
Dissertation

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By

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Abstract

The concept of singular shocks was introduced in a series of papers in the 1980s, by Keyfitz and Kranzer, in order to solve Riemann problems for a class of equations which cannot be solved using classical solutions. Classical solutions for Riemann problems are measurable functions composed of regular shocks and rarefactions, and singular shocks are distributions involving delta measures that are weak limits of approximate viscous solutions. During the past decades, many abstract theories of singular shocks were developed, and many examples of this type of solution in problems modeling physical phenomena were discovered.

We study singular shocks as self-similar zero-viscosity limits via the viscous regularization $u_t + f(u)_x = \epsilon u_{xx}$ for two systems of conservation laws. The first system models incompressible two-phase fluid flow in one space dimension, and the second one is the Keyfitz-Kranzer system. Singular shocks for both systems have been analyzed in the literature, and the results are enhanced in this dissertation.

We improve and apply theorems from Geometric Singular Perturbation Theory, including Fenichel’s Theorems, the Exchange Lemma, and the Corner Lemma, to prove existence and convergence of viscous profiles for singular shocks for those two examples. We also derive estimates for the growth rates of the unbounded viscous solutions. In particular, it is demonstrated that, although viscous solutions for these two systems both have shock layers of widths of order $\epsilon$, they tend to infinity in
quantitatively different manners. For the two-phase flow model, the maximum value of the solution is of order $\log(1/\epsilon)$, while for the Keyfitz-Kranzer system, the maximum value is of order $1/\epsilon^2$. 
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Chapter 1: Overview

A system of conservation laws in one space dimension is a time dependent system of partial differential equations of the form

$$u_t + f(u)_x = 0, \quad (1.1)$$

where $t \geq 0, \ x \in \mathbb{R}, \ f : \mathbb{R}^n \to \mathbb{R}^n, \ n \geq 1,$ and the unknown $u(x, t) \in \mathbb{R}^n$ is a vector-valued function. Each component $u_j$ of the state variable $u$ is called a *conserved quantity*, for the reason that $\int_{-\infty}^{\infty} u_j(x, t) \, dx$ should be constant with respect to $t$. In fluid dynamics problems, the conserved quantities are usually mass, momentum, and energy. An underlying assumption in (1.1) is that knowing the value of $u$ at a given point in space and time allows us to determine the rate of flow, or flux, of the state variable at $(x, t)$. The flux of the $j$th component is given by some function $f_j(u(x, t))$. The vector-valued function $f(u)$ with $j$th component $f_j(u)$ is called the *flux function*. Systems of conservation laws naturally arise in a wide variety of applications. Examples include the study of explosions and blast waves, the propagation of waves in solids, the flow of glaciers, and the separation of chemical species by chromatography.

A simple example that illustrates the interesting behavior of solutions to conservation laws is the “shock tube problem” of gas dynamics, originally formulated, and solved, by Riemann [Rie60]. The physical set-up is a tube filled with gas, initially
divided by a membrane into two sections. The gas has higher density and pressure in one half of the tube than in the other half, with zero velocity everywhere. At time $t = 0$, the membrane is suddenly removed or broken, and the gas allowed to flow. Intuitively, one might expect a gradual transition pressure. However, it was observed that there are discontinuities in some of the state variables. A shock wave propagates into the region of lower pressure, across which the density and pressure suddenly jump to higher values.

In a real experimental shock tube, the state variables would not be discontinuous across the shock wave because of effects such as viscosity and heat conduction. However, these smooth solutions would be nearly discontinuous, in the sense that the rise in density would occur over a distance that is microscopic compared to the natural length scale of the shock tube. This suggests the following approach to solving a system of conservation laws: introduce a diffusive term into the equations to obtain an equation with a unique smooth solution, and then let the coefficient of this term go to zero. In many cases, this vanishing viscosity method enables us to define mathematically a unique solution that is the physically correct inviscid limit.

Solutions to the shock tube problem were successfully described by the method of shock and rarefaction wave curves; this early research is surveyed in Courant and Friedrichs [CF48]. The distillation of that work led to the solution, by Lax [Lax57], of the Riemann problem, with weak waves, for strictly hyperbolic systems of conservation laws with characteristic families that are genuinely nonlinear or linearly degenerate. However, it has been observed by various authors [KK78, HL96, CL03] that Riemann problems for certain equations from nonlinear elasticity and gas dynamics cannot be solved using those classical solutions. For that reason, the notions
of a delta shock wave and a singular shock wave were introduced and employed, and it was shown that a large class of Riemann problems can be solved globally with these additional building blocks. Roughly speaking, a singular shock is a distribution involving Dirac delta functions which is the weak limit of a sequence of approximate viscous solutions.

A derivation of singular shocks as limits of approximations at the level of formal asymptotics was presented by [KK95, Sev07]. The derivation was improved, using geometric singular perturbation theory (GSPT), in pioneering work of Schecter [Sch04], to a theorem that demonstrates the existence of approximate solutions and makes precise their properties.

GSPT is the geometric approach to studying singularly perturbed problems with multiple time scales

\[
\epsilon x' = f(x, y, \epsilon) \\
y' = g(x, y, \epsilon).
\]

Surveys on this topic can be found in [Jon95, Kap99, KJ01, RT02]. Singularly perturbed equations arise in a variety of applications: two-point boundary value problems, traveling wave problems in reaction diffusion equations, chemical pattern formation, the propagation of action potentials in neurophysiology, coupled mechanical oscillators, perturbed Hamiltonian systems, combustion, friction modeling, fluid particle motion, and celestial mechanics.

The spirit of GSPT is to first study a set of subsystems which forms a decomposition of a system, and then use the information from the subsystems to conclude results for the original system. A system with multiple time scales has a natural decomposition into each of these respective time scales. Since these reduced equations will be lower-dimensional, one can expect situations under which the system can be
well analyzed. The full motion of the system will then in principle be recovered by piecing together trajectories from each of these subsystems.

The difficulty in GSPT lies in patching the pieces together to produce a real trajectory. Theorems from GSPT which we will discuss include Fenichel’s Theorems, the Exchange Lemma, and the Corner Lemma. Fenichel’s Theorems, developed in [Fen74, Fen77, Fen79], are concerned with persistence of hyperbolic structure of invariant manifolds under perturbations. The Exchange Lemma was developed by Jones and Kopell [JK94] in a study of relaxation-oscillations for the FitzHugh-Nagumo model. The idea is to follow the tangent space to a collection of trajectories that are passing near the invariant manifold in question. The conclusion is that the transversality of the tracking manifold to the stable manifold of a slow manifold is exchanged for closeness to part of the unstable manifold. Similar in flavor to the Exchange Lemma, the Corner Lemma is a tool to track a manifold past a normally hyperbolic manifold consisting of equilibria.

The principal contribution of this thesis is to improve and apply tools from GSPT to prove existence of viscous profiles of singular shocks for two examples. Besides proving existence and convergence of viscous profiles, we also derive estimates for the growth rates of the unbounded viscous solutions. The viscous solutions for these two systems tend to infinity in quantitatively different manners: the growth rates of viscous solutions in Chapters 4 and 5 are of order \( \log(1/\epsilon) \) and \( 1/\epsilon^2 \), respectively.

The detailed organization of this thesis is as follows. In Chapter 2 we review some knowledge of conservation laws. In Chapter 3, we present and give new proofs for a version of the Exchange Lemma and the Corner Lemma. In Chapters 4 and 5, we study viscous profiles for singular shocks for two different systems.
Chapter 2: Systems of Conservation Laws

A system of conservation laws in one space dimension, as described in Chapter 1, is a system of the form

$$u_t + f(u)_x = 0,$$

(2.1)

where $t \geq 0$, $x \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, and the unknown $u(x,t) \in \mathbb{R}^n$ is a vector-valued function. The system (2.1) is said to be strictly hyperbolic if the matrix $Df(u)$ has a complete set of real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$ for each $u \in \mathbb{R}^n$. The eigenvalues $\lambda_i(u)$ are called the characteristic speeds. A basic feature of nonlinear hyperbolic systems is that, even with smooth or analytic initial data, the solutions may lose smoothness after a finite amount of time. The theory of hyperbolic conservation laws was elaborated in a number of textbooks and specialized monographs [Bre00, Daf10, Hör97, LeV92, Smo83, Ser99, Ser00].

Because of the nature of loss of regularity, to describe solutions of (2.1) for all time $t \geq 0$, it is necessary to use the framework of weak solutions. A weak solution of the Cauchy problem (2.1) with initial data $u_0(x)$ is a locally integrable function $u(x,t)$ such that $f(u)$ is locally integrable and

$$\int_0^\infty \int_{\mathbb{R}} \{u \varphi_t + f(u) \varphi_x\} \, dx \, dt + \int_{\mathbb{R}} u_0(x) \varphi(x,0) \, dx = 0$$

(2.2)

for all compactly supported smooth functions $\varphi \in C^\infty_c(\mathbb{R} \times [0, \infty))$. 

5
2.1 Shock Waves and Riemann Problems

A Riemann problem is an initial value problem (2.1) with Cauchy data of the form

\[ u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \]

(2.3)

where \( u_L \) and \( u_R \) are constant. Since smooth Cauchy data can be approximated by a sequence of Riemann problems, understanding the structure of solutions of Riemann problems helps understanding Cauchy problems. A typical solution of a Riemann problem consists of discontinuities called \textit{shocks} and continuous transitions called \textit{rarefaction waves}. In this thesis we are interested in shocks, so here we skip the discussion of rarefaction waves.

A shock wave solution with shock speed \( s \), where \( s \in \mathbb{R} \), is a solution \( u(x,t) \) to the Riemann problem for which discontinuities occur on the line \( x = st \). It is easy to see that a function \( u(x,t) \) of the form

\[ u(x,t) = \begin{cases} u_L, & x < st, \\ u_R, & x > st, \end{cases} \]

(2.4)

where \( s \in \mathbb{R} \), is a weak solution for (2.1) if and only if the Rankine-Hugoniot jump condition

\[ f(u_R) - f(u_L) = s(u_R - u_L) \]

(2.5)

is satisfied. Note that in (2.4) we ignore the value of \( u \) on the line \( \{x = st\} \) since measurable functions are equivalent up to measure zero sets.

To give an idea of the role which strict hyperbolicity plays in conservation laws, we consider, as a simple example, the linear system (see e.g. [LeV92, Chapter 6])

\[ u_t + Au_x = 0, \]

(2.6)
where \( A \) is a constant matrix with real distinct eigenvalues \( \lambda_1 < \cdots < \lambda_n \). Let \( r_i \), \( i = 1 \ldots n \), be eigenvectors for \( \lambda_i \). If the given Riemann data satisfy that \( u_R - u_L \) is a scalar multiple of \( r_i \) for some \( i \in \{1, \ldots, n\} \), then

\[
Au_R - Au_L = \lambda_i(u_R - u_L).
\]

This means that the Rankine-Hugoniot condition (2.5) is satisfied with \( s = \lambda_i \). Hence (2.4) gives a shock solution for (2.6) with \( s = \lambda_i \). In this case the solution is called an \( i \)-shock, or a shock in the \( i \)-th family. For arbitrary Riemann data \((u_L, u_R)\), write

\[
u_L = \sum_{i=1}^n a_ir_i, \quad u_R = \sum_{i=1}^n b_ir_i,
\]

and set

\[
\omega_k = u_L + \sum_{i=1}^k (b_i - a_i)r_i, \quad k = 1, \ldots, n.
\]

Then

\[
u(x, t) = \begin{cases} 
  u_L, & x < \lambda_1t \\
  \omega_i, & \lambda_1t < x < \lambda_{i+1}t, \quad 1 \leq i \leq (n - 1) \\
  u_R, & x > \lambda_nt
\end{cases}
\] (2.7)

gives a solution for the Riemann problem. This means that Riemann solutions of hyperbolic linear systems (2.6) are compositions of \( n \) families of waves.

For nonlinear hyperbolic systems, where \( f \) is nonlinear and hence \( A = Df(u) \) is non-constant, if the system is strictly hyperbolic, it can be shown that there are also \( n \) families of shock waves, and \( i \)-shocks are defined by the \textit{Lax admissibility condition} [Lax57]:

\[
\lambda_{i-1}(u_L) < s < \lambda_i(u_L), \quad \lambda_i(u_R) < s < \lambda_{i+1}(u_R).
\] (2.8)

The shock (2.4) is said to be \textit{compressive} if it satisfies (2.8) for some \( i \). Another way to express compressivity is choosing \( j, k \in \{1, \ldots, n\} \) such that

\[
\lambda_{j-1}(u_L) < s < \lambda_j(u_L), \quad \lambda_k(u_R) < s < \lambda_{k+1}(u_R),
\] (2.9)
and it is clear that the shock is compressive if and only if $j = k$. The shock is called \textit{under-compressive} if $j < k$, and is called \textit{over-compressive} if $j > k$. For instance, when $n = 2$, over-compressibility means $\lambda_2(u_R) < s < \lambda_1(u_L)$.

\section*{2.2 Viscous Profiles}

Weak solutions for a Riemann problem are not unique: Consider for example the Burgers’ Equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with Riemann data

$$u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

(2.11)

It is easy to see that both the rarefaction wave

$$u(x,t) = \begin{cases} 0, & x < 0 \\ x/t, & 0 < x < t \\ 1, & x > t \end{cases}$$

(2.12)

and the shock wave

$$u_\alpha(x,t) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

(2.13)

are weak solutions of (2.10) and (2.11). One way to distinguish an admissible solution from the set of weak solutions is the \textit{vanishing viscosity method}, which means to consider the regularized equation

$$u_t + f(u)_x = \epsilon u_{xx},$$

(2.14$\epsilon$)

and then to study the limiting behavior of the solutions as $\epsilon$ approaches 0. It can be shown that solutions of (2.14$\epsilon$) and (2.11), if they exist, converge weakly to the function given in (2.12); see e.g. [Daf10, Theorem 8.6.1]. The general form of a viscous
system is
\[ u_t + f(u)_x = \epsilon(Bu_x)_x, \]  
where \( B = B(t, u) \) is an \( n \times n \) matrix-valued function. Note that (2.14e) corresponds to the case that \( B \) is the identity matrix. Typical choices of \( B \) are positive definite matrices, so that (2.15e) renders parabolic PDEs. Some other characterizations of appropriate viscosity matrices \( B \) were studied in [CS70, MP85, Liu85, SC72]. A family of solutions of (2.15e) and (2.3) for some \( B \) is called a **viscous profile**.

To find viscous profiles corresponding to (2.14e), a standard method (see e.g. [Daf10, Section 8.6]) is to consider solutions of the form \( u(x, t) = u(x - st) \), and then the system reduces to
\[ \frac{d}{d\tau}(-su + f(u)) = \frac{d^2}{d\tau^2}u, \]  
where \( \tau = (x - st)/\epsilon \). Integrating (2.16) formally from \(-\infty\) to any finite time, we obtain
\[ \frac{d}{d\tau}u = f(u) - su - \left( f(u_L) - su_L \right). \]  
Note that \( u_L \) is an equilibrium for (2.17). If the Rankine-Hugoniot condition (2.5) holds, then \( u_R \) is also an equilibrium. In this case, to find a viscous profile, it suffices to find a trajectory for (2.17) joining \( u_L \) with \( u_R \). Consider \( n = 2 \) for example, the compressive condition (2.8) implies that one of \( u_L \) and \( u_R \) is a saddle, and the other is a node, so the viscous profile is a saddle-node connection. On the other hand, over-compressivity implies that the viscous profile is a node-node connection, and it corresponds to a one-parameter family of trajectories [LX92].

The above approach, using the regularization (2.14e), gives an effective method to find viscous profiles for shock waves. However, when dealing with other types of Riemann solutions, such as rarefaction waves, difficulties occur in this approach.
because the system (2.14$\epsilon$) is not invariant under uniform stretching of time and space. As a remedy to those difficulties, the regularization

$$u_t + f(u)_x = \epsilon tu_{xx}$$

(2.18$\epsilon$)

was introduced, which corresponds to $B = tI_n$ in (2.15$\epsilon$), so that the regularized system is invariant under the transformation $(x, t) \mapsto (\alpha x, \alpha t)$. The study of self-similar solutions of (2.18$\epsilon$) was initialized independently by Kalasnikov [Kal59], Tupciev [Tup66, Tup73], and Dafermos [Daf73]. Using the self-similar variable $\xi = x/t$, the system (2.18$\epsilon$) becomes

$$-\xi \frac{d}{d\xi} u + \frac{d}{d\xi} (f(u)) = \epsilon \frac{d^2}{d\xi^2} u,$$

(2.19$\epsilon$)

and the initial condition (2.3) becomes

$$u(-\infty) = u_L, \quad u(+\infty) = u_R.$$ 

(2.20)

In a series of papers [Daf73, Daf74, DD76], Dafermos and DiPerna proved the existence and convergence of viscous profiles for some systems. Their proof was mainly based on the Leray-Schauder fixed point theorem. Using GSPT, the existence theory was extended to a larger class of systems by Schecter [Sch02], Schecter and Szmolyan [SS04, SS09], and Weishi Liu [Liu04].

To study viscous profiles for self-similar solutions of (2.1), we will consider the regularized system (2.18$\epsilon$), and hence (2.19$\epsilon$). We call a family of solutions of (2.19$\epsilon$) and (2.20), following [Sch04], a Dafermos profile.

2.3 Singular Shocks

For a Riemann problem (2.3) for a hyperbolic system of conservation laws, if the right state $u_R$ is sufficiently close to the left state $u_L$, by standard theory (see e.g. [Sch04], a Dafermos profile.
[Daf10, Chapter 9]) an admissible solution can be uniquely defined as a composition of rarefaction waves and shocks, and the solution is a bounded measurable function.

When \( u_R \) is not close to \( u_L \), there is no general theory for existence of solutions. In a series of papers [KK78, KK80, KK89a, KK89b, KK95], Keyfitz and Kranzer considered some Riemann problems with no classical solutions, and the concept of singular shocks was introduced to solve the problem. A singular shock is a measure which contains Dirac delta functions, and is the weak limit of a sequence of approximate solutions. A more precise definition was given in [Sev07]:

**Definition 2.1.** A singular shock solution of (2.1) is a measure of the form

\[
 u(x, t) = \tilde{u}(x, t) + \sum_i M_i(t)\delta(x - x_i(t)),
\]

where \( \tilde{u} \) is a weak solution of (2.1) in the complement of \( \bigcup x_i(t) \); the curve segments \( \{x_i\} \) supporting the singularities are \( W^{1,\infty} \) functions; and the singular masses \( M_i \) are \( L^\infty_{\text{loc}} \) (vector-valued) functions. In addition, \( u \) is the pointwise limit of approximations \( u_\varepsilon \) satisfying

1. \( u_\varepsilon \to u \) in \( L^1_{\text{loc}} \)

2. \( u_\varepsilon(\cdot, t) \rightharpoonup u(\cdot, t) \) weakly in the space of measures on \( \mathbb{R} \), pointwise in \( t \)

3. for some positive definite matrix \( B \),

\[
 \partial_t u_\varepsilon(\cdot, t) + \partial_x f(u_\varepsilon(\cdot, t)) - \varepsilon \partial_x \left( B \partial_x u_\varepsilon(\cdot, t) \right) \to 0
\]

weakly in the space of measures on \( \mathbb{R} \), pointwise in \( t \).

Note that a singular shock may not be a solution of (2.1) in the sense of (2.2) because \( f(u) \) is in general not defined for measures \( u \) containing delta functions.
If $f(u_{\epsilon})$ are locally bounded measures, then $u$ is also called a delta-shock. A detailed presentation can be found in the memoir by Sever [Sev07]. Some abstract treatments of delta shocks or singular shocks were made by D. Tan, T. Zhang and Y.X. Zheng [TZZ94], F. Huang [Hua97], J. Hu [Hu97], Ercole [Erc00], Danilov and Shelkovich [DS05a, DS05b], Panov and Shelkovich [PS06], Danilov and Mitrovic [DM08], Shelkovich [She06, She08], H. Yang and Y. Zhang [YZ12, YZ14].

There are also examples of singular shocks existing in problems modeling physical phenomena. A system modeling incompressible two-phase fluid flow was analyzed in [KSS03, KSZ04]. The vanishing pressure limit of solutions to the Euler equations modeling gas dynamics has also been extensively studied [Yan01, LY01, CL03, CL04, CY11a]. Examples in two-component chromatography were discovered by Mazzotti [Maz07, Maz09, MTC+10], and there are related models analyzed by [Sun11, Wan13, CY11b]. Recently, delta shocks for systems modeling thin films were discovered and analyzed by L. Wang, Bertozzi and their coworkers [WB14, BRSW15].

Note that Definition 2.1, when applied to regular shocks, does not distinguish the admissibility condition (2.8). This motivates further study of viscous approximations and their limits. A proof of existence of viscous singular shocks for the Keyfitz-Kranzer system was given by Schecter [Sch04]. An analogous approach was adopted by Keyfitz and Tsikkou [KT12] for some systems closely related to the Keyfitz-Kranzer system. In Chapter 4 we prove existence and convergence of viscous profiles for singular shocks for a system modeling incompressible fluid two-phase flow. In Chapter 5 we improve Schecter’s proof in [Sch04] and obtain the convergence of the viscous profiles and estimates for the growth rate of the unbounded family of viscous solutions.
Chapter 3: Geometric Singular Perturbation Theory

Our main goal is to solve the boundary value problem (2.19\(\epsilon\)) and (2.20). Note that (2.19\(\epsilon\)) is a \textit{singularly perturbed equation} because the perturbation \(\epsilon \frac{d^2}{d\xi^2} u\) has a higher order derivative than the other terms in the equation. To deal with singularly perturbed equations, we will apply \textit{Geometric Singular Perturbation Theory}.

The idea of geometric singular perturbation theory is to first study a set of subsystems which forms a decomposition of a system, and then to use the information for the subsystems to conclude results for the original system. Some prototypical examples are relaxation oscillations for forced Van der Pol Equations [DR96, KS01a, KS01b] and FitzHugh-Nagumo Equations [JKL91, KSS03, LVV06]. Surveys on this topic can be found in [Jon95, Kap99, KJ01, RT02].

In Section 3.1 and 3.2, we recall some fundamental theorems in geometric singular perturbation theory. In Section 3.3 and 3.4 we state and give new proofs for a version of the Exchange Lemma and the Corner Lemma.

3.1 Fenichel’s Theory for Fast-Slow Systems

To study the singularly perturbed equation (2.19\(\epsilon\)), we consider the Liénard-type variable, following [Sch02, Liu04], defined by

\[
w = \epsilon \frac{d}{d\xi} u - f(u) + \xi u.
\]

(3.1)
The system (2.19ε) becomes

\[
\epsilon \frac{d}{d\xi} u = f(u) - \xi u - w \\
\frac{d}{d\xi} w = -u. 
\]

(3.2ε)

Up to a rescaling of time, this is equivalent to

\[
\dot{u} = f(u) - \xi u - w \\
\dot{w} = -\epsilon u \\
\dot{\xi} = \epsilon. 
\]

(3.3ε)

Here we avoided explicitly expressing the time variable in (3.3ε) since it is implicitly defined by the equation for \( \dot{\xi} \).

Note that (3.3ε) is a fast-slow system; this means that the system is of the form

\[
\dot{x} = f(x, y, \epsilon) \\
\dot{y} = \epsilon g(x, y, \epsilon). 
\]

(3.4ε)

where \((x, y) \in \mathbb{R}^n \times \mathbb{R}^l\), and \( \epsilon \) is a parameter. In order to deal with fast-slow systems, Fenichel’s Theory was developed in [Fen74, Fen77, Fen79]. Some expositions for that theory can be found in [Wig94, Jon95].

An important feature of a fast-slow system is that the system can be decomposed into two subsystems: the limiting fast system and the limiting slow system. The limiting fast system is obtained by taking \( \epsilon = 0 \) in (3.4ε); that is,

\[
\dot{x} = f(x, y, 0) \\
\dot{y} = 0. 
\]

(3.5)

On the other hand, note that the system (3.4ε) can be converted to, after a rescaling of time,

\[
\epsilon x' = f(x, y, \epsilon) \\
y' = g(x, y, \epsilon). 
\]

(3.6ε)
Taking \( \epsilon = 0 \) in (3.6), we obtain the limiting slow system
\[
0 = f(x, y, 0) \tag{3.7}
\]
\[
y' = g(x, y, 0).
\]
Note that the limiting slow system (3.7) describes dynamics on the set of critical points of the limiting fast system (3.5), so we will need to piece together the information of the limiting fast system and the limiting slow system in the vicinity of the set of critical points. To piece this information together, \textit{normal hyperbolicity} defined below will be a crucial condition.

**Definition 3.1.** A critical manifold \( S_0 \) for (3.5) is an \( l \)-dimensional manifold consisting of critical points of (3.5). A critical manifold is \textit{normally hyperbolic} if \( D_xf(x, y, 0)|_{S_0} \) is hyperbolic. That is, at any point \((x_0, y_0) \in S_0\), all eigenvalues of \( D_xf(x, y, 0)|_{(x_0, y_0)} \) have nonzero real part.

Now we turn to discussing normal hyperbolicity for general systems
\[
\dot{z} = F(z), \tag{3.8}
\]
where \( z \in \mathbb{R}^N, \ N \geq 1 \). A manifold \( S \subset \mathbb{R}^N \) is \textit{locally invariant} if for any point \( p \in S \setminus \partial S \), there exist \( t_1 < 0 < t_2 \) such that \( p \cdot (t_1, t_2) \in S \), where \( \cdot \) denotes the flow for (3.8). In the vicinity of a locally invariant manifold, under certain conditions the system can be decomposed into lower-dimensional subsystems. For instance, when \( S = \{p_0\} \) is an isolated hyperbolic equilibrium for (3.8), the stable and unstable manifolds \( W^s(p_0) \) and \( W^u(p_0) \) exist according to the Hartman-Grobman Theorem [Har64], and the union of their tangent spaces at \( p_0 \) spans \( \mathbb{R}^N \).

A locally invariant \( C^r \) manifold \( S \subset \mathbb{R}^N, \ r \geq 1 \), is normally hyperbolic for the system (3.8) if the growth rate of vectors transverse to the manifold dominates the
growth rate of vectors tangent to the manifold. (Note that this is consistent with Definition 3.1.) In this case, from the standard theory for normally hyperbolic manifolds (see, for example, [HPS77, VvG87, CL88]) it is assured that stable and unstable manifolds $W^s(S)$ and $W^u(S)$ are defined.

For a locally invariant manifold $\Lambda \subset \mathbb{R}^N$ for (3.8) which is not necessarily normally hyperbolic, a center manifold is a normally hyperbolic locally invariant manifold, with the smallest possible dimension, containing $\Lambda$. In classical cases, $\Lambda = \{p_0\}$ is an isolated non-hyperbolic equilibrium, and a center manifold for $p_0$ has dimension equal to the number of generalized eigenvalues of $DF(p_0)$ with zero real part. For instance, the planar system

$$\dot{x} = x^3, \quad \dot{y} = y,$$

has a non-hyperbolic isolated equilibrium $p_0 = (0,0)$, and the $x$-axis is a center manifold for $p_0$. For general invariant sets $\Lambda$, we refer to [CLY00a, CLY00b].

Fenichel’s Theory is a center manifold theory for fast-slow systems. For a normally hyperbolic critical manifold $S_0$ for (3.5), the stable and unstable manifolds $W^s(S_0)$ and $W^u(S_0)$ can be defined in the natural way. We denote them by $W^s_0(S_0)$ and $W^u_0(S_0)$ to indicate their invariance under (3.4$\epsilon$) with $\epsilon = 0$. Fenichel’s Theory assures that the hyperbolic structure of $S_0$ persists under perturbation (3.4$\epsilon$). Below we state three fundamental theorems of Fenichel’s Theory following [Jon95].

**Theorem 3.1** (Fenichel’s Theorem 1). *Consider the system* $(3.4\epsilon)$, *where* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^l$, *and* $f, g$ *are* $C^r$ *for some* $r \geq 2$. *Let* $S_0$ *be a compact normally hyperbolic manifold for* (3.5). *Then for any small* $\epsilon \geq 0$ *there exist locally invariant* $C^r$ *manifolds, denoted by* $S_\epsilon$, $W^s(\epsilon)$ *and* $W^u(\epsilon)$, *which are* $C^1$ $O(\epsilon)$-*close to* $S_0$, $W^s_0(S_0)$ *and* $W^u_0(S_0)$, *respectively. Moreover, for any continuous families of compact sets*
\( I_\epsilon \subset W^u(\mathcal{S}_\epsilon), \ J_\epsilon \subset W^s(\mathcal{S}_\epsilon), \ \epsilon \in [0, \epsilon_0], \) there exist positive constants \( C \) and \( \nu \) such that

\[
\text{dist}(z \cdot t, \mathcal{S}_\epsilon) \leq Ce^{\nu t} \quad \forall \ z \in I_\epsilon, \ t \leq 0 \\
\text{dist}(z \cdot t, \mathcal{S}_\epsilon) \leq Ce^{-\nu t} \quad \forall \ z \in J_\epsilon, \ t \geq 0,
\]

where \( \cdot \) denotes the flow for (3.4\( \epsilon \)).

**Proof.** See [Jon95, Theorem 3]. \( \square \)

**Remark 3.1.** If \( \mathcal{S}_0 \) is locally invariant under (3.4\( \epsilon \)) for each \( \epsilon \), then the \( \mathcal{S}_\epsilon \) can be chosen to be \( \mathcal{S}_0 \) because of the construction in the proof of [Jon95, Theorem 3].

Note that \( W^u(\mathcal{S}_\epsilon) \) and \( W^s(\mathcal{S}_\epsilon) \) can be interpreted as a decomposition in a neighborhood of \( \mathcal{S}_0 \) in \((x, y)\)-space. The following theorem asserts that this induces a change of coordinates \((a, b, c)\) such that \( W^u(\mathcal{S}_\epsilon) \) and \( W^s(\mathcal{S}_\epsilon) \) correspond to \((a, c)\)-space and \((b, c)\)-space, respectively.

**Theorem 3.2** (Fenichel’s Theorem 2). Suppose the assumptions in Theorem 3.1 hold. Then under a \( C^r \) \( \epsilon \)-dependent coordinate change \((x, y) \mapsto (a, b, c)\), the system (3.4\( \epsilon \)) can be brought to the form

\[
\dot{a} = A^u(a, b, c, \epsilon)a \\
\dot{b} = A^s(a, b, c, \epsilon)b \\
\dot{c} = \epsilon(h(c) + E(a, b, c, \epsilon))
\]

in a neighborhood of \( \mathcal{S}_\epsilon \), where the coefficients are \( C^{r-2} \) functions satisfying

\[
\inf_{\lambda \in \text{Spec}A^u(a, b, c, 0)} \text{Re} \lambda > 2\nu, \quad \sup_{\lambda \in \text{Spec}A^s(a, b, c, 0)} \text{Re} \lambda < -2\nu
\]

for some \( \nu > 0 \) and

\[
E = 0 \quad \text{on} \ \{a = 0\} \cup \{b = 0\}.
\]
Proof. See [Jon95, Section 3.5] or [JT09, Proposition 1].

The family of trajectories for (3.7) forms a foliation of \( S_0 \). The following theorem says that this induces a foliation of \( W^u_\epsilon(S_\epsilon) \) and \( W^s_\epsilon(S_\epsilon) \).

**Theorem 3.3** (Fenichel’s Theorem 3). Suppose the assumptions in Theorem 3.1 hold. Let \( \Lambda_0 \) be a submanifold in \( S_0 \) which is locally invariant under (3.7). Then there exist locally invariant manifolds \( \Lambda_\epsilon \), \( W^s_\epsilon(\Lambda_\epsilon) \), and \( W^u_\epsilon(\Lambda_\epsilon) \) for (3.4\( \epsilon \)) which are \( C^{r-2} \) \( O(\epsilon) \)-close to \( \Lambda_0 \), \( W^s_0(\Lambda_0) \), and \( W^u_0(\Lambda_0) \), respectively. Moreover, for any continuous families of compact sets \( \mathcal{I}_\epsilon \subset W^u_\epsilon(\Lambda_\epsilon) \), \( \mathcal{J}_\epsilon \subset W^s_\epsilon(\Lambda_\epsilon) \), \( \epsilon \in [0,\epsilon_0] \), there exist positive constants \( C \) and \( \nu \) such that (3.9) holds with \( S_\epsilon \) replaced by \( \Lambda_\epsilon \). Suppose in addition that \( S_0 \) is invariant under (3.4\( \epsilon \)) for each \( \epsilon \). Then \( \Lambda_\epsilon \) can be chosen to be \( \Lambda_0 \).

Proof. Using Fenichel’s coordinates \((a,b,c)\) in Theorem 3.2 for the splitting of \( S_0 \), we can take \( W^u_\epsilon(\Lambda_\epsilon) \) and \( W^s_\epsilon(\Lambda_\epsilon) \) to be the pre-images of the sets \( \{(a,b,c) : a = 0, c \in \Lambda_0\} \) and \( \{(a,b,c) : b = 0, c \in \Lambda_0\} \), respectively, in \((x,y)\)-space. From (3.11) we obtain (3.9) with \( S_\epsilon \) replaced by \( \Lambda_\epsilon \). Suppose \( S_0 \) is invariant under (3.4\( \epsilon \)) for each \( \epsilon \), then from the remark after Theorem 3.1, we can take \( S_\epsilon = S_0 \) and hence \( \Lambda_\epsilon = \Lambda_0 \).

The system (3.10\( \epsilon \)) is called a **Fenichel normal form** for (3.4\( \epsilon \)), and the variables \((a,b,c)\) are called **Fenichel coordinates**.

### 3.2 Silnikov Boundary Value Problem

We have seen in Section 3.1 that fast-slow systems (3.4\( \epsilon \)) can locally be converted into normal forms (3.10\( \epsilon \)), where \( A^u \) and \( A^s \) satisfy the gap condition (3.11), and \( E \) is a small term satisfying (3.12). If we append the system with the equation \( \dot{\epsilon} = 0 \)
Figure 3.1: Trajectories in the rectangle \( \{0 \leq a \leq a^1, 0 \leq b \leq b^0\} \) can be parametrized in \( T \geq 0 \) by \( a(T) = a^1, b(0) = b^0 \).

and then replace \( c \) by \( \tilde{c} = (c, \epsilon) \), we obtain a system of the form

\[
\begin{align*}
\dot{a} &= A^u(a, b, \tilde{c})a \\
\dot{b} &= A^s(a, b, \tilde{c})b \\
\dot{\tilde{c}} &= \tilde{h}(\tilde{c}) + E(a, b, \tilde{c}),
\end{align*}
\]  

for which (3.11) and (3.12) are satisfied with \( E \) replaced by \( \tilde{E} \). For convenience, we will drop the tilde notation in (3.13) in the remaining discussion.

A Silnikov problem is the system (3.13) along with boundary data of the form

\[
(b, c)(0) = (b^0, c^0), \quad a(T) = a^1,
\]  

where \( T \geq 0 \). This boundary value problem was posed in [Sil67] to study homoclinic bifurcation. A heuristic reason for the existence of solutions of a Silnikov problem is illustrated in Fig 3.1. Consider the simple case \( \dot{a} = a, \dot{b} = -b \) and \( \dot{c} = 0 \). There
are infinitely many trajectories contained in the box \( \{0 \leq a \leq a^1, 0 \leq b \leq b^0 \} \). We may parametrize the set of trajectories in \( T \geq 0 \) by \( b(0) = b^0 \) and \( a(T) = a^1 \). On the \( a \)-axis and \( b \)-axis, the trajectories tend to the origin in backward and forward time, respectively. This suggests that trajectories near the axes can stay for an arbitrarily long time in the box, which implies that for any large \( T \) there exists a trajectory satisfying \( b(0) = b^0 \) and \( a(T) = a^1 \). When \( T \) grows to infinity, the trajectories approach the axes. In the general case \( \dot{a} = A^a a \) and \( \dot{b} = A^b b \) in arbitrary dimension, both \( a \)- and \( b \)-spaces consist of solutions tending to the origin in forward or backward time, so we have the same conclusion.

The critical manifold for (3.13) is \( \{a = 0, b = 0\} \), on which the system is governed by the limiting slow system

\[
\dot{c} = h(c). \tag{3.15}
\]

For a solution \((a(t), b(t), c(t))\) to the Silnikov boundary value problem (3.13) and (3.14), from conditions (3.11) and (3.12), it is natural to expect that \( a(t) \) and \( b(t) \) decay to 0 in backward time and forward time, respectively, and that \( c(t) \) is approximately the solution of (3.15). A theorem from [Sch08b] asserts that this is the case:

**Theorem 3.4** (Generalized Deng’s Lemma [Sch08b]). Consider the system (3.13) satisfying (3.11) and (3.12) with \( C^r \) coefficients, \( r \geq 1 \), defined on the closure of a bounded open set \( B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l \), where \( B_{k,\Delta} = \{a \in \mathbb{R}^k : |a| < \Delta\} \), \( \Delta > 0 \), and \( V \) is a bounded open set in \( \mathbb{R}^l \).

Let \( K_0 \) and \( K_1 \) be compact subsets of \( V \) such that \( K_0 \subset \text{Int}(K_1) \). For each \( c^0 \in K_0 \) let \( J_{c^0} \) be the maximal interval such that \( \phi(t, c^0) \in \text{Int}(K_1) \) for all \( t \in J_{c^0} \), where \( \phi(t, c^0) \) is the solution of (3.15) with initial value \( c^0 \). Let \( \nu > 0 \) be the number in
Suppose there exists \( \beta > 0 \) such that \( \tilde{\nu} := \nu - r \beta > 0 \) and

\[
|\phi(t, c^0)| \leq Me^{\beta |t|} \quad \forall t \in J_\omega.
\]

Then there is a number \( \delta_0 > 0 \) such that if \( |a^1| < \delta_0, |b^0| < \delta_0, c^0 \in V_0, \) and \( T > 0 \) is in \( J_\omega, \) then the Silnikov boundary value problem (3.13) and (3.14) has a solution \((a, b, c)(t, T, a^1, b^0, c^0)\) on the interval \( 0 \leq t \leq T. \) Moreover, there is a number \( K > 0 \) such that for all \((t, T, a_1, b_0, c_0)\) as above and for all multi-indices \( \mathbf{i} \) with \( |\mathbf{i}| \leq r, \)

\[
|D_\mathbf{i}a(t, T, a^1, b^0, c^0)| \leq Ke^{\tilde{\nu}(T-t)}
\]

\[
|D_\mathbf{i}b(t, T, a^1, b^0, c^0)| \leq Ke^{\tilde{\nu}t}
\]

(3.16)

\[
|D_\mathbf{i}c(t, T, a^1, b^0, c^0) - D_\mathbf{i}\phi(t, c^0)| \leq Ke^{-\tilde{\nu}T}.
\]

Sketch of Proof. Here we sketch the proof in [Sch08a]. Write (3.13) as

\[
\dot{a} = \tilde{A}^u(t, c^0)a + f(t, c^0, a, b, z)
\]

\[
\dot{b} = \tilde{A}^s(t, c^0)b + g(t, c^0, a, b, z)
\]

\[
\dot{z} = \tilde{A}^c(t, c^0)z + \theta(t, c^0, z) + \tilde{E}(t, c^0, a, b, z),
\]

where

\[
\tilde{A}^i(t, c^0) = A^i(0, 0, \phi(t, c^0)), \quad i = u, s,
\]

\[
\tilde{A}^c(t, c^0) = Dh|_{\phi(t, c^0)}
\]

and

\[
\tilde{E}(t, c^0, a, b, z) = E(a, b, \phi(t, c^0) + z).
\]

Let \( \Phi^i(t, s, c^0) \) be the solution operator for \( \tilde{A}^i(t, c^0), \) \( i = u, s, c. \) Then \((a(t), b(t), c(t))\) is a solution of Silnikov problem (3.13) and (3.14) if and only if \( c(t) = \phi(t, c^0) + z(t) \) and \( \eta(t) = (a(t), b(t), z(t)) \) satisfies

\[
a(t) = \Phi^u(t, T, c^0)a^1 - \int_t^T \Phi^u(t, s, c^0)f(s, c^0, \eta(s)) \, ds
\]

\[
b(t) = \Phi^s(t, 0, c^0)b^0 + \int_0^t \Phi^s(t, s, c^0)g(s, c^0, \eta(s)) \, ds
\]

(3.17)

\[
z(t) = \int_0^t \Phi^c(t, s, c^0)(\theta(s, c^0, z(s)) + \tilde{E}(s, c^0, \eta(s))) \, ds.
\]
Define an linear operator $L$ by the right-hand side of (3.17) for functions $\eta(t) = (a(t), b(t), z(t))$. It can be shown that the restriction of $L$ on a neighborhood of $0$ in the space of functions $\eta(t) = (a(t), b(t), z(t))$ equipped with the norm

$$
\|\eta\|_j = \sup_{0 \leq t \leq T} \left( e^{\bar{\nu}(T-t)} |a(t)| + e^{\bar{\nu}t} |b(t)| + e^{\bar{\nu}T} |z(t)| \right)
$$

is a contraction mapping. Hence the existence of solution of (3.13) and (3.14) follows from the standard Banach fixed point theorem.

**Remark 3.2.** Theorem 3.4 is a generalization of the *Strong $\lambda$-Lemma* in Deng [Den90], and *$C^r$-Inclination Theorem* in Brunovsky [Bru99]. In Deng’s work, the boundary data lie near an equilibrium that may nonhyperbolic. In Brunovsky’s work, the boundary data lie near a solution of a rectifiable slow flow on a normally hyperbolic invariant manifold. Schecter’s work allows considering more general flows on normally hyperbolic invariant manifolds.

### 3.3 The Exchange Lemma

Consider (3.10$\epsilon$) as a special case of (3.13), and recall that (3.10$\epsilon$) is the normal form of fast-slow systems (3.4$\epsilon$). We will use Theorem 3.4 to analyze Silnikov problems for fast-slow systems (3.4$\epsilon$). The result turns out to be a variation of the $(k + \sigma)$-Exchange Lemma [JT09, Tin94].

The Silnikov problem for (3.10$\epsilon$) corresponds to the boundary data

$$
a(\tau/\epsilon) = a^1, \quad (b, c)(0) = (b^0, c^0),
$$

with given $(a^1, b^0, c^0) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$ and $\tau > 0$. It can be interpreted as finding trajectories for (3.10$\epsilon$) connecting the sets $\{b = b^0, c = c^0\}$ and $\{a = a^1\}$, with prescribed time interval $0 \leq t \leq \tau/\epsilon$; see Fig 3.2. Note that the set $\{b = b^0, c = c^0\}$ is
of dimension $k$. The Exchange Lemma is a tool tracking the $(k + 1)$-manifold $I^*_\epsilon$ that evolves from a $k$-manifold $I_\epsilon$ which is transverse to the center-stable manifold $\{a = 0\}$. The theory of Exchange Lemma was first developed in [JKL91, JK94, JKK96] to study singularly perturbed systems near a normally hyperbolic, locally invariant manifold. Some generalizations of the Exchange Lemma for a broader class of systems were given by W. Liu [Liu00] and Schecter [Sch08b].

Another generalization, given by Tin [Tin94], is the $(k + \sigma)$-Exchange Lemma, $1 \leq \sigma \leq l$, which tracks the $(k + \sigma)$-manifold $I^*_\epsilon$ which evolves from a $(k + \sigma - 1)$-manifold $I_\epsilon = \{b = b^0, c^0 \in \Lambda\}$, where $\Lambda$ is a $(\sigma - 1)$-manifold. A major difference between the $(k + \sigma)$-Exchange Lemma and the general Exchange Lemma in [Sch08b] is that the estimates (3.16) for the derivatives in slow variables were not considered in [Sch08b].

We analyze Silnikov problems for fast-slow systems in normal form (3.10) in Lemma 3.1, and then, in Theorem 3.5, return to (3.4) to present a version of the $(k + \sigma)$-Exchange Lemma.

**Lemma 3.1.** Consider a system of the form (3.10) satisfying (3.11) and (3.12) defined on the closure of a bounced open set $B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$, where the coefficients are $C^r$ for some integer $r \geq 0$. Let $\Lambda \subset V$ be a $(\sigma - 1)$-dimensional manifold, $1 \leq \sigma \leq l$ and $\tau_0 > 0$. Suppose

$$c \circ [0, \tau_0] \subset V \quad \forall c \in \Lambda,$$

where $\circ$ denotes the flow for the limiting slow system (3.7). Let $J \subset (0, \tau_0)$ be a closed interval and $A \subset B_{k,\Delta} \setminus \{0\}$ be a compact set. Then for each small $\epsilon > 0$ and $(a^1, c^0, \tau) \in A \times \Lambda \times J$, the boundary value problem (3.4) and (3.18) has a unique
solution, denoted by \((a, b, c)(t; \tau, a^1, b^0, c^0, \epsilon), t \in [0, \tau/\epsilon]\). Moreover, if we set
\[
p_\epsilon = (a, b, c)(0; \tau, a^1, b^0, c^0, \epsilon), \quad q_\epsilon = (a, b, c)(\tau/\epsilon; \tau, a^1, b^0, c^0, \epsilon),
\]
then
\[
\|p_\epsilon - (0, b^0, c^0)\|_{C^r(A \times \Lambda \times J)} + \|q_\epsilon - (a^1, 0, c^0 \circ \tau)\|_{C^r(A \times \Lambda \times J)} \leq C e^{-\tilde{\nu}/\epsilon}
\]
for some positive constants \(\tilde{C}\) and \(\tilde{\nu}\). See Fig 3.2.

**Sketch of Proof.** Existence of solutions follows directly from Theorem 3.4, so it remains to prove (3.20). Write \(p_\epsilon = (a^{in}_\epsilon, b^0, c^0)\) and \(q_\epsilon = (a^1, \hat{b}_\epsilon, \hat{c}_\epsilon)\), then (3.20) is equivalent to
\[
\|(a^{in}_\epsilon, \hat{b}_\epsilon, \hat{c}_\epsilon - c^0 \circ \tau)\|_{C^r(A \times \Lambda \times J)} \leq \tilde{C} e^{-\tilde{\nu}/\epsilon}.
\]
The estimate of the derivatives in \((a^1, c^0) \in A \times \Lambda\) in (3.21) follows directly from (3.16). To prove the estimate of the derivatives in \(\tau \in J\), note that from (3.17) we
Figure 3.3: The \((k + \sigma)\)-Exchange Lemma asserts that \(\mathcal{I}'_\epsilon\) is \(C^1\)-close to \(W^u_0(\bar{\Lambda})\) in a neighborhood of \(q_0\).

As in the proof of Theorem 3.4, it can be shown that the derivatives of the integrands in (3.22) are exponentially small, so we obtain (3.21).

The following theorem is a modification of the \((k + \sigma)\)-Exchange Lemma. The main difference is that in this version we assert the existence of certain trajectories, while in the original version those trajectories were assumed to exist. The proof of the original theorem [Tin94] is based on tracking tangent spaces to an invariant manifold using linearized differential equations in terms of differential forms, while
the approach we present below relies on estimates for solution operators, following closely to the proof of the general Exchange Lemma in [Sch08a].

**Theorem 3.5.** Consider a system of the form (3.4ε) where \((x,y) \in \mathbb{R}^n \times \mathbb{R}^l\), and \(f\) and \(g\) are \(C^r\) functions for some \(r \geq 2\). Let \(S_0\) be a normally hyperbolic critical manifold for (3.5), and suppose \(D_xf|_{S_0}\) has a splitting of \(k\) unstable eigenvalues and \(m\) stable eigenvalues, \(k + m = n\). Let \(\bar{q}_0 \in W^u_0(S_0) \setminus S_0\), \(\bar{p}_0 \in W^s_0(S_0) \setminus S_0\), \(\bar{\tau}_0 > 0\), and assume

\[
\pi^s(\bar{p}_0) \circ [0, \bar{\tau}_0] \subset S_0 \quad \text{and} \quad \pi^u(\bar{q}_0) = \pi^s(\bar{p}_0) \circ \tau_0, \tag{3.23}
\]

where \(\circ\) denotes the flow for the limiting slow system (3.7), and \(\pi^{s,u}\) are the projections into \(S_0\) along stable/unstable fibers with respect to the limiting fast system (3.5). Let \(\{I_\varepsilon\}_{\varepsilon \in [0,\epsilon_0]}\) be a \(C^r\) family of \((k+\sigma-1)\)-dimensional manifolds, \(1 \leq \sigma \leq l\), and suppose

- (T1) \(I_0\) is transverse to \(W^s(S_0)\) at \(p_0\), and \(\Lambda := \pi^s(I_0 \cap W^s(S_0))\) is of dimension \((\sigma - 1)\).
- (T2) the slow flow (3.7) is not tangent to \(\Lambda\) at \(\pi^s(\bar{p}_0)\).
- (T3) The trajectory \(\pi^s(p_0) \circ [0,\tau_0]\) is rectifiable and not self-intersecting.

Let

\[
I^*_\varepsilon = I_\varepsilon \cdot [0,\infty), \tag{3.24}
\]

where \(\cdot\) denotes the flow for (3.4ε). Choose a compact interval \(J \subset (0,\infty)\) containing \(\bar{\tau}_0\) satisfying \(\Lambda \circ J \subset S_0\), and set \(\tilde{\Lambda} = \Lambda \circ J\). Then there exists a neighborhood \(V_0\) of \(\bar{q}_0\) such that

\[
I^*_\varepsilon \cap V_0 \text{ is } C^{r-2} O(\varepsilon)\text{-close to } W^u_0(\tilde{\Lambda}) \cap V_0. \tag{3.25}
\]
See Fig 3.3. Moreover, given any sequence \( \bar{q}_\epsilon \in T_\epsilon^* \cap V_0 \) such that \( \bar{q}_\epsilon \to \bar{q}_0 \), there exists a sequence \((\bar{p}_\epsilon, \bar{r}_\epsilon) \in T_\epsilon \times J \) which converges to \((\bar{p}_0, \bar{r}_0)\) and satisfies that, setting \( T_\epsilon = \bar{r}_\epsilon / \epsilon \),

\[
\bar{q}_\epsilon = \bar{p}_\epsilon \cdot T_\epsilon \quad \forall \epsilon > 0,
\]

and

\[
T_\epsilon = (\tau_0 + o(1)) \epsilon^{-1}.
\]

**Proof.** Under the assumption (3.35), from [Den90, Lemma 2.2], after a \( C^{r-2} \) change of coordinates, we can convert (3.4\epsilon) to (3.10\epsilon), and, from (T1), we may assume

\[
T_\epsilon = B_{k,\Delta} \times \{b^0\} \times \Lambda \quad \text{(3.28)}
\]

for some constant \( b^0 \in B_{m,\Delta} \setminus \{0\} \).

Since \( \bar{q}_0 \in W^u_0(S_0) \setminus S_0 \), we have \( a(\bar{q}_0) \neq 0 \) and \( b(\bar{q}_0) = 0 \), where \( a(\bar{q}_0) \) and \( b(\bar{q}_0) \) denote the \( a \)- and \( b \)-coordinates of \( \bar{q}_0 \). Set

\[
A = \{ a \in \mathbb{R}^k : |a - a(\bar{q}_0)| < \Delta_1 \}
\]

for some positive number \( \Delta_1 < \frac{1}{2} \min \{ \Delta, |a(\bar{q}_0)| \} \), so that \( A \subset B_{k,\Delta} \setminus \{0\} \). Let \( p_\epsilon \) and \( q_\epsilon \) be the functions of \( (a^1, c^0, \tau) \in A \times \Lambda \times J \) defined by (3.19). From (3.28) we see that \( (p_\epsilon, \tau) \) parametrizes \( I_\epsilon \times J \) in a neighborhood of \((p_0, \tau_0)\). Hence \( q_\epsilon \) parametrizes \( I_\epsilon^* \) in neighborhoods of \( \bar{q}_0 \). The estimate (3.20) holds with \( r \) replaced by \( r - 2 \). In particular,

\[
\|q_\epsilon - (a^1, 0, c^0 \circ \tau)\|_{C^{r-2}(A \times \Lambda \times J)} \leq C e^{-\tilde{\nu}/\epsilon}.
\]

Note that

\[
W^u(\tilde{\Lambda}) = \{(a, b, c) : b = 0, c \in \tilde{\Lambda}\}
\]

\[
= \{(a, b, c^0 \circ \tau) : b = 0, c^0 \in \Lambda, \tau \in J\},
\]

27
so we obtain (3.25).

Next we consider the sequence $\bar{q}_\epsilon \in \mathcal{I}_\epsilon$ given in the statement. Choose $(a^1_\epsilon, c^0_\epsilon, \tau_\epsilon) \in \mathcal{A} \times \Lambda \times J$ such that $\bar{q}_\epsilon = q_\epsilon(a^1_\epsilon, c^0_\epsilon, \tau_\epsilon)$, and set $\bar{p}_\epsilon = p_\epsilon(a^1_\epsilon, c^0_\epsilon, \tau_\epsilon)$. Then by definition $\bar{q}_\epsilon = \bar{p}_\epsilon \cdot (\bar{\tau}_\epsilon/\epsilon)$. From (T2) and (T3), $\bar{\Lambda}$ is a $\sigma$-dimensional manifold, and for any $c^1 \in \bar{\Lambda}$, there exists unique $(c^0, \tau_0) \in \Lambda \times J$ such that $c^1 = c^0 \circ \tau_0$. Hence (3.26) uniquely determines $\bar{p}_0 \in \mathcal{I}_0 \cap W^s(\Lambda)$ and $\bar{\tau}_0 \in J$. To show $p_\epsilon \rightarrow \bar{p}_0$ and $\tau_\epsilon \rightarrow \bar{\tau}_0$, since $(p_\epsilon, \tau_\epsilon)$ lies in the compact set $\Lambda \times J$, it suffices to show that every convergent subsequence of $\{(c_\epsilon, \tau_\epsilon)\}$ converges to $(\bar{c}^0, \bar{\tau}_0)$. Note that from the equation for $\hat{c}_\epsilon$ in (3.22), we have

$$\hat{c}_\epsilon = c^0_\epsilon \circ \tau_\epsilon + o(1). \tag{3.31}$$

Since $q_\epsilon \rightarrow \bar{q}_0 \equiv (\bar{a}^1, 0, \bar{c}^1)$, given any convergent subsequence $(c_{\epsilon j}, \tau_{\epsilon j})$ of $(c_\epsilon, \tau_\epsilon)$, say $(c_{\epsilon j}, \tau_{\epsilon j}) \rightarrow (\bar{c}^0, \bar{\tau}_0)$, from (3.31) we obtain $\bar{c}^1 = \bar{c}^0 \circ \bar{\tau}_0$. From (3.23) we have $\bar{c}^1 = \bar{c}^0 \circ \bar{\tau}_0$. Hence $(\bar{c}^0, \bar{\tau}_0) = (\bar{c}^0, \bar{\tau}_0)$. This completes the proof. \hfill \Box

### 3.4 The Corner Lemma

The Corner Lemma was first asserted in [Sch04], but its author later pointed out [Sch08a, Remark 2.4] that the proof was flawed and needed to be reworked. In Theorem 3.7 we modify both the statement and the proof of the original lemma. In our modified version, the required assumptions are more restricted, but they are already enough for our purpose in Chapter 5.

First we state the special case of Theorem 3.4 with $h \equiv 0$ in (3.13) as follows.
Figure 3.4: The Corner Lemma with $a = (\beta, \kappa)$, $b = r$ and $c = (w_1, w_2, \xi)$. The projections of the dynamics for this 6D system in $(\beta, \kappa, r)$- and $(\kappa, r, w_1)$-spaces are illustrated in (a) and (b), respectively. Note that $I^*$ expands exponentially in the $\beta$-direction, but in the $w_1$-direction it changes only mildly.

**Theorem 3.6.** Consider a system of the form

\[
\begin{align*}
\dot{a} &= A^u(a, b, c)a \\
\dot{b} &= A^s(a, b, c)b \\
\dot{c} &= E(a, b, c),
\end{align*}
\tag{3.32}
\]

satisfying (3.11) and (3.12) with $C^r$ coefficients, $r \geq 1$, defined on the closure of a bounded open set $B = B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$. Then for any $(a^1, b^0, c^0) \in B$ and $T \geq 0$, the Silnikov boundary value problem (3.32) and (3.14) has a unique solution, denoted by $(a, b, c)(t; T, a^1, b^0, c^0)$, $t \in [0, T]$. Moreover, if we set

\[
p_T = (a, b, c)(0; T, a^1, b^0, c^0), \quad q_T = (a, b, c)(T; T, a^1, b^0, c^0)
\]
and write $p_T = (a_T^{in}, b^0, c^0)$ and $q_T = (a^1, \hat{b}_T, \hat{c}_T)$, then

$$
\| (a_T^{in}, \hat{b}_T, \hat{c}_T - c^0) \|_{C^r(B)} \leq \tilde{C} e^{-\mu T}
$$

(3.33)

for some positive constants $\tilde{C}$ and $\mu$.

We will consider special cases of the system (3.32) for which there is an invariant manifold of codimension 1 which is transverse to an unstable direction. For definiteness, we assume $\{a_k = 0\}$ to be invariant under (3.32), and the matrix-valued function $A^u(a, b, c)$ is of the form

$$
A^u = \begin{pmatrix} A_0^u * & \ast \\ 0 & \lambda_k \end{pmatrix}
$$

(3.34)

where $A_0^u$ is a $(k - 1) \times (k - 1)$ matrix function and $\lambda_k$ is a positive scalar function.

Using Theorem 3.6, we will prove the following:

**Theorem 3.7** (Corner Lemma). Consider (3.32) defined on the closure of a bounded open set $B_{k, \Delta} \times B_{m, \Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$, where the coefficients $A^u$, $A^s$ and $E$ are $C^r$ for some $r \geq 3$, and $A^u$ is of the form (3.34). Assume (3.11) and

$$
E(a, b, c) = 0 \quad \text{on} \quad \{a = 0\} \cap \{b = 0\}.
$$

(3.35)

Let $\Lambda \subset V$ be a $\sigma$-dimensional $C^r$ manifold, $0 \leq \sigma \leq l$, and let $\mathcal{I}$ be a $C^r$ manifold of the form

$$
\mathcal{I} = \{(a, b, c) : |a| < \Delta_1, b = b^0, c = c^0 + \theta(a, c^0)a, c^0 \in \Lambda\},
$$

(3.36)

where $0 < \{\Delta_1, |b^0|\} < \Delta$, and $\theta$ is a $(l \times k)$-matrix function. Let $\mathcal{I}_\epsilon = \mathcal{I} \cap \{a_k = \epsilon\}$. Denote $\mathcal{I}_\epsilon^* = \mathcal{I}_\epsilon \cdot [0, \infty)$. Then the following holds: Fix any $q_0 \in W^u(\Lambda)$ with positive $a_k$-coordinate. Then there exists a neighborhood $V_0$ of $q_0$ satisfying that

$$
\mathcal{I}_\epsilon^* \cap V_0 \text{ is } C^{r-3} \text{ close to } W^u(\Lambda) \cap V_0
$$

(3.37)
as $\epsilon \to 0$. See Fig 3.4.

Furthermore, given any sequence of points $q_\epsilon \in I^*_\epsilon \cap V_0$, $\epsilon \in [0, \epsilon_0]$, which converges to a point $q_0 \in W^u(\Lambda_L)$ as $\epsilon \to 0$, let $p_\epsilon \in I_\epsilon$ and $T_\epsilon > 0$ be such that $q_\epsilon = p_\epsilon \cdot T_\epsilon$, and let $p_0$ be the unique point in $I_0$ satisfying $\pi^s(p_0) = \pi^u(q_0)$, where $\pi^s,\pi^u$ are the projections along stable/unstable fibers. Then $p_\epsilon \to p_0$ as $\epsilon \to 0$, and

$$\tilde{C}^{-1} \log \frac{1}{\epsilon} \leq T_\epsilon \leq \tilde{C} \log \frac{1}{\epsilon}$$

(3.38)

for some $\tilde{C} > 0$.

Proof. Under the assumption (3.35), from [Den90, Lemma 2.2], there exists a $C^{r-2}$ change of variables of the form $(a, b, c) \mapsto (a, b, \hat{c})$ so that the new system converted from (3.32), still denoted by (3.32), satisfies (3.12). The change of coordinate is a modification only on $c$, so $I$ is still parametrized as (3.36) in the new coordinates. Therefore, by dropping the hat in $\hat{c}$, we assume (3.12) holds for the system (3.32), and the coefficients are $C^{r-2}$ functions.

The stable/unstable manifolds for (3.32) are

$$W^s(\Lambda) = \{(a, b, c) : b = 0\}, \quad W^u(\Lambda) = \{(a, b, c) : a = 0\}.$$  

(3.39)

From (3.12), the slow variable $c$ is constant on $\{a = 0\} \cup \{b = 0\}$, which implies

$$\pi^u(a, 0, c) = (0, 0, c), \quad \pi^s(0, b, c) = (0, 0, c).$$

(3.40)

Let

$$\mathcal{A} = \{a \in \mathbb{R}^k : |a - a(q_0)| < \Delta_2\}$$

(3.41)

for some positive number $\Delta_2 < \frac{1}{2} \min\{\Delta, |a(q_0)|, a_k(q_0)\}$, so that $\mathcal{A} \subset B_{k,\Delta}$, where $a(q_0)$ and $a_k(q_0)$ denote the $a$- and $a_k$-coordinates of $q_0$. Choose a smooth real-valued
function $\chi(b)$ so that $\chi(b^0) = 1$ and $\chi(0) = 0$. Let

$$\tilde{c} = c - \chi(b)\theta(a, c^0)a.$$  \hspace{1cm} (3.42)

Then from $\chi(b^0) = 0$ we have

$$\tilde{c} = c - \theta(a, c^0)a \quad \text{on} \quad \{b = b^0\} \hspace{1cm} (3.43)$$

and from $\chi(0) = 0$ we have

$$\tilde{c} = c \quad \text{on} \quad \{a = 0\} \cup \{b = 0\}. \hspace{1cm} (3.44)$$

From (3.43), the image of $I$ in $(a, b, \tilde{c})$-space is

$$\tilde{I} = \{(a, b, \tilde{c}) : |a| < \Delta_1, b = b^0, \tilde{c} = c^0, c^0 \in \Lambda\}. \hspace{1cm} (3.45)$$

From (3.11) we know

$$\tilde{C}^{-1} < \lambda_k < \tilde{C} \hspace{1cm} (3.46)$$

for some positive constant $\tilde{C}$. In $(a, b, \tilde{c})$-coordinates, the system (3.32) is converted to, after dividing the equation by $\lambda_k$,

$$a' = \begin{pmatrix} \tilde{A}_0^u & * \\ 0 & 1 \end{pmatrix} a, \quad b' = \tilde{A}^s b, \quad \tilde{c}' = \tilde{E},$$  \hspace{1cm} (3.47)

for some $C^{r-3}$ coefficients $\tilde{A}_0^u$, $\tilde{A}^s$ and $\tilde{E}$, where $'$ denotes the derivative with respect to the time variable $\zeta$ defined by

$$d\zeta/d\sigma = \lambda_k,$$  \hspace{1cm} (3.48)

where $\sigma$ is the time variable for (3.32). Clearly (3.11) holds with $A^u$, $A^s$ and $\nu$ replaced by $\tilde{A}_0^u$, $\tilde{A}^s$ and $\tilde{\nu} := \nu/\tilde{C}$. Note that the condition (3.12) means $c$ is constant.
on \( \{a = 0\} \cup \{b = 0\} \). From (3.44) we see that \( \tilde{c} \) is also constant on that set. Hence (3.12) holds with \( E \) replaced by \( \tilde{E} \). Thus Theorem 3.6 can be applied to (3.47).

By Theorem 3.6, for any sufficiently large number \( T \) and any \((a^1, c^0) \in A \times \Lambda\), we can set \((a, b, \tilde{c})(t; T, a^1, b^0, c^0), t \in [0, T]\), to be the solution of (3.47) satisfying
\[
(b, \tilde{c})(0) = (b^0, c^0), \quad a(T) = a^1.
\]

Since the equation for \( a_k \) in (3.47) is \( a_k' = a_k \), by choosing \( T = \zeta = \log(a_k^1/\epsilon) \), where \( a_k^1 \) is the \( a_k \)-coordinate of \( a^1 \), the solution corresponding to (3.49) satisfies \( a_k(0) = \epsilon \).

We set
\[
\tilde{p}_\epsilon = (a, b, \tilde{c})(0; \zeta, a^1, b^0, c^0), \quad \tilde{q}_\epsilon = (a, b, \tilde{c})(\zeta; \zeta, a^1, b^0, c^0),
\]
and let \( p_\epsilon \) and \( q_\epsilon \) be the images of \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \), respectively, in \((a, b, c)\)-space. From (3.45) we see that \( \tilde{p}_\epsilon \in \tilde{I} \), and hence \( p_\epsilon \in I \). Since the \( a_k \)-coordinate of \( p_\epsilon \) is \( a_k(0) = \epsilon \), we conclude that \( p_\epsilon \in I_\epsilon \).

Regarding \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \) as functions of \((a^1, c^0) \in A \times \Lambda\), using (3.33) with \( T \) and \( \nu \) replaced by \( \zeta \) and \( \tilde{\nu} \), we have
\[
\|\tilde{p}_\epsilon - (0, b^0, c^0)\|_{C^{r-3}(A \times \Lambda)} + \|\tilde{q}_\epsilon - (a^1, 0, c^0)\|_{C^{r-3}(A \times \Lambda)} \leq C\epsilon^{\tilde{\nu}}. \tag{3.50}
\]

From (3.44) it follows that the \( \tilde{c} \)-coordinates of \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \) are \( O(\epsilon^{\tilde{\nu}}) \)-close to \( c^0 \) in \( C^{r-2} \)-norm. Hence (3.50) holds with \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \) replaced by \( p_\epsilon \) and \( q_\epsilon \). Since \( p_\epsilon \) and \( q_\epsilon \) parametrize \( I_\epsilon \) and \( I_\epsilon^* \) in neighborhoods of \( p_0 \) and \( q_0 \), by (3.39) this proves (3.37).

Next we consider the sequences \( q_\epsilon \) and \( p_\epsilon \) described in the statement. Write
\[
p_\epsilon = (a_{\epsilon}^1, b_{\epsilon}^1, c_{\epsilon}^1), \quad q_\epsilon = (\hat{a}_{\epsilon}, \hat{b}_{\epsilon}, \hat{c}_{\epsilon}),
\]
and \( q_0 = (a^1, 0, c^0) \) in \((a, b, c)\)-coordinates. By the definition of \( I \), we have \( b_{\epsilon}^1 = b^0 \).

The assumption \( q_\epsilon \to q_0 \) gives \( \hat{c}_{\epsilon} \to c^0 \), and then by (3.40) the assumption \( \pi^u(q_0) = \)
\( \pi^*(p_0) \) implies \( p_0 = (0, b^0, c^0) \). From (3.50) we have \( a_{\epsilon}^{in} = o(1) \) and \( c_{\epsilon}^{in} = \hat{c}_{\epsilon} + o(1) \). It follows that \( c_{\epsilon}^{in} \rightarrow c^0 \), and hence \( p_{\epsilon} \rightarrow p_0 \).

Let \( T_\epsilon > 0 \) be the number such that \( q_{\epsilon} = p_{\epsilon} \cdot T_\epsilon \). Since \( p \in I_\epsilon \), the \( a_k \)-coordinate of \( p_{\epsilon} \) equals \( \epsilon \), so from (3.48) we have

\[
T_\epsilon = \int_0^{\zeta_\epsilon} \frac{1}{\lambda_k} \, d\zeta , \quad \text{where} \quad \zeta_\epsilon = \log \frac{a_k(q_{\epsilon})}{\epsilon} = \log \frac{a_k(q_0) + o(1)}{\epsilon} .
\] (3.51)

Inserting (3.46) in (3.51), we then obtain (3.38). \( \square \)
Chapter 4: Singular Shocks for the Two-Phase Flow System

4.1 Introduction

Keyfitz et al [KSS03, KSZ04] considered the system of conservation laws

\[
\begin{align*}
\beta_t + (v B_1(\beta))_x &= 0 \\
v_t + (v^2 B_2(\beta))_x &= 0
\end{align*}
\]

(4.1)

where \( t \geq 0, x \in \mathbb{R}, v \in \mathbb{R}, \beta \in [\rho_1, \rho_2] \) with \( \rho_2 < \rho_1 \) and

\[
B_1(\beta) = \frac{(\beta - \rho_1)(\beta - \rho_2)}{\beta}, \quad B_2(\beta) = \frac{\beta^2 - \rho_1 \rho_2}{2\beta^2}.
\]

(4.2)

For Riemann problems with data in feasible regions, they constructed uniquely defined admissible solutions. It can be readily shown that this system is not everywhere hyperbolic, and hence standard methods does not apply [Smo83, Daf10]. To resolve this problem, along with rarefaction waves and regular shocks, the concept of singular shocks was adopted.

To justify singular shocks for (4.1), in [KSS03] some heuristic arguments were made, and in [KSZ04] approximate solutions for the Dafermos regularization were constructed. In this chapter, rather than approximate solutions, we prove existence of exact solutions for the Dafermos regularization; also we give descriptions of their limiting behavior including weak convergence and growth rates.
The system (4.1) is equivalent to a two-fluid model for incompressible two-phase flow [DP99, p.248] of the form

\[
\begin{align*}
\partial_t (\alpha_i) + \partial_x (\alpha_i u_i) &= 0 \\
\partial_t (\alpha_i \rho_i u_i) + \partial_x (\alpha_i \rho_i u_i^2) + \alpha_i \partial_x p_i &= F_i, & i = 1, 2,
\end{align*}
\]

where the drag terms \( F_i \) are neglected and the pressure terms satisfy \( p_1 = p_2 \). To reduce (4.3) to (4.1), in [KSS03] the volume fractions \( \alpha_1 \) and \( \alpha_2 = 1 - \alpha_1 \) have been replaced by a density-weighted volume element \( \beta = \rho_2 \alpha_1 + \rho_1 \alpha_2 \) and the momentum equations replaced by a single equation for the momentum difference \( v = \rho_1 u_1 - \rho_2 u_2 - (\rho_1 - \rho_2) K \), where \( K = \alpha_2 u_1 + \alpha_2 u_2 \) is taken to be zero. This is a simple example of continuous model for two-phase flow, but it shares with other continuous models the property of changing type – that is, it is not hyperbolic for some (in this case, most) states.

The purpose of this study is to shed light on the mathematical properties of the change-of-type system that appear in continuous models of two-phase flow. The original studies [KSS03, KSZ04] showed the existence of self-similar solutions with reasonable properties. Specifically, the singular shocks that appear can be considered to be propagating phase boundaries. In this study, we focus on viscous profiles of singular shocks and unveil some of their limiting behavior.

In Section 4.2, we state our main result, and in Section 4.3 the validity of the assumptions of the theorem is discussed, with some proofs for the sufficient conditions postponed to Section 4.7. Section 4.4 is devoted to describing the structure of the system. We complete the proof of the main theorem in Section 4.5, and some numerical simulations are shown in Section 4.6.
4.2 Main Result

In standard notation for conservation laws, we write (4.1) as

\[ u_t + f(u)_x = 0, \quad (4.4) \]

where \( u = (\beta, v) \), and write Riemann data for Riemann problems in the form

\[ u(x, 0) = u_L + (u_R - u_L)H(x), \quad (4.5) \]

where \( H(x) \) is the step function taking value 0 if \( x < 0 \); 1 if \( x > 0 \). As seen in Section 2.2, viscous profiles for Dafermos regularization

\[ u_t + f(u)_x = \epsilon u_{xx} \quad (4.6\epsilon) \]

correspond to the boundary value problem

\[ -\xi \frac{d}{d\xi} u + \frac{d}{d\xi} (f(u)) = \epsilon \frac{d^2}{d\xi^2} u, \quad (4.7\epsilon) \]

and

\[ u(-\infty) = u_L, \quad u(+\infty) = u_R. \quad (4.8) \]

As seen in Section 3.1, the system (4.7\epsilon) is equivalent to

\[ u_\xi = f(u) - \xi u - w \quad (4.9\epsilon) \]

\[ w_\xi = -\epsilon u \]

or, up to a rescaling of time,

\[ \dot{u} = f(u) - \xi u - w \quad (4.10\epsilon) \]

\[ \dot{w} = -\epsilon u \]

\[ \dot{\xi} = \epsilon. \]
The time variable in (4.10) is implicitly defined by the equation of \( \dot{\xi} \). When \( \epsilon = 0 \), (4.10) is reduced to
\[
\dot{u} = f(u) - \xi u - w \tag{4.11}
\]
\[
\dot{w} = 0, \quad \dot{\xi} = 0.
\]
Returning to the \((\beta, v)\) notation, the system (4.10) is written as
\[
\dot{\beta} = -B_1(\beta)v - \xi \beta - w_1
\]
\[
\dot{v} = -B_2(\beta)v^2 - \xi v - w_2
\]
\[
\dot{w}_1 = -\epsilon \beta
\]
\[
\dot{w}_2 = -\epsilon v
\]
\[
\dot{\xi} = \epsilon,
\]
and (4.11) becomes
\[
\dot{\beta} = -B_1(\beta)v - \xi \beta - w_1
\]
\[
\dot{v} = -B_2(\beta)v^2 - \xi v - w_2 \tag{4.13}
\]
\[
\dot{w}_1 = 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0.
\]
The linearization at any equilibrium \((\beta, v, w_1, w_2, \xi)\) for (4.13) has eigenvalues \( \lambda_{\pm}(\beta, v) - \xi \), where
\[
\lambda_{\pm}(u) = 2vB_2(\beta) \pm v\sqrt{B_1(\beta)B'_2(\beta)}. \tag{4.14}
\]
Note that \( \text{Re}(\lambda_{\pm}(u)) = 2vB_2(\beta) \) since \( B_1(\beta)B'_2(\beta) \leq 0 \) when \( \rho_2 \leq \beta \leq \rho_1 \). Moreover, the system is nonhyperbolic everywhere in the physical region except on the union of the lines \( \{\beta = \rho_1\}, \{\beta = \rho_2\}, \) and \( \{v = 0\} \).

An over-compressive shock region is a region where the condition (H1) defined below holds. It was shown in [KSZ04] that any data in an over-compressive shock region admits a singular shock solution, and the shock speed \( s \) is defined by (4.15) below. Our main theorem confirms Dafermos profiles in a subset of this region.

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Main Theorem. In the Riemann problem (4.1), (4.5), let \( u_L = (\beta_L, v_L) \) and \( u_R = (\beta_R, v_R) \) be two points in \([\rho_1, \rho_2] \times (0, \infty)\) with \( \beta_R \neq \beta_L \). Let

\[
\begin{align*}
    s &= \frac{v_L B_1(\beta_L) - v_R B_1(\beta_R)}{\beta_L - \beta_R} \\
    w_L &= f(u_L) - s u_L, \quad w_R = f(u_R) - s u_R \\
    e_0 &= w_{2L} - w_{2R}
\end{align*}
\] (4.15)

where we denote \( w_L = (w_{1L}, w_{2L}) \) and \( w_R = (w_{1R}, w_{2R}) \). Assume

(H1) \( \text{Re}(\lambda_r(u_R)) < s < \text{Re}(\lambda_r(u_L)) \), where \( \lambda_r(u) \) are defined in (4.14).

(H2) \( e_0 > 0 \).

(H3) For the system (4.13), there exists a trajectory joining \((\beta_L, v_L, w_L, s)\) and \((\rho_1, +\infty, w_L, s)\), and a trajectory joining \((\beta_R, v_R, w_R, s)\) and \((\rho_2, +\infty, w_R, s)\).

Then there is a singular shock with Dafermos profile for the Riemann data \((u_L, u_R)\).

That is, for each small \( \epsilon > 0 \), there is a solution \( \tilde{u}(\xi) \) of (4.7\( \epsilon \)) and (4.8), and \( \tilde{u}(\xi) \) becomes unbounded as \( \epsilon \to 0 \). Indeed,

\[
\max_{\xi} \left( \epsilon \log \tilde{u}(\xi) \right) = \frac{(\rho_1 - \rho_2)(w_{2L} - w_{2R})}{\rho_1 + \rho_2} + o(1) \quad \text{as} \quad \epsilon \to 0. \quad (4.18)
\]

Moreover, if we set \( u_\epsilon(x, t) = \tilde{u}(x/t) \), then \( u_\epsilon(x, t) \) is a solution of (4.6\( \epsilon \)) and

\[
\begin{align*}
    \beta_\epsilon &\to \beta_L + (\beta_R - \beta_L)H(x - st) \quad (4.19a) \\
    v_\epsilon &\to v_L + (v_R - v_L)H(x - st) + \frac{e_0}{\sqrt{1 + s^2}} t \delta_{x=st} \quad (4.19b)
\end{align*}
\]

in the sense of distributions as \( \epsilon \to 0 \).

The trajectories in (H3) are illustrated in Fig 4.1.
Figure 4.1: Phase portraits for \( \dot{u} = f(u) - su - w \) with fixed \( s \) and \( w \). The singular trajectories in (H3) are denoted by \( \gamma_1 \) and \( \gamma_2 \).

**Remark 4.1.** A similar result holds if \( v_L < 0 \) and \( v_R < 0 \). In that case, the assumption \( e_0 > 0 \) in (H2) is replaced by \( e_0 < 0 \), and \( +\infty \) in (H3) is replaced by \( -\infty \).

The notation \( t\delta_{\{x=st\}} \) in (4.19b) denotes, following [TZZ94, CL03], the functional on \( C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \) defined by

\[
\langle t\delta_{\{x=st\}}, \varphi \rangle = \int_0^\infty t \varphi(st, t) \sqrt{1 + s^2} \, dt. \tag{4.20}
\]

The weight \( \sqrt{1 + s^2} \) in the integral is to normalize the functional so that it is independent of parametrization of the line \( \{x = st\} \).

The estimate (4.18) confirms the asymptotic behavior conjectured in [KSS03]. In [KSZ04], some approximate solutions for the Dafermos regularization were constructed, but they were not exact solutions to (4.7). The results in the main theorem can also be compared to [Sch04] and [KT12], where Dafermos profiles were
constructed for a system motivated by gas dynamics. Those authors obtained families of unbounded solutions to (4.7ε), but they did not give descriptions of asymptotic behaviors the of solutions.

The assumption (H3) says that there exist solutions of (4.13) of the form
\[\gamma_1 = (\beta_1(\xi), v_1(\xi), w_{1L}, w_{2L}, s), \gamma_2 = (\beta_2(\xi), v_2(\xi), w_{1R}, w_{2R}, s)\] (4.21)
satisfying
\[\lim_{\xi \to -\infty} (\beta_1(\xi), v_1(\xi)) = (\beta_L, v_L), \quad \lim_{\xi \to \infty} (\beta_1(\xi), v_1(\xi)) = (\rho_1, +\infty)\] (4.22)
and
\[\lim_{\xi \to -\infty} (\beta_2(\xi), v_2(\xi)) = (\rho_2, +\infty), \quad \lim_{\xi \to \infty} (\beta_2(\xi), v_2(\xi)) = (\beta_R, v_R).\] (4.23)

A local analysis for (4.13) with \((w, \xi) = (w_{1L}, s)\) and \((w, \xi) = (w_{1R}, s)\), respectively, at \((\rho_1, +\infty)\) and \((\rho_2, +\infty)\) shows that the trajectories in (H3), if they exist, are unique.

A sample set of data for which (H1)-(H3) holds is, following [KSS03],
\[\rho_1 = 2, \rho_2 = 1, \quad u_L = (1.9, 1.0), \quad u_R = (1.1, 1.1/1.9).\] (4.24)
This will be verified in the next subsection.

### 4.3 Sufficient Conditions for (H1)-(H3)

The regions at which (H1) holds, or the over-compressive shock regions, can be described by the following

**Proposition 4.1.** In the Riemann problem (4.1), (4.5), let \(u_L = (\beta_L, v_L)\) and \(u_R = (\beta_R, v_R)\) be two points in \([\rho_1, \rho_2] \times (0, \infty)\). Then (H1) holds if and only if \(u_R\) lies in the interior of a cusped triangular region bounded by the curves
\[v = v_L \left( \frac{B_1(\beta_L) - 2B_2(\beta_L)(\beta_L - \beta)}{B_1(\beta)} \right), \quad \rho_2 < \beta_R < \beta_L,\] (4.25)
Figure 4.2: The over-compressive shock region for \( \rho_1 = 2, \rho_2 = 1, U_L = (1.1, 1.1/1.9) \).

and

\[
v = v_L \left( \frac{B_1(\beta_L)}{B_1(\beta) + 2B_2(\beta)(\beta_L - \beta)} \right), \quad \rho_2 < \beta_R < \beta_L.
\]

(4.26)

On the boundary segment (4.25), \( s = \text{Re}(\lambda_\pm(u_L)) \), and on (4.26), \( s = \text{Re}(\lambda_\pm(u_R)) \).

The curves defined by (4.25) and (4.26) and the region where over-compressive shock solution exist are illustrated in Fig 4.2.

**Proof.** This follows from a direct calculation. See [KSS03, Corollary 3.1].

The following proposition asserts that (H2) is implied by (H1).

**Proposition 4.2.** In the Riemann problem (4.1), (4.5), if the Riemann data lie in an over-compressive shock region in \([\rho_1, \rho_2] \times (0, \infty)\), then (H2) holds.

**Proof.** See [KSS03, Section 3.1].

The assumption (H3) is a condition on dynamics of 2-dimensional systems. Analyzing phase portraits we have the following
**Proposition 4.3.** Given Riemann data in an over-compressive shock region in \([\rho_1, \rho_2] \times (0, \infty)\), if \(\beta_R < \sqrt{\rho_1\rho_2} < \beta_L\), \(w_{1L} < 0, w_{2R} < 0 < w_{2L}\), and \(|s|\) is sufficiently small, then (H3) holds.

*Proof.* See Section 4.7. \(\square\)

Proposition 4.2 says that (H2) holds whenever (H1) holds, so the Main Theorem requires only (H1) and (H3). The author believes that (H3) is also a consequence of (H1). This needs further work to be verified.

For the sample set of data (4.24), we have

\[
(w_{1L}, w_{2L}) = (-.05, .22), \quad (w_{1R}, w_{2R}) = (-.05, -.11), \quad s = 0.
\]

(4.27)

Since \(w_{2R} < w_{2L}\), (H2) holds. From Proposition 4.1 and 4.3, (H1) and (H3) also hold. Hence the main theorem applies. Note that the conditions (H1)-(H3) persist under perturbation of the Riemann data \((u_L, u_R)\), so those assumptions still hold for any data close to (4.24).

### 4.4 Singular Configuration

The fast-slow system (4.10\(\epsilon\)) has multiple limiting subsystems corresponding to different time scales. In this section we will find trajectories, called singular trajectories, for those subsystems such that the union of those trajectories joins the end states \(u_L\) and \(u_R\). The union of those singular trajectories is called a singular configuration. In later sections we will show that there are solutions of (4.10\(\epsilon\)) close to the singular configuration.
4.4.1 End States $\mathcal{U}_L$ and $\mathcal{U}_R$

The system (4.13) has a normally hyperbolic critical manifold

$$S_0 = \{(u, w, \xi) : f(u) - \xi u - w = 0, \xi \neq \text{Re}(\lambda_{\pm}(u))\},$$  \hspace{1cm} (4.28)

where $\lambda_{\pm}(u)$ are the eigenvalues of $Df(u)$, defined in (4.14). The limiting slow system for (4.12) is

$$0 = f(u) - \xi u - w$$

$$w' = -u$$

$$\xi' = 1.$$  \hspace{1cm} (4.29)

From (H1) we have $s < \text{Re}(\lambda_{\pm}(u_L))$, so $(u_L, w_L, s) \in S_0$. Choose $\delta > 0$ so that $s + 2\delta < \text{Re}(\lambda_{\pm}(u_L))$, and set

$$\mathcal{U}_L = (u_L, w_L, s) \bullet (-\infty, \delta]$$

$$= \{(u, w, \xi) : u = u_L, w = w_L - \alpha_1 u_L, \xi = s + \alpha_1, \alpha_1 \in (-\infty, \delta]\},$$  \hspace{1cm} (4.30)

where $\bullet$ denotes the flow for (4.29). It is clear that $\mathcal{U}_L \subset S_0$ is normally hyperbolic with respect to (4.13), and is locally invariant with respect to (4.10).

Note that each point in $\mathcal{U}_L$ is a hyperbolic equilibrium for the 2-dimensional system (4.11), and the unstable manifold $W_{0u}(\mathcal{U}_L)$ is naturally defined.

**Proposition 4.4.** Assume (H1). Let $\mathcal{U}_L$ be defined by (4.30). Fix any $k \geq 1$. There exists a family of invariant manifolds $W_{\epsilon}^u(\mathcal{U}_L)$ which are $C^k O(\epsilon)$-close to $W_{0u}^u(\mathcal{U}_L)$ such that for any continuous family $\{\mathcal{I}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ of compact sets $\mathcal{I}_\epsilon \subset W_{\epsilon}^u(\mathcal{U}_L)$,

$$\text{dist}(p \bullet t, \mathcal{U}_L) \leq C e^{\mu t} \quad \forall p \in \mathcal{I}_\epsilon, \ t \leq 0, \ \epsilon \in [0, \epsilon_0],$$  \hspace{1cm} (4.31)

for some positive constants $C$ and $\mu$. 44
Proof. This follows from Theorem 3.3 by taking $U_L$ to be $U_0$. Although $U_L$ is not compact, it is uniformly normally hyperbolic since $\xi - \text{Re}(\lambda_{\pm}(u_L)) < -\delta$ on $U_L$, and the proof of Theorem 3.3 in [Jon95, Theorem 4] is still valid. \hfill \Box

**Remark 4.2.** Proposition 4.4 was also asserted in [Sch04, Liu04, KT12].

From (H1) we also have, by decreasing $\delta$ if necessary, $s - 2\delta > \text{Re}(\lambda_{\pm}(u_R))$, and hence a similar result holds for the set $U_R$ defined by

$U_R = (u_R, w_R, s) \cdot (-\delta, \infty)$

$$= \{(u, w, \xi) : u = u_R, w = w_R - \alpha_2 u_R, \xi = s + \alpha_2, \alpha_2 \in [-\delta, \infty)\}. \quad (4.32)$$

**Proposition 4.5.** Assume (H1). Let $U_R$ be defined by (4.32). Fix any $k \geq 1$. There exists a family of invariant manifolds $W^s_\epsilon(U_R)$ which are $C^k O(\epsilon)$-close to $W^s_0(U_R)$ such that for any continuous family $\{J_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ of compact sets $J_\epsilon \subset W^s_\epsilon(U_R)$,

$$\text{dist}(p \cdot t, U_R) \leq Ce^{-\mu t} \quad \forall \ p \in J_\epsilon, \ t \geq 0, \epsilon \in [0, \epsilon_0], \quad (4.33)$$

for some positive constants $C$ and $\mu$.

**4.4.2 Intermediate States $P_L$ and $P_R$**

Consider the system (4.12$\epsilon$). In order to study the dynamics at $\{v = +\infty\}$, we set $r = 1/v$ and $\kappa = \epsilon \log(1/r)$. Then (4.12$\epsilon$) is converted, after multiplying the equations by $r$, to

$$\dot{\beta} = B_1(\beta) - \xi \beta r - w_1 r$$

$$\dot{r} = -r B_2(\beta) + \xi r^2 + w_2 r^3$$

$$\dot{w}_1 = -\epsilon \beta r$$

$$\dot{w}_2 = -\epsilon$$

$$\dot{\xi} = \epsilon r$$

$$\dot{\kappa} = \epsilon (B_2(\beta) + \xi r + w_2 r^2). \quad (4.34\epsilon)$$

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Note that the time variable in (4.34) is different from that of (4.12). We use the same dot symbol to denote derivatives, but there should be no ambiguity since the different time scales can be distinguished by comparing the term $\dot{\xi}$.

The limiting fast system for (4.34) is

$$
\dot{\beta} = B_1(\beta) - \xi \beta r - w_1 r \\
\dot{r} = -r B_2(\beta) + \xi r^2 + w_2 r^3 \\
\dot{w}_1 = 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0, \quad \dot{\kappa} = 0.
$$

(4.35)

The obvious equilibria for (4.35), besides $(\beta_L, r_L, w_1L, w_2L, s)$ and $(\beta_R, r_R, w_1L, w_2L, s)$, where $r_L = 1/v_L$ and $r_R = 1/v_R$, are

$$
P_L = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_1, r = 0\}, \quad (4.36)$$

$$
P_R = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_2, r = 0\}. \quad (4.37)
$$

The limiting slow system on $P_L$ is

$$
w_1' = 0, \quad w_2' = -1, \quad \xi' = 0, \quad \kappa' = B_2(\rho_1),
$$

(4.38)

and on $P_R$ is

$$
w_1' = 0, \quad w_2' = -1, \quad \xi' = 0, \quad \kappa' = B_2(\rho_2)
$$

(4.39)

The Fenichel coordinates near $P_L$ can be described as follows.

**Proposition 4.6.** Let $W^u(s)(P_L)$ be the $C^k$ unstable/stable manifolds of $P_L$ for (4.34), $k \geq 1$. Then there exists a $C^k$ function $\hat{\beta} = \hat{\beta}(\beta, r, w_1, w_2, \xi, \kappa, \epsilon)$ such that

$$
\hat{\beta} = \beta \quad \text{when } r = 0
$$

(4.40)

and $(\hat{\beta}, r, w_1, w_2, \xi, \kappa)$ is a change of coordinates near $P_L$ satisfying

$$
W^u(s)(P_L) = \{ (\hat{\beta}, r, w_1, w_2, \xi, \kappa) : \hat{\beta} = \rho_1 \}
$$

(4.41)

$$
W^u(s)(P_L) = \{ (\hat{\beta}, r, w_1, w_2, \xi, \kappa) : r = 0 \}
$$

(4.42)
Moreover, the projection $\pi_s^{e, P_L}$ into $P_L$ along stable fibers with respect to (4.34) is

$$\pi_s^{e, P_L}(\rho_1, r, w_1, w_2, \xi, \kappa) = (\rho_1, 0, w_1, w_2, \xi, \kappa) \quad (4.43)$$
in $(\hat{\beta}, r, w_1, w_2, \xi, \kappa)$-coordinates.

**Proof.** The linearization of (4.35) at $P_L$ corresponds to the matrix

$$
\begin{pmatrix}
B'_1(\rho_1) & -\xi \rho_1 \\
0 & -B_2(\rho_1)
\end{pmatrix} = 
\begin{pmatrix}
1 - \frac{\rho_2}{\rho_1} & -\xi \rho_1 \\
0 & 1 - \frac{\rho_2}{\rho_1}
\end{pmatrix}
\quad (4.44)
$$
which has one positive and one negative eigenvalue. Note that $P_L$ is invariant under (4.34$\epsilon$) for each $\epsilon$. From Theorem 3.1 and the remark following it, $W_s^e(P_L)$ and $W_u^u(P_L)$ are well defined and both have dimension 1, and we may take $W_u^u(P_L) = \{r = 0\}$. Note that $\{\beta = \rho_1\}$ is transverse to $W_s^e(P_L)$, so we can choose Fenichel coordinates $(a, b, c)$ corresponding to this splitting with $b = r$ and

$$a = \beta - \rho_1 + \phi(w_1, w_2, \xi, \kappa, \epsilon, r)r$$
for some $C^k$ function $\phi$. Let $\hat{\beta} = a + \rho_1$. Then the desired result follows. \qed

An analogous result holds for $P_R$. We omit it here.

### 4.4.3 Transversal Intersections

Fix small $r^0 > 0$ such that $\gamma_1$ intersects $\{r = r^0\}$ at a unique point. Denote this point by $p_0^{\text{in}}$. That is,

$$p_0^{\text{in}} = \gamma_1 \cap \{r = r^0\}. \quad (4.45)$$

We set

$$\mathcal{I}_\epsilon = W_u^u(U_L) \cap \{r = r^0\} \cap V_1, \quad (4.46)$$
where $V_1$ is an open neighborhood of $p_0^{\text{in}}$ such that $\mathcal{I}_\epsilon$ can be parametrized as

$$\mathcal{I}_\epsilon = \{ (\hat{\beta}, r, w_1, w_2, \xi, \kappa) : r = r^0, \kappa = \epsilon \log(1/r^0),$$

$$(w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-\beta_L, -v_L, 1) + \epsilon \theta(a, \alpha_1, \epsilon),$$

$$|a| < \Delta_1, |\alpha_1| < \Delta_1 \} \quad \text{(4.47)}$$

where the coordinates $(\hat{\beta}, r, w_1, w_2, \xi, \kappa)$ are defined in Proposition 4.6. From (4.41) we see that $\mathcal{I}_0$ and $W^s_0(\mathcal{P}_L)$ intersect transversally at $p_0^{\text{in}}$, and if we set

$$\Lambda_L = \pi_{0, P_L}^s (\mathcal{I}_0 \cap W^s_0(\mathcal{P}_L)), \quad \text{(4.48)}$$

where $\pi_{0, P_L}^s$ is the projection into $\mathcal{P}_L$ along stable fibers with respect to (4.35), then

$$\Lambda_L = \{ (\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_1, r = 0, \kappa = 0,$$

$$(w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-\beta_L, -v_L, 1),$$

$$|\alpha_1| < \Delta_1 \} \quad \text{(4.49)}$$

Similarly, by shrinking $r^0$ if necessary, $\gamma_2$ intersects $\{ r = r^0 \}$ at a unique point

$$p_0^{\text{out}} = \gamma_2 \cap \{ r = r^0 \}. \quad \text{(4.50)}$$

Set

$$\mathcal{J}_\epsilon = W^s_\epsilon (\mathcal{U}_R) \cap \{ r = r^0 \} \cap V_2, \quad \text{(4.51)}$$

where $V_2$ is an open neighborhood of $p_0^{\text{out}}$ such that $\mathcal{J}_\epsilon$ has a parametrization analogous to (4.47). Then $\mathcal{J}_0$ is transverse to $W^u_0(\mathcal{P}_R)$ at $p_0^{\text{out}}$, and we set

$$\Lambda_R = \pi_{0, P_R}^u (\mathcal{J}_0 \cap W^u_0(\mathcal{P}_R)), \quad \text{(4.52)}$$

where $\pi_{0, P_R}^u$ is the projection into $\mathcal{P}_R$ along unstable fibers with respect to (4.35).

To connect $p_0^{\text{in}}$ and $p^{\text{out}}$, we have the following
Proposition 4.7. The system (4.35) has a trajectory

\[ \gamma_0 = \{(\beta, 0, w_{1L}, w_{20}, s, \kappa_0) : \beta \in (\rho_2, \rho_1)\}, \tag{4.53} \]

which joins the points

\[ \pi^s_{P_R}(p^{\text{out}}_0) \bullet \tau_{10} \in \mathcal{P}_L \quad \text{and} \quad \pi^u_{P_L}(p^{\text{in}}_0) \bullet (-\tau_{20}) \in \mathcal{P}_R, \]

where

\[ w_{20} = w_{2L} + \frac{\rho_1}{\rho_1 + \rho_2}(w_{2L} - w_{2R}), \quad \kappa_0 = \frac{\rho_1(\rho_1 - \rho_2)}{2\rho_2(\rho_1 + \rho_2)}, \tag{4.54} \]

and

\[ \tau_{10} = \frac{\rho_2}{\rho_1 + \rho_2}(w_{2L} - w_{2R}), \quad \tau_{20} = \frac{\rho_1}{\rho_1 + \rho_2}(w_{2L} - w_{2R}). \tag{4.55} \]

Moreover, if we set

\[ \tilde{\Lambda}_L = \Lambda_L \bullet \tau_{10} \quad \text{and} \quad \tilde{\Lambda}_R = \Lambda_R \bullet (-\tau_{20}), \tag{4.56} \]

where \( \tau_{1-} < \tau_{10} < \tau_{1+} \) and \( \tau_{2-} < \tau_{20} < \tau_{2+} \), then \( W^u_0(\tilde{\Lambda}_L) \) and \( W^s_0(\tilde{\Lambda}_R) \) intersect transversally along \( \gamma_0 \) in the space \( \{r = 0\} \).

Proof. Note that the restriction of the system (4.35) on \( \{r = 0\} \) is simply \( \dot{\beta} = B_1(\beta) \), so every trajectory of (4.35) joins \( \{\beta = \rho_1\} \) and \( \{\beta = \rho_2\} \). Also note that

\[ \pi^s_{P_R}(p^{\text{out}}_0) \bullet \tau = (\rho_1, 0, w_{1L}, w_{2L}, s, 0) + \tau(0, 0, -1, 0, B_2(\rho_1)) \]

\[ \pi^u_{P_L}(p^{\text{in}}_0) \bullet \tau = (\rho_2, 0, w_{1R}, w_{2R}, s, 0) + \tau(0, 0, 0, -1, 0, B_2(\rho_2)), \quad \forall \tau \in \mathbb{R}, \]

in \( (\beta, r, w_1, w_2, \xi, \kappa) \)-coordinates. Hence \( \gamma_0 \) defined in (4.53) joins \( \pi^s_{P_R}(p^{\text{out}}_0) \bullet \tau_{10} \)

and \( \pi^u_{P_L}(p^{\text{in}}_0) \bullet (-\tau_{20}) \) if

\[ w_{20} = w_{2L} - \tau_{10} = w_{2R} + \tau_{20}, \quad \kappa_0 = B_2(\rho_1)\tau_{10} = -B_2(\rho_2)\tau_{20}, \tag{4.57} \]

which gives (4.54) and (4.55).
Let \( \tilde{\Lambda}_L \) and \( \tilde{\Lambda}_R \) be defined in (4.56). From the parameterizations (4.42) and (4.49), we have

\[
W^u_0(\tilde{\Lambda}_L) = \{ (\beta, r, w_1, w_2, \xi, \kappa) : r = 0, (w_1, w_2, \xi, \kappa) = (w_{1L}, w_{2L}, s, 0) + \alpha_1(-\beta, -v, 1, 0) + \tau_1(0, -1, 0, B_2(\rho_1)), \beta \in (\rho_2, \rho_1), |\alpha_1| < \Delta_1, \tau_1 \in [\tau_{1-}, \tau_{1+}] \}
\]

and

\[
W^s_0(\tilde{\Lambda}_R) = \{ (\beta, r, w_1, w_2, \xi, \kappa) : r = 0, (w_1, w_2, \xi, \kappa) = (w_{1R}, w_{2R}, s, 0) + \alpha_2(-\beta, -v, 1, 0) - \tau_2(0, -1, 0, B_2(\rho_2)), \beta \in (\rho_2, \rho_1), |\alpha_2| < \Delta_1, \tau_2 \in [\tau_{2-}, \tau_{2+}] \}
\]

Fix any \( q_0 \in \gamma_0 \), we have

\[
T_{q_0}W^u_0(\tilde{\Lambda}_L) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\beta \\ -v \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ B_2(\rho_1) \end{pmatrix} \right\}
\]

and

\[
T_{q_0}W^s_0(\tilde{\Lambda}_R) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\beta \\ -v \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ B_2(\rho_2) \end{pmatrix} \right\}
\]

Hence \( T_{q_0}W^u_0(\tilde{\Lambda}_L) \) and \( T_{q_0}W^s_0(\tilde{\Lambda}_R) \) span the space \( \{ r = 0 \} \). This means \( W^u_0(\tilde{\Lambda}_L) \) and \( W^s_0(\tilde{\Lambda}_R) \) intersect transversally in the space \( \{ r = 0 \} \) at \( q_0 \). \( \square \)

Let \( \gamma_0 \) be the trajectory defined in Proposition 4.7. We set

\[
q_0 = \gamma_0 \cap \Gamma,
\]

where

\[
\Gamma = \{ (\beta, r, w_1, w_2, \xi, \kappa) : \beta = \frac{\rho_1 + \rho_2}{2} \}.
\]
Figure 4.3: The singular configuration $\gamma_1 \cup \sigma_1 \cup \gamma_0 \cup \sigma_2 \cup \gamma_2$ connects $U_L$ and $U_R$.

Then

$$\pi_{PL}^u(q_0) = \pi_{PL}^s(p_0^{in}) \cdot \tau_{10}, \quad \pi_{PR}^u(q_0) = \pi_{PR}^s(p_0^{out}) \cdot (-\tau_{20}).$$

Let

$$\sigma_1 = \pi_{PL}^s(p_0^{in}) \cdot [0, \tau_{10}]$$

$$= \{(\rho_2, 0, w_{1L}, w_{2L} - \tau, s, B_2(\rho_1)\tau) : \tau \in [0, \tau_{10}]\}$$

and

$$\sigma_2 = \pi_{PL}^u(p_0^{in}) \cdot [-\tau_{20}, 0]$$

$$= \{(\rho_1, 0, w_{1R}, w_{2R} + \tau, s, -B_2(\rho_2)\tau) : \tau \in [0, \tau_{20}]\}$$

in $(\beta, r, w_1, w_2, \xi, \kappa)$-coordinates. Then we obtain the singular configuration

$$\gamma_1 \cup \sigma_1 \cup \gamma_0 \cup \sigma_2 \cup \gamma_2$$

connecting $U_L$ and $U_R$. See Fig 4.3.
4.5 Completing the Proof of the Main Theorem

We split the proof of the main theorem into two parts. In the first subsection we prove the existence of solutions of the boundary value problem (4.7ε), (4.8), and show that (4.18) holds. In the second subsection we derive the weak limit (4.19).

4.5.1 Existence of Trajectories

**Proposition 4.8.** Assume (H1)-(H3). Let \( p^\text{in}_0, p^\text{out}_0, q_0, \mathcal{I}_\epsilon, \mathcal{J}_\epsilon \) and \( \Sigma \) be defined in Section 4.4.3. Then for each small \( \epsilon > 0 \), there exist \( p^\text{in}_\epsilon \in \mathcal{I}_\epsilon, p^\text{out}_\epsilon \in \mathcal{J}_\epsilon, q_\epsilon \in \Gamma \) and \( T_{1\epsilon}, T_{2\epsilon} > 0 \) such that

\[
p^\text{in}_\epsilon = q_\epsilon \cdot (-T_{1\epsilon}), \quad p^\text{out}_\epsilon = q_\epsilon \cdot T_{2\epsilon},
\]

where \( \cdot \) denotes the flow of (4.34ε), satisfying

\[
(p^\text{in}_\epsilon, p^\text{out}_\epsilon, q_\epsilon) = (p^\text{in}_0, p^\text{out}_0, q_0) + o(1)
\]

and

\[
T_{1\epsilon} = \left( \tau_{10} + o(1) \right) \epsilon^{-1}, \quad T_{2\epsilon} = \left( \tau_{20} + o(1) \right) \epsilon^{-1},
\]

as \( \epsilon \to 0 \), where \( \tau_{10} \) and \( \tau_{20} \) are defined in (4.55). Moreover, if we set \( \kappa_\epsilon(\sigma) \) to be the \( \kappa \)-coordinate of \( q_\epsilon \cdot \sigma, \sigma \in [-T_{1\epsilon}, T_{2\epsilon}] \), then

\[
\max_{\sigma \in [-T_{1\epsilon}, T_{2\epsilon}]} \kappa_\epsilon(\sigma) = \kappa_0 + o(1),
\]

where \( \kappa_0 \) is defined in (4.54).

**Proof.** We will apply the Exchange Lemma (Theorem 3.5) with \( (k, m, l, \sigma) = (1, 1, 3, 1) \). From (4.47) and (4.41), we know that the \( (k + \sigma) \)-manifold \( \mathcal{I}_0 \) is transverse to the \( (m + l) \)-manifold \( W^s_0(\mathcal{P}_L) \) at \( p^\text{in}_0 \), and the image of the projection

\[
\pi^s_{\mathcal{P}_L}(\mathcal{I}_0 \cap W^s_0(\mathcal{P}_L)) = \Lambda_L
\]
is $\sigma$-dimensional, so (T1) in the Exchange Lemma holds. The limiting slow system on $\mathcal{P}_L$ is governed by (4.38), and by the parametrization (4.49) of $\Lambda_L$ it follows that (T2) holds. Also it is clear that (T3) holds with $\tau_0 = \tau_{10}$, where $\tau_{10}$ is defined in (4.55). Theorem 3.5 implies that there exists a neighborhood $V_0$ of $q_0$ such that

$$I^*_\epsilon \cap V_0 \text{ is } C^1 O(\epsilon)\text{-close to } W^u_0(\widetilde{\Lambda}_L) \cap V_0, \quad (4.70)$$

where $I^*_\epsilon = I_\epsilon \cdot [0, \infty)$. Similarly,

$$J^*_\epsilon \cap V_0 \text{ is } C^1 O(\epsilon)\text{-close to } W^s_0(\widetilde{\Lambda}_R) \cap V_0, \quad (4.71)$$

where $J^*_\epsilon = J_\epsilon \cdot (-\infty, 0]$. From Proposition 4.7, it follows that the projections of $I^*_\epsilon$ and $J^*_\epsilon$ in the 5-dimensional space $\{r = 0\}$ intersect transversally at a unique point in $\Gamma$ near $q_0$. For the relation $r = \exp(-\kappa/\epsilon)$, we then recover a unique intersection point

$$q_\epsilon \in I^*_\epsilon \cap J^*_\epsilon \cap \Gamma$$

in $(\beta, r, w_1, w_2, \xi, \kappa)$-space. By construction we have (4.66) and (4.67). The estimates (4.68) follows from (3.27). Note that

$$\max_{\sigma_1 \cup \gamma_0 \cup \sigma_2} \kappa = \kappa_0,$$

where $\sigma_1$, $\sigma_2$ and $\gamma_0$ are defined in Section 4.4.3, so we obtain (4.69). $\square$

**Proposition 4.9.** Assume (H1)-(H3) hold. Let $q_\epsilon = (\beta_\epsilon^0, r_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0, \kappa_\epsilon^0) \in \Gamma$ be defined in Proposition 4.8. Let $v_\epsilon^0 = \exp(\kappa_\epsilon^0/\epsilon)$ and

$$((\tilde{\beta}_\epsilon, \tilde{\nu}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon})(\xi) = (\beta_\epsilon^0, v_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0)_{(4.9\epsilon)} (\xi - \xi_\epsilon^0), \quad (4.72)$$

or equivalently,

$$((\tilde{\beta}_\epsilon, \tilde{\nu}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon}, \xi) = (\beta_\epsilon^0, v_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0)_{(4.12\epsilon)} \left(\frac{\xi - \xi_\epsilon^0}{\epsilon}\right). \quad (4.73)$$
Then \((\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)\) is a solution of (4.7\(\epsilon\)) and (4.8), and it satisfies (4.18).

**Proof.** Since (4.7\(\epsilon\)) and (4.9\(\epsilon\)) are equivalent, and \((\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_1\epsilon, \tilde{w}_2\epsilon)(\xi)\) is a solution of (4.9\(\epsilon\)), we know \((\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)\) is a solution of (4.7\(\epsilon\)).

Let \(T_{1\epsilon}, T_{2\epsilon} \in \mathbb{R}\) be defined in Proposition 4.8. Then

\[
q_{\epsilon} \bullet \left(-T_{1\epsilon}/\epsilon\right) \in \mathcal{I}_\epsilon, \quad q_{\epsilon} \bullet \left(T_{2\epsilon}/\epsilon\right) \in \mathcal{J}_\epsilon.
\]

Since \(\mathcal{I}_\epsilon \subset W^u_\epsilon(U_L)\) and \(\mathcal{J}_\epsilon \subset W^s_\epsilon(U_R)\), from (4.31) and (4.33) we have

\[
\lim_{t \to -\infty} \text{dist}(q_{\epsilon} \bullet t, U_L) = 0, \quad \lim_{t \to \infty} \text{dist}(q_{\epsilon} \bullet t, U_R) = 0,
\]

which implies (4.8). Since \(\kappa_\epsilon = \epsilon \log(v_\epsilon)\), from (4.69) we obtain (4.18). \(\square\)

### 4.5.2 Convergence of Trajectories

Based on the results in Proposition 4.8, we first derive some estimates for the self-similar solution \(\tilde{u}_\epsilon(\xi)\).

**Proposition 4.10.** Let \(\tilde{u}_\epsilon = (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)\) be the solution of (4.7\(\epsilon\)) and (4.8) in Proposition 4.9. Let \(p^{\text{in}}_\epsilon\) and \(p^{\text{out}}_\epsilon\) be defined in Proposition 4.8. Then

\[
|\xi^{\text{in}}_\epsilon - s| + |\xi^{\text{out}}_\epsilon - s| = o(1) \quad (4.74)
\]

\[
\int_{-\infty}^{\xi^{\text{in}}_\epsilon} |\tilde{u}(\xi) - u_L| \, d\xi + \int_{\xi^{\text{out}}_\epsilon}^{\infty} |\tilde{u}(\xi) - u_R| \, d\xi = o(1) \quad (4.75)
\]

\[
\int_{\xi^{\text{in}}_\epsilon}^{\xi^{\text{out}}_\epsilon} \tilde{u}(\xi) \, d\xi = (0, e_0) + o(1) \quad (4.76)
\]

as \(\epsilon \to 0\), where \(\xi^{\text{in}}_\epsilon\) and \(\xi^{\text{out}}_\epsilon\) are \(\xi\)-coordinates of \(p^{\text{in}}_\epsilon\) and \(p^{\text{out}}_\epsilon\), respectively.

**Proof.** Note that \(s\) is the \(\xi\)-coordinate of \(p^0_\epsilon\), so

\[
|\xi^{\text{in}}_\epsilon - s| \leq |p^{\text{in}}_\epsilon - p^0_\epsilon|,
\]

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which tends to zero by (4.67). Similarly, \(|\xi^\text{out} - s| \to 0\). This gives (4.74).

Since every point in \(U_L\) has \(u\)-coordinate equal to \(u_L\),

\[
|\tilde{u}(\xi) - u_L| \leq \operatorname{dist}((\tilde{u}(\xi), \tilde{w}(\xi), \xi), U_L) = \operatorname{dist}((u^0_\epsilon, w^0_\epsilon, \xi^0_\epsilon, U_L) \times \frac{\xi - \xi^0_\epsilon}{\epsilon}, U_L),
\]

where the last equality follows from (4.73). Using (4.31), the last term is \(\leq C \exp\left(\frac{\nu(\xi - \xi^0_\epsilon)}{\epsilon}\right)\).

Since \(\xi^\text{in} < \xi^0_\epsilon\), it follows that

\[
\int_{-\infty}^{\xi^\text{in}} |\tilde{u}(\xi) - u_L| \, d\xi \leq \int_{-\infty}^{\xi^\text{in}} C \exp\left(\frac{\nu(\xi - \xi^0_\epsilon)}{\epsilon}\right) \, d\xi \leq \int_{-\infty}^{\xi^0_\epsilon} C \exp\left(\frac{\nu(\xi - \xi^0_\epsilon)}{\epsilon}\right) \, d\xi = \frac{\epsilon}{\nu} C,
\]

A similar inequality holds for \(\int_{\xi^\text{out}}^{\infty} |\tilde{u}(\xi) - u_R| \, d\xi\), so we obtain (4.75).

Since \(\tilde{\beta}_\epsilon(\xi)\) is uniformly bounded in \(\epsilon\), its integral between \(\xi^\text{in}_\epsilon\) and \(\xi^\text{out}_\epsilon\) is \(o(1)\) by (4.74), and this proves the first part of (4.76). From the equation of \(\dot{\xi}\) in (4.34\(\epsilon\)), denoting the time variable by \(\zeta\), we can write \(\xi = \xi(\zeta)\) by

\[
\xi(0) = \xi^0_\epsilon, \quad \frac{d\xi}{d\zeta} = \epsilon\tilde{r}_\epsilon(\xi), \quad (4.77)
\]

where \(\tilde{r}_\epsilon(\xi) = 1/\tilde{v}_\epsilon(\xi)\). From (4.66) we have

\[
\xi(-T_1\epsilon) = \xi^\text{in}_\epsilon, \quad \xi(T_2\epsilon) = \xi^\text{out}_\epsilon. \quad (4.78)
\]

From (4.77) and (4.78) it follows that

\[
\int_{\xi^\text{in}}^{\xi^\text{out}} \tilde{v}(\xi) \, d\xi = \int_{\xi^\text{in}}^{\xi^\text{out}} \frac{1}{\tilde{r}(\xi)} \, d\xi = \int_{-T_1\epsilon}^{T_2\epsilon} \epsilon \, d\zeta = \epsilon(T_1\epsilon + T_2\epsilon),
\]

which converges to \(w_{2L} - w_{2R} = e_0\) by (4.68). This proves (4.76).

From the estimates in Proposition 4.10, we can derive the weak convergence of \(\tilde{u}(\xi)\) as follows.
Proposition 4.11. Let \( \tilde{u}_\epsilon = (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon) \) be the solution of (4.7\( \epsilon \)) and (4.8) given in Proposition 4.9. Then

\[
\tilde{u}_\epsilon \rightharpoonup u_L + (u_R - u_L)H(\xi - s) + (0, e_0)\delta_0(\xi - s)
\]

(4.79)
in the sense of distributions as \( \epsilon \to 0 \).

Proof. Let \( \psi \in C_\infty^c(\mathbb{R}) \) be a smooth function with compact support. From (4.75) we have

\[
\left| \int_{-\infty}^{\xi_{in}} \psi(\xi) (\tilde{u}(\xi) - u_L) \, d\xi \right| \leq \|\psi\|_{L^\infty} \int_{-\infty}^{\xi_{in}} |\tilde{u}(\xi) - u_L| \, d\xi \leq \|\psi\|_{L^\infty} C \epsilon,
\]

which implies

\[
\int_{-\infty}^{\xi_{in}} \psi(\xi) \tilde{u}(\xi) \, d\xi = \left( \int_{-\infty}^{\xi_{in}} \psi(\xi) \, d\xi \right) u_L + o(1) = \left( \int_{-\infty}^{s} \psi(\xi) \, d\xi \right) u_L + o(1).
\]

A similar inequality holds for \( \int_{\xi_{out}}^{\infty} \psi u \, d\xi \), so

\[
\int_{\mathbb{R} \setminus [\xi_{in}, \xi_{out}]} \psi(\xi) \tilde{u}(\xi) \, d\xi = \left( \int_{-\infty}^{s} \psi(\xi) \, d\xi \right) u_L + \left( \int_{s}^{\infty} \psi(\xi) \, d\xi \right) u_R + o(1).
\]

(4.80)

On the other hand, from (4.74) and (4.76) we have

\[
\int_{\xi_{in}}^{\xi_{out}} |(\psi(\xi) - \psi(s))\tilde{u}_\epsilon(\xi)| \, d\xi \leq \left( \max_{\xi \in [\xi_{in}, \xi_{out}]} |\psi(\xi) - \psi(s)| \right) \left( \int_{\xi_{in}}^{\xi_{out}} \tilde{u}_\epsilon(\xi) \, d\xi \right)
\]

\[
= o(1) \left( (0, e_0) + o(1) \right) = o(1)
\]

which implies

\[
\int_{\xi_{in}}^{\xi_{out}} \psi(\xi) \tilde{u}_\epsilon(\xi) \, d\xi = \psi(s) \int_{\xi_{in}}^{\xi_{out}} \tilde{u}_\epsilon(\xi) \, d\xi + o(1) = \psi(s)(0, e_0) + o(1).
\]

(4.81)

Combining (4.80) and (4.81) we obtain

\[
\int_{-\infty}^{\infty} \psi(\xi) \tilde{u}_\epsilon(\xi) \, d\xi = u_L \int_{-\infty}^{s} \psi(\xi) \, d\xi + u_R \int_{s}^{\infty} \psi(\xi) \, d\xi + (0, e_0)\psi(s) + o(1).
\]

(4.82)

This holds for all \( \psi \), so we have (4.79). \( \square \)
Converting the results of Proposition 4.11 from self-similar variables to physical space variables, we obtain the following

**Proposition 4.12.** Let \( \tilde{u}_\epsilon = (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon) \) be the solution of (4.7\(\epsilon \)) and (4.8) given in Proposition 4.9. Let \( u_\epsilon(x,t) = \tilde{u}_\epsilon(x/t) \). Then the weak convergence (4.19a) and (4.19b) holds.

**Proof.** Let \( \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}_+) \). From (4.79) we have

\[
\int_0^\infty \int_{-\infty}^\infty \varphi(x,t)u_\epsilon(x,t) \, dx \, dt = \int_0^\infty \int_{-\infty}^\infty \varphi(t\xi, t)\tilde{u}_\epsilon(\xi) \, d\xi \, dt
\]

\[
= \int_0^\infty t \left\{ u_L \int_{-\infty}^s \varphi(t\xi, t) \, d\xi + (0, e_0, 0)\varphi(st, t) + u_R \int_s^\infty \varphi(t\xi, t) \, d\xi \right\} \, dt + o(1)
\]

\[
= u_L \int_0^\infty \int_{-\infty}^st \varphi(x, t) \, dx \, dt + u_R \int_0^\infty \int_{st}^\infty \varphi(x, t) \, dx \, dt + (0, e_0) \int_0^\infty t \varphi(st, t) \, dt + o(1).
\]

From (4.20), this means (4.19) holds.

Now Propositions 4.9 and 4.12 complete the proof of the Main Theorem.

### 4.6 Numerical Simulations

Some numerical solutions for (4.1) using the Lax-Friedrichs scheme are shown in Figure 4.4. The solutions appear to grow unboundedly as the number of steps increases.

Also some numerical approximations for (4.6\(\epsilon \)) are shown in Fig 4.5. The algorithm was a shooting method following the descriptions in [KSS03]. Since \( w_1 \) and \( \xi \) are essentially constant near the shock, we project the trajectories in the \((\beta, r, w_2)\) space. Note that \( w_2(\xi) \) does not converge as \( \xi \to \pm\infty \) while \( x_2 = w_2 + (\xi - s)v \) converges, we replace \( w_2 \) by \( x_2 \) (again, following [KSS03]). Note that \( x_2 \) is a mild modification of \( w_2 \) near the shock since within the \( \epsilon \)-neighborhood of \( \xi = s \) the difference between \( x_2 \) and \( w_2 \) is of order \( o(1) \).
Figure 4.4: Lax-Friedrichs Scheme up to 20,000 steps with CFL= 0.05

As $\epsilon$ decreases, the minimal value of $r$-coordinate on the trajectories in Fig 4.5 tends to zero. This means the maximum of $v$ tends to infinity. Also observe that the change of the value of $x_2$ concentrates in the vicinity of $\beta = \rho_1$ and $\beta = \rho_2$. This is consistent with our proof for the main theorem.

4.7 Proof of Proposition 4.3

If the section we prove the sufficiency of the conditions in Proposition 4.3 for (H3), which says that there is a trajectory for (4.13) connecting $(u_L, w_L, s)$ and $\{\beta = \rho_1, v = +\infty\}$, and a trajectory connecting $(u_R, w_R, s)$ and $\{\beta = \rho_2, v = +\infty\}$. We will focus on finding the first trajectory while finding the second one is similar.

We will switch back and forth between $(\beta, v)$- and $(\beta, r)$-coordinates, where $r = 1/v$. The system (4.13) is converted to (4.35) in $(\beta, r)$-coordinates. It suffices to find trajectories connecting $u_L = (\beta_L, r_L)$ and $p_L \equiv (\rho_1, 0)$ for (4.35) with $(w, \xi) = (w_L, s)$. From (H1) we know that $u_L$ is a source, and we will also see that $p_L$ is a saddle. Our
strategy is to construct a negatively invariant region in which every trajectory goes backward to \( u_L \), and one of those trajectories goes forward to \( p_L \). See Fig 4.6.

To construct a such region, we first study the flow on the boundary of the feasible region \( \{ \rho_2 \leq \beta \leq \rho_1 \} \). The equation of \( \dot{\beta} \) in (4.13) with \((w, \xi) = (w_L, s)\) is

\[
\dot{\beta} = -s \rho_1 - w_{1L} \quad \text{on} \quad \{ \beta = \rho_1 \}.
\]

Hence the proposition below implies that the region \( \{ \beta \leq \rho_1 \} \) is negatively invariant. Similarly, for (4.13) with \((w, \xi) = (w_R, s)\), the region \( \{ \beta \geq \rho_2 \} \) is positively invariant.

**Lemma 4.13.** *If (H1) holds, then*

\[
s \rho_1 + w_{1L} < 0 \quad \text{and} \quad s \rho_2 + w_{1R} < 0,
\]

*where \( s, w_{1L} \) and \( w_{1R} \) are as defined in (4.15) and (4.16).*

Figure 4.5: Trajectories for \( \epsilon u'' = (Df(u) - s) u' \) as \( \epsilon \) decreases from 1 to 0.01. The variable \( x_2 \) is a modification of \( w_2 \).
Figure 4.6: Phase portraits for (4.84) in (β, v) space and (4.87) in (β, r) space. The shaded region V is a backward invariant region in which every backward trajectory tends to uL, and γ1 is the unique trajectory in V which tends to pL = (ρ1, 0).

Proof. By definition of s and w1L we have
\[ sρ_1 + w_{1L} = sρ_1 + v_L B_1(β_L) - sβ_L \]
\[ = \frac{v_L B_1(β_L) - v_R B_1(β_R)}{β_L - β_R} (ρ_1 - β_L) + v_L B_1(β_L). \]

From Proposition 4.1, we know (H1) implies βR < βL and
\[ v_R \leq v_L \left( \frac{B_1(β_L) - 2B_2(β_L)(β_L - β_R)}{B_1(β_R)} \right). \]

Since B1(βL) < 0, it follows that
\[ sρ_1 + w_{1L} \leq \frac{ρ_1 - β_L}{β_L - β_R} v_L \left( 2B_2(β_L)(β_L - β_R) \right) + v_L B_1(β_L) \]
\[ = \left( \frac{(ρ_1 - β_L)(β_L^2 - ρ_1 ρ_2)}{β_L^2} + \frac{(ρ_1 - β_L)(ρ_2 - β_L)}{β_L} \right) v_L \]
\[ = -ρ_2(ρ_1 - β_L)^2 v_L < 0. \]

Similarly, using \( sρ_2 + w_{1R} = sρ_2 + v_R B_1(β_R) - sβ_R, \) one obtains \( sρ_2 + w_{1R} < 0. \)

Proposition 4.3. Suppose (H1) holds. If \( β_R < \sqrt{ρ_1 ρ_2} < β_L, \) \( w_{10} < 0, \) \( w_{2R} < 0 < w_{2L}, \)
and \(|s|\) is sufficiently small, then (H3) holds.
Proof. We focus on $u_L$ while the proof for $u_R$ is similar. As mentioned at the beginning of this section and Fig 4.6, we first construct a negatively invariant region in which every trajectory goes backward to $u_L$, and then show that one of those trajectories goes forward to $p_L$.

Consider (4.13) with $(w, \xi) = (w_L, s)$. That is,

$$\dot{\beta} = v B_1(\beta) - s \beta - w_{1L}$$
$$\dot{v} = v^2 B_2(\beta) - sv - w_{2L}.$$  \hfill (4.84)

The null-clines for this system are

$$\dot{\beta} = 0 : \quad v = \vartheta_1(\beta) := \frac{\beta(s \beta + w_{1L})}{(\beta - \rho_1)(\beta - \rho_2)} \quad (4.85)$$
$$\dot{v} = 0 : \quad v = \vartheta_2(\beta) := \frac{s \beta + \sqrt{s^2 \beta^2 + 2w_{2L}(\beta^2 - \rho_1 \rho_2)}}{\beta^2 - \rho_1 \rho_2} \beta. \quad (4.86)$$

When $|s|$ is small, it can be readily seen that $\vartheta_1$ is increasing and $\vartheta_2$ is decreasing on the interval $(\sqrt{\rho_1 \rho_2}, \rho_1)$. Also we have $\vartheta_2(\rho_1) > 0$ since $w_{2L} > 0$.

Let $\sigma(\tau)$ be the solution to (4.84) with initial condition $\sigma(0) = (\rho_1, v_L)$. By the monotonicity of $\vartheta_1$ and $\vartheta_2$, we know that $\sigma$ hits the half-line $l_1 = \{ (\beta_L, v) : v \geq v_L \}$ at some time $\tau_- < 0$. Let $l_2 = \{ (\rho_1, v) : v \geq v_L \}$ and $V$ be the region enclosed by the curves

$$l_1 \cup \{ \sigma(\tau) : \tau_- \leq \tau \leq 0 \} \cup l_2.$$

Then $V$ forms a backward invariant region. See Fig 4.6.

We claim that $u_L$ attracts every point in $V$ in backward time. Note that

$$\frac{\partial}{\partial \beta}(v B_1(\beta) - s \beta - w_{10}) + \frac{\partial}{\partial v}(v^2 B_2(\beta) - sv - w_{2L})$$
$$= v B'_1(\beta) - s + 2v B_2(\beta) - s$$
$$= 4v B_2(\beta) - 2s,$$
which is positive when \( \beta \in (\sqrt{\rho_1\rho_2},\rho_1) \) and \( s \) is small. In the last equality we used \( B'_1(\beta) = 2B_2(\beta) \). By Bendixson’s negative criterion, the system has no periodic orbit inside \( V \). Since \( V \) is backward invariant and \( u_L \) is the only equilibrium on the closure of \( V \), it follows from the Poincaré-Bendixson Theorem that every trajectory in \( V \) tends to \( u_L \) in backward time.

It remains to show that there is a trajectory in \( V \) tending to \( \{\beta = \rho_1, v = \infty\} \).

Let \( r = 1/v \). Then (4.84) is converted to, after multiplying by \( r \),

\[
\begin{align*}
\dot{\beta} &= B_1(\beta) - s\beta r - w_{10}r \\
\dot{r} &= -rB_2(\beta) + sr^2 + w_{2L}r^3.
\end{align*}
\tag{4.87}
\]

At the equilibrium \( p_L = (\rho_1,0) \), the eigenvalues of the linearized system are \( \lambda_+ = 1 - \frac{\rho_2}{\rho_1} \) and \( \lambda_- = \frac{1}{2}(1 - \frac{\rho_2}{\rho_1}) \), and the corresponding eigenvectors are \( y_+ = (1,0) \) and \( y_- = \left(\frac{2\rho_1(w_{1L} + s\rho_1)}{3(\rho_1 + \rho_2)}, 1\right) \), so \( p_L \) is a hyperbolic saddle, and hence there exists a trajectory, denoted by \( \gamma_1 \), which tends to \( p_L \). The trajectory of \( \gamma_1 \) is tangent to the line \( \{p_L + ty_\gamma : t \in \mathbb{R}\} \) at \( p_L \). Since \( s\rho_1 + w_{1R} < 0 \) by Lemma 4.13, we know \( p_L + ty_1, t \geq 0 \), lies in the region \( \{\beta < \rho_1, r \geq 0\} \). Therefore, converting (4.87) back to (4.84), the solution converted from \( \gamma_1(\tau) \), also denoted by \( \gamma_1(\tau) \), lies in \( V \). Now we conclude that \( \gamma_1(\tau) \) approaches \( \{\beta = \rho_1, v = \infty\} \) in forward time and approaches \( u_L \) in backward time. \( \square \)
Chapter 5: Singular Shocks for the Keyfitz-Kranzer System

5.1 Introduction

The Keyfitz-Kranzer system

\[ \begin{align*}
    u_{1,t} + (u_1^2 - u_2)_x &= 0 \\
    u_{2,t} + (\frac{1}{3}u_1^3 - u_1)_x &= 0
\end{align*} \]

was first introduced in [KK89a, KK90]. It is a strictly hyperbolic, genuinely nonlinear system of conservation laws. A significant feature is that this model provides an example for singular shocks. In an attempt to justify singular shocks for (5.1), Keyfitz and Kranzer considered the Dafermos regularization for (5.1),

\[ \begin{align*}
    u_{1,t} + (u_1^2 - u_2)_x &= \epsilon t u_{1,xx} \\
    u_{2,t} + (\frac{1}{3}u_1^3 - u_1)_x &= \epsilon t u_{2,xx}
\end{align*} \]

(5.2\(\epsilon\))

with small \(\epsilon > 0\). They provided both analytic and numerical evidence for their conjecture that, for certain Riemann data

\[ (u_1, u_2)(x, 0) = \begin{cases} 
    (u_{1L}, u_{2L}), & x < 0, \\
    (u_{1R}, u_{2R}), & x > 0,
\end{cases} \]

(5.3)

the regularized system (5.2\(\epsilon\)) has a family of smooth solutions \((u_{1\epsilon}, u_{2\epsilon})(x, t)\) that converges to a step function away from the discontinuity as \(\epsilon \to 0\), and approaches a combination of delta functions near the discontinuity.
Parts of that conjecture were proved in [Sch04] using Geometric Singular Perturbation Theory. In that work, solutions of (5.2$\epsilon$) and (5.3) were proved to exist and approach infinity near the discontinuity as $\epsilon \to 0$, but convergence of solutions was not considered. We enhance that pioneering work in the following respects: First, we simplify the process of blowing-up in [Sch04], and construct the solutions in a more intuitive way. Second, we prove the weak convergence of the solutions, which confirms the conjecture in [KK90].

The system (5.1) can be derived from a single space dimensional model for isentropic gas dynamics equations

$$
\rho_t + (\rho u)_x = 0
$$

$$
(\rho u)_t + (\rho u^2 + \rho \gamma)_x = 0 \tag{5.4}
$$

with $\gamma = 1$, which corresponds to isothermal gas dynamics. By subtracting $u$ times the first equation in (5.4) from the second equation, one obtains (5.1) with $u_1 = u$ and $u_2 = \frac{1}{2} u^2 - \log \rho$ (see [Key11]). This means that (5.1) is equivalent to the isothermal gas dynamics (5.4) for smooth solutions, but conservation of mass and momentum has been replaced by conservation of velocity and a quantity that is an entropy for the original system.

The system (5.4) with any $\gamma$ between 1 and $5/3$ was considered in [KT12], and the existence of viscous profiles for singular shock was also proved. Some other generalizations of (5.1) were systematically analyzed in [Sev07].

In Section 5.2, we state our main result. In Sections 5.3 and 5.4, we sketch the construction of the solutions, and the proof for the Main Theorem is given in Section 5.5.
5.2 Main Result

First we make some reductions for the problem as in Section 4.2. In standard notation for conservation laws, we write (5.1) as

\[ u_t + f(u)_x = 0, \tag{5.5} \]

where \( u = (u_1, u_2) \), and write Riemann data for Riemann problems in the form

\[ u(x, 0) = u_L + (u_R - u_L)H(x), \tag{5.6} \]

where \( H(x) \) is the step function taking value 0 if \( x < 0 \); 1 if \( x > 0 \). As seen in Section 2.2, viscous profiles for Dafermos regularization

\[ u_t + f(u)_x = \epsilon u_{xx} \tag{5.7} \]

correspond to the boundary value problem

\[ -\xi \frac{d}{d\xi} u + \frac{d}{d\xi} (f(u)) = \epsilon \frac{d^2}{d\xi^2} u, \tag{5.8} \]

and

\[ u(-\infty) = u_L, \quad u(+\infty) = u_R. \tag{5.9} \]

As seen in Section 3.1, the system (5.8) is equivalent to

\[ u_\xi = f(u) - \xi u - w \tag{5.10} \]

\[ w_\xi = -\epsilon u \]

or, up to a rescaling of time,

\[ \dot{u} = f(u) - \xi u - w \tag{5.11} \]

\[ \dot{w} = -\epsilon u \]

\[ \dot{\xi} = \epsilon. \]
Figure 5.1: Numerical solutions for the Riemann data $u_L = (1, 6)$ and $u_R = (-1.6, 4.56)$ using a Lax-Friedrichs scheme up to 50,000 steps with CFL = 0.05.

The time variable in (5.11ε) is implicitly defined by the equation of $\dot{\xi}$. When $\epsilon = 0$, (5.11ε) is reduced to

$$\begin{align*}
\dot{u} &= f(u) - \xi u - w \\
\dot{w} &= 0, \quad \dot{\xi} = 0.
\end{align*}$$

(5.12)

Returning to the $(u_1, u_2)$ notation, the system (5.11ε) is written as

$$\begin{align*}
\dot{u}_1 &= u_1^2 - u_2 - \xi u_1 - w_1 \\
\dot{u}_2 &= \frac{1}{3}u_1^3 - u_1 - \xi u_2 - w_2 \\
\dot{w}_1 &= -\epsilon u_1, \quad \dot{w}_2 = -\epsilon u_2, \quad \dot{\xi} = \epsilon.
\end{align*}$$

(5.13ε)

and (5.12) becomes

$$\begin{align*}
\dot{u}_1 &= u_1^2 - u_2 - \xi u_1 - w_1 \\
\dot{u}_2 &= \frac{1}{3}u_1^3 - u_1 - \xi u_2 - w_2 \\
\dot{w}_1 &= 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0.
\end{align*}$$

(5.14)

At any equilibrium $u_0 = (u_{10}, u_{20})$ of (5.14), the eigenvalues for the linearized system are

$$\lambda_-(u_0) = u_{10} - 1, \quad \lambda_+(u_0) = u_{10} + 1.$$
Main Theorem. Consider the Riemann problem (5.1) and (5.6). Let

\[ s = \frac{f_1(u_L) - f_1(u_R)}{u_{1L} - u_{1R}} \]  
\[ w_L = f(u_L) - su_L, \quad w_R = f(u_R) - su_R \]  
\[ e_0 = w_{2L} - w_{2R} \]

Assume

(H1) \( \text{Re}(\lambda_{\pm}(u_R)) < s < \text{Re}(\lambda_{\pm}(u_L)) \).

(H2) \( e_0 > 0 \).

Then there exists a Dafermos profile for a singular shock from \( u_L \) to \( u_R \). That is, for each small \( \epsilon > 0 \), there is a solution \( \tilde{u}_\epsilon(\xi) \) of (5.8\( \epsilon \)) and (5.9), and this solution becomes unbounded as \( \epsilon \to 0 \). Indeed,

\[ \max_{\xi} \pm \tilde{u}_1(\xi) = \left( \omega_0 + o(1) \right) \epsilon^{-1} \]  
\[ \max_{\xi} \tilde{u}_2(\xi) = \left( \kappa_0^2 + o(1) \right) \epsilon^{-2} \]

as \( \epsilon \to 0 \), where \( \kappa_0 \) and \( \omega_0 \) are positive constant defined later in (5.36) and (5.70). Moreover, if we set \( u_\epsilon(x, t) = \tilde{u}_\epsilon(x/t) \), then \( u_\epsilon(x, t) \) is a solution of (5.2\( \epsilon \)) and

\[ u_{1\epsilon} \to u_{1L} + (u_{1R} - u_{1L})H(x - st) \]  
\[ u_{2\epsilon} \to u_{2L} + (u_{2R} - u_{2L})H(x - st) + \frac{e_0}{\sqrt{1 + s^2}} t\delta_{\{x=st\}} \]

in the sense of distributions.

The notation \( t\delta_{\{x=st\}} \) in (5.18b) denotes the linear functional defined by

\[ \langle t\delta_{\{x=st\}}, \varphi \rangle = \int_0^\infty t\varphi(st, t)\sqrt{1 + s^2} \, dt. \]  

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The weight \( \sqrt{1 + s^2} \) is the arc length of the parametrized line \( \{x = st\} \), so that the definition of the functional is independent of parametrizations.

A set of sample data for which (H1) and (H2) hold is

\[
    u_L = (2, 6), \quad u_R = (-1.6, 4.56),
\]

for which (5.16) gives \( s = 0 \) and \( \epsilon_0 = 0.423 \). A numerical solution for this Riemann data using a finite difference scheme is shown in Fig 5.1. Observe that both \( u_1 \) and \( u_2 \) appear to grow unboundedly near the shock. This is consistent with the theorem.

### 5.3 Compactification and Desingularization

To find solutions of (5.13ε) connecting \( u_L \) and \( u_R \), we first consider the limiting system (5.14) with \((w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) \) and \((w_{1R}, w_{2R}, s) \), where \( s, w_L \) and \( w_R \) are as defined in (5.16).

**Proposition 5.1.** Assume (H1). Then there exists a unique solution of (5.14) of the form \( \gamma_1(\sigma) = (u^{(1)}(\sigma), w_L, s) \) satisfying

\[
    \lim_{\sigma \to -\infty} u^{(1)}(\sigma) = u_L, \quad \lim_{\sigma \to 0^-} \left( u^{(1)}_2, \frac{u^{(1)}_1}{\sqrt{u^{(1)}_2}} \right)(\sigma) = \left( +\infty, \sqrt{3 - \sqrt{3}} \right)
\]

and a unique solution of the form \( \gamma_2(\sigma) = (u^{(2)}(\sigma), w_R, s) \) satisfying

\[
    \lim_{\sigma \to +\infty} u^{(1)}(\sigma) = u_R, \quad \lim_{\sigma \to 0^+} \left( u^{(2)}_2, \frac{u^{(2)}_1}{\sqrt{u^{(2)}_2}} \right)(\sigma) = \left( +\infty, -\sqrt{3 - \sqrt{3}} \right).
\]

**Proof.** See [SSS93, Theorem 3.1].

Motivated by Proposition 5.1, we compactify the state space by defining

\[
    \beta = \frac{u_1}{\sqrt{u_2}}, \quad r = \frac{1}{\sqrt{u_2}} \quad (5.21)
\]
In this definition we have assumed $u_2$ to be positive. This is just for convenience and has no loss of generality. In general cases, since the value of $u_2$ is bounded from below along $\gamma_1$ and $\gamma_2$, say $u_2 > -M$, we may replace $u_2$ by $u_2 + M$.

In $(\beta, r, w_1, w_2, \xi, \epsilon)$-coordinates, (5.13$\epsilon$) becomes, after multiplying by $r$,

\[
\begin{align*}
\dot{\beta} &= -\frac{1}{6}(\beta^4 - 6\beta^2 + 6) + r\left(-\frac{\beta\xi}{2} + r\left(\frac{\beta^2}{2} - w_1\right) + \frac{r^2}{2}\beta w_2\right) \\
\dot{r} &= -\frac{\beta^3}{6}r + \frac{r^2}{2}(\xi + r\beta + r^2w_2) \\
\dot{w}_1 &= -\beta\epsilon \\
\dot{w}_2 &= \frac{\epsilon}{r} \\
\dot{\xi} &= r\epsilon \\
\dot{\epsilon} &= 0.
\end{align*}
\]

(5.22)

Note that the time scale in (5.22) is different from that of (5.13$\epsilon$), but we use the same notation $\cdot$ to denote derivatives in time. This should cause no ambiguity since the time scales can be distinguished by the equations for $\dot{\xi}$.

In (5.22), the equation for $\dot{w}_2$ is not defined when $r = 0$. To make sense of it, one naive way is to multiply the system by $r$, but this will make the set $\{r = 0\}$ non-normally hyperbolic. To avoid this degeneracy, our remedy is to replace $\epsilon$ by $\kappa = \epsilon/r$. Then the system (5.22) becomes

\[
\begin{align*}
\dot{\beta} &= -\frac{1}{6}(\beta^4 - 6\beta^2 + 6) + r\left(-\frac{\beta\xi}{2} + r\left(\frac{\beta^2}{2} - w_1\right) + \frac{r^2}{2}\beta w_2\right) \\
\dot{r} &= -\frac{\beta^3}{6}r + \frac{r^2}{2}(\xi + r\beta + r^2w_2) \\
\dot{w}_1 &= -\kappa\beta r \\
\dot{w}_2 &= -\kappa \\
\dot{\xi} &= \kappa r^2 \\
\dot{\kappa} &= \frac{\beta^3}{6}\kappa + \frac{\beta}{2}\left(-\kappa\xi - \beta\kappa + r^2\kappa w_2\right)
\end{align*}
\]

(5.23)

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Note that the first two equations in (5.22) and (5.23) are identical.

The sets \{u_2 = +\infty\} and \{\epsilon = 0\} correspond to \{r = 0\} and \{\kappa = 0\}. Taking \(r = 0\) and \(\kappa = 0\), the system (5.23) reduces to a single equation for \(\beta\), namely

\[
\dot{\beta} = \frac{1}{6}(\beta^4 - 6\beta^2 + 6).
\]

(5.24)

For this equation, the equilibria are \(\beta = \rho_j, j = 1, \ldots, 4\), where

\[
\rho_1 = -\sqrt{3 + \sqrt{3}}, \quad \rho_2 = -\sqrt{3 - \sqrt{3}}, \quad \rho_3 = \sqrt{3 - \sqrt{3}}, \quad \rho_4 = \sqrt{3 + \sqrt{3}}.
\]

(5.25)

Let

\[
P_L = \{(\beta, r, \kappa, w_1, w_2, \xi) : \beta = \rho_3, r = 0, \kappa = 0\}
\]

(5.26)

\[
P_R = \{(\beta, r, \kappa, w_1, w_2, \xi) : \beta = \rho_2, r = 0, \kappa = 0\}.
\]

(5.27)

The trajectory \(\gamma_1\) given in Proposition 5.1 connects \(u_L\) and \(P_L\), and \(\gamma_2\) connects \(u_R\) and \(P_R\). Next we shall find connections between \(P_L\) and \(P_R\).

We will find a trajectory on \(\{r = 0\}\) connecting \((\beta, \kappa, w_1, w_2, \xi) = (\rho_3, 0, w_{1L}, w_{2L}, s)\) and \((\rho_2, 0, w_{1R}, w_{2R}, s)\). When \(r = 0\), the system reduces to

\[
\dot{\beta} = \frac{1}{6}(\beta^4 - 6\beta^2 + 6)
\]

(5.28a)

\[
\dot{\kappa} = \frac{\beta^3}{6}\kappa
\]

(5.28b)

\[
\dot{w}_2 = -\kappa
\]

(5.28c)

\[
\dot{w}_1 = 0, \quad \dot{\xi} = 0.
\]

(5.28d)

Observe that the system (5.28) is only weakly coupled, so we can solve it by integration:

**Proposition 5.2.** There exist positive smooth functions \(\iota_1, \iota_2\) and \(\iota_3\) which satisfy the following: For any parameters \((\bar{\kappa}, \bar{w}_1, \bar{w}_2, \bar{\xi})\), the system (5.28) with boundary
Figure 5.2: Trajectories for (5.28) starting at \((\rho_3, 0, w_{1L}, w_{2L}, s)\), and those ending at \((\rho_2, 0, w_{1R}, w_{2R}, s)\).

conditions

\[(\beta, \kappa)(0) = (0, \bar{\kappa}), \quad (w_1, w_2, \xi)(-\infty) = (\bar{w}_1, \bar{w}_2, \bar{\xi}),\]  

has a unique solution

\[(\beta^-, \kappa^-, w_1^-, w_2^-, \xi^-)(\sigma) = (\iota_1(\sigma), \bar{\kappa}_2(\sigma), \bar{w}_1 + \bar{\kappa}_3(\sigma), \bar{\xi}).\]  

For any parameters \((\hat{\kappa}, \hat{w}_1, \hat{w}_2, \hat{\xi})\), the system (5.28) with boundary conditions

\[(\beta, \kappa)(0) = (0, \hat{\kappa}), \quad (w_1, w_2, \xi)(+\infty) = (\hat{w}_1, \hat{w}_2, \hat{\xi}),\]  

has a unique solution

\[(\beta^+, \kappa^+, w_1^+, w_2^+, \xi^+)(\sigma) = (\iota_1(-\sigma), \bar{\kappa}_2(-\sigma), \hat{w}_1 - \hat{\kappa}_3(\sigma), \hat{\xi}).\]  

Proof. First we solve (5.28a) by setting

\[\iota_1(\sigma)\] to be the solution of (5.28a) satisfying \(\iota_1(0) = 0.\)
Figure 5.3: $\gamma_1$, $\gamma_2$ and $\gamma_0$ displayed in $(\beta, r, w_2)$-space.

Let

$$\nu_2(\sigma) = \exp \left( \int_0^\sigma \frac{\nu_1(\tau)^3}{6} d\tau \right), \quad \nu_3(\sigma) = \int_{-\infty}^\sigma \nu_2(\tau) d\tau. \quad (5.34)$$

Then a direct calculation shows that (5.30) and (5.32) are solutions of (5.28) satisfying (5.29) and (5.31).

See Fig 5.2 for the trajectories given in Proposition 5.2. Note that $\nu_1(\sigma)$ defined in (5.33) satisfies $\nu_1(-\infty) = \rho_3$ and $\nu_2(+\infty) = \rho_2$.

**Proposition 5.3.** If we set

$$(\bar{\kappa}, \bar{\omega}_1, \bar{\omega}_2, \bar{\xi}) = (\kappa_0, w_{1L}, w_{2L}, s), \quad (\bar{\kappa}, \bar{\omega}_1, \bar{\omega}_2, \bar{\xi}) = (\kappa_0, w_{1R}, w_{2R}, s), \quad (5.35)$$

with

$$\kappa_0 = \frac{w_{2R} - w_{2L}}{2\nu_3(0)}, \quad (5.36)$$
where $\iota_3(\sigma)$ is as defined in (5.34), then (5.30) and (5.32) coincide, and this gives a solution of (5.28), denoted by $\gamma_0(\sigma)$, satisfying

$$\gamma_0(-\infty) = (\rho_3, 0, w_{1L}, w_{2L}, s), \quad \gamma_0(+\infty) = (\rho_2, 0, w_{1R}, w_{2R}, s). \quad (5.37)$$

Proof. First we set

$$(\bar{w}_1, \bar{w}_2, \bar{\xi}) = (w_{1L}, w_{2L}, s), \quad (\tilde{w}_1, \tilde{w}_2, \tilde{\xi}) = (w_{1R}, w_{2R}, s).$$

From the definitions in (5.16a) and (5.16b) we have $w_{1L} = w_{1R}$. Solving

$$(\beta^-, \kappa^-, w^-, \tilde{w}^-)(0) = (\beta^+, \kappa^+, w^+, \tilde{w}^+)(0)$$

in (5.30) and (5.32) for $\bar{\kappa}$ and $\tilde{\kappa}$, we obtain the solution $\bar{\kappa} = \tilde{\kappa} = \kappa_0$ as defined in (5.36). This gives a trajectory $\gamma_0(\sigma)$ satisfying (5.37). From the uniqueness of solutions of boundary value problems, this trajectory is unique. \hfill \Box
We will show that for the system (5.23) there are trajectories close to $\gamma_1 \cup \gamma_0 \cup \gamma_2$ lying in hyper-surfaces $\{rk = \epsilon\}$, $\epsilon > 0$. See Fig 5.3 and 5.4.

For solutions $(u_1, u_2)\xi$ of (5.8) and (5.9), from the equation for $\xi$ in (5.13), we know the $\xi$-interval corresponding to any compact segment of $\gamma_1$ or $\gamma_2$ has length of order $O(\epsilon)$. We will see at the end of Section 5.5.1 that the length of the $\xi$-interval corresponding to any compact segment of $\gamma_0$ is of order $O(\epsilon^2)$.

5.4 Singular Configuration

As in Section 4.4, we will find trajectories of limiting subsystems of the target fast-slow system, in this case (5.13), such that the union of those trajectories forms a singular configuration joining the end states $u_L$ and $u_R$.

5.4.1 End States $U_L$ and $U_R$

The treatment in this section is analogous to that in Section 4.4.1. First observe that the system (5.14) has a normally hyperbolic critical manifold

$$S_0 = \{(u, w, \xi) : f(u) - \xi u - w = 0, \xi \neq \text{Re}(\lambda_{\pm}(u))\}, \quad (5.38)$$

where $\lambda_{\pm}(u)$ are the eigenvalues of $Df(u)$, as defined in (5.15). The limiting slow system for (5.13) is

$$0 = f(u) - \xi u - w$$

$$w' = -u$$

$$\xi' = 1.$$  \quad (5.39)

From $(H1)$ we have $s < \text{Re}(\lambda_{\pm}(u_L))$, so $(u_L, w_L, s) \in S_0$. Choose $\delta > 0$ so that $s + 2\delta < \text{Re}(\lambda_{\pm}(u_L))$, and set

$$U_L = (u_L, w_L, s) \bullet (\infty, \delta)$$

$$= \{(u, w, \xi) : u = u_L, w = w_L - \alpha_1 u_L, \xi = s + \alpha_1, \alpha_1 \in (\infty, \delta)\}. \quad (5.40)$$
where \( \bullet \) denotes the flow for (5.39). It is clear that \( U_L \subset S_0 \) is normally hyperbolic with respect to (5.14), and is locally invariant with respect to (5.11\( \epsilon \)).

Note that each point in \( U_L \) is a hyperbolic equilibrium for the 2-dimensional system (5.12), and the unstable manifold \( W^u_0(U_L) \) is naturally defined.

**Proposition 5.4.** Assume (H1). Let \( U_L \) be defined in (5.40). Fix any \( r \geq 1 \). There exists a family of invariant manifolds \( W^u_\epsilon(U_L) \) which are \( C^K O(\epsilon) \)-close to \( W^u_0(U_L) \) such that for any continuous family \( \{ \mathcal{I}_\epsilon \}_{\epsilon \in [0, \epsilon_0]} \) of compact sets \( \mathcal{I}_\epsilon \subset W^u_\epsilon(U_L) \),

\[
\text{dist}(p \cdot t, U_L) \leq C e^{\mu t} \quad \forall \ p \in \mathcal{I}_\epsilon, \ t \leq 0, \epsilon \in [0, \epsilon_0],
\]

for some positive constants \( C \) and \( \mu \).

**Proof.** This follows from Theorem 3.3 by taking \( U_L \) to be \( U_0 \). Although \( U_L \) is not compact, it is uniformly normally hyperbolic since \( \xi - \text{Re}(\lambda_\pm(u_L)) < -\delta \) on \( U_L \), and the proof of Theorem 3.3 in [Jon95, Theorem 4] is still valid. \( \Box \)

**Remark 5.1.** Proposition 5.4 was also asserted in [Sch04, Liu04, KT12].

From (H1) we also have, by decreasing \( \delta \) if necessary, \( s - 2\delta > \text{Re}(\lambda_\pm(u_L)) \), and hence a similar result holds for for the set \( U_R \) defined by

\[
U_R = (u_R, w_R, s) \cdot (-\delta, \infty)
\]

\[= \{ (u, w, \xi) : u = u_R, w = w_R - \alpha_2 u_R, \xi = s + \alpha_2, \alpha_2 \in [-\delta, \infty) \}. \]

**Proposition 5.5.** Assume (H1). Let \( U_R \) be defined by (5.42). Fix any \( k \geq 1 \). There exists a family of invariant manifolds \( W^s_\epsilon(U_R) \) which are \( C^K O(\epsilon) \)-close to \( W^s_0(U_R) \) such that for any continuous family \( \{ \mathcal{J}_\epsilon \}_{\epsilon \in [0, \epsilon_0]} \) of compact sets \( \mathcal{J}_\epsilon \subset W^s_\epsilon(U_R) \),

\[
\text{dist}(p \cdot t, U_R) \leq C e^{-\mu t} \quad \forall \ p \in \mathcal{J}_\epsilon, \ t \geq 0, \epsilon \in [0, \epsilon_0],
\]

for some positive constants \( C \) and \( \mu \).
5.4.2 Intermediate States $\mathcal{P}_L$ and $\mathcal{P}_R$

It is easy to see that $\mathcal{P}_L$ defined in (5.26) is a normally hyperbolic critical manifold for (5.23), so $C^k$ unstable and stable manifolds $W^u(\mathcal{P}_L)$ and $W^s(\mathcal{P}_L)$ of $\mathcal{P}_L$ exist for any fixed $k \geq 1$. Note that $\{r = 0\}$ and $\{\kappa = 0\}$ are invariant under (5.23) while $\{\beta = \rho_3\}$ is not. We can straighten $W^u(\mathcal{P}_L)$ and $W^s(\mathcal{P}_L)$ by modifying $\beta$:

**Proposition 5.6.** Let $W^{u,s}(\mathcal{P}_L)$ be $C^k$ unstable/stable manifolds of $\mathcal{P}_L$ for (5.23), $k \geq 1$. There exists a $C^k$ function $\hat{\beta} = \hat{\beta}(\beta, r, w_1, w_2, \xi)$ such that

$$\hat{\beta} = \beta \quad \text{when} \quad r = 0 \quad (5.44)$$

and $(\hat{\beta}, r, \kappa, w_1, w_2, \xi)$ is a change of coordinates near $\mathcal{P}_L$ satisfying

$$W^s(\mathcal{P}_L) = \{(\hat{\beta}, r, \kappa, w_1, w_2, \xi) : \hat{\beta} = \rho_3, \kappa = 0\} \quad (5.45)$$

$$W^u(\mathcal{P}_L) = \{(\hat{\beta}, r, \kappa, w_1, w_2, \xi) : r = 0\}. \quad (5.46)$$

Hence (5.23) is converted into

$$\dot{\hat{\beta}} = \left(\frac{2}{3} \rho_3 (\rho_3^2 - 3) + h_1\right)(\hat{\beta} - \rho_3)$$

$$\dot{\hat{r}} = \left(-\frac{\rho_3^2}{6} + h_2\right) r$$

$$\dot{\hat{\kappa}} = \left(\frac{\rho_3^2}{6} + h_3\right) \kappa$$

$$\dot{\hat{w}}_1 = h_4$$

$$\dot{\hat{w}}_2 = -\kappa$$

$$\dot{\hat{\xi}} = \kappa r^2,$$

where $h_i$ are $C^{k-1}$ functions satisfying $h_1, h_2, h_3 = O(|(\hat{\beta}, r, \kappa)|)$ and $h_4 = O(|r| \cdot |(\hat{\beta}, \kappa)|)$ as $|(\hat{\beta}, \kappa, r)| \to 0$, and the projection $\pi^s_{\mathcal{P}_L}$ into $\mathcal{P}_L$ along stable fibers is

$$\pi^s_{\mathcal{P}_L}(0, r, 0, w_1, w_2, \xi) = (0, 0, 0, w_1, w_2, \xi). \quad (5.48)$$
Proof. At each point of \( P_L \) defined in (5.26), the linearized system corresponds to the matrix represented in \((\beta, r, \kappa)\)-coordinates as
\[
\begin{pmatrix}
-\frac{2}{3} \rho_3 (\rho_3^2 - 3) & -\frac{1}{6} \rho_3 \xi & 0 \\
0 & \frac{1}{6} \rho_3^3 & 0 \\
0 & 0 & \frac{1}{6} \rho_3^3
\end{pmatrix},
\] (5.49)
which has eigenvalues
\[
-\frac{2}{3} \rho_3 (\rho_3^2 - 3) > 0, \quad -\frac{1}{6} \rho_3^3 < 0, \quad \frac{1}{6} \rho_3^3 > 0,
\] (5.50)
and eigenvectors
\[
(1, 0, 0)^\top, \quad \left( \frac{1}{2} \rho_3 \xi, -\frac{2}{3} \rho_3 (\rho_3^2 - 3) + \frac{1}{6} \rho_3^3, 0 \right)^\top, \quad (0, 0, 1)^\top.
\] (5.51)
Since the sets \( \{ r = 0 \} \) and \( \{ \kappa = 0 \} \) are invariant under (5.23), it follows that
\[
W^u(P_L) = \{ (\beta, r, \kappa, w_1, w_2) : r = 0 \}
\] (5.52)
and \( W^s(P_L) \) can be parameterized as
\[
W^s(P_L) = \{ (\beta, r, \kappa, w_1, w_2, \xi) : \kappa = 0 \text{ and } \beta = \rho_3 + \phi(r, w_1, w_2, \xi)r \}
\] (5.53)
where \( \phi \) is a \( C^k \) function satisfying
\[
\phi(r, w_1, w_2, \xi) = -\frac{2}{3} \rho_3 (\rho_3^2 - 3) + \frac{1}{6} \rho_3^3 + O(r).
\] (5.54)
Set
\[
\hat{\beta} = \beta - \phi(r, w_1, w_2, \xi)r.
\] (5.55)
Then (5.54) implies (5.44). Now (5.45) follows from (5.53) and (5.55), and (5.46) follows from (5.52). \( \square \)

A similar result holds for \( P_R \). We omit it here.
Figure 5.5: The 1D intervals $\Lambda_L$ and $\Lambda_R$ are projections of $I_0$ and $J_0$, respectively, on the critical manifolds $P_L$ and $P_R$. In the 5D space $\{r = 0\}$, the 3D manifolds $W^u(\Lambda_L)$ and $W^s(\Lambda_R)$ intersect transversally at $q_0$, and their intersection is the curve $\gamma_0$, which is transversal to $\Gamma$.

5.4.3 Transversal Intersections

To prove the Main Theorem, we need to find trajectories for (5.23) connecting $U_L$ and $U_R$ in $(\beta, r, \kappa, w_1, w_2, \xi)$-space satisfying $r\kappa = \epsilon$ for each small $\epsilon > 0$. Note that the trajectories $\gamma_1$ and $\gamma_2$ given in Proposition 5.1 satisfy $\gamma_1 \subset W^u_0(U_L) \cap W^s(P_L)$ and $\gamma_2 \subset W^s_0(U_R) \cap W^u(P_R)$. Our strategy is to first define 2-dimensional manifolds $I_\epsilon$ and $J_\epsilon$, $\epsilon \in [0, \epsilon_0]$, contained in $W^u_\epsilon(U_L)$ and $W^s_\epsilon(U_R)$, respectively, such that $\cup_\epsilon I_\epsilon$ and $\cup_\epsilon J_\epsilon$ are transverse to $\gamma_1$ and $\gamma_2$, and then track forward/backward trajectories from $I_\epsilon$ and $J_\epsilon$. An illustration with $\epsilon = 0$ is shown in Fig 5.5.

To track trajectories evolving from $I_\epsilon$ and $J_\epsilon$, we will apply the Corner Lemma stated in Section 3.4. The key idea is to show that the manifolds that evolve from $I_\epsilon$
and $\mathcal{J}_\epsilon$, denoted by $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$, respectively, are $C^1$ close to $W^u(\Lambda_L)$ and $W^s(\Lambda_R)$, where $\Lambda_L \subset \mathcal{P}_L$ and $\Lambda_R \subset \mathcal{P}_R$ are projections of $\mathcal{I}_0$ and $\mathcal{J}_0$. Hence transversal intersection of $W^u(\Lambda_L)$ and $W^s(\Lambda_R)$ will imply that of $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$.

Fix a small $r^0 > 0$ so that $\gamma_1$ intersects $\{r = r^0\}$ at a unique point. Denote this point by $p^0_{\text{in}}$. That is,

$$p^0_{\text{in}} = \gamma_1 \cap \{r = r^0\}. \quad (5.56)$$

We set

$$\mathcal{I}_\epsilon = W^u(\mathcal{U}_L) \cap \{r = r^0\} \cap V_1, \quad (5.57)$$

where $V_1$ is an open neighborhood of $p^0_{\text{in}}$ to be specified below: From the expression (5.40), $\mathcal{U}_L$ is 1-dimensional, so from (H1) we see that $W^u(\mathcal{U}_L)$ is 3-dimensional. Hence we can choose $V_1$ so that $\mathcal{I}_\epsilon$ is parametrized, in $(\hat{\beta}, r, \kappa, w_1, w_2, \xi)$-coordinates given in Proposition 5.6, by

$$\mathcal{I}_\epsilon = \{(\hat{\beta}, r, \kappa, w_1, w_2, \xi) : r = r^0, \kappa = \epsilon/r^0, \quad (w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-u_{1L}, -u_{2L}, 1) + \epsilon\theta(\hat{\beta}, \alpha_1, \epsilon), \quad (5.58)$$

$$|\hat{\beta} - \rho_3| < \Delta_1, |\alpha_1| < \Delta_1 \}$$

for some $\Delta_1 > 0$ and some $C^4$ function $\theta$. (The order of differentiability of $\theta$ is chosen so that the Corner Lemma applies.) Note that $\mathcal{I}_0$ is a affine surface, and $\mathcal{I}_\epsilon$ can be viewed as a perturbation of $\mathcal{I}_0$.

Let

$$\mathcal{I} = \bigcup_{\epsilon \in [0, \epsilon_0]} \mathcal{I}_\epsilon. \quad (5.59)$$

Since $p^0_{\text{in}} \in \gamma_1 \subset W^u(\mathcal{U}_L) \cap W^s(\mathcal{P}_L)$, from (5.53) and (5.58) we see that $\mathcal{I}$ and $W^s(\mathcal{P}_L)$ intersect transversally at $p^0_{\text{in}}$, and the projection into $\mathcal{P}_L$ of their intersection along
stable fibers is, by (5.48),
\[ \Lambda_L = \{ (\beta, r, \kappa, w_1, w_2, \xi) : \beta = \rho_3, r = 0, \kappa = 0, \\
(w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-u_{1L}, -u_{2L}, 1), \] \tag{5.60}
\]
\[ |\alpha_1| < \Delta_1 \}. \]

Also we let \( p^\text{out}_0, J, J^e, J^f \) and \( \Lambda_R \) be analogously defined.

Since \( \Lambda_L \) is a subset of the normally hyperbolic critical manifold \( P_L \) for (5.47), the unstable manifold \( W^u(\Lambda_L) \) can be defined in the natural way. From (5.46) we see that \( W^u(\Lambda_L) \subset \{ r = 0 \} \). Similarly, \( \Lambda_R \) and \( W^s(\Lambda_R) \) are defined, and \( W^s(\Lambda_R) \subset \{ r = 0 \} \).

Note that the trajectory \( \gamma_0 \) given in Proposition 5.3 satisfies
\[ \gamma_0 \subset W^u(\Lambda_L) \cap W^s(\Lambda_R). \]

To track the intersection of \( W^u(\Lambda_L) \) and \( W^s(\Lambda_R) \) along \( \gamma_0 \), we fix a hyperplane
\[ \Gamma = \{ (\beta, r, \kappa, w_1, w_2, \xi) : \beta = 0 \} \] \tag{5.61}
and set
\[ q_0 = \gamma_0 \cap \Gamma. \] \tag{5.62}

**Proposition 5.7.** \( W^u(\Lambda_L) \) and \( W^s(\Lambda_R) \) intersect transversally at \( q_0 \) in the space \( \{ r = 0 \} \), and their intersection near \( q_0 \) is a portion of the curve \( \gamma_0 \) given in Proposition 5.3, and hence is transverse to \( \Gamma \) at \( \gamma_0 \).

**Proof.** From Proposition 5.2 and 5.3, we have
\[ T_{q_0}W^u(\Lambda_L) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ * \\ 0 \\ 0 \\ * \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2\nu(0) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u_{1L} \\ -u_{2L} \\ 1 \end{pmatrix} \right\} \] \tag{5.63}
and

\[
T_{q_0} W^s(\Lambda_R) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ * \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2t_3(0) \\ * \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -u_{1R} \\ -u_{2R} \\ 1 \end{pmatrix} \right\}
\]

(5.64)

in \((\beta, r, \kappa, w_1, w_2, \xi)\) coordinates, where \(t_3(\sigma)\) is the positive function defined in (5.34).

Since \(\varepsilon(0) \neq 0\) and \(u_{1L} \neq u_{1R}\), from (5.63) and (5.64) we see that \(T_{q_0} W^u(\Lambda_L)\) and \(T_{q_0} W^s(\Lambda_R)\) span \((\beta, \kappa, w_1, w_2, \xi)\)-space and they have a 1-dimensional intersection which is transversal to \(\Gamma\). Since \(q_0 \in \gamma_0 \subset W^u(\Lambda_L) \cap W^s(\Lambda_R)\), the desired result follows.

\[\square\]

Now we have obtained the singular configuration \(\gamma_1 \cup \gamma_0 \cup \gamma_2\), which joins the end states \(u_L\) and \(u_R\). In the next section we will show that there are solutions of (5.11\(\epsilon\)) which are close to the singular configuration.

### 5.5 Completing the Proof of the Main Theorem

We split the proof of the main theorem into two parts. In Section 5.5.1 we prove the existence of solutions of the boundary value problem (5.8\(\epsilon\)) and (5.9). In Section 5.5.2 we derive the weak convergence (5.18).

#### 5.5.1 Existence of Trajectories

**Proposition 5.8.** Let \(p_{0}^{in}, p_{0}^{out}, q_{0}, I_{\epsilon}, J_{\epsilon}\) and \(\Gamma\) be defined in Section 5.4.3. For each small \(\epsilon > 0\), there exist \(p_{\epsilon}^{in} \in I_{\epsilon}, p_{\epsilon}^{out} \in J_{\epsilon}, q_{\epsilon} \in \Gamma\) and \(T_{1\epsilon}, T_{2\epsilon} > 0\) such that

\[
q_{\epsilon} = p_{\epsilon}^{in} \cdot T_{1\epsilon}, \quad q_{\epsilon} = p_{\epsilon}^{out} \cdot (-T_{2\epsilon}),
\]

(5.65)

where \(\cdot\) denotes the flow for (5.23), satisfying

\[
(p_{\epsilon}^{in}, p_{\epsilon}^{out}, q_{\epsilon}) = (p_{0}^{in}, p_{0}^{out}, q_{0}) + o(1)
\]

(5.66)
as $\epsilon \to 0$, and

$$C^{-1} \log \frac{1}{\epsilon} \leq T_{i\epsilon} \leq C \log \frac{1}{\epsilon}, \quad i = 1, 2,$$

(5.67)

for some $C > 0$. Moreover, if we set $\beta_\epsilon(\sigma)$ and $\kappa_\epsilon(\sigma)$ to be the $\beta$- and $\kappa$-coordinates of $q_\epsilon \cdot \sigma$, $\sigma \in [-T_1\epsilon, T_2\epsilon]$, then

$$\max_{\sigma \in [-T_1\epsilon, T_2\epsilon]} \kappa_\epsilon(\sigma) = \kappa_0 + o(1)$$

(5.68)

and

$$\max_{\sigma \in [-T_1\epsilon, T_2\epsilon]} \pm \beta_\epsilon(\sigma) \kappa_\epsilon(\sigma) = \omega_0 + o(1),$$

(5.69)

as $\epsilon \to 0$, where $\kappa_0$ is defined in (5.36), and

$$\omega_0 = \kappa_0 \iota_2(\sigma_0),$$

(5.70)

where $\sigma_0$ is the unique number such that $\iota_1(\sigma_0) = 1$, and $\iota_1(\sigma), \iota_2(\sigma)$ are positive functions defined in (5.33) and (5.34).

Proof. Let $\mathcal{I} = \bigcup_{\epsilon} \mathcal{I}_\epsilon$. Since $\mathcal{I}_\epsilon \subset \{ r = r^0 \}$, from the relation $\kappa = \epsilon / r$ we have

$$\mathcal{I}_\epsilon = \mathcal{I} \cap \{ \kappa = \epsilon / r^0 \}.$$

From the construction of $p^\text{in}_0, p^\text{out}_0$ and $q_0$, we have

$$\pi^u_{\mathcal{P}_L}(q_0) = \pi^s_{\mathcal{P}_L}(p^\text{in}_0) = (0, 0, 0, w_{1L}, w_{2L}, s) \in \Lambda_L$$

$$\pi^s_{\mathcal{P}_R}(q_0) = \pi^u_{\mathcal{P}_R}(p^\text{out}_0) = (0, 0, 0, w_{1R}, w_{2R}, s) \in \Lambda_R$$

in ($\beta, r, \kappa, w_1, w_2, \xi$)-coordinates, where $\pi^s_{\mathcal{P}_{L,R}}$ is the projection onto $\mathcal{P}_{L,R}$ along stable/unstable fibers. For the system (5.47), the conditions for the Corner Lemma are satisfied. Hence there exists a neighborhood $V_0$ of $q_0$ such that

$$\mathcal{I}_\epsilon^* \cap V_0 \text{ is } C^1 \text{ close to } T_{q_0} W^u(\Lambda_L) \cap V_0,$$

(5.71)
where $\mathcal{I}_\epsilon^* = \mathcal{I}_\epsilon \cdot [0, \infty)$. Similarly, setting $\mathcal{J}_\epsilon^* = \mathcal{J}_\epsilon \cdot (-\infty, 0]$, we have

$$\mathcal{J}_\epsilon^* \cap V_0 \text{ is } C^1 \text{ close to } T_{q_0} W^s(\Lambda_R) \cap V_0. \tag{5.72}$$

From (5.71), (5.72) and Proposition 5.7, it follows that the projections of $\mathcal{I}_\epsilon^*$ and $\mathcal{J}_\epsilon^*$ in the 5-dimensional space $\{r = 0\}$ intersect transversally at a unique point in $\Gamma$ near $q_0$. From the relation $r = \epsilon/\kappa$, we then recover a unique intersection point

$$q_\epsilon \in \mathcal{I}_\epsilon^* \cap \mathcal{J}_\epsilon^* \cap \Gamma \tag{5.73}$$

in $(\beta, r, \kappa, w_1, w_2, \xi)$-space. By the construction we have (5.65) and (5.66). The estimate (5.67) follows from (3.38).

The unstable fiber containing $q_0$ in $W^u(\mathcal{P}_L)$ is the trajectory $\gamma_0$ defined in Proposition 5.3. The $\beta$- and $\kappa$-coordinates on $\gamma_0$ are $\iota_1(\sigma)$ and $\kappa_0 \iota_2(\sigma)$, respectively. From (5.34) we know $\iota_2(\sigma) \leq \iota_2(0) = 1$. Hence (5.68) follows. To prove (5.69), by symmetry of $\gamma_0$, it suffices to show that

$$\max_{\sigma \in (-\infty, 0]} \iota_1(\sigma) \iota_2(\sigma) = \iota_2(\sigma_0), \tag{5.74}$$

where $\sigma_0$ is defined by $\iota_1(\sigma_0) = 1$. Note that the values of $\iota_1(\sigma)$ and $\iota_2(\sigma)$ are positive on $(-\infty, 0)$, and

$$\iota_1(0) \iota_2(0) = 0 \cdot 1 = 0, \quad \iota_1(-\infty) \iota_2(-\infty) = \rho_3 \cdot 0 = 0.$$  

By taking the derivative of $\iota_1(\sigma) \iota_2(\sigma)$ it can be readily seen that the maximum of this function occurs at a unique number $\sigma_0$ satisfying $\iota_1(\sigma_0) = 1$. Indeed, from the definition (5.33) and (5.34), we have

$$\frac{d}{d\sigma} (\iota_1(\sigma) \iota_2(\sigma)) = \iota_2(\sigma) \left[ \dot{\iota}_1(\sigma) + \frac{1}{6} \iota_1(\sigma)^3 \right]$$

$$= \frac{1}{6} \iota_2(\sigma) \left[ - (\beta^4 - 6\beta^2 + 6) + \beta^3 \right],$$

83
where we write $\beta = \iota_1(\sigma)$. Since $0 < \iota_1(\sigma) < \rho_3$ for $\sigma \in (-\infty, 0)$, this derivative has a unique zero, which occurs when $\beta = 1$. This proves (5.74) and hence (5.69).

**Proposition 5.9.** Let $q_\epsilon = (\beta_\epsilon^0, r_\epsilon^0, \kappa_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0) \in \Gamma$ be defined in Proposition 5.8. Let $(u_{1\epsilon}^0, u_{2\epsilon}^0) = (\beta_\epsilon^0/r_\epsilon^0, 1/(r_\epsilon^0)^2)$ and

$$
(\ddot{u}_{1\epsilon}, \ddot{u}_{2\epsilon}, \ddot{w}_{1\epsilon}, \ddot{w}_{2\epsilon})(\xi) = (u_{1\epsilon}^0, u_{2\epsilon}^0, w_{1\epsilon}^0, w_{2\epsilon}^0) \bullet (\xi - \xi_\epsilon^0),
$$

or equivalently,

$$
(\ddot{u}_{1\epsilon}, \ddot{u}_{2\epsilon}, \ddot{w}_{1\epsilon}, \ddot{w}_{2\epsilon}, \xi) = (u_{1\epsilon}^0, u_{2\epsilon}^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0) \bullet \left(\frac{\xi - \xi_\epsilon^0}{\epsilon}\right),
$$

(5.75)

Then $(\ddot{u}_{1\epsilon}, \ddot{u}_{2\epsilon})$ is a solution of (5.8$\epsilon$) and (5.9), and it satisfies (5.17).

**Proof.** Since (5.8$\epsilon$) and (5.10$\epsilon$) are equivalent and $(\ddot{u}_{1\epsilon}, \ddot{u}_{2\epsilon}, \ddot{w}_{1\epsilon}, \ddot{w}_{2\epsilon})(\xi)$ is a solution of (5.10$\epsilon$), we know $(\ddot{u}_{1\epsilon}, \ddot{u}_{2\epsilon})$ is a solution of (5.8$\epsilon$).

Let $T_{1\epsilon}$ and $T_{2\epsilon}$ be as defined in Proposition 5.8. Then

$$
q_\epsilon \cdot (T_{1\epsilon}) \in \mathcal{I}_\epsilon, \quad q_\epsilon \cdot (T_{2\epsilon}) \in \mathcal{I}_\epsilon,
$$

where $\cdot$ denotes the flow for (5.23). Since $\mathcal{I}_\epsilon \subset W^u(\mathcal{U}_L)$ and $\mathcal{J}_\epsilon \subset W^s(\mathcal{U}_R)$, from (5.41) and (5.43) it follows that

$$
\lim_{t \to -\infty} \text{dist}(p_\epsilon^0 \cdot t, \mathcal{U}_L) = 0, \quad \lim_{t \to \infty} \text{dist}(p_\epsilon^0 \cdot t, \mathcal{U}_R) = 0,
$$

which implies (5.9). Since $\ddot{u}_{2\epsilon} = (1/r_\epsilon)^2 = (\bar{\kappa}_\epsilon/\epsilon)^2$ and $\ddot{u}_{1\epsilon} = \dddot{\beta}_\epsilon/r_\epsilon = \dddot{\beta}_\epsilon \bar{\kappa}_\epsilon/\epsilon$, from (5.68) and (5.69) we obtain (5.17).

Here we justify the assertion made at the end of Section 5.3. From the equation for $\dot{\xi}$ in (5.23), we have $\dot{\xi} = \bar{\kappa}_\epsilon \bar{r}_\epsilon^2 = \epsilon^2/\bar{\kappa}_\epsilon$. Since the integral of $1/\kappa$ along any compact segment of $\gamma_0$ is finite, the change in $\xi$ near such a segment is of order $O(\epsilon^2)$.
5.5.2 Convergence of Trajectories

**Proposition 5.10.** Let \( \tilde{u}_\epsilon = (\tilde{u}_{1\epsilon}, \tilde{u}_{2\epsilon}) \) be the solution of (5.8 ε) and (5.9) given in Proposition 5.9. Let \( p_{\epsilon}^{in} \) and \( p_{\epsilon}^{out} \) be defined in Proposition 5.8, and \( s \) defined in (5.16a). Then

\[
|\xi_{\epsilon}^{in} - s| + |\xi_{\epsilon}^{out} - s| = o(1) \tag{5.77}
\]

\[
\int_{-\infty}^{\xi_{\epsilon}^{in}} |\tilde{u}_\epsilon(\xi) - u_L| \, d\xi + \int_{\xi_{\epsilon}^{out}}^{\infty} |\tilde{u}_\epsilon(\xi) - u_R| \, d\xi = o(1) \tag{5.78}
\]

\[
\int_{\xi_{\epsilon}^{in}}^{\xi_{\epsilon}^{out}} \tilde{u}_\epsilon(\xi) \, d\xi = (0, c_0) + o(1) \tag{5.79}
\]

as \( \epsilon \to 0 \), where \( \xi_{\epsilon}^{in, out} \) and are the \( \xi \)-coordinates of \( p_{\epsilon}^{in, out} \).

**Proof.** Note that \( s \) is the \( \xi \)-coordinate of \( p_{0}^{in} \). From the triangular inequality we have

\[
|\xi_{\epsilon}^{in} - s| \leq |p_{\epsilon}^{in} - p_{0}^{in}| = o(1),
\]

where the second inequality follows from (5.66). A similar inequality holds for \( \xi_{\epsilon}^{out} \), so we obtain (5.77).

Since every point in \( U_L \) has \( u \)-coordinate equal to \( u_L \),

\[
|\tilde{u}(\xi) - u_L| \leq \text{dist}((\tilde{u}(\xi), \tilde{w}(\xi), \xi, U_L) = \text{dist}((u_{\epsilon}^{0}, w_{\epsilon}^{0}, \xi_{\epsilon}^{0}) \bullet \frac{\xi_{\epsilon}^{0} - \xi_{\epsilon}}{\epsilon}, U_L),
\]

where the last equality follows from (5.75). Using (5.41), the last term is \( \leq C \exp \left( \mu \frac{\xi_{\epsilon}^{0} - \xi_{\epsilon}}{\epsilon} \right) \).

Since \( \xi_{\epsilon}^{in} < \xi_{\epsilon}^{0} \), it follows that

\[
\int_{-\infty}^{\xi_{\epsilon}^{in}} |\tilde{u}(\xi) - u_L| \, d\xi \leq \int_{-\infty}^{\xi_{\epsilon}^{in}} C \exp \left( \mu \frac{\xi_{\epsilon}^{0} - \xi_{\epsilon}}{\epsilon} \right) \, d\xi \leq \int_{-\infty}^{\xi_{\epsilon}^{in}} C \exp \left( \mu \frac{\xi_{\epsilon}^{0} - \xi_{\epsilon}^{in}}{\epsilon} \right) \, d\xi = \frac{\epsilon}{\mu} C.
\]

A similar inequality holds for \( \int_{\xi_{\epsilon}^{out}}^{\infty} |\tilde{u}(\xi) - u_R| \, d\xi \), so we obtain (5.78).

From the equation for \( \dot{\xi} \) in (5.23), denoting the time variable by \( \sigma \), we can write

\[ \dot{\xi} = \xi(\sigma) \]

by

\[ \xi(0) = \xi_{\epsilon}^{0}, \quad \frac{d\xi}{d\sigma} = \kappa_{\epsilon}(\xi)\bar{r}_{\epsilon}(\xi)^2, \tag{5.80} \]
From (5.65) we have
\[ \xi(-T_1 \epsilon) = \xi_{\epsilon}^\text{in}, \quad \xi(T_2 \epsilon) = \xi_{\epsilon}^\text{out}. \] (5.81)

From (5.80) and (5.81), using the equation for \( \dot{w}_2 \) in (5.23), it follows that
\[ \int_{\xi_{\epsilon}^\text{in}}^{\xi_{\epsilon}^\text{out}} \tilde{u}_{2 \epsilon}(\xi) \, d\xi = \int_{\xi_{\epsilon}^\text{in}}^{\xi_{\epsilon}^\text{out}} \frac{1}{(\tilde{r}_\epsilon(\xi))^2} \, d\xi = \int_{-T_1 \epsilon}^{T_2 \epsilon} \tilde{\kappa}_\epsilon(\sigma) \tilde{r}_\epsilon(\sigma)^2 \, d\sigma \]
\[ = - \int_{-T_1 \epsilon}^{T_2 \epsilon} \dot{\tilde{w}}_2(\sigma) \, d\sigma = w_2(p_{\epsilon}^\text{out}) - w_2(p_{\epsilon}^\text{in}) = w_{2L} - w_{2R} + o(1) \] (5.82)

where \( w_2(p) \) denotes the \( w_2 \)-coordinate of \( p \), and that
\[ \int_{\xi_{\epsilon}^\text{in}}^{\xi_{\epsilon}^\text{out}} |\tilde{u}_{1 \epsilon}(\xi)| \, d\xi = \int_{\xi_{\epsilon}^\text{in}}^{\xi_{\epsilon}^\text{out}} |\tilde{\beta}_\epsilon(\xi)| \, d\xi = \int_{-T_1 \epsilon}^{T_2 \epsilon} |\tilde{\beta}_\epsilon(\sigma)| \tilde{\kappa}_\epsilon(\sigma) \tilde{r}_\epsilon(\sigma) \, d\sigma \]
\[ = \epsilon \int_{-T_1 \epsilon}^{T_2 \epsilon} |\tilde{\beta}_\epsilon(\sigma)| \, d\sigma \leq C \epsilon (T_1 \epsilon + T_2 \epsilon) \leq \tilde{C} (\epsilon \log \frac{1}{\epsilon}) = o(1), \] (5.83)

where the last inequality follows from (5.67). Now (5.82) and (5.83) give (5.79). \[ \square \]

The remaining part of this section is analogous to that in Section 4.5.2, so we only sketch the proofs briefly.

**Proposition 5.11.** Let \( \tilde{u}_\epsilon = (\tilde{u}_{1 \epsilon}, \tilde{u}_{2 \epsilon}) \) be the solution of (5.8e) and (5.9) given in Proposition 5.9. Then
\[ \tilde{u}_\epsilon \rightharpoonup u_L + (u_R - u_L)H(\xi - s) + (0, e_0) \delta(\xi - s) \] (5.84)
in the sense of distributions as \( \epsilon \to 0 \).

**Proof.** Given any smooth function \( \psi \in \mathcal{C}_c^\infty(\mathbb{R}) \) with compact support, using Proposition 5.10 it can be readily seen that
\[ \int_{-\infty}^{\infty} \psi(\xi) \tilde{u}_\epsilon(\xi) \, d\xi = u_L \int_{-\infty}^{s} \psi(\xi) \, d\xi + u_R \int_{s}^{\infty} \psi(\xi) \, d\xi + (0, e_0) \psi(s) + o(1). \]
This holds for all \( \psi \), so (5.84) holds. \[ \square \]
**Proposition 5.12.** Let \( \tilde{u}_\epsilon = (\tilde{u}_{1\epsilon}, \tilde{u}_{2\epsilon}) \) be the solution of (5.8e) and (5.9) given in Proposition 5.9. Let \( u_\epsilon(x,t) = \tilde{u}_\epsilon(x/t) \). Then the weak convergence (5.18) holds.

**Proof.** Given any \( \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}_+) \), using Proposition 5.11 we have

\[
\int_0^\infty \int_{-\infty}^\infty \varphi(x,t)u_\epsilon(x,t) \, dx \, dt = \int_0^\infty t \int_{-\infty}^\infty \varphi(t\xi,t)\tilde{u}_\epsilon(\xi) \, d\xi \, dt \\
= \int_0^\infty t \left\{ u_L \int_{-\infty}^s \varphi(t\xi,t) \, d\xi + (0,e_0)\varphi(st,t) + u_R \int_s^\infty \varphi(t\xi,t) \, d\xi \right\} dt + o(1) \\
= u_L \int_0^\infty \int_{-\infty}^{st} \varphi(x,t) \, dx \, dt + u_R \int_0^\infty \int_{st}^\infty \varphi(x,t) \, dx \, dt + (0,e_0) \int_0^\infty t\varphi(st,t) \, dt + o(1).
\]

By the definition (5.19), this implies (5.18). \( \square \)

Proposition 5.9 and 5.12 complete the proof of the Main Theorem.
Bibliography


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