THE Riemann Integral

A Thesis Presented for the Degree of Master of Art

by

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INTRODUCTION

It is the purpose of this thesis to formulate a precise definition of an integral and to inquire into the conditions under which an integral exists.

In the first chapter we define and discuss the properties of the concepts involved in the definition of the Riemann integral. We then define Riemann integration of bounded functions over a finite interval. The theorems of this chapter prove the conditions under which a function is Riemann integrable, the most important of which is: A necessary and sufficient condition that a function be Riemann integrable is that the points of discontinuity form a set of measure zero. We have included that portion of Jordan content and Lebesgue measure necessary to this important theorem in order that this thesis may be complete. We then discuss the properties of the Riemann integral of a bounded function over a finite interval.

In the second chapter, we define and discuss the properties of the Indefinite integral. Our most important theorem is the theorem which proves what conditions are necessary in order that the operations of differentiation and of integration are inverse operations, the fundamental theorem of the Integral Calculus.
In the third chapter, we define the Riemann integrals of unbounded functions over a finite interval. We also inquire into how far the properties of the Riemann integral of a bounded function over a finite interval may be extended to the Riemann integral of unbounded functions over a finite interval.

In the fourth chapter, we define the Riemann integrals of bounded functions over an unbounded interval and express some specific conditions to determine whether a function has a Riemann integral. We also inquire into how far the properties of the Riemann integral of a bounded function over a finite interval may be extended to the Riemann integral of a bounded function over an unbounded interval.

In the fifth chapter we define the Riemann integral of unbounded functions over an unbounded interval.

Chapters four and five contain some convergence tests and examples for the two types of infinite integrals.
CHAPTER I

BOUNDED FUNCTIONS OVER A FINITE INTERVAL

Before defining the Riemann Integral, we wish to discuss the properties of the concepts involved in the definition of the Riemann integral.

Let \( I \) be the fixed finite linear interval \((a, b)\), \( a \leq x \leq b \). Let the linear interval be divided into a number of non-overlapping closed subintervals, \( a = x_0 < x_1 < \cdots < x_n = b \), where \( x_1, x_2, \ldots, x_{n-1} \) are points of division. Designate this subdivision by \( D \) and a particular interval of the subdivision \( x_i \leq x \leq x_{j} \) by \( \Delta \) and let \(|\Delta|\) be the length of the interval \( \Delta \).

By a refinement of \( D \) we mean any subdivision which contains all the points of division of \( D \).

Let \( D_1 \) and \( D_2 \) be any two subdivisions. By the product of these two subdivisions, \( D_1D_2 \), we mean the subdivision which contains all the points of division of \( D_1 \) and also all the points of division of \( D_2 \).

Note: If \( D = D_1D_2 \), then \( D \) is a refinement of \( D_1 \) and \( D \) is also a refinement of \( D_2 \).
Define $\|D\|$ to be the largest subinterval in the subdivision $D$. This is called the norm of the subdivision $D$.

For a function $y = f(x), x \in I : a \leq x \leq b$ and a subinterval $\Delta$ of $I$, we define:

$$\overline{B}(\Delta, f) = \text{l.u.b. } f(x), x \in \Delta$$
$$\underline{B}(\Delta, f) = \text{g.l.b. } f(x), x \in \Delta,$$

where the symbol "$\in$" means included in.

Define:

$$\Phi^*(f, D) = \sum_{\Delta \in D} |\Delta| \overline{B}(\Delta, f)$$
$$\Phi^*(f, D) = \sum_{\Delta \in D} |\Delta| \underline{B}(\Delta, f).$$

1.1 Lemma: $\Phi^*(f, D) \geq \Phi_*(f, D)$ for any subdivision $D$.

Proof:

Since $\overline{B}(\Delta, f) \geq \underline{B}(\Delta, f)$ we have

$$\Phi^*(f, D) = \sum_{\Delta \in D} |\Delta| \overline{B}(\Delta, f) \geq \sum_{\Delta \in D} |\Delta| \underline{B}(\Delta, f) = \Phi_*(f, D)$$

1.2 Lemma: If $D'$ is a refinement of $D$, then:

a) $\Phi^*(f, D') \leq \Phi^*(f, D)$

b) $\Phi_*(f, D') \geq \Phi_*(f, D)$

Proof:

Since $D'$ is a refinement of $D$, $D'$ is by definition a subdivision which contains all the points of division of $D$. Consider any interval of the subdivision $D$, $\Delta : x_i \leq x \leq x_j$. This interval of $D$ is either an interval of $D'$ or it is the sum of a finite number of intervals of $D'$. 

2.
1.2 (cont.)

a) Case 1. This interval of $D$ is an interval of $D'$. It follows that $\sum_{\Delta \in B} |B(\Delta, \xi)| = |A|$, where $\sum_{\Delta \in D} |\Delta_n| = |A|$. This is true for every interval of $D$. Therefore

$$\sum_{\Delta \in D} |B(\Delta, \xi)| = \sum_{\Delta \in D} |B(\Delta, \xi)| \text{ or } \Phi_x(\xi, D') \leq \Phi_x(\xi, D)$$

b) Case 1. This interval of $D$ is an interval of $D'$. It follows that $\sum_{\Delta \in D} |B(\Delta, \xi)| = |A|$, where $\sum_{\Delta \in D} |\Delta_n| = |A|$. This is true for every interval of $D$. Therefore

$$\sum_{\Delta \in D} |B(\Delta, \xi)| = \sum_{\Delta \in D} |B(\Delta, \xi)| \text{ or } \Phi_x(\xi, D') \leq \Phi_x(\xi, D)$$

1.3 Lemma: If $D_1$, $D_2$ are any two subdivisions, then

$$\Phi_x(\xi, D_2) \leq \Phi_x(\xi, D_1)$$

Proof:

Set $D = D_1, D_2$. It then follows from lemmas 1.1 and 1.2 that:

$$\Phi_x(\xi, D_2) \leq \Phi_x(\xi, D) \leq \Phi_x(\xi, D) \leq \Phi_x(\xi, D_1)$$

Define: $\int f(x) \, d\alpha = \text{g.l.b. } \Phi_x(\xi, D)$ where g.l.b. is taken with respect to all subdivisions $D$.

Define: $\int f(x) \, d\alpha = \text{l.u.b. } \Phi_x(\xi, D)$ where l.u.b. is taken with respect to all subdivisions $D$. 

3.
1.4 Lemma: For any subdivision $D$, we have:

a) $\Phi^*(\xi, D) \leq \int f(x) \, dx$

b) $\Phi^1(\xi, D) \geq \int f(x) \, dx$

Proof:

From lemma 1.3, $\Phi^*(\xi, D) \leq \Phi^1(\xi, D)$ for every $D$.

Hence $\Phi^*(\xi, D)$ is a lower bound of $\Phi^1(\xi, D)$ for all subdivisions $D$, and $\Phi^*(\xi, D) \leq \text{g.l.b. } \Phi^1(\xi, D) = \int f(x) \, dx$, where g.l.b. is taken with respect to all subdivisions $D$.

From lemma 1.3, $\Phi^*(\xi, D) \leq \Phi^*(\xi, D)$ for every $D$.

Hence $\Phi^*(\xi, D)$ is an upper bound of $\Phi^1(\xi, D)$ for all subdivisions $D$, and $\Phi^*(\xi, D) \geq \text{l.u.b. } \Phi^1(\xi, D) = \int f(x) \, dx$, where l.u.b. is taken with respect to all subdivisions $D$.

1.5 Theorem: $\int f(x) \, dx \geq \int f(x) \, dx$

Proof:

From lemma 1.4, $\Phi^*(\xi, D) \leq \int f(x) \, dx$ for every $D$.

Therefore l.u.b. $\Phi^*(\xi, D) \leq \int f(x) \, dx$

but l.u.b. $\Phi^*(\xi, D) = \int f(x) \, dx$ and therefore

$\int f(x) \, dx \leq \int f(x) \, dx$

1.6 Lemma: Let $\Delta = \Delta' + \Delta''$ where $|\Delta| = x_j - x_i$, $x_i < x_j$, $|\Delta'| = x_k - x_i$, $|\Delta''| = x_j - x_k$, $x_i < x_k < x_j$, $\Delta \subset \Omega$.

Then $|\Delta| \mid B(\Delta_j) - |\Delta'\mid B(\Delta'_i) + |\Delta''| B(\Delta''_j) \leq 2M ||D||$

4.
1.6 (cont.) Proof:
\[ |\Delta| \bar{B}(\xi) = |\Delta'| \bar{B}(\xi) + |\Delta''| \bar{B}(\xi) + \]
\[ |\Delta'| \left[ \bar{B}(\xi) - \bar{B}(\xi') \right] + |\Delta''| \left[ \bar{B}(\xi) - \bar{B}(\xi') \right], \text{ but} \]
\[ \bar{B}(\xi) - \bar{B}(\xi') \leq |\bar{B}(\xi) - \bar{B}(\xi')| \leq |\bar{B}(\xi) + |\bar{B}(\xi')| \leq 2M \]
where M is equal to the upper bound of \(|f(x)|\)
over \((a, b)\), and \(\bar{B}(\xi) - \bar{B}(\xi') \leq |\bar{B}(\xi) - \bar{B}(\xi')| \leq \]
\[ |\bar{B}(\xi)| + |\bar{B}(\xi')| \leq 2M \]
Therefore \[ |\Delta| \bar{B}(\xi) - |\Delta'| \bar{B}(\xi') + |\Delta''| \bar{B}(\xi') \leq \]
\[ |\Delta'| 2M + |\Delta''| 2M = 2M |\Delta| \leq 2M \parallel D \parallel \]

1.7 Lemma: \( \Phi^*(\xi, D) \leq \Phi^*(\xi, D_n) + 2M \parallel D \parallel \)

Proof:
Let \( D \) be a subdivision of a closed interval.
Let \( D_n \) be the subdivision obtained from \( D \) by
the addition of one point: \( D_1 \) - obtained from
\( D_1 \) by the addition of one point: \( \ldots \)
\( D_n \) - obtained from \( D_{n-1} \) by the addition of
one point. Therefore, it is true that \( \parallel D \parallel \geq \)
\( \parallel D_1 \parallel \geq \ldots \geq \parallel D_n \parallel \). It follows from lemma 1.2 that
\( \Phi^*(\xi, D_1) \leq \Phi^*(\xi, D) \), but from lemma 1.8, it
follows that:

1) \( \Phi^*(\xi, D) \leq \Phi^*(\xi, D_1) + 2M \parallel D \parallel \)
1.7 (cont.)

2) $\Phi^*(f, D) \leq \Phi^*(f, D_2) + 2M \|D\| + 2M \|D_2\|
   \leq \Phi^*(f, D_2) + 2M \cdot 2 \cdot \|D\|

3) $\Phi^*(f, D) \leq \Phi^*(f, D_3) + 2M \|D\| + 2M \|D_2\|
   + 2M \|D_3\| \leq \Phi^*(f, D_3) + 2M \cdot 3 \cdot \|D\|

\vdots

n) $\Phi^*(f, D) \leq \Phi^*(f, D_n) + 2M \cdot n \cdot \|D\|

which was to be proved.

The limit superior means by definition l.u.b. of the limit points of a sequence, and the limit inferior means by definition g.l.b. of the limit points of a sequence.

1.8 Theorem: For any sequence of subdivisions with $\|D_n\| \to 0$, we have $\Phi^*(f, D_n) \to \int f(x) \, dx$

Proof:

Limit Inferior $\Phi^*(f, D_n) \geq \int f(x) \, dx$

From lemma 1.7, it follows that $\Phi^*(f, D_n) \leq \Phi^*(f, D \cdot D_n) + 2M \cdot \|D\| \cdot \|D_n\|$ where $D$ is any fixed subdivision and $m$ is the number of subdivisions of $D$. Since $D \cdot D_n$ is a refinement of $D$, it follows from lemma 1.2 that $\Phi^*(f, D \cdot D_n) \leq \Phi^*(f, D_n)$.
1.8 \text{(cont.)} \\

Therefore \( \Phi^*(f, D_n) + 2M_m \| D_n \| \leq \Phi^*(f, D) + 2M_m \| D_n \| \) \\
and \( \Phi^*(f, D_n) \leq \Phi^*(f, D) + 2M_m \| D_n \| \) . \\

For \( \| D_n \| \to 0 \) , \( 2M_m \| D_n \| \to 0 \) . Therefore \( \Phi^*(f, D_n) \leq \Phi^*(f, D) \) as \( \| D_n \| \to 0 \) , from which we 

conclude that Limit Superior \( \Phi^*(f, D, n) \leq \Phi^*(f, D) \) 

for every subdivision \( D \) , as \( \| D_n \| \to 0 \).

Since g.l.b. \( \Phi^*(f, D) = \int f(x) \, dx \) , it follows that 

Limit Superior \( \Phi^*(f, D, n) \leq \int f(x) \, dx \) as \( \| D_n \| \to 0 \).

Therefore, Limit Superior \( \Phi^*(f, D, n) \leq \int f(x) \, dx \leq \) 

Limit Inferior \( \Phi^*(f, D) \) . From which we conclude:

Limit Superior \( \Phi^*(f, D, n) \to \int f(x) \, dx \) \( \Phi^*(f, D, n) \) 

which is the condition that \( \Phi^*(f, D, n) \to \int f(x) \, dx \) 

as \( \| D_n \| \to 0 \).

1.9 Lemma: Let \( \Delta = \Delta' + \Delta'' \) where \( |\Delta| = x_j - x_i , x_i < x_j \) , \\
|\Delta'| = x_k - x_i , |\Delta''| = x_j - x_k , x_i < x_k < x_j , \Delta \in D \) 

Then \( |\Delta'| B(\Delta', f) + |\Delta''| B(\Delta'', f) = 2M \| \Delta \| D \| \) 

Proof: 

\( |\Delta'| B(\Delta', f) + |\Delta''| B(\Delta'', f) = |\Delta| B(\Delta, f) \) + 

\( |\Delta'| [B(\Delta', f) - B(\Delta, f)] + |\Delta''| [B(\Delta'', f) - B(\Delta, f)] \) 

but \( B(\Delta', f) - B(\Delta, f) \leq |B(\Delta', f)| + |B(\Delta, f)| \leq 2M \) 

where \( M \) is equal to the upper bound of \( |f(x)| \) 

over \( (a, b) \), and \( B(\Delta', f) - B(\Delta, f) \leq 

\( |B(\Delta', f)| + |B(\Delta, f)| \leq 2M \)
Therefore \(|\Delta'| \bar{B}(\Delta', \xi') + |\Delta''| \bar{B}(\Delta'', \xi') - |\Delta| \bar{B}(\Delta, \xi) \leq 0.9  \Delta |2 M + |\Delta''| 2 M = 2 M |\Delta| \leq 2 M \|D\|.

1.10 \textbf{Lemma:} \(\Phi_x(\xi, D) \geq \Phi_x(\xi, D_n) - 2 M \cdot n \|D\|\)

\textbf{Proof:}

Let \(D\) be a subdivision of a closed interval.

Let \(D_1\) be the subdivision obtained from \(D\) by the addition of one point: \(D_2\) obtained from \(D_1\) by the addition of one point: \(\cdots\)

\(D_n\) obtained from \(D_{n-1}\) by the addition of one point. Therefore, it is true that \(\|D\| \geq \|D_1\| \geq \cdots \|D_n\|\). It follows from lemma 1.2 that \(\Phi_x(\xi, D_1) \geq \Phi_x(\xi, D)\), but from lemma 1.9, it follows that:

1) \(\Phi_x(\xi, D) \geq \Phi_x(\xi, D_1) - 2 M \|D\|\)

2) \(\Phi_x(\xi, D) \geq \Phi_x(\xi, D_2) - 2 M \|D\| - 2 M \|D, n\| = \Phi_x(\xi, D_2) - 2 M \cdot 2 \cdot \|D\|\)

3) \(\Phi_x(\xi, D) \geq \Phi_x(\xi, D_3) - 2 M \|D\| - 2 M \|D, n\| - 2 M \|D_n\| = \Phi_x(\xi, D_3) - 2 M \cdot 3 \|D\|\)

\vdots

\(n) \Phi_x(\xi, D) \geq \Phi_x(\xi, D_n) - 2 M \cdot n \|D\|

which was to be proved.
1.11 **Theorem:** For any sequence of subdivisions with \( \|D_n\| \rightarrow 0 \), we have \( \Phi_* (f, D_n) \rightarrow \int f(x) \, dx \).

**Proof:**

Limit Superior \( \Phi_* (f, D_n) \leq \int f(x) \, dx \)

From lemma 1.10, it follows that \( \Phi_* (f, D_n) \geq \Phi_* (f, D : D_n) - 2 M \, \|D_n\| \) where \( D \) is any fixed subdivision and \( m \) is the number of subdivisions of \( D \). Since \( D : D_n \) is a refinement of \( D \), it follows from lemma 1.2 that \( \Phi_* (f, D : D_n) \geq \Phi_* (f, D) \) and \( \Phi_* (f, D : D_n) - 2 M \, \|D_n\| \geq \Phi_* (f, D) - 2 M \, \|D_n\| \).

Therefore \( \Phi_* (f, D_n) \geq \Phi_* (f, D) - 2 M \, \|D_n\| \).

For \( \|D_n\| \rightarrow 0 \), \( 2 M \, \|D_n\| \rightarrow 0 \). Therefore \( \Phi_* (f, D_n) \geq \Phi_* (f, D) \) as \( \|D_n\| \rightarrow 0 \), from which we conclude that Limit Inferior \( \Phi_* (f, D_n) \geq \Phi_* (f, D) \) for every subdivision \( D \), as \( \|D_n\| \rightarrow 0 \).

Since \( f \) is bounded, \( \Phi_* (f, D) = \int f(x) \, dx \), it follows that Limit Inferior \( \Phi_* (f, D_n) \geq \int f(x) \, dx \) as \( \|D_n\| \rightarrow 0 \).

Therefore, Limit Superior \( \Phi_* (f, D_n) \leq \int f(x) \, dx \leq \) Limit Inferior \( \Phi_* (f, D_n) \). From which we conclude:

Limit Superior \( \Phi_* (f, D_n) = \int f(x) \, dx = \) Limit Inferior \( \Phi_* (f, D_n) \) which is the condition that \( \Phi_* (f, D_n) \rightarrow \int f(x) \, dx \) as \( \|D_n\| \rightarrow 0 \).

---

1.12 **Definition of the Riemann Integral**

We are now ready to define the Riemann integral of a bounded function over a finite interval.
A function is said to be Riemann integrable if \( \int f(x) \, dx = \tilde{\int} f(x) \, dx \).

Define fluctuation of \( f(x) \) over the interval \( \Delta \) to be \( \overline{\omega}(\Delta, f) - \underline{\omega}(\Delta, f) \). Call this \( \omega(\Delta, f) \). Then
\[
\sum_{\Delta \in D_n} |\Delta| \omega(\Delta, f) = \overline{\Phi}(f, D_n) - \underline{\Phi}(f, D_n).
\]

1.13 Theorem: For any sequence of subdivisions with \( \|D_n\| \to 0 \) we have
\[
\sum_{\Delta \in D_n} |\Delta| \omega(\Delta, f) \to \int f(x) \, dx - \tilde{\int} f(x) \, dx.
\]

Proof:
The fact that \( \overline{\Phi}(f, D_n) \to \int f(x) \, dx \) means by definition that for any \( \varepsilon > 0 \) it is true that there exists \( N_1(\frac{\varepsilon}{\overline{\Phi}}) \) such that for any \( n > N_1(\frac{\varepsilon}{\overline{\Phi}}) \) it follows that \( |\overline{\Phi}(f, D_n) - \int f(x) \, dx| < \frac{\varepsilon}{\overline{\Phi}} \).
The fact that \( \underline{\Phi}(f, D_n) \to \int f(x) \, dx \) means by definition that for any \( \varepsilon > 0 \) it is true that there exists \( N_2(\frac{\varepsilon}{\sqrt{\alpha}}) \) such that for any \( n > N_2(\frac{\varepsilon}{\sqrt{\alpha}}) \) it follows that \( |\underline{\Phi}(f, D_n) - \int f(x) \, dx| < \frac{\varepsilon}{\sqrt{\alpha}} \).

Choose \( N(\varepsilon) = N_1(\frac{\varepsilon}{\overline{\Phi}}) + N_2(\frac{\varepsilon}{\sqrt{\alpha}}) \). Then for any \( \varepsilon > 0 \) it is true that there exists \( N(\varepsilon) \) such that for any \( n > N(\varepsilon) \) it follows that
\[
|\overline{\Phi}(f, D_n) - (\int - \int) + (\int - \underline{\Phi})(f, f)| < \varepsilon,
\]
which is the condition that
\[
\sum_{\Delta \in D_n} |\Delta| \omega(\Delta, f) = \overline{\Phi}(f, D_n) - \underline{\Phi}(f, D_n) \to \int f(x) \, dx - \tilde{\int} f(x) \, dx.
\]

1.14 Corollary: A necessary and sufficient condition that a function \( f(x) \) be Riemann integrable in the interval is that for any sequence of subdivisions with \( \|D_n\| \to 0 \), we have
\[
\sum_{\Delta \in D_n} |\Delta| \omega(f, \Delta) \to 0.
\]

10.
1.15 Corollary: A sufficient condition that a function \( f(x) \) be Riemann integrable in the interval is that the function be continuous.

Note: This corollary also follows directly from theorem 1.29. It has been placed here to show that if one is working with continuous functions only, the discussion up to the present is sufficient.

Some knowledge of Jordan Content and Lebesgue Measure is essential in the proofs of subsequent theorems. Therefore, we wish to define these concepts of measure and to prove those theorems on content and measure essential to the Riemann Integral.

1.16 Jordan Content:

A point \( x \) is called a limit point of a set if in every interval, \((x-\epsilon, x+\epsilon)\) however small, there are an infinite number of points of the set.

A point \( x \) is called an inner point of a set, \( E \) if it may be included in an open interval \((x-\epsilon, x+\epsilon), \epsilon > 0\) such that all points of this interval are points of \( E \).

Let \( E \) be any set on the interval \((a, b)\).

Let \( \overline{E} \) designate the set plus all its limit points.
Let \( Q(E) \) designate the set containing only inner points of \( E \).

Let \( \mathcal{R} \) be a finite system of closed intervals over the interval \((a, b)\).

Define the exterior content (Jordan), \( J_e(E) \), to be \( \text{g.l.b.} \sum_{\Delta \in \mathcal{R}} |\Delta| \) where g.l.b. is taken with respect to all systems \( \mathcal{R} \) such that \( \sum_{\Delta \in \mathcal{R}} \Delta \supseteq \overline{E} \), where "\( \supseteq \)" means includes.

Define the interior content (Jordan), \( J_i(E) \), to be \( \text{l.u.b.} \sum_{\Delta \in \mathcal{R}} |\Delta| \) where l.u.b. is taken with respect to all systems \( \mathcal{R} \) such that \( \sum_{\Delta \in \mathcal{R}} \Delta \subseteq \mathcal{J}(E) \), where the symbol "\( \subseteq \)" means is included in.

A set has Jordan content \( [\mathcal{J}(E)] \) if \( J_e(E) = J_i(E) \) and \( \mathcal{J}(E) = J_e(E) \).

A set is called "discrete" if \( \mathcal{J}(E) = 0 \).

It follows from the definitions of \( J_e(E) \) and \( J_i(E) \) that \( J_e(E) \geq J_i(E) \).

1.17 Lebesgue Measure:

Define \( \mathcal{E}(E) \) to be the interval \((a, b)\) minus \( E \).

Let \( \mathcal{R} \) be any system of open intervals over \((a, b)\).

Define the exterior measure (Lebesgue), \( \mathcal{M}_e(E) \), to be \( \text{g.l.b.} \sum_{\Delta \in \mathcal{R}} \text{|\Delta|} \) where g.l.b. is taken with respect to all systems \( \mathcal{R} \) such that \( \sum_{\Delta \in \mathcal{R}} \Delta \supseteq \overline{E} \).

A set is Lebesgue measureable \( [\mathcal{M}(E)] \) if \( \mathcal{M}_e(E) + \mathcal{M}_e[\mathcal{J}(E)] = \mathcal{M}_e(I) \) and \( \mathcal{M}(E) = \mathcal{M}_e(E) \).
1.18 **Theorem:** If $\bigcup_{e}(E)=\emptyset$, then $\bigcup_{e}(E)=\emptyset$.

**Proof:**

$\emptyset \subseteq \bigcup_{i}(E) \subseteq \bigcup_{e}(E) = \emptyset$

Therefore $\emptyset = \bigcup_{i}(E) = \bigcup_{e}(E) = \emptyset$.

1.19 **Theorem:** $\mathcal{M}_{e}(\mathcal{I}) = b - a$

**Proof:**

$\mathcal{M}_{e}(b-a) = g \cdot I \cdot B \cdot \sum_{\Delta \in \mathcal{R}} |\Delta|$

Let $\mathcal{R}$ be any system of open intervals covering $(a,b)$. If the intervals are infinite in number, by the Heine-Borel Theorem, they may be replaced by a finite number with the same property.

There exists an interval $\Delta_{1}'$ which contains $a$ and is the longest interval which contains $a$ - right end point $\alpha_{1}'$. $|\Delta_{1}'| \geq \alpha_{1}' - a$

There exists an interval $\Delta_{2}'$ which contains $\alpha_{1}'$ and is the longest interval which contains $\alpha_{1}'$ - right end point $\alpha_{2}'$. $|\Delta_{2}'| \geq \alpha_{2}' - \alpha_{1}'$

$\vdots$

Since the interval covering $(a,b)$ are finite in number, there exists an interval $\Delta_{n}'$ which contains $b$. $|\Delta_{n}'| \geq b - \alpha_{n-1}$. $|\Delta_{1}'| + |\Delta_{2}'| + \cdots + |\Delta_{n}'| \geq (\alpha_{1}' - a) + (\alpha_{2}' - \alpha_{1}') + \cdots + (b - \alpha_{n-1}) = b - a$

$\sum_{\Delta \in \mathcal{R}} |\Delta| \geq b - a$, $\mathcal{R} \supset (b-a)$

But there exists an interval equal to $(b-a)$.
1.19 (cont.)

Therefore: \[ \mathcal{M}_e (b-a) = g_1.b.\sum_{\Delta \in \mathcal{R}} |\Delta| = b-a, \text{ which } \mathcal{R} > b-a \]

1.20 Theorem: \[ \mathcal{M}_e (E) + \mathcal{M}_e [\mathcal{L}_0 (E)] \geq b-a \]

Proof:

Let \( \Delta \) be the intervals covering \( E \).
Let \( \Delta' \) be the intervals covering \( \mathcal{L}_0 (E) \).
Let \( \Delta \) and \( \Delta' \) be from one of the systems \( \mathcal{R} \).

From Theorem 1.19, it follows that:
\[ \sum_{\Delta \in \mathcal{R}} |\Delta| + \sum_{\Delta' \in \mathcal{R}} |\Delta'| \geq b-a \]

Therefore:
\[ g_1.b.\sum_{\Delta \in \mathcal{R}} |\Delta| + g_1.b.\sum_{\Delta' \in \mathcal{R}} |\Delta'| \geq b-a \]

\[ \mathcal{M}_e (E) + \mathcal{M}_e [\mathcal{L}_0 (E)] \geq b-a \]

1.21 Theorem: If \( \mathcal{M}_e (E) = 0 \), then \( \mathcal{M} (E) = 0 \).

Proof:

1) From Theorem 1.19, we have \( \mathcal{M}_e (I) = b-a \)
2) \( \mathcal{M}_e [\mathcal{L}_0 (E)] \leq b-a \)
3) \( \mathcal{M}_e (E) = 0 \)

Adding (2) and (3), we have:
4) \( \mathcal{M}_e (E) + \mathcal{M}_e [\mathcal{L}_0 (E)] \leq b-a \).

But from Theorem 1.20, we have:
5) \( \mathcal{M}_e (E) + \mathcal{M}_e [\mathcal{L}_0 (E)] \geq b-a \).

Therefore, \( \mathcal{M}_e (E) + \mathcal{M}_e [\mathcal{L}_0 (E)] = b-a \)

and \( E \) is Lebesgue measureable. \( \mathcal{M} (E) = \mathcal{M}_e (E) = 0 \).
1.22 Theorem: If Jordan content equals zero, then the Lebesgue measure equals zero.

Proof:
It follows from the definitions of content and measure that \( J_c(E) \geq M_c(E) \geq 0 \), but \( J_c(E) = 0 \). Therefore \( M_c(E) = 0 \). From Theorem 1.21, it follows that \( \mathcal{M}(E) = 0 \).

A bounded set is a set whose points lie in a finite interval.

1.23 Theorem: If \( F \) is a measurable bounded closed exterior set, then measure equals content.

Proof:
Let \( \Delta \) be the open intervals which cover \( F \).
By the Heine-Borel Theorem, we may select a finite number of these intervals which will cover \( F \). Therefore \( \mathcal{M}(F) = J_c(F) \).

1.24 Theorem: Every denumerable set is of measure zero.

Proof:
Let \( \{x_1, x_2, \ldots, x_n, \ldots \} \) be the points of the set. Cover each point by \( \Delta_1, \Delta_2, \ldots, \Delta_n, \ldots \) respectively. Choose \( |\Delta_n| = \frac{\varepsilon}{2^n} \), then \( \sum |\Delta_n| \leq \varepsilon \).
Since \( \varepsilon \) is arbitrarily small, \( M_c(E) = 0 \) and \( \mathcal{M}(E) = 0 \).
1.25 **Theorem:** If \( E = (E_1 + E_2 + \cdots + E_n + \cdots) \) then
\[
\sum_{n=1}^{\infty} M_\varepsilon(E_n) \geq M_\varepsilon(E)
\]

**Proof:**

For every set \( E_n \), there exists a set of intervals \( R_n \) such that:

1) \( \sum_{\Delta \in R_n} \Delta \supset E_n \)

2) \( \sum_{\Delta \in R_n} \vert \Delta \vert < M_\varepsilon(E_n) + \varepsilon/2^n \)

It is true that \( \sum_{n=1}^{\infty} \sum_{\Delta \in R_n} \vert \Delta \vert \supset E \)

From (2), it follows that \( \sum_{n=1}^{\infty} M_\varepsilon(E_n) > \sum_{n=1}^{\infty} (\sum_{\Delta \in R_n} \vert \Delta \vert - \varepsilon/2^n) \)

which is equal to \( \left( \sum_{n=1}^{\infty} \sum_{\Delta \in R_n} \vert \Delta \vert \right) - \varepsilon \)

which is greater than or equal to \( M_\varepsilon(E) - \varepsilon \)

Since \( \varepsilon \) is arbitrarily small, it follows that
\[
\sum_{n=1}^{\infty} M_\varepsilon(E_n) \geq M_\varepsilon(E)
\]

1.26 **Corollary:** If \( E = \sum E_n \), \( M_\varepsilon(E) = 0 \), then \( M(E) = 0 \).

1.27 **Theorem:** A necessary and sufficient condition that a function \( f(x) \) be Riemann integrable is that for any pair of numbers \( \varepsilon > 0 \), \( \eta > 0 \) there exists a subdivision \( D_{\varepsilon, \eta} \) such that \( \sum \vert \Delta \vert < \varepsilon \) for those \( \Delta \) of \( D_{\varepsilon, \eta} \) on which \( \omega(\Delta, f) > \eta \).
1.27 (cont.)

Proof:

Necessary:
Define $\sum |\Delta'|$ to be the sum of the subintervals in $D_{\epsilon,\eta}$ for which the fluctuation $\geq \eta$.
Define $\sum |\Delta''|$ to be the sum of the subintervals in $D_{\epsilon,\eta}$ for which the fluctuation $< \eta$.

Given: For every $\epsilon^0$ and $\eta^0$ it is true that there exists $D_{\epsilon,\eta}$ such that $\sum_{\Delta \in D_{\epsilon,\eta}} |\Delta| \omega(\Delta, f) < \eta \epsilon$

To show: $\sum |\Delta'| < \epsilon$ for subdivision $D_{\epsilon,\eta}$.
$\sum_{\Delta \in D_{\epsilon,\eta}} |\Delta| \omega(\Delta, f) \geq \eta \sum |\Delta'| + m \sum |\Delta''| = \eta \sum |\Delta'|$

where $m$ is the smallest oscillation over $\Delta''$ intervals.
Therefore $\eta \sum |\Delta'| < \eta \epsilon$ and $\sum |\Delta'| < \epsilon$

Sufficient:

Given: For every $\epsilon^0$ and $\eta^0$ there exists $D_{\epsilon,\eta}$ such that $\sum |\Delta'| < \epsilon$.

To show: $\sum_{\Delta \in D_{\epsilon,\eta}} |\Delta| \omega(\Delta, f) < \epsilon'$
$\sum_{\Delta \in D_{\epsilon,\eta}} |\Delta| \omega(\Delta, f) \leq \eta (b-a) + M \sum |\Delta'|

where $M$ is the largest oscillation over interval $(a,b)$.

Therefore $\sum_{\Delta \in D_{\epsilon,\eta}} |\Delta| \omega(\Delta, f) \leq \eta (b-a) + M \epsilon$

Choose $\epsilon' = \eta (b-a) + M \epsilon$. Since $\eta$ and $\epsilon$ are arbitrarily small, $\epsilon'$ is an arbitrarily small number.
Saltus is defined to be the excess of the greatest over the least of \( f(\alpha + 0), f(\alpha + \delta), f(\alpha - 0), f(\alpha - \delta), f(\alpha) \); where \( f(\alpha + 0) \) is the upper right hand limit of \( f(x) \) as \( x \to \alpha \), \( f(\alpha + \delta) \) is the lower right hand limit of \( f(x) \) as \( x \to \alpha \), \( f(\alpha - 0) \) is the upper left hand limit of \( f(x) \) as \( x \to \alpha \), \( f(\alpha - \delta) \) is the lower left hand limit of \( f(x) \) as \( x \to \alpha \), and \( f(\alpha) \) is the functional value at \( \alpha \), where \( \alpha \leq \alpha \leq b \).

1.28 Corollary: A necessary and sufficient condition for the Riemann integration of a function is that the points of discontinuity for which the saltus is greater than or equal to an arbitrarily small number \( \gamma \) form a discrete set.

1.29 Lemma: Given any function, the set of points \( E_n \) for which the function has discontinuities with saltus \( \geq \frac{1}{n} \) is a closed set.

Proof:
Assume \( P \) is a limit point of the set of points \( E_n \) for which the function has discontinuities with saltus \( \geq \frac{1}{n} \). Assume that \( P < \frac{1}{n} \). Then there exists a neighborhood of \( P \) such that every point of the set in the neighborhood is \( < \frac{1}{n} \). Therefore this neighborhood can not contain any points of \( E_n \); but this is contrary to the hypothesis that \( P \) is a limiting point.
1.30 **Theorem:** A necessary and sufficient condition that a function \( f(x) \) be Riemann integrable is that the set of points of discontinuity of \( f(x) \) form a set of measure zero.

**Proof:**

**Necessary:**

**Given:** That the function is Riemann integrable.

**To show:** That the points of discontinuity of \( f(x) \) form a set of measure zero.

Let \( E_n \) be the set of points for which the function has discontinuities with saltus \( \geq \frac{1}{n} \).

\[
E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots
\]

By 1.26, the content of \( E_n = 0 \). Therefore, by 1.22, the measure of \( E_n = 0 \). Let \( E = E_1 + E_2 + \cdots + E_n + \cdots \)

All the points of discontinuity, \( E \), are included in this sum. Consequently, from 1.25, \( \uparrow_{\epsilon}(E) = \)

\[
\uparrow_{\epsilon}(E_1 + E_2 + \cdots + E_n + \cdots) \leq \uparrow_{\epsilon}(E_1) + \uparrow_{\epsilon}(E_2) + \cdots +
\]

\[
\downarrow_{\epsilon}(E_n) + \cdots = 0 + 0 + \cdots + 0 + \cdots = 0
\]

and \( \downarrow_{\epsilon}(E) = 0 \). Therefore by 1.21, \( \downarrow(E) = 0 \).

**Sufficient:**

**Given:** \( \uparrow(E) = 0 \)

**To show:** \( E_n \) forms a discrete set; where \( E_n \) is the set of points for which the function has discontinuities with saltus \( \geq \frac{1}{n} \).
1.30 (cont.)

By hypothesis, \( \mathcal{M}(E) = 0 \). Therefore, since \( E_\infty \) is a subset of \( E \), \( \mathcal{M}(E_\infty) = 0 \). \( E_\infty \), by 1.29, is a bounded closed set. Therefore, from 1.23, we have that the measure of \( E_\infty \) is equal to the content of \( E_\infty \), which is equal to zero. Therefore, \( E_\infty \) forms a discrete set, which is, by 1.28, the condition that a function be Riemann integrable.

1.31 Corollary: If the function \( f(x) \) is continuous, then the function is Riemann integrable.

1.32 Theorem: If the function \( f(x) \) has a finite or denumerably infinite number of discontinuities, it is Riemann integrable.

Proof:

It follows from 1.24, that every denumerable set is of measure zero. Therefore, by 1.30, the function is Riemann integrable.

1.33 Example: A function which is continuous on the irrationals and discontinuous on the rationals.

\[
 f(x) = \begin{cases} 
 1 & \text{for } x \text{ irrational} \\
 1 - \frac{1}{n} & \text{for } x = \frac{p}{q}, \text{ where } p, q \text{ are relatively prime integers.}
\end{cases}
\]
1.33 (cont.)

This function is continuous on the irrationals because, for every \( \epsilon > 0 \), there exists a \( q_0 \) such that \( \frac{1}{q_0} < \epsilon \), and is discontinuous on the rationals. The rationals are denumerably infinite; therefore the function is Riemann integrable.

1.34 **Example:** \( f(x) = \frac{1}{2^n} \) for \( \frac{1}{2^n} \leq x < \frac{1}{2^{n+1}} \)

\[ f(0) = 0, \quad f(1) = \frac{1}{2} \]

This function has points of discontinuity at \( x = \frac{1}{2^n} \). There are a denumerably infinite number of points of discontinuity; therefore this function is Riemann integrable.

1.35 **Theorem:** If the function \( f(x) \) is integrable in \( (a, b) \), it is integrable in any subinterval of \( (a, b) \).

**Proof:**

The points of discontinuity of the function in \( (a, b) \) form a set of measure zero. In any subinterval of \( (a, b) \), the points of discontinuity necessarily form a set of measure zero. Therefore the function is Riemann integrable in any subinterval of \( (a, b) \).
1.36 **Theorem:** The additivity of the Riemann integral over a finite interval.

\[
\int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{a}^{c} f(x) \, dx, \quad a < b < c
\]

**Proof:**

Choose \( b \) as an endpoint of one subinterval of every subdivision of \( (a, c) \), where \( a < b < c \).

\[
\sum_{\Delta \in (a, c)} |\Delta| \sum_{\Delta \in (a, b)} \overline{B} (\Delta, f) + \sum_{\Delta \in (b, c)} |\Delta| \overline{B} (\Delta, f) \\
\lim_{|\Delta| \to 0} \sum_{\Delta \in (a, c)} |\Delta| \overline{B} (\Delta, f) = \lim_{\Delta \to 0} \sum_{\Delta \in (a, b)} |\Delta| \overline{B} (\Delta, f) + \lim_{\Delta \to 0} \sum_{\Delta \in (b, c)} |\Delta| \overline{B} (\Delta, f)
\]

Therefore:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx
\]

Likewise:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx, \text{ but } \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx
\]

Therefore:

\[
\int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{b}^{c} f(x) \, dx
\]

1.37 **Theorem:** If \( f_1(x) \) and \( f_2(x) \) are Riemann integrable,

then

\[
\int_{a}^{b} f_1(x) \, dx + \int_{a}^{b} f_2(x) \, dx = \int_{a}^{b} (f_1(x) + f_2(x)) \, dx
\]

**Proof:**

\[
\overline{B} (\Delta, f_1 + f_2) \leq \overline{B} (\Delta, f_1) + \overline{B} (\Delta, f_2)
\]

\[
\sum_{\Delta \in D} |\Delta| \overline{B} (\Delta, f_1 + f_2) \leq \sum_{\Delta \in D} |\Delta| \left[ \overline{B} (\Delta, f_1) + \overline{B} (\Delta, f_2) \right]
\]

\[
\int_{a}^{b} (f_1 + f_2) \, dx \leq \int_{a}^{b} f_1 \, dx + \int_{a}^{b} f_2 \, dx
\]

22.
1.37 (cont.)

Likewise:

\[
\int_a^b (f_1 + f_2) \, dx \geq \int_a^b f_1 \, dx + \int_a^b f_2 \, dx
\]
\[
\int_a^b f_1 \, dx + \int_a^b f_2 \, dx \geq \int_a^b (f_1 + f_2) \, dx \geq \int_a^b f_1 \, dx + \int_a^b f_2 \, dx
\]

but \( \int_a^b f_1 \, dx = \int_a^b f_1 \, dx \), \( \int_a^b f_2 \, dx = \int_a^b f_2 \, dx \)

Therefore:

\[
\int_a^b (f_1 + f_2) \, dx = \int_a^b (f_1 + f_2) \, dx = \int_a^b f_1 \, dx + \int_a^b f_2 \, dx
\]

1.38 Theorem: If \( f(x) \) is Riemann integrable in \((a, b)\),
then so is \( |f(x)| \) and \( |\int_a^b f(x) \, dx| = \int_a^b |f(x)| \, dx \).

Proof:

Part I:

Define \( f_1(x) = f(x) \) when \( f(x) > 0 \)
\[= 0 \quad \text{otherwise.} \]

Define \( f_2(x) = -f(x) \) when \( f(x) \leq 0 \)
\[= 0 \quad \text{otherwise.} \]

Case I: If \( f(x) > 0 \) then
\[
f_1(x) = f(x)
\]
\[
f_2(x) = 0
\]

a) \( f_1(x) - f_2(x) = f(x) \)

b) \( f_1(x) + f_2(x) = |f(x)| \)
1.38 (cont.)

Case 2: If \( f(x) \leq 0 \) then
\[
\begin{align*}
& f'(x) = 0 \\
& f''(x) = -f(x)
\end{align*}
\]

a) \( f'(x) - f''(x) = f(x) \)
b) \( f''(x) = |f(x)| \)
c) \( f'(x) + f''(x) = |f(x)| \)

Therefore we conclude:
\[
\begin{align*}
& f'(x) - f''(x) = f(x) \\
& f'(x) + f''(x) = |f(x)|
\end{align*}
\]

1) If \( f(x) > 0 \) at \( x_0 \) and is continuous at \( x_0 \),
then \( f'(x_0) > 0 \) and is continuous at \( x_0 \). \( f''(x) = 0 \)
and is continuous at \( x_0 \).
2) If \( f(x) > 0 \) at \( x_1 \) and is discontinuous at \( x_1 \),
then \( f'(x_1) > 0 \) and is discontinuous at \( x_1 \).
\( f''(x) = 0 \) and may be continuous or discontinuous.
3) If \( f(x) \leq 0 \) at \( x_2 \) and is continuous at \( x_2 \), then
\( f'(x_2) = 0 \) and is continuous. \( f''(x_2) = -f(x_2) \)
and is continuous.
4) If \( f(x) \leq 0 \) at \( x_3 \) and is discontinuous at \( x_3 \),
then \( f'(x_3) = 0 \) and may be continuous or discontinuous.
\( f''(x) = -f(x_3) \) and is discontinuous.

In conclusion: \( f(x) \) is discontinuous at a point in a
closed interval \((a,b)\) only if \( f(x) \) is discontinuous.
\( f'(x) \) is discontinuous at a point in a closed interval
\((a,b)\) only if \( f(x) \) is discontinuous.
1.36 (cont.)

Part I (cont.)

$f(x)$ is Riemann integrable and therefore by Theorem 1.30, the points of discontinuity form a set of measure zero.

$f_1(x)$ and $f_2(x)$ have discontinuities which form sets of measure zero respectively.

Therefore: $f_1(x)$ is Riemann integrable

$f_2(x)$ is Riemann integrable

$f_1(x) + f_2(x) = |f(x)|$ is Riemann integrable.

Part II:

It follows from Theorem 1.37 that $\int f(x) \, dx = \int [f_1(x) - f_2(x)] \, dx = \int f_1(x) \, dx - \int f_2(x) \, dx$

It is also true that $|\int f_1(x) \, dx| \leq |\int f_1(x) \, dx| + |\int f_2(x) \, dx|$

but $f_1(x)$ and $f_2(x)$ are never negative. Therefore:

$|\int f_1(x) \, dx| \leq \int f_1(x) \, dx + \int f_2(x) \, dx = \int [f_1(x) + f_2(x)] \, dx = \int |f(x)| \, dx$.

1.39 Lemma: $|a - b| \geq |a| - |b|$

Proof: $|a - b| \geq |a| - |b|$

$|a - b| + |b| \geq |a| \quad \text{or} \quad |a - b| \geq |a| - |b|$

$|a - b| - |a| \geq |b| \quad \text{or} \quad |a - b| \geq |b| - |a| = |b| - |a|$

Therefore: $|a - b| \geq |a| - |b|$
1.40 Theorem: If $f(x)$ and $\varphi(x)$ are both continuous at $x_0$, it follows that $F(x) = f(x) \varphi(x)$ is continuous at $x_0$.

Proof:

By hypothesis, $f(x)$ is continuous at $x_0$, which means by definition that for every $\varepsilon^o$, it is true that there exists a $\delta_1(\varepsilon)$ such that for $|x-x_0| < \delta_1(\varepsilon)$ it follows that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$.

By hypothesis, $\varphi(x)$ is continuous at $x_0$, which means by definition that for every $\varepsilon^o$, it is true that there exists a $\delta_2(\varepsilon)$ such that for $|x-x_0| < \delta_2(\varepsilon)$ it follows that $|\varphi(x) - \varphi(x_0)| < \frac{\varepsilon}{2}$.

Choose $\delta(\varepsilon)$ to be the smaller of $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$.

Consider $|\varphi(x)f(x) - \varphi(x_0)f(x_0)|$, where

$|x-x_0| < \delta(\varepsilon)$. $|\varphi(x)f(x) - \varphi(x_0)f(x_0)| =

|\varphi(x)f(x) - \varphi(x_0)f(x) + \varphi(x_0)f(x) - \varphi(x_0)f(x_0)| \

\leq |\varphi(x)f(x) - \varphi(x_0)f(x)| + |\varphi(x_0)f(x) - \varphi(x_0)f(x_0)| \

\leq |f(x)| |\varphi(x) - \varphi(x_0)| + |\varphi(x_0)||f(x) - f(x_0)| \

\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

But from lemma 1.39, $|f(x) - f(x_0)| \leq |f(x) - f(x_0)| < \frac{\varepsilon}{2}$

Therefore, $|f(x)| < \frac{\varepsilon}{2} + |f(x_0)|$. Now

$|\varphi(x_0)| < \frac{\varepsilon}{2} \leq \left(|f(x)| + \frac{\varepsilon}{2}\right) \frac{\varepsilon}{2} + |\varphi(x_0)| \frac{\varepsilon}{2}$.

Since $(|f(x)| + \frac{\varepsilon}{2}) \frac{\varepsilon}{2} + |\varphi(x_0)| \frac{\varepsilon}{2}$ is an arbitrarily small number, this is the condition that $F(x) = f(x) \cdot \varphi(x)$ is continuous at $x_0$. 

26.
1.41 **Theorem:** If \( f(x) \) and \( \varphi(x) \) are Riemann integrable respectively, then \( F(x) = \int f(x) \varphi(x) \) is Riemann integrable.

**Proof:**

It follows from Theorem 1.40 that \( F(x) \) is continuous at every point at which \( f(x) \) and \( \varphi(x) \) are continuous. Therefore the only points of discontinuity are those of the functions \( f(x) \) and \( \varphi(x) \) each of which has measure zero. Hence, the set of points of discontinuity of \( F(x) \) has measure zero, and thus \( F(x) \) is Riemann integrable.

1.42 **Theorem:** If a sequence of functions \( \int f_n(x) \) are integrable (Riemann), for each \( n, n = 1, 2, \ldots \) in the interval \( (a,b) \) and if they converge uniformly in the interval to a bounded function \( f(x) \) then \( \int f(x) \) is Riemann integrable and

\[
\int_a^b f_n(x) \, dx \to \int_a^b f(x) \, dx
\]

**Proof:**

**Part I:**

Since \( f_n(x) \) is Riemann integrable, the set of points of discontinuity, \( E_n \), form a set of measure zero for each \( n, n = 1, 2, \ldots \). Then by 1.26, the set of all points \( E = \bigcup E_n \) for which any of them have discontinuities is of measure zero.
Let \( \chi_i \) be a point not in \( E \).

\[ \int_{\chi_i} f_n(x) \text{ converges uniformly. Therefore:} \]

1) \( | \int_{\chi_i} f_n(x) - f(x) | < \frac{\epsilon}{2} \quad \text{for } n > N(\epsilon) \text{ and for any } x. \)

For this \( \epsilon \) there exists a \( \delta(\frac{\epsilon}{2}) \) such that

2) \( | \int_{\chi_i} f_n(x) - f_n(x_i) | < \frac{\epsilon}{3}, |x - x_i| < \delta, \) because \( \chi_i \) is not in \( E \) and is therefore a point at which the function is continuous. Therefore \( | \int_{\chi_i} f(x) - \int_{\chi_i} f(x) | = | \int_{\chi_i} f(x) - f_n(x) + f_n(x) - f_n(x) | \leq | \int_{\chi_i} f(x) - f_n(x) | + | \int_{\chi_i} f(x) - f_n(x) | + | \int_{\chi_i} f_n(x) - f_n(x) | < \epsilon \)

for \( n > N(\frac{\epsilon}{3}), |x - x_i| < \delta, \) for from (1) \( | \int_{\chi_i} f(x) - f_n(x) | < \frac{\epsilon}{3}, n > N(\frac{\epsilon}{3}) \)

and \( | \int_{\chi_i} f_n(x) - f(x) | < \frac{\epsilon}{3}, |x - x_i| < \delta(\frac{\epsilon}{3}) \)

Since \( \epsilon \) is arbitrarily small, \( \chi_i \) is a point of continuity of \( f(x) \). Therefore, \( f(x) \) is continuous at all points at which all of the functions \( f_n(x) \)

are continuous. Therefore the points of discontinuity of \( f(x) \) form a set of measure zero.

Part II:

For \( \epsilon > 0 \), have \( N(\epsilon) \) such that \( | \int_{\chi_i} f_n(x) - f(x) | < \epsilon \)

for \( n > N(\epsilon) \).

\[ | \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx | \leq \int_a^b | f_n(x) - f(x) | \, dx < \epsilon(b - a) \]

which is the condition that: \( \int_a^b f_n(x) \, dx \rightarrow \int_a^b f(x) \, dx \).
The following are examples which show that 1.40 would not be true if the condition of uniform convergence were omitted.

1.43 Example: The sequence of functions \( f_n(x) \) are each Riemann integrable, in the interval \((a,b)\) and they converge in the interval to a function \( f(x) \) but the function \( f(x) \) is not Riemann integrable. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \) be rational numbers written as a sequence.

\[
  f(x) = \begin{cases} 
    0 & \text{on rationals} \\
    1 & \text{on irrationals}
  \end{cases}
\]

\[
  f_n(x) = \begin{cases} 
    0 & \text{for } x = \lambda_1, \lambda_2, \ldots, \lambda_n, \text{ for } m > n, f_n(\lambda_m) = 1 \\
    1 & \text{everywhere else.}
  \end{cases}
\]

Each \( f_n(x) \) is Riemann integrable but \( f(x) \) is discontinuous at every point.

1.44 Example: The sequence of functions \( f_n(x) \) are each Riemann integrable, in the interval \((a,b)\) and they converge in the interval to a function \( f(x) \) which is Riemann integrable, but

\[
  \int_a^b f_n(x) \, dx
\]

does not converge to \( \int_a^b f(x) \, dx \).
1.44 (cont.)

\[ f_1(x) = \begin{cases} 0, & x = 0, x = 1 \\ 1, & 0 < x < 1 \end{cases} \]

\[ f_2(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, x = 1 \\ 2, & \frac{1}{2} < x < 1 \end{cases} \]

\[ \vdots \]

\[ f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{n-1}{n}, x = 1 \\ n, & \frac{n-1}{n} < x < 1 \end{cases} \]

\[ f_n(x) \to 0 \text{ at every point}. \] Therefore \[ f(x) = 0, 0 \leq x \leq 1 \].

\[ \int_0^1 f_n(x) \, dx = \int_{\frac{n-1}{n}}^1 n \, dx = n \left[ 1 - \frac{n-1}{n} \right] = n \cdot \frac{1}{n} = 1 \]

\[ \int_0^1 f(x) \, dx = 0 \]

1.45 Theorem: If \( f(x) \) is a bounded, integrable (Riemann) function in the interval \((a, b)\), then

\[ m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \]

where \( m = \text{g.l.b. of } f(x) \) in \((a, b)\), and

where \( M = \text{l.u.b. of } f(x) \) in \((a, b)\).

Proof:

\[ m(b-a) \leq \Phi_x(f, D) \leq \Phi^*(f, D) \leq M(b-a) \]

for every subdivision \( D \). Therefore, \( m(b-a) \leq \text{l.u.b. } \Phi_x(f, D) \leq \text{g.l.b. } \Phi^*(f, D) \leq M(b-a) \), but

\[ \text{l.u.b. } \Phi_x(f, D) = \int_a^b f(x) \, dx \text{ and g.l.b. } \Phi^*(f, D) = \int_a^b f(x) \, dx. \]
Therefore \[ M(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \]

This result is called the First Law of the Mean for the Integral Calculus.
CHAPTER II

THE INDEFINITE INTEGRAL

2.1 Definition of the Indefinite Integral
If \( \int f(x) \) is Riemann integrable in the interval \((a,b)\), then by Theorem 1.35 of the previous chapter, we have \( \int_a^x f(u) \, du \) exists, where \( x \) is any intermediate value between \( a \) and \( b \). This integral is called the indefinite integral and defines a function of \( x \).
\[
\Phi(x) = \int_a^x f(u) \, du
\]

2.2 Theorem: If \( f(x) \) is Riemann integrable and bounded in the interval \((a,b)\) then \( \Phi(x) = \int_a^x f(u) \, du \) is a continuous function of \( x \) in this interval.
Proof:
\[
\Phi(x+\Delta x) - \Phi(x) = \int_a^{x+\Delta x} f(u) \, du - \int_a^x f(u) \, du = \int_x^{x+\Delta x} f(u) \, du
\]
By the additivity of the Riemann integral, we have
\[
\Phi(x+\Delta x) - \Phi(x) = \int_x^{x+\Delta x} f(u) \, du.
\]
From 1.38 and 1.43 of the previous chapter, \( \left| \int_x^{x+\Delta x} f(u) \, du \right| \leq \int_x^{x+\Delta x} |f(u)| \, du \leq M \cdot \Delta x \) where \( M = \text{l.u.b.} |f(x)| \). Therefore,
\[
\left| \Phi(x+\Delta x) - \Phi(x) \right| \leq M \cdot \Delta x
\]
2.2 (cont.)

For any $\varepsilon$, positive number, choose $\Delta x' < \frac{\varepsilon}{M}$.

Then for all $\Delta x < \Delta x'$, we have:

$$|\Phi(x + \Delta x) - \Phi(x)| < \varepsilon,$$

which is the condition of the continuity of $\Phi(x)$.

2.3 **Theorem:** $\int_{a}^{x} f(u) \, du$ and $\int_{a}^{x} f(u) \, du$ are continuous functions of $x$.

Proof:

Part I:

Define $\Phi(x)$ to be $\int_{a}^{x} f(u) \, du$.

$$\Phi(x + \Delta x) - \Phi(x) = \int_{a}^{x + \Delta x} f(u) \, du - \int_{a}^{x} f(u) \, du$$

but this is equal to $\int_{a}^{x} f(u) \, du + \int_{x}^{x + \Delta x} f(u) \, du$ which is equal to $\int_{x}^{x + \Delta x} f(u) \, du$. It follows from the definition of $\int_{x}^{x + \Delta x} f(u) \, du$ that $|\int_{x}^{x + \Delta x} f(u) \, du| \leq \int_{x}^{x + \Delta x} |f(u)| \, du \leq M \cdot \Delta x$

where $M = \text{l.u.b.} \, |f(x)|$. Therefore,

$$|\Phi(x + \Delta x) - \Phi(x)| \leq M \cdot \Delta x$$

For any $\varepsilon$, positive number, choose $\Delta x' < \frac{\varepsilon}{M}$.

Then for all $\Delta x < \Delta x'$, we have $|\Phi(x + \Delta x) - \Phi(x)| < \varepsilon$ which is the condition of continuity.
2.3 (cont.)

Part II:

Define \( \Phi (x) \) to be \( \int_{a}^{x} f(u) \, du \).

\[ \Phi (x + \Delta x) - \Phi (x) = \int_{x}^{x + \Delta x} f(u) \, du - \int_{a}^{x} f(u) \, du. \]

But this is equal to \( \int_{x}^{x + \Delta x} f(u) \, du \) which is equal to \( \int_{a}^{x + \Delta x} f(u) \, du \).

It follows from the definition of \( \int_{a}^{x} f(u) \, du \) that \( |\int_{x}^{x + \Delta x} f(u) \, du| \leq \int_{x}^{x + \Delta x} |f(u)| \, du \leq M \cdot \Delta x \)

where \( M = \text{l.u.b.} |f(x)| \).

Therefore,

\[ |\Phi (x + \Delta x) - \Phi (x)| \leq M \cdot \Delta x \]

For any \( \epsilon > 0 \), choose \( \Delta x' < \frac{\epsilon}{M} \).

Then for all \( \Delta x \leq \Delta x' \), we have \( |\Phi (x + \Delta x) - \Phi (x)| < \epsilon \) which is the condition of continuity.

2.4 Theorem: Let \( f(x) \) be bounded and integrable in the interval \((a, b)\) and suppose \( \Phi(x) \) to be a non-negative bounded monotone decreasing function.

Then, there exists a value \( c \) such that:

\[ \int_{a}^{b} f(x) \, \Phi(x) \, dx = \Phi(x) \int_{a}^{c} f(x) \, dx, \quad a \leq c \leq b. \]

Proof:

It follows from the conditions of the theorem that \( \Phi(x) \) is integrable and hence, by 1.39 of the previous chapter, the product \( f(x) \, \Phi(x) \) is integrable.
2.4 (cont.)

\[ \int_a^b f(x) \varphi(x) \, dx = \lim_{\|D_n\| \to 0} \sum_{n \in D_n} |A_k| \, \overline{B}(\Delta_k) \, \overline{B}(\Delta, \varphi). \]

|Δ₀| = x₁ - x₀, |Δ₁| = x₂ - x₁, ..., |Δₖ| = xₖ₊₁ - xₖ, ..., |Δₙ₋₁| = xₙ - xₙ₋₁ where x₀ = a, xₙ = b.

Define \( \sum_{n=0}^{n=k} |A_k| \, \overline{B}(\Delta_k, \varphi) \) to be \( A_n \). Then

\[ \sum_{n=0}^{n=k} |A_k| \, \overline{B}(\Delta_k, \varphi) \, \overline{B}(\Delta, \varphi) = A_0 \, \overline{B}(\Delta_0, \varphi) + A_1 \, \overline{B}(\Delta_1, \varphi) + \cdots \]

\[ + A_{n-1} \, \overline{B}(\Delta_{n-1}, \varphi) \, \overline{B}(\Delta_{n-1}, \varphi) + A_n \, \overline{B}(\Delta_n, \varphi) + \cdots \]

which is equal to

\[ \sum_{n=0}^{n=k} A_{k-1} \left[ \overline{B}(\Delta_{k-1}, \varphi) - \overline{B}(\Delta_k, \varphi) \right] + A_n \, \overline{B}(\Delta_n, \varphi) \]

Choose \( \mu \) equal to the largest of \( A_0, A_1, \ldots, A_{n-1} \)

Choose \( \gamma \) equal to the smallest of \( A_0, \), \( A_n, \ldots, A_{n-1} \)

From this it follows that:

\[ \gamma \, \overline{B}(\Delta_0, \varphi) \leq \sum_{k=0}^{n=k} |A_k| \, \overline{B}(\Delta_k, \varphi) \leq \mu \, \overline{B}(\Delta, \varphi) \]

but \( \overline{B}(\Delta, \varphi) = \varphi(x) \). Therefore,

\[ \varphi(x) \leq \sum_{k=0}^{n=k} |A_k| \, \overline{B}(\Delta_k, \varphi) \leq \varphi(a) \mu \]

Recalling the values of \( \mu \) and \( \gamma \), take the limit as \( \|D_n\| \to 0 \) then, \( \varphi(a) \leq b \cdot \int_a^b f(x) \, dx \leq \int_a^b f(x) \varphi(x) \, dx \leq \varphi(a) 1 \cdot \text{b} \cdot \int_a^b f(x) \, dx \)

where \( a \leq x \leq b \).
2.4 (cont.)

Since \( F(x) = \int_a^x f(x) \, dx \) is a continuous function of \( x \), then \( F(x) \) takes on all values between l.u.b. and g.l.b. Therefore there exists an \( x = f \) such that \( F(a) \int_a^b f(x) \, \varphi(x) \, dx \).

We have then, \( \int_a^b f(x) \, \varphi(x) \, dx = \varphi(a) \int_a^f f(x) \, dx \)
where \( a < f < b \).

This result is called the Second Law of the Mean for the Integral Calculus.

2.5 Theorem: If \( f(x) \) is Riemann integrable in \((a, b)\) and \( \varphi(x) = \int_a^x f(x') \, dx' \), \( a \leq x \leq b \)

then \( \varphi'(x) \) exists and is equal to \( f(x) \) at every point at which \( f(x) \) is continuous.

Proof:

Take any point in the interval at which \( f(x) \) is continuous and designate it by \( x_0 \).

From the definition of \( \varphi(x) \) we have,

\[ \varphi(x_0 + \Delta x) - \varphi(x_0) = \int_{x_0}^{x_0 + \Delta x} f(x') \, dx' \]

From 1.43 of the previous chapter, we have,

\[ | \varphi(x_0 + \Delta x) - \varphi(x_0) | \leq \int_{x_0}^{x_0 + \Delta x} |f(x')| \, dx' \]

where:

\[ \Delta : x_0 \leq x \leq x_0 + \Delta x, \Delta x > 0 \]

\[ \Delta : x_0 + \Delta x \leq x < x_0, \Delta x < 0 \]
2.5 (cont.)

Dividing by $|\Delta|$, we have:

$$
\frac{\Phi(x_0+\Delta x) - \Phi(x_0)}{|\Delta|} \leq \frac{1}{|\Delta|} \int_{x_0}^{x_0+\Delta x} \Phi'(\mu) d\mu \leq \overline{B}(x_0+\Delta x, f)
$$

But \( \lim_{|\Delta| \to 0} \overline{B}(x_0+\Delta x, f) = f(x_0) \)

and \( \lim_{|\Delta| \to 0} \underline{B}(x_0+\Delta x, f) = f(x_0) \)

Therefore, \( \lim_{|\Delta| \to 0} \frac{\Phi(x_0+\Delta x) - \Phi(x_0)}{|\Delta|} = \Phi'(x_0) = f(x_0) \)

Since \( x_0 \) is any point of continuity in the interval \((a,b)\), the theorem follows.

2.6 Corollary: If \( f(x) \) is continuous in the interval \((a,b)\), and \( \Phi(x) \) denotes the indefinite integral \( \int_a^x f(\mu) d\mu \), then at every point in \((a,b)\), \( \Phi(x) \) has a derivative and \( \Phi'(x) = f(x) \).

2.7 Theorem: At any point \((a,b)\) at which \( f(x) \) is continuous, the upper and lower integrals \( \overline{\int}_a^x f(\mu) d\mu \), \( \int_a^x f(\mu) d\mu \), each possess a derivative which is equal to \( f(x) \).

Proof:

Part I:

Take any point in the interval at which \( f(x) \) is continuous and designate it by \( x_0 \). From the definition of \( \overline{\Phi}(x) \), \( \overline{\Phi}(x_0+\Delta x) - \overline{\Phi}(x_0) = \int_{x_0}^{x_0+\Delta x} \overline{\Phi'}(\mu) d\mu \)
2.7 (cont.)

Let $\Delta'$ be any subdivision over $\Delta$. Then
\[ B(x_0 + \Delta x, f) / |\Delta| \leq \Phi^*(f, \Delta') \leq B(x_0 + \Delta x, f) / |\Delta| \]
where:
\[ \Delta : x_0 < x < x_0 + \Delta x, \ \Delta x > 0 \]
\[ \Delta : x_0 + \Delta x < x < x_0, \ \Delta x < 0 \]

This is true for every $\Delta'$. Therefore,
\[ B(x_0 + \Delta x, f) / |\Delta| \leq \text{g.l.b.} \Phi^*(f, \Delta') \leq B(x_0 + \Delta x, f) / |\Delta| \]

but \( \text{g.l.b.} \Phi^*(f, \Delta') = \int_{x_0}^{x_0 + \Delta x} f(u) \, du \). Now we have
\[ B(x_0 + \Delta x, f) / |\Delta| = \int_{x_0}^{x_0 + \Delta x} f(u) \, du = \Phi(x_0 + \Delta x) - \Phi(x_0) \leq B(x_0 + \Delta x, f) / |\Delta| \]

Dividing by $|\Delta|$ we have:
\[ B(x_0 + \Delta x, f) = \frac{1}{|\Delta|} \int_{x_0}^{x_0 + \Delta x} f(u) \, du = \frac{\Phi(x_0 + \Delta x) - \Phi(x_0)}{|\Delta|} \leq B(x_0 + \Delta x, f) \]

But \( \lim_{|\Delta| \to 0} B(x_0 + \Delta x, f) = f(x_0) \)

and \( \lim_{|\Delta| \to 0} B(x_0 + \Delta x, f) = f(x_0) \)

Therefore \( \lim_{|\Delta| \to 0} \frac{\Phi(x_0 + \Delta x) - \Phi(x_0)}{|\Delta|} = \Phi'(x_0) = f(x_0) \)

Part II:

Take any point in the interval at which $f(x)$ is continuous and designate it by $x_0$. From the definition of $\Phi(x)$, $\Phi(x_0 + \Delta x) - \Phi(x_0) = \int_{x_0}^{x_0 + \Delta x} f(u) \, du$

Let $\Delta'$ be any subdivision over $\Delta$. Then
\[ B(x_0 + \Delta x, f) / |\Delta| \leq \Phi^*(f, \Delta') \leq B(x_0 + \Delta x, f) / |\Delta| \]

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2.7 (cont.)

where:  
\[ \Delta: x_0 < x < x_0 + \Delta x, \Delta x > 0 \]
\[ \Delta: x_0 - \Delta x < x < x_0, \Delta x < 0 \]

This is true for every \( \Delta' \). Therefore,

\[ B(x_0 + \Delta x, f) \Delta \leq \text{l.u.b. } \Phi_x(f, \Delta') \leq B(x_0 + \Delta x, f) \Delta \]

but \( \text{l.u.b. } \Phi_x(f, \Delta') = \int_{x_0}^{x_0 + \Delta x} f(u) \, du \). Now we have

\[ B(x_0 + \Delta x, f) \Delta \leq \int_{x_0}^{x_0 + \Delta x} f(u) \, du = \frac{\Phi(x_0 + \Delta x) - \Phi(x_0)}{\Delta} \leq B(x_0 + \Delta x, f) \Delta \]

Dividing by \( \Delta \) we have:

\[ B(x_0 + \Delta x, f) \leq \int_{x_0}^{x_0 + \Delta x} f(u) \, du \leq \frac{\Phi(x_0 + \Delta x) - \Phi(x_0)}{\Delta} \leq B(x_0 + \Delta x, f) \]

But \( \lim_{\Delta \to 0} B(x_0 + \Delta x, f) = f(x_0) \)

and \( \lim_{\Delta \to 0} B(x_0 + \Delta x, f) = f(x_0) \)

Therefore \( \lim_{\Delta \to 0} \frac{\Phi(x_0 + \Delta x) - \Phi(x_0)}{\Delta} = \Phi'(x_0) = f(x_0) \)
CHAPTER III

RIEMANN INTEGRALS OF UNBOUNDED FUNCTIONS

OVER A FINITE INTERVAL

Previous to this, we have considered the Riemann integral of a function \( f(x) \) in a finite linear interval where \( f(x) \) was bounded. We now wish to extend this definition to cases in which \( f(x) \) has infinite discontinuities.

3.1 Assume that \( c, a < c < b \), is such that, in an arbitrarily small neighborhood, the absolute values of the function have no upper limit. Also assume that in every sub-interval of \((a, b)\) which does not contain \( c \) within it or at an end, the function is Riemann integrable.

\[
\int_{a}^{c-e} f(x) \, dx \quad \text{and} \quad \int_{c+e'}^{b} f(x) \, dx \quad \text{both exist for } e, e' \text{ arbitrarily small.}
\]

If \( \lim_{e \to 0} \int_{a}^{c-e} f(x) \, dx \) exists and \( \lim_{e' \to 0} \int_{c+e'}^{b} f(x) \, dx \) exists, then the sum \( \lim_{e \to 0} \int_{a}^{c-e} f(x) \, dx + \lim_{e' \to 0} \int_{c+e'}^{b} f(x) \, dx \) is defined to be the improper integral of \( f(x) \) in the interval \((a, b)\) and is denoted by \( \int_{a}^{b} f(x) \, dx \).
In case \( a \) is a point of infinite discontinuity, then if \( \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x) \, dx \) exists, this is defined to be the improper integral \( \int_{a}^{b} f(x) \, dx \) of \( f(x) \) in the interval \((a,b)\). Likewise for the case where \( b \) is a point of infinite discontinuity.

We have considered the case where we have only one point of infinite discontinuity.

3.2 Assume now that \( c_1 \) and \( c_2 \) are endpoints of an interval and furthermore that they are points of infinite discontinuity. \( \int_{c_1}^{c_2} f(x) \, dx \) is defined as the improper integral and is equal to

\[
\lim_{\epsilon \to 0} \int_{c_1+\epsilon}^{c} f(x) \, dx + \lim_{\epsilon' \to 0} \int_{c}^{c_2-\epsilon'} f(x) \, dx
\]

if these limits exist and if \( c_1 < c < c_2 \).

The definition of improper integrals may now be extended to the case where there are a finite number of points of infinite discontinuity in a finite interval.

Assume \( c_1, c_2, \ldots, c_n \) are such points taken in order from left to right in the interval \((a,b)\). Then if the improper integrals

\[
\int_{c}^{c_1} f(x) \, dx, \quad \int_{c_1}^{c_2} f(x) \, dx, \quad \ldots, \quad \int_{c_{n-1}}^{c_n} f(x) \, dx
\]

all exist, their sum is defined to be the improper integral.
The first derived set, \( E_1 \), is by definition the set of points consisting of the limit points of the set under consideration.

3.3 **Theorem:** \( E_1 \) is a closed set.

**Proof:**

Assume that \( P \) is a limit point of \( E_1 \). Then, in any neighborhood of \( P \) there exists an infinite number of points of \( E_1 \). But in the neighborhood of every point of \( E_1 \), there are infinitely many points of the set under consideration. Therefore, there are an infinite number of points of the set under consideration in any neighborhood of \( P \) and \( P \) is a limit point of this set. Therefore, \( P \subseteq E_1 \).

Since \( P \) is any limit point, the theorem follows.

The limit points of the first derived set constitute the second derived set, \( E_2 \). \( \cdots \) \( E_n \) consists of the limit points of \( E_n \), and constitutes the \( n^\text{th} \) derived set. It follows from theorem 3.3 that \( E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots \).

3.4 We may now extend the definition to the case where there is an infinite number of points of infinite discontinuity where the first derived set consists of a finite number of points.
3.4 (cont.)

Let $\alpha, \beta$ be any two consecutive points of this first derived set. There can only be a finite number of points of infinite discontinuity in the interval $(\alpha + \epsilon, \beta - \epsilon')$. Our previous definition takes care of this finite number of points.

If the improper integral \[ \int_{\alpha + \epsilon}^{\beta - \epsilon'} f(x) \, dx \]
exists and has a definite limit as $\epsilon$ and $\epsilon' \to 0$ and if this is also true for every consecutive pair of points of the derived set, then the improper integral \[ \int_{\alpha}^{\beta} f(x) \, dx \]
is said to exist and is equal to \[ \lim_{\epsilon \to 0} \int_{\alpha + \epsilon}^{\beta - \epsilon'} f(x) \, dx. \]

3.5 We now see that this definition may be extended to the case where the infinite points of discontinuity are such that some one of the derived sets contains only a finite number of points.

3.6 If \[ \int_{\alpha}^{\beta} f(x) \, dx \]
exists by these definitions, then \[ \int_{a}^{b} f(\omega) \, d\omega \]
exists, $a < x < b$, and this integral is a continuous function of $x$. 

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3.7 If the limit defining this improper integral exists, the integral is said to converge. Otherwise, the integral is divergent. If the integral still converges when the absolute value of the integrand is taken, the integral is said to be absolutely convergent; otherwise, it is conditionally convergent.

3.8 The limit of a function may be defined as follows:
If \( c \) is a point of infinite discontinuity and if \( c^- \) it is true that there exists a \( \delta_\varepsilon \) such that \( \delta < \delta_\varepsilon \) it follows that \( \left| \int_\alpha^x f(x) \, dx - A \right| < \varepsilon \) then \( A \) is said to be the limit of \( \int_\alpha^x f(w) \, dw \) as \( x \to c \).

3.9 In the finite case, if \( f(x) \) is Riemann integrable, then so is \( \left| f(x) \right| \) but this is not true for the improper integral.

3.10 **Example:** \( \int_0^1 \frac{1}{x} \sin \frac{1}{x} \, dx \) exists, but is not absolutely convergent.

Proof:
Integrating by parts, we find that \( \int_0^1 \frac{1}{x} \sin \frac{1}{x} \, dx \) is equal to \( \left[ x \cos \frac{1}{x} \right]_0^1 - \int_0^1 \cos \frac{1}{x} \, dx \) as \( \varepsilon \to 0 \).
3.10 (cont.)

But $\int_0^\infty \frac{1}{x} \sin \frac{1}{x} \, dx$ is bounded and has only one point of discontinuity. Therefore $\lim$ exists, and

$$\int_0^1 \frac{1}{x} \sin \frac{1}{x} \, dx \text{ exists.}$$

Now, consider $\int_0^1 \frac{1}{x} \sin \frac{1}{x} \, dx$ taken over the set of intervals $\left( \frac{1}{2n\pi + \frac{\pi}{2}}, \frac{1}{2(n+1)\pi} \right)$ where $n = 1, 2, \ldots, m$

1) $\int_0^\frac{1}{\pi} \frac{1}{x} \sin \frac{1}{x} \, dx = \int \frac{1}{x} \sin \int \frac{1}{x} \, dx$

2) $\int \frac{dx}{x} = \log \frac{2n\pi + \frac{\pi}{2}}{2(n+1)\pi} = \log \frac{2n+2}{2(n+1)} = \log (1 + \frac{1}{2n+1})$

3) $|\sin \frac{1}{x}| > \frac{1}{\sqrt{2}}$ in the interval $\left( \frac{1}{2n\pi + \frac{\pi}{2}}, \frac{1}{2(n+1)\pi} \right)$.

Using the fact that:

4) $\int_a^b |f(x)| \, dx > c \int_a^b |g(x)| \, dx \quad \text{if} \quad c \leq |f(x)|$

$a \leq x \leq b$, (1) is greater than $\frac{1}{\sqrt{2}} \log (1 + \frac{1}{2n+1})$.

Hence the integral taken over the set of intervals is $> \frac{1}{\sqrt{2}} \log \left( \prod_{n=1}^{m} \left( 1 + \frac{1}{2n+1} \right) \right) = \frac{1}{\sqrt{2}} \log \left( \frac{2m+2}{2(m+1)} \right)$

As $m$ increases indefinitely, the value of the integral does also, therefore $|\frac{1}{x} \sin \frac{1}{x}|$ can not be integrable in the interval $(0,1)$.

3.11 **Theorem:** The sum of two integrals is the integral of the sum but the converse does not hold.
3.11 (cont.)

Proof:

Part I: If \( \int_{a}^{b} f_{1}(x) \, dx \) converges and \( \int_{a}^{b} f_{2}(x) \, dx \) converges, then \( \int_{a}^{b} [f_{1}(x) + f_{2}(x)] \, dx \) converges.

Proof:

It follows from the laws of limits that at least two of the three limits defining the integrals must exist in order for the third to exist.

Part II: If \( \int_{a}^{b} [f_{1}(x) + f_{2}(x)] \, dx \) converges, it is not necessarily true that \( \int_{a}^{b} f_{1}(x) \, dx \) and \( \int_{a}^{b} f_{2}(x) \, dx \) exist.

Proof:

One example will be sufficient to show this.

Example:

1) \( \int_{a}^{b} \frac{x \sin x - l}{x} \, dx \) does not converge.

2) \( \int_{a}^{b} \frac{d\alpha}{x} \) does not converge.

The sum which is equal to \( \int_{a}^{b} \sin x \, dx \)

converges to 1.
CHAPTER IV

RIEMANN INTEGRALS OF BOUNDED FUNCTIONS

OVER AN UNBOUNDED INTERVAL

4.1 \[ \int_a^\infty f(x) \, dx \] is Riemann integrable if

1) \[ \int_a^{x_1} f(x) \, dx, \int_a^{x_2} f(x) \, dx, \cdots, \int_a^{x_n} f(x) \, dx, \cdots \] are each

Riemann integrable in the interval \((a,x_n)\), where

\[ x_1, x_2, \cdots, x_n, \cdots \] are a sequence of increasing numbers having no upper limit.

2) The sequence defined in (1) has a definite limit "A", independent of the sequence of \( \{x_n\} \) chosen, and the value of the integral is equal to "A".

4.2 \[ \int_{-\infty}^b f(x) \, dx \] is Riemann integrable if

1) \[ \int_{x_1}^b f(x) \, dx, \int_{x_2}^b f(x) \, dx, \cdots, \int_{x_n}^b f(x) \, dx, \cdots \] are each

Riemann integrable in the interval \((x_n,b)\), where

\[ x_1, x_2, \cdots, x_n, \cdots \] are a sequence of decreasing numbers having no lower limit.

2) The sequence defined in (1) has a definite limit "B" independent of the sequence of \( \{x_n\} \) chosen, and the value of the integral is equal to "B".

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4.3 \( \int_{-\infty}^{\infty} f(x) \, dx \) is Riemann integrable if \( \int_{-\infty}^{0} f(x) \, dx \) and \( \int_{0}^{\infty} f(x) \, dx \) both exist as defined, and their sum is equal to \( \int_{-\infty}^{\infty} f(x) \, dx \), and the value of the integral is equal to the value of \( \int_{-\infty}^{0} f(x) \, dx \) plus the value of \( \int_{0}^{\infty} f(x) \, dx \).

4.4 It follows from the definition of the Riemann integral of a function \( f(x) \) over an unbounded interval that \( \int_{a}^{x} f(\mu) \, d\mu \) exists, \( x > a \), and this integral is a continuous function of \( x \).

If the limit defining this integral exists, the integral is said to converge. Otherwise, the integral is divergent. If the integral still converges when the absolute value of the integrand is taken, the integral is said to be absolutely convergent; otherwise, it is said to be conditionally convergent.

4.5 \( \lim_{x \to \infty} \int_{a}^{x} f(\mu) \, d\mu = A \) may be defined as follows: for every \( \varepsilon > 0 \), it is true that there exists an \( N(\varepsilon) \) such that \( n > N(\varepsilon) \) it follows that:

\[ | \int_{a}^{n} f(\mu) \, d\mu - A | < \varepsilon \]
4.6 In the finite case, if the function \( f(x) \) is Riemann integrable, then so is \( |f(x)| \) but this is not true for the Riemann integral of a function over an unbounded integral. An example will be sufficient to show this.

Example:

\[
\begin{cases}
  1, & \text{for } 0 \leq x \leq 1 \\
  0, & \text{for } n-1 < x \leq n - \frac{1}{n} \\
  (-1)^{n}, & \text{for } n - \frac{1}{n} < x \leq n \quad \text{where } n = 2, 3, 4 \ldots
\end{cases}
\]

1) \[ \int_{0}^{\infty} f(x) \, dx = \int_{0}^{1} f(x) \, dx + \int_{1}^{1 + \frac{1}{2}} f(x) \, dx + \int_{1 + \frac{1}{2}}^{\infty} f(x) \, dx \cdots = 1 - 0 - \frac{1}{2} + 0 + \frac{1}{3} + 0 - \frac{1}{4} + 0 + \cdots \]

\[ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ converges.} \]

2) \[ \int_{0}^{\infty} |f(x)| \, dx = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \text{ diverges.} \]

Graph of the function.
CHAPTER V

RIEMANN INTEGRALS OF UNBOUNDED FUNCTIONS

OVER AN UNBOUNDED INTERVAL

We may now extend \( \int_{a}^{\infty} f(x) \, dx \), and \( \int_{-\infty}^{b} f(x) \, dx \)

to the case in which \( f(x) \) has points of infinite discontinuity.

5.1 If the improper integral \( \int_{a}^{\infty} f(x) \, dx \)

exists for every value of \( X > a \), and if it converges to a definite limit as \( X \) increases indefinitely, then this limit defines \( \int_{a}^{\infty} f(x) \, dx \).
CHAPTER VI

CONVERGENCE TESTS FOR THE INTEGRALS OF UNEBOUNDED
FUNCTIONS OVER A FINITE INTERVAL

6.1 Theorem: Given the function \( f(x) \) over the interval (a, b) and having an infinite discontinuity at \( c \), the condition that the integral \( \int_a^b f(x) \, dx \) converges is that for every \( \varepsilon > 0 \) there exists a \( \delta_\varepsilon > 0 \) such that for every \( \delta < \delta_\varepsilon \), it is true that \( \left| \int_{c-\delta}^{c+\delta} f(x) \, dx \right| < \varepsilon \)
where \( 0 < \delta < 1 \).

Proof:

Given that the integral converges to a value "A" means by definition that for every \( \varepsilon > 0 \), it is true that there exists a \( \delta_\varepsilon > 0 \) such that for every \( \delta < \delta_\varepsilon \), it follows that \( \left| \int_a^b f(x) \, dx - A \right| < \varepsilon \)

For that same \( \varepsilon \) and \( \delta \), it follows that

\[
\left| A - \int_{c-\delta}^{c+\delta} f(x) \, dx \right| < \varepsilon
\]

Then

\[
\left| \int_{c-\delta}^{c+\delta} f(x) \, dx \right| = \left| \int_{c-\delta}^{c} f(x) \, dx - \int_{c}^{c+\delta} f(x) \, dx \right| = \\
\left| \int_{c-\delta}^{c} f(x) \, dx - A + A - \int_{c}^{c+\delta} f(x) \, dx \right| \leq \\
\left| \int_{c-\delta}^{c} f(x) \, dx - A \right| + \left| A - \int_{c}^{c+\delta} f(x) \, dx \right| < \varepsilon
\]
which was to be shown.

This condition is equivalent to the condition that

\[
\lim_{\delta \to 0} \int_{c-\delta}^{c+\delta} f(x) \, dx \to 0
\]

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6.2 **Theorem:** Let \( F(x) \) be positive for all values of \( x \) in \((a, b)\) and suppose \( \int_a^b F(x) \, dx \) converges. If the given function \( f(x) \) satisfies the condition that 
\[ |f(x)| \leq F(x) \quad \text{for all values of } x \text{ in the deleted neighborhoods of the points of infinite discontinuity of } f(x) \text{ then the integral } \int_a^b f(x) \, dx \text{ also converges.} \]

**Proof:**

Given that \( \int_a^b F(x) \, dx \) converges, it follows that if \( c^+ \) is any point of infinite discontinuity, then for every \( \varepsilon > 0 \) it is true that there exists a \( \delta_c \) such that for every \( \delta < \delta_c \) it follows that
\[ |\int_{c-\delta}^{c} F(x) \, dx| < \varepsilon \quad \text{where } 0 < \delta < 1. \]
But
\[ |\int_{c-\delta}^{c} f(x) \, dx| \leq \int_{c-\delta}^{c} |f(x)| \, dx \leq \int_{c-\delta}^{c} F(x) \, dx < \varepsilon \]
which is the condition of the convergence of \( \int_a^b f(x) \, dx \).

6.3 **Example:** \( \int_0^1 \log x \, dx \)

This function has an infinite discontinuity at \( x = 0 \).

We know that \( \lim_{x \to 0^+} x^{\frac{1}{2}} \log x = 0 \). Hence we may write
\[ x^{\frac{1}{2}} \log x < a \quad \text{(finite) or } |\log x| < \frac{a}{x^{\frac{1}{2}}}. \]

The integral \( \int_0^1 \frac{1}{x^{\frac{1}{2}}} \, dx \) is convergent.

Hence, from the above theorem, \( \int_0^1 \log x \, dx \)
is also convergent.
6.4 **Theorem:** If \( f(x) \) is non-negative, bounded monotone decreasing function, then \( \int_a^b f(x) \, \varphi(x) \, dx \) is convergent if \( \int_a^b \varphi(x) \, dx \) converges.

**Proof:**

1) Let \( M = \sup \{ f(x) \mid c - \delta \leq x \leq c \} \) for \( c - \delta \leq x \leq c \).

2) For every \( \varepsilon > 0 \), it is true that there exists a \( \delta_{\varepsilon M} \) such that for any \( \delta < \delta_{\varepsilon M} \), it follows that

\[
\int_{c - \delta}^{c + \delta} |\varphi(x)| < \frac{\varepsilon}{M} \quad \text{where} \quad 0 < \delta < 1, \quad \text{and} \quad c \text{ is any point of infinite discontinuity.}
\]

By the second law of the mean for integral calculus:

3) \( \left| \int_{c - \delta}^{c + \delta} f(x) \, \varphi(x) \, dx \right| = \left| \int_{c - \delta}^{c + \delta} f(x) \, \varphi(x) \, dx \right| < M \cdot \frac{\varepsilon}{M} = \varepsilon \)

where \( c - \delta \geq f \geq c - \delta \), which is the condition of convergence.

6.5 **Example:**

Test \( \int_0^1 e^{-\frac{1}{x^2}} \log x \, dx \). Since \( \int_0^1 \log x \, dx \) converges and in the given interval \( e^{-\frac{1}{x^2}} \) is bounded and monotone decreasing, also positive, it follows from the theorem that \( \int_0^1 e^{-\frac{1}{x^2}} \log x \, dx \) must converge.

It is not true that if the integral of each of two functions converges, that the integral of their product converges. An example will show this.
Example: \( \int f(x) \, dx = \varphi(x) = \frac{1}{x} \). Then the two integrals
\[
\int_0^1 f(x) \, dx = \int_0^1 \varphi(x) \, dx = \int_0^1 \frac{dx}{x^\mu} \quad \text{both converge.}
\]
However, \( \int_0^1 f(x) \cdot \varphi(x) \, dx = \int_0^1 \frac{dx}{x} \) does not converge.

6.6 Theorem: If there exists a number \( \mu \) where \( 0 < \mu < 1 \) such that for all values of \( x \) in the deleted neighborhood of \( a \), the point of infinite discontinuity, we have \( (x-a)^\mu |f'(x)| < M \) (finite),
then \( \int_a^b f(x) \, dx \) converges for \( x = a \).

Proof:
Put \( (x-a)^\mu |f'(x)| = \psi(x) \), then
\[
|\int_a^{a+\delta} f(x) \, dx| \leq \int_a^{a+\delta} \psi(x) \, dx = \int_a^{a+\delta} \frac{\psi(x) \, dx}{x^\mu} \leq \\
M \int_a^{a+\delta} \frac{dx}{(x-a)^\mu} = \frac{M}{\mu+1} \left\{ \frac{1}{(a+\delta-a)^{\mu-1}} - \frac{1}{(a-\delta-a)^{\mu-1}} \right\} = \\
\frac{M}{\mu+1} \left\{ \frac{1}{\delta^{\mu-1}} - \frac{1}{(\delta)^{\mu-1}} \right\} = \frac{M}{\mu+1} \left\{ \delta^{-\mu} - (\delta)^{-\mu} \right\}
\]
But we have \( \lim_{\delta \to 0} \frac{M}{\mu+1} \left\{ \delta^{-\mu} - (\delta)^{-\mu} \right\} = 0 \)
Therefore we may write \( \lim_{\delta \to 0} \int_a^{a+\delta} f(x) \, dx \to 0 \)
and \( \int_a^b f(x) \, dx \) converges for \( x = a \).

6.7 Example: Test \( \int_0^1 \frac{\sin \frac{x}{\sqrt{x}} \, dx}{x^\mu} \). For \( \mu = \frac{1}{2} \),
\( x^\mu |f'(x)| = |\sin \frac{x}{\sqrt{x}}| < 1 \). Therefore \( \int_0^1 \frac{\sin \frac{x}{\sqrt{x}} \, dx}{x^\mu} \)
must converge for \( x = 0 \).
6.8 **Theorem:** The integral \( \int_a^b f(x) \, dx \) is convergent if the integral \( \int_a^b |f(x)| \, dx \) is convergent.

**Proof:**

1) \[ |\int_a^b f(x) \, dx| \leq \int_a^b |f(x)| \, dx \]

For \( c \) any point of infinite discontinuity in \((a, b)\), it follows that:

2) for any \( \varepsilon > 0 \) it is true that there exists a \( \delta_c \)
such that for any \( \delta < \delta_c \) it is true that

\[ |\int_c^{c+\delta} f(x) \, dx| \leq \int_c^{c+\delta} |f(x)| \, dx < \varepsilon \]

from (1) and the condition that \( \int f(x) \, dx \) is convergent.

Since \( c \) is any point of infinite discontinuity in \((a, b)\), it follows that (2) is the condition that \( \int_a^b f(x) \, dx \) converges.

6.9 **Example:** Consider \( \int_0^1 \frac{\cos \frac{1}{x}}{\sqrt{1-x^2}} \, dx \).

According to the foregoing theorem, this integral converges if \( \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx \) converges.

We have

\[
\int_0^1 \frac{|\cos \frac{1}{x}|}{\sqrt{1-x^2}} \, dx \leq \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x \bigg|_0^1 = \frac{\pi}{2}
\]

As this last integral exists, \( \int_0^1 \frac{|\cos \frac{1}{x}|}{\sqrt{1-x^2}} \, dx \) must converge, and hence the given integral converges.
CHAPTER VII

CONVERGENCE TESTS FOR THE INTEGRALS OF BOUNDED FUNCTIONS OVER AN UNBOUNDED INTERVAL

7.1 Theorem: A condition that the integral \( \int_a^x f(u) \, du \) converges is that for every \( \varepsilon > 0 \) it is true that there exists an \( x_1, x_2 \) such that for every \( x' > x_1 \), \( x'' > x_2 \) it follows that \( \left| \int_{x''}^{x'} f(u) \, du \right| < \varepsilon \).

Proof:

Given that for every \( \varepsilon > 0 \) it is true that there exists an \( x(\varepsilon) \) such that \( x' > x(\varepsilon) \) it follows that \( \left| \int_a^{x'} f(u) \, du - A \right| < \frac{\varepsilon}{2} \) and also \( \left| A - \int_a^{x''} f(u) \, du \right| < \frac{\varepsilon}{2} \).

where \( x'' > x(\varepsilon) \).

\[
\left| \int_{x''}^{x'} f(u) \, du \right| = \left| \int_a^{x'} f(u) \, du - \int_a^{x''} f(u) \, du \right| = \left| \int_a^{x'} f(u) \, du - A + A - \int_a^{x''} f(u) \, du \right|
\]

\[
\leq \left| \int_a^{x'} f(u) \, du - A \right| + \left| A - \int_a^{x''} f(u) \, du \right| < \varepsilon
\]

which was to be shown.

This condition is equivalent to the condition

\[
\lim_{x' \to \infty, x'' \to \infty} \int_{x''}^{x'} f(u) \, du = 0.
\]

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7.2 **Theorem:** Let \( F(x) \) be positive for all values of \( x \) in the interval \((a, \infty)\), and suppose that the integral \( \int_{a}^{\infty} F(x) \, dx \) converges. If for all values of \( x > a \), we have the relation \( |f(x)| \leq F(x) \), then the integral \( \int_{a}^{\infty} f(x) \, dx \) converges.

**Proof:**

From the given conditions, we have:

1) \( \left| \int_{x_i}^{x_2} f(u) \, du \right| \leq \int_{x_i}^{x_2} F(u) \, du \), \( a < x_i < x_2 \), but

2) \( \lim_{x_i \to \infty} \int_{x_i}^{x_2} F(u) \, du \to 0 \)

Therefore from (1) it follows that

\[
\lim_{x_i \to \infty} \left| \int_{x_i}^{x_2} f(u) \, du \right| \to 0 \quad \text{and} \quad \lim_{x_i \to \infty} \int_{x_i}^{x_2} f(u) \, du \to 0.
\]

7.3 **Example:** Consider \( \int_{0}^{\infty} \frac{\sin x}{1 + x^2} \, dx \)

\( F(x) = \frac{1}{1 + x^2} \). Since \( |\sin x| < 1 \), \( \left| \int_{0}^{\infty} \frac{\sin x}{1 + x^2} \, dx \right| \)

\[
\leq \int_{0}^{\infty} \frac{dx}{1 + x^2} = \lim_{x \to \infty} \arctan x \bigg|_{0}^{\infty} = \frac{\pi}{2}
\]

which is the condition that \( \int_{0}^{\infty} \frac{\sin x}{1 + x^2} \, dx \) converges.

7.4 **Theorem:** If \( f(x) \) is non-negative, bounded and monotonic decreasing function such that

\( \lim_{x \to \infty} f(x) \to 0 \) then the integral \( \int_{a}^{\infty} f(x) \varphi(x) \, dx \)

converges, provided \( \left| \int_{a}^{\infty} \varphi(x) \, dx \right| \) is bounded as \( \varphi(x) \) takes all finite values.
7.4 (cont.)

Proof:
Let \( M = \text{l.u.b.} \left| \int_a^x \phi(u) \, du \right| \) as \( x \to \infty \) takes all finite values. Since \( f(x) \to 0 \) monotonically as \( x \to \infty \) indefinitely, we can choose a value \( x_0 \) such that we have for all values of \( x \geq x_0 \), \( |f(x)| < \frac{\varepsilon}{2M} \).

By the second law of the mean for the integral calculus, we have, for \( x_3 > x_2 \geq x_0 \):

\[
\left| \int_{x_2}^{x_3} f(u) \phi(u) \, du \right| = \left| f(x_2) \int_{x_2}^{x_3} \phi(u) \, du \right| \leq \int_{x_2}^{x_3} \phi(u) \, du \leq \frac{\varepsilon}{2M} \cdot 2M = \varepsilon
\]

Hence we obtain \( \lim_{x \to \infty} \int_{x_0}^{x_2} f(u) \phi(u) \, du = 0 \), and the integral converges.

7.5 Example: Consider \( \int_{x_0}^{\infty} \frac{1}{\sqrt{x}} \sin x \, dx \), \( x > 0 \)

Let \( f(x) = \frac{1}{\sqrt{x}} \). Then \( \lim_{x \to \infty} f(x) = 1 \cdot \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0 \)

We have also \( \left| \int_{x_0}^{x} \phi(x) \, dx \right| \leq \left| \int_{x_0}^{x} \sin x \, dx \right| = |\cos x - \cos x_0| \leq 2 \).

Therefore the integral converges.

7.6 Theorem: If a number \( \mu > 1 \) exists such that

\( x \mu \left| f(x) \right| < M \) (finite) for all values of \( x > x_0 \), then \( \int_a^x f(u) \, du \) converges as \( x \to \infty \).
7.6 (cont.)

Proof:

Put \( x^\mu |f(x)| = \Psi(x) \). We then have:

\[
|\int_{x_0}^{x} f(x) \, dx| \leq \int_{x_0}^{x} |\Psi(x)| \, dx = \int_{x_0}^{x} \frac{\Psi(x)}{x^\mu} \, dx
\]

\[
\leq M \int_{x_0}^{x} \frac{dx}{x^\mu} = \frac{M}{\mu - 1} \left( \frac{1}{x_0^{\mu - 1}} - \frac{1}{x^{\mu - 1}} \right)
\]

But we know that \( \lim_{x \to \infty} \frac{M}{\mu - 1} \left( \frac{1}{x_0^{\mu - 1}} - \frac{1}{x^{\mu - 1}} \right) = 0 \)

Therefore, we may write \( \lim_{x \to \infty} \int_{x_0}^{x} f(x) \, dx = 0 \)

which is the condition that \( \int_{x_0}^{x} f(x) \, dx \)

converges.

7.7 Theorem: \( \int_{a}^{\infty} f(x) \, dx \) converges if \( \int_{a}^{\infty} |f(x)| \, dx \) converges.

Proof:

\[
|\int_{a}^{x_1} f(x) \, dx| \leq \int_{a}^{x_1} |f(x)| \, dx , \quad a < x < x_1
\]

From theorem 7.2 and the convergence of \( \int_{a}^{\infty} |f(x)| \, dx \)
we have: \( \lim_{x_1 \to \infty} |\int_{a}^{x_1} f(x) \, dx| = 0 \)

Therefore \( \lim_{x_1 \to \infty} \int_{x_1}^{\infty} f(x) \, dx = 0 \)

which is the condition that \( \int_{a}^{\infty} f(x) \, dx \) is convergent.
7.8 Example: Consider \( \int_{1}^{\infty} \frac{\cos x}{x^2} \, dx \)

\[
\int_{1}^{\infty} \left| \frac{\cos x}{x^2} \right| \, dx = \lim_{x \to \infty} \int_{1}^{x} \frac{\left| \cos x \right|}{x^2} \, dx
\]

\[
\leq \lim_{x \to \infty} \int_{1}^{x} \frac{dx}{x^2} = \lim_{x \to \infty} \left\{ \left[ -\frac{1}{x} \right] \right\} = 1
\]

which is the condition that \( \int_{1}^{\infty} \frac{\cos x}{x^2} \, dx \)

converges.
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