ON APPROXIMATE DERIVATIVES

A Thesis Presented for the
Degree of Master of Arts

By

Chow Shu-er

THE OHIO STATE UNIVERSITY
1946

Approved by:

[Signature]
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1. Introduction and Definitions. If \( f(x) \) is any real function, the four Dini derivatives at the point \( \bar{z} \), denoted by \( D^+(\bar{z}), D^-(\bar{z}), D^-(-\bar{z}), D^+(-\bar{z}) \), are defined as follows:

\[
D^+(\bar{z}) = \lim_{x \to \bar{z}^+} \sup Q(f, \bar{z}, x)
\]

\[
D^-(-\bar{z}) = \lim_{x \to \bar{z}^-} \inf Q(f, \bar{z}, x)
\]

\[
D^-(\bar{z}) = \lim_{x \to \bar{z}^-} \sup Q(f, \bar{z}, x)
\]

\[
D^+(-\bar{z}) = \lim_{x \to \bar{z}^+} \inf Q(f, \bar{z}, x)
\]

where

\[
Q(f, \bar{z}, x) = \frac{f(x) - f(\bar{z})}{x - \bar{z}}
\]

The celebrated theorem of Denjoy on these derivatives\([1]\)\* is as follows:

If \( f(x) \) is any continuous function, (defined in the whole continuum, or on a given interval) then at every point \( \bar{z} \), except for a set of measure zero, we have one of the three following cases:

\(\alpha\) The directional angle is \(90^\circ\). This means that the

\*The numbers cited have reference to the Bibliography appended below.
four Dini derivatives are finite and equal; in other words, that the derivative exists and is finite.

β) The directional angle is 180°. This means that the lower Dini derivative on one side and the upper Dini derivative on the other side (such a pair is called "opposite derivatives") are finite and equal; while the other two Dini derivatives are ±∞ respectively. Specifically, either $D^+(3)$ and $D^-(3)$ are finite and equal, $D^+(3) = +∞$ and $D^-(3) = -∞$; or the alternative holds with sides reversed.

γ) The directional angle is 360°. This means that the two upper Dini derivatives are $+∞$ and the other two $-∞$.

This theorem has been generalized in different ways.

G.C. Young[2] established this result for measurable functions.

S. Saks[3,1] generalized the theorem to arbitrary functions $f(x)$, defined on a set $E$, by defining a monotone function which coincides with $f(x)$ on a subset of $E$. H.E. Hanson[4] has the simplest proof for the case of unrestricted function.

His argument shows that this general case calls for no more laborious reasoning than that for a measurable function.

Another line of generalization is concerned not with the extension of the domain of functions, but with the use of approximate derivatives, which we define below.

In the sequel we shall always assume $f(x)$ to be a real, one-valued function defined on an interval $I: (a, b)$. All sets
we shall speak of will be supposed to lie in \( I \).

**Definition 1.** The upper right approximate limit of \( f(x) \) at \( x \), denoted by \( u^+(x) \), or simply by \( u(x) \) if there is no ambiguity, is the g.l.b. of the numbers \( k \) for which the set

\[ E_x [\exists \varepsilon > 0, x > x + \varepsilon] \]

has 0 right hand exterior metric density at \( x \). Similarly, the lower right approximate limit of \( f(x) \) at \( x \), denoted by \( l^+(x) \), or by \( l(x) \) --- is defined as the l.u.b. of the numbers \( k \) for which the set

\[ E_x [\exists \varepsilon > 0, x < x + \varepsilon] \]

has 0 right hand exterior metric density at \( x \). The two left hand limits, \( u^-(x) \) and \( l^-(x) \), are defined similarly.

Without reference to the directions of approach, we define the upper and lower approximate limits of \( f(x) \) at \( x \) respectively as:

\[ u(x) = \max[u^+(x), u^-(x)] \]

and

\[ l(x) = \min[l^+(x), l^-(x)] \]

**Definition 2.** The upper right approximate derivative of \( f(x) \) at \( x \), which we shall denote by \( A^+(x) \), or simply by \( A^+(x) \) --- is the upper right approximate limit of the quotient function \( Q(x, x) \). Similarly, we define the other three approximate derivatives, the lower right \( A^- (x) \), upper left \( A^-(x) \), and lower left \( A^- (x) \), as the respective approximate limits of \( Q(x, x) \).

*The functions \( u(x) \) and \( l(x) \) are called the "upper" and "lower metric boundaries" of the function \( f(x) \) by Professor Blumberg [5].*
If all these four approximate derivatives are equal, the common value is called the approximate derivative of \( f(x) \) at \( \bar{z} \), and denoted by \( A(\bar{z}) \), or by \( A(\bar{z}) \). If, furthermore, \( A(\bar{z}) \) is finite, then \( f(x) \) is said to possess an approximate derivative, or be \textit{approximately derivable}, at \( \bar{z} \).

The approximate derivatives, as defined in Definition 2, are the basis of generalizations of Denjoy's theorem in articles by J.C. Burkill and U.S. Haslam-Jones (and others). They proved the following result[6,(1)]:

\textbf{If} \( f(x) \) \textbf{is a measurable function, the directional angle, in respect to approximate derivation, of the curve} \( y = f(x) \), \textbf{is almost everywhere either} \( 0 \) \textbf{or} \( 360 \) °. \textbf{Consequently, if one of the approximate derivatives is finite at every point of a set} \( E \), \textbf{then} \( f(x) \) \textbf{is approximately derivable at almost every point of} \( E \).

A.J. Ward[7] obtained some related results for the case of an arbitrary function. In particular, he showed that

\textbf{For any finite function} \( f(x) \), \textbf{the set of points} \( x \) \textbf{for which} \( A^+(x) > A^-(x) \) \textbf{is of measure zero}.

Later on, Burkill and Haslam-Jones[6,(2)] proved that

\textbf{If, for any function} \( f(x) \), \( A^+(x) \) \textbf{is finite at the points} \( x \) \textbf{of the set} \( E \), \textbf{then} \( A^+(x) = A^-(x) \) \textbf{almost everywhere on} \( E \).

In his "Theory of Integrals", Saks[3,(2)], too, proves the results of Burkill and Haslam-Jones on the approximate derivatives of measurable functions. He makes a general remark that this result may be extended to finite non-measurable functions, but he gives no explicit proof. Moreover,
he requires an additional condition for the validity in question; namely, that to each point of the set $E$ (which he substitutes for the interval $I$) there correspond a certain measurable set. However, as we shall show, and without the use of any additional condition, the Denjoy analogue for approximate derivatives in the case of non-measurable functions is different from the literature result for measurable functions.

The main result of the present thesis is the definite solution of the problem of obtaining the analogue, for approximate derivatives, of the Denjoy theorem in the case where the function is arbitrary. As we just indicated, the results for arbitrary functions do not completely correspond to those for measurable functions. For the latter, only cases $a)$ (directional angle $= 0^0$) and $\gamma$) (directional angle $= 360^0$) can occur --- except in a set of measure 0. In the case of arbitrary functions, however, these two cases and also case $\beta$) (directional angle $= 180^0$) may occur in a set of positive exterior measure.

The principal means employed for obtaining this solution is the theorem of Professor Blumberg on the "measurable boundaries of an arbitrary function." This theorem is as follows:

**Theorem (Blumberg).** With every real function $f(x)$, one-or many-valued, defined on an interval $I$, there are uniquely associated two functions $u(x)$ and $l(x)$, defined in $I$, having the following properties:
1) \( u(x) \) and \( l(x) \) are measurable.

ii) The set of points \( x \) for which \( f(x) > u(x) \) or \( < l(x) \)
is of measure 0.

iii) The points \( (x, u(x)) \) and \( (x, l(x)) \) are positively approached (see Explanation below) by the curve \( y = u(x) \), \( y = l(x) \) respectively, for every \( x \); these points are fully approached by the curve \( y = f(x) \) (see Explanation below) for almost every \( x \), and positively approached for every \( x \).

Explanation. The "full" and "positive approach" of this theorem has reference to exterior metric density. The point \( (\xi, \eta) \) is fully approached by the "curve" or function \( y = f(x) \) if for every pair \( k, l \) of real numbers, with \( k < \eta < l \), the set \( E_{\eta, l} = E_{\eta} [k < f(x) < l] \) is of exterior metric density 1 at \( \xi \). The curve \( y = f(x) \) approaches \( (\xi, \eta) \) positively, if for every pair \( k, l \), with \( k < \eta < l \), the upper exterior metric density of \( E_{\eta, l} \) is positive at \( \xi \).

This theorem gives a structural representation of an arbitrary, real function \( f(x) \) which shows that \( f(x) \) is necessarily built on the scaffolding of two measurable functions, \( u(x) \) and \( l(x) \), the "measurable boundaries" of \( f(x) \). It is this structure which permits the transfer of various theorems on measurable functions to arbitrary functions. The memoir of Professor Blumberg cites a number of diverse cases where this transfer can be effected. One of these cases is that of the Denjoy theorem as generalized to unconditioned functions.

The writer has succeeded in making such a transfer for
approximate derivatives. It is only in this way that he has managed to solve the problem in question.

2. **Lemmas.** In this section, we prove two lemmas which we shall utilize in the proof of the theorems of Section 3.

**Lemma 1.** For every real function \( f(x) \), the set \( E = E_x [u^+(x) \neq u^-(x)] \) is of measure zero. A like result holds for \( l^+(x) \) and \( l^-(x) \).

**Proof.** For a rational number \( r \), let \( E_r = E_x [u^+(x) > r u^-(x)] \), and \( E' = E_x [u^+(x) > u^-(x)] = \bigvee \mathcal{E}_r \), where \( r \) ranges over the set of rational numbers. On account of the definition of \( u^\pm(x) \), and the condition for the points of \( E_r \), the set

\[
E_x [f(x) > r, x > \mathfrak{y}]
\]

has positive upper exterior metric density at every point \( \mathfrak{y} \) of \( E_r \); and the set

\[
E_x [f(x) > r, x < \mathfrak{y}]
\]

has 0 exterior metric density at \( \mathfrak{y} \). Hence the set

\[
E_x [f(x) > r]
\]

has positive upper exterior metric density at the right and 0 exterior metric density at the left of every point \( \mathfrak{y} \) of \( E_r \). Now an arbitrary set has its exterior metric density either 0 or 1 almost everywhere (as follows readily by the use of its measurable envelope). Hence \( E_r \), and therefore \( E' \), is of measure 0. Similarly, the set of points \( x \) for which \( u^+(x) < u^-(x) \) is of measure 0. We conclude that \( E \) is of measure 0.
It follows similarly that the set \( E_\varepsilon \{ 1^+(x) \neq 1^-(x) \} \)
im of measure 0.

**Lemma 2.** For every real function \( f(x) \), if \( u(3) = f(3) \),
then \( A^+(u,3) = A^+(\frac{1}{3},3) \) and \( A^-(u,3) = A^-(\frac{1}{3},3) \). Similarly, if
\( 1^-(3) = f(3) \), then \( A^-(\frac{1}{3},3) = A^-(\frac{1}{3},3) \) and \( A^+(\frac{1}{3},3) = A^+(\frac{1}{3},3) \).

**Proof.** Suppose \( u(3) = f(3) \). Since \( u(x) \geq f(x) \)
for almost every \( x \), we have
\[ Q(u,3,x) \geq Q(\frac{1}{3},3,x) \quad \text{for a.e. } x > 3 \]
Hence
\[ A^+(u,3) \geq A^+(\frac{1}{3},3) \]
On the other hand, suppose \( A^+(\frac{1}{3},3) = k \). We may assume
that \( k < +\infty \); for if \( k = +\infty \), we have \( A^+(u,3) \geq A^+(\frac{1}{3},3) = +\infty \),
whence \( A^+(u,3) = A^+(\frac{1}{3},3) \). For a given \( c > 0 \), let \( L \) be
the straight line through \( [3,\frac{1}{3}(\delta)] := [3,u(3)] \) with slope \( k + c \).
Then, by the definition of \( A^+(\frac{1}{3},3) \), the set \( E \) of numbers
\( x > 3 \) for which the point \( (x, f(x)) \) lies above \( L \) has
exterior metric density 0 at 3. Hence for every \( \eta > 0 \),
there exists a positive number \( h \) such that
\[ \mu^*(IE) < \eta l \]
for every interval \( I \) with 3 as left end point and length \( l < h \).

If \( \overline{E} = I - IE \), we have
\[ \mu^*(\overline{E}) > (1-\eta) l \]
where \( \mu^*(\overline{E}) \) denotes the interior Lebesgue measure of \( \overline{E} \).

Let \( T \) be a measurable subset of \( \overline{E} \) of measure \( > (1-\eta)l \) and
such that the metric density of \( T \) is 1 at each of its points.
For every \( x \) of \( \overline{E} \), the point \( (x, f(x)) \) lies on or below the
line L. Since by the theorem of Blumberg, the point \((x, u(x))\) is positively approached by \(y = f(x)\) for every \(x\), it follows that for every \(x\) of \(T\), the point \((x, u(x))\) also lies on or below \(L\); i.e.,

\[Q(u, \bar{z}, x) \leq k + \epsilon\]

for every \(x\) in \(T\).

Hence, if we denote the set

\[E \left\{ Q(u, \bar{z}, x) > k + \epsilon, x > \bar{z} \right\}\]

by \(E_I\), then \(m_\nu(E_I) < \eta \cdot \ell\). Since this holds for every \(\eta\) however small, and for every \(I\) of length \(\ell < \eta\), we conclude that the exterior metric density of \(E_I\) is 0 at the point \(\bar{z}\). This shows that \(A^+(u, \bar{z}) \leq k + \epsilon\), and since \(\epsilon\) is arbitrary, it follows that

\[A^+(u, \bar{z}) \leq k = A^+(f, \bar{z})\]

Therefore, by (2.1),

\[A^+(u, \bar{z}) = A^+(f, \bar{z})\]

Other parts of the Lemma can be proved similarly.

3. Theorems. Theorem I states some properties of approximate derivatives which hold for both measurable and non-measurable functions. These results are in the literature, but our proof is different and simpler. Theorem II gives sufficient conditions (which in view of Theorem III are also necessary) for the approximate derivability of arbitrary functions. Theorem III states the principal result of this thesis, namely the extension of Denjoy's theorem to approximate derivatives of arbitrary functions.

**Theorem I.** If \(f(x)\) is any finite function, then

1) The set of points for which the upper approximate
derivative on one side is less than the lower approximate
derivative on the other side is of measure 0.

ii) If one of the approximate derivatives is finite
at every point of a set $E$, it is equal to its opposite
derivative at almost every point of $E$.

iii) The set of points where the upper approximate
derivative is $-\infty$, or the lower approximate derivative is
$+\infty$, is of measure $0$.

Proof. 1) It is sufficient to reason about the set

$$E = E_x[A^+(x) < A_-(x)]$$

since the corresponding set with the left and right sides
interchanged may be treated similarly. If $x$ is a point
of $E$, $A^+(x)$ cannot be $+\infty$. It follows that the set

$$E_x[u(x) > f(x), x \in E]$$

is of measure $0$. For by the Theorem of Blumberg, the
point $(x, u(x))$ is fully approached by the curve $y = f(x)$
at almost every $x$. Hence the inequality $u(x) > f(x)$ would
imply, for almost every $x$ of $E$, that $A^+(x) = +\infty$. Accordingly,$u(x) \leq f(x)$ for almost every point $x$ of $E$. But by the
Theorem of Blumberg, the set $E_x[u(x) < f(x)]$ is of measure $0$.
Consequently,

$$u(x) = f(x) \quad \text{a.e. in } E$$

Now we apply Lemma 2, obtaining the equalities

$$(3.1) \quad A^+(x)[= A^+[f, x)] = A^+(u, x)$$

$$(3.2) \quad A_-(x)[= A_-(f, x)] = A_-(u, x)$$

for almost every $x$ in $E$. Since $u(x)$ is measurable, the set

$$E_x[A^+(u, x) < A_-(u, x)]$$

is of measure $0$. (according to the results
cited in the Introduction for the approximate derivatives of measurable functions). Hence \( E = E_x [A^+(x) < A_-(x)] \) is of measure 0.

ii) Suppose \( A^+(x) \) is finite at every point of a set \( E \). As before, we must have
\[
u(x) = f(x) \quad \text{a.e. in } E
\]
and (3.1) applies. By the results for measurable functions just referred to, the measurable function \( u(x) \) has a finite approximate derivative \( A(u,x) \) almost everywhere on \( E \). Hence
\[
A^+(u,x) = A_-(u,x) \quad \text{a.e. in } E
\]
Consequently, by (3.1)
\[
A^+(x) = A_-(x) \quad \text{a.e. in } E.
\]
The reasoning for the cases where the other approximate derivatives are finite is similar.

iii) It is sufficient to reason about the set
\[
E = E_x [A^+(x) = -\infty]
\]
since the other cases may be similarly treated. By similar reasoning to that for i), we have
\[
u(x) \leq f(x) \quad \text{a.e. in } E
\]
Since the set of points for which \( u(x) < f(x) \) is of measure 0, it follows that
\[
u(x) = f(x) \quad \text{a.e. in } E
\]
We may consequently apply (3.1), and conclude that
\[
A^+(u,x) = A^+(x) = -\infty \quad \text{a.e. in } E
\]
But for the measurable function \( u(x) \), the set of points for which \( A^+(u,x) = -\infty \) is of measure 0. Hence \( E = E_x [A^+(x) = -\infty] \) is of measure 0.

Theorem II. If \( f(x) \) is a given finite function, let \( E \) be the set of abscissas at which either two approximate
derivatives on the same side, or two upper (or lower) approximate derivatives are finite. Then \( f(x) \) is approximately derivable at almost every point of \( E \).

**Proof.** Let \( E \) be the subset of \( E \) at which \( A^+(x) \) and \( A_-(x) \) are finite. By Theorem I case ii), we have,

\[
A^+(x) = A_-(x), \quad A_+(x) = A^-(x) \quad \text{a.e. in } E,
\]

But

\[
A^+(x) \geq A_+(x), \quad A^-(x) \geq A_-(x) \quad \text{for every } x \text{ in } E.
\]

Therefore \( A_-(x) \geq A^-(x) \), so that \( A_-(x) = A^-(x) \). The four approximate derivatives are consequently equal and finite, hence \( f(x) \) is approximately derivable at almost every point of \( E \). The other three possibilities \( (A^-(x), A_-(x) \text{ finite}; A^+(x), A^-(x) \text{ finite}; A_+(x), A_-(x) \text{ finite}) \) may be treated similarly.

**Theorem III.** For any finite function \( f(x) \), the directional angle, in respect to approximate derivation, of the curve \( y = f(x) \) is, at almost every \( x \), either \( 0^\circ \), or \( 180^\circ \), or \( 360^\circ \).

**Proof.** The set of \( x \) for which any one of the relations

\[
f(x) > u(x) \quad f(x) < l(x) \\
u^+(x) \neq u^-(x) \quad l^+(x) \neq l^-(x)
\]

holds is of measure 0 (by the Theorem of Blumberg and Lemma 1). Therefore we need only consider those points \( x \) at which

\[
u^+(x) = u^-(x) = u(x) \\
l^+(x) = l^-(x) = l(x)
\]

and \( u(x) \geq f(x) \geq l(x) \).

**Case (a).** \( u(x) > f(x) > l(x) \). From \( u(x) > f(x) \), it follows that
\[ A^+(x) = +\infty \quad \quad A_-(x) = -\infty \quad a.e. \]

(cf. proof of Theorem I). Likewise, it follows from \( f(x) > l(x) \) that

\[ A^-(x) = +\infty \quad \quad A_+(x) = -\infty \quad a.e. \]

Hence in this case the directional angle is \( 360^\circ \) almost everywhere on the set \( E_x[u(x) > f(x) > l(x)] \).

**Case (b).** \( u(x) = f(x) > l(x) \). (The relation \( u(x) > f(x) = l(x) \) may be treated similarly.) From \( f(x) > l(x) \), it follows, as before, that

\[ A^-(x) = +\infty \quad \quad A_+(x) = -\infty \quad a.e. \]

From \( u(x) = f(x) \), it follows, by Lemma 2, that

\[ A^+(x) = A^+(u,x) \quad \quad A_-(x) = A_-(u,x) \]

If \( A^+(u,x) \) is finite for almost every \( x \) in a set \( E \), then, on account of the measurability of \( u(x), A^+(u,x) = A_-(u,x) \) almost everywhere in \( E \). Hence \( A^+(x) \) and \( A_-(x) \) are finite and equal almost everywhere in \( E \). The directional angle is, therefore, \( 130^\circ \) almost everywhere in \( E \). If \( A^+(u,x) = +\infty \) for almost every \( x \) in \( E \), then \( A_-(u,x) = -\infty \) at almost every point of \( E \) (again on account of the results for measurable functions). Hence \( A^+(x) = +\infty \) and \( A_-(x) = -\infty \) almost everywhere in \( E \). Therefore the directional angle is \( 360^\circ \) almost everywhere in \( E \).

**Case (c).** \( u(x) = f(x) = l(x) \). By Lemma 2, we have at almost every point of \( E = E_x[u(x) = f(x) = l(x)] \).

\[
\begin{align*}
A^+(x) &= A^+(u,x) \\
A^-(x) &= A^-(1,x) \\
A_-(x) &= A_-(u,x) \\
A_+(x) &= A_+(1,x)
\end{align*}
\]

(3.2)

Now since \( u(x) \geq 1(x) \) for every \( x \), it follows that

\[
\begin{align*}
A^+(u,x) &\geq A^+(1,x) \\
A_-(u,x) &\leq A_-(1,x)
\end{align*}
\]

(3.3)
On the other hand, \( u(x) \) and \( l(x) \) are both measurable. The directional angle is, therefore, either \( 0^\circ \) or \( 360^\circ \) at the points of \( y = u(x) \) and \( y = l(x) \), for almost every \( x \). Consequently,

\[
\begin{align*}
A^+(u, x) &= A^-(u, x) = \overline{A}(u, x), \\
A^+(l, x) &= A^-(l, x) = \overline{A}(l, x), \\
A_+(u, x) &= A_-(u, x) = \underline{A}(u, x), \\
A_+(l, x) &= A_-(l, x) = \underline{A}(l, x),
\end{align*}
\]

(3.4)

From (3.3) and (3.4), we obtain

\[
\overline{A}(u, x) = \overline{A}(1, x) \quad \underline{A}(u, x) = \underline{A}(1, x)
\]

(3.5)

Combining (3.2) and (3.5), we obtain, for almost every point \( x \) of \( E \),

\[
A^+(x) = \overline{A}(u, x) = \overline{A}(1, x) = A^-(x) \quad \underline{A}(u, x) = \underline{A}(1, x) = A_-(x)
\]

Hence the directional angle of \( y = f(x) \) is, almost everywhere in \( E \), the same as that of \( y = u(x) \) or \( y = l(x) \); i.e., either \( 0^\circ \) or \( 360^\circ \). This completes the proof.

**Remark 1.** A measurable function \( f(x) \) is approximately continuous almost everywhere, and therefore

\[ u(x) = f(x) = l(x) \]

Thus Case (c) (of Theorem III) occurs almost everywhere.

Following is a simple example which shows that case \( \beta \) (directional angle = \( 180^\circ \)) may be realized for a non-measurable function on a set of positive measure, indeed, at every point of the interval of definition of the function:
Let $I = E_1 \cup E_2$ be a decomposition of the unit interval $I: (0, 1)$ into two disjoint, non-measurable sets, each of exterior measure 1. (As is well known, such a decomposition is possible). Define $f(x) = 1$ in $E_1$, $0$ in $E_2$. It follows that case $\beta$ is valid at every point of $I$.

Remark 2. The results of this thesis on approximate derivatives, in particular Theorems I, II and III, are readily extensible to the case where $x$ ranges over an arbitrary set instead of an interval. They are also easily extended to many-valued functions.
BIBLIOGRAPHY


