric on a noncommutative space implemented by the operator $D$, a good notion of intrinsic curvatures comes from the density functions of the heat kernel coefficients $V_j(\cdot, D^2)$ in (1.0.1). This approach was carried out in great depth on noncommutative two torus. The technical tool for the computation is the pseudo differential calculus associated to a $C^*$-dynamical system which was developed in Connes’ seminal paper [6]. The first application of such calculus is the Gauss-Bonnet theorem for noncommutative two torus [13]. The major progress occurred in Connes and Moscovici’s recent work [12], in which they generalized several geometric functionals, such as second heat coefficients, the Ray-Singer analytic torsion, on Riemann surfaces to noncommutative two tori, more importantly, the closed formulas for the associated density functions were computed explicitly. In contrast to the classical geometry, the appearance of the modular curvature functionals in those closed formulas gives vivid reflections of the noncommutativity. An independent calculation of the expressions of the modular curvature functions was carried out in [17]. Their works made use of different CAS (computer algebra system) to avoid algebraic mistakes during the lengthy computation. Recently, in [21], the computation was extended to Heisenberg modules over noncommutative two torus and the whole calculation was greatly simplified so that CAS was no longer need.

It is natural to investigate how to implement the program for other noncommutative manifolds. An interesting class of examples comes from deformation of classical Riemannian manifolds, such is the Connes-Landi deformations, also called
toric noncommutative manifolds (see [11] [3]). The underlying deformation theory, called $\theta$-deformation in the literature, was developed in Rieffel’s work [23].

In the spirit of the previous work on noncommutative tori (cf. [12], [15]), the technical tool used to compute the heat kernel coefficients is a pseudo differential calculus for toric noncommutative manifolds. Its construction is the first main outcome of this paper. Our pseudo differential calculus is designed to handle two families of noncommutative manifolds simultaneously: tori and spheres obtained by the Connes-Landi deformation. In contrast to the tori family, the noncommutative four sphere is different in two essential ways:

1. The dimension of the action torus (which is two) is less than the dimension of the underlying manifold (which is four);
2. The underlying manifold is not parallelizable.

The first one implies that the torus action is not transitive, hence the correspondent $C^*$-dynamical system will not be able to reveal the entire geometry. The second fact says that one should expect a more sophisticated asymptotic formula for the product of two symbols than the one appears in Connes’ construction. The method taken in this paper is to apply the deformation theory not only to the algebra of smooth functions on the underlying Riemannian manifold, but also to the whole pseudo differential calculus. The resulting pseudo differential calculus blends the commutative and the noncommutative coordinates in a simplest fashion.

In order for the deformation theory to apply is that both the symbol map and the quantization map in the calculus have to be equivariant with respect to the
torus action. This leads us to work with global pseudo differential calculus on closed manifolds in which all the ingredients are given in a coordinate-free way. Such calculus, which appeared first in Widom’s work [27] and [28], turned out to be the perfect tool to develop the deformation process.

In the rest of the paper, we devote the attention to applications. The discussion is parallel to the series work on noncommutative tori (cf. [12], [15] and [21]). Similar to the conformal changes of metric (cf. [12, sec. 1.5]), we perturb the classical metric, that is the scalar Laplacian operator via a Weyl factor $k$ (cf. definition 7.2.2), the resulting geometric operator is denoted by $\pi^\Theta(P_k)$. The first consequence of the pseudo differential calculus is the existence of the asymptotic expansion (1.0.1) or the meromorphic extension of the zeta function with respect to $\pi^\Theta(P_k)$.

To carry out explicit computation, we take the simplest but totally nontrivial perturbation: $k\Delta$, here $k$ is a Weyl factor as before, and $\Delta$ is the scalar Laplacian associated to the Riemannian metric. As a generalization of [12, Theorem 2.2], we prove that the zeta function at zero is independent of the conformal perturbation, namely:

$$\zeta_{k\Delta}(0) = \zeta_{\Delta}(0). \quad (1.0.2)$$

Due to the simplified set up, we define the modular curvature to be the density of functional associated to the second heat coefficient $f \mapsto V_2(f, k\Delta)$ in (1.0.1). The main result is the following, the modular curvature $\mathcal{R} \in C^\infty(M_{\Theta})$ is of the form:

$$\mathcal{R}(k) = \left(k^{-m/2}K(\Delta)(\nabla^2 k) + k^{-(m+2)/2}G(\Delta(1), \Delta(2)) (\nabla k \nabla k) \right) g^{-1} + ck^{-(m/2-1)}S_\Delta. \quad (1.0.3)$$
Let us explain the notations. First, \( k \in C^\infty(M_\Theta) \) is a Weyl factor, \( m \) is an even integer that stands for dimension of the manifold, \( g^{-1} \) is the metric tensor on the cotangent bundle and \( \nabla \) is the Levi-Civita connection so that the contraction \((\nabla^2 k)g^{-1}\) is equal to \( \Delta k \) and \((\nabla k \nabla k)g^{-1}\) generalizes the Dirichlet quadratic form appeared in [12, Eq. (0.1)]. The scalar curvature function \( S_{\Delta} \) associated to the metric \( g \) appears naturally if the metric is nonflat, the coefficient \( c \) is a constant depends only on the dimension. The triangle \( \triangle \) (compare to \( \Delta \), the Laplacian operator) is the modular operator (see (8.3.5)), while for \( j = 1, 2 \), \( \triangle^{(j)} \) indicates that the operator \( \Delta \) applied only to the \( j \)-th factor. The modular curvature functions \( \mathcal{K} \) and \( \mathcal{G} \) are computed explicitly in the last section. A crucial property of the modular curvature functions is that they can be written as linear combinations of the modified logarithm \( \mathcal{L}_0 = \log s/(s - 1) \), which is the generating function of Bernoulli numbers after the substitution \( s \mapsto e^s \).

As pointed out in [20], this fact serves as one of those “conceptual explanations” of the two novel aspects of the modular curvature functions \( \tilde{K}_0 \) and \( \tilde{H}_0 \) appeared in the expression of the Gaussian curvature \( \text{grad}_h F \) for noncommutative two tori [7, Thm. 4.8]:

1. \( \tilde{K}_0 \) is (upto a factor 1/8) the generating function of Bernoulli numbers;

2. \( \tilde{H}_0 \) is sum of divided differences of \( \tilde{K}_0 \).

The second main outcome of this paper is obtained by specializing the result above onto dimension two. We show that the expressions of \( \mathcal{K} \) and \( \mathcal{G} \) agree with the result in [21, Theorem 3.2] which gives further validation for our pseudo differential
calculus and the computation performed in the last section as in [21] and in contrast to [12], the computation does not require aid from CAS.

In dimension four, we obtain another validation for our algorithm by recovering the modular functions in [15]. In fact, both $\mathcal{K}$ and $\mathcal{G}$ are vanished if the heat operator is just $k\Delta$. Since $k\Delta$ is the leading part of the Laplacian adapted in [15], the non-zero contributions to the modular functions come from the symbols of degree one and zero.

We end this introduction with an outline of the paper. Section 2 consists of functional analytic backgrounds of the deformation theory. The main construction is that any toric equivariant binary operations, such as multiplications, left(right) module structures, can be deformed via a manner of twisted convolution (see (2.1.9)). We split the discussion into two parts: deformation of algebras and deformation of operators according to their roles as “symbols” and “operators” in the general framework of pseudo differential calculi. All the examples of $\mathbb{T}^n$-modules in this paper arise from dualizing the following geometric set up: closed Riemannian manifolds whose isometry group contains an $n$-torus $\mathbb{T}^n$.

In section 3, we apply the deformation theory to the tensor calculus over the underlying manifold, the resulting deformed tensor calculus serves as the backbone of the symbol calculus in our pseudo differential operator theory. The tensor calculus consists of three operations: a connection $\nabla$ which is characterized by the Leibniz property with respect to the pointwise tensor product $\otimes$ and the pointwise contraction $\cdot$ between tensor fields. As a generalization of the deformation of the pointwise multiplication between functions, we follow equation (2.1.9) to defined the $\Theta$-tensor ($\otimes_\Theta$)
and $\Theta$-contraction($\cdot\Theta$). The Leibniz property for the Levi-Civita connection $\nabla$ still holds due to the equivariant property of $\nabla$ (proposition 3.2.4). As an example, we see that the Dirichlet quadratic form appeared in [12, Eq. (0.1)] with respect to the complex structure $\sqrt{-1}$ has the following counterpart in term of the deformed tensor calculus:

$$\square_R(h) = (\nabla h \otimes_\theta \nabla h) \cdot_\theta g^{-1},$$  \hspace{1cm} (1.0.4)

where $g^{-1}$ the metric tensor on the cotangent bundle. Another crucial ingredient for our pseudo differential calculus is the phase functions $\ell(\xi, y) \in C^\infty(T^*M \times M)$ which is a manifold counterpart of the phase function $\langle \xi, y - x \rangle$ in the Fourier theory on functions on $\mathbb{R}^n$. Even though such functions are by no means unique, we show that there exists an $T^n$-invariant one that makes the whole symbol calculus $T^n$-equivariant. In the last part of this section, we discuss the deformation of tensor calculus on the cotangent bundle. Thanks to the phase function $\ell$, we can lift the Levi-Civita connection to a covariant derivative on $T^*M$, still denoted by $\nabla$, which plays the role of the $D_x$ in the pseudo differential calculus in local coordinates $(x, \xi)$. The generalization of $D_\xi$ is straightforward, still denoted by $D$. The $T^n$-equivariant property for both $\nabla$ and $D$ gives rise to the Leibniz rule in the deformed setting.

Section 4 consists of the deformation of the algebra of pseudo differential operators. Since pseudo differential operators can be unbounded, it is necessary to discuss the Fréchet topology in detail to ensure that the torus action is smooth in the sense of section 2. In section 5, we briefly recall Widom’s symbol calculus for pseudo differential operators acting only on smooth functions. Since the theory is well-explained in detail in [27], we often only provide partial proofs to clarify the notations.
In section 6, we prove the $T^n$-equivariant property for both the symbol map and the quantization map and the asymptotic symbol product for deformed pseudo differential operators follows easily. The asymptotic formula keeps the same form as its classical counterpart, only the tensor calculus is replaced by the deformed version.

The remaining two sections are devoted to applications. Following the work on noncommutative tori, a noncommutative metric is implemented by a perturbation of the scalar Laplacian $\Delta$ by a Weyl factor $k$, the resulting operator is denoted by $\pi^\Theta(P_k)$. As in the commutative case, most of the results on pseudo differential operators we developed before hold in the parametric situation which allow us to proceed the resolvent approximation. Arguing like in [19] and [4], we establish the asymptotic expansion of the heat kernels and meromorphic extension of the associated zeta functions. As a quick consequence, we prove the stability of the value of the zeta function $\zeta_{\pi^\Theta(P_k)}(s)$ at zero (i.e., (1.0.2)). Those are the main results in section 7. Section 8 consists of explicit computation of the modular curvature in (1.0.3). The efficiency of our algorithm comes from the tensor calculus that reduces tons of cancellations which were carried out by CAS in the previous work.
CHAPTER 2
DEFORMATION ALONG $\mathbb{T}^N$

2.1 Deformation of Fréchet algebras

In this section, we will provide the functional analytic framework which is necessary for our later discussion on toric noncommutative manifold. We do not intend to give a thorough discussion in this topic and refer to Rieffel’s monograph [23] for more details. The author also learned this topic from [10], [18], [3], [29], [5]. We first recall the Peter-Weyl theory with respect to the group $\mathbb{T}^n$. We always assume the topological vectors spaces used in this paper are over the complex numbers $\mathbb{C}$.

**Definition 2.1.1.** Let $V$ be Fréchet space whose topology is defined by an increasing family of semi-norms $\|\cdot\|_k$. We say $V$ is a smooth $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ module if $V$ admits an $n$-torus action $\alpha_t : V \rightarrow V$ such that the function $t \mapsto \alpha_t(v)$ belongs to $C^\infty(\mathbb{T}^n, V)$ for all $v \in V$, moreover, we require the action is strongly continuous in the following sense: $\forall v \in V$, given a multi-index $\mu$, we can find another integer $j'$ such that

$$\|\partial_\mu^\alpha \alpha_t(v)\|_j \leq C_{j'} \|v\|_{j'}, \quad t \in \mathbb{T}^n,$$  \hspace{1cm} (2.1.1)

where the constant $C_{j'}$ depends on $j'$ and the vector $v$. 


Due to the duality between $\mathbb{Z}^n$ and $\mathbb{T}^n$, Fourier theory tells us that all smooth $\mathbb{T}^n$-module in definition 2.1.1 are $\mathbb{Z}^n$-graded:

$$V = \bigoplus_{r \in \mathbb{Z}^n} V_r,$$

where $V_r$ is the image of the projection $p_r : V \to V$:

$$p_r(v) = \int_{\mathbb{T}^n} \alpha_t(v) e^{-2\pi ir \cdot t} dt, \quad v \in V. \quad (2.1.2)$$

Namely, any vector in $V$ admits a isotypical decomposition:

$$v = \sum_{r \in \mathbb{Z}^n} v_r, \text{ with } v_r = p_r(v) \text{ as above.} \quad (2.1.3)$$

The sequence $\{v_r\}_{r \in \mathbb{Z}^n}$ is of rapidly decay in $r$ in the following sense:

**Proposition 2.1.1.** Let $V$ be a smooth $\mathbb{T}^n$-module as in definition 2.1.1, whose topology is given by an countable increasing family of semi-norms $\| \cdot \|_j$ with $j \in \mathbb{N}$. Then for any element $v = \sum_{r \in \mathbb{Z}^n} v_r \in V$ with its isotypical decomposition, then the $j$-th semi-norms $\|p_r(v)\|_j$ is of rapidly decay in $r \in \mathbb{Z}^n$. More precisely, for any integer $k, j > 0$, there exist a degree $k$ polynomial $Q_k(x_1, \cdots, x_n)$ and another large integer $j'$ such that $\forall v \in V$,

$$\|p_r(v)\|_j \leq \frac{C_{k,j'}}{|Q_k(r)|} \|\alpha_t(v)\|_{j'} \quad (2.1.4)$$

In particular, the isotypical decomposition $\sum_{r \in \mathbb{Z}^n} v_r$ converges absolutely to $v$.

**Proof.** Let $\Delta$ be the Laplacian operator on $\mathbb{T}^n$. For $r \in \mathbb{R}^n$, let $Q(r)$ be the degree two polynomial such that $\Delta(e^{2\pi ir \cdot t}) = Q(r)e^{2\pi ir \cdot t}$. For any integer $k > 0$, apply integration by parts $k$ times on the integral in (2.1.2), we get:

$$p_r(v) = \int_{\mathbb{T}^n} e^{-2\pi ir \cdot r} \alpha_t(v) dt = \int_{\mathbb{T}^n} \frac{1}{Q_k(r)} e^{-2\pi ir \cdot r} \Delta^k \alpha_t(v) dt.$$
Due to the compactness of the torus and the strong continuity of the action (2.1.1), there exists a large index $j' \in \mathbb{N}$ so that:

$$\|\alpha_t(v)\|_j \leq C_{j'} \|v\|_{j'}$$

Hence

$$\|p_r(v)\|_j \leq \frac{C_{k,j'}}{|Q^k(r)|} \|\alpha_t(v)\|_{j'}.$$  

\(\square\)

Conversely, suppose $V$ admits a smooth $\mathbb{Z}^n$ grading: $V = \bigoplus_{r \in \mathbb{Z}^n} V_r$, then the $T^n$ action is given by on each homogeneous component $V_r$,

$$t \cdot v_r = e^{2\pi i r \cdot t} v_r, \quad t \in T^n. \quad (2.1.5)$$

A vector $v = \sum_{r \in \mathbb{Z}^n} v_r \in V$ is smooth respect to the torus action if and only if for each semi-norm $\|\cdot\|_j$, the sequence $\|v_r\|_j$ decays faster than any polynomial in $r$, that is for each semi-norm $\|\cdot\|_j$ and integer $k$, there is an integer $l$ and a constant $C_{j,k}$, such that

$$\|v_r\|_j \leq C_{j,k} \frac{\|v\|_l}{r^k}. \quad (2.1.6)$$

**Definition 2.1.2.** A $T^n$ smooth algebra $A$ is a smooth $T^n$ module as in definition 2.1.1 such that the multiplication map $A \times A \to A$ is $T^n$-equivariant and jointly continuous, that is for every $j$, there is a $k$ and a constant $C_j$ such that

$$\|ab\|_j \leq C_k \|a\|_k \|b\|_k, \quad \forall a, b \in A. \quad (2.1.7)$$

Additionally, if $A$ is a $*$-algebra, we required the $*$-operator is continuous and $T^n$-equivariant. Similarly, a $T^n$-smooth left(right) $A$-module $V$ is a $T^n$-smooth module
while the left(right) module structure is $\mathbb{T}^n$-equivariant and the jointly continuous as
in (2.1.7).

**Definition 2.1.3.** Let $\mathcal{A}$ be a $\mathbb{T}^n$ smooth algebra as above. Fix any skew symmetric
$n \times n$ matrix $\Theta$, we denote the corresponding bi-character:

$$\chi_{\Theta}(r, l) = e^{\pi i \langle r, \Theta l \rangle}, \quad r, l \in \mathbb{Z}^n,$$

(2.1.8) where the pairing $\langle \cdot, \cdot \rangle$ is the usual dot product in $\mathbb{R}^n$. The deformation of $\mathcal{A}$ is a
family of algebras $\mathcal{A}_{\Theta}$ parametrized by $\Theta$, whose underlying topological vector space
is equal to $\mathcal{A}$ while the multiplication $\times_{\Theta}$ is deformed as follow:

$$a \times_{\Theta} b = \sum_{r, s \in \mathbb{Z}^n} \chi_{\Theta}(r, l) a_{r} b_{s}, \quad \forall a, b \in \mathcal{A},$$

(2.1.9) and $a = \sum r a_r, b = \sum s b_s$ are the isotypical decomposition as in (2.1.2).

Each $\mathcal{A}_{\Theta}$ inherits the smooth $\mathbb{T}^n$ module structure from $\mathcal{A}$ and since $\chi_{\Theta}(r, l)$ in
(2.1.9) are complex numbers of unit length, the new multiplication is jointly continuous
as well. To sum up, for all skew symmetric $n \times n$ matrices, the deformation $\mathcal{A}_{\Theta}$ are
all smooth $\mathbb{T}^n$ algebra as in definition 2.1.2.

**Proposition 2.1.2.** The deformed product $\times_{\Theta}$ on $\mathcal{A}_{\Theta}$ is associative. That is, for
any $a, b, c \in \mathcal{A}$,

$$(a \times_{\Theta} b) \times_{\Theta} c = a \times_{\Theta} (b \times_{\Theta} c).$$

(2.1.10)

If the algebra $\mathcal{A}$ is a $\ast$-algebra, the deformation $\mathcal{A}_{\Theta}$ are $\ast$-algebras as well with respect
to the original $\ast$-operator, that is $\forall a, b \in \mathcal{A}$,

$$(a \times_{\Theta} b)^\ast = b^\ast \times_{\Theta} a^\ast.$$

(2.1.11)
Proof. Let \( a, b, c \in \mathcal{A}_\Theta \) with their isotypical decomposition: 
\[
a = \sum_r a_r, \quad b = \sum_s b_s, \quad c = \sum_l c_l,
\]
where \( r, s, l \) are summed over \( \mathbb{Z}^n \). We compute the left hand side of (2.1.10),
\[
(a \times_\Theta b) \times_\Theta c = \sum_{k,l} \chi_\Theta(k, l) \left( \sum_{r+s=k} \chi_\Theta(r, s) a_r b_s \right) c_l
\]
\[
= \sum_{r,s,l} \chi_\Theta(r + s, l) \chi_\Theta(r, s) a_r b_s c_l
\]
\[
= \sum_{r,s,l} \chi_\Theta(r, l) \chi_\Theta(s, l) \chi_\Theta(r, s) a_r b_s c_l,
\]
here we have used the estimate (2.1.6) to exchange the order of summation. Similar computation gives us the right hand side:
\[
a \times_\Theta (b \times_\Theta c) = \sum_{r,s,l} \chi_\Theta(r, s) \chi_\Theta(r, l) \chi_\Theta(s, l) a_r b_s c_l.
\]
Thus we have proved the associativity. Notice that we have not yet used the skew-symmetric property of \( \Theta \). In fact, the skew-symmetric property is only necessary for the \( \ast \)-operator to survive after deformation. In particular, it implies that for the bi-character \( \chi_\Theta \) defined in (2.1.8),
\[
\chi_\Theta(r, l) = \chi_\Theta(l, r)^\ast, \quad \forall r, l \in \mathbb{Z}^n,
\]
here the \( \ast \) operator is the conjugation on complex numbers. Since the \( \ast \) operator is \( \mathbb{T}^n \)-equivariant, it flips the \( \mathbb{Z}^n \)-grading of \( \mathcal{A} \), that is it sends the \( r \) component to the \(-r\) component: \( (a_r)^\ast = a_{-r}^\ast \), where \( a = \sum_{r \in \mathbb{Z}^n} a_r \in \mathcal{A} \). Indeed,
\[
(a_r)^\ast = \left( \int_{\mathbb{T}^n} e^{-2\pi ir \cdot t} \alpha_t(a) dt \right)^\ast = \int_{\mathbb{T}^n} e^{2\pi ir \cdot t} \alpha_t(a^\ast) dt = a_{-r}^\ast.
\]
Therefore:

\[(a \times_\Theta b)^* = \left( \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r, l)a_r b_l \right)^* = \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r, l)^*(b_l)^*(a_r)^* \]

\[= \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(l, r)b_l^*a_r^* = \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(-l, -r)b_l^*a_r^* \]

\[= \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(l, r)b_l^*a_r^* = b^* \times_\Theta a^*. \]

The proof is complete. \(\square\)

**Proposition 2.1.3.** Let \( \phi : A \to B \) be a \(\mathbb{T}^n\)-equivariant continuous algebra homomorphism, where \( A, B \) are two \(\mathbb{T}^n\) smooth algebras which admit deformation as above. If we identify \( A \) and \( A_\Theta \), \( B \) and \( B_\Theta \) by the identity maps respectively, then

\[ \phi : A_\Theta \to B_\Theta \] (2.1.12)

is still an \(\mathbb{T}^n\)-equivariant algebra homomorphism with respect to the new product \( \times_\Theta \).

**Proof.** For any \( a, a' \in A \) with the isotypical decomposition \( a = \sum_r a_r \), \( b = \sum_l b_l \), thanks to the equivariant property of \( \phi \), we have \( \phi(a_r) = \phi(a)_r \) and \( \phi(b_l) = \phi(b)_l \) for any \( r, l \in \mathbb{Z}^n \). Use the continuity of \( \phi \), we compute:

\[ \phi(a \times_\Theta a') = \phi \left( \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r, l)a_r a'_l \right) = \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r, l)\phi(a_r)\phi(a'_l) \]

\[= \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r, l)\phi(a)_r\phi(a')_l \]

\[= \phi(a) \times_\Theta \phi(a') \]

\(\square\)

The next proposition shows that any \(\mathbb{T}^n\)-equivariant trace on a smooth \(\mathbb{T}^n\) algebra \( A \) extends naturally to a trace on all the deformations \( A_\Theta \).
Proposition 2.1.4. Let $\tau : \mathcal{A} \to \mathbb{C}$ be a $T^n$-equivariant trace and $\Theta$ is an $n \times n$ skew-symmetric matrix as before. Then $\tau : \mathcal{A}_\Theta \to \mathbb{C}$ is a continuous linear functional for the deformation $\mathcal{A}_\Theta$ and $\mathcal{A}$ are identical as topological vector spaces. However, $\tau$ is indeed a trace on $\mathcal{A}_\Theta$, that is $\forall a, b \in \mathcal{A}_\Theta$

\[
\tau(a \times_\Theta b) = \tau(ab) = \tau(ba) = \tau(b \times_\Theta a).
\] (2.1.13)

Proof. Due to the duality of $\mathbb{Z}^n$ and $T^n$, a linear map between two $T^n$ modules is $T^n$-equivariant if and only if it preserves the $\mathbb{Z}^n$-grades. Take $\mathbb{C}$ as a trivial smooth $T^n$ module, the equivariant assumption on $\tau$ can be rephrased in terms of the $\mathbb{Z}^n$ grading as follows, for any $a = \sum a_r \in \mathcal{A}$ with its isotypical decomposition,

\[
\tau(a) = \tau(\sum_{r \in \mathbb{Z}^n} a_r) = \tau(a_0)
\]

Since $\Theta$ is skew-symmetric, we get

\[
\tau(a \times_\Theta b) = \sum_{r \in \mathbb{Z}^n} \chi_\Theta(r, -r) \tau(a_r b_{-r}) = \sum_{r \in \mathbb{Z}^n} \tau(a_r b_{-r}) = \tau(ab).
\]

Similar computation gives that $\tau(ba) = \tau(b \times_\Theta a)$. Therefore if $\tau$ is a trace on $\mathcal{A}$, then it is a trace on $\mathcal{A}_\Theta$ as well. \hfill $\Box$

The deformation can be extended to certain limit of Fréchet algebras. Explicitly, we consider a filtrated algebra $\mathcal{A}$ with a filtration:

\[
\cdots \subset \mathcal{A}_{-j} \subset \cdots \subset \mathcal{A}_0 \subset \cdots \mathcal{A}_j \cdots \subset \mathcal{A},
\] (2.1.14)

where each $\mathcal{A}_j$ ($j \in \mathbb{Z}$) is a smooth $T^n$-module as defined before, in particular, a Fréchet space. As a topological vector space, the total space $\mathcal{A}$ is a countable strict
inductive limit of \( \{ A_j \}_{j \in \mathbb{Z}} \), the topology is just called strict inductive limit topology (cf. for instance [26, sec. 13] for more details). This topology is never metrizable unless the filtration is stabilized started from some \( A_j \), therefore it is not Fréchet. We repeat the process above to produce a family of deformations. For a fixed skew symmetric matrix \( \Theta \), the deformation \( A_\Theta \) is identical to \( A \) as a filtrated topological vector space while the multiplication is replace by a new one, denoted by \( \times_\Theta \). Assume that on each \( A_j \), the topology is defined by a countable family of increasing semi-norms \( \{ \| \cdot \|_{l,j} \}_{l \in \mathbb{N}} \). The multiplication preserves the filtration:

\[
m : A_{j_1} \times A_{j_2} \to A_{j_1 + j_2} \quad (2.1.15)
\]
such that the continuity condition holds: for fixed \( j_1, j_2 \) and a positive integer \( l \), one can find a integer \( k \) and constant \( C_{k,j_1,j_2} \) such that

\[
\| m(a_1a_2) \|_{l,j_1+j_2} \leq C_{k,j_1,j_2} \| a_1 \|_{k,j_1} \| a_2 \|_{k,j_2}, \quad \forall a_1 \in A_{j_1}, \ a_2 \in A_{j_2}. \quad (2.1.16)
\]

The multiplication \( m \) is deformed in a similar fashion as in (2.1.9):

\[
m_\Theta : A_{j_1} \times A_{j_2} \to A_{j_1 + j_2} \quad (2.1.17)
\]

\[
(a_1, a_2) \mapsto \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r,l)m((a_1)_r(a_2)_l). \quad (2.1.18)
\]

\section{2.2 Deformation of operators}

The associativity of the \( \times_\Theta \) multiplication proved in proposition 2.1.2 is a special instance of certain “functoriality” in the categorical framework explained in [3]. Following the spirit, we shall discuss deformation of \( \mathbb{T}^n \)-equivariant representations in this section.
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces which are both strongly continuous unitary representation of $\mathbb{T}^n$, denoted by $t \mapsto U_t \in B(\mathcal{H}_1)$ and $t \mapsto \tilde{U}_t \in B(\mathcal{H}_2)$ respectively, where $t \in \mathbb{T}^n$. If no confusions arise, both representations will be denoted by $U_t$. Then $B(\mathcal{H}_1, \mathcal{H}_2)$, the space of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, becomes a $\mathbb{T}^n$-module via the adjoint action:

$$P \in B(\mathcal{H}_1, \mathcal{H}_2) \mapsto \text{Ad}_t(P) \triangleq \tilde{U}_t P U_{-t}, \ t \in \mathbb{T}^n.$$ (2.2.1)

Let $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$ be the space of all the $\mathbb{T}^n$ smooth vectors in $B(\mathcal{H}_1, \mathcal{H}_2)$. It is a Fréchet space on which the topology is defined by the semi-norms $\{\|\cdot\|_j\}_{j \in \mathbb{N}}$:

$$q_j(P) \triangleq \sum_{|\beta| \leq j} \frac{1}{\beta!} \left\| \partial_\beta \text{Ad}_t(P) \right\|_{B(\mathcal{H}_1, \mathcal{H}_2)}. \quad (2.2.2)$$

The semi-norms above are built in such a way that the continuity estimate (2.1.1) for the torus action holds automatically, hence $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$ is a smooth $\mathbb{T}^n$ module as in definition 2.1.1. Following from proposition 2.1.1, we see that any $\mathbb{T}^n$-smooth operator $P$ admits an isotypical decomposition $P = \sum_{r \in \mathbb{Z}^n} P_r$, where the operator norms $\{\|P_r\|\}_{r \in \mathbb{Z}^n}$ decays faster than polynomial in $r$, in particular the converges of the infinite sum is absolute with respect to the operator norm in $B(\mathcal{H}_1, \mathcal{H}_2)$.

Let $\mathcal{T} = \{T_1, \cdots, T_n\}$ be a basis of the Lie algebra of $\mathbb{T}^n$ acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ as closed operators by differentiating the torus action. Then $\text{ad}(\mathcal{T}) = \{\text{ad}(T_1), \cdots, \text{ad}(T_n)\}$ are derivations from $A$ to $B(\mathcal{H})$. Given a multi-index $\beta = (\beta_1, \cdots, \beta_n)$, denote $\text{ad}(T)^\beta = \text{ad}(T_1)^{\beta_1} \cdots \text{ad}(T_1)^{\beta_n}$, then the Fréchet topology on $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$ (all smooth vectors with respect to the torus action) is given by the semi-norms

Denoted by $\mathcal{D}_{\mathbb{T}^n}$ the completion of $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$ with respect to the $q_j$-norm. Follow the same proof as in proposition 2.1.1, we have
Lemma 2.2.1. Let $T \in B(H_1, H_2)_\infty$ be a smooth operator. Define

$$T_r = \int_{\mathbb{T}^n} e^{-2\pi i t \cdot r} \text{Ad}_t(T) dt.$$  \hspace{1cm} (2.2.3)

Then for a fixed integer $k \geq 0$, the sequence $\{q_k(T_r)\}_{r \in \mathbb{T}^n}$ is of rapidly decay in $r$. More precisely, for any integers $j > 0$ the $q_j$ norm of $T_r$ has the estimate: for any degree $k$ polynomial $Q_k(r_1, \cdots, r_n)$, there exists another integer $l > 0$ and a constant $C_{j,k}$ such that

$$q_j(T_r) \leq C_{j,k} |Q_k(r)|_{q_l}.$$  \hspace{1cm} (2.2.4)

Let $T = \{T_1, \cdots, T_n\}$ be a basis of the Lie algebra of $\mathbb{T}^n$, then $T$ is a collection of closed derivations on $B(H_1, H_2)$. For a multi-index $\beta = (\beta_1, \cdots, \beta_k)$, $T^\beta$ denotes the operator $T_{\beta_k} \cdots T_{\beta_1}$, define

$$D^i_{\mathbb{T}^n} = \bigcap_{|\beta| \leq j} \text{Dom}(T^\beta).$$  \hspace{1cm} (2.2.5)

From the Fourier theory of operator-valued functions, we have the following lemma:

Lemma 2.2.2. If $T \in D^{n+1}_{\mathbb{T}^n}$, then the infinite sum $\sum_{r \in \mathbb{Z}^n} T_r$ converges to $T$ absolutely with respect to the operator norm, where $T_r$ is defined in (2.2.3).

Now we are ready to define the deformation map $\pi^\Theta : B(H_1, H_2)_\infty \to B(H_1, H_2)_\infty$.

Definition 2.2.1. Let $H_1$ and $H_2$ be two Hilbert space with strongly continuous unitary $\mathbb{T}^n$ actions as above, that is $\forall v \in H_j$ ($j = 1, 2$), $t \mapsto t \cdot v$ is continuous in $t \in \mathbb{T}^n$. We denote the actions by $t \mapsto U_t$ and $t \mapsto \tilde{U}_t$ respectively. For a fixed $n \times n$ skew symmetric matrix $\Theta$, we recall the associated bi-character $\chi_\Theta(r,l) = e^{\pi i (r, \Theta l)}$. **18**
Then the deformation map \( \pi^\Theta : B(\mathcal{H}_1, \mathcal{H}_2)_\infty \to B(\mathcal{H}_1, \mathcal{H}_2)_\infty : P \mapsto \pi^\Theta(P) \) is defined as follows,

\[
\pi^\Theta(P)(f) = \sum_{r,l \in \mathbb{Z}^n} \chi_{\Theta}(r, l) P_r(f_l), \quad P \in B(\mathcal{H}_1, \mathcal{H}_2)_\infty,
\]

where \( P = \sum_{r \in \mathbb{Z}^n} P_r \) and \( f = \sum_{l \in \mathbb{Z}^n} f_l \in \mathcal{H}_1 \) are the isotypical decomposition. We have assume that \( f \) is a \( \mathbb{T}^n \)-smooth vector so that \( \sum_{l \in \mathbb{Z}^n} f_l \) converges in \( \mathcal{H}_1 \), however the subspace of all \( \mathbb{T}^n \)-smooth vectors in dense in \( \mathcal{H}_1 \), hence the operator \( \pi^\Theta(P) \) is completely determined by (2.2.6). Alternatively, \( \pi^\Theta(P) \) is given by

\[
\pi^\Theta(P) = \sum_{r \in \mathbb{Z}^n} P_r U_{r \cdot \Theta/2},
\]

here \( r \cdot \Theta/2 \) denotes the matrix multiplication between a row vector \( r \) and \( \Theta \), the results defines a point in \( \mathbb{T}^n \).

**Remark.** The deformed operator \( \pi^\Theta(P) \) belongs to \( B(\mathcal{H}_1, \mathcal{H}_2)_\infty \) because the each isotypical component of \( P \) is perturbed by a unitary operator \( U_{r \cdot \Theta/2} \), therefore the sum in the right hand side of (2.2.7) is a sum of rapidly decay sequence, which implies the smoothness of \( \pi^\Theta(P) \).

**Lemma 2.2.3.** Let \( B(\mathcal{H}_1, \mathcal{H}_2)_\infty \) denote the Fréchet algebra of \( \mathbb{T}^n \)-smooth vectors in \( B(\mathcal{H}_1, \mathcal{H}_2) \) whose topology is given by the seminorms in (2.2.2). Then the deformation \( \pi^\Theta : B(\mathcal{H}_1, \mathcal{H}_2)_\infty \to B(\mathcal{H}_1, \mathcal{H}_2)_\infty \) is a continuous linear map with respect to the Fréchet topology with the estimate: for any multi-index \( \mu \), one can find any integer \( l \) large enough such that

\[
\partial_t^\mu (\text{Ad}_t(\pi^\Theta(P))) \leq C_\mu q_l(P)
\]
Proof. Given \( P \in B(\mathcal{H}_1, \mathcal{H}_2)_{\infty} \) a \( \mathbb{T}^n \)-smooth operator with the isotypical decomposition \( P = \sum_{r \in \mathbb{Z}^n} P_r \). From the definition, \( \pi^\Theta(P) \) has the isotypical decomposition \( \pi^\Theta(P) = \sum_{r \in \mathbb{Z}^n} P_r U_r \Theta/2 \), thus

\[
\text{Ad}_t(\pi^\Theta(P)) = \sum_{r \in \mathbb{Z}^n} \text{Ad}_t(P_r U_r \Theta/2) = \sum_{r \in \mathbb{Z}^n} e^{2\pi i r \cdot t} P_r U_r \Theta/2.
\]

If we let \( h(r) \) be the polynomial in \( r \) such that \( \partial_t^\mu(e^{2\pi i r \cdot t}) = h(r)e^{2\pi i r \cdot t} \), the degree of \( h(r) \) is equal to \( |\mu| \), compute

\[
\partial_t^\mu(\text{Ad}_t(\pi^\Theta(P))) = \sum_{r \in \mathbb{Z}^n} \partial_t^\mu(e^{2\pi i r \cdot t}) P_r U_r \Theta/2 = \sum_{r \in \mathbb{Z}^n} h(r)e^{2\pi i r \cdot t} P_r U_r \Theta/2 \quad (2.2.9)
\]

Form lemma 2.2.1, one can find an integer \( l \) large enough such that the operator norm
\[
\|P_r\| \leq \frac{C q_l(P)}{|r|^{n+1}}
\]

Therefore
\[
\|\partial_t^\mu(\text{Ad}_t(\pi^\Theta(P)))\| \leq \left( \sum_{r \in \mathbb{Z}^n} \frac{C}{|r|^{n+1}} \right) q_l(P).
\]

Lemma 2.2.4. Let \( \Theta \) and \( \Theta' \) be two \( n \times n \) skew symmetric matrices and for any \( P \in B(\mathcal{H}_1, \mathcal{H}_2)_{\infty} \), we have

\[
\pi^\Theta \circ \pi^{\Theta'}(P) = \pi^{\Theta+\Theta'}(P).
\]

Proof. Given \( P = \sum_{r \in \mathbb{Z}^n} P_r \), \( \pi^\Theta(T) = \sum_{r \in \mathbb{Z}^n} P_r U_r \Theta/2 \) is the isotypical decomposition of \( \pi^\Theta(T) \), therefore

\[
\pi^\Theta' \left( \pi^\Theta(P) \right) = \sum_{r \in \mathbb{Z}^n} P_r U_r \Theta/2 U_r \Theta'/2 = \sum_{r \in \mathbb{Z}^n} P_r U_r (\Theta+\Theta')/2 = \pi^{\Theta+\Theta'}(P).
\]

\[\square\]
In particular, we see that the deformation process is invertible.

**Corollary 2.2.5.** For a fixed skew symmetric matrix $\Theta$, the deformation map $\pi^{\Theta}$ is invertible with its inverse $\pi^{-\Theta}$.

The $\ast$ operator is preserved during the deformation.

**Lemma 2.2.6.** Let $P \in B(\mathcal{H}_1, \mathcal{H}_2)_\infty$, then its adjoint $P^\ast \in B(\mathcal{H}_1, \mathcal{H}_2)_\infty$ as well, we have

$$
\pi^{\Theta}(P^\ast) = \pi^{\Theta}(P)^\ast \quad (2.2.10)
$$

**Proof.** Since the torus action is unitary, the adjoint operation is equivariant:

$$(\text{Ad}_t(P))^\ast = (U_tP U_{-t})^\ast = U_tP^\ast U_{-t} = \text{Ad}_t(P^\ast),$$

therefore for the isotypical components, $P_r^\ast = (P_{-r})^\ast$ for all $r \in \mathbb{Z}^n$,

$$
\pi^{\Theta}(P^\ast) = \sum_{r \in \mathbb{Z}^n} P_r^\ast U_{r, \Theta/2} = \sum_{r \in \mathbb{Z}^n} (P_{-r})^\ast (U_{-r, \Theta/2})^\ast
$$

$$
= \sum_{r \in \mathbb{Z}^n} (U_{-r, \Theta/2} P_{-r})^\ast = \sum_{r \in \mathbb{Z}^n} (P_{-r} U_{-r, \Theta/2})^\ast
$$

$$
= \sum_{r \in \mathbb{Z}^n} (P_r U_{r, \Theta/2})^\ast
$$

$$
= (\pi^{\Theta}(P))^\ast,
$$

here we have used the facts that $U_{-r, \Theta/2}$ and $T_r$ commute. \qed

If we take $\mathcal{H}_1$ and $\mathcal{H}_2$ above to be the same Hilbert space, $B(\mathcal{H}_\infty)$ becomes an $\mathbb{T}^n$ smooth algebra as in definition 2.1.2. Following definition 2.1.3, we obtain a family of deformed algebras $(B(\mathcal{H}_\infty, \times_\Theta)$ parametrized by skew symmetric matrices $\Theta$. The
multiplication map is obviously $\mathbb{T}^n$-equivariant, that is $\text{Ad}_t(P_1)\text{Ad}_t(P_2) = \text{Ad}_t(P_1P_2)$, for all $t \in \mathbb{T}^n$ and for any $P_1, P_2 \in B(\mathcal{H}_\infty)$. The associativity of the $\times_\Theta$ multiplication has the following analogy.

**Proposition 2.2.7.** Keep the notations as above. The deformation map 

$$\pi_\Theta : (B(\mathcal{H}_\infty), \times_\Theta) \to \pi_\Theta(B(\mathcal{H}_\infty)) \subset B(\mathcal{H})$$

is an algebra isomorphism, namely, for any $P_1, P_2 \in B(\mathcal{H}_\infty)$, 

$$\pi_\Theta(P_1)\pi_\Theta(P_2) = \pi_\Theta(T_1 \times_\Theta T_2), \quad (2.2.11)$$

recall that the deformed product $\times_\Theta$ is defined in (2.1.9).

**Proof.** The invertibility of $\pi_\Theta$ is proved in corollary 2.2.5. It remains to show that it is an algebra morphism, that is for any $\mathbb{T}^n$-smooth vector $v \in \mathcal{H}$, we have 

$$\pi_\Theta(P_1)\left(\pi_\Theta(P_2)(v)\right) = \pi_\Theta(P_1 \times_\Theta P_2)(v). \quad (2.2.12)$$

Observe that $\pi_\Theta(P)(v)$ can be formally written as $P \times_\Theta v$ according to (2.2.6). Therefore the left hand side and the right hand side of (2.2.12) becomes $P_1 \times_\Theta (P_2 \times_\Theta v)$ and $(P_1 \times_\Theta P_2) \times_\Theta v$ respectively, thus equation (2.2.12) is exactly the same as the associativity of the $\times_\Theta$-multiplication proved in (2.1.10).

We have seen that the isotypical decomposition of an operator $P = \sum_r P_r$ converges with respect to operator norms. The normality of the trace somehow allows itself to pass the summation, namely $\text{Tr}(P) = \sum_r \text{Tr}(P_r)$ whenever $P$ is traceable. We shall prove this in the lemma.
Lemma 2.2.8. Let $\mathcal{H}$ be a separable Hilbert space with a strongly continuous unitary $\mathbb{T}^n$ action and $P = \sum r P_r \in B(\mathcal{H})_\infty$ is $\mathbb{T}^n$-smooth operator with its isotypical decomposition. Then $P$ is a trace-class operator if and only if the $\mathbb{T}^n$-invariant part $P_0$ is a trace-class operator. because they have the same $\mathbb{T}^n$-invariant part according to (2.2.7).

Proof. Since $\mathcal{H}$ is a strongly continuous unitary representation of $\mathbb{T}^n$, it admits a orthonormal decomposition

$$\mathcal{H} = \bigoplus_{l \in \mathbb{Z}^n} H_l,$$

in which each $H_l$ consists of eigenvector of the torus action:

$$H_l = \{v \in \mathcal{H}| t \cdot v = e^{2\pi i t \cdot l} v\}.$$

For each $H_l$, one can pick a orthonormal basis $\{\varepsilon_{k,l}\}_{k \in \mathbb{N}}$, then $\{\varepsilon_{k,l}\}_{l \in \mathbb{Z}^n, k \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. Since $\sum_{r \in \mathbb{Z}^n} T_r$ convergence absolutely in the operator norm,

$$\sum_{l \in \mathbb{Z}^n, k \in \mathbb{N}} \left( \sum_{r \in \mathbb{Z}^n} T_r(\varepsilon_{k,l}), \varepsilon_{k,l} \right) = \sum_{l \in \mathbb{Z}^n, k \in \mathbb{N}} \sum_{r \in \mathbb{Z}^n} \langle T_r(\varepsilon_{k,l}), \varepsilon_{k,l} \rangle,$$

observe that for all $r \in \mathbb{Z}^n$, $T_r(H_l) \subset H_{r+l}$, therefore $\langle T_r(\varepsilon_{k,l}), \varepsilon_{k,l} \rangle = 0$ except the case when $r = 0$. We continue the computation above:

$$\sum_{l \in \mathbb{Z}^n, k \in \mathbb{N}} \sum_{r \in \mathbb{Z}^n} \langle T_r(\varepsilon_{k,l}), \varepsilon_{k,l} \rangle = \sum_{l \in \mathbb{Z}^n, k \in \mathbb{N}} \langle T_0(\varepsilon_{k,l}), \varepsilon_{k,l} \rangle.$$

Therefore $T$ is traceable if and only of $T_0$ is traceable and they are the same trace. □

From (2.2.7) and (2.1.9), we can see that when the operations $\pi^\infty(\cdot)$ and $\times_{\Theta}$ are well-defined, $T$ and $\pi^\Theta(T)$, $T_1 T_2$ and $T_1 \times_{\Theta} T_2$ have the same $\mathbb{T}^n$-invariant part. Thus we have the following corollary.

23
**Corollary 2.2.9.** Let $\mathcal{H}$ be a separable Hilbert space. Let $T, T_1, T_2 \in B(\mathcal{H})$ be $\mathbb{T}^n$-smooth vectors. Then $T$ is traceable if and only if $\pi^\Theta(T)$ is of trace-class, $T_1 T_2$ is traceable if and only if $T_1 \times_\Theta T_2$ is traceable, moreover

$$\operatorname{Tr}(T) = \operatorname{Tr}(\pi^\Theta(T)), \quad \operatorname{Tr}(T_1 T_2) = \operatorname{Tr}(T_1 \times_\Theta T_2).$$

Combine the two equations above with (2.2.11), we obtain:

$$\operatorname{Tr} \left( \pi^\Theta(T_1) \pi^\Theta(T_2) \right) = \operatorname{Tr}(T_1 T_2).$$
CHAPTER 3
DEFORMATION OF RIEMANNIAN GEOMETRY

In this section, we will discuss how the whole tensor calculus on a toric manifolds is deformed following the machinery developed in the previous section. The deformed tensor calculus includes not only the deformation of the algebra of coordinate functions but also provides the foundation of the whole pseudo differential calculus. Most important result is that the Levi-Civita connection still serves a covariant derivative for the deformed calculus in the sense that the Leibniz formula still holds after the deformation.

**Definition 3.0.2.** A toric Riemannian manifolds $M$ is a closed (compact without boundary) Riemannian manifolds whose isometry group contains a $n$-torus. In other word, $M$ admits an $\mathbb{T}^n$-action as isometries.

**Example 3.0.1.** Let $M = \mathbb{T}^n$ be the $n$-torus with the usual flat metric, while the $n$-torus acts on itself by translations. The deformation gives the well-known family called noncommutative $n$-torus.
**Example 3.0.2.** Consider the two torus $\mathbb{T}^2$ acts on the four sphere $S^4$ by embedding $\mathbb{T}^2$ into $\text{SO}(5)$ as rotations in the first four coordinates:

$$(t_1, t_2) \in \mathbb{R}^2 \mapsto \begin{pmatrix} e^{-\pi i t_1} \\ e^{-\pi i t_2} \\ 1 \end{pmatrix},$$

where we identify $\mathbb{R}^5$ with $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$. Thus any Riemannian metric which is invariant under the rotations above provides an instance of a toric manifold, for example, the Robertson-Walker metrics with a cosmic scale factor $a(t)$:

$$ds^2 = dt^2 + a(t)^2 d\omega^2,$$

where $d\omega^2$ is the round metric on $S^3$, when $a(t) = \sin t$, we recover the round metric on $S^4$.

### 3.1 Deformation of $C^\infty(M)$

Given an isometry $\varphi : M \to M$, denoted by $U_\varphi : C^\infty(M) \to C^\infty(M)$ the pull-back action:

$$U_\varphi(f)(x) = f(\varphi^{-1}(x)), \quad f \in C^\infty(M), \quad (3.1.1)$$

which is $\ast$-automorphism of $C^\infty(M)$.

**Lemma 3.1.1.** Let $M$ be a toric manifold as in definition 3.0.2. Equipped with the action in (3.1.1), $C^\infty(M)$ is a smooth $\mathbb{T}^n$-module with respect to the usual Fréchet semi-norms of local partial derivatives.
Proof. For any \( t = (t_1, \ldots, t_n) \in \mathbb{T}^n \) viewed as an isometry on \( M \), the associated operator in is denoted by \( U_t \). In any local coordinate system and \( f \in C^\infty(M) \),

\[
\partial_{t_j} U_t(f) = U_t(X_j(f))
\]

where \( X_j \) is the vector field generated by the torus action along the \( t_j \) coordinate. For higher order derivatives,

\[
\partial^\alpha_t U_t(f) = U_t(X^\alpha(f)),
\]

where \( \alpha \) is a multi-index. Since the right hand side still belongs to \( C^\infty(M) \), we see that \( \partial^\alpha_t U_t(f) \) belongs to \( C^\infty(\mathbb{T}^n, C^\infty(M)) \) for any multi-index \( \alpha \). The verification of the continuity estimate (2.1.1) is straightforward. \( \square \)

**Lemma 3.1.2.** The integration map for the Riemannian measure of the metric \( g \)

\[
\int_M : C^\infty(M) \to \mathbb{C} \tag{3.1.2}
\]

is \( \mathbb{T}^n \)-invariant.

**Proof.** Again, this is an straightforward consequence of the isometry-action assumption. \( \square \)

**Corollary 3.1.3.** The action (3.1.1) makes the Hilbert space \( \mathcal{H} = L^2(M) \) of \( L^2 \) functions on \( M \) into a unitary representation of the \( n \)-torus. Moreover, the corresponding Sobolev spaces \( \mathcal{H}^s \), \( s \in \mathbb{R} \) are all unitary representations of \( \mathbb{T}^n \).

**Definition 3.1.1.** Let \( M \) be a closed Riemannian manifold whose isometry group contains an \( n \)-torus. For a fixed skew-symmetric \( n \times n \) matrix \( \Theta \), we denote by
\( C^\infty(M_\Theta) = \pi^\Theta(C^\infty(M)) \subset B(H) \), the image of \( C^\infty(M) \) under the deformation map \( \pi^\Theta \), that serves as an analogy of the smooth functions on the noncommutative space of \( M_\Theta \).

Remark. In terms of proposition 2.2.7, \( C^\infty(M_\Theta) \) is identical to \( C^\infty(M) \) as a topological vector space while the multiplication is replaced by \( \times_\Theta \), because \( \pi^\Theta \) is an algebra isomorphism:

\[
\pi^\Theta : (C^\infty(M), \times_\Theta) \to C^\infty(M_\Theta)
\]

We are ready to see some examples.

**Example 3.1.1 (Noncommutative Two Torus).** Let \( M = T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) with the induce flat metric and \((x_1, x_2)\) be the coordinates on \( T^2 \), put \( e_1(x_1, x_2) = e^{2\pi i x_1} \), \( e_2(x_1, x_2) = e^{2\pi i x_2} \). By elementary Fourier theory on \( T^2 \), \( \bigoplus e_k^1 e_l^2 \) serves as basis for the \( C^\infty(T^2) \), that is

\[
f = \sum_{(k,l) \in \mathbb{Z}^2} f_{(k,l)} e_k^1 e_l^2, \quad f \in C^\infty(T^2), \quad (3.1.3)
\]

moreover the Fourier coefficients \( \{f_{(k,l)}\} \) are of rapidly decay in \((k,l)\). Given \( t = (t_1, t_2) \in \mathbb{R}^2 \), the torus action is given by

\[
\alpha_t(e_k^1 e_l^2) = e^{2\pi i t_1 k + t_2 l} e_k^1 e_l^2 \quad (3.1.4)
\]

Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), denote \( \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \). The deformed algebra \( C^\infty(T^2_\Theta) \) is identical to \( C^\infty(T^2) \) as a topological vector space with the deformed product

\[
f \times_\Theta g = \sum_{r,s \in \mathbb{Z}^2} e^{\pi i (r_1 s_2 - r_2 s_1)} f_r g_s e_1^{r_1 + s_1} e_2^{r_2 + s_2} = \sum_{r,s \in \mathbb{Z}^2} e^{-\pi i \theta (r_1 s_2 - r_2 s_1)} f_r g_s e_1^{r_1 + s_1} e_2^{r_2 + s_2}, \quad (3.1.5)
\]
where \( r = (r_1, r_2), s = (s_1, s_2), f_r \) and \( g_s \) are the Fourier coefficients of \( f \) and \( g \). Take \( f = e_1, g = e_2 \) we see that

\[
e_1 \times_\theta e_2 = e^{-2\pi i \theta} e_2 \times_\theta e_1.
\]

Also the \(*\)-operator on \( C^\infty(T^2_\theta) \) is the usual one by taking conjugate of a function in \( T^2 \). Let \( \int \) be the integration against the volume form (with respect to the flat metric) on \( T^2 \) which is still a trace on \( C^\infty(T^2_\theta) \) by proposition 2.1.4, we define a pairing \( \phi : C^\infty(T^2_\theta) \times C^\infty(T^2_\theta) \to \mathbb{C} \):

\[
\phi(f, g) = \int (g^* \times_\theta f).
\]

Observe that the completion of the pre-Hilbert space \( (C^\infty(T^2_\theta), \phi) \) is exactly the space of \( L^2 \)-functions on \( T^2 \), we denote it by \( \mathcal{H}_d \). The left multiplication \( f \mapsto L^\theta_f \) makes \( C^\infty(T^2_\theta) \) a subalgebra of \( B(\mathcal{H}_d) \). Let \( C(T^2_\theta) \) be the \( C^* \)-algebra of the operator norm completion of \( C^\infty(T^2_\theta) \) which serves as the analogy of continuous functions on the noncommutative spaces \( T^2_\theta \).

Recall that the noncommutative torus \( \mathcal{A}_\theta \) with \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) is the universal \( C^* \)-algebra generated by two unitary elements \( u \) and \( v \) subjecting to the relation \( uv = e^{-2\pi i \theta} vu \). The smooth noncommutative torus \( \mathcal{A}_\theta^\infty \) is the subalgebra of \( \mathcal{A}_\theta \) consists of formal sums \( \sum_{(k,l) \in \mathbb{Z}^2} a_{(k,l)} u^k v^l \) with complex coefficients and the sequence \( \{a_{(k,l)}\} \) is of rapidly decay in the index \( (k,l) \). There exists a canonical trace \( \varphi : \mathcal{A}_\theta^\infty \to \mathbb{C} \) defined by

\[
\varphi \left( \sum_{(k,l) \in \mathbb{Z}^2} a_{(k,l)} u^k v^l \right) \triangleq a_{(0,0)}.
\]

The completion of the pre-Hilbert space \( (\mathcal{A}_\theta^\infty, \varphi) \) is denoted by \( \mathcal{H}_c \).
We define an algebra homomorphism $W : A^\infty_\theta \to C^\infty(T_\theta)$ by sending the generator $u$ and $v$ to $e_1$ and $e_2$ respectively, that is

$$W : A^\infty_\theta \to C^\infty(T_\theta) \quad (3.1.6)$$

$$a = \sum_{r \in \mathbb{Z}^2} a_r V^r \mapsto W(a) = \sum_{r \in \mathbb{Z}^2} a_r E^r,$$

where $E^r = e_1^{r_1} \times_\theta e_2^{r_2} \in C^\infty(T_\theta)$ and $V^r = u^{r_1} v^{r_2} \in A^\infty_\theta$, with $r = (r_1, r_2) \in \mathbb{Z}^2$.

**Proposition 3.1.4.** Keep the notations above, the map $W : A^\infty_\theta \to C^\infty(T_\theta)$ extends to a $C^*$-algebra isomorphism $W : A_\theta \to C(T_\theta)$.

**Proof.** First, $W$ is surjective because the image of $W$ contains $C^\infty(T^2_\theta)$ which is a dense subalgebra in $C(T^2_\theta)$. For the injectivity, we need the fact that $A_\theta$ is a simple algebra for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, thus the kernel of $W$ is trivial. \qed

Since $\{E^r\}_{r \in \mathbb{Z}^2}$ and $\{V^r\}_{r \in \mathbb{Z}^2}$ are orthonormal basis of $H_d$ and $H_c$ respectively, then $W$ extends to a unitary map between $H_d$ and $H_c$, denoted by $\tilde{W}$.

The two torus $\mathbb{R}^2/\mathbb{Z}^2$ action on $A_\theta$ is given by

$$\alpha_t(V^r) \triangleq e^{2\pi i t \cdot r} V^r, \quad t \in \mathbb{R}^2, r \in \mathbb{Z}^2.$$

The basic derivations representing infinitesimal generators to the above group automorphisms are given by the defining relations:

$$\begin{align*}
\delta_1(u) &= u, \quad \delta_1(v) = 0 \\
\delta_2(v) &= v, \quad \delta_2(u) = 0.
\end{align*} \quad (3.1.7)$$
While on \( \mathcal{H}_d \), we have two unbounded operators \( \partial_{x_1} \) and \( \partial_{x_2} \), which act on generators \( e_{r_1}^1 \times_{\theta} e_{r_2}^2 \) in the following way:

\[
\partial_{x_1}(e_{r_1}^1 \times_{\theta} e_{r_2}^2) = 2\pi i r_1 e_{r_1}^1 \times_{\theta} e_{r_2}^2,
\]

\[
\partial_{x_2}(e_{r_1}^1 \times_{\theta} e_{r_2}^2) = 2\pi i r_2 e_{r_1}^1 \times_{\theta} e_{r_2}^2,
\]

and then extended to \( \mathcal{H}_d \) by linearity.

**Lemma 3.1.5.** Keep the notations as above. For \( j = 1, 2 \), the unbounded \( 2\pi i \delta_j \) and \( \partial_{x_2} \) are are unitary equivalent with respect to the unitary map \( \bar{W} \).

**Example 3.1.2** (Noncommutative \( n \)-Torus). The general noncommutative \( n \)-torus is given by the deformation of \( n \)-torus acting on itself via translations as in example 3.0.1. Let \( \Theta = (\theta)_{ij} \) be a skew symmetric matrix of dimension \( n \) and then the deformed algebra \( C^\infty(\mathbb{T}_\Theta^n) \) is generated by \( e_j = e^{\pi i t_j}, j = 1, \ldots, n \) subject to the relations

\[
e_j \times_{\Theta} e_k = e^{2\pi i \theta_{jk}} e_k \times_{\Theta} e_j,
\]

where \( t = (t_1, \ldots, t_n) \) are the coordinates on \( \mathbb{T}^n \). Elements in \( C^\infty(\mathbb{T}_\Theta^n) \) is of the form

\[
f = \sum_{r \in \mathbb{Z}^n} f_r E^r,
\]

where \( f_r \in \mathbb{C} \) and \( E^r = e_{r_1}^1 \cdots e_{r_n}^n \) with \( r = (r_1, \ldots, r_n) \), while the sequence \( \{f_r\}_{r \in \mathbb{Z}^n} \) is of rapidly decay in \( r \).

**Example 3.1.3** (Noncommutative Four Sphere). Let \( M = S^4 \) be the unit four sphere endowed with the Robertson-Walker metrics and the two-torus rotation action defined in example 3.0.2. Let \( \mathbb{R}^5 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R} \) with coordinates \( (z_1, z_2, x) \), then the polynomial functions on \( S^4 \) is generated by the coordinate functions \( z_1, z_2 \) and \( x \) with the relation

\[
z_1 z_1^* + z_2 z_2^* + x^2 = 1.
\]
While the pull-back action on functions on generators is given by

\[ z_1 \mapsto e^{2\pi it_1} z_1, \ z_2 \mapsto e^{2\pi it_2} z_2, \ x \mapsto x, \]

where \((e^{2\pi it_1}, e^{2\pi it_2}) \in \mathbb{T}^2\). Choose \(\Theta\) to be

\[ \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \ \theta \in \mathbb{R}, \]

then the resulting multiplication \(\times_\Theta\) is presented by the relations on generators:

\[ z_1 \times_\Theta z_2 = e^{2\pi i\theta} z_2 \times_\Theta z_1 \text{ while } x_0 \text{ is central.} \]

Since we only focus on pseudo differential operators on scalar functions, the study of the deformation of toric spectral triples is postpone to the future publications. For the constructions of the spectral triples for toric noncommutative manifolds, We refer to [11], [9], [29] and [5].

### 3.2 Deformation of tensor fields

Recall that the tensor bundle \(T^r_s M\) over \(M\) of covariant rank \(s\) and contravariant rank \(r\) is obtained by tensoring the cotangent and the tangent bundle \(s\) and \(r\) time respectively. There are two basic operations between smooth tensor fields (i.e., smooth section of tensor bundles): the pointwise tensor product and the pointwise contraction, they both admits a deformation by a twisted convolution as in equation (2.1.9). As before, \(\Theta\) always stands for a skew symmetric matrix that has the same dimension of the torus, the associated bi-character \(\chi_\Theta\) on \(\mathbb{Z} \times \mathbb{Z}\) is defined by \(\chi_\Theta(\mu, \nu) = e^{\pi i(\mu \cdot \Theta \nu)}.\)
We first extend the pull-back action of an isometry $\varphi : M \to M$ to tensor fields over $M$. For any $f \in C^\infty(M)$, $X \in \Gamma^\infty(TM)$, $\omega \in \Gamma^\infty(T^*M)$, we define:

$$U_\varphi(X)|_x = \varphi_*(X|_{\varphi^{-1}(x)}), \quad U_\varphi(\omega)|_x = \varphi^*(\omega|_{\varphi^{-1}(x)}),$$

(3.2.1)

where $\varphi_* : TM \to TM$ and $\varphi^* : T^*M \to T^*M$ are the induced diffeomorphisms with the property:

$$\varphi^* = (\varphi_*^{-1})^t.$$

(3.2.2)

A straightforward consequence of (3.2.1) and (3.2.2) is that the contraction map:

$\cdot : \Gamma(TM) \times \Gamma(T^*M) \to C^\infty(M)$

is equivariant, namely, for any vector fields $X$ and one-forms $\omega$,

$$U_\varphi (X \cdot \omega) = U_\varphi(X) \cdot U_\varphi(\omega).$$

(3.2.3)

Apply the construction in section 2, the deformed version $\times_\Theta : \Gamma(TM) \times \Gamma(T^*M) \to C^\infty(M)$ is given by

$$X \times_\Theta \omega = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu)(X)_\mu \cdot (\omega)_\nu.$$

(3.2.4)

When acting on tensor fields, denote by $(\varphi_*)^\otimes j$ and $(\varphi^*)^\otimes j$ the diffeomorphisms between tensor bundles, $U_{(\varphi_*)^\otimes j}$ and $U_{(\varphi^*)^\otimes j}$ are the pull-back action on smooth sections as before. However, we shall simply use $U_\varphi$ when no ambiguity arises. Therefore the pointwise tensor product between tensor fields:

$$\otimes : \Gamma(T^r_s M) \times \Gamma(T^r'_s M) \to \Gamma(T^r_s M \otimes T^r'_s M)$$

(3.2.5)
is $\mathbb{T}^n$-equivariant. Similar to the contraction operation, we define $\otimes:\Gamma(\mathcal{T}_s^r M) \times \Gamma(\mathcal{T}_{s'}^r M) \to \Gamma(\mathcal{T}_s^r M \otimes \mathcal{T}_{s'}^r M)$ by for any $s_1 \in \Gamma(\mathcal{T}_s^r M)$ and $s_2 \in \Gamma(\mathcal{T}_{s'}^r M)$,

$$(s_1 \otimes s_2)(x) = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu)(s_1)_{\mu|x} \otimes (s_2)_{\nu|x}, \quad (3.2.6)$$

where the indices $\mu, \nu$ indicate the isotypical component with respect to the torus action.

The pairing $\times_\Theta$ and tensor $\otimes_\Theta$ can be extended to contractions and tensors to tensor fields of all rank in a canonical way. The next step is to show that the exterior derivative $d$ and the Levi-Civita connection $\nabla$ are both equivariant. As the crucial consequence, the Leibniz’s rule still holds. Recall that using the parallel transport, the covariant derivatives can be computed in the following way:

**Lemma 3.2.1.** Let $s \in \Gamma(\mathcal{T}_t^r M)$ be a tensor field. Given two points $x, y \in M$ that are close enough such that the parallel transport along the unique geodesic joining them with respect to the Levi-Civita connection is well-defined, denoted by $\mathcal{T}_x^y : (\mathcal{T}_t^r)_{yM} \to (\mathcal{T}_t^r)_{xM}$. The $j$-the covariant derivative can be computed in the following way: $Y_1, \ldots, Y_j \in T_x M$:

$$(\nabla^j s)(X_1 \otimes \cdots \otimes X_j) = \frac{d}{dv_1|_{v_1=0}} \cdots \frac{d}{dv_j|_{v_j=0}} T_x^{(v_1, \ldots, v_j)}(s), \quad (3.2.7)$$

where

$$r(v_1, \ldots, v_j) = \exp_x (v_1 Y_1 + \cdots + v_j Y_j) \quad (3.2.8)$$

Similar results holds for the symmetrized covariant derivative $\partial^i$: 

34
**Corollary 3.2.2.** Let \( \partial^j = \text{Sym} \circ \nabla^j \) be the \( j \)-th symmetrized covariant derivative, then for \( s \in \Gamma(\mathcal{T}^r l M) \), \( Y_1, \ldots, Y_j \in T_x M \):

\[
((\partial^j s)|_x) \cdot (Y_1 \otimes \cdots \otimes Y_j) = \frac{d}{dt} \bigg|_{t=0} \mathcal{T}_x^{r(l)}(s),
\]

where

\[
r(t) = \exp_x(tY_1 + \cdots + tY_j)
\]

Recall that \( \partial^j \triangleq \text{Sym} \circ \nabla^j \), with

\[
\text{Sym}(x_1 \otimes x_j) \triangleq \frac{1}{j!} \sum \sigma(x_1) \otimes \sigma(x_j),
\]

where the sum is over all the permutations in \( S_j \). Now we are ready to prove the \( \mathbb{T}^n \)-equivariant property.

**Lemma 3.2.3.** Let \( \varphi : M \to M \) be an isometry. We abbreviate \( \varphi^* \) or \( \varphi_* \) to \( \varphi \) when no confusion will arise. Given \( Y_1, \ldots, Y_j \in T_x M \), let \( v = (v_1, \ldots, v_j) \in \mathbb{R}^j \) belong to a small neighborhood of the origin, denote:

\[
r(v) = r(v_1, \ldots, v_j) = \exp_x (v_1 Y_1 + \cdots v_j Y_j),
\]

\[
\tilde{r}(v) = \tilde{r}(v_1, \ldots, v_j) = \exp_{\varphi^{-1}(x)} (v_1 \varphi^{-1} Y_1 + \cdots v_j \varphi^{-1} Y_j).
\]

For any tensor field of rank \((r, l)\): \( s \in \Gamma(\mathcal{T}^r l M) \), denoted by \( \mathcal{I}_x^y : (\mathcal{T}^r l)_y M \to (\mathcal{T}^r l)_x M \) the parallel transport with respect to the Levi-Civita connection, then

\[
\varphi \left( \mathcal{I}_{\varphi^{-1}(x)}^y (s) \right) = \mathcal{I}_x^{\tilde{r}(v)} \left( \varphi(s|_{\varphi^{-1}(x)}) \right)
\]
Proposition 3.2.4. We fixed an isometry $\varphi : M \to M$, and $U_\varphi$ is the pull-back operator on sections of tensors fields defined above. We have the following commutative diagram:

$$
\begin{array}{ccc}
\Gamma(T^*_x M) & \xrightarrow{\nabla^j} & \Gamma(\otimes^j T^* M \otimes T^*_x M) \\
\downarrow U_\varphi & & \downarrow U_\varphi \\
\Gamma(T^*_x M) & \xrightarrow{\nabla^j} & \Gamma(\otimes^j T^* M \otimes T^*_x M)
\end{array}
$$

Proof. We need to show that for any $Y_1, \ldots, Y_j \in T_x M$

$$(\nabla^j U_\varphi(s)|_x) \cdot (Y_1 \otimes \cdots \otimes Y_j) = U_\varphi \left( \nabla^j s \right) |_x \cdot (Y_1 \otimes \cdots \otimes Y_j). \quad (3.2.12)$$

We introduce some abbreviations as before:

$$r(v) = r(v_1, \ldots, v_j) = \exp_x \left( v_1 Y_1 + \cdots + v_j Y_j \right),$$

$$\tilde{r}(v) = \tilde{r}(v_1, \ldots, v_j) = \exp_{\varphi^{-1}(x)} \left( v_1 \varphi^{-1} Y_1 + \cdots + v_j \varphi^{-1} Y_j \right)$$

$$\left. \frac{d}{dv} \right|_{v=0} = \left. \frac{d}{dv_1} \right|_{v_1=0} \cdots \left. \frac{d}{dv_j} \right|_{v_j=0}.$$

According to the definition of $U_\varphi$ and lemma 3.2.1, the right hand side of (3.2.12) is equal to

$$U_\varphi \left( \nabla^j s \right) |_x \cdot (Y_1 \otimes \cdots \otimes Y_j)$$

$$= \varphi \left( (\nabla^j s)_{\varphi^{-1}(x)} \cdot (\varphi^{-1} Y_1 \otimes \cdots \otimes \varphi^{-1} Y_j) \right)$$

$$= \varphi \left( \left. \frac{d}{dv} \right|_{v=0} T_{\varphi^{-1}(x)}^r(v) \left( s \right) \right)$$

$$= \left. \frac{d}{dv} \right|_{v=0} \varphi \left( T_{\varphi^{-1}(x)}^r(v) \left( s \right) \right).$$

36
while for the left hand side of (3.2.12), by lemma 3.2.1 again,

\[
(\nabla^j U_\varphi(s))|_x : (Y_1 \otimes \cdots \otimes Y_j)
\]

\[
= \frac{d}{dv}|_{v=0} T^{(v)}_x(U_\varphi(s))
\]

\[
= \frac{d}{dv}|_{v=0} T^{(v)}_x \left( \varphi(r^{-1}(r(s))) \right)
\]

To finish the proof, we need

\[
T_x^{(v)} \left( \varphi(s|r^{-1}(r(s))) \right) = \varphi(T_x^{\hat{r}}(v)(s))
\]

which is exactly the statement of lemma 3.2.3. ∎

**Proposition 3.2.5.** Let the subscript \((\cdot)_\mu\) denote isotypical decomposition with respect to the torus action defined in (2.1.2). Then for any tensor field \(s \in \Gamma(T^r M)\),

\[
\nabla^j s_\mu = (\nabla^j s)_\nu \quad j \in \mathbb{N}.
\]

**Proof.** For \(j \in \mathbb{N}, s \in \Gamma(T^r M)\):

\[
\nabla^j s_\mu = \nabla^j \left( \int_{T^n} e^{-2\pi i t \cdot \mu} U_t(s) dt \right) = \int_{T^n} e^{-2\pi i t \cdot \mu} (\nabla^j U_t(s)) dt
\]

\[
= \int_{T^n} e^{-2\pi i t \cdot \mu} U_t(\nabla^j s) dt = (\nabla^j s)_\mu,
\]

the interchange of the order of the covariant derivative and integration is valid because of the well-known fact that \(\nabla^j : \Gamma(T^* M) \to \Gamma(\otimes^j T^* M \otimes T^r M)\) is continuous with respect to the Fréchet topologies coming from the smooth structure of the manifold. ∎
Proposition 3.2.6 (Leibniz Rule). Given two tensor fields on \( M \), \( s_1 \in \Gamma(\mathcal{T}_s^r M) \) and \( s_2 \in \Gamma(\mathcal{T}_s^r M) \), we have

\[
\nabla(s_1 \otimes_\Theta s_2) = \nabla s_1 \otimes_\Theta s_2 + s_1 \otimes_\Theta \nabla s_2.
\]

(3.2.14)

For the contraction map \( \times_\Theta \), given a vector field \( X \) and a one-form \( \omega \), we get

\[
d(X \times_\Theta \omega) = (\nabla X) \times_\Theta \omega + X \times_\Theta \nabla \omega.
\]

(3.2.15)

Proof. The proof of (3.2.14) and (3.2.15) are quite similar, thus we only show the first one.

\[
\nabla(s_1 \otimes_\Theta s_2) = \nabla \left( \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu)(s_1)_\mu \otimes (s_2)_\nu \right) = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu)\nabla ((s_1)_\mu \otimes (s_2)_\nu) \\
= \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu)(\nabla(s_1)_\mu) \otimes (s_2)_\nu + (s_1)_\mu \otimes \nabla(s_1)_\mu,
\]

apply equation (3.2.13), we continue:

\[
\nabla(s_1 \otimes_\Theta s_2) = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu) ((\nabla s_1)_\mu \otimes (s_2)_\nu + (s_1)_\mu \otimes (\nabla s_1)_\mu) \\
= \nabla s_1 \otimes_\Theta s_2 + s_1 \otimes_\Theta \nabla s_2.
\]

We can switch \( \nabla \) and the sum \( \sum_{\mu, \nu \in \mathbb{Z}^n} \) for \( \nabla \) is a continuous map with respect to the smooth Fréchet topologies. \( \square \)
3.3 Phase functions with respect to a given connection.

Given on $\mathbb{R}^n$ with symbol $p(x, \xi) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, the pseudo differential operator $P$ is given by

$$ (Pf)(x) = \int_{\mathbb{R}^n} e^{-i\xi \cdot (x-y)} p(x, \xi) f(y) dy d\eta $$

The function $l(x_0, \xi, y) = \xi \cdot (y - x)$ plays a significant role in the quantization map above. The generalization to manifolds is a smooth function $\ell(\xi_x, y) \in C^\infty(T^*M \times M)$. The linearity in $\xi$ becomes the linearity of $\ell$ on each fiber of $T^*M$, but, the linearity in $x$ has no straightforward analogy. However, when the manifold $M$ is equipped with a connection $\nabla$ (on the cotangent bundle of $M$), the linearity at $x_0$ can be described as the vanishing of the symmetrized covariant derivative (along the $y$ variable) $D^k \ell(\xi_x, y)$ at $x = y$ for any $k \geq 2$.

**Definition 3.3.1.** Let $M$ be a smooth manifold with a connection $\nabla$ (on the cotangent) bundle and let $\partial^j$ be the $j$-symmetrized covariant derivative. A phase function with respect to a given connection is a real-valued smooth function $\ell(\xi_x, y) \in C^\infty(T^*M \times M)$ such that for fixed base point $x_0$, $\ell$ is linear in $\xi \in T_{x_0}^*M$ and such that for all $\xi_x$, the symmetrized covariant derivative (along $y$):

$$ \partial^j \ell(\xi_x, y) = \begin{cases} 
\xi_x, & j = 1, \\
0, & k \neq 1. 
\end{cases} $$

The existence of such functions was proved in, [28, Proposition 2.1]. A typical example of $\ell$ near by a point $x$ is $\ell(\xi_x, y) = \langle \xi_x, \exp^{-1}_x y \rangle$, where $\exp$ is the exponential map defined by the connection $\nabla$. Phase functions defined by (3.3.2) are by no means
unique, nevertheless, they all give rise to the same pseudo differential calculus later. Therefore we would like to choose a $\mathbb{T}^n$-invariant one to build our pseudo differential calculus if that is possible. Indeed, equation (3.3.2) is invariant under the torus action in the following sense. The pull-back action $U_\varphi$ is defined in a similar way as before: for $x, y \in M, \xi_x \in T^*_x M$ and $f \in C^\infty(T^* M \times M)$:

$$U_\varphi(f)(\xi_x, y) = f((\varphi^{-1})^*\xi_x, \varphi^{-1}y), \quad (3.3.3)$$

where $\varphi : M \to M$ is an isometry.

**Lemma 3.3.1.** Let $\ell \in C^\infty(T^* M \times M)$ be a phase function as in definition 3.3.1, then for any isometry $\varphi$, $U_\varphi(\ell)$ is still a phase function, that is equation (3.3.2) holds for $U_\varphi(\ell)$.

**Proof.** First, $U_\varphi(\ell)(\xi_x, x) = \ell((\varphi^{-1})^*\xi_x, \varphi^{-1}x) = 0$ is easy to verify. For the first covariant derivative: we have to show that for any $Y \in T_x M$

$$\nabla_Y U_\varphi(\ell)(\xi_x, y)\bigg|_{y=x}$$

Indeed,

$$\nabla_Y U_\varphi(\ell)(\xi_x, y)\bigg|_{y=x} = \frac{d}{dt}\bigg|_{t=0} U_\varphi(\ell)(\xi_x, \exp_x tY)$$

$$= \frac{d}{dt}\bigg|_{t=0} \ell((\varphi^{-1})^*\xi_x, \varphi^{-1}\exp_x tY)$$

$$= \frac{d}{dt}\bigg|_{t=0} \ell((\varphi^{-1})^*\xi_x, \exp_{\varphi^{-1}(x)} t(\varphi^{-1})_*Y)$$

$$= \langle (\varphi^{-1})^*\xi_x, (\varphi^{-1})_*Y \rangle = \langle \xi_x, \varphi_*(\varphi^{-1})_*Y \rangle$$

$$= \langle \xi_x, Y \rangle .$$

40
When \( j \geq 2 \), apply corollary 3.2.2, for any, \( X_1, \ldots, X_j \in T_x^* M \),

\[
(\partial^j U_\varphi(\ell))(\xi_x, y) \bigg|_{y=x} (X_1 \otimes \cdots \otimes X_j)
\]

\[
= \frac{d^j}{dt^j} \bigg|_{t=0} U_\varphi(\ell) (\xi_x, \exp_x t(X_1 + \cdots + X_j))
\]

\[
= \frac{d^j}{dt^j} \bigg|_{t=0} \ell \left( (\varphi^{-1})_* \xi_x, \exp_x t((\varphi^{-1})_* X_1 + \cdots + (\varphi^{-1})_* X_j) \right)
\]

\[
= \partial^j \ell((\varphi^{-1})_* \xi_x, y) \bigg|_{y=(\varphi^{-1})_* \xi_x} ((\varphi^{-1})_* X_1 \otimes \cdots \otimes (\varphi^{-1})_* X_j)
\]

\[
= 0.
\]

Therefore we have proved that the property (3.3.2) is \( T^n \)-invariant. \( \square \)

Therefore, start with any phase function, we can produce and \( T^n \)-invariant one by averaging it over the torus.

**Proposition 3.3.2.** Start with any phase function \( \ell \)

\[
\int_{T^n} U_t(\ell) dt
\]  

(3.3.4)

defines a \( T^n \)-invariant one.

From now on, we assume that the phase function \( \ell \) is \( T^n \)-invariant. The following lemme is crucial for latter discussion.

**Lemma 3.3.3.** Let \( \varphi : M \to M \) be an isometry and \( \ell(\xi_x, y) \) is a \( U_\varphi \)-invariant phase function. Denote by \( (\omega \big|_{\xi_x}) (y) = d_y \ell(\xi_x, y) \), the one-form supported near by the point \( x \in M \), where \( d_y \) is the exterior derivative on the \( y \)-variable.

\[
(\varphi^{-1})^* \left( (\omega \big|_{\xi_x}) (y) \right) = (\omega \big|_{(\varphi^{-1})_* \xi_x}) (\varphi^{-1}(y))
\]

(3.3.5)
Proof. We first prove that for any \( Y \in T^*_{\varphi^{-1}(y)} M \), where \( y \) belongs to a small neighborhood of \( x \):

\[
(\omega\big|_{\xi_x})(y) \cdot \varphi_* Y = \omega\big|_{(\varphi^{-1})^* \xi_x} (\varphi^{-1} y) \cdot Y
\]

(3.3.6)

The right hand side, by definition, is equal to

\[
\frac{d}{dt}\Big|_{t=0} \ell((\varphi^{-1})^* \xi_x, \exp_{\varphi^{-1}(y)} t Y) = \frac{d}{dt}\Big|_{t=0} \ell(\xi_x, \varphi(\exp_{\varphi^{-1}(y)} t Y))
= \frac{d}{dt}\Big|_{t=0} \ell(\xi_x, \exp_{\varphi} t \varphi_* Y)
= \omega(\xi_x)|_{\varphi_* Y}
\]

To finish the proof,

\[
(\varphi^{-1})^* \left( (\omega\big|_{\xi_x})(y) \right) \cdot Y = (\omega\big|_{\xi_x})(y) \cdot \varphi_* Y = \omega\big|_{(\varphi^{-1})^* \xi_x} (\varphi^{-1} y) \cdot Y.
\]

\[\square\]

### 3.4 Tensor calculus on \( T^* M \)

The symbol calculus of pseudo differential operator is base on the tensor calculus on the cotangent bundle \( T^* M \), in which the whole tensor bundles over \( T^* M \) is only the pull-back tensor bundles over the base manifold \( M \) via the projection \( \pi : T^* M \rightarrow M \), namely we denote by

\[
\mathcal{B}^r_s M \overset{\Delta}{=} \pi^* \mathcal{T}^r_s M,
\]

(3.4.1)

a tensor bundle of rank \((r, s)\), where \( r, s \) are the contravariant and the covariant rank respectively as before. From analytical point of view sections of \( \mathcal{B}^r_s M \) are differ from
ordinary tensor fields on $M$ in the only way that their base point coordinates depend on $(x, \xi)$.

The torus action is defined in a similar fashion as the case for tensor fields on $M$. Start with an isometry $\varphi : M \to M$ with the induced diffeomorphism $\varphi^* : T^*M \to T^*M$, the pull-back action $U_\varphi$ act on functions

$$U_\varphi(p)(\xi_x) = p\left( (\varphi^{-1})^*\xi_x \right), \quad \xi_x \in T^*_xM, \ p \in C^\infty(T^*M) \tag{3.4.2}$$

while for vector fields $X \in \Gamma(B^1_0M)$ and one-forms $\omega \in \Gamma(B^0_1M)$, the actions are

$$U_\varphi(X)(\xi_x) = \varphi_*(X|_{(\varphi)^{-1}\xi_x}), \quad U_\varphi(\omega)(\xi_x) = \varphi^*(\omega|_{(\varphi)^{-1}\xi_x}). \tag{3.4.3}$$

**Definition 3.4.1.** Let $M$ be a closed Riemannian manifold. The algebra of symbols $S\Sigma(M) \subset C^\infty(T^*M)$ is a filtered subalgebra of smooth functions on the cotangent bundle:

$$S\Sigma = \bigcup_{j=\infty}^\infty S\Sigma^j(M),$$

where each $S\Sigma^j(M)$ consists of smooth functions with the estimate in any local coordinates $(x, \xi)$,

$$\left| \partial^\alpha_x \partial^\beta_\xi p(x, \xi) \right| \leq C_{\alpha,\beta}(1 + |\xi|)^j - |\beta|, \tag{3.4.4}$$

the optimized constants $C_{\alpha,\beta}$ define a family of semi-norms that makes $S\Sigma(M)$ into a Fréchet space. The smoothing symbols $S\Sigma^{-\infty}$ is the intersection:

$$S\Sigma^{-\infty} = \bigcap_{j=-\infty}^\infty S\Sigma^j(M),$$

and the quotient $\text{CL} = S\Sigma/S\Sigma^{-\infty}$ is called the space of complete symbols.
Given $t \in T^m$ viewed as an isometry on $M$, let $p \in S\Sigma^j(M)$ be a symbol of order $j$, $t \mapsto U_t(p)$ is a smooth functions valued in $C^\infty(M)$, the partial derivatives in $t$ can be written as a finite sum in local coordinates:

$$\partial_t^i U_t(p) = U_t \left( \sum \partial_x^\alpha \partial_\xi^\beta p \right),$$

this shows that $U_t(p)$ still belongs to $S\Sigma^j(M)$ and the torus action on $S\Sigma^j(M)$ is smooth as in definition 2.1.1 and the estimate (2.1.1) holds. Therefore we can twist the pointwise multiplication on the filtrated algebra $S\Sigma(M)$ as explained at the end of section 2, the deformed version is denoted by

$$S\Sigma(M_\Theta) = (S\Sigma(M), \times_\Theta),$$

where $\times_\Theta$ is given in (2.1.17).

For our intrinsic pseudo differential calculus, we need to replace the local partial derivatives $\partial_x^\alpha$ and $\partial_\xi^\beta$ in equation (3.4.4) by those who are defined in a coordinates-free way. The replace for $\partial_\xi$ is straightforward to obtain:

**Definition 3.4.2.** The vertical derivative $D$ is the differential along the fibers of on $T^*M$. For any $x \in M$, $p \in C^\infty(T^*M)$, the $j$-th derivative $D^j p$ evaluate at point $\xi_x \in T^*_x M$ gives rise to a $j$-linear function on $T^*_x M$, thus $D^j p$ is a contravariant $k$-tensor. The precise definition is given as follows: for an integer $j \geq 1$, $D^j : C^\infty(T^*M) \to \Gamma(B^j_0 M)$

$$\begin{align*}
(D^j p)|_{\xi_x} \cdot (\omega_1 \otimes \cdots \otimes \omega_j) &= \left. \frac{d}{ds_1} \right|_{s_1=0} \cdots \left. \frac{d}{ds_j} \right|_{s_j=0} p(\xi_x + s_1 \omega_1 + \cdots + s_j \omega_j), \\
& \quad (3.4.5) \end{align*}$$

where $p \in C^\infty(T^*M)$, $\omega_1, \ldots, \omega_j \in T^*_x M$. 

44
Proposition 3.4.1. The vertical differential $D$ is equivariant with respect to diffeomorphisms of $M$. Namely, let $\varphi : M \rightarrow M$ be a diffeomorphism, $U_\varphi$ is the induced pull-back action, then

$$D^j U_\varphi(p) = D^j (U_\varphi(p)) \quad (3.4.6)$$

Proof. Start with the right hand side, let $\omega_1, \ldots, \omega_j \in T^*_x M$,

$$\left. (D^j U_\varphi(p)) \right|_{\xi_x} \cdot \omega_1 \otimes \cdots \otimes \omega_j$$

$$= \left. \frac{d}{ds_1} \frac{d}{ds_2} \cdots \frac{d}{ds_j} \right|_{s_1=0} \cdots \frac{d}{ds_j} \bigg|_{s_j=0} U_\varphi(p) (\xi_x + s_1 \omega_1 + \cdots + s_j \omega_j)$$

$$= \left. \frac{d}{ds_1} \frac{d}{ds_2} \cdots \frac{d}{ds_j} \right|_{s_1=0} \cdots \frac{d}{ds_j} \bigg|_{s_j=0} p \left( (\varphi^{-1})^* (\xi_x + s_1 \omega_1 + \cdots + s_j \omega_j) \right)$$

while, the right hand side:

$$U_\varphi \left( (D^j p) \right) \bigg|_{\xi_x} \cdot \omega_1 \otimes \cdots \otimes \omega_j$$

$$= \left. (D^j p) \right|_{(\varphi^{-1})^* \xi_x} \cdot (\varphi^{-1})^* \omega_1 \otimes \cdots \otimes (\varphi^{-1})^* \omega_j$$

$$= \left. \frac{d}{ds_1} \frac{d}{ds_2} \cdots \frac{d}{ds_j} \right|_{s_1=0} \cdots \frac{d}{ds_j} \bigg|_{s_j=0} p \left( (\varphi^{-1})^* \xi_x + s_1 (\varphi^{-1})^* \omega_1 \otimes \cdots \otimes s_j (\varphi^{-1})^* \omega_j \right)$$

Notice that $(\varphi^{-1})^*$ is linear on the fibers on $T^* M$, then the proof is complete. \(\square\)

The replacement for $\partial_x$ is not obvious, in fact, in order to satisfy the coordinate free requirement, we link the coordinates $x$ and $\xi$ through the phase function $\ell$, as a consequence, the horizontal covariant derivative has to involve both $\partial_x$ and $\partial_\xi$ in local coordinates.

Definition 3.4.3. For a fixed phase function $\ell$, the horizontal covariant derivative, still denoted by $\nabla$, is given by

$$(\nabla p)(\xi_x) = \nabla_y p((\ell|_{\xi_x})(y))|_{y=x}, \quad p(\xi_x) \in C^\infty(T^* M) \quad (3.4.7)$$
where we have adopted the notations in lemma 3.3.3, for each \( \xi_x \in T^* M \), \((\omega|_{\xi_x})(y)\) is a one-form on \( M \) supported in a small neighborhood of \( x \):

\[
(\omega|_{\xi_x})(y) = d_y \ell(\xi_x, y).
\]

**Remark.**

1) When evaluating at \( y = x \), the value in the right hand side of (3.4.7) does not depend on the choice of the phase function \( \ell \) as long as the property (3.3.2) is fulfilled.

2) The vertical and horizontal derivatives \( D \) and \( \nabla \) can be apply on all pull-back tensor fields (cf. (3.4.1)): \( p(\xi_x) \in C^\infty(\mathcal{B}_s^r M) \), where \((r, s)\) is the rank.

As an analogy of lemma 3.2.1,

**Lemma 3.4.2.** Keep the notations in lemma 3.3.3. Let \( p \in C^\infty(T^* M) \), then the \( j \)-the vertical covariant derivative is given by

\[
(\nabla^j p)|_{\xi_x} \cdot (Y_1 \otimes \cdots \otimes Y_j) = \frac{d}{ds_1}_{|s_1=0} \cdots \frac{d}{ds_j}_{|s_j=0} p((\omega|_{\xi_x})(\exp_x(s_1 Y_1 + \cdots + s_j Y_j)))
\]

(3.4.8)

**Proposition 3.4.3.** For any integer \( j \geq 1 \), the \( j \)-th vertical covariant derivative \( \nabla^j \) is equivariant with respect to the groups of isometries of \( M \). Namely, for any isometry \( \varphi : M \to M \):

\[
U_\varphi \left( \nabla^j p \right) = \nabla^j U_\varphi(p),
\]

(3.4.9)

where \( p \in \mathcal{B}_s^r M \) is a pull-back tensor field over \( T^* M \) of rank \((r, s)\).
Proof. The proof is a slightly modification of the proof of proposition 3.2.4. Let us set up some notations first. When presenting induced diffeomorphisms on the pull back tensor bundles, we often abbreviate $\varphi_*$ and $\varphi^*$ to $\varphi$ when no confusion arise. For $x, y \in M$ closed enough, $T^y_x : (T^y_x)|_M \rightarrow (T^y_x)|_M$ is the parallel transport along the unique geodesic joining the two points. As in the proof of proposition 3.2.4, we denote:

$$r(v) = r(v_1, \ldots, v_j) = \exp_x (v_1 Y_1 + \cdots v_j Y_j),$$

$$\tilde{r}(v) = \tilde{r}(v_1, \ldots, v_j) = \exp_{\varphi^{-1}(x)} (v_1 \varphi^{-1} Y_1 + \cdots v_j \varphi^{-1} Y_j),$$

$$\left. \frac{d}{dv} \right|_{v=0} = \left. \frac{d}{dv_1} \right|_{v_1=0} \cdots \left. \frac{d}{dv_j} \right|_{v_j=0}.$$

We would like to show that for any $Y_1, \ldots, Y_j \in T_x M$,

$$U_\varphi \left( \nabla^j p \right) |_{\xi_x} \cdot (Y_1, \ldots, Y_j) = (\nabla^j U_\varphi(p))|_{\xi_x} \cdot (Y_1, \ldots, Y_j) \quad (3.4.10)$$

For the right hand side,

$$\left. (\nabla^j U_\varphi(p))|_{\xi_x} \cdot (Y_1 \otimes \cdots \otimes Y_j) \right|_{v=0} = \left. \frac{d}{dv} \right|_{v=0} \left( \nabla^j (U_\varphi(p))|_{\xi_x} \cdot (r(v)) \right)$$

$$= \left. \frac{d}{dv} \right|_{v=0} \left( T^{(v)}_x \nabla^j (U_\varphi(p))|_{\xi_x} \cdot (r(v)) \right)$$

$$= \left. \frac{d}{dv} \right|_{v=0} \left( \nabla^j (U_\varphi(p))|_{\xi_x} \cdot (\tilde{r}(v)) \right),$$

to obtain the last equal sign, we have used lemma 3.3.3. For the left hand side,

$$\left. U_\varphi \left( \nabla^j p \right) \right|_{\xi_x} \cdot (Y_1 \otimes \cdots \otimes Y_j)$$

$$= \varphi \left( \left. (\nabla^j p) \right|_{\varphi^{-1} \xi_x} \cdot (\varphi^{-1} Y_1 \otimes \cdots \otimes \varphi^{-1} Y_j) \right)$$

$$= \left. \frac{d}{dv} \right|_{v=0} \varphi \left( T^{(v)}_{\varphi^{-1} x} p((\omega|_{\varphi^{-1} \xi_x})(\tilde{r}(v))) \right).$$

47
To finish the proof, we use the fact that for any isometry $\varphi$,

$$\varphi \circ T^{(v)}_{\varphi^{-1}x} = T^{(v)}_x \circ \varphi$$

which has been discussed in proposition 3.2.4.

The Leibniz rule (proposition 3.2.6) with respect to the deformed tensor product and contraction follows from the $T^n$-invariance of $D$ and $\nabla$ immediately.

**Proposition 3.4.4.** Given two pull back tensor fields $s_1 \in B^r_s M$, $s_2 \in B^{r'}_{s'}$ of rank $(r, s)$ and $(r', s')$, we define the twisted tensor product $\otimes_{\Theta}$ and contraction $\times_{\Theta}$ as in (3.2.6)

$$\nabla(s_1 \otimes_{\Theta} s_2) = \nabla s_1 \otimes_{\Theta} s_2 + s_1 \otimes_{\Theta} \nabla s_2,$$

$$D(s_1 \otimes_{\Theta} s_2) = Ds_1 \otimes_{\Theta} s_2 + s_1 \otimes_{\Theta} Ds_2.$$

(3.4.11)

Same results holds for the deformed contraction (cf. (3.2.4)).
CHAPTER 4

PSEUDO DIFFERENTIAL OPERATORS ON
NONCOMMUTATIVE MANIFOLDS

4.1 Pseudo differential operators on $\mathbb{R}^m$ and Fréchet topolo-
gies

In this subsection, following [1, section 1], we will defined two Fréchet topologies on
zero order pseudo differential operators on $\mathbb{R}^m$, one comes from the symbol, the other
comes from the operator norms on Sobolev spaces. We simply name them the symbol
topology and the operator norm topology respectively. As a main result, we show
that the two topologies are equivalent.

Let $p(x, \xi)$ be a smooth function on $\mathbb{R}^m \times \mathbb{R}^m$, for multi-indices $\alpha$, $\beta$, we denote
\[ p_{(\alpha)}^{(\beta)} = \partial_x^\beta \partial_{\xi}^\alpha p, \] (4.1.1)
the associated pseudo differential operator $P = p(x, D)$ is given by
\[ (Pf)(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi) d'\xi, \] (4.1.2)
where the density $d'\xi = (2\pi)^{-m}d\xi$, and the function $f(x)$ can be any function whose Fourier transform $\hat{f}$

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x) dx,$$

belongs to $D(\mathbb{R}^m)$, the space of all test functions. One can recover the symbol from its operator via

$$p(x, \xi) = e_{-\xi}(x)(Pe_\xi)(x) \text{ with } e_\xi(x) = e^{ix\cdot\xi}. \quad (4.1.3)$$

Set $\rho(\xi) = (1 + |\xi|^2)^{1/2}$. The symbol spaces $S\Sigma^d(\mathbb{R}^m)$ of order $d \in \mathbb{Z}$ consists of all smooth functions $p(x, \xi)$ such that

$$\left| \frac{\partial^{\alpha} p(x, \xi)}{\partial \xi^\beta} \right| \leq C_{\alpha, \beta} \rho(\xi)^{d-|\alpha|}. \quad (4.1.4)$$

The optimized constants $C_{\alpha, \beta}$ above gives rise a family of semi-norms on $S\Sigma^d(\mathbb{R}^m)$ that makes $S\Sigma^d(\mathbb{R}^m)$ into a Fréchet space. The correspondent operators $P = p(x, D)$ with $p(x, \xi) \in S\Sigma^d(\mathbb{R}^m)$ constitute pseudo differential operators of order $d$, denoted by $\Psi^d(\mathbb{R}^m)$. We call the inherited Fréchet topology simply the will the symbol topology, compare with the operator norm topology that is about be explained.

Let $\mathcal{S}(\mathbb{R}^m)$ be the space of Schwartz functions on $\mathbb{R}^m$ with its dual space $\mathcal{S}'(\mathbb{R}^m)$, the space of all tempered distributions. For any $s \in \mathbb{R}$, the Sobolev space $H_s$ contains all distributions $f \in \mathcal{S}(\mathbb{R}^m)$ such that $\rho^s \hat{f} \in L^2(\mathbb{R}^m)$, where the pairing $\langle f, h \rangle_{H_s}$ is defined to be $\langle \rho^s \hat{f}, \rho^s \hat{g} \rangle_{L^2}$.

It is well-known that operators in $\Psi^d(\mathbb{R}^m)$ send $\mathcal{S}(\mathbb{R}^m)$ to $\mathcal{S}(\mathbb{R}^m)$, and extent to a bounded operator $H_{s+d} \rightarrow H_s$ for all $s \in \mathbb{R}$ (cf.[19, lemma 1.21]).
For $1 \leq j \leq m$, let $D_j = i\partial_{x_j}$ and let $x_j$ denote the multiplication by the coordinate function $x_j$. For an operator $P : S(\mathbb{R}^m) \to S'(\mathbb{R}^m)$, we introduce

$$\text{ad}_{x_j}(B) = -[ix_j, B] = -(ix_j B - iB x_j), \quad \text{ad}_{D_j}(B) = [D_j, B] = D_j B - BD_j.$$  

(4.1.5)

For multi-orders $\alpha, \beta \in \mathbb{Z}_+^n$, put

$$P_{(\beta)}^{(\alpha)} \triangleq \text{ad}_{x_1}^{\alpha_1} \ldots \text{ad}_{x_m}^{\alpha_m} \text{ad}_{D_1}^{\beta_1} \ldots \text{ad}_{D_m}^{\beta_m}(P),$$

(4.1.6)

while $P_{(0)}^{(0)} = P$ as it should be. For two pseudo differential operators $P, Q$ with order $d_1$ and $d_2$ respectively, then the commutator (when it is well-defined) is of order $d_1 + d_2 - 1$. In particular, for any $P \in \Psi^0(\mathbb{R}^m)$, $P_{(\beta)}^{(\alpha)} \in \Psi^{-|\alpha|}$. Then operator norm gives rise a another family of semi-norms indexed by $\alpha$ and $\beta$. The associated Fréchet topology is called operator norm topology. The fact that the two topologies coincides on $\Psi^0(\mathbb{R}^m)$ follows immediately from the proposition below ([1, Theorem 1.4]). In fact, the symbol class considered in [1, Theorem 1.4] is the one with the estimate

$$\left| p_{(\beta)}^{(\alpha)} \right| \leq C_{\alpha, \beta} \rho(\xi)^{d-|\alpha|+|\beta|}.$$

For completeness, we reproduce the proof of [1, Theorem 1.4] (with the correction [2]) with respect to the symbol class in our setting.

**Proposition 4.1.1.** Let $P : S(\mathbb{R}^m) \to S'(\mathbb{R}^m)$ be a linear operator. The following are equivalent:

i) $P$ is a pseudo differential operator of order zero.

ii) For any $\alpha, \beta \in \mathbb{Z}_+^m$, the operator $P_{(\beta)}^{(\alpha)}$ defined in (4.1.6) extends to a bounded operator from $H_{-|\alpha|}$ to $H_0 = L^2$. 

51
Proof. Suppose $P = p(x, D)$ belong to $\Psi^0(\mathbb{R}^m)$, from the symbol map (4.1.3), one can quickly verify that $P_\beta^{(\alpha)} = p_\beta^{(\alpha)}(x, D)$. Since $p_\beta^{(\alpha)} \in S\Sigma^{-|\alpha|}(\mathbb{R}^m)$, we conclude that $P_\beta^{(\alpha)} \in B(H_{-|\alpha|}, H_0)$.

Conversely, assume that $P_\beta^{(\alpha)} \in B(H_{-|\alpha|}, H_0)$ for any $\alpha, \beta \in \mathbb{Z}_m$ and $P$ maps $\mathcal{S}(\mathbb{R}^m)$ to $\mathcal{S}'(\mathbb{R}^m)$. Pick $\psi \in \mathcal{S}(\mathbb{R}^m)$ with $\psi(0) = 1$, $\text{supp } \hat{\psi}(\xi) \subset \{ |\xi| \leq 1 \}$ and $\psi(x) = \psi(-x)$. Denote $\psi_x(y) = \psi(y - x)$. For $f \in \mathcal{S}(\mathbb{R}^m)$, $f(x)$ can be rewritten as

$$
f(x) = f(x)\psi_x(x) = \int \int e^{i(x-y)\cdot \xi}\psi_x(y)f(y)dyd'\xi
$$

where $e_\xi(x) = e^{ix\cdot \xi}$ as before. We compute formally:

$$
(Pf)(x) = \int \int e^{-iy\cdot \xi}P(e_\xi(x)\psi_y)(x)f(y)dyd'\xi
$$

$$
= \int \int e^{-i(x-y)\cdot \xi}p_0(x, y, \xi)f(y)dyd'\xi
$$

$$
= \int \int e^{-ix\cdot \xi}p(x, \xi) \hat{f}(\xi)d'\xi,
$$

where

$$
p_0(x, y, \xi) = e_{-\xi}(x)P(e_\xi(x)\psi_y)(x), \quad (4.1.7)
$$

$$
p(x, \xi) = \int e^{i(x-y)\cdot (\eta-\xi)}p_0(x, y, \eta)d'\eta dy. \quad (4.1.8)
$$

The computation is valid for localized $P_\varepsilon = \varphi_\varepsilon P \varphi_\varepsilon$, where $\varphi \in \mathcal{D}(\mathbb{R}^m)$ with $\varphi(x) = 1$ near the origin and $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ for $0 < \varepsilon \leq 1$. If we can show that the associated symbol $p_\varepsilon(x, \xi)$ has the desired estimate (4.1.3), then

$$
Pf = \lim_{\varepsilon \to 0} p_\varepsilon(x, D)f = p(x, D)f, \quad \forall f \in \mathcal{S}(\mathbb{R}^m).
$$
The converges was explained in [1, Theorem 1.4]. Let us focus on deriving the desired estimate for the symbol $p$. We start with the $L^2$-norm $\| \cdot \|$ of $p_0$ and its derivatives as a functions in the $x$-variable. Since $P \in B(H)$,

$$\|p_0(\cdot, y, \xi)\| \leq C \| e_\xi \psi_y \| = C \| \psi_y \| ,$$

the constant $C$ is independent of $y$ and $\xi$. We claim the $L^2$ norm estimate

$$\left\| D^\beta_x D^\gamma_y D^\alpha_\xi p_0 \right\| \leq C_{\alpha, \beta, \gamma} \sum_{|\eta|+m=-|\alpha|+|\gamma|} \| e_\xi D^\eta \psi_y \|_m .$$

(4.1.9)

Indeed, for the $y$-derivatives,

$$\left\| \partial_y p_0 \right\| = \| e_\xi P(e_\xi D_j \partial_y \psi_y) \| \leq C \| e_\xi \partial_y \psi_y \| = C \| e_\xi \partial_x \psi_y \| .$$

Consider the $x$-derivatives,

$$D_j p_0 = D_j (e_\xi P(e_\xi(x)\psi_y)) = e_\xi (D_j - \xi_j) P(e_\xi(x)\psi_y)$$

$$= e_\xi \text{ad}_{D_j}(P)(e_\xi(x)\psi_y) + e_\xi P(e_\xi D_j \psi_y),$$

by the assumption, $\text{ad}_{D_j}(P)$ and $P$ are bounded on $L^2$, thus

$$\| D_j p_0 \| \leq C(\| \psi_y \| + \| D_j \psi_y \|).$$

For the $\xi$-derivatives, let $\partial_j = i \partial_{\xi_j}$ and $\| \cdot \|_s$ be the norm on $B(H_s)$ for $s \in \mathbb{R}$, a similar computation gives us

$$\partial_j p_0 = e_\xi \text{ad}_{x_j}(P)((e_\xi(x)\psi_y),$$

and $\text{ad}_{x_j}(P) \in B(H_{-1}, H_0)$, so

$$\| \partial_j p_0 \| \leq C \| e_\xi \psi_y \|_{-1} .$$

53
The (4.1.9) follows from induction. We continue to estimate the right hand side of (4.1.9), by Peeter’s Inequality (cf.\cite[Lemma 1.1.8]{19}):\[
\rho(x + y)^s \leq \rho(y)^s \rho(x)^{|s|}, \quad x, y \in \mathbb{R}^m, \quad s \in \mathbb{R},
\]
and the fact that \(\hat{\psi}\) has compact support, we compute:
\[
\left| \rho(\eta)^n (e_\xi \psi_y)(\eta) \right| = \rho(\eta)^n \left| \hat{\psi}(\eta - \xi) \right| \leq C \rho(\xi)^m.
\]
Therefore from (4.1.9), we obtain the \(L^2\)-norm estimate (for fixed \(\psi\))
\[
\left\| D^\beta_x D^\gamma_y D^\alpha_\xi p_0 \right\| \leq C_{\alpha\beta\gamma} \rho(\xi)^{-|\alpha|+|\gamma|}.
\] (4.1.10)
To prove the \(L^\infty\)-norm estimate, we follow the argument in \cite{2}. For \(t > 0\), define the dilation operator \((V_t f)(x) = t^{-m/2} f(t^{-1} x)\), put \(P_t = V_t^{-1} P V_t\). One can check
\[
(P_t)^{(\alpha)}_{(\beta)} = t^{|\beta|-|\alpha|} (P_0^{(\alpha)})_{(\beta)} t.
\]
Denote
\[
p_{0,t}(x, y, \xi) = e_{-\xi} P_t (e_\xi \psi_y)(x).
\]
Notice that the \(s\)-th Sobolev norm of \(V_t (e_\xi \psi_y) y\) is controlled by \(\rho(\xi/t)^s\), uniformly with respect to \(y, t\) and \(\xi\). Therefore the dilated symbol \(p_{0,t}\) has the \(L^2\)-norm estimate:
\[
\left\| D^\beta_x D^\gamma_y D^\alpha_\xi p_{0,t}(\cdot, y, \xi) \right\| \leq C_{\alpha\beta\gamma} t^{|\beta|-|\alpha|} \rho(\xi/t)^{-|\alpha|+|\gamma|}.
\] (4.1.11)
After the substitution \(\xi \mapsto t \xi\) with \(t = 1/\rho(x)\), we get
\[
\left\| D^\beta_x D^\gamma_y D^\alpha_\xi p_{0,t}(\cdot, y, t\xi) \right\| \leq C_{\alpha\beta\gamma} \rho^{|\beta|}.
\] (4.1.12)
Quoting the Sobolev embedding theorem, we conclude that the same estimate above holds for the $L^\infty$-norm (with the new constants $C_{\alpha\beta\gamma}$). Notice that

$$p_0(x, y, \xi) = p_{0,t}(x/t, y/t, t\xi),$$

we have arrived at the $L^\infty$-norm estimate,

$$\left| D_\xi^\beta D_y^\gamma D_\xi^\alpha p_0(x, y, \xi) \right| \leq C_{\alpha\beta\gamma} \rho(\xi)^{-|\alpha|}. \tag{4.1.13}$$

Apply [19, lemma 1.2.2], the symbol function $p(x, \xi)$ (defined in (4.1.8)) has the desired estimate

$$\left| p^{(\alpha)}_{(\beta)} \right| \leq C_{\alpha\beta} \rho(\xi)^{-|\alpha|}$$

where the constant $C_{\alpha\beta}$ depends only on the operator norm of $P^{(\alpha)}_{(\beta)}$. \hfill \rightline{□}

## 4.2 Deformations of pseudo differential operators

Let $M$ be a toric Riemannian manifold as in definition 3.0.2. We would like to apply the construction in section 2.2 to $\Psi(M)$, the algebra of pseudo differential operators on $C^\infty(M)$. Namely, the algebra of pseudo differential operators on the noncommutative manifold is just the image of $\Psi(M)$ under the deformation map $\pi^\Theta$ (see definition 2.2.1) can be apply. Since half of the pseudo differential operators are unbounded, the convergence of the twisted convolution in equation (2.1.9) needs to be re-investigate in the Fréchet topology discussed in the previous section.

**Definition 4.2.1.** Let $M$ be a closed manifold of dimension $m$. For any $d \in \mathbb{Z}$, denote by $\Psi^d(M)$, the space of pseudo differential operators of order $d$, consists of
all continuous operators $P : C^\infty(M) \to C^\infty(M)$ such that $P = \sum P_\alpha$ is a finite sum in which each $P_\alpha$ is a pseudo differential operator of order $d$ on $\mathbb{R}^m$ discussed in the previous section. More precisely, $P_\alpha = p(x, D) : C^\infty(O_\alpha) \to C^\infty(O_\alpha)$ (see (4.1.2)), where $O_\alpha \subset M$ is an open subset with a coordinate system $x : O_\alpha \to \mathbb{R}^m$, and the symbol $p(x, \xi)$, which is a smooth function satisfies the estimate (4.1.4). As usual, we put

$$\Psi(M) = \bigcup_{d \in \mathbb{Z}} \Psi^d(M), \quad \Psi^{-\infty}(M) = \bigcap_{d \in \mathbb{Z}} \Psi^d(M). \quad (4.2.1)$$

The symbol topology and the operator norm topology $\Psi^d(\mathbb{R}^m)$ discussed in previous section can be pasted together to Fréchet topologies on $\Psi^d(M)$, which are equivalent as shown in the previous section. Indeed, we choose an finite open cover $\{O_j\}$ for $M$ with a partition of unity $\{\varphi_j\}$, such that $\forall P \in \Psi(M)$, $P = \sum P_j = \sum \varphi_j P \varphi_j$, while each $P_j$ is a pseudo differential operator on $O_j$, thus the symbol and the operators semi-norms (cf. (4.1.4) and (4.1.6)) can be apply to $P$ as follows, for any $\alpha, \beta \in \mathbb{Z}_n^+$ (as in (4.1.6)),

$$\|P\|_{\alpha, \beta} = \sum \|P_j\|_{\alpha, \beta}.$$

Of course, the semi-norms depends on the choice of the open cover and the partition of unity, however the resulting Fréchet topology does not.

We now give an coordinate free way to describe the operator norm topology on each $\Psi^d(M)$. The following proposition was stated in [25], but the proof goes back to [1] and [14].

**Proposition 4.2.1.** Let $M$ be a closed manifold and $\Psi^0(M)$ be the space of all zero order pseudo differential operators. Given a operator $P : C^\infty(M) \to C^\infty(M)$, then
$P \in \Psi^0(M)$ if and only if for any finite collection of first order differential operators $\mathcal{F} = \{F_1, \cdots, F_l\}$, we have

$$\text{ad}F_l \cdots \text{ad}F_1(P) \in B(L^2(M))$$

(4.2.2)

where $\text{ad}F_j(P) = [F_j, P]$, $1 \leq j \leq l$.

**Proof.** The problem is local as explained above. In fact, choose an finite open cover $\{O_j\}$ for $M$ with a partition of unity $\{\varphi_j\}$, then $P = \sum_j P_j = \sum_j \varphi_j P \varphi_j$. We apply proposition 4.1.1 on each $P_j$ to show that $P_j$ is a pseudo differential operator on $\mathbb{R}^m$. Then the sum constitute a pseudo differential operator on the manifold $M$. 

The characterization can be generalized to pseudo differential operators of all integer orders.

**Corollary 4.2.2.** Let $M$ be a compact Riemannian manifold with associated Sobolev spaces $\{\mathcal{H}_s\}_{s \in \mathbb{R}}$ and $\Psi^s(M)$ be the space of pseudo differential operators of order $s$ in definition 4.2.1. A bounded operator $P : \mathcal{H}_s \to \mathcal{H}_0$ belongs to $\Psi^s(M)$ if and only if for any finite collection of first order differential operators $\mathcal{F} = \{F_1, \cdots, F_l\}$, we have

$$\text{ad}F_l \cdots \text{ad}F_1(P) \in B(\mathcal{H}_s, \mathcal{H}_0).$$

(4.2.3)

**Proof.** Let us fix an elliptic operator of order 1, called $Q$ which is self-adjoint with positive eigenvalues so that $Q^s$ is well-defined for all $s \in \mathbb{R}$. Write $P = PQ^{-s}Q^sP$, it remains to show that $PQ^{-s}$ is a zero order pseudo differential operator. By proposition 4.2.1 above, we need to prove that for any finite collection of first order differential operators $\mathcal{F} = \{F_1, \cdots, F_l\}$, $\text{ad}F_l \cdots \text{ad}F_1(PQ^{-s})$ is bounded on $\mathcal{H}_0$. An
easy induction argument let us reduce to prove only for one first order differential operator $F$, $\text{ad}F(PQ^{-s}) \in B(H_0)$. In fact,

$$\text{ad}F(PQ^{-s}) = \text{Pad}F(Q^{-s}) + \text{ad}F(P)Q^{-s},$$

since $\text{ad}F(Q^{-s})$ is a pseudo differential operator of order $-s$, thus $\text{ad}F(Q^{-s}) \in B(H_0, H_s)$, thus $\text{Pad}F(Q^{-s}) \in B(H_0)$ for $P \in B(H_s, H_0)$. The second term $\text{ad}F(P)Q^{-s}$ belongs to $B(H_0)$ as well, indeed, $\text{ad}F(P) \in B(H_s, H_0)$ by assumption, and $Q^{-s} \in B(H_0, H_s)$. Hence we have shown that $\text{ad}F(PQ^{-s}) \in B(H_0)$.

Let $s \in \mathbb{R}$, given $P \in \Psi^s(M)$ a pseudo differential operator of order $s$, and a finite collection of first order differential operators: $\mathfrak{H} = (F_1, \cdots, F_k)$. Observe that the iterated commutator: $[F_k, \cdots, [F_1, P]]$ still belong to $\Psi^0$, thus defines a bounded operator on $H_s \to H_0$. We define a family of semi-norms indexed by the pair $(s, \mathfrak{H})$:

$$\|P\|_{(s, \mathfrak{H})} = \|[F_k, \cdots, [F_1, P]]\|_{s, 0},$$

(4.2.4)

where on the right hand side, $\|\cdot\|_{s, 0}$ is the operator norm from $H_s(M) \to H_0(M)$.

**Proposition 4.2.3.** For each $d \in \mathbb{Z}$, the semi-norms $\|\cdot\|_{(s, \mathfrak{H})}$ defined in (4.2.4) make $\Psi^d(M)$ into a Fréchet space. In particular, if $d_1 < d_2$, then the inclusion $(\Psi^{d_1}(M), \|\cdot\|_{(d_1, \mathfrak{H})}) \to (\Psi^{d_2}(M), \|\cdot\|_{(d_1, \mathfrak{H})})$ is continuous.

**Proof.** Since the manifold is compact, we can pick an increasing countable sub-family inside $\|\cdot\|_{(s, \mathfrak{H})}$ to determine the topology. The completeness is guaranteed by 4.2.2.

\[\Box\]
Proposition 4.2.4. Let $\Psi^\infty = \bigcap_{s \in \mathbb{R}} \Psi^k$ be the space of smoothing operators. The for each $s \in \mathbb{R}$, $\Psi^\infty \subset \Psi^s$ is a closed subspace with respect to the Fréchet topology described above.

The well-know Kuranishi’s lemma states that the leading symbols of pseudo differential operators are are invariantly defined on the cotangent bundle, as a consequence, the spaces $\Psi^d(M)$ for all $d \in \mathbb{Z}$, are stable under the action of diffeomorphisms. In particular, given any diffeomorphism $\varphi : M \to M$, let $U_\varphi : C^\infty(M) \to C^\infty(M)$ be the pull-back operator, for any $P \in \Psi^d(M)$, then the conjugation $U_\varphi PU_\varphi^{-1}$ still belongs to $\Psi^d(M)$. Apply this fact to our toric noncommutative manifold $M$, we obtained that the Fréchet spaces $\Psi^d(M)$ ($d \in \mathbb{Z}$) become $T^n$-modules by the adjoint action: $orall t \in T^n, P \in \Psi^d(M)$

\[ t \cdot P \triangleq \text{Ad}_t(P) = U_tPU_{-t}, \tag{4.2.5} \]

where $U_t : C^\infty(M) \to C^\infty(M)$ is the pull-back action (cf. (3.1.1)) with respect to the isometry $t$. Before apply the Peter-Weyl theory in section 2, we have to show that the $T^n$-action is smooth on each $\Psi^d(M)$.

Proposition 4.2.5. Let $M$ be a compact Riemannian manifold with a $n$-torus action as before and $\mathcal{H}_v$ ($v \in \mathbb{R}$) be the family of Sobolev spaces of smooth functions on $M$. Let $t = (t_1, \ldots, t_n)$ be a coordinate on the torus $T^n$, and then $\{\partial_{t_1}, \ldots, \partial_{t_n}\}$ constitute a basis of its Lie algebra, the associated vector fields on $M$ are denoted by $\{X_1, \ldots, X_n\}$. Given $P \in \Psi^d(M)$ be a pseudo differential operator of order $d$, the operator-valued function $t \to \text{Ad}_t(P)$ is smooth in $t$. More precisely, for any finite collection of first order differential operators $\mathcal{F} = \{F_1, \cdots, F_k\}$ and any multi-index
\[ \mu = (\mu_1, \ldots, \mu_j), \text{ one can find and a constant } C \text{ and another finite collection of first order operators } \mathfrak{F}' \text{ depending on } \mathfrak{F} \text{ and } \mu \text{ such that:} \]

\[ \| \partial_t^\mu \text{Ad}_t(P) \|_{m, \mathfrak{F}} \leq C \| P \|_{(s, \mathfrak{F}')} \quad t \in \mathbb{T}^n. \quad (4.2.6) \]

**Proof.** Apply the product rule onto (4.2.5), we see that

\[ \partial_t^i (\text{Ad}_t(P)) = \text{Ad}_t(\text{ad}(X_i)(P)), \]

where \( \text{ad}(X_i) \) is the commutator: \( \text{ad}(X_i)(P) = [X_i, P] \). Similarly, the higher order partial derivatives are given by:

\[ \partial_{t_\mu_1} \cdots \partial_{t_\mu_j} (\text{Ad}_t(P)) = \text{Ad}_t \left( \text{ad}(X_{\mu_1}) \cdots \text{ad}(X_{\mu_j}) \cdot P \right) \quad (4.2.7) \]

For \( \mathfrak{F} = \{ F_1, \ldots, F_l \} \), a finite collection of vector fields on \( M \), one can quickly verify that

\[ \text{ad}(F_1) \cdots \text{ad}(F_l) (\partial_t^\mu \text{Ad}_t(P)) \]

\[ = \text{ad}(F_1) \cdots \text{ad}(F_l) \left( \text{Ad}_t \left( \text{ad}(X_{i_1}) \cdots \text{ad}(X_{i_j}) \cdot P \right) \right) \]

\[ = \text{Ad}_t \left( \text{ad}(\text{Ad}_t(F_1)) \cdots \text{ad}(\text{Ad}_t(F_l)) \text{ad}(X_{\mu_1}) \cdots \text{ad}(X_{\mu_j}) \cdot P \right) \]

Since the torus action is unitary,

\[ \| \partial_t^\mu \text{Ad}_t(P) \|_{s, \mathfrak{F}} \leq C \| P \|_{s, \mathfrak{F}'} \]

with

\[ \mathfrak{F}' = \{ F_1, \ldots, F_l, X_{\mu_1}, \ldots, X_{\mu_k} \}. \]

Same argument works as well when \( \partial_{i_1 \cdots i_k} \) is replaced by \( \partial^\mu \) for any multi-index \( \mu \). \( \square \)
Follows from the smoothness of the torus action, the right hand side of the isotypical decomposition

\[ P = \sum_{r \in \mathbb{Z}^n} P_r, \quad \forall P \in \Psi^d(M), \]

with \( P_r = \int_{\mathbb{T}^n} \text{Ad}_t(P)e^{-2\pi ir \cdot t} dt \), converges to \( P \) with respect to the Fréchet topology. Moreover, for each semi-norm \( \|\cdot\|_{\psi,\mathbb{B}} \), the sequence \( \|P_r\|_{\psi,\mathbb{B}} \) is of rapidly decay in \( r \).

Fixed a skew symmetric \( n \times n \) matrix \( \Theta \), for any pseudo differential operator of order \( d \), we define the deformation map \( \pi^{\Theta} \) as in definition 2.2.1:

\[ \pi^{\Theta}(P) \triangleq \sum_{r \in \mathbb{Z}^n} P_r U_{r,\Theta/2}, \tag{4.2.8} \]

where \( r \cdot \Theta/2 \) denotes the matrix multiplication in which \( r \) is a row vector. Of course, the description in (2.2.6) is still valid for pseudo differential operators.

For each \( d \in \mathbb{Z} \), we denote by \( \Psi^d(M_{\Theta}) \) the image of \( \Psi^d(M) \) under \( \pi^{\Theta} \), \( \Psi(M_{\Theta}) \) and \( \Psi^{-\infty}(M_{\Theta}) \) are the union and intersection as in (4.2.1) respectively. Due to the continuity of the map \( \pi^{\Theta} \) (cf. (2.2.8)) we see that the order of \( P \) is stable under the deformation:

**Lemma 4.2.6.** Let \( P \in \Psi^d(M) \) is a pseudo differential operator on \( M \) of order \( d \in \mathbb{Z} \). Then \( \pi^{\Theta}(P) : C^{\infty}(M) \to C^{\infty}(M) \) define in (4.2.8) extends to a bounded operator from \( H_s \to H_{s-d} \) for all \( s \in \mathbb{R} \).

We summarize some crucial properties of \( \Psi(M_{\Theta}) \) in the following proposition.

**Proposition 4.2.7.** Let \( \Theta \) be a \( n \times n \) skew symmetric matrix. The filtrated algebra \( \Psi(M) \) of all pseudo differential operators admits the following deformation. For
any pseudo differential operators $P$ and $Q$, order $d_1$ and $d_2$ respectively, the $\times_\Theta$ multiplication

$$\times_\Theta : \Psi^{d_1} \times \Psi^{d_2} \rightarrow \Psi^{s_1+s_2} \quad (4.2.9)$$

$$(P, Q) \mapsto P \times_\Theta Q = \sum_{r,l \in \mathbb{Z}^n} e^{\pi i \langle r, \Theta l \rangle} P_r Q_l \quad (4.2.10)$$

is well-defined. Due to the skew symmetric property of $\Theta$, the $\times_\Theta$ multiplication is compatible with the original $\ast$-operation in $\Psi(M)$, namely:

$$(P \times_\Theta Q)^\ast = Q^\ast \times_\Theta P^\ast. \quad (4.2.11)$$

Therefore $(\Psi(M), \times_\Theta)$ is a filtrated $\ast$-algebra. Follows from lemma 2.2.6 and proposition 2.2.7, we obtain that the deformation map $\pi^\Theta$ makes $(\Psi(M), \cdot)$ into a filtrated $\ast$-algebra, where $\cdot$ denotes the composition between operators. More explicitly, we have for any $P, Q \in \Psi(M)$,

$$\pi^\Theta(P \times_\Theta Q) = \pi^\Theta(P) \pi^\Theta(Q), \quad \pi^\Theta(P^\ast) = \pi^\Theta(P)^\ast. \quad (4.2.12)$$

Proof. Notice that composition of operators

$$\Psi^{s_1} \times \Psi^{s_2} \rightarrow \Psi^{s_1+s_2} : (P, Q) \mapsto PQ, \quad s_1, s_2 \in \mathbb{R},$$

satisfies the jointly continuity (cf. (2.1.7)) with respect to the operator norms, plus the rapidly decay property in the components of the isotypical decomposition, we conclude that the infinite sum in the right hand side of (4.2.10) converges with respect to the Fréchet topology, moreover, the limit is indeed a pseudo differential operator via corollary 4.2.2. The rest of the proposition follows from the deformation theory in section 2, for instance, (4.2.12) is a straightforward generalization of lemma 2.2.6 and
proposition 2.2.7. At last, the deformation \( \pi^\Theta \) has an inverse \( \pi^{-\Theta} \) (cf. 2.2.4), therefore, it is an filtrated \(*\)-algebra isomorphism between \( (\Psi(M), \times_\Theta) \) and \( (\Psi(M_\Theta), \cdot) \).

\[ \square \]

### 4.3 Deformation of classical pseudo-differential operators

The deformation \( \pi^\Theta(P) \) of a pseudo differential operator is not pseudo differential in general. Indeed, \( \pi^\Theta(P) \) is not pseudo-local (cf. [19, lemma 1.2.7]) in general due to the appearance of the unitary operators \( U_{r,\Theta/2} \) in (4.2.8). Nevertheless, the deformation map \( \pi^\Theta \) maps \( \Psi^{-\infty}(M) \) into itself. Therefore, modular smoothing operators has the same meaning in both the commutative and noncommutative settings.

**Lemma 4.3.1.** Let \( M \) be a compact smooth manifold without boundary. Given \( P : C^\infty(M) \to C^\infty(M) \) a continuous (with respect to the Fréchet topology given by the partial derivatives in local coordinate) linear operator with the smoothing property that for any \( s, t \in \mathbb{R} \), \( P \) can be extended to a bounded operator between associated Sobolev spaces: from \( \mathcal{H}^s \) to \( \mathcal{H}^t \). In other words, there exits a constant \( C_{s,t} \) such that for all \( f \in C^\infty(M) \)

\[ \| Pf \|_s \leq C_{s,t} \| f \|_t . \]

Then the the distributional kernel of \( P \), \( K(x, y) \) over \( M \times M \), belongs to \( C^\infty(M \times M) \).

**Proof.** The proof can be achieved by applying the Sobolev lemma to the estimate in [19, Lemma 1.2.9]

\[ \square \]

Follows from the continuity of the deformation map \( \pi^\Theta \) on \( B(\mathcal{H}_s, \mathcal{H}_{s'}) \) for any \( s, s' \in \mathbb{R} \), we have

63
Proposition 4.3.2. Let $\mathcal{H}$ be the Hilbert space of $L^2$-functions on a toric manifold $M$. As a $\mathbb{T}^n$-smooth subspace of $B(\mathcal{H})$, $\Psi^{-\infty}(M)$ is stable under the deformation map $\pi^\Theta$, that is

$$\pi^\Theta \left( \Psi^{-\infty}(M) \right) \subset \Psi^{-\infty}(M)$$

We define the deformation of classical pseudo differential operators $\text{CL}(M_\Theta)$ to be the quotient:

$$\text{CL}(M_\Theta) = \Psi^\infty(M_\Theta)/\Psi^{-\infty}(M) = \pi^\Theta(\Psi^\infty(M))/\Psi^{-\infty}(M) \quad (4.3.1)$$

$$= \pi^\Theta \left( \Psi^\infty(M)/\Psi^{-\infty}(M) \right) = \pi^\Theta(\text{CL}(M)) \quad (4.3.2)$$
CHAPTER 5
WIDOM’S PSEUDO DIFFERENTIAL CALCULUS

In this section, we will briefly recall Widom’s symbol calculus. Most of the proofs in this section are served as a remarks to explain the notations. The complete version is provided in Widom’s work [28] and [27].

5.1 The symbol calculus

We start with some basic fact about the symmetrized covariant derivatives \( \partial \). Let Sym denote the symmetrization of tensors in call their indices, and put the symmetric tensor \( \otimes \):

\[
    u \otimes v \triangleq \text{Sym}(u \otimes v) \quad (5.1.1)
\]

then

\[
    \text{Sym}(\nabla^k (u \otimes v)) = \sum_{j=0}^{k} \binom{k}{i} \text{Sym}(\nabla^j u) \otimes \text{Sym}(\nabla^{k-j} v). \quad (5.1.2)
\]

Finally, we will denote the symmetrized covariant derivative by \( \partial \), that is

\[
    \partial^k u = \text{Sym}(\nabla^k u). \quad (5.1.3)
\]
We have the higher order product rule:

\[
\partial^k(u_1 \cdots u_r) = \sum_{i_1 \cdots i_r} \frac{k!}{i_1 \cdots i_r} \partial^{i_1} u_1 \otimes \cdots \otimes \partial^{i_r} u_r,
\]  

(5.1.4)

where the sum is over \( i_1 + \cdots + i_r = k \).

Pseudo differential operators on a closed (compact without boundary) manifold is reviewed in the previous section while the space of symbols is recalled in definition 3.4.1. Moreover, to keep track of the degree, we new parameter \( t \in \mathbb{R} \), whose power determines the degree of the symbol.

**Definition 5.1.1.** A dilation of a pseudo differential operator \( P \) is a family of pseudo differential operators \( P_t \) with \( t \in [1, \infty) \) given by dilation of the symbols functions, in local coordinate:

\[
p_t(x, \xi) = p(x, \xi/t).
\]  

(5.1.5)

We now define the symbol map \( \sigma \).

**Definition 5.1.2.** Let \( M \) be a smooth manifold with a linear connection \( \nabla \), and \( \ell(x, y) \in C^\infty(T^*M \times M) \) be a phase function in definition 3.3.1 with respect to \( \nabla \). Denote by \( \psi_\Delta \in C^\infty(M \times M) \) a cut-off function such that \( \psi_\Delta = 1 \) is equal to 1 on a small neighborhood of the diagonal and also \( \text{supp} \psi_\Delta \) is still closed to the diagonal so that \( \psi_\Delta(x, y) \neq 0 \) implies \( d_y \ell(\xi_x, y) \neq 0 \) for all \( \xi_x \neq 0 \). For any pseudo differential operator \( P : C^\infty(M) \to C^\infty(M) \), the symbol \( \sigma(P) \) of \( P \) is a smooth function on \( T^*M \):

\[
\sigma(P)(\xi) = P \psi_\Delta(x, y) e^{i\ell(\xi_x, y)} \bigg|_{y=x},
\]  

(5.1.6)

where the operator \( P \) acts on the \( y \) variable.
Remark. Up to a smoothing operator, $\sigma(P)$ is independent of the choice of the cut-off function $\psi_\Delta$ and the phase function $\ell$.

Equipped with a phase function $\ell$, we can establish a Taylor’s expansion formula for smooth functions on $M$. If we fixed a point $y \in M$, consider $\ell(\xi, y)$, which is a linear functional on the fiber $T^*_y M$. Therefore, $\ell(\cdot, y)$ can be thought as vector field on $M$ (supported near by the point $y$). With the interpretation, we denote:

$$\ell(\cdot, y)^k = \ell(\cdot, y) \otimes \cdots \otimes \ell(\cdot, y), \quad (5.1.7)$$

which is a symmetric $k$-th order contravariant tensor field and so the pairing

$$\nabla^k f(x_0) \cdot \ell(x_0, x)^k \quad (5.1.8)$$

is well-defined and, by the symmetry of $\ell(x_0, x)^k$,

$$\nabla^k f(x_0) \cdot \ell(x_0, x)^k = D^k f(x_0) \ell(x_0, x)^k, \quad (5.1.9)$$

where $\partial^k$ is the symmetrization of $\nabla^k$ as before. Now we are ready to state the Taylor’s expansion formula.

**Proposition 5.1.1.** For each point $x_0 \in M$ and each integer $N$ the function

$$f(x) - \sum_{j=0}^{N} \frac{1}{j!} \nabla^j f(x_0) \cdot \ell(x_0, x)^j$$

vanished to order $N + 1$ at $x_0$.

For a function $p(x, \xi) \in C^\infty(T^*M)$, we define $D^k p$ to be the $k$-th derivative of $p$ in the direction of the fibers of $T^*M$. We think of $p(x, \cdot)$ as a function on $T^*_x M$ when the base point $x$ is fixed, then the $k$-th derivative $D^k f$ gives rise to a $k$-linear
functional on $T^*_x M$ and so can be identified with an element of $\otimes_k T_{x_0} M$. Hence $D^k p$ is a contravariant $k$-tensor. This is the analogue of $\partial^k_\xi$. Indeed, the analogy can be demonstrated by an example. Given $f \in C^\infty(M)$, $\nabla^k f$ is a covariant $k$-tensor, thus the pairing $\nabla^k f \cdot D^k p$ produces a scalar. In a local coordinate $(x_1, \ldots, x_n)$, then $\xi_x = \sum_{j=0}^n \xi_j dx_j |_x$, we denote $\ell_j(x, y) = \ell(dx_j |_x, y)$, then for any indices $j_1, \ldots, j_k$:

$$k! \partial_{j_1} \cdots \partial_{j_k} p(\xi_x) = \partial_y^k \ell_{j_1}(x, y) \cdots \ell_{j_k}(x, y) \cdot D^k p(\xi_y) \bigg|_{y=x}.$$  \hfill (5.1.11)

The analogue of $\partial_x$ is not that obvious. Indeed, we define a covariant derivative, still denoted by $\nabla$, $\nabla : C^\infty(T^* M) \to C^\infty(B^1_0 M)$:

$$\nabla^j p(\xi_x) \triangleq \nabla^j p(d_y \ell(\xi_x, y)) \bigg|_{y=x}, \quad j \in \mathbb{N}. \hfill (5.1.12)$$

Although, the function $\ell$ is not unique, all derivatives at $y = x$ are the same. Thus $\nabla^j p$ in (5.1.12) is a well-defined covariant $j$-tensor. Observe that

$$d_y \ell(\xi_x, y) \bigg|_{y=x} = \xi_x,$$  \hfill (5.1.13)

It is not difficult to see that the horizontal and the vertical derivatives $\nabla$ and $D$ in definitions 3.4.3 and 3.4.2 commute, thus for any smooth functions $p \in C^\infty(T^* M)$, the mixed derivatives

$$\nabla^k D^j p(\xi_x) \triangleq \nabla_y^k D^j p(d_y \ell(\xi_x, y)) \bigg|_{y=x}.$$  \hfill (5.1.14)

gives rise to a tensor of type $(j, k)$. Here the subscript $y$ in $\nabla_y$ and $d_y$ indicates the variable with respect to which the derivatives are taken.

The symbol calculus are based on the following two lemmas. The proofs provided contain only the formal calculation that leads us to the asymptotic expansion. We
refer to [27] for the complete version. The first one is so-called “the Leibniz” formula for pseudo differential operators.

**Lemma 5.1.2.** Let $f \in C^\infty(M)$ supported in a small neighborhood of $x \in M$ and $P$ be a pseudo differential operator with symbol $\sigma(P) \in C^\infty(T^*M)$.

$$P \left( f(y) e^{it\ell(x,y)} \right) \bigg|_{y=x} \sim \sum_{m=0}^{\infty} \frac{(it)^{-m}}{m!} \nabla^m f(x) D^m \sigma(P)(\xi_x).$$

(5.1.15)

Since $D^m \sigma(P)(\xi_x)$ is a symmetric tensor, the $\nabla^m f$ above can be replaced by its symmetrization $\partial^m f$.

**Proof.** Near a point $x \in M$, recall that $\sigma(P)(\xi_x)$ can be computed by:

$$\sigma(P)(\xi_x) = (P_y e^{it\ell(x,y)}) \bigg|_{y=x}.$$

and:

$$D^m e^{it\ell(x,y)} = (it)^k e^{it\ell(x,y)} \ell(x,y)^m$$

(5.1.16)

thus

$$D^m \sigma(P) (t \xi_x) = D^m (P_y e^{it\ell(x,y)}) \bigg|_{y=x} = (P_y e^{it\ell(x,y)})(it)^m \ell(x,y)^m \bigg|_{y=x}$$

(5.1.17)

Apply the Taylor expansion to $f$ near the point $x$:

$$f(y) \sim \sum_{m=0}^{N} \nabla^m f(x) \ell(x,y)^m + g_M(x,y),$$

where the function $g_M(x, \cdot)$ vanishes to order $N + 1$ at $x$. Hence:

$$P_y \left( f(y) e^{it\ell(x,y)} \right) \bigg|_{y=x} = \sum_{m=0}^{M} P_y(\nabla^m f(x), \ell(x,y)^m) e^{it\ell(x,y)}$$

$$+ P_y(g_M(x,y)) e^{it\ell(x,y)}.$$

69
Now apply (5.1.17)
\[ P_y(\langle \nabla^m f(x), \ell(x,y)^m \rangle e^{it\ell(x,y)}) = (it)^{-m} \langle \nabla^m f(x), P_y(e^{it\ell(x,y)}\ell(x,y)^m) \rangle = (it)^{-m} \nabla^m f(x) \cdot D^m \sigma(P)(\xi_x), \]
which gives the expansion (5.1.15). It remains to show that the remainder belong to symbols of order \( p - N \), where \( p \) is the order of \( \sigma(P) \). That is, in the local coordinate \((x,dx)\) around \( x \),
\[ \frac{\partial^j}{\partial x^j} \frac{\partial^k}{\partial \xi^k} P_y(g(x,y)e^{it\ell(x,y)}) \bigg|_{y=x} = O(t^{-M}(1 + |\xi|^{p-M-|k|})). \]
This is proved in [27, Prop. 2.5].

Main technical lemma:

**Lemma 5.1.3.** Let \( f, \varphi \in C^\infty(M) \), with \( \varphi \) real-valued such that \( d\varphi \neq 0 \) on \( \text{supp} \ f \), and \( P \) is a pseudo differential operator with symbol \( p \), if the connection \( \nabla \) is torsion free, then
\[ e^{-it\varphi(x)} P_y \left( f(y)e^{it\varphi(y)} \right) \bigg|_{y=x} \sim \sum_{k,m_1,\ldots,m_k \geq 2} \frac{(it)^{k-m}}{k!m_0!\cdots m_k!} \nabla^{m_0} f \nabla^{m_1}(t\varphi) \cdots \nabla^{m_k}(t\varphi) D^m p(t\varphi), \quad (5.1.18) \]
where \( m = m_0 + \cdots + m_k \). Since \( D\sum m_i p(t\varphi) \) is a symmetric tensor, \( \nabla^{m_0} f \nabla^{m_1}(t\varphi) \cdots \nabla^{m_k}(t\varphi) \) can be replaced by the symmetrization:
\[ \partial^{m_0} f \boxtimes \partial^{m_1}(t\varphi) \boxtimes \cdots \boxtimes \partial^{m_k}(t\varphi), \]
here \( \boxtimes \) is the symmetrized tensor product:
\[ s \boxtimes t \triangleq \text{Sym}(s \otimes t). \quad (5.1.19) \]
Remark. The fact that the summation is over $m_1, \ldots, m_k \geq 2$ is crucial in applications.

Proof. Let $L(x, y) = \varphi(y) - \varphi(x) - \ell(d\varphi|_{x, y})$, use the linearity of $\ell$ in the first variable,
\[
t(\varphi(y) - \varphi(x)) = tL(x, y) + \ell(td\varphi|_{x, y}).
\]
Recall Taylor expansion:
\[
e^s = \sum_{k=0}^{N} \frac{s^k}{k!} + \frac{s^N}{(N-1)!} \int_0^1 u^{N-1} e^{(1-u)s} du.
\]
\[
P_y \left( f(y) e^{it(\varphi(y) - \varphi(x))} \right) = \sum_{k=0}^{N} (it)^k \frac{1}{k!} P_y \left( f(y)L(x, y)^k e^{it(t\varphi|{x,y})} \right) + R_N,
\]
where the remainder $R_N$ is given by
\[
\frac{(it)^N}{(N-1)!} \int_0^1 u^{N-1} P_y \left( f(y)L(x, y)^{N} e^{i(1-u)tL(x, y)+it(\ell(d\varphi|_{x,y}))} \right) du
\]
By previous lemma:
\[
P_y \left( f(y)L(x, y)e^{it\ell(t\varphi|_{x,y})} \right) \bigg|_{y=x} \sim \sum_{m=0}^{\infty} \frac{(it)^{m}}{m!} \partial^m \left( f(y)L(x, y)^k \right) \bigg|_{y=x} D^m p^{(td\varphi|_{x})}
\]
Product rule of $\partial^m$:
\[
\partial^m \left( f(y)L(x, y)^k \right) = \sum_{j_0 + \ldots + j_k = m} \frac{m!}{j_0! j_1! \ldots j_k!} \partial^{j_0} f(y) \partial^{j_1} L(x, y) \ldots \partial^{j_k} L(x, y).
\]
Finally, notice that
\[
L(y, y) = 0, \quad d_y L(x, y) \big|_{y=x} = 0,
\]
hence any term with any of $j_1, \ldots, j_k$ equal to 0 or 1 vanishes. Moreover, when $j_k \geq 2$, we have $\partial^{j_k} L(x, y) = \partial^{j_k} \varphi(y)$ for $\partial^{j} \ell(x, y) \big|_{y=x} = 0$ when $j \geq 2$. \qed
Given two pseudo differential operators $P$ and $Q$ with symbols $p$ and $q$ respectively, the symbol of the composition $PQ$ is given by an asymptotic product $\sum_j a_j(p, q)$, where the $a_j(\cdot, \cdot)$ are bi-differential operators, we first describe the symbols of $a_j$. Denoted by $(\alpha)_k, (\beta)_k, \ldots$, the $k$-tuples with positive integer components, namely $(\alpha)_k = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k$, with

$$|\alpha)_k| = \alpha_1 + \cdots + \alpha_k, \quad (\alpha)_k! = \alpha_1! \cdots \alpha_k!,$$

$(\alpha)_k \geq s$ means $\alpha_1 \geq s, \ldots, \alpha_k \geq s$.

The symbols of $a_j$ are linear combinations of the covariant tensors:

$$\rho_{(\alpha)_k, (\beta)_k} = \partial^{\alpha_1} \partial^{\beta_1} \ell(\xi, x) \boxtimes \cdots \boxtimes \partial^{\alpha_k} \partial^{\beta_k} \ell(\xi, x). \quad (5.1.21)$$

It is very important to keep in mind that

$$\partial^{i+k} \neq \partial^i \partial^k,$$

in general, thus there are two groups of indices above, namely, the $\alpha$’s and the $\beta$’s, the symmetric tensor $\boxtimes$ product denotes the symmetrization inside those groups.

**Proposition 5.1.4.** Keep the notations as above, for any two pseudo differential operators $P$ ans $Q$ with $p = \sigma(P)$ and $q = \sigma(Q)$, then $\sigma(PQ) = p \star q$ has the following asymptotic form:

$$p \star q \sim \sum_{j=0}^{\infty} a_j(p, q), \quad (5.1.22)$$

where $a_j(\cdot, \cdot)$ are bi-differential operators reducing the total degree by $j$, namely, for any $d, d' \in \mathbb{Z}$,

$$a_j(\cdot, \cdot) : S\Sigma^d(M) \times S\Sigma^{d'}(M) \to S\Sigma^{d+d'-j}(M).$$
Therefore \( \simeq \) means that if we truncate the sum to the first \( K \) terms, then the remainder belongs to symbol of order \( d + d' - K \), where \( d \) and \( d' \) are the order of \( P \) and \( Q \) respectively.

Taking the symbols \( \rho_{(\alpha)_{k},(\beta)_{k}} \) in equation (5.1.21) into account, the \( a_{j} \)s are given by:

\[
a_{j}(p, q) = \sum_{k \geq 0, (\beta)_{k} \geq 2, j = k-l-|\alpha|_{k}-|\beta|_{k} \geq 2} \frac{i^{j}}{k!(\alpha)_{k}!(\beta)_{k}!!}(D^{j+|\alpha|_{k}}p)(\partial^{j}D^{(\beta)_{k}}q) \cdot \rho_{(\alpha)_{k},(\beta)_{k}} \tag{5.1.23}
\]

The contraction is indicated by the indices, that is, \( \partial^{j} \) paired with \( D^{j} \), while \( D^{(\beta)_{k}} \) and \( D^{(\alpha)_{k}} \) contract with the \( \beta \) and \( \alpha \) parts, respectively, in \( \rho_{(\alpha)_{k},(\beta)_{k}} \).

Remark.

1) In (5.1.23), the order of the multiplication of \( p, q \) and \( \rho \) is arranged in such a way that it is works for pseudo differential operators on vector bundles.

2) Since \( \partial^{j}l(\xi, y)|_{y=x} = 0 \) for \( j \geq 1 \), the \( \rho_{(\alpha)_{k},(\beta)_{k}} \) is equal to zero if one of the \( \alpha \) components is zero. Hence the summation in (5.1.23) is over \( (\alpha)_{k} \in \mathbb{Z}_{+}^{k} \).

3) The first a few \( a_{j} \) are

\[
a_{0}(p, q) = pq, \tag{5.1.24}
\]
\[
a_{1}(p, q) = -iDp\partial q, \tag{5.1.25}
\]
\[
a_{2}(p, q) = -\frac{1}{2}D^{2}p\partial^{2}q - \frac{1}{2}\partial\partial^{2}D\partial^{2}q. \tag{5.1.26}
\]

Proof. We would like to compute \( (P_{t}Q_{t})_{y}e^{it\ell(\xi, y)}|_{y=x} \), where \( P_{t}, Q_{t} \) are dilations of \( P \) and \( Q \) in the definition 5.1.1,

\[
(P_{t})_{y} \left( ((Q_{t})_{z}e^{it\ell(\xi, z)-\ell(\xi, y)})|_{z=y}e^{it\ell(\xi, y)} \right)|_{y=x} \tag{5.1.27}
\]
denote by $h(y) = Q_x e^{it(\ell(x,z) - \ell(x,y))}|_{z=y}$, apply lemma 5.1.2, (5.1.27) above becomes:

$$\sum_{j=0}^{\infty} \frac{(it)^j}{j!} \partial^j h(y) D^m p(t\xi_x)$$  \hspace{1cm} (5.1.28)

while $h(y)$ has a expansion according to lemma 5.1.3 (with $f(x) = 1$ and $\varphi(z) = \ell(\xi_x, z)$):

$$h(y) \sim \sum_{m_1, \ldots, m_k \geq 2} \frac{(it)^{k-\sum m_i}}{k! m_1! \cdots m_k!} \partial^{m_1} \ell(\xi_x, y) \otimes \cdots \otimes \partial^{m_k} \ell(\xi_x, y) D^{\sum m_i} q(t d\ell(\xi_x, y)),$$

again by the higher order product rule of the symmetrized covariant derivative $\partial$ (cf. (5.1.4)), we arrive at the desired expansion, put $j = j_0 + \cdots + j_k$, $m = m_1 + \cdots + m_k$:

$$\sum_{m_1, \ldots, m_k \geq 2} \frac{(it)^{k-\sum m-j}}{k! j_0! \cdots j_k! m_0! \cdots m_k!} \partial^{j_1} \partial^{m_1} \ell(\xi_x, y) \otimes \cdots \otimes \partial^{j_k} \partial^{m_k} \ell(\xi_x, y) \otimes \partial^{j_0} D^m q(d_y \ell(\xi_x, y)) \big|_{y=x} D^j p(\xi_x).$$

5.2 Quantization map $\text{Op}$ and the Schwartz Kernels

**Definition 5.2.1.** The map sending symbols to pseudo differential operators is called the quantization map:

$$\text{Op} : S^\Sigma^d(M) \to \Psi^d(M), \ \forall d \in \mathbb{Z}.$$
Let $\psi_\Delta$ be the cut-off function used in definition 5.1.2. For any \( f \in C^\infty(M) \), put \( y = \exp_x Y \),
\[
(\text{Op}(p)f)(x) = \frac{1}{(2\pi)^m} \int_{T_x^*M} \int_{T_x^*M} e^{-i\langle \xi_x, Y \rangle} p(\xi_x) \psi_\Delta(x, y) f(\exp_x Y) dY d\xi,
\]
where \( m \) is the dimension of the manifold \( M \), \( \langle \xi_x, Y \rangle \) is the canonical pairing: \( T_x^*M \times T_x^*M \to \mathbb{R} \), and \( dY, d\xi \) denote the densities that are dual to each other. Different choices of the cut-off function \( \psi_\Delta \) give rise the same quantization map modular smoothing operators.

The connection between the Schwartz kernel and of an pseudo differential operator and its symbol is given in [27, Theorem 5.7], moreover the proof explains the fact that the symbol map \( \sigma \) and the quantization map \( \text{Op} \) are inverse to each other upto smoothing operators(or symbols). Let us recall only the result here.

Let \( (M, g) \) be a \( m \) dimensional closed Riemannian manifold with the metric tensor \( g \). The canonical \( 2m \)-form on the cotangent bundle \( T^*M \) is given in local coordinates \( (x, \xi) \) by
\[
\Omega = dx_1 \wedge \cdots \wedge dx_m \wedge d\xi_1 \wedge \cdots \wedge d\xi_m,
\]
while \( \Omega = dg d\xi_{x,g^{-1}} \) with the volume form \( dg = (\det g) dx_1 \wedge \cdots dx_m \) and
\[
d\xi_{x,g^{-1}} = (\det g)^{-1} d\xi_1 \wedge \cdots \wedge d\xi_m
\]
defines a measure on the fiber \( T^*_x M \).

**Proposition 5.2.1.** Keep the notations as above, let \( P \) be pseudo differential operator \( P \) whose order is less than \( m \), the dimension of the manifolds such that the Schwartz
kernel function \( k_P(x, y) \) exists, then the Schwartz kernel of the dilation \( P_t \) (see definition 5.1.1) on the diagonal is given by

\[
k_{P_t}(x, x) = \frac{t^m}{(2\pi)^m} \int_{T^*_M} \sigma(P) d\xi_{g^{-1}} + O(t^{-N}), \; \forall N \in \mathbb{N}. \tag{5.2.4}
\]

In particular, we obtain the trace formula for \( P_t \):

\[
\text{Tr}(P_t) = \frac{t^m}{(2\pi)^m} \int_{T^*_M} \sigma(P) \Omega + O(t^{-N}), \; \forall N \in \mathbb{N}, \tag{5.2.5}
\]

where \( \Omega \) is the canonical \( 2m \)-form defined in (5.2.2).

We refer to [27, Theorem 5.7] for the proof.
CHAPTER 6
DEFORMATION OF THE SYMBOL CALCULUS

Let $M$ be a compact Riemannian manifold whose isometry group contains a $n$-torus and $\Psi(M)$ is the algebra of pseudo differential operators action on $C^\infty(M)$. We have seen that the deformation $\Psi(M_\Theta)$ is the image under map $\pi^\Theta$. Given two deformed pseudo differential operators $\pi^\Theta(P)$ and $\pi^\Theta(Q)$ with symbols $p$ and $q$ respectively, the main result of this section is to deformed the asymptotic symbol product $*$ (defined in equation (5.1.28)) into $*_{\Theta}$, such that

$$\pi^\Theta(P)\pi^\Theta(Q) - \pi^\Theta(\text{Op}(p \ast_{\Theta} q))$$

belongs to $\Psi^{-\infty}(M)$, where $\text{Op}$ is the quantization map in definition 5.2.1. By a formal calculation, we claim that

$$\sigma(P \times_{\Theta} Q) = p \ast_{\Theta} q \sim \sum_{j=0}^{\infty} a_j(p, q)_{\Theta},$$

where

$$a_j(p, q)_{\Theta} = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_{\Theta}(\mu, \nu)a_j(p_\mu, q_\nu), \text{ with } \chi_{\Theta}(\mu, \nu) = e^{2\pi i (\mu, \Theta \nu)}. \quad (6.0.1)$$
Indeed,

\[
\sigma(P \times_\Theta Q) = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu) \sigma(P_\mu Q_\nu) \sim \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu) p_\mu \ast q_\mu \\
\sim \sum_{\mu, \nu \in \mathbb{Z}^n} \sum_{j=0}^{\infty} \chi_\Theta(\mu, \nu) a_j(p_\mu, q_\nu) \sim \sum_{j=0}^{\infty} \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_\Theta(\mu, \nu) a_j(p_\mu, q_\nu) \\
= \sum_{j=0}^{\infty} a_j(p, q)_\Theta.
\]

The summation \( \sum_{j=0}^{\infty} \) simply means if we truncate the sum to first \( N \) terms, the remainder belongs to symbol of order \(-J_N\), where \( J_N \to \infty \) as \( N \to \infty \). Therefore \( \sum_{j=0}^{\infty} \) behaves like a finite sum, thus we have no problem to exchange the order of \( \sum_{\mu, \nu \in \mathbb{Z}^n} \) and \( \sum_{j=0}^{\infty} \). With respect to the topologies of the symbols \( S\Sigma(M) \) and pseudo differential operators \( \Psi(M) \) discussed in section 4.1, the symbol map \( \sigma \) and the quantization map \( \text{Op} \) are continuous, thus it is valid to move \( \sigma \) and \( \text{Op} \) in or out of the summation \( \sum_{\mu, \nu \in \mathbb{Z}^n} \). It remains to show that The isotypical decomposition of the operators and the symbols are compatible, namely \( \sigma(P_\mu) = p_\mu \) and \( \sigma(Q_\nu) = q_\nu \). This follows from the equivariant properties of \( \sigma \) and \( \text{Op} \) which will be proved immediately.

**Proposition 6.0.2.** Let \( \varphi : M \to M \) be an isometry, \( U_\varphi : C^\infty(M) \to C^\infty(M) \) is the pull-back action as in (3.1.1), for any pseudo differential operator \( P \), we denote \( \Psi(M) \) by \( \text{ad}_\varphi(P) = U_\varphi PU_\varphi^{-1} \). For any symbol \( p \in S\Sigma(M) \), we have \( \text{ad}_\varphi(\text{Op}(p)) - \text{Op}(U_\varphi(p)) \) is a smoothing operator, where \( U_\varphi(p)(\xi_x) = p((\varphi^{-1})^*\xi_x) \) as in (3.4.2).

**Proof.** Since \( \varphi \) is an isometry, we have seen before that \( \forall f \in C^\infty(M), Y \in T^*_{\varphi^{-1}x}M \) which is so close to the origin that the exp map is a diffeomorphism,

\[
(U_\varphi f)(\exp_{\varphi^{-1}x} Y) = f(\varphi(\exp_{\varphi^{-1}x} Y)) = f(\exp_x \varphi_* Y)
\]
Let $P = \text{Op}(p)$, put $x' = \varphi^{-1}x$, $z = \exp_{x'}Z$, 

\[
(U_{\varphi}PU_{\varphi^{-1}}f)(x) = (PU_{\varphi^{-1}}f)(\varphi^{-1}x)
\]

\[
= \frac{1}{(2\pi)^m} \int_{T^*_xM} \int_{T^*_xM} e^{-i(\eta_{x'Z})p(\eta_{x'})\psi_{\Delta}(x', z)}(U_{\varphi^{-1}}f)(\exp_{x'}Z)dZd\eta_{x'}
\]

\[
= \frac{1}{(2\pi)^m} \int_{T^*_xM} \int_{T^*_xM} e^{-i(\eta_{x'Z})p(\eta_{x'})\psi_{\Delta}(x', z)}f(\exp_{x'}\varphi Z)dZd\eta_{x'}.
\]

Consider the substitution $\xi_x = \varphi^*\eta_{x'}$, and $Y = \varphi^*Z$ (with $Z \in T^*_xM$, thus $\langle \xi_x, Y \rangle = \langle \eta_{x'}, Z \rangle$ for $\varphi$ is an isometry. Also we have $dYd\xi_x = dZd\eta_{x'}$ because they are pairs of dual densities. Therefore the computation above can be continued:

\[
(U_{\varphi}PU_{\varphi^{-1}}f)(x) = \frac{1}{(2\pi)^m} \int_{T^*_xM} \int_{T^*_xM} e^{i(\xi_x, Y)p((\varphi^{-1})^*\xi_x)\tilde{\psi}_{\Delta}(x, y)}f(\exp_xY)dYd\xi_x
\]

\[
= \frac{1}{(2\pi)^m} \int_{T^*_xM} \int_{T^*_xM} e^{i(\xi_x, Y)U_{\varphi}(p)(\xi_x)\tilde{\psi}_{\Delta}(x, y)}f(\exp_xY)dYd\xi_x
\]

\[
\sim (\text{Op}(U_{\varphi}(p))f)(x)
\]

where $y = \exp_xY$ and

\[
\tilde{\psi}_{\Delta}(x, y) \triangleq \psi_{\Delta}(\varphi^{-1}x, \varphi^{-1}y)
\]

which is a valid cut-off function for the quantization map. This explains the appearance of the $\sim$ (upto smoothing) in the last line of the calculation above. 

The construction of the whole symbol calculus is completed after expressing each $a_j(p, q)_{\Theta}$ via our deformed tensor calculus developed before.
Proposition 6.0.3. Keep the notations in proposition 5.1.4. Let $M$ be a closed Riemannian manifold with a $n$-torus action so that the previous deformation machinery applies. Let $\pi^\Theta(P)$ and $\pi^\Theta(Q)$ be the deformation of pseudo differential operators. Denote $p = \sigma(P)$ and $q = \sigma(Q)$, then

$$\pi^\Theta(P)\pi^\Theta(Q) \sim \pi^\Theta(\text{Op}(p \otimes q)),$$

with

$$p \otimes q \sim \sum_{j=0}^\infty a_j(p, q),$$

where the bi-differential operators $a_j(\cdot, \cdot)$ are the deformation of $a_j(\cdot, \cdot)$ in proposition 5.1.4:

$$a_j(\cdot, \cdot)_\Theta = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi^\Theta(\mu, \nu) a_j(\mu, \nu), \quad \chi^\Theta(\mu, \nu) = e^{i\langle \mu, \Theta \nu \rangle}.$$

Moreover, $a_j(p, q)_\Theta$ can be obtained from $a_j(p, q)$ (in equation (5.1.23)) by replace the pointwise tensor product and contraction by the deformed version $\otimes$ and $\times_\Theta$ (see (3.2.6) and (3.2.4)), therefore

$$a_j(p, q)_\Theta = \sum_{k \geq 0, (\beta)_k \geq 2} \frac{j^i}{k!(\alpha)_k!(\beta)_k}! \rho_{(\alpha)_k, (\beta)_k} \otimes_\Theta \partial^{|(\alpha)_k|} q \otimes_\Theta \partial^{|(\beta)_k|} p,$$

(6.0.2)

here we use $\otimes$ to denote both deformed tensor product and contraction between mixed tensors. Also as the same as in the commutative situation, for each $j \geq 0$, $a_j(p, q)_\Theta$ reduce the total degree of $p$ and $q$ by $j$.

Remark. Recall from proposition 5.1.4:

$$\rho_{(\alpha)_k, (\beta)_k} = \partial^{\alpha_1} \partial^{\beta_1} \ell(\xi, x) \otimes \cdots \otimes \partial^{\alpha_k} \partial^{\beta_k} \ell(\xi, x),$$
since the phase function $\ell$ is $\mathbb{T}^n$-invariant (see proposition 3.3.2), so is each $\partial^{\alpha_1} \partial^{\beta_1} \ell(\xi_x, x)$ above. Hence the symmetric tensor $\otimes$ can be replace by its deformation $\otimes \Theta$. The great advantage of doing this is to make all the “multiplication type” operation (the deformed tensor product and contraction) associative. Therefore, the summands in (6.0.2) can be simply written as

$$\partial^{\alpha_1} \partial^{\beta_1} \ell(\xi_x, x) \ldots \partial^{\alpha_k} \partial^{\beta_k} \ell(\xi_x, x)(\partial^l D^{(\beta)_k}|q)(D^l D^{(\alpha)_k}|p)$$

which is formally identical to the its commutative analogies, however the pointwise tensor product and the contraction are deformed along the torus action.

Proof. By definition,

$$a_j(p, q)_\Theta = \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_{\Theta}(\mu, \nu) a_j(p, q).$$

Use the $\mathbb{T}^n$-invariant property of the vertical and the horizontal derivatives, and the phase function $\ell$, we see that the tensor fields $\rho^{(a)_{k}, (\beta)_{k}}$ are $\mathbb{T}^n$-invariant, moreover, if we fix the indices $\alpha, \beta, j, k, l$ in (6.0.2),

$$\sum_{\mu, \nu \in \mathbb{Z}^n} \chi_{\Theta}(\mu, \nu) \rho^{(a)_{k}, (\beta)_{k}}(\partial^l D^{(\beta)_k}|q_{\mu})(D^l D^{(\alpha)_k}|p_{\nu})$$

$$= \rho^{(a)_{k}, (\beta)_{k}} \sum_{\mu, \nu \in \mathbb{Z}^n} \chi_{\Theta}(\mu, \nu) (\partial^l D^{(\beta)_k}|q_{\mu})(D^l D^{(\alpha)_k}|p_{\nu})$$

$$= \rho^{(a)_{k}, (\beta)_{k}} \times_{\Theta} (\partial^l D^{(\beta)_k}|q) \times_{\Theta} (D^l D^{(\alpha)_k}|p).$$

Hence (6.0.2) is proved. \qed
CHAPTER 7
HEAT KERNEL ASYMPTOTIC

7.1 Pseudo differential operators with parameter

Let \( \Psi(M, \Lambda) \) denote the space of pseudo differential operators with a parameter \( \lambda \in \Lambda \). It is more than just a map from \( \Lambda \rightarrow \Psi(M) \), for instance, in our application, we require that our families of operators is holomorphic in the parameter, moreover, when parametric symbols are considered, differentiating in the parameter reduces the order of the operator.

**Definition 7.1.1.** Let \( \Lambda \subset \mathbb{C} \) be a cone region centered at the origin. We denote by \( S\Sigma^d(M, \Lambda) \) the class of symbols \( p(\xi, \lambda) \) such that for fixed \( \lambda \in \Lambda \), \( p(\lambda) \) is a smooth function on \( T^*M \), moreover \( p(\lambda) \) is holomorphic in \( \lambda \) when viewed as a \( C^\infty(T^*M) \)-valued function on \( \lambda \). In any local coordinate \( (x, \xi) \), the estimate holds:

\[
|D_\xi^\alpha D_\lambda^\gamma D_\lambda^j p(x, \xi, \lambda)| \leq C_{\alpha, \beta, j} \left(1 + |\xi|^2 + |\lambda|\right)^{(d-|\beta|-2j)/2}.
\] (7.1.1)

The best constants \( C_{\alpha, \beta, j} \) give rise a family of semi-norms that makes \( S\Sigma^d(M, \Lambda) \) a Fréchet space. More over, since the estimate (7.1.1) is stable under change of
coordinates, $S\Sigma^d(M, \Lambda)$ becomes a smooth $\mathbb{T}^m$-module equipped with the same torus action as the nonparametric case.

As before:

$$S\Sigma^{-\infty}(M, \Lambda) = \bigcap_{d \in \mathbb{Z}} S\Sigma^d(M, \Lambda), \quad S\Sigma^\infty(M, \Lambda) = \bigcup_{d \in \mathbb{Z}} S\Sigma^d(M, \Lambda)$$

and

$$S\Sigma(M, \Lambda) = S\Sigma^{-\infty}(M, \Lambda)/S\Sigma^\infty(M, \Lambda)$$

whose multiplication admits a deformation along the torus with respect to a skew-symmetric matrix $\Theta$, the resulting algebra is denoted by $S\Sigma(M_\Theta, \Lambda)$.

Quantization of $S\Sigma^d(M, \Lambda)$ gives rise the associated pseudo differential operators, denoted by $\Psi^d(M, \Lambda)$.

The operator norm estimate in terms of $\lambda$ is essential for our discussion. Recall [19, Lemma 1.7.1 (b)]:

**Proposition 7.1.1.** Given any positive integer $k$, one can find another integer $K > 0$ such that for any $Q(\lambda) \in \Psi^{-K}(M, \Lambda)$, the operator norm of $Q(\lambda) \in B(\mathcal{H}^{-k}, \mathcal{H}^k)$ has the estimate:

$$\|Q(\lambda)\|_{-k,k} \leq C(1 + |\lambda|)^{-K}. \quad (7.1.2)$$

The same estimate holds for the deformed operator.

**Proposition 7.1.2.** Let $Q(\lambda) \in \Psi^d(M, \Lambda)$, the such that the operator norm $\|\cdot\|_{s,s'}$ in $B(\mathcal{H}_s, \mathcal{H}_{s'})$ is of order $O(|\lambda|^N)$ for some $N \in \mathbb{Z}$, and $s, s' \in \mathbb{R}$. Then $\|\pi^\Theta(Q(\lambda))\|_{s,s'}$ is still of order $O(|\lambda|^N)$. 83
Proof. From the definition of $\pi^\Theta(Q)$,

$$\|\pi^\Theta(Q(\lambda))\| \leq C \sum_{|\mu|=n+1} \left\| \partial_t^\mu \right\|_{t=0} \| \text{Ad}_t(Q(\lambda)) \|,$$

here $\partial_t^\mu$ are partial derivatives on the torus $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$. While each $\left\| \partial_t^\mu \right\|_{t=0} \text{Ad}_t(Q(\lambda))$ is dominated by the associated symbol semi-norms, which are the optimized constant in the estimate (7.1.1). According to the calculation in the proof of proposition 4.1.1, in local coordinates $(x, \xi)$, the symbol of $\partial_t^\mu \text{Ad}_t(Q(\lambda))$ can be obtained from applying certain finite sum of partial derivatives $\partial_x, \partial_\xi$ on the symbol of $Q(\lambda)$. The crucial fact is that the partial derivatives $\partial_x, \partial_\xi$ do not increase the power of $\lambda$ on the right hand side of (7.1.1), hence $\left\| \partial_t^\mu \right\|_{t=0} \text{Ad}_t(Q(\lambda))$ is of order $O(|\lambda|^N)$ if $\|Q(\lambda)\|$ is. $\square$

### 7.2 Perturbation of the scalar Laplacian $\Delta$ via a Weyl factor

As mentioned at the beginning, in the framework of noncommutative geometry, the metric of a noncommutative manifold is implemented by the Dirac operator in a spectral triple. Since we restrict ourselves to pseudo differential acting only on coordinate functions, only the Laplacian is available in general. Following the works on noncommutative tori (cf.[13], [12], [17], [15]), we first consider a conformal perturbation of the Laplacian by a Weyl factor in the noncommutative coordinate algebra.

**Definition 7.2.1.** Let $M$ be a toric Riemannian manifold as in definition (3.0.2) and $C^\infty(M_\Theta) = \pi^\Theta(C^\infty(M)) \subset B(\mathcal{H})$ is the algebra of smooth functions on its deformation $M_\Theta$ with respect to a skew-symmetric $n \times n$ matrix. Here, $\mathcal{H}$ is the
Hilbert space of $L^2$-functions on $M$ as before. Denote by $C(M_\Theta)$ the $C^*$-algebra of the operator norm completion of $C^\infty(M_\Theta)$. A Weyl factor $\pi^\Theta(k)$ is an element in $C^\infty(M_\Theta) \subset C(M_\Theta)$ (that is $k \in C^\infty(M)$) which is invertible and positive.

**Definition 7.2.2.** Fixed a Weyl factor $\pi^\Theta(k)$ with $k \in C^\infty(M)$, a deformed differential operator $\pi^\Theta(P_k)$ is called of perturbed Laplacian type if its is spectrum is contain in $[0, \infty)$, and $P_k \in \Psi^2(M)$ is a differential operator whose symbol is of the form $\sigma(P_k) = p_2 + p_1 + p_0$, when $j = 0, 1$, $p_j(\xi_x) \in S\Sigma^j(M)$ is polynomial in $\xi$ of degree $j$.

For $j = 2$, we require the leading symbol is of the form:

$$p_2(\xi_x) = k \times_\Theta |\xi|^2 = k |\xi|^2.$$

The second equal sign holds for the function $|\xi|^2$ is $T^n$-invariant so that $\times_\Theta$ coincides with the usual pointwise multiplication between functions.

**Lemma 7.2.1.** Let $M_\Theta$ be a toric noncommutative manifold whose spin structure is toric equivariant as well. The $C^*$-algebra $C(M_\Theta)$ is the obtained by completing of $C^\infty(M_\Theta)$ with respect to the operator norm inside $B(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space of $L^2$-spinor sections. Then $C^\infty(M_\Theta)$ is holomorphically closed inside $C(M_\Theta)$.

**Proof.** Let $\slashed{D}$ be the Dirac operator. Let $\delta$ be the closed derivation $T \to \langle [\slashed{D}], T \rangle$.

Let $C(M)$ be the space of continuous functions on $M$. The $C^\infty(M) \subset C(M)$ is characterized by the property: given $f \in C(M)$, $f$ belongs to $C^\infty(M)$ if and only if both $f$ and $[\slashed{D}, f]$ lie in the domain of $\delta^j$ for any $j \in \mathbb{N}$. In other words, the smooth Fréchet topology on $C^\infty(M)$ is determined by the following family of semi-norms (cf. [22]):

$$q_j(f) = \|\delta^j(f)\| + \|\delta^j([D, f])\|, \quad j \in \mathbb{N}. \quad (7.2.1)$$
Observe that the Dirac operator $\mathcal{D}$ is $T^m$-invariant with respect to the torus action. It follows that for any $T^m$-smooth operator $T \in B(\mathcal{H})$

$$\delta(\pi^\Theta(T)) = \pi^\Theta(\delta(T)), \quad \delta([\mathcal{D}, \pi^\Theta(T)]) = \pi^\Theta(\delta([\mathcal{D}, T])). \tag{7.2.2}$$

Due to the continuity of $\pi^\Theta$ (lemma 2.2.3), the deformed algebra $C^\infty(M_\Theta) = \pi^\Theta(C^\infty(M))$ is still regular in the sense that both $\pi^\Theta(f)$ and $[\mathcal{D}, \pi^\Theta(f)]$ are bounded on $\mathcal{H}$, given $f \in C^\infty(M)$. Therefore, we conclude that given $f \in C(M_\Theta)$, $f$ belongs to $C^\infty(M_\Theta)$ if and only if $q_j(f)$ is finite for all $j \in \mathbb{N}$. The semi-norms $\{q_j\}$ defines a Fréchet topology on $C^\infty(M_\Theta)$ such that $\pi^\Theta$ is a topological algebra isomorphism. At last, it is well-known (cf. [22, Lemma 16]) that semi-norms in the form of (7.2.1) is a very common way to produce holomorphically closed subalgebras.

Let $s \geq 1$, consider the resolvent $(s\pi^\Theta(k) - \lambda)^{-1}$ in $C(M_\Theta)$, it has been shown in [23, Chapter 7] that $\pi^\Theta(C^\infty(M)) \subset C(M_\Theta)$ is exactly the subalgebra of all $T^m$-smooth vectors. Therefore the subalgebra $\pi^\Theta(C^\infty(M))$ is spectrum invariant, in particular, we can conclude that $(s\pi^\Theta(k) - \lambda)^{-1}$ belong to $\pi^\Theta(C^\infty(M))$. So there exists a unique smooth function, denoted by $(sk - \lambda)^{-1}$ (which distinguishes from the reciprocal of a function) as it has been suggested, such that $\pi^\Theta((sk - \lambda)^{-1}) = (s\pi^\Theta(k) - \lambda)^{-1}$. Denote by

$$r(s, \lambda) = \rho(s)(sk - \lambda)^{-1}$$

where $\rho(s)$ is a cut-off function:

$$\rho(s) = \begin{cases} 
0 & \text{for } t \leq 1/2 \\
1 & \text{for } t \geq 1.
\end{cases} \tag{7.2.3}$$

86
Finally, we construct the inverse of $p_2$ as follows:

$$b_0(\xi_x, \lambda) = r(|\xi_x|^2, \lambda), \quad (7.2.4)$$

which is well-defined as a smooth functions on $T^* M$. We would like to show that $b_0(\xi_x, \lambda)$ defined above is a inverse of $p_2(\xi_x, \lambda) = k|\xi|^2 - \lambda$ in the symbol algebra $S\Sigma(M_\Theta, \lambda)$. By the construction above, the function $r(s, \lambda)$ is the inverse of $sk - \lambda$ in the algebra $C^\infty(M_\Theta)$, to pass to the symbol algebra $S\Sigma(M_\Theta, \Lambda)$, the $T^n$-invariant property of the function $|\xi|^2$ is crucial.

**Lemma 7.2.2.** Denote by $C^\infty(\mathbb{R}, C^\infty(M_\Theta))$ the smooth functions on $\mathbb{R}$ valued in the deformed algebra $C^\infty(M_\Theta)$. Consider the subalgebra $C^\infty(T^* M)^{T^n}$ consists of functions on this form $F(f(\xi_x))$ where $F \in C^\infty(\mathbb{R}, C^\infty(M_\Theta))$ and $f \in C^\infty(T^* M)$ that is $T^n$-invariant. We denote in multiplication in $C^\infty(T^* M)$ and $C^\infty(M)$ by $\times_\Theta$ and $*_{\Theta}$ respectively, then due to the $T^n$-invariant property, $\times_\Theta$ is reduced to $*_{\Theta}$, namely, if we put $\tilde{F} = F(f(\xi_x))$ and $\tilde{G} = G(g(\xi_x))$, where $F,G \in C^\infty(\mathbb{R}, C^\infty(M_\Theta))$ and $f,g \in C^\infty(T^* M)$ which are $T^n$-invariant, then

$$\tilde{F} \times_\Theta \tilde{G}(\xi_x) = F(f(\xi_x)) *_{\Theta} G(g(\xi_x))|_x \quad (7.2.5)$$

Apply the lemma with $F$ and $G$ being the function $sk - \lambda$, and $r(s, \lambda)$ as above, and $f = g = |\xi|^2$ which is $T^n$-invariant, we conclude that

**Proposition 7.2.3.** The function $b_0(\xi_x, \lambda)$ defined in (7.2.4) belongs to $S\Sigma^{-2}(M_\Theta, \lambda)$ and serves as the inverse of $p_2(\xi_x, \lambda) = k|\xi|^2 - \lambda$ in the deformed symbol algebra $S\Sigma(M_\Theta, \Lambda)$, that is $p_2 \times_\Theta b_0 - 1$ and $b_0 \times_\Theta p_2 - 1$.
7.3 Resolvent approximation

Let $\pi^\Theta(P_k)$ be perturbed Laplacian as in definition 7.2.2.

Based on the asymptotic expansion of the product of two symbols:

$$p(\lambda) \star_\Theta q(\lambda) \sim \sum_{j=0}^{\infty} a_j^\Theta(p(\lambda), q(\lambda)),$$

where the deformed bi-differential operators decrease the degree by $j$. In particular, let $p(\lambda)$ and $q(\lambda)$ be homogeneous of degree $m_1$ and $m_2$ in $(\xi, \sqrt{\lambda})$ respectively, then the resulting symbol $a_j^\Theta(p, q)$ is of degree $m_1 + m_2 - j$.

The construction of resolvent approximation can be proceed as in the commutative case. Following for instance [19], we start with the equation

$$(p_2(\lambda) + p_1 + p_0) \star_\Theta (b_0(\lambda) + b_1(\lambda) + \cdots) \sim 1.$$

The first a few terms of the left hand side can be grouped in terms of the degree:

\begin{align*}
p_2(\lambda) \times_\Theta b_0(\lambda) \\
+ a_1^\Theta(p_2(\lambda), b_0(\lambda)) + p_1 \times_\Theta b_0(\lambda) + p_2(\lambda) \times_\Theta b_1(\lambda) \\
+ a_2^\Theta(p_2(\lambda), b_0(\lambda)) + a_1^\Theta(p_1, b_0(\lambda)) + p_0 \times_\Theta b_0(\lambda) + p_1 \times_\Theta b_1(\lambda) + a_1^\Theta(p_2, b_1(\lambda)) + p_2(\lambda) \times_\Theta b_2(\lambda) \\
+ \cdots
\end{align*}

Therefore $b_0(\lambda)$ should be the $\times_\Theta$ inverse of $p_2(\lambda)$ which is defined in equation (7.2.4), the rest are obtained by the recurrence formula:

$$b_\kappa(\lambda) = -b_0(\lambda) \times_\Theta \left( \sum_{\substack{-\kappa = \mu - 2 - \nu - j \\
\nu < \kappa}} a_j^\Theta(p_\mu(\lambda), b_\nu(\lambda)) \right) \quad (7.3.1)$$
or

\[
b_\kappa(\lambda) = - \left( \sum_{\kappa = \mu - 2 - \nu - j} a_j^\Theta (b_\nu(\lambda), p_\mu(\lambda)) \right) \times_\Theta b_0(\lambda), \tag{7.3.2}\]

where \( \mu = 0, 1, 2 \) and \( \nu = 0, 1, 2, \ldots \). For example, we write down \( b_1 \) and \( b_2 \) explicitly, which will be needed later:

\[
b_1 = ((-i)Db_0 \times_\Theta \partial p_2 + b_0 \times_\Theta p_1) \times_\Theta (-b_0) \tag{7.3.3}\]

\[
b_2 = (b_0 \times_\Theta p_0 + p_1 \times_\Theta b_1(\lambda) + (-i)Db_0 \times_\Theta \partial p_1 + (-i)Db_1 \times_\Theta \partial p_2 \]
\[
- \frac{1}{2} D^2 b_0 \times_\Theta \partial^2 p_2 - \frac{1}{2} (\partial \partial^2 \ell) Db_0 \times_\Theta D^2 p_2 \times_\Theta (-b_0) \tag{7.3.4}\]

In general, similar to the commutative situation, each \( b_\kappa \) is a sum of terms of the form:

\[
b^{j_1}_0 Q_1 b^{j_2}_0 Q_2 \cdots b^{j_l}_0 Q_l b^{j_{l+1}}_0,\]

where the multiplication is the deformed one, the functions \( \{Q_1, \ldots, Q_{l+1}\} \) are obtained by taking the deformed tensor \( \otimes_\Theta \) between horizontal and vertical derivatives of the symbols \( p_2, p_1 \) and \( p_0 \) and then performing the contraction between contravariant and covariant tensors. As an analogy of [19, Lemma 1.7.2], we have the following estimates for the resolvent approximation. The proof is omitted because it is almost identical to the proof of [19, Lemma 1.7.2] up to a deformation map \( \pi^\Theta \), which can be handled by proposition 7.1.2.

**Proposition 7.3.1.** Given a perturbed Laplacian \( \pi^\Theta(P_k) \) via a Weyl factor \( \pi^\Theta(k) \) as in definition 7.2.2. Let \( b_j(\xi_x, \lambda) \in S\Sigma^{-2-j}(M, \Lambda) \) with \( j = 0, 1, 2, \ldots \) be the sequence of symbols defined above, then each \( b_j \) is homogeneous in \( (\xi_x, \sqrt{\lambda}) \) of degree \(-2 - j\),
and let $R_j(\lambda)$ be the pseudo differential operator with symbol $b_0 + \cdots + b_j$, then for any $K \in \mathbb{N}$, one can choose a large $j$ such that both $(\pi^\Theta(P_k) - \lambda)\pi^\Theta(R_j(\lambda)) - I$ and $\pi^\Theta(R_j(\lambda))(\pi^\Theta(P_k) - \lambda) - I$ belong to $\Psi^{-K}(M, \lambda)$. As a consequence, if we put $\pi^\Theta(R(\lambda)) = (\pi^\Theta(P_k) - \lambda)^{-1}$, then

$$
\|\pi^\Theta(R(\lambda)) - \pi^\Theta(R_j(\lambda))\|_{-K,K} \leq C_K(1 + |\lambda|)^{-K}. \quad (7.3.5)
$$

$$
\|R(\lambda) - R_j(\lambda)\|_{-K,K} \leq C_K(1 + |\lambda|)^{-K}, \quad (7.3.6)
$$

where $\|\cdot\|_{-K,K}$ is the Sobolev norm from $\mathcal{H}^{-K}$ to $\mathcal{H}^K$.

### 7.4 Heat kernels and its variations

Now we are ready to establish the heat kernel expansion of the perturbed Laplacian $\pi^\Theta(P_k)$. By the assumption the spectrum of $\pi^\Theta(P_k)$ is contained in $[0, \infty)$, thus the heat operator $e^{-t\pi^\Theta(P_k)}$ is given by the Cauchy integral formula:

$$
e^{-t\pi^\Theta(P_k)} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\pi^\Theta(P_k) - \lambda)^{-1} d\lambda, \quad (7.4.1)
$$

where $C$ is a curve in the complex plane that circle around $[0, \infty)$ in such a way that

$$
e^{-ts} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(s - \lambda)^{-1} ds \quad s \geq 0.
$$

Denote by $\pi^\Theta(R(\lambda, k)) = (\pi^\Theta(P_k) - \lambda)^{-1}$, and put

$$
H(t, k) = \frac{1}{2\pi i} \int_C e^{-t\lambda} R(\lambda, k) d\lambda. \quad (7.4.2)
$$

Therefore, $\pi^\Theta(H(t, k)) = e^{-t\pi^\Theta(P_k)}$. From corollary 2.2.9: $\forall f \in C^\infty(M),

$$
\text{Tr}(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)}) = \text{Tr}(fH(t, k)),
$$
Hence the asymptotic expansion of $\text{Tr}(\pi^{\Theta}(f)e^{-t\pi^{\Theta}(P_k)})$ is given by the Schwartz kernel function of the operator $H(t, k)$. Notice that $f$ and $H(t, k)$ are both pseudo differential operators on $M$. We can follow the classical way, for instance [19, sec. 1.7], to establish the heat kernel asymptotic. Following the Cauchy integral in (7.4.1), we denote

$$B_j(\xi_x, t) = \frac{1}{2\pi i} \int_C e^{-t\lambda} b_j(\xi_s, \lambda) d\lambda,$$

(7.4.3)

where $j = 0, 1, 2, \ldots$. Integration by parts repeatedly on (7.4.3) tells us that for a fixed $t > 0$, $B_j(\xi_x, t)$ are smoothing symbols. Let $H_j(k, t)$ be the pseudo differential operator with symbol $B_0(\xi_x, t) + \cdots + B_j(\xi_x, t)$. After the contour integral, the estimate in $\lambda$ (cf. (7.3.6)) becomes estimate in $t$: for any $l \in \mathbb{N}$, we can find a large enough $j$ such that the Sobolev norm estimate holds:

$$\|H(t, k) - H_j(t, k)\|_{-l,l} \leq C_t^l.$$

(7.4.4)

In terms of the Schwartz kernel functions:

$$\left|K_H(t, x, y) - K_{H_j}(t, x, y)\right|_{\infty,l} \leq C t^l, \text{ for } 0 < t < 1,$$

(7.4.5)

where the $\cdot|_{\infty,l}$ is defined as follows:

$$\sup \left|\partial_x^\alpha \partial_y^\beta K(x, y)\right|, \forall K(x, y) \in C^\infty(M \times M).$$

The sup is taken over all $x, y \in M$ and all multi-indices $\alpha, \beta$ such that $|\alpha| \leq l$ and $|\beta| \leq l$. We have proved that $K_{H_j}$ approximates $K_H$ to arbitrarily high jets as $t \to 0$, it remains to compute $K_{H_j}(t, x, y)$. In the parametric case, the Schwartz kernel of an operator and its symbol a linked in a similar way as in proposition 5.2.1.
Lemma 7.4.1. Let $b(\xi, \lambda) \in S^{\Sigma_{-d}}(M, \Lambda)$ be a classical (i.e., homogeneous in $(\xi, \sqrt{\lambda})$) parametric symbol of order $-d$. Assume that $d > m$, the dimension of $M$ so that the operator $\text{Op}(b)$ has a continuous Schwartz kernel, denoted by $K_b(x, y, \lambda)$, then

$$K_b(x, x, \lambda) = \frac{1}{(2\pi)^m} \int_{T^*_x M} b(\xi, \lambda) d\xi_{x, \lambda^{-1}} + O(|\lambda|^{-N}), \forall N \in \mathbb{N}, \quad (7.4.6)$$

here $m$ is the dimension of the underlying manifold $M$.

Therefore, after the contour integral, we obtain:

Corollary 7.4.2. Keep the notations as above.

$$K_{E_k}(t, x, x) = \sum_{j=0}^K \frac{1}{(2\pi)^m} \int_{T^*_x M} B_j(\xi, t) d\xi_{x, t^{-1}} + o(t^N) \quad \forall N \in \mathbb{N}. \quad (7.4.7)$$

Let us compute $B_j$. Due to the homogeneity of $b_j$, that is, for $t > 0$

$$b_j(\xi_x/\sqrt{t}, \lambda/t) = tb_j(\xi_x, \lambda),$$

change of variables $\lambda \mapsto t\lambda$ and $\xi_x \mapsto t^{-1/2}\xi_x$ gives us:

$$\int_{T^*_x M} B_j(\xi, t) d\xi = \frac{1}{2\pi i} \int_C e^{-t\lambda} b_j(\xi_x, \lambda) d\lambda d\xi$$

$$= t^{-m-1} \int_{T^*_x M} \frac{1}{2\pi i} \int_C e^{-t\lambda} b_j(t^{-1/2}\xi_x, t^{-1}\lambda) d\lambda d\xi$$

$$= t^{-m-1 + \frac{1}{4}j} \int_{T^*_x M} \frac{1}{2\pi i} \int_C e^{-\lambda} b_j(\xi_x, \lambda) d\lambda d\xi$$

$$= t^{\frac{m}{2}} V_j(x),$$

where $m$ is the dimension of the manifold $M$, and

$$V_j(x) = \int_{T^*_x M} \frac{1}{2\pi i} \int_C e^{-\lambda} b_j(\xi_x, \lambda) d\lambda d\xi \quad (7.4.7)$$

92
Sum over \( j \geq 0 \), we have arrived at

\[
K_H(t, x, x) \sim \sum_{j=0}^{\infty} t^{j-m} V_j(x), \text{ as } t \to 0^+.
\] (7.4.8)

We summarize the discussion above to the following theorem.

**Theorem 7.4.3.** For any \( f \in C^\infty(M) \), viewed as a zero-order pseudo differential operator by left-multiplication, then

\[
\Tr \left( \pi^\Theta(f) e^{-t\pi^\Theta(P_k)} \right) \sim \sum_{j=0}^{\infty} t^{(j-m)/2} V_j(f, \pi^\Theta(P_k)),
\] (7.4.9)

where

\[
V_j(f, \pi^\Theta(P_k)) = \int_M f(x) V_j(x) dg = \int_{T^\ast M} \left( \frac{1}{2\pi i} \int_\mathbb{C} e^{-\lambda} b_j(\xi_x, \lambda) d\lambda \right) \Omega,
\] (7.4.10)

\[
= \int_M \int_{T^\ast_\mathbb{C} M} \left( \frac{1}{2\pi i} \int_\mathbb{C} e^{-\lambda} b_j(\xi_x, \lambda) d\lambda \right) d\xi_x g^{-1} dg,
\]

where \( \Omega \) is the canonical volume form on \( T^\ast M \) defined in (5.2.2) and \( \mathcal{C} \) is the contour that defines the heat operator.

Let \( \pi^\Theta(P_k(s)) \) be a smooth family of perturbed Laplacian with \( s \in [0, \varepsilon) \), namely, for fixed \( s \), \( \pi^\Theta(P_k(s)) \) is a perturbed Laplacian as in definition 7.2.2. Put \( R(\lambda, s) = (\pi^\Theta(P_k(s)) - \lambda)^{-1} \) and \( R_j(\lambda, s) \) be the resolvent approximation constructed before, following the argument in [4, eq. (3.12-3.14)], we can prove the estimate: for any \( l \in \mathbb{N} \), one can find a large enough \( j \) such that

\[
\left\| \frac{d}{ds} (R(\lambda, s) - R_j(\lambda, s)) \right\|_{-l, l} \leq C_l (1 + |\lambda|)^{-l}.
\]

After the contour integral, we get

\[
\| H(t, k, s) - H_j(t, k, s) \|_{-l, l} \leq C_l t^{-l},
\]

93
for the Schwartz kernel functions

\[ |K_{H(t,k,s)} - K_{H_j(t,k,s)}|_{\infty,t} \leq C_t t^{-l}. \]

We have arrived at the following proposition (cf. [4, Theorem 3.3]).

**Proposition 7.4.4.** Let \( \pi_\Theta(P_k(s)) \) be a family of perturbed Laplacian as in definition 7.2.2 parametrized by \( s \in [0, \varepsilon) \). Then for any \( \pi_\Theta(f) \in C^\infty(M_\Theta) \), \( \text{Tr}(\pi_\Theta(f)e^{-t\pi_\Theta(P_k(s))}) \) is smooth in \( s \) and its derivative has an asymptotic expansion obtained by term by term differentiation of the right hand side of (7.4.9):

\[
\frac{d}{ds} \text{Tr}(\pi_\Theta(f)e^{-t\pi_\Theta(P_k(s))}) \sim \sum_{j=0}^{\infty} t^{(j-m)/2} \frac{d}{ds} V_j(\pi_\Theta(f), \pi_\Theta(P_k(s)))
\]

(7.4.11)

Now we are ready to prove an analogy of “conformal Indy” on Riemannian manifolds [4] and on noncommutative two tori [12]. Since the prove involves only operators, we shall simplify the \( \pi_\Theta(k) \), \( \pi_\Theta(f) \), etc. in \( C^\infty(M_\Theta) \) to \( k \), \( f \), etc. Let \( k = \exp(h) \) be a Weyl factor, with its logarithm \( h \), again, the the exponential exp is taken with respect to the deformed product in \( C^\infty(M_\Theta) \). For \( s \in [0,1] \), we define a family of perturbed Laplacians \( P_s = \exp(sh)\Delta \).

**Proposition 7.4.5.** Keep the notations as above. Let \( m \) be an even integer that stands for the dimension of the underlying manifold. For the family of perturbed Laplacians \( P_s = \exp(sh)\Delta \) with \( s \in [0,1] \), The the \( m \)-th heat coefficient \( V_m(P_s) \) is independent of \( s \):

\[
\frac{d}{ds} V_m(P_s) = 0.
\]

(7.4.12)

In particular, we get:

\[
V_m(k\Delta) = V_m(\Delta).
\]

(7.4.13)
Proof. The differentiation of the family of heat operators $\frac{d}{ds}(e^{-tP_s})$ can be handled by Duhamel’s formula

$$
\frac{d}{ds} e^{-tP_s} = - \int_0^1 e^{-xtP_s} \frac{d}{ds}(tP_s)e^{-(1-x)tP_s} dx,
$$

where

$$
\frac{d}{ds} P_s = h \exp(sh) \Delta = hP_s.
$$

Plus the trace property, we obtain:

$$
\frac{d}{ds} \text{Tr}(e^{-tP_s}) = -t \text{Tr}(hP_s e^{-tP_s}).
$$

Observe that

$$
- \text{Tr}(hP_s e^{-tP_s}) = \frac{d}{dt} \text{Tr}(he^{-tP_s}),
$$

hence we have arrived at:

$$
\frac{d}{ds} \text{Tr}(e^{-tP_s}) = t \frac{d}{dt} \text{Tr}(he^{-tP_s}). \tag{7.4.14}
$$

Apply the asymptotic expansions in (7.4.11) and (7.4.9) to the left and right respectively, we obtain:

$$
\frac{d}{ds} V_j(P_s) = (j - m)V_j(h, P_s), \ j \in \mathbb{Z}_{\geq 0}. \tag{7.4.15}
$$

In particular, we obtain

$$
\frac{d}{ds} V_m(P_s) = 0
$$

$\Box$
7.5 Zeta functions and conformal indices

Let $\pi^\Theta(P_k)$ be a perturbed Laplacian with respect to a Weyl factor $\pi^\Theta(k)$ (see definition 7.2.2), then the spectrum of $\pi^\Theta(P_k)$ is contained in $[0, \infty)$. Let $\mathcal{P}_{\Delta_k}$ be the projection onto the ker $\pi^\Theta(P_k)$, One can formally define the complex power of $\pi^\Theta(P_k)$ via a contour integral

$$
\pi^\Theta(P_k)^z = \frac{1}{2\pi i} \int_C \lambda^z (\pi^\Theta(P_k)(I - \mathcal{P}_{\Delta_k}) - \lambda)^{-1} d\lambda,
$$

(7.5.1)

where $z \in \mathbb{C}$, $\lambda^z$ is holomorphic for all $\lambda \in \mathbb{C} \setminus [0, \infty)$, and the contour $C$ can be taken to be the one that defines the heat operator. For $\Re z < 0$, the contour integral (7.5.1) converges to a bounded operator on the Sobolev space $\mathcal{H}_s$ for any $s \in \mathbb{R}$, and is holomorphic in $z$. Moreover, the following two properties hold (cf. [24, theorem 10.1]):

1. For $\Re z < 0, \Re w < 0$, $(\pi^\Theta(P_k)^z \pi^\Theta(P_k)^w = \pi^\Theta(P_k)^{z+w}$;

2. For $j \in \mathbb{Z}_+$, $\pi^\Theta(P_k)^{-j} = (\pi^\Theta(P_k)(I - \mathcal{P}_{\Delta_k}))^j$.

Thus one can define for any $z \in \mathbb{C}$, the complex power of $\pi^\Theta(P_k)$ (more precisely the complex power of $\pi^\Theta(P_k)(I - \mathcal{P}_{\Delta_k})$) by

$$
\pi^\Theta(P_k)^z = \pi^\Theta(P_k)^j \frac{1}{2\pi i} \int_C \lambda^{z-j}(\pi^\Theta(P_k)(I - \mathcal{P}_{\Delta_k}) - \lambda)^{-1} d\lambda,
$$

(7.5.2)

where $j$ can be any integer that is greater than $\Re z$. The associated zeta function is the trace of the complex power:

$$
\zeta_{P_k}(z) = \text{Tr}(\pi^\Theta(P_k)^{-z}),
$$

(7.5.3)
which is well-defined for \( \Re z >> 0 \) and, thank to the heat kernel asymptotic, has a meromorphi
c extension to the whole complex plane with simple poles. The zeta function and the heat kernel are linked by the Mellin transform:

\[
\int_0^\infty t^{s-1}e^{-t\lambda}dt = \lambda^{-s}\Gamma(s).
\]

Taking the projection \( P_{\Delta_k} \) in the integral (7.5.2) into account and consider the zeta function with respect to \( \pi^\Theta(f) \in C^\infty(M_\Theta) \):

\[
\zeta_{P_k}(\pi^\Theta(f), s) \triangleq \text{Tr}(\pi^\Theta(f)\pi^\Theta(P_k)^{-s})
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}\left(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)(I-P_{\Delta_k})}\right) dt
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\text{Tr}\left(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)}\right) - \text{Tr}(\pi^\Theta(f)P_{\Delta_k})\right) dt
\]

(7.5.4)

We break the integral into two parts: \( \int_0^1 + \int_1^\infty \). It is easy to show that \( \text{Tr}(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)}) - \text{Tr}(\pi^\Theta(f)P_{\Delta_k}) \) is exponential decay in \( t \) as \( t \to \infty \), plus the fact that the Gamma function is never zero, thus

\[
\frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \left(\text{Tr}\left(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)}\right) - \text{Tr}(\pi^\Theta(f)P_{\Delta_k})\right) dt
\]
gives rise an entire function in \( s \). While for the integral from 0 to 1, we replace \( \text{Tr}(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)}) \) by the by the first \( N \) terms of the asymptotic expansion (7.4.9) (keep in mind that \( m \) is the dimension of the manifold):

\[
\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\text{Tr}\left(\pi^\Theta(f)e^{-t\pi^\Theta(P_k)}\right) - \text{Tr}(\pi^\Theta(f)P_{\Delta_k})\right) dt
= \frac{1}{\Gamma(s)} \left(\sum_{j=0}^N \frac{1}{s - \frac{m-j}{2}}V_j(\pi^\Theta(f), \pi^\Theta(P_k)) - \frac{\text{Tr}(\pi^\Theta(f)^2)}{s} + R_N(s)\right),
\]

97
where the remainder $R_N(s)$ is holomorphic in $s$ for $\Re s > -(N - m)/2$. Notice that the Gamma function $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$, we have proved the following proposition:

**Proposition 7.5.1.** Let $m$ be the dimension of the underlying noncommutative manifold. The extended zeta function $\zeta_{P_k}(\pi^\Theta(f), s)$ defined in (7.5.4) is well-defined for $\Re s > 0$ and has a meromorphic extension to the whole complex plane with isolated simple poles at $z_j = (m - j)/2$ with $j = 0, 1, 2 \ldots$ with residue

$$\text{Res}_{s=z_j} \zeta_{P_k}(\pi^\Theta(f), s) = V_j(\pi^\Theta(f), \pi^\Theta(P_k))/\Gamma(z_j).$$

When $z_j \in \mathbb{Z}_{\leq 0}$, then the zeta function is indeed regular at $z_j$ with value

$$\zeta_{P_k}(\pi^\Theta(f), z_j) = \begin{cases} 
V_m(\pi^\Theta(f), \pi^\Theta(P_k)) + \text{Tr}(\pi^\Theta(f)\pi^\Theta(P_k)), & z_j = 0, \\
V_j(\pi^\Theta(f), \pi^\Theta(P_k))(\text{Res}_{s=z_j} \Gamma(s)), & z_j = -1, -2, \ldots
\end{cases}$$

We have shown in proposition 7.4.5 that $V_m(k\Delta) = V_m(\Delta)$, hence

$$\zeta_{k\Delta}(0) = V_m(k\Delta) - \dim \ker(k\Delta) = V_m(\Delta) - \dim \ker \Delta = \zeta_{\Delta}(0)$$

Hence we have proved the following analogy of [12, Theorem 2.2].

**Theorem 7.5.2.** Let $k$ be a Weyl factor, then the zeta function of $k\Delta$ at zero is independent of $k$. More precisely,

$$\zeta_{k\Delta}(0) = \zeta_{\Delta}(0).$$

(7.5.5)
CHAPTER 8

MODULAR CURVATURE

Let $\pi^\Theta(P_k)$ be a perturbed Laplacian via a Weyl factor $\pi^\Theta(k)$, (see definitions 7.2.2 and 7.2.1). Recall that the modular scalar curvature $R \in C^\infty(M_\Theta)$ is the density of the second heat coefficient $V_2(\cdot, \pi^\Theta(P_k))$ in the heat kernel expansion. We have shown in the previous section that $P_k$ is a pseudo differential operator on $M$ whose resolvent is approximated by a sequence of symbols $\{b_j(\lambda)\}_{j=0}^\infty$ with $b_j(\lambda) \in S\Sigma^{-j-2}(M, \lambda)$. According to (7.4.10), the density function is given by

$$\tilde{R}(x) = \pi^\Theta(R) \quad \text{with} \quad R \in C^\infty(M):$$

$$R = \int_{T^*_x M} \frac{1}{2\pi i} \int_C e^{-\lambda b_2(\xi_x, \lambda)} d\lambda d\xi_x g^{-1}, \quad x \in M.$$  

The order of integration is proceed as follows:

$$\tilde{R}(x) = \int_0^\infty \int_{S^*_x M} \left( \frac{1}{2\pi i} \int_C e^{-\lambda b_2(\xi_x, \lambda)} d\lambda \right) d\omega_{x,g^{-1}}(r^{m-1} dr), \quad (8.0.1)$$

where $d\omega_{x,g^{-1}}$ is the volume form on the unit cosphere inside $T^*_x M$ determined by the metric $g^{-1}$ and $m$ is the dimension of the underlying manifold. Since the unit cosphere in each $T^*_x M$ is compact, we perform integration against $d\omega_{x,g^{-1}}$ first, while $d\lambda dr$ is handled by lemma 8.3.1 and the rearrangement lemma developed in [13], [12] and [20].
At the end, we showed that the results agree with the known ones in [21], [12] and [15]. In the rest of the computation, the tensor calculus is always the deformed version, we will suppress all $\otimes_\Theta$, $\times_\Theta$ to simplify the notations.

8.1 Computation of $b_2$

We perform the calculation with respect to a perturbed Laplacian $\pi^\Theta(P_k)$ (cf. definition 7.2.2) whose symbol is of the form:

$$
\sigma(\pi^\Theta(P_k)) = k |\xi|^2 + p_1(\xi_x) + p_0(\xi_x). 
$$

(8.1.1)

Here $p_1(\xi_x)$ and $p_0(\xi_x)$ are the degree one and zero parts respectively whose explicit expressions will be determined later when we focus on specific examples. Let us consider, at first, the simplest perturbation $\pi^\Theta(k)\Delta$. To calculate its symbol, we only need the first three terms in the asymptotic expansion of $\star_\Theta$:

$$
b \star_\Theta q \sim b \times_\Theta q + (-i) Db \times_\Theta \partial q - \frac{1}{2} D^2 b \times_\Theta \partial^2 q - \frac{1}{2} (\partial \partial^2 \ell) D^2 p \times_\Theta D q
$$

(8.1.2)

+ ... 

Lemma 8.1.1. Let $\pi^\Theta(P_k) = \pi^\Theta(k)\Delta$ be a perturbed Laplacian with $P_k \in \Psi^2(M)$, a differential operator on $M$. Then the symbol of $P_k$ is equal to $k \times_\Theta |\xi|^2 = k |\xi|^2$.

Proof. Since the function $|\xi|^2$ is $\mathbb{T}^n$-invariant, $k \times_\Theta |\xi|^2 = k |\xi|^2$. According to the asymptotic expansion $\star_\Theta$,

$$
\sigma(\pi^\Theta(k)\Delta) = k \times_\Theta \sigma(\Delta).
$$
Hence it remains to show that $\sigma(\Delta) = |\xi|^2$. Indeed, in a local coordinate,

$$\Delta f = - (\nabla^2 f)_{ij} g^{ij}$$  \hspace{1cm} (8.1.3)$$

where $(g^{ij})$ is the metric tensor on $T^* M$. To compute the symbol:

$$\nabla^2 e^{i\ell(x,y)} = e^{i\ell(x,y)} \left( i\nabla^2 \ell(x,y) - \nabla \ell(x,y) \nabla \ell(x,y) \right).$$

Note that $\nabla$ is torsion-free, thus $\nabla^2$ is equal to its symmetrization $\partial^2$. Therefore by the definition of $\ell$. $\nabla^2 \ell = 0$ and $\nabla_j \ell(x,y) = \xi_j$. Pair $\nabla^2 e^{i\ell(x,y)}$ with $g = g^{ij}$, we obtain:

$$\sigma(\Delta)(\xi_x) = \xi_i \xi_j g^{ij} = |\xi|^2.$$  \hspace{1cm} (8.1.4)$$

In the rest of the section, we denote by

$$p_2(\xi_x, \lambda) = k |\xi|^2 - \lambda$$  \hspace{1cm} (8.1.5)$$

and

$$b_0(\xi_x, \lambda) = (k |\xi|^2 - \lambda)^{-1},$$

where the inverse of $p_2(\xi_x, \lambda)$ is taken in the algebra $C^\infty(T^* M_{\Theta})$. The existence has been discussed (cf. Eq. (7.2.4)).

Recall the recurrence formulas (7.3.3) and (7.3.4):

$$b_1 = (b_0 p_1 + (-i) Db_0 \partial p_2)(-b_0)$$  \hspace{1cm} (8.1.6)$$

$$b_2 = \left( b_0 p_0 + b_1 p_1 + (-i) Db_1 \partial p_2 - \frac{1}{2} D^2 b_0 \partial^2 p_2 - \frac{1}{2} (\partial \partial^2 \ell) Db_0 D^2 p_2 \right)(-b_0)$$  \hspace{1cm} (8.1.7)$$
The horizontal derivatives of \( p_2 \) are given by:

\[
\partial k |\xi|^2 = (\partial k) |\xi|^2, \quad (8.1.8)
\]
\[
\partial^2 k |\xi|^2 = (\partial^2 k) |\xi|^2 + k(\partial^2 |\xi|^2), \quad (8.1.9)
\]

they are straightforward consequences of lemma 8.2.1. Differentiating \( b_0 \) yields:

\[
Db_0 = -(b_0 b_0) Dp_2 = -(b_0^2 k) D |\xi|^2, \quad (8.1.10)
\]

and

\[
D^2 b_0 = D(-b_0 b_0 Dp_2)
\]
\[
= -b_0 b_0 D^2 p_2 + 2b_0 b_0 b_0 Dp_2 Dp_2 \quad (8.1.11)
\]
\[
= -b_2^2 kD^2 |\xi|^2 + 2b_0^2 k^2 (D |\xi|^2)^2,
\]

Combine (8.1.6), (8.1.10) and (8.1.8), we get:

\[
b_1 = -ib_0^2 k(D |\xi|^2 \cdot \partial k) b_0 |\xi|^2 - b_0 p_1 b_0. \quad (8.1.12)
\]

**Proposition 8.1.2.** The \( b_2 \) term is given by:

\[
b_2 = 2b_0^3 k^2 (\partial k) b_0 (\partial k) b_0 |\xi|^4 (D |\xi|^2)^2
\]
\[
- b_0^2 k(\partial k) b_0 (\partial k) b_0 \left(|\xi|^2 (D |\xi|^2)^2 + |\xi|^4 D^2 |\xi|^2 \right)
\]
\[
+ b_0^2 k(\partial k) b_0^2 k(\partial k) b_0 |\xi|^4 (D |\xi|^2)^2
\]
\[
- \frac{1}{2} (b_0^2 k^2)(\partial^2 \ell)(D |\xi|^2)(D^2 |\xi|^2) b_0 - \frac{1}{2} b_0^2 k^2 D^2 |\xi|^2 \partial^2 |\xi|^2 b_0
\]
\[
- \frac{1}{2} b_0^2 k(\partial^2 k) b_0 |\xi|^2 D^2 |\xi|^2 + b_0^3 k^2 (\partial^2 k) b_0 |\xi|^2 (D |\xi|^2)^2
\]
\[
- iD(b_0 p_1 b_0)(\partial p_2) b_0 - b_0 p_0 b_0 - b_1 p_1 b_0. \quad (8.1.13)
\]
Proof. We first compute the last two terms (terms with $1/2$ in front) in equation (8.1.7). From (8.1.11) and (8.1.9):

$$D^2 b_0 \times_\Theta \partial^2 p_2$$

$$= (-b_2^2 k D^2 |\xi|^2 + 2b_0^2 k^2 (D |\xi|^2)^2)((\partial^2 k) |\xi|^2 + k(\partial^2 |\xi|^2))$$

$$= -b_0^2 k (D^2 |\xi|^2 \cdot \partial^2 k) |\xi|^2 - b_0^2 k^2 D^2 |\xi|^2 \partial^2 |\xi|^2$$

$$+ 2b_0^2 k^2 ((D |\xi|^2)^2 \partial^2 k) |\xi|^2 + 2b_0^2 k^3 (D |\xi|^2)^2 \partial^2 |\xi|^2.$$  \hspace{1cm} (8.1.14)

Combine (8.1.10) and (8.1.9):

$$(\partial \partial^2 \ell) D b_0 D^2 p_2 = (\partial \partial^2 \ell)(-b_0^2 k)(D |\xi|^2)k(D^2 |\xi|^2)$$

$$= (-b_0^2 k^2)(\partial \partial^2 \ell)(D |\xi|^2)(D^2 |\xi|^2).$$  \hspace{1cm} (8.1.15)

Apply $D$ onto (8.1.12),

$$Db_1 = -i \left[ D(b_0^2 k)(D|\xi|^2 \cdot \partial k)b_0 \xi^2 + b_0^2 k D(D |\xi|^2 \cdot \partial k)b_0 |\xi|^2 \right.$$

$$+ b_0^2 k (D |\xi|^2 \cdot \partial k)(D b_0) |\xi|^2 + b_0^2 k(D |\xi|^2 \cdot \partial k)b_0 D |\xi|^2 \big] - D(b_0 p_1 b_0)$$

$$= -i \left[ -2b_0^3 k^2 D |\xi|^2 (D |\xi|^2 \cdot \partial k)b_0 |\xi|^2 + b_0^2 k(D |\xi|^2 \cdot \partial k)b_0 D |\xi|^2 \right.$$

$$+ b_0^2 k(D^2 |\xi|^2 \cdot \partial k)b_0 |\xi|^2 - b_0^2 k(D |\xi|^2 \cdot \partial k)b_0^2 k(D |\xi|^2) |\xi|^2 \big]$$

$$- D(b_0 p_1 b_0),$$

103
Thus

\[ Db_1 \partial p_2 = -i \left[ -2b_0^3 k^2 (D |\xi|^2 \cdot \partial k) b_0 (D |\xi|^2 \cdot \partial k) |\xi|^4 \\
+ b_0^2 k (D |\xi|^2 \cdot \partial k) b_0 (D |\xi|^2 \cdot \partial k) |\xi|^2 \\
+ b_0^2 k (D^2 |\xi|^2 \cdot \partial k) b_0 (\partial k) |\xi|^4 \\
- b_0^2 k (D |\xi|^2 \cdot \partial k) b_0^2 k (D |\xi|^2 \cdot \partial k) |\xi|^4 \right] \]

(8.1.16)

We substitute (8.1.14), (8.1.15) and (8.1.16) into (8.1.7):

\[ b_2 = 2b_0^3 k^2 (D |\xi|^2 \cdot \partial k) b_0 (D |\xi|^2 \cdot \partial k) b_0 |\xi|^4 \\
- b_0^2 k (D |\xi|^2 \cdot \partial k) b_0 (D |\xi|^2 \cdot \partial k) b_0 |\xi|^2 \\
- b_0^2 k (D^2 |\xi|^2 \cdot \partial k) b_0 (\partial k) b_0 |\xi|^4 \\
+ b_0^2 k (D |\xi|^2 \cdot \partial k) b_0^2 k (D |\xi|^2 \cdot \partial k) b_0 |\xi|^4 \\
+ \frac{1}{2} (-b_0^2 k^2) (\partial \partial^2 \ell) (D |\xi|^2) (D^2 |\xi|^2) b_0 \\
- \frac{1}{2} b_0^2 k (D^2 |\xi|^2 \cdot \partial^2 k) b_0 |\xi|^2 \\
- \frac{1}{2} b_0^2 k^2 D^2 |\xi|^2 \partial^2 |\xi|^2 b_0 \\
+ b_0^3 k^2 (D |\xi|^2)^2 \partial^2 k) b_0 |\xi|^2 \\
- i D (b_0^2 b_1 b_0) (\partial p_2) b_0 - b_0 p_0 b_0 - b_1 p_1 b_0. \]

To arrive at equation (8.1.13), we just have to move the vertical and the horizontal derivatives of $|\xi|^2$ to the right end for each summand above. This operation is valid because of the $\mathbb{T}^n$-invariant property of the function $|\xi|^2$. \hfill \Box
8.2 Integration on the sphere $S^{m-1}$

In this section, $m$ is the dimension of the underlying manifold $M$ which is even. The integration is carried out on each fiber $T^*_x M$ with $x \in M$, hence we are free to choose local coordinates. To simplify the notations, we introduce tensor fields:

\[ \rho^j_\ell = \partial^j \partial^\ell, \quad j, l \geq 1, \]  
(8.2.1)

in terms of tensor indices, let $\alpha = \langle v_1, \ldots, v_j \rangle$, $\beta = \langle u_1, \ldots, u_l \rangle$,

\[ \rho^\alpha_\beta = (\partial^j \partial^\ell \rho^j_\ell)_{\alpha,\beta} = (\partial^j \partial^\ell \rho^j_\ell)_{v_1 \ldots v_j , u_1 \ldots u_l}. \]  
(8.2.2)

From [27, Eq. (4.16), (4.17)]

\[ \rho^{2,1}_\ell = \rho^{(ij),(l)}_\ell = (\partial^2 \rho^j_\ell)_{ijkl} = -\frac{1}{3} \xi^p \left( R^p_{i(js)} + R^p_{j(i)s} \right), \]  
(8.2.3)

\[ \rho^{1,2}_\ell = \rho^{(l),(ij)}_\ell = (\partial^2 \rho^\ell_{ij})_{ijkl} = -\frac{1}{6} \xi^p \left( R^p_{j(i)l} + R^p_{i(js)} \right). \]  
(8.2.4)

**Lemma 8.2.1.** The horizontal and the vertical derivatives of $|\xi|^2$ are given by:

\[ \partial |\xi|^2 = 0 \]  
(8.2.5)

\[ (\partial^2 |\xi|^2)_{ij} = 2\xi^a (\partial^2 \rho^\ell_{ij})_{ijkl} g^{ia} = 2\xi^a \rho^{(l),(ij)}_\ell g^{ia} \]  
(8.2.6)

while

\[ (D^2 |\xi|^2)^{ij} = 2g^{ij} \quad (D |\xi|^2)^{ij} = 2\xi^i g^{ij}, \]  
(8.2.7)

where $g^{ij}$ are the tensor components of $g^{-1}$, which is the metric tensor on $T^* M$. 

105
Proof. Set $d\ell = d_y \ell(\xi_x, y)$, we rewrite the metric pairing $\langle d\ell, \ell \rangle_{g^{-1}}$ as $d\ell \otimes d\ell \cdot g^{-1}$. Following the definition of the horizontal derivative, plus the fact that $\nabla g^{-1} = 0$ and the Leibniz’s rule, we compute:

$$\partial |\xi|^2 = \nabla (d\ell \otimes d\ell \cdot g^{-1}) = 2\nabla d\ell \otimes d\ell \cdot g^{-1} = \nabla^2 \ell \otimes \ell \cdot g^{-1} = 0,$$

the last = holds because $\nabla$ is torsion-free,

$$\nabla^2 \ell = \partial^2 \ell = 0.$$

However, the second covariant derivative is nonzero:

$$\partial^2 |\xi|^2 = \partial^2 ((d\ell \otimes d\ell) \cdot g^{-1}) = 2(\partial^2 \partial \ell \otimes d\ell) \cdot g^{-1},$$

where the contraction is implemented as follows:

$$(\partial^2 \partial \ell)_{ij}(d\ell)_{a}g^{aj}.$$ 

Thus we have proved (8.2.5) and (8.2.6).

The vertical derivative $D$ is the partial derivatives in the $\xi$ variable, thus the calculation of (8.2.7) is straightforward. 

\[\Box\]

**Lemma 8.2.2.** In local coordinates:

$$\rho^{(ij),(l)}_l (D^2 |\xi|^2)^{ij} (D |\xi|^2)^{ij} = -\frac{4}{3} \xi_a \xi_p (R^p_{ij} + R^p_{ji}) g^{ij} \quad (8.2.8)$$

$$(D^2 |\xi|^2)_{ij}(\partial^2 |\xi|^2)_{ij} = \frac{2}{3} \xi_a \xi_p (R^p_{ij} + R^p_{ji}) g^{ij} \quad (8.2.9)$$

$$((D |\xi|^2)^2)^{ij}(\partial^2 |\xi|^2)_{ij} = 0 \quad (8.2.10)$$

106
Proof. Equation (8.2.8) follows immediately from (8.2.3) and (8.2.7).

Substitute (8.2.4) into (8.2.6):

\[
(\partial^2 |\xi|^2)_{ij} = 2\xi_a\rho^{(ij)_\ell} g^{al} = \frac{1}{3} \xi_a \xi_p \left( R^p_{ji\ell} + R^p_{ij\ell} \right) g^{al} = \frac{1}{3} \xi_a \xi_p (R^p_{i\ j} + R^p_{j\ i}),
\]

pair it with \((D^2 |\xi|^2)_{ij} = 2g^{ij}\), we have reached (8.2.9).

From (8.2.7), we get

\[
((D |\xi|^2)^2)_{ij} = 4\xi_s \xi_l g^{is} g^{jl},
\]

hence

\[
((D |\xi|^2)^2)_{ij} (\partial^2 |\xi|^2)_{ij} = \frac{4}{3} \xi_a \xi_p \xi_s \xi_l (R^p_{i\ j} + R^p_{j\ i}) g^{is} g^{jl} = \frac{4}{3} \xi_a \xi_p \xi_s \xi_l (R^{psal} + R^{plas}) = 0.
\]

The last “=” sign above is due to the anti-symmetries of the curvature tensor. Therefore the last equality is proved.

One can prove the following lemma by the symmetries of polynomials \(\xi_a \xi_s\) on \(\mathbb{R}^m\):

**Lemma 8.2.3.** Let \((x, \xi)\) be a local coordinate system around \(x\) such that the metric tensor \(g^{ij} = \delta_{ij}\) at \(x\). Thus the unit sphere coincides with the Euclidean unit sphere in \(\mathbb{R}^m\), we denote the induced measure by \(d\sigma_{S^{m-1}}\) and \(\xi_s, \xi_a\) are components of \(\xi\), then

\[
\int_{S^{m-1}} \xi_s \xi_a d\sigma_{S^{m-1}} = \begin{cases} 
0 & \text{when } s \neq a \\
\frac{|\xi|^2}{m} \frac{\text{Vol}(S^{m-1})}{\text{Vol}(S^{m-1})} & \text{when } s = a.
\end{cases} \tag{8.2.11}
\]
Corollary 8.2.4. Let $S^{m-1} \subset T^*_xM \cong \mathbb{R}^m$ be the unit sphere with respect to the Riemannian metric $g^{-1}$ and $d\sigma_{S^{m-1}}$ be the induced measure as before. Then

\[ \int_{S^*_xM} (D|\xi|^2)^2 d\omega_{x,g^{-1}} = \frac{4}{m} |\xi|^2 \operatorname{Vol}(S^{m-1})g^{-1} \bigg|_x \]  

(8.2.12) 

\[ \int_{S^*_xM} D^2 |\xi|^2 d\omega_{x,g^{-1}} = 2\operatorname{Vol}(S^{m-1})g^{-1} \bigg|_x . \]  

(8.2.13)

As a consequence:

\[ \int_{S^*_xM} D^2 |\xi|^2 \cdot \partial^2 k d\omega_{g^{-1}} = 2\operatorname{Vol}(S^{m-1})\Delta k \]  

(8.2.14)

\[ \int_{S^*_xM} (D|\xi|^2)^2 \cdot \partial^2 k d\omega_{g^{-1}} = \frac{4}{m} |\xi|^2 \operatorname{Vol}(S^{m-1})\Delta k \]  

(8.2.15)

Proof. Since $D^2 |\xi|^2 = 2g^{-1}$, (8.2.13) follows immediately. To show (8.2.12), we choose an orthonormal basis at $T^*_xM$ as before, then

\[ g^{-1} = \sum_{j=1}^m \left( \frac{\partial}{\partial x_j} \right)^2 \]

\[ |\xi|^2 = \sum_{j=1}^m \xi_j^2, \]

\[ D|\xi|^2 = \sum_{j=1}^m \partial^j |\xi|^2 \frac{\partial}{\partial x_j} = 2\xi_j \frac{\partial}{\partial x_j}, \]

thus

\[ (D|\xi|^2)^2 = D|\xi|^2 \otimes D|\xi|^2 = 4 \sum_{j=1}^m \xi_j^2 \left( \frac{\partial}{\partial x_j} \right)^2 + 4 \sum_{j \neq k} \xi_j \xi_k \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_k}. \]

Apply lemma 8.2.3, the contribution of first term is exactly the right hand side of (8.2.12), while the second term vanishes after integration.

Since

\[ (\partial^2 k) \cdot g^{-1} = (\nabla^2 k) \cdot g^{-1} = \Delta k, \]

the equations (8.2.14) and (8.2.15) follow immediately. \(\square\)
Corollary 8.2.5. Suppose that \((x, \xi)\) is a local coordinate system in which the metric tensor \(g_{ij}|_x = \delta_{ij}\) and \(S^*_x M \subset T^*_x M \cong \mathbb{R}^m\) be the unit sphere with respect to the Riemannian metric \(g^{-1}\) with \(d\sigma_{S^{m-1}}\) the induced measure.

\[
\int_{S^*_x M} \xi_s \xi_a (R^s g^a)_{j_i} + (R^s g^a)_{i_j} g^{ij} d\omega x, g^{-1} = \frac{2}{m} \text{Vol}(S^{m-1}) |\xi|^2 S_\Delta, \tag{8.2.16}
\]

where \(S_\Delta \in C^\infty(M)\) is the scalar curvature function.

Proof. Since \(g^{ij} = \delta_{ij}\), we see that

\[
\xi_s \xi_a (R^s g^a)_{j_i} + (R^s g^a)_{i_j} g^{ij} = 2 \sum_{s,a,j} \xi_s \xi_a R^s g^a_{j_j} = 2 \sum_{s,a,j} \xi_s \xi_a R^s g^a_{s,a}
\]

\[
= 2 \sum_{s,a} \xi_s \xi_a R_{sa},
\]

where \(R_{sa} = R_{siaj} g^{ij}\) is the Ricci curvature tensor. By lemma 8.2.3,

\[
\int_{S^{m-1}} 2 \sum_{s,a} \xi_s \xi_a R_{sa} d\sigma_{S^{m-1}} = \int_{S^{m-1}} 2 \sum_a \xi_a \xi_a R_{aa} d\sigma_{S^{m-1}}
\]

\[
= \frac{2}{m} \text{Vol}(S^{m-1}) |\xi|^2 (\sum_a R_{aa})
\]

\[
= \frac{2}{m} \text{Vol}(S^{m-1}) |\xi|^2 S_\Delta,
\]

where

\[
S_\Delta = \sum_a R_{aa} \in C^\infty(M)
\]

is the scalar curvature function.

We substitute the results in corollary 8.2.4 and 8.2.5 into \(b_2\) given in equation (8.1.13)

109
Proposition 8.2.6. Keep the notations as above. Assume that the lower order symbols of the Laplacian is zero, that is $p_1 = p_0 = 0$, substitute the results in corollary 8.2.4 and 8.2.5 into $b_2$ (in equation (8.1.13)), we obtain that $\int_{S^{m-1}} b_2 d\sigma_{S^{m-1}}$ is equal to, up to an overall factor $\text{Vol}(S^{m-1})$:

\[
\begin{align*}
&\frac{4}{m} 2 b_0^3 k^2 (\partial k) b_0 (\partial k) b_0 |\xi|^6 g^{-1} - (2 + \frac{4}{m}) b_0^2 k (\partial k) b_0 (\partial k) b_0 |\xi|^4 g^{-1} \\
&+ \frac{4}{m} b_0^2 k (\partial k) b_0^2 k (\partial k) b_0 |\xi|^6 g^{-1} - b_0^2 k (\Delta k) b_0 |\xi|^2 + \frac{4}{m} b_0^3 k^2 (\Delta k) b_0 |\xi|^4 \\
&+ \frac{2}{m} \frac{2}{3} b_0^2 k^2 S_{\Delta b_0} |\xi|^2 ,
\end{align*}
\]  

(8.2.17)

where $m$ is the dimension of the manifold.

8.3 Integration in $\lambda$ and the rearrangement lemma

Let $m$ be the dimension of the underlying manifold, we still denote by

\[
b_2(r, \lambda) \triangleq \int_{S^{m-1}} b_2 d\sigma_{S^{m-1}},
\]

(8.3.1)

where $r \in [0, \infty)$ and $\lambda \in \Lambda$, a cone region in $\mathbb{C}$ in which $\sqrt{\lambda}$ is well-defined. We would like to switch the order of $d\lambda (r^{m-1}dr)$ in (8.0.1). When $m = 2$, this was explained in [12, section 6], it is valid to swap the order, moreover, due to the homogeneity of $b_2$, we have

\[
\frac{1}{2\pi i} \int_C e^{-t\lambda} (\int_0^\infty b_2(r, \lambda) r dr) d\lambda = \int_0^\infty b_2(r, -1) r dr. 
\]
For higher dimensions, \( \int_0^\infty b_2(r, \lambda)r^{m-1}dr \) diverges, however, integration by parts gives us:

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} b_2(r, \lambda)d\lambda(r^{m-1}dr) = \int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} \frac{d^j}{d\lambda^j} b_2(r, \lambda)d\lambda(r^{m-1}dr), \ \forall j \in \mathbb{Z}_{\geq 0},
\]

in which the homogeneity of \( \frac{d^j}{d\lambda^j} b_2(r, \lambda) \) is equal to \(-4 - 2j\).

**Lemma 8.3.1.** Let \( m \) be the dimension of the manifold \( M \) which is even, \( r \in [0, \infty) \) and the resolvent parameter \( \lambda \) belong to a conic region \( \Lambda \subset \mathbb{C} \) in which \( \sqrt{\lambda} \) is well-defined. Given a function \( b_2(r, \lambda) \) which is homogeneous in \((r, \sqrt{\lambda})\) is the following way, for any integer \( j \geq 0 \),

\[
\frac{d^j}{d\lambda^j} b_2(\xi_x, \lambda)
\]

is homogeneous in \((r, \sqrt{\lambda})\) of degree \(-4 - 2j\). Denoted by \( j_0 \) the smallest integer such that \( 4 + 2j > m \), which assures the existence of the integral

\[
B_{j_0}(\lambda) = \int_0^\infty \frac{d^j}{d\lambda^j} b_2(r, \lambda)r^{m-1}dr,
\]

(8.3.2)

then

\[
\frac{1}{2\pi i} \int_C e^{-\lambda} B_{j_0}(\lambda)d\lambda = B_{j_0}(-1)
\]

(8.3.3)
Proof. Claim that $B_{j_0}$ is homogeneous in $\lambda$ of degree $-1$, indeed, for a parameter $c > 0$,

$$B_{j_0}(c\lambda) = \int_0^\infty \frac{d^{j_0}}{d\lambda^{j_0}} b_2(\sqrt{c} \frac{r}{\sqrt{c}}, c\lambda) r^{m-1} dr$$

$$= \int_0^\infty e^{-\frac{4-2j_0}{2}} \frac{d^{j_0}}{d\lambda^{j_0}} b_2(\frac{\xi}{\sqrt{c}}, \lambda) r^{m-1} dr$$

$$= c^{-2-j_0+m/2} \int_0^\infty \frac{d^{j_0}}{d\lambda^{j_0}} b_2(\lambda, \lambda) r^{m-1} dr.$$

Since $m$ is even, the way in which $j_0$ is chosen implies that $-2 - j_0 + m/2 = -1$. The contour $C$ around $[0, \infty)$ is picked in such a way that

$$\frac{1}{2\pi i} \int_C e^{-t\lambda} \frac{1}{s-\lambda} d\lambda = e^{-ts}, \forall s, t \geq 0,$$

and $B_{j_0}(\lambda)$ is holomorphic on the half plane $\{ \Re \lambda < 0 \}$, for any $\varepsilon > 0$ and $K > 0$, we can choose $C$ to be a contour which is so closed to $(-\infty, -1]$ such that

$$|B_{j_0}(\lambda/|\lambda|) - B_{j_0}(-1)| < \varepsilon, \forall |\lambda| < K,$$

therefore

$$\frac{1}{2\pi i} \int_C e^{-t\lambda} B_{j_0}(\lambda) d\lambda = \frac{1}{2\pi i} \int_C -e^{-t\lambda} \frac{1}{\lambda} d\lambda B_{j_0}(-1) = B_{j_0}(-1).$$

Integration in $r$ is handle by the rearrangement lemma which will be explained below. We will feel free to use the notations in [21] and [20]. From now on, the parameter $\lambda$ is taken to be $-1$. Put $r = |\xi|$. After a substitution $r \mapsto r^2$, the summands in (8.2.17) contain two types:

$$kf_0(rk)\rho f_1(rk) \text{ or } kf_0(rk)\rho_1 f_1(rk)\rho_2 f_2(rk), \quad (8.3.4)$$

112
here $k$ is the Weyl factor and $f_j$’s are some smooth functions on $\mathbb{R}_+$, while $\rho_j$’s are tensor fields over $M$ on which $C^\infty(M_\Theta)$ acts from both sides. Introduce the modular operator $\triangle$:

$$\triangle(\rho) \triangleq k^{-1}\rho k,$$

then the rearrangement lemma (cf. [12, Lemma 6.2], [20, Corollary 3.9]) yields:

$$\int_0^\infty k f_0(rk) \rho f_1(rk) dr = K(\triangle)(\rho), \text{ with }$$

$$K(s) = \int_0^\infty f_0(r) f_1(rs) dr, \ s \in (0, \infty).$$

For the second type,

$$\int_0^\infty k f_0(rk) \rho_1 f_1(rk) \rho_2 f_2(rk) dr = G(\triangle_{(1)}, \triangle_{(2)})(\rho_1 \cdot \rho_2), \text{ with}$$

$$G(s_1, s_2) = \int_0^\infty f_0(r) f_1(rs_1) f_2(rs_2) dr, \ s_1, s_2 \in (0, \infty),$$

where $\triangle_{(j)}$ indicates that $\triangle$ acts on the $j$-th factor with $j = 1, 2$. The functions $f_j$ is of the form

$$f_j(s) = s^l(1 + s)^{-\alpha_j - 1}, \ l, \alpha_j \in \mathbb{Z}_{\geq 0}.$$ 

More precisely, we shall need the following family of explicit integrals. Let $p, l \in \mathbb{Z}_+$, and $\alpha \in \mathbb{Z}_+^p$ a multiindex, for $s \in (\mathbb{C} \setminus \mathbb{R}_-)^p$ we consider

$$H_\alpha^{(p)}(s, l) = \int_0^\infty x^l(1 + x)^{-\alpha_0 - 1} \prod_{j=1}^p (x + s_j)^{-\alpha_j - 1} dx$$

$$= \int_0^\infty x^{|\alpha| + p - 1 - l} (1 + x)^{-\alpha_0 - 1} \prod_{j=1}^p (1 + s_j x)^{-\alpha_j - 1} dx$$

$$= (-1)^{|\alpha| + p - 1 - l}[1^{\alpha_0 + 1}, s_1^{\alpha_1 + 1}, \ldots, s_p^{\alpha_p + 1}]id^l \log.$$
Here the notation $[1^{\alpha_0+1}, s_1^{\alpha_1+1}, \ldots, s_p^{\alpha_p+1}]$ stands for the divided difference of order $\alpha + p + 1$ of the function $\text{id}^l \log$ and $\text{id}^l \log$ is the product of two functions: the log function and the function $x \mapsto x^l$. We shall only need some special cases, in which $l = 0$. When $p = 1$, we get the “modified logarithm” in [13, sec. 3 and 6]:

$$L_0 = [1, s] \log = \frac{\log s}{s - 1}$$

(8.3.11)

which is the generating function of the Bernoulli numbers up to a substitution $s \mapsto e^s$.

While for $j \geq 0$,

$$L_j(s) = H^{(1)}_{(1,0)}(s) = (-1)^j [1^{j+1}, s] \log$$

$$= (-1)^j [1^j, s] L_0 = \frac{1}{j!} \partial_s^j (s^j L_0(s)).$$

(8.3.12)

For $p = 2$,

$$H^{(2)}_{(\alpha_0, \alpha_1, \alpha_2)}(s, t) = (-1)^{\alpha_1+1} [1^{\alpha_0+1}, s^{\alpha_1+1}, t^{\alpha_2+1}] \log$$

$$= (-1)^{\alpha_1+1} [1^{\alpha_0}, s^{\alpha_1}, t^{\alpha_2}] L_0$$

$$= \frac{(-1)^{\alpha_1+\alpha_0+1}}{\alpha_1! \alpha_2!} \partial_s^{\alpha_1} \partial_t^{\alpha_2} \frac{1}{t - s} (L_{\alpha_0}(t) - L_{\alpha_0}(s)).$$

(8.3.13)

8.4 Modular curvature on noncommutative two tori

Theorem 8.4.1. Let $M_\Theta = \mathbb{T}_\Theta^2$, the noncommutative two torus and $\pi^\Theta(P_k) = \pi^\Theta(k) \Delta$ be a perturbed Laplacian with respect to a Weyl factor $\pi^\Theta(k)$. Then the second heat coefficient of $\pi^\Theta(P_k)$ can be express in terms of the modular scalar curvature $R(k) \in C^\infty(M)$ in the following way: $\forall f \in C^\infty(M)$,

$$V_2(\pi^\Theta(f), \pi^\Theta(P_k)) = \int_M f \times_\Theta R dg$$

(8.4.1)
where

\[
\mathcal{R}(k) = \left( k^{-1} \mathcal{K}(\triangle)(\partial^2 k) + k^{-2} \mathcal{G}(\triangle^{(1)}, \triangle^{(2)})(\partial k \partial k) \right) g^{-1},
\]

(8.4.2)

the modular curvature functions \( \mathcal{K} \) and \( \mathcal{G} \) can be written as linear combinations of simple\(^1\) divided differences of the log function. The explicit expression are given in (8.4.5) and (8.4.9).

Remark.

(1) Originally, the integrand of the right hand side of (8.4.1) should be \( \mathcal{R} \times_{\Theta} f \), but since \( \mathcal{R} \) is self-adjoint and \( \int_M \) is a trace with respect to the deformed product, we see that

\[
\int_M \mathcal{R} \times_{\Theta} f \, dg = \int_M f \times_{\Theta} \mathcal{R} \, dg.
\]

(2) One can verify that the functions \( \mathcal{K} \) and \( \mathcal{G} \) agree with the modular curvature functions \(-F_{0,0}(s)\) and \( G_{0,0}^{\Theta}(s,t) \) in [21, Thm 3.2] in the following way:

\[
\mathcal{K}(s) = -F_{0,0}(s), \quad \mathcal{G}(s,t) = \frac{1}{s} G_{0,0}^{\Theta}(s,t).
\]

The appearance of the negative sign in front of \( F_{0,0} \) is due to the fact that \((\partial^2 k)g^{-1} = -\Delta k\). In [21, Thm 3.2], the quadratic form is defined as \((k^{-1} \partial k)(k^{-1} \partial k)g^{-1}\), compare to our quadratic form \( k^{-2}(\partial k)(\partial k)g^{-1} \). The factor \(1/s\) in front of \( G_{0,0}^{\Theta} \) that stands for the modular operator \( \triangle^{-1} = k^{-1}(\cdot)k \) is exactly the price to pay to move \( k^{-1} \) in front of \( \partial k \).

\(^1\)That means at most the third divided difference occurs.
(3) Section 3 and 4 in [21] shows that when we rephrase the formula for $R$ in terms of $h = \log k$, the resulting modular curvature functions agrees with those with respect to the the degree zero Laplacian with complex structure $\sqrt{-1}$ appeared in [12]. Therefore our calculation gives a new confirmation of the result on noncommutative two tori which is independent of the aid of CAS.

Proof. According to lemma 8.3.1, when $\dim M = 2$.

$$R = \int_{0}^{\infty} \int_{S^{m-1}} b_{2}(\xi, -1) d\sigma_{S^{m-1}}(rdr),$$

where $\int_{S^{m-1}} b_{2}d\sigma_{S^{m-1}}$ has been calculated in equation (8.2.17). Since the Laplacian $\Delta$ is taken with respect to the flat metric, the scalar curvature term in (8.2.17) is vanished. The contribution to the one variable function $K$ comes from $-b_{0}^{2}k(\Delta k)b_{0}|\xi|^{2}$ and $2b_{0}^{3}k^{2}(\Delta k)b_{0}|\xi|^{4}$. Replace $|\xi|$ by $r$, and $\lambda$ by $-1$, then $b_{0}$ becomes $1/(kr^{2} + 1)$.

Here, of course, the inverse is taken in the deformed algebra $C^{\infty}(T^{2}_{\Theta})$. Perform a substitution $r \mapsto r^{2}$ and apply the rearrangement lemma:

$$\int_{0}^{\infty} 2b_{0}^{3}k^{2}(\Delta k)b_{0}|\xi|^{4} rdr = \int_{0}^{\infty} \frac{(kr)^{2}}{(kr + 1)^{3}} \Delta k \frac{1}{kr + 1} dr = k^{-1}K_{1}(\Delta)(\Delta k)$$

$$\int_{0}^{\infty} -b_{0}^{2}k(\Delta k)b_{0}|\xi|^{2} rdr = -\frac{1}{2} \int_{0}^{\infty} \frac{kr}{(kr + 1)^{2}} \Delta k \frac{1}{kr + 1} dr = k^{-1}K_{2}(\Delta)(\Delta k)$$

where

$$K_{1}(s) = \int_{0}^{\infty} \frac{u^{2}}{(u + 1)^{3}} \frac{1}{su + 1} du, \quad K_{2}(s) = -\frac{1}{2} \int_{0}^{\infty} \frac{u}{(u + 1)^{2}} \frac{1}{su + 1} du. \quad (8.4.3)$$

Using divided difference notations, we rewrite:

$$K_{1}(s) = H_{(2,0)}^{(1)}(s, 0), \quad K_{2}(s) = -\frac{1}{2} H_{(1,0)}^{(1)}(s, 0). \quad (8.4.4)$$

116
Therefore:

\[
\mathcal{K}(s) = (\mathcal{K}_1 + \mathcal{K}_2)(s) \\
= \frac{s^2 - 4s + 2\log(s) + 3}{(s - 1)^3} - \frac{1}{2}\frac{s - \log(s) - 1}{(s - 1)^2} \\
= -2s + (s + 1)\log(s) + 2 \\
\]

Perform similar calculation for the remaining terms in (8.2.17). The summand

\[4b_0^3k^2(\partial k)b_0(\partial k)b_0|\xi|^6\] yields:

\[
2\int_0^\infty \frac{k^2}{(kr + 1)^3}(\partial k)\frac{1}{kr + 1}(\partial k)\frac{1}{kr + 1}r^3dr = k^{-2}\mathcal{G}_1(\triangle^{(1)}, \triangle^{(2)}) (\partial k\partial k)
\]

with

\[
\mathcal{G}_1(s_1, s_2) = 2\int_0^\infty \frac{r^3}{(r + 1)^3} \frac{1}{s_1r + 1} \frac{1}{s_1s_2r + 1} dr \\
= 2H_{(2,0,0)}^{(2)}(s_1, s_2, 0).
\]

The summand \(-4b_0^3k(\partial k)b_0(\partial k)b_0|\xi|^4\) yields:

\[
-2\int_0^\infty \frac{k}{(kr + 1)^2}(\partial k)\frac{1}{kr + 1}(\partial k)\frac{1}{kr + 1}r^2dr = k^{-2}\mathcal{G}_2(\triangle^{(1)}, \triangle^{(2)}) (\partial k\partial k)
\]

where

\[
\mathcal{G}_2(s_1, s_2) = -2\int_0^\infty \frac{r^2}{(r + 1)^2} \frac{1}{s_1r + 1} \frac{1}{s_1s_2r + 1} dr \\
= -2H_{(1,0,0)}^{(2)}(s_1, s_2, 0).
\]

The last one \(2b_0^2k(\partial k)b_0^2k(\partial k)b_0|\xi|^6\) yields:

\[
\int_0^\infty \frac{k}{(kr + 1)^2}(\partial k)\frac{k}{(kr + 1)^2}(\partial k)\frac{1}{kr + 1}r^3dr = k^{-2}\mathcal{G}_3(\triangle^{(1)}, \triangle^{(2)}) (\partial k\partial k)
\]
where

$$G_3(s_1, s_2) = \int_0^\infty \frac{r^2 r s_1}{(r + 1)^2 (s_1 r + 1)^2 s_1 s_2 r + 1} \, dr$$

$$= s_1 H^{(2)}_{(1,1,0)}(s_1, s_2, 0).$$

Sum up:

$$G(s,t) = (G_1 + G_2 + G_3)(s,t)$$

$$= (st-1)^3 \log(s) - (s-1)(t-1)(st-1)(st-2)(st-1) + (s-1)(2t-1)(st-1) \log(st)$$

$$= (s-1)^2 s(t-1)(st-1)^3$$

(8.4.9)

$\square$

8.5 Modular curvature for even dimensional toric noncommutative manifolds

The main difference dimension two and higher dimensions is that we have to differentiate the $b_2$ term in (8.2.17) in $\lambda$ to certain amount of times according to lemma 8.3.1. Another difference is that when the dimension of the manifold is strictly greater than the dimension of the action torus, such as the non-commutative four sphere $S^4_\theta$, it is impossible to rephrase the modular curvature in terms of $h \triangleq \log k$ and $\nabla \triangleq \log \Delta = -[h, \cdot]$. Therefore we keep our theorems in agreement with the notations in [21, Thm. 3.2] instead of [21, Cor. 3.3].

A significant aspect of the modular curvature function holds for general $M_\Theta$ is that both $K$ and $G$ can be written as linear combinations of divided differences of the modified logarithm $L_0 = \log s/(s-1)$ which is the generating function of Bernoulli
numbers after the substitution $s \mapsto e^s$. The importance of this fact was explained in [20]: it is one of the “conceptual explanations” behind the two novel aspects of the modular curvature functions $\tilde{K}_0$ and $\tilde{H}_0$ appeared in the expression of the Gaussian curvature $\text{grad}_h F$ for noncommutative two tori [7, Thm. 4.8]:

(1) $\tilde{K}_0$ is (upto a factor $1/8$) the generating function of Bernoulli numbers;

(2) $\tilde{H}_0$ is sum of divided differences of $\tilde{K}_0$.

Let us start to prove this property. The summands in (8.2.17) are of the form:

$$b_0^l \rho_1 b_0^l_1 \rho_2 b_0^l_2 \cdots \rho_j b_0^l_j,$$

in which only the $b_0$ factors carry $\lambda$. Also for $l \in \mathbb{Z}_+$,

$$\frac{d}{d\lambda} b_0^l = -l b_0^{l+1}.$$ 

As a result

$$\frac{d}{d\lambda} \left( b_0^l \rho_1 b_0^l_1 \rho_2 b_0^l_2 \cdots \rho_j b_0^l_j \right)$$

$$= (-l_0) b_0^{l+1} \rho_1 b_0^l_1 \rho_2 b_0^l_2 \cdots \rho_j b_0^l_j + (-l_1) b_0^l_1 \rho_1 b_0^{l+1} \rho_2 b_0^l_2 \cdots \rho_j b_0^l_j + \ldots$$

$$+ (-l_j) b_0^l \rho_1 b_0^l_1 \rho_2 b_0^l_2 \cdots \rho_j b_0^{l+1}.$$

After applying the rearrangement lemma, each summand gives rise a modular curvature function in the family $H_{\alpha}^{(\rho)}(s,0)$ defined in (8.3.10). Let us carry out the computation in dimension four. Start with, for instance, the term $2b_0^3 k^2 (\Delta k) b_0 |\xi|^4$ that gives rise to a modular curvature function $H_{(2,0)}^{(1)}(s,0)$ in dimension two, it becomes

$$-6b_0^4 k^2 (\Delta k) b_0 |\xi|^4 - 2b_0^3 k^2 (\Delta k) b_0^2 |\xi|^4$$

119
after \(d/d\lambda\). Take the factor \(r^{(d-2)/2} dr = rdr\) and the definition of \(H^{(p)}_{\alpha}(s, m)\) into account. The new modular curvature function is given by

\[3H^{(1)}_{(3,0)}(s, 0) + H^{(1)}_{(2,1)}(s, 0).\]

Let us look at another example, say the summand associated to \(H^{(2)}_{(1,0,0)}(s_1, s_2, 0)\) (the \(\mathcal{G}_2\) term in (8.4.7) up to a factor \(1/2\)), it will contribute three terms after applying \(d/d\lambda\):

\[2H^{(2)}_{(2,0,0)}(s_1, s_2, 0) + H^{(2)}_{(1,1,0)}(s_1, s_2, 0) + H^{(2)}_{(1,0,1)}(s_1, s_2, 0).\]

Apply the argument to all summands in (8.2.17) that involve \(K\) and \(G\), we obtained that in dimension four:

\[
\mathcal{K}(s, 4) = \frac{1}{2} \left( 3H^{(1)}_{(3,0)}(s, 0) + H^{(1)}_{(2,1)}(s, 0) \right) - \frac{1}{2} \left( 2H^{(1)}_{(2,0)}(s, 0) + H^{(1)}_{(1,1)}(s, 0) \right).
\] (8.5.1)

and

\[
\mathcal{G}(s_1, s_2, 4) = 3H^{(2)}_{(3,0,0)}(s_1, s_2, 0) + H^{(2)}_{(2,1,0)}(s_1, s_2, 0) + H^{(2)}_{(2,0,1)}(s_1, s_2, 0)
- \frac{3}{2} (2H^{(2)}_{(2,0,0)}(s_1, s_2, 0) + H^{(2)}_{(1,1,0)}(s_1, s_2, 0) + H^{(2)}_{(1,0,1)}(s_1, s_2, 0))
+ \frac{1}{2} s_1 \left( 2H^{(2)}_{(2,1,0)}(s_1, s_2, 0) + 2H^{(2)}_{(1,2,0)}(s_1, s_2, 0) + H^{(2)}_{(1,1,1)}(s_1, s_2, 0) \right).
\] (8.5.2)

By repeating this argument, we conclude that for any even dimensional toric non-commutative manifold \(M_\theta\), the modular curvature functions \(\mathcal{K}\) and \(\mathcal{G}\) are linear combinations of the family \(H^{(p)}_{\alpha}(s, 0)\).

From (8.3.10), we can see that the explicit expressions of \(H^{(p)}_{\alpha}(s, 0)\) are quite lengthy, indeed, there are tons of cancellations between the terms appeared in the right hand side of (8.5.1) and (8.5.2). An easier way to calculate the explicit
expressions of $K$ and $G$ is to express them as integrals (cf. (8.5.5) and (8.5.9)), and
then leave the calculation (both differentiation and integration) to a computer algebra system.

**Theorem 8.5.1.** Let $M_\Theta$ be a noncommutative toric manifold whose dimension is
an even integer $m$ and $\pi^\Theta(P_k) = \pi^\Theta(k)\Delta$. Then the associated scalar curvature $R(k)$
(the density function of the second heat coefficient functional, cf. (8.4.1)), is of the form:

$$R(k) = \left( k^{-m/2}K(\Delta, m)(\partial^2 k) + k^{-(m+2)/2}G(\Delta^{(1)}, \Delta^{(2)}, m) (\partial k \partial k) \right) g^{-1}$$  \hspace{1cm} (8.5.3)

$$+ c_m k^{-\left(\frac{m}{2} - 1\right)} S_{\Delta},$$

where $c_m$ is a constant depending only on the dimension of the manifold and $S_{\Delta} \in C^\infty(M)$ is the scalar curvature function. The modular curvature functions $K$ and $G$
are given by (8.5.5) and (8.5.9) respectively. As before, they are linear combinations
of simple divided differences of the log function.

**Proof.** As explained in lemma 8.3.1, we have to differentiate $b_2$ in the $\lambda$ variable
$j$-times before performing the integration in $\int_0^\infty (\cdot) dr$, where $j$ is the smallest integer
such that $-4 - 2j$ is greater than $m$. Since only the $b_0$ factors in (8.2.17) involves
$\lambda$, we can apply the $\frac{d^j}{d \lambda^j}$ directly onto the modular functions before integrate in $r$
variable. Namely, to compute $K(s, m)$, we take the integrands in (8.4.3) and replace
all the “+1”s in the denominators by $-\lambda$:

$$T_1^{(1)}(s, r, \lambda) = \frac{2}{m (r - \lambda)^2} \frac{1}{sr - \lambda}$$  \hspace{1cm} T_2^{(1)}(s, r, \lambda) = \frac{1}{2} \frac{r}{(r - \lambda)^2} \frac{1}{sr - \lambda}$$  \hspace{1cm} (8.5.4)
then
\[ K(s, m) = \int_0^{\infty} \frac{d^j}{d\lambda^j} \bigg|_{\lambda=-1} \left( T_1^{(1)}(s, r, \lambda) + T_2^{(1)}(s, r, \lambda) \right) r^{(m-2)/2} dr, \quad (8.5.5) \]

the factor \( r^{(m-2)/2} \) appears because the volume form \( d\xi_g^{-1} = r^{m-1} dr \sigma_{S^{m-1}} \) and the substitution \( r \mapsto r^2 \). Similarly, for the two-variables modular function \( G(s_1, s_2) \), we define:

\[ T_1^{(2)}(s, r, \lambda) = \frac{4}{m} \frac{r^3}{(r-\lambda)^3} \frac{1}{s_1 r - \lambda s_2 r - \lambda} \quad (8.5.6) \]
\[ T_2^{(2)}(s, r, \lambda) = -(1 + \frac{2}{m}) \frac{r^2}{(r-\lambda)^2} \frac{1}{s_1 r - \lambda s_2 r - \lambda} \quad (8.5.7) \]
\[ T_3^{(2)}(s, r, \lambda) = \frac{2}{m} \frac{r^2}{s_1 r - \lambda s_2 r - \lambda} \quad (8.5.8) \]

according to (8.4.6), (8.4.7) and (8.4.8). Therefore

\[ G(s_1, s_2, m) = \int_0^{\infty} \frac{d^j}{d\lambda^j} \bigg|_{\lambda=-1} \left( \sum_{j=1}^{3} T_j^{(2)}(s, r, \lambda) \right) r^{(m-2)/2} dr. \quad (8.5.9) \]

At last, we handle the scalar curvature term. We would like to compute

\[ \int_0^{\infty} \frac{d^j}{d\lambda^j} \bigg|_{\lambda=-1} (t_0^2 k^2 S_{\Delta} b_0 |\xi|^2)(r^{m-1} dr), \]

where \( j \) is the smallest integer such that \( 3 + j > m/2 \). After moving \( S_{\Delta} \) out of the integral (since it is \( \mathbb{T}^n \)-invariant) and a substitution \( r \mapsto r^2 \), the integral above becomes:

\[ \left( \frac{1}{2} k^{-\frac{m}{2} - 2} \int_0^{\infty} \frac{d^j}{d\lambda^j} \bigg|_{\lambda=-1} \frac{(kr)^{m/2}}{(kr - \lambda)^2} dr \right) S_{\Delta} = k^{-\frac{m}{2} - 2} f(L_k)(S_{\Delta}), \]

122
where $L_k$ is the left multiplication by $k$. By an operator substitution lemma (for instance, cf. [20, Lem. 2.3]) the function $f(s)$ is equal to

$$f(s) = \frac{1}{2} \int_0^\infty \frac{d^j}{d\lambda^j} \bigg|_{\lambda=-1} \frac{(sr)^{m/2}}{(sr - \lambda)^j} dr = \left( \frac{1}{2} \int_0^\infty \frac{d^j}{d\lambda^j} \bigg|_{\lambda=-1} \frac{r^{m/2}}{(r - \lambda)^j} dr \right) \frac{1}{s}.$$

Therefore

$$\left( \frac{1}{2} \int_0^\infty \frac{d^j}{d\lambda^j} b_0^2 k^2 r^{m/2} dr \right) S_\Delta = c k^{-\left(\frac{m}{2} - 1\right)} S_\Delta,$$

where the constant $c$ depends only on $m$. \hfill \Box

**Corollary 8.5.2.** Let $M_\Theta$ be a toric noncommutative manifold of dimension four. For the perturbed Laplacian $\pi^\Theta(P_k) = \pi^\Theta(k) \Delta$, the modular scalar curvature is simply:

$$R(k) = c k^{-1} S_\Delta. \quad (8.5.10)$$

**Proof.** Using (8.5.5) and (8.5.9), or (8.5.1) and (8.5.2), we conclude that both modular functions $K(s)$ and $G(s_1, s_2)$ are zero, therefore only the scalar curvature term survives. \hfill \Box

### 8.6 Comparison with [15] and [16]

To compare with [15, lemma 3.3], we consider the perturbed Laplacian $\pi^\Theta(P_k)$ whose symbol is given by $\sigma(P_k) = p_2 + p_1 + p_0$ with

$$p_2 = k |\xi|^2, \quad p_1 = -\frac{i}{2} \partial k D |\xi|^2, \quad p_0 = -\Delta(k) + ((\partial k) k^{-1}(\partial k)) g^{-1}, \quad (8.6.1)$$

where the Laplacian $\Delta$ is associated to the flat metric on $T^4$. Based on the previous computation, the contribution to the modular curvature functions $K$ and $G$ from $p_2$...
to the modular functions is zero, and the scalar curvature function $S_\Delta$ vanishes as well. It remains to count the contribution from $p_1$ and $p_0$ which is in the last line of the $b_2$ term in (8.1.13):

$$-iD(b_0 p_1 b_0)(\partial p_2)b_0 - b_0 p_0 b_0 - b_1 p_1 b_0. \quad (8.6.2)$$

Plug in the expressions in (8.6.1), we obtain:

$$-iD(b_0 p_1 b_0)(\partial p_2)b_0 = -\frac{1}{2} \left( b_0^2 k(\partial k)b_0(\partial k)b_0(D|\xi|^2)^2|\xi|^2 + b_0(\partial k)b_0^2 k(\partial k)b_0(D|\xi|^2)^2|\xi|^2 \right), \quad (8.6.3)$$

$$+ \frac{1}{2}(b_0(\partial k)b_0(\partial k)b_0(D^2|\xi|^2)|\xi|^2)$$

$$- b_1 p_1 b_0$$

$$= \left( -\frac{1}{2} b_0^2 k(\partial k)b_0(\partial k)b_0(D|\xi|^2)^2|\xi|^2 + \frac{1}{4} b_0(\partial k)b_0(\partial k)b_0(D|\xi|^2)^2 \right), \quad (8.6.4)$$

$$- b_0 p_0 b_0$$

$$= b_0(\Delta k)b_0 - b_0(\partial k)k^{-1}(\partial k)b_0 g^{-1}. \quad (8.6.5)$$

After integration on the sphere, they yield:

$$\int_{S^2_M} (-b_0 p_0 b_0)d\omega_{x,g}$$

$$= \text{Vol}(S^3) \left( b_0(\Delta k)b_0 - b_0(\partial k)k^{-1}(\partial k)b_0 g^{-1} \right),$$

and

$$\int_{S^2_M} (-iD(b_0 p_1 b_0)(\partial p_2)b_0)d\omega_{x,g}$$

$$= -\frac{1}{2} \text{Vol}(S^3) \left( b_0^2 k(\partial k)b_0(\partial k)b_0 |\xi|^4 + b_0(\partial k)b_0^2 k(\partial k)b_0 |\xi|^4 \right) g^{-1}$$

$$+ \text{Vol}(S^3)b_0(\partial k)b_0(\partial k)b_0 |\xi|^2 g^{-1},$$

124
and

\[
\int_{S^3} (-b_1 p_1 b_0) d\omega_{x,g} = \text{Vol}(S^3) \left( -\frac{1}{2} b_0^2 k(\partial k)b_0(\partial k) b_0 |\xi|^4 + \frac{1}{4} b_0(\partial k)b_0(\partial k) b_0 |\xi|^2 \right) g^{-1}.
\]

As before, we perform a substitution $|\xi| = r \mapsto r^2$, and apply the rearrangement lemma,

\[
\int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} (b_0(\Delta k)b_0)(rdr/2) = k^{-2} K(\Delta)(\Delta k),
\]

\[
\int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} (b_0^2 k(\partial k)b_0(\partial k)r^2)(rdr/2) = k^{-3} \tilde{G}_1(\triangle^{(1)}, \triangle^{(2)})(\partial k \partial k),
\]

\[
\int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} (b_0(\partial k)b_0^2 k(\partial k)b_0(\partial k)r^2)(rdr/2) = k^{-3} \tilde{G}_2(\triangle^{(1)}, \triangle^{(2)})(\partial k \partial k),
\]

\[
\int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} (b_0(\partial k)b_0(\partial k)b_0^{2} r^2)(rdr/2) = k^{-3} \tilde{G}_3(\triangle^{(1)}, \triangle^{(2)})(\partial k \partial k),
\]

\[
\int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} (b_0(\partial k)k^{-1}(\partial k)b_0)(rdr/2) = k^{-3} \tilde{G}_4(\triangle^{(1)}, \triangle^{(2)})(\partial k \partial k)
\]

with

\[
K(s) = \frac{1}{2} \int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} \left( \frac{1}{r - \lambda} \frac{1}{sr - \lambda} \right) rdr = \frac{1}{2s}, \quad (8.6.6)
\]

\[
\tilde{G}_1(s, t) = \frac{1}{2} \int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} \left( \frac{r^3}{(r - \lambda)^2 (sr - \lambda)(str - \lambda)} \right) dr = \frac{1}{2s^2 t}, \quad (8.6.7)
\]

\[
\tilde{G}_2(s, t) = \frac{1}{2} \int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} \left( \frac{sr^3}{(r - \lambda)(sr - \lambda)^2 (str - \lambda)} \right) dr = \frac{1}{2s^2 t}, \quad (8.6.8)
\]

\[
\tilde{G}_3(s, t) = \frac{1}{2} \int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} \left( \frac{r^2}{(r - \lambda)(sr - \lambda)(str - \lambda)} \right) dr = \frac{1}{2s^2 t}, \quad (8.6.9)
\]

\[
\tilde{G}_4(s, t) = \frac{1}{2s} \int_{0}^{\infty} \left. \frac{d}{d\lambda} \right|_{\lambda=-1} \left( \frac{1}{r - \lambda}(sr - \lambda) \right) dr = \frac{1}{2s^2 t}. \quad (8.6.10)
\]

We summarise the result as follows.
Theorem 8.6.1. Let $M_\Theta = \mathbb{T}_\Theta^4$ be a noncommutative four torus. With respect to the noncommutative metric defined by the perturbed Laplacian $\pi_\Theta(P_k)$ given in Eq. (8.6.1), the modular curvature is of the form:

$$R = k^{-2}K(\Delta)(\Delta k) + k^{-3}G(\Delta^{(1)}, \Delta^{(2)})(\partial k \partial k),$$

with

$$K(s) = \frac{1}{2s}, \quad G(s, t) = -\frac{5}{8s^2t}. \quad (8.6.11)$$

The result should be compared with [16, Eq. (1)].
BIBLIOGRAPHY


