NEW ASYMPTOTIC METHODS FOR THE GLOBAL
ANALYSIS OF ORDINARY DIFFERENTIAL
EQUATIONS AND FOR NON-SELFADJOINT SPECTRAL
PROBLEMS

DISSERTATION

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ABSTRACT

In this thesis we employ new asymptotic methods to study some problems in classical analysis and mathematical physics. The first problem is finding the global behavior in \( \mathbb{C} \) of a class of analytic functions from the Taylor coefficients. Several explicit examples are presented, from which it is clear that combined with Laplace transformation and Borel transformation, the method has great potential in analyzing the solutions of differential equations. In the second problem we use methods originating from exponential asymptotics to study a non-selfadjoint spectral problem in mathematical physics. In the third problem, tronquée solutions and tritronquée solutions of the third and fourth Painlevé equation are studied using the theory of Borel summation for analytic nonlinear rank one system of ODE’s, which has been studied extensively in [2], [3], [5] and other works.
To my family
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Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

This thesis consists of four chapters. In each of Chapter 2, 3 and 4 a problem is studied using new asymptotic methods.

In the first chapter some basic definitions, propositions, lemmas and theorems that are necessary for later chapters are introduced.

In the second chapter we analyze the conditions on the Taylor coefficients of an analytic function to admit global analytic continuation, complementing a recent paper of Breuer and Simon ([24]) on general conditions for natural boundaries to form. A new summation method is introduced to convert a relatively wide family of infinite sums and local expansions into integrals. The integral representations yield global information such as analytic continuability, position of singularities, asymptotics for large values of the variable and asymptotic location of zeros.

In the third chapter we study the fundamental self-similar solution to the SU(2) sigma model, found by Shatah and Turok-Spergel. We give a rigorous proof for its mode stability. Based on earlier results by the second author Roland Donninger, the proof in this chapter constitutes the last building block for a completely rigorous proof of the nonlinear stability of the Shatah-Turok-Spergel wave map.
In the last chapter, we obtain global information about the tronquée and tritronquée solutions of the third and fourth Painlevé equations. We find their sectors of analyticity, their Borel summed representations in these sectors as well as the asymptotic position of the singularities near the boundaries of the analyticity sectors. We also correct slight errors in a recent paper on this subject by Lin, Dai and Tibboel.

1.2 Preliminaries

Most definitions and results in this section are taken from Prof. Costin’s book [1] with slight modifications.

1.2.1 Asymptotic expansions

Some basic definitions are listed below.

**Definition 1.2.1** (Asymptotic expansion). An asymptotic expansion at \( \infty \) is a formal series \( \sum_{k=0}^{\infty} f_k(t) \) of simple functions \( f_k(t) \) in which each successive term \( f_{k+1}(t) \) is much smaller than its predecessors \( f_k(t) \) as \( t \to \infty \), that is,

\[
\lim_{t \to \infty} \frac{f_{k+1}(t)}{f_k(t)} = 0
\]  

(1.2.1)

**Definition 1.2.2.** A function \( f \) is asymptotic to an asymptotic expansion \( \sum_{k=0}^{\infty} f_k(t) \) as \( t \to \infty \) if

\[
\lim_{t \to \infty} \frac{f(t) - \sum_{k=0}^{N} f_k(t)}{f_N(t)} = 0, \quad \forall N \in \mathbb{N}
\]  

(1.2.2)

**Definition 1.2.3** (Asymptotic power series). A function is asymptotic to a series as \( z \to 0 \), in the sense of power series if

\[
f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}), \quad (\forall N \in \mathbb{N}) \text{ as } z \to 0
\]  

(1.2.3)
1.2.2 Laplace transform

Laplace transformation is widely used in studying analytic functions and differential equations. Its definition and some basic properties are listed in this subsection.

**Definition 1.2.4.** Let $F \in L^1(\mathbb{R}^+)$, the Laplace transform of the function $F$ is defined to be a function $f(s)$ of a complex variable as follows:

$$(\mathcal{L}F)(s) = f(s) := \int_0^\infty F(t)e^{-st}dt$$  \hspace{1cm} (1.2.4)

**Proposition 1.2.5.** Let $F \in L^1(\mathbb{R}^+)$ and $f(s) = (\mathcal{L}F)(s)$. Then $f(s)$ is an analytic function in the open right half-plane $\mathbb{H} := \{ s : \text{Re}(s) > 0 \}$ and

$$f^{(k)}(s) = \int_0^\infty (-t)^k F(t)e^{-st}dt \quad \text{Re}(s) > 0$$ \hspace{1cm} (1.2.5)

The following theorem gives a set of sufficient conditions when the operation $\mathcal{L}$ can be inverted.

**Theorem 1.2.1.** Assume that over the half-plane $\text{Re}(s) \geq a$, $f(s)$ is analytic and there exists $G_a(t) \in L^1(\mathbb{R})$ such that $|f(a + it)| \leq G_a(t)$. Then the following integral over the vertical line $\text{Re}(s) = a$

$$F(t) := \int_{a-i\infty}^{a+i\infty} f(s)e^{st}ds \hspace{1cm} (1.2.6)$$

exists and $\mathcal{L}^{-1}f := F$ is a continuous function for all $t \in \mathbb{R}$. Moreover, $F(t) = 0$ for $t < 0$ and $\mathcal{L}F = f$ at least for $\text{Re}(s) \geq a$.

Another set of sufficient conditions is also provided:

**Theorem 1.2.2.** (i) Assume $f$ is analytic in an open sector $\mathbb{H}_\delta = \{ s : |\text{arg} \, s| < \pi/2 + \delta \}$, $\delta \geq 0$ and is continuous on $\partial \mathbb{H}_\delta$, and that for some $K > 0$ and any $x \in \mathbb{H}_\delta$ we have

$$|f(s)| \leq K(|s|^2 + 1)^{-1}$$ \hspace{1cm} (1.2.7)
Then $L^{-1}f$ is well defined by

$$F(p) = (L^{-1}f)(p) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} f(s)e^{sp}ds$$

(1.2.8)

and $LF = f$. We have $\|L^{-1}f\|_\infty \leq K/2$ and $L^{-1}f \to 0$ as $p \to \infty$.

(ii) If $\delta > 0$, then $F = L^{-1}F$ is analytic in the sector $S = \{p \neq 0 : |\arg(p)| < \delta\}$.

In addition, $\sup_S |F| \leq K/2$ and $F(p) \to 0$ as $p \to \infty$ along rays in $S$.

Watson’s lemma is a vital tool in the study of formal and actual solutions of differential equations:

**Lemma 1.2.6** (Watson’s Lemma). Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants $\beta_i$ with $\Re(\beta_i) > 0$, $i = 1, 2$. Then, for $a \leq \infty$,

$$f(x) = \int_0^a e^{-xp}F(p)dp \sim \sum_{k=0}^{\infty} c_k \Gamma(k\beta_1 + \beta_2)x^{-k\beta_1-\beta_2}$$

(1.2.9)

along any ray $e^{i\theta}\mathbb{R}^+$ with $\cos(\theta) > 0$.

### 1.2.3 Transseries

The classical asymptotic power series introduced in §1.2.1 are useful in describing the asymptotic behavior of functions (along certain directions), but they have their shortcomings. A simple example illustrates well the incompleteness of the classical asymptotic power series.

Consider the differential equation

$$\frac{d}{dx} y(x) + y(x) = x^{-1}$$

(1.2.10)

The general solution of (1.2.10) is

$$y(x; C) = Ce^{-x} + e^{-x}Ei(x)$$

(1.2.11)
For each $C \in \mathbb{C}$ we have $y(x; C) \sim \sum_{k=0}^{\infty} k!x^{-k-1} \ (x \to \infty)$. This implies that the classical asymptotic power series are unable to distinguish between functions whose difference is exponentially small.

Transseries is a generalization of series. The class of transseries is the closure of series under a number of operations, including

- Algebraic operations: addition, multiplication and their inverses
- Differentiation and integration
- Composition and functional inversion
- Solution of fixed point problems of formally contractive mappings

For strict construction of transseries, please see Chapter 4 in [1].

1.2.4 Borel transform and Borel summation

It is straightforward to check that for $m > 1$, the inverse Laplace transform of $x^{-m}$ is

$$\mathcal{L}^{-1}x^{-m} = \frac{p^{m-1}}{\Gamma(m)}$$

**Definition 1.2.7.** The Borel transform of a formal series

$$\tilde{f} = \sum_{k=1}^{\infty} c_k x^{r-k}, \ r \in (0, 1) \quad (1.2.12)$$

is by definition the formal series

$$B\tilde{f} = : = p^{-r} \sum_{k=0}^{\infty} \frac{c_{k+1}}{\Gamma(k-r+1)} p^k \quad (1.2.13)$$

A formal series $\tilde{f}$ is said to be Borel summable along a ray $e^{i\phi}\mathbb{R}^+$ if the following conditions are satisfied:
(a) The series $\mathcal{B}\tilde{f}$ has a nonzero radius of convergence;

(b) The analytic function $F$ to which $\mathcal{B}\tilde{f}$ converges can be analytically continued along the ray $e^{i\phi}\mathbb{R}^+$ and

(c) The analytic continuation $F$ grows at most exponentially along the ray and is therefore Laplace transformable along $e^{i\phi}\mathbb{R}^+$.

Under these circumstances, Borel summation in the classical sense is defined as follows:

**Definition 1.2.8.** The Borel sum of a series $\tilde{f}$ in the direction $\phi$ (arg $x = \phi$), $(\mathcal{L}\mathcal{B})_\phi \tilde{f}$ is by convention, the Laplace transform of $\mathcal{B}\tilde{f}$ along the ray $xp \in \mathbb{R}^+$, that is arg($p$) = $-\phi$:

$$(\mathcal{L}\mathcal{B})_\phi \tilde{f} = \int_0^\infty e^{-px} F(p) dp = \mathcal{L}_{-\phi} F = \mathcal{L} F(e^{-i\phi}) \quad (1.2.14)$$

**1.2.5 Banach space and Contractive mapping theorem**

A Banach space is a complete normed vector space. Examples of Banach spaces include the following:

- Euclidean spaces $\mathbb{R}^n$ with Euclidean norm,

- Given a measure space $(X, \mathcal{M}, \mu)$ and $1 \leq p \leq \infty$, define norm

  $$\|f\|_p = \left[ \int_X |f|^p d\mu \right]^{1/p} \quad p \in [1, \infty)$$

  $$\|f\|_\infty = \inf \{ a \geq 0 : \mu(\{ x : |f(x)| > a \}) = 0 \}$$

  Then

  $$L^p(X, \mathcal{M}, \mu) := \{ f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty \}$$

  is a Banach space.
• The space $C([a,b])$ of continuous function on the interval $[a,b]$ equipped with the uniform norm $\| \cdot \|_u$.

Note that in the second example if $\mu$ is the counting measure then $L^p(X, M, \mu)$ is also denoted by $l^p(X)$. In particular if $X = \mathbb{N}$, then $l^p$ is a space of sequences with norm $\| \cdot \|_p$.

Let $\mathcal{B}$ be a Banach space and $S$ be a closed subset of $\mathcal{B}$. In the induced topology, $S$ is also a Banach space with the same norm. Assume $\mathcal{M} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\| \mathcal{M}(y) - \mathcal{M}(x) \| \leq \lambda \| y - x \|$$

with $\lambda < 1$. Such operators are called contractive. A powerful tool for error estimation in the study of differential equations is the contractive mapping theorem.

**Theorem 1.2.3.** Assume $\mathcal{M} : S \mapsto S$, where $S$ is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation

$$x = \mathcal{M}(x)$$

has a unique solution in $S$.

### 1.2.6 Linear difference equations and Pincherle’s Theorem

Consider a class of $k$th-order linear homogeneous difference equation

$$x(n+k) + p_1(n)x(n+k-1) + \ldots + p_k(n)x(n) = 0$$

where $p_i(n)$ satisfy

$$\lim_{n \to \infty} p_i(n) = p_i \in \mathbb{R}, \quad 1 \leq i \leq k.$$
An equation of the form (1.2.17), (1.2.18) is called a difference equation of Poincaré type ([10]). The characteristic equation associated with (1.2.17) is

\[ \lambda^k + p_1 \lambda^{k-1} + \ldots + p_k = 0. \]  

(1.2.19)

We have the following results ([10]):

**Theorem 1.2.4** (Poincaré’s Theorem). Suppose that condition (1.2.18) holds and the characteristic roots \( \lambda_1, \lambda_2, \ldots, \lambda_k \) of (1.2.19) have distinct moduli. If \( x(n) \) is a solution of (1.2.17), then either \( x(n) = 0 \) for all large \( n \) or

\[ \lim_{n \to \infty} \frac{x(n+1)}{x(n)} = \lambda_i \]  

(1.2.20)

for some \( i, 1 \leq i \leq k \).

**Theorem 1.2.5** (Perron’s First Theorem). Assume that \( p_k(n) \neq 0 \) for all \( n \in \mathbb{Z}^+ \) and the assumptions of Theorem 1.2.4 hold. Then (1.2.17) has a fundamental set of solutions (i.e. a set of \( k \) linearly independent solutions) \( \{x_1(n), x_2(n), \ldots, x_k(n)\} \) such that

\[ \lim_{n \to \infty} \frac{x_i(n+1)}{x_i(n)} = \lambda_i, \quad 1 \leq i \leq k. \]  

(1.2.21)

**Definition 1.2.9** (Continued fraction). Let \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=0}^\infty \) be two sequences of complex numbers. A continued fraction is of the form

\[ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \]  

(1.2.22)

or, in a compact notation,

\[ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \]  

(1.2.23)
Define the $n$th approximant of the continued fraction (1.2.22) by its truncation

$$C(n) = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n}$$

(1.2.24)

Then the continued fraction (1.2.22) is said to converge if there exists a number $L \in \mathbb{C}$ such that

$$\lim_{n \to \infty} C(n) = L.$$

Consider a second-order difference equation of the following form,

$$x(n) + p_1(n)x(n-1) + p_2(n)x(n-2) = 0$$

(1.2.25)

A solution $\phi(n)$ of (1.2.25) is called a minimal solution if for any solution $x(n)$ of (1.2.25) that is not a multiple of $\phi(n)$ we have

$$\lim_{n \to \infty} \phi(n)/x(n) = 0.$$

Assume in addition that $p_2(n) \neq 0$ for $n \in \mathbb{Z}^+$. We have the following theorem

**Theorem 1.2.6** (Pincherle). The continued fraction

$$\frac{p_2(1)}{p_1(1)} - \frac{p_2(2)}{p_1(2)} - \frac{p_2(3)}{p_1(3)} - \ldots, \quad n = 1, 2, 3\ldots$$

(1.2.26)

converges if and only if (1.2.25) has a minimal solution $\phi(n)$, with $\phi(0) \neq 0$. In case of convergence, moreover, one has

$$-\frac{\phi(n-1)}{\phi(n-2)} = \frac{p_2(n)}{p_1(n)} - \frac{p_2(n+1)}{p_1(n+1)} - \frac{p_2(n+2)}{p_1(n+2)} - \ldots$$

(1.2.27)

For proofs of the theorems in this section and more details please see [10].
Chapter 2

FROM TAYLOR SERIES OF ANALYTIC FUNCTIONS
TO THEIR GLOBAL ANALYSIS

2.1 Introduction

Finding the global behavior of an analytic function in terms of its Taylor coefficients is a notoriously difficult problem, in fact one which is impossible in full generality since undecidable statements can be formulated in these terms. However, a very interesting and quite general criterion for the disk of convergence of a Taylor series to coincide with its maximal domain of analyticity was recently discovered by Breuer and Simon [24] (see also [27]). The present paper complements this result by finding criteria on the Taylor coefficients, say at zero, for the associated analytic function not to have natural boundaries and to belong to the class $\mathcal{M}$ of functions analytic in the complex plane with finitely many cuts and with algebraic behavior at infinity (see Definition 2.2.1 below). To the best of our knowledge, our approach is new (see however footnote 2 on p. 5).

Our condition is that the coefficients $c_k$ admit generalized Borel summable (or Ecalle-Borel summable, EB) transseries in $k$. Many general classes of problems in analysis are known to have EB transseries solutions. For the general theory of generalized Borel summability, transseries and resurgence see [26, 30, 32, 33, 34].

In particular it is known [25, 32, 26] that, if the $c_k$ are solutions of generic linear
or nonlinear recurrence relations of arbitrary but finite order with analytic coefficients, then they are EB-summable. Recurrence relations exist for instance when the coefficients are obtained by solving differential equations by power series. In a forthcoming paper we show that the Taylor coefficients of the Borel transform of solutions of generic systems of linear or nonlinear ODEs (in the setting of [41]) also admit EB summable transseries. The global analytic structure of the Borel transform is crucial in understanding the monodromy of solutions of such equations.

We also extend our procedure to analyze the global behavior of entire functions, and to formal series, giving criteria directly on the coefficients for the formal series to be Borel summable.

Globally reconstructing function from its Taylor coefficients, when these admit EB summable transseries is effective, constructive and explicit -in the sense of producing integral representations far easier to analyze than the sums; this provides a new summation procedure, generalizing in some ways the Poisson summation formula.

We recently used this approach to analyze a class of linear PDEs with variable coefficients, [29]. One can obtain explicit integral representations for solutions of ODEs not known to be solvable such as

\[ A\eta^2 f^{(4)} + 2A\eta f''' + \frac{1}{2}\eta f'' - (1 + a)f = 0 \]  

(2.1.1)

arising as the scaling pinching profile \( h \sim (t_c - t)f(x(t_c - t)^{-1/2}) \) of the thin film equation,

\[ h_t + (h_{xxx}h)_x = 0, \ h \sim (t_c - t)f(x(t_c - t)^{-1/2}) \]

where \( t_c \) is the singularity time, where the interest is in the solution analytic at zero, \( f_0 \). Let \( A \neq 0 \), and \( a + \frac{1}{2} \notin \mathbb{N} \). While it is not clear how to obtain representations of \( f_0 \) itself, the Taylor coefficients of \( f_0 \) are explicit. From this, our technique introduced in
§2.2.1 by first considering the Laplace transform \( F(p) = \mathcal{L}f(p) \), which is expressible in terms of integrals of Whittaker functions: with
\[
F(p) = Ce^{-\frac{p^2}{8A}} p^{-3/2} \left[ M_{-a-\frac{3}{4}, \frac{1}{4}} \left( \frac{p^2}{4A} \right) - \frac{a\Gamma(a)}{2\sqrt{\pi}} W_{-a-\frac{3}{4}, \frac{1}{4}} \left( \frac{p^2}{4A} \right) \right]
\]
we have \( f = \mathcal{L}^{-1}F \). The proof of (2.1.2) is sketched in §2.4.3.

In particular, if \( c_k = \varphi(k) \) where the function \( \varphi \), defined in the right half plane, is inverse Laplace transformable and its inverse Laplace transform \( \mathcal{L}^{-1}\varphi \) can be calculated in closed form, the function \( f \) has integral representations in terms of \( \varphi \). We will use some particularly simple examples for illustration. For the first one, the generalized Hurwitz zeta function, our procedure quickly yields one of the known integral representations. For the other three, our procedure yields integral representations while the global behavior of \( \sum_k c_k z^k \) does not follow in any other obvious way:
\[
c_k^{[1]} = \frac{1}{(k+a)^b}; \quad c_k^{[2]} = \frac{1}{k^b + \ln k}, \quad c_k^{[3]} = \frac{1}{k^{a+1}}, \quad c_k^{[4]} = e^{\sqrt{k}}, \quad (a, b > 0) \quad (2.1.3)
\]
We find that
\[
f_1(z) := \sum_{k=1}^{\infty} c_k^{[1]} z^k = \frac{z}{\Gamma(b)} \int_0^\infty \frac{[\ln(1+t)]^{b-1} dt}{(1+t)^{a+1}(t-(z-1))}
\]
(2.1.4)

On the first Riemann sheet \( f_1 \) has only one singularity, at \( z = 1 \), of logarithmic type, and \( f_1 = o(z) \) for large \( z \). General Riemann surface information and monodromy follow straightforwardly from (2.1.4). A similar complex analytic structure is shared by \( f_2(z) = \sum_{k=1}^{\infty} c_k^{[2]} z^k \), which has one singularity at \( z = 1 \) where it is analytic in \( \ln(1-z) \) and \( (1-z) \); more precisely,
\[
f_2(z) = -\frac{1}{2\pi i} \ln \ln z \int_0^\infty \frac{e^{-u \ln(z)}}{(-u)^b + \ln(-u)} du + E(z)
\]
see Definition 2.2.2, where \( E \) is entire.
The function \( f_3(z) = \sum_{k=1}^{\infty} c_k^3 z^k \) is entire; questions answered regard say the behavior for large negative \( z \) (certainly not obvious from the series) or the asymptotic location of zeros. It will follow that \( f_3 \) can be written as

\[
f_3(z) = \int_0^\infty (1 + u)^{-1} G(\ln(1 + u)) \left[ \exp \left( \frac{z e^{-1}}{1 + u} \right) - 1 \right] \, du \quad (2.1.6)
\]

where \( G(p) = s_2'(1+p) - s_1'(1+p) \) and \( s_{1,2} \) are two branches of the functional inverse of \( s - \ln s \), cf. §2.4.2. Using the integral representation of \( f_3 \), its behavior for large \( z \) can be obtained from (2.1.6) by standard asymptotics methods; in particular, for large negative \( z \), \( f_3 \) behaves like a constant plus \( z^{-1/2} e^{-z/e} \) times a factorially divergent series (whose terms can be calculated).

For \( c_k^4 \) we find

\[
f_4(z) = \sum_{k=1}^{\infty} c_k^4 z^k = -\frac{z}{2\sqrt{\pi}} \int_{C_1} \frac{p^{-3/2} e^{-\frac{1}{4} p \theta}}{e^p - z} \, dp \quad (2.1.7)
\]

where \( C_1 \) is a spiral \( S_1 \) followed by \([1, \infty)\), where \( S_1 \) starts at 0 and ends at 1, and is given in polar coordinates by \( r = \theta e^{2\pi i \theta} \), \((\theta \in [0, 1])\).

We also show that Borel summation of divergent series or transseries of resurgent functions with finitely many Borel-plane singularities, as well as the Abel-Plana version of the Euler-Maclaurin summation formula (see also [28]) can be derived by a natural extension of our analysis. Another illustration is obtaining the closed form Borel summed formula for \( \ln \Gamma \), cf. (2.2.25) below.

A separate category is represented by lacunary series. Their coefficients do not satisfy our assumption; however a slightly different approach allows for a detailed study of the associated functions as the natural boundary is approached, [31].
2.2 Main results

A first class of problems is finding the location and type of singularities in $\mathbb{C}$ and the behavior for large values of the variable of functions given by series with finite radius of convergence (Theorem 2.2.1), such as the first, second and fourth in (2.1.3).

The second class of problems amenable to the techniques presented concerns the behavior at infinity (growth, decay, asymptotic location of zeros etc.) of entire functions presented as Taylor series (Theorem 2.2.2).

The third class of problems is essentially the converse of the two above: given a function that has analytic continuation on some Riemann surface, how is this reflected on $c_k$? (Theorem 2.2.1.)

The fourth class of problems is to determine Borel summability of series with zero radius of convergence such as

$$\tilde{f}_5 = \sum_{n=0}^{\infty} n^{n+1} z^n \quad (2.2.1)$$

in which the coefficients of the series are analyzable (Theorem 2.2.3).

**Definition 2.2.1.** Let $\{a_j : 1 \leq j \leq N\}$ be a set of nonzero complex numbers with distinct arguments. Let $\mathcal{M}$ consist of the functions algebraically bounded at $\infty$ and analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{N} \{a_j t : t \geq 1\}$. By dividing by a power of $z$ and subtracting out the principal part (i.e., the negative powers of $z$) we can assume that $f \in \mathcal{M}' = \{f \in \mathcal{M} : f(z) = o(z) \text{ as } |z| \to \infty\}$.

This is one of the simplest settings often occurring in applications. We can see later from the proof that the approach is more general.

**Definition 2.2.2.** Assume $g(s)$ is analytic in $U_\delta \setminus [0, \infty)$ for some $\delta > 0$, where $U_\delta = \{z : |\text{Im}(z)| \leq \delta, \text{Re}(z) \geq -\delta\}$ and $g(s) \to 0$ uniformly in $U_\delta$, as $\text{Re}(s) \to \infty$. Assume $\epsilon \leq \delta$. We define $\Gamma_\epsilon$ to be the contour around $\mathbb{R}^+$ consisting of two rays $l_{1,\epsilon}, l_{2, \epsilon}$
and a semicircle $\gamma_\epsilon$, where $l_{1,\epsilon} = \{x - \epsilon i : a \in [0, \infty)\}$ oriented towards the left, $l_{2,\epsilon} = \{x + \epsilon i : a \in [0, \infty)\}$ oriented towards the right; $\gamma_\epsilon$ is the left semicircle centered at origin oriented clockwise. Assume also that $g(s)$ is absolutely integrable over $\Gamma_\epsilon$ for some $\epsilon$. We denote by
\[
\oint_0^\infty g(s) ds
\]
(2.2.2)
the integral of $g$ over $\Gamma_\epsilon$. Since $g(s)$ vanishes at $\infty$, the integral is independent of the choice of $\epsilon$ as long as it is small enough.

The following observations will simplify our proofs.

**Note 2.2.3.** Let $g(s)$, $U_\delta$ and $\Gamma_\epsilon$ be as in Definition 2.2.2. $\Gamma_\epsilon$ separates $\mathbb{C} \setminus \Gamma_\epsilon$ into two regions. We denote the region containing $\mathbb{R}^+$ by $S_1$ and the other by $S_2$. Let
\[
G_1(z) = \int_{\Gamma_\epsilon} \frac{g(s) ds}{s - z} \quad (z \in S_1)
\]
(2.2.3)
\[
G_2(z) = \int_{\Gamma_\epsilon} \frac{g(s) ds}{s - z} \quad (z \in S_2)
\]
(2.2.4)

Then $G_1$ is analytic in $S_1$ and $G_2$ is analytic in $S_2$. By slightly deforming $\Gamma_\epsilon$ we are able to see that each $G_i$ can be analytically continued to $S_i \cup \Gamma_\epsilon$, $i = 1, 2$. On $\Gamma_\epsilon$ their analytic continuations satisfy
\[
G_2(z) - G_1(z) = 2\pi i g(z) \quad (z \in \Gamma_\epsilon)
\]
(2.2.5)
Hence $G_2(z)$ can be analytically continued to $\mathbb{C} \setminus [0, \infty)$ and $G_1(z)$ can be analytically continued to at least $U_\delta$ and in regions where $g$ is analytic. For each $z \in \mathbb{C} \setminus [0, \infty)$
\[
G_2(z) = \oint_0^\infty \frac{g(s) ds}{s - z}
\]
(2.2.6)

**Note 2.2.4.** A representation of the form (2.2.2) exists for Hilbert-transform-like integrals such as $h(t) = \int_0^\infty (s - t)^{-1} H(s) ds$ with $H$ analytic at zero, for instance $h(t) = -(2\pi i)^{-1} \oint_0^\infty (s - t)^{-1} H(s) \ln s \, ds$. 

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Note 2.2.5. Consider the composition of $g$ with $s \mapsto \ln(1 + s)$, the branch cut of which is chosen to be $(-\infty, -1]$. If $g$ is analytic in $U_\delta \setminus [0, \infty)$, then $g(\ln(1 + s))$ is analytic in the set $-1 + \exp(U_\delta \setminus [0, \infty))$. If in addition we have the decay condition $g(\ln(1 + s)) = o(|s|^{-\alpha})$ for some $\alpha > 0$ as $|s| \to \infty$, then there exists a $\tilde{\delta}$ small enough such that $U_\delta \subseteq -1 + \exp(U_\delta \setminus [0, \infty))$. It is easy to see from the decay condition that

$$\int_{\Gamma_\delta} g(p) dp = \int_{\exp(\Gamma_\delta) - 1}^\infty \frac{g(\ln(1 + s))}{1 + s} ds = \int_{\Gamma_{\tilde{\delta}}} g(\ln(1 + s)) ds$$

and thus we can make the change of variable

$$\int_0^\infty g(p) dp = \int_0^\infty \frac{g(\ln(1 + s))}{1 + s} ds \quad (2.2.7)$$

While providing integral formulae in terms of functions with known singularities which are often rather explicit, the following result can also be interpreted as a duality of resurgence. 2.

Theorem 2.2.1. (i) Assume that $f(z) = \sum_{k=0}^\infty c_k z^k$ is a series with positive, finite radius of convergence, with $c_k$ having Borel sum-like representations of the form

$$c_k = \sum_{j=1}^N a_j^{-k} \int_0^\infty e^{-kp} F_j(p) dp \quad (k \geq 1) \quad (2.2.8)$$

with $a_j$ as in Definition 2.1, $F_j$ analytic in $U_\delta \setminus [0, \infty)$ for some $\delta > 0$ and algebraically bounded at $\infty$. Then, $f$ is given by

$$f(z) = f(0) + z \int_0^\infty \sum_{j=1}^N \frac{F_j(\ln(1 + s)) ds}{(1 + s)((1 + s)a_j - z)} \quad (2.2.9)$$

(ii) Furthermore, $f \in \mathcal{M}'$. The behavior of $f$ at $a_j$ and is of the same type as the behavior of $F_j(\ln(1 + s))$ at 0. More precisely, for small $z \notin [0, \infty)$,

$$f(a_j(z + 1)) = 2\pi i F_j(\ln(1 + z)) + G(z) \quad (2.2.10)$$

2 After developing these methods, it has been brought to our attention that a duality between resurgent functions and resurgent Taylor coefficients has been noted in an unpublished manuscript by Écalle. This will be further explored in a forthcoming paper.
where $G(z)$ is analytic at 0.

(iii) Conversely, assume $f \in \mathcal{M}'$, and has finitely many singularities located at 
\( \{a_j t_{j,l}\} \), \( 1 \leq j \leq N, 1 \leq l \), with 1 = $t_{j,1}$ and $t_{j,l}$ < $t_{j,l+1}$ for all $j,l$. Let 
$c_k = f^{(k)}(0)/k!$; then $c_k$ have Borel sum-like representations of the form

\[
c_k = \frac{1}{2\pi i} \sum_{j=1}^{N} (a_j)^{-k} \int_{0}^{\infty} e^{-ks} f(a_j e^s) \, ds, \quad k \geq 1
\]  

(2.2.11)

The behavior at $a_j$ and at $\infty$ will follow from the proof.

As it will be clear from the proofs, the method and results would apply, with minor adaptations to functions of several complex variables.

### 2.2.1 Entire functions

We restrict the analysis to entire functions of exponential order one, with complete information on the Taylor coefficients. Such functions include of course the exponential itself, or expressions such as $f_3$. It is useful to start with $f_3$ as an example. The analysis is brought to the case in Theorem 2.2.1 by first taking a Laplace transform. Note that

\[
\int_{0}^{\infty} e^{-xz} f(z) \, dz = \frac{1}{x} \sum_{n=1}^{\infty} \frac{n!}{n^{n+1} x^n}
\]  

(2.2.12)

The study of entire functions of exponential order one likely involves the factorial, and then a Borel summed representation of the Stirling formula is needed; this is provided in the Appendix.

**Theorem 2.2.2.** Assume that the entire function $f$ is given by

\[
f(z) = \sum_{k=1}^{\infty} \frac{c_k z^k}{k!}
\]  

(2.2.13)

with $c_k$ as in Theorem 2.2.1 (i). Then,

\[
f(z) = \int_{0}^{\infty} \sum_{j=1}^{N} \left[ \left( e^{a_j z + s} - 1 \right) \frac{F_j(\ln(1+s))}{(1+s)} \right] \, ds
\]  

(2.2.14)
As in the simple example, the behavior at infinity follows from the integral representation by classical means.

### 2.2.2 Borel summation

We obtain from Theorem 2.2.1, in the same way as above, the following.

**Theorem 2.2.3.** Consider the formal power series

\[
\tilde{f}(z) = \sum_{k=1}^{\infty} c_k k! x^{-k-1}
\]  

(2.2.15)

with coefficients \(c_k\) as in Theorem 2.2.1 (i). Then the series (2.2.15) is (generalized) Borel summable to

\[
\int_0^\infty dp e^{-pz} \sum_{j=1}^{N} \int_0^{\infty} \frac{F_j(\ln(1+s))}{(1+s)(a_j s + a_j - p)} ds
\]

\[= \sum_{j=1}^{N} \int_0^{\infty} \frac{F_j(\ln(1+s))}{1+s} \left(-\frac{1}{x} + a_j(s+1)e^{-a_j(s+1)x} \text{Ei}(a_j(s+1)x)\right) ds
\]  

(2.2.16)

The proof proceeds as in the previous sections, taking now a Borel transform followed by Laplace transform.

### 2.2.3 Other applications; the examples in the introduction

**Other growth rates**

Series with coefficients with growth rates precluding a straightforward inverse Laplace transform can be accommodated, for instance by analytic continuation. We have for positive \(\gamma\),

\[e^{-\gamma \sqrt{\pi}} = \frac{\gamma}{2\sqrt{\pi}} \int_0^{\infty} p^{-3/2} e^{-\frac{p^2}{2\gamma}} e^{-np} dp \]  

(2.2.17)

which can be analytically continued in \(\gamma\). We note first that the contour cannot be, for this function, detached from zero. Instead, we keep \([1, \infty)\) as part of the
original contour fixed and, deform the part \([0, 1]\) by simultaneously rotating \(\gamma\) and \(p\) to maintain \(\gamma^2/p\) real and positive near the origin. We get

\[
e^{\sqrt{n}} = -\frac{1}{2\sqrt{\pi}} \int_{C_1} p^{-3/2}e^{-\frac{1}{4p}}e^{-np}dp \tag{2.2.18}
\]

and (2.1.7) follows, for the same reason Theorem 2.2.1 (i) holds. In particular,

\[
\lim_{z \to -1^+} \sum_{n=1}^{\infty} e^{\sqrt{n}z^n} = \frac{1}{2\sqrt{\pi}} \int_{C_1} p^{-3/2}e^{-\frac{1}{4p}}e^{-p}dp \tag{2.2.19}
\]

The sum (2.2.19) is unwieldy numerically, while the integral (2.2.17) can be evaluated accurately by standard means. In a similar way we get

\[
\sum_{k=0}^{\infty} \frac{e^{i \sqrt{k}}}{k^a} = -\gamma 2^{a-1/2}/\sqrt{\pi} \int_{C} e^{-\frac{1}{4p}}U(2a + 1/2; \frac{1}{\sqrt{2p}})p^{a-1}(e^p + 1)dp \tag{2.2.20}
\]

for \(a > 1/2\) for which the series converges. Here \(C\) is a contour consisting of \(S_2\) followed by \([1, \infty)\), where \(S_2\) starts at 0 and ends at 1, and is given by \(r = 1 - \theta/\pi\), \((\theta \in [0, \pi])\) in polar coordinates. \(U\) is the parabolic cylinder function [23].

The coefficients \(c_k^{[1]}\) in (2.1.3). We have

\[
\mathcal{L}^{-1} \left[ \frac{1}{(n + a)^b} \right] = \Gamma(b)^{-1}p^{b-1}e^{-ap} \tag{2.2.21}
\]

The rest follows in the same way (2.1.7) was obtained, after changing variables to \(1 + t = e^p\).

The coefficients \(c_k^{[2]}\). We let \(x = n\) and take the inverse Laplace transform in \(x\):

\[
G(p) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{xp} \frac{dx}{x^b + \ln x} \tag{2.2.22}
\]

where the contour can be bent backwards for \(p \in \mathbb{R}^+\), to hang around \(\mathbb{R}^-\). Then, with the change of variable \(x = -u\) (2.2.22) becomes

\[
G(p) = \frac{1}{2\pi i} \int_{0}^{\infty} e^{-up} \frac{e^{-up}}{(-u)^b + \ln(-u)}du \tag{2.2.23}
\]

and thus

\[
c_k = (\mathcal{L}G)(k) = \int_{0}^{\infty} G(p)e^{-kp}dp = \int_{0}^{\infty} \left[ \frac{-G(p)\ln p}{2\pi i} \right] e^{-kp}dp
\]
We see that $F_1(p) = (-G(p) \ln p)/2\pi i$ and by Theorem 2.1

$$f_2(z) = z \oint_0^\infty \frac{\tilde{G}(s)}{s - (z - 1)} ds$$

where $\tilde{G}(s) = F_1(\ln(1 + s))/(1 + s)$. Hence the singularity of $f_2(z)$, at $z = 1$ on the first Riemann sheet, according to Note 2.2.3 is that of $\phi(z) = 2\pi i \tilde{G}(z - 1)$, as in (2.1.5).

The example of the coefficients $c^{[3]}_k$ is studied in a similar way as Theorem 2.2.2; related calculations can be found in §2.4.2.

The coefficients $c^{[4]}_k$ were treated at the beginning of this section.

Another example is provided by the log of the Gamma function, $\ln \Gamma(n) = \sum_{k=1}^{n} \ln k$.

It is convenient to first subtract out the leading behavior of the sum to arrange that the summand is inverse Laplace transformable. With

$$g_n = \ln \Gamma(n + 1) - \left((n + 1) \ln(n + 1) - n - \frac{1}{2} \ln(n + 1)\right)$$

we get

$$g_N = \sum_1^{N} \left[1 - \left(\frac{1}{2} + n\right) \ln \left(1 + \frac{1}{n}\right)\right]$$

$$= \sum_1^{N} \int_0^\infty e^{-np} \frac{1 - \frac{p}{2} - \left(p + 1\right)e^{-p}}{p^2} dp$$

(2.2.24)

where $\mathcal{L}^{-1}$ of the summand in the middle term is most easily obtained by noting that its second derivative is a rational function. Summing as usual $e^{-np}$ we get

$$\ln \Gamma(n) = n(\ln n - 1) - \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) +$$

$$\int_0^\infty \frac{1 - \frac{p}{2} - \left(p + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-np} dp$$

(2.2.25)

Obviously, if the behavior of the coefficients is of the form $A^k c_k$ where $c_k$ satisfies the conditions in the paper, one simply changes the independent variable to $z' = Az$. 20
2.3 Proof of Theorem 2.2.1

If $f \in \mathcal{M}'$ we write the Taylor coefficients in the form

$$c_k = \frac{1}{2\pi i} \oint \frac{f(p)dp}{p^{k+1}} \quad (k \geq 1) \quad (2.3.1)$$

where the contour of integration is a small circle of radius $r$ around the origin. We attempt to increase $r$ without bound. In the process, the contour will hang around the singularities of $f$ as shown in Figure 1. Each integral over a curve that wraps around a ray $\{a_je^t : t \geq 1\}$ converges by the decay assumptions and the contribution of the arcs at large $r$ vanishes, since $f(z) = o(z)$ as $z \to \infty$. To be more precise, let $C_{j,\epsilon}$ be the part of the image of $\Gamma_\epsilon$ under the mapping $s \to a_je^s$, let $C_{j,\epsilon,R}$ be the part of $C_{j,\epsilon}$ inside the disk $|s| \leq R$, and $C_R$ be the part of the contour on $|s| = R$. Then
for $\epsilon$ small enough and $R$ large enough we have

$$c_k = \frac{1}{2\pi i} \oint f(p)dp = \frac{1}{2\pi i} \left( \sum_{j=1}^{N} \int_{C_j, R} \frac{f(p)dp}{p^{k+1}} + \int_{C_R} \frac{f(p)dp}{p^{k+1}} \right)$$

By the change of variable $p = a_j e^{s}$ and letting $R \to \infty$ we get

$$\int_{a_jC_j, R} \frac{f(p)dp}{p^{k+1}} = \int_{\Gamma} e^{-kp} F_j(p)dp = \int_{\Gamma} e^{-kp} F_j(p)dp$$

and the integral over $C_R$ vanishes as $R \to \infty$ by decay condition for $k \geq 1$.

In the opposite direction, first let $\epsilon$ be small enough so that for all $j$ and $k = 1$

$$\int_{0}^{\infty} e^{-kp} F_j(p)dp = \int_{\Gamma} e^{-kp} F_j(p)dp$$

Then for all $1 \leq j \leq N, k \geq 1$ (2.3.2) is true. Also let $z$ be small so that

$$|a_j^{-1} e^{-p} z| \leq \delta^2 < 1$$

for all $j$ and $p \in \Gamma_c$.

Then, by the dominated convergence theorem (which applies in this case, see (2.3.5)) we have

$$f(z) - f(0) = \sum_{k=1}^{\infty} c_k z^k = \sum_{k=1}^{\infty} \sum_{j=1}^{N} \left( \int_{\Gamma} e^{-kp} F_j(p)dp \right) z^k$$

as stated. The third equality holds because we have, in view of (2.3.3),

$$\left| \sum_{j=1}^{N} (a_j^{-1} e^{-p} z)^k F_j(p) \right| \leq \sum_{j=1}^{N} |a_j^{-1} e^{-p} z|^k \left( |a_j^{-1} e^{-p} z|^k |F_j(p)| \right)$$

$$\leq \sum_{j=1}^{N} \delta^k \left( |a_j^{-1} e^{-p} z|^k |F_j(p)| \right)$$

(2.3.5)
For each $j$, $|a_j^{-1}e^{-pz}|^{h/2}F_j(p)$ is integrable over $\Gamma_\epsilon$ since $F_j$ is algebraically bounded at $\infty$, so we may interchange the order of integration and summation over $k$. The last equality holds because $F_j((1+s))/(sa_j + a_j - z) = o(|s|^{\alpha})$ for some $\alpha > 1$ so we can make the change of variable $p = \ln(1+s)$, see Note 2.2.5. Hence (2.2.8) holds for $z$ small. Given $j \in \{1, \ldots, N\}$ we may write

$$I_j(z) := \oint_0^\infty \frac{F_j(\ln(1+s))}{(1+s)(sa_j + a_j - z)} \frac{(a_j(1+s))}{(s - (z/a_j - 1))} \, ds$$

(2.3.6)

Since $F_j$ is analytic in $C \setminus [0, \infty)$ and is algebraically bounded at $\infty$, the function $F_j(\ln(1+s))/\left(\frac{a_j(1+s)}{(s - (z/a_j - 1))}\right)$ is analytic in $C \setminus [0, \infty)$ and vanishes uniformly as $\text{Re}(s) \to \infty$. Then it becomes obvious from Note 2.2.3 that the integral $I_j$ in (2.3.6) is analytic in $C \setminus \{a_j t : t \geq 1\}$. Thus $f(z)$ can be analytically continued to $C \setminus N \cup \{a_j t : t \geq 1\}$.

To see that $f(z) = o(z)$ as $|z| \to \infty$, it suffices to show that for each $j$, $I_j(z) = o(1)$. Assume $3\epsilon < \delta$. We use the contours $\exp(\Gamma_\epsilon) - 1$ and $\exp(\Gamma_{3\epsilon}) - 1$. If $|\arg(z/a_j)| \geq 2\epsilon$ and $|z/a_j|$ is large enough we write:

$$I_j(z) = \oint_{\exp(\Gamma_\epsilon) - 1} \frac{F_j(\ln(1+s))}{\left(\frac{a_j(1+s)}{(s - (z/a_j - 1))}\right)} \, ds$$

(2.3.7)

Then there exist some positive number $\rho_1$ such that $|s - (z/a_j - 1)| \geq \rho_1|s|$ for each $s \in \exp(\Gamma_\epsilon) - 1$, so we can use dominated convergence to obtain $I_j(z) \to 0$ as $|z| \to \infty$.

If $|\arg(z/a_j)| \leq 2\epsilon$ we use Note 2.2.3 to write:

$$I_j(z) = \int_{\exp(\Gamma_{3\epsilon})} \frac{F_j(\ln(1+s))}{\frac{a_j(1+s)}{(s - (z/a_j - 1))}} \, ds + 2\pi i F_j(\ln(z/a_j))/z$$

(2.3.8)

Then there exist some positive number $\rho_2$ such that $|s - (z/a_j - 1)| \geq \rho_2|s|$ for each $s \in \exp(\Gamma_{3\epsilon}) - 1$, so we can use dominated convergence to prove the integral in the right hand side of (2.3.8) is $o(1)$. By assumption it is obvious that $F_j(\ln(z/a_j))/z = o(1)$, so in this case we also have $I_j(z) \to 0$ as $|z| \to \infty$. 

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The nature of the singularities of $f$ is derived from Note 2.2.3. Let $z \notin [0, \infty)$ be small, then for each $j \in \{1, ..., N\}$

$$f((z + 1)a_j)$$

$$= f(0) + ((z + 1)a_j) \sum_{l \neq j} I_l((z + 1)a_j) + ((z + 1)a_j)I_j((z + 1)a_j)$$

$$= f(0) + ((z + 1)a_j) \sum_{l \neq j} I_l((z + 1)a_j)$$

$$+ ((z + 1)a_j) \int_0^{\infty} \frac{F_j(\ln(1+s))/(a_j(1+s))}{s-z} \, ds$$

$$= f(0) + ((z + 1)a_j) \sum_{l \neq j} I_l((z + 1)a_j) + A(t) + 2\pi i F_j(\ln(1+z))$$

where $A(t)$ is analytic at $z = 0$. The last equality is obtained by Note 2.2.3. It is obvious from Note 2.2.3 that each $I_l((z + 1)a_j)$ ($l \neq j$) is analytic on $[0, \infty)$. Thus

$$f((z + 1)a_j) = \tilde{G}(z) + 2\pi i F_j(\ln(1+z)) \quad (2.3.9)$$

where $\tilde{G}(z)$ is analytic at $z = 0$. Hence (2.2.10) follows.

### 2.4 Appendix

#### 2.4.1 Simple integral representation of $1/n!$ [23]

We have

$$\frac{1}{\Gamma(z)} = -\frac{ie^{-\pi iz}}{2\pi} \int_0^{\infty} s^{-z} e^{-s} ds = -\frac{ie^{-\pi iz}z^{-z}}{2\pi} \int_0^{\infty} s^{-z} e^{-zs} ds \quad (2.4.1)$$

with our convention of contour integration. From here, one can proceed as in §2.4.2.
2.4.2 The Gamma function and Borel summed Stirling formula

We have

\[ n! = \int_0^\infty t^n e^{-t} dt = n^{n+1} \int_0^\infty e^{-n(s-\ln s)} ds \]

\[ = n^{n+1} \int_0^1 e^{-n(s-\ln s)} ds + n^{n+1} \int_1^\infty e^{-n(s-\ln s)} ds \quad (2.4.2) \]

On \((0, 1)\) and \((1, \infty)\) separately, the function \(s - \ln(s)\) is monotonic and we may write, after inverting \(s - \ln(s) = t\) on the two intervals to get \(s_{1,2} = s_{1,2}(t)\)

\[ n! = n^{n+1} \int_1^\infty e^{-nt}(s'_2(t) - s'_1(t)) dt = n^{n+1} e^{-n} \int_0^\infty e^{-np} G(p) dp \quad (2.4.3) \]

where \(G(p) = s'_2(1 + p) - s'_1(1 + p)\). From the definition it follows that \(G\) is bounded at infinity and \(p^{1/2}G\) is analytic in \(p\) at \(p = 0\). Using now \((2.4.3)\) and Theorem 2.2.1 in \((2.2.12)\) we get

\[ \int_0^\infty e^{-xz} f_3(z) dz = \frac{1}{x^2} \int_0^\infty \frac{G(\ln(1 + t))}{(te + (e - x^{-1}))(t + 1)} dt \quad (2.4.4) \]

Upon taking the inverse Laplace transform we obtain \((2.1.6)\).

2.4.3 Solution of \((2.1.1)\)

Assume the solution to \((2.1.1)\) which is analytic at \(\eta = 0\) has Taylor expansion

\[ f(\eta) = \sum_{k=0}^\infty c_k \eta^k. \]

Then

\[ c_k = \frac{c_1 (-1)^{k/2-1/2} \Gamma(k/2 - 1 - a)k}{\Gamma(-1/2 - a) (\Gamma(k + 1))^2} \quad (k \text{ is odd}) \]

\[ c_k = 0 \quad (k \text{ is even}) \quad (2.4.5) \]

\[ \text{(2.4.6)} \]

---

3The functions \(s_{1,2}\) are given by branches of \(-W(-e^t)\), where \(W\) is the Lambert function.
It is obvious that $f(\eta)$ is entire. Consider the Laplace transform $F(p) = \mathcal{L}f(p)$. Then $F(p) = \sum_{k=0}^{\infty} \Gamma(k+1)c_k p^{-k-1}$. Let $G(p) = F(1/p)$; then $G(z)$ is a solution to the differential equation

$$G''(z) + \left(-\frac{4}{z} + \frac{z}{2A}\right) G'(z) + \left(\frac{6}{z^2} - \frac{3/2 + a}{A}\right) G(z) = 0$$

(2.4.7)

and $G(z)$ is analytic at $z = 0$.

The normalization transformation $G(z) = z^{3/2} e^{-z^2/A} h(z^2/A)$ (cf. [41] for a general description of normalization methods) yields

$$h'' - \frac{7}{4} h' + \frac{1}{16} \left(12 - \frac{3 + 4a}{s} + \frac{3}{s^2}\right) h = 0$$

(2.4.8)

solvable in terms of Whittaker functions [23]. Substituting back, by straightforward algebra, this yields (2.1.2).
Chapter 3
A PROOF FOR THE MODE STABILITY OF A
SELF-SIMILAR WAVE MAP

3.1 Introduction

A map \( u : \mathbb{R}^{1,d} \rightarrow M \) from \((d + 1)\)-dimensional Minkowski space to a Riemannian manifold \( M \) is called a wave map if it satisfies

\[
\partial_\mu \partial^\mu u^a + \Gamma^a_{bc}(u) \partial_\mu u^b \partial^\mu u^c = 0
\]

where \( \Gamma^a_{bc} \) are the Christoffel symbols on \( M \). Wave maps are natural geometric non-linear generalizations of the wave equation and they attracted a lot of interest in the recent past. We refer the reader to the survey article [19] and the references in [11, 12] for more background.

In the special case \( d = 3 \) and \( M = S^3 \), the three-sphere, the wave maps equation reduces to the well-known SU(2) sigma model from particle physics [18, 20]. In hyperspherical coordinates on \( S^3 \) and for so-called co-rotational maps of the form \( u(t, r, \theta, \varphi) = (\psi(t, r), \theta, \varphi) \), where \( (t, r, \theta, \varphi) \) are the standard spherical coordinates on Minkowski space, the SU(2) sigma model is described by the single semilinear wave equation

\[
\psi_{tt} - \psi_{rr} - \frac{2}{r} \psi_r + \frac{\sin(2\psi)}{r^2} = 0.
\]
Eq. (3.1.1) is a supercritical wave equation and it exhibits finite-time blowup. This was demonstrated by Shatah [21, 14] who proved the existence of a self-similar solution to Eq. (3.1.1) which was later found in closed form by Turok and Spergel [22],

$$\psi^T(t, r) = 2 \arctan \left( \frac{r}{T - t} \right),$$

where $T$ is a free parameter (the blowup time). The relevance of $\psi^T$ for generic time evolutions depends on its stability. Numerical investigations by Bizoń, Chmaj and Tabor [13] suggest that $\psi^T$ describes a stable blowup regime.

### 3.1.1 Mode stability

A first, heuristic stability analysis (which was already performed in [13]) addresses the question of *mode stability* and proceeds as follows. One introduces a new coordinate system $(\tau, \rho)$ which is adapted to self-similarity and given by

$$\tau = -\log(T - t), \quad \rho = \frac{r}{T - t}.$$

It is natural to restrict the analysis to the backward lightcone of the blowup point $(T, 0)$ which yields the coordinate domain $\tau \geq -\log T$ and $\rho \in [0, 1]$ (note that $t \to T -$ corresponds to $\tau \to \infty$). Eq. (3.1.1) transforms into

$$\phi_{\tau\tau} + \phi_\tau + 2\rho \phi_\rho - (1 - \rho^2) \phi_{\rho\rho} - 2 \frac{1 - \rho^2}{\rho} \phi_\rho + \frac{\sin(2\phi)}{\rho^2} = 0$$  \hspace{1cm} (3.1.2)

where $\psi(t, r) = \phi(\tau, \rho)$. The point of this transformation is of course that the self-similar Shatah-Turok-Spergel solution $\psi^T$ becomes static in the new coordinates and is simply given by $2 \arctan \rho$. In order to obtain information on its stability one inserts the mode ansatz

$$\phi(\tau, \rho) = 2 \arctan(\rho) + e^{\lambda \tau} u_\lambda(\rho), \quad \lambda \in \mathbb{C}$$

and linearizes in $u_\lambda$ which yields the ODE spectral problem

$$-(1 - \rho^2) u''_\lambda + 2 \frac{1 - \rho^2}{\rho} u'_\lambda + 2\lambda \rho u'_\lambda + \lambda(\lambda + 1) u_\lambda + \frac{V(\rho)}{\rho^2} u_\lambda = 0$$  \hspace{1cm} (3.1.3)
where
\[
V(\rho) = 2 \cos(4 \arctan \rho) = \frac{2(1 - 6\rho^2 + \rho^4)}{(1 + \rho^2)^2}.
\] (3.1.4)

Eq. (3.1.3) has the two regular singular points 0 and 1 and we call a (nontrivial) solution \( u_\lambda \) of Eq. (3.1.3) \textit{admissible} if \( u_\lambda \in C^\infty[0,1] \). It follows from the functional framework developed in [12] that one may restrict oneself to smooth functions here.

Admissible solutions of Eq. (3.1.3) with \( \text{Re}\lambda \geq 0 \) are called \textit{unstable modes}. Obviously, the existence of unstable modes is expected to indicate instabilities of \( \psi^T \). In fact, it is easy to see that there exists an unstable mode \( u_\lambda(\rho) = 2\rho \frac{d}{d\rho} \arctan \rho = \frac{2\rho}{1 + \rho^2} \) with \( \lambda = 1 \). However, this instability is a coordinate effect (a “symmetry mode”) and it is related to the freedom of choosing the blowup time \( T \) (see [12] for a thorough discussion on this). Consequently, the solution \( \psi^T \) is called \textit{mode stable} if there do not exist unstable modes except for the symmetry mode.

In and by itself, the study of mode solutions does not yield any information on the \textit{nonlinear} stability of \( \psi^T \). However, the second author proved that \textbf{mode stability of} \( \psi^T \) \textit{implies nonlinear stability} [11].

### 3.1.2 Known results

There are convincing numerical studies that exclude the existence of unstable modes [13, 9, 17]. Unfortunately, it seems to be very challenging to prove the mode stability of \( \psi^T \) rigorously. The spectral problem (3.1.3) is nonstandard since \( \lambda \) appears in the coefficient of \( u'_\lambda \). Thus, in principle it is possible that there exist unstable modes for nonreal \( \lambda \) which complicates matters tremendously. Of course, one may remove the first derivative by transforming Eq. (3.1.3) into Liouville normal form. This transformation, however, involves the spectral parameter and it turns out that it is only admissible if \( \text{Re}\lambda > 1 \) [15, 16, 12]. It is thus easy to exclude unstable modes with \( \text{Re}\lambda > 1 \) by Sturm oscillation theory [16] but the domain \( 0 \leq \text{Re}\lambda \leq 1 \) remains open.
In [15] it is proved that there do not exist unstable modes for \( \lambda \in (0, 1) \). Furthermore, in [12] the existence of unstable modes for \( \text{Re}\lambda \geq \frac{1}{2} \) and \( \lambda \neq 1 \) is excluded. Finally, also in [12] there is given an argument why there cannot exist unstable modes for \( \lambda \) far away from the real axis.

### 3.1.3 Content of the paper

As a first step, we quantify the constants in the argument from [12] to show that there are no unstable eigenvalues \( \lambda \) with \( |\text{Im}\lambda| \geq 380 \). The remaining compact region \( \{ \lambda \in \mathbb{C} : 0 \leq \text{Re}\lambda \leq \frac{1}{2}, |\text{Im}\lambda| \leq 380 \} \) is then studied by a new approach which is in some sense “brute force”. This means that we work with suitable approximations to solutions of the equation, e.g. truncated Chebyshev expansions, and a rigorous version of a method used by Bizoń [9] which relates the existence of eigenvalues to the convergence of a certain continued fraction. We provide rigorous error estimates for all our approximations. Furthermore, we emphasize that all computations are free of rounding errors since we work exclusively with rational numbers. As a consequence, our result is completely rigorous although we have to admit that it seems unrealistic to perform all our computations without the aid of some computer algebra system.

In the present paper we prove the following result.

**Theorem 3.1.1.** The Shatah-Turok-Spergel wave map is mode stable, i.e., there does not exist a nontrivial solution \( u_\lambda \in C^\infty[0, 1] \) to Eq. (3.1.3) if \( \text{Re}\lambda \geq 0 \) and \( \lambda \neq 1 \).

We reiterate that Theorem 3.1.1 in conjunction with the result in [11] provides a completely rigorous proof of the nonlinear stability of \( \psi^T \) in the sense of [11].

In view of our method of proof it is natural to split the problem as follows.

**Theorem 3.1.2.** There are no unstable modes for \( \lambda \) with \( \text{Re}\lambda \in [0, \frac{1}{2}] \) and \( |\text{Im}\lambda| \geq 380 \).
**Theorem 3.1.3.** There are no unstable modes for $\lambda$ with $\text{Re}\lambda \in [0, \frac{1}{2}]$ and $|\text{Im}\lambda| \leq 10$.

**Theorem 3.1.4.** There are no unstable modes for $\lambda$ with $\text{Re}\lambda \in [0, \frac{1}{2}]$ and $10 \leq |\text{Im}\lambda| \leq 380$.

### 3.2 Proof of Theorem 3.1.2

#### 3.2.1 Functional setup

We use the functional framework from [12]. Recall the following spaces used in [12].

Let

$$
\tilde{H}_1 := \{ u \in C^2[0,1] : u(0) = u'(0) = 0 \} \\
\tilde{H}_2 := \{ u \in C^1[0,1] : u(0) = 0 \}
$$

and

$$
\|u\|_1^2 := \int_0^1 |u'(\rho)|^2 \frac{d\rho}{\rho^2}; \quad \|u\|_2^2 := \int_0^1 |u'(\rho)|^2 d\rho.
$$

We denote by $H_j$ the completion of $\tilde{H}_j$ with respect to $\|\cdot\|_j$, $j \in \{1, 2\}$. We define two linear operators (one unbounded, the other one bounded) acting on $H := H_1 \times H_2$,

$$
L_0u(\rho) := \begin{pmatrix}
-\rho u'_1(\rho) + u_1(\rho) + \rho u'_2(\rho) - u_2(\rho) \\
\frac{1}{\rho} u'_1(\rho) - \rho u'_2(\rho)
\end{pmatrix},
$$

$$
L'u(\rho) := \begin{pmatrix}
-V_1(\rho) \int_0^\rho s u_2(s) ds \\
0
\end{pmatrix},
$$

where $V_1(\rho) = -\frac{16}{(1+\rho^2)^2}$. The precise domain of $L_0$ is given in [12] but irrelevant for our purposes. We only note that $L_0$ is a closed operator. We also set $L := L_0 + L'$.

A straightforward computation shows that $(\lambda - L)u = 0$ is equivalent to

$$
-(1-\rho^2)u'' - 2\frac{1-\rho^2}{\rho}u' + 2\lambda \rho u' + \lambda(\lambda+1)u + \frac{V(\rho)}{\rho^2}u = 0
$$
where
\[ u_1(\rho) = \rho^2 u_2(\rho) + (\lambda - 2) \int_0^\rho s u_2(s) ds \]
\[ u(\rho) = \frac{1}{\rho^2} \int_0^\rho s u_2(s) ds \]
and
\[ V(\rho) = \rho^2 V_1(\rho) + 2 = \frac{2(1 - 6\rho^2 + \rho^4)}{(1 + \rho^2)^2}. \]
Thus, our goal is to show that \( \lambda \) is not an eigenvalue of \( L \) if \( \text{Re} \lambda \geq 0 \) and \( |\text{Im} \lambda| \geq 380 \).

### 3.2.2 Resolvent bounds

In order to show absence of eigenvalues, we construct the resolvent. By the Birman-Schwinger principle we have
\[ R_L(\lambda) = R_{L_0}(\lambda)[I - L'R_{L_0}(\lambda)]^{-1}. \]
Thus, we need to estimate \( \|L'R_{L_0}(\lambda)\| \) where
\[ \|u\|^2 := \|u_1\|^2 + \|u_2\|^2. \]
If \( \lambda \) is such that \( \|L'R_{L_0}(\lambda)\| < 1 \) then \( [I - L'R_{L_0}(\lambda)]^{-1} \) exists (Neumann series) and thus, \( \lambda \) is not an eigenvalue. By recalling the definition of \( L' \) this boils down to estimating
\[ \|V_1 K[R_{L_0}(\lambda)f]_2\|_1 \]
in terms of \( f \), where \( Ku(\rho) := \int_0^\rho s u(s) ds \). To this end, we use Hardy’s inequality,
\[ \left\| \frac{u}{s} \right\|_{L^2} \leq 2\|u'\|_{L^2}, \quad u \in H_1, \]
and the estimate \( \|u\|_{L^2} \leq \frac{1}{\sqrt{2}} \|u'\|_{L^2} \) for \( u \in H_2 \), which is a consequence of Cauchy-Schwarz (all function spaces are on the interval \((0, 1)\), i.e., \( L^2 = L^2(0, 1) \)). Furthermore, we use the bound \( \|R_{L_0}(\lambda)\| \leq \frac{1}{\text{Re} \lambda + \frac{1}{2}} \) which follows from semigroup theory.
First, we estimate the operator norm of $V_1$, viewed as a map from $H_1$ to $H_1$. We have

$$
\|V_1 u\|_1 = \left\| \frac{(V_1 u')'}{\cdot} \right\|_{L^2} \leq \left\| \frac{V_1 u'}{\cdot} \right\|_{L^2} + \left\| \frac{V_1' u}{\cdot} \right\|_{L^2}
$$

\leq \|V_1\|_{L^\infty} \|u\|_1 + 2\|V_1'\|_{L^\infty} \|u'\|_{L^2}

\leq 50\|u\|_1.

It remains to estimate $\|K[R_{L_0}(\lambda)f]_2\|_1$. If we set $u := R_{L_0}(\lambda)f$, we obtain $(\lambda - L_0)u = f$ which implies

$$u_1(\rho) = \rho^2 u_2(\rho) + (\lambda - 2)Ku_2(\rho) - Kf_2(\rho)$$

as a straightforward computation shows. Consequently, we find

$$|\lambda - 2|\|Ku_2\|_1 \leq \|u_1\|_1 + \|\cdot |^2 u_2\|_1 + \|K f_2\|_1

= \|u_1\|_1 + \|\cdot |^{-1}(\cdot |^2 u_2)'\|_{L^2} + \|\cdot |^{-1}(K f_2)'\|_{L^2}

\leq \|u_1\|_1 + 2\|u_2\|_{L^2} + \|u_2'\|_{L^2} + \|f_2\|_{L^2}

\leq \|u_1\|_1 + (\frac{2}{\sqrt{2}} + 1)\|u_2\|_2 + \frac{1}{\sqrt{2}}\|f_2\|_2

\leq (2 + \sqrt{2})\|u\| + \frac{1}{\sqrt{2}}\|f\|

and this yields

$$|\lambda - 2|\|K[R_{L_0}(\lambda)f]_2\|_1 \leq \frac{2 + \sqrt{2}}{\text{Re}\lambda + \frac{\sqrt{2}}{2}}\|f\| + \frac{1}{\sqrt{2}}\|f\| < 7.6\|f\|$$

provided $\text{Re}\lambda \geq 0$. In summary, we find

$$\|L' R_{L_0}(\lambda)\| < \frac{50 \cdot 7.6}{|\lambda - 2|} = \frac{380}{|\lambda - 2|}$$

and this shows that there are no eigenvalues with $\text{Re}\lambda \geq 0$ and $|\text{Im}\lambda| \geq 380$. 

[33]
3.3 Proof of Theorem 3.1.3

3.3.1 Preparatory remarks

A number $\lambda$ for which equation (3.1.3), (3.1.4) has a solution $u_\lambda \in C^\infty[0,1]$ will be simply called an eigenvalue of the equation on $[0,1]$.

Note that $\rho = 0$ is a regular singular point with indices 1 and $-2$, and $\rho = 1$ is a regular singular point with indices 0 and $1 - \lambda$. For $\lambda$ noninteger, by Fuchsian theory, a solution is $C^\infty[0,1]$ if and only if it is analytic on $[0,1]$.

It is convenient to substitute

$$F(t) = u_\lambda(\sqrt{t})/\sqrt{t}, \quad G(t) = F(1-t)$$

(3.3.1)

(where $\sqrt{t} > 0$ for $t > 0$) which transforms (3.1.3), (3.1.4) to

$$t(1-t)G''(t) + \left[ -\frac{5}{2}t + \lambda(1-t) \right] G'(t)$$

$$+ \left[ -\frac{\lambda^2 + 3\lambda}{4} + \frac{1}{2} + \frac{t(4-t)}{(2-t)^2} \right] G(t) = 0$$

(3.3.2)

Remark. A number $\lambda$ with $0 \leq \text{Re}(\lambda) \leq 1/2$ is an eigenvalue of (3.1.3), (3.1.4) on $[0,1]$ if and only if $\lambda$ is an eigenvalue of (3.3.2) on $[0,1]$.

Indeed, it is easy to check using Taylor series at 0 that a solution $u_\lambda$ analytic at $\rho = 0$ has the form $\rho F(\rho^2)$ with $F$ analytic at 0, hence $u_\lambda$ is analytic on $[0,1]$ if and only if $F$, and therefore $G$, are analytic on $[0,1]$.

Denote

$$R = \{ \lambda \in \mathbb{C} \mid 0 \leq \text{Re}\lambda \leq \frac{1}{2}, -10 \leq \text{Im}\lambda \leq 10 \}$$

The proof is slightly different on each of the three subsets $R = R_1 \cup R_2 \cup R_3$ where

$$R_1 = \{ \lambda \mid 0 \leq \text{Re}\lambda \leq \frac{1}{2}, \mid \text{Im}\lambda \mid \leq \frac{1}{2} \}$$

(3.3.3)

$$R_2 = \{ \lambda \mid 0 \leq \text{Re}\lambda \leq \frac{1}{2}, \frac{1}{2} \leq \mid \text{Im}\lambda \mid \leq 4 \}$$

(3.3.4)
\[ R_3 = \{ \lambda \mid 0 \leq \Re \lambda \leq \frac{1}{2}, \ 4 \leq |\Im \lambda| \leq 10 \} \] (3.3.5)

Note that \( \lambda \) is an eigenvalue of (3.3.2) if and only if \( \bar{\lambda} \) is an eigenvalue. Therefore it suffices to consider \( \lambda \) with \( \Im \lambda \geq 0 \). Denote the upper half of \( R_i \) by \( S_i \), \( (i = 1, 2, 3) \).

### 3.3.2 Absence of eigenvalues in \( S_1 \)

Let

\[ S_1 = \{ \lambda \in R_1 \mid \Im \lambda \geq 0 \} \]

be the upper half of the rectangle (3.3.3). In this section we show that no \( \lambda \) in \( S_1 \) is an eigenvalue.

**Outline of the proof** \( t = 0 \) and \( t = 1 \) are regular singularities of eq. (3.3.2) with Frobenius indices \( \{0, 1 - \lambda\} \) at \( t = 0 \) and \( \{-3/2, 0\} \) at \( t = 1 \). For a fixed eigenvalue \( \lambda \), if \( G(t) \) is a solution analytic at \( t = 0 \) and \( H(t) \) is a solution analytic at \( t = 1 \), then \( G(t) \) and \( H(t) \) must be linearly dependent, in other words, their Wronskian \( W(G, H) \equiv 0 \). Thus it suffices to show that \( W(G, H)(1/2) \neq 0 \) if \( \lambda \in S_1 \).

We will prove this by first constructing approximations \( G_a(t) \) of \( G(t) \) and \( H_a(t) \) of \( H(t) \), showing that the Wronskian \( W(G_a, H_a) \) of the approximations is nonzero at \( t = 1/2 \) and that the difference \( |W(G_a, H_a) - W(G, H)|(1/2) \) is smaller than \( |W(G_a, H_a)| \). The differences between actual solutions and the chosen approximations are proved rigorously to be small.

**Proof** (I) *Approximation.*

The way the approximation is obtained is irrelevant for the proof. However we describe the method to motivate the approach.

We would like to construct an approximation of \( G(t) \) on \([0, 1/4]\) and on \((1/4, 1/2]\),
and an approximation of \( H(t) \) on \([3/4, 1]\) and on \([1/2, 3/4]\). We explain below the process of constructing an approximation of \( G(t) \) on interval \([0, 1/4]\).

Let \( G(t) \) be the solution which is analytic at \( t = 0 \) satisfying \( G(0) = \lambda \neq 0 \), and \( H(t) \) be the solution which is analytic at \( t = 1 \) satisfying \( H(1) = 1 \). They have series expansions

\[
G(t) = \sum_{n=0}^{\infty} c_n t^n \quad H(t) = \sum_{n=0}^{\infty} d_n (t - 1)^n
\]  

(3.3.6)

Both series have radii of convergence at least one, since the only singularities in \( \mathbb{C} \) of (3.3.2) are 0, 1 and 2. It is straightforward to see that each \( c_n \) is a rational function of \( \lambda \) and analytic in \( S_1 \), the denominators of \( c_n \) vanishes only for \( \lambda \in \mathbb{Z}^- \), and each \( d_n \) is a polynomial in \( \lambda \). We obtain an approximation \( G_a \) of \( G(t) \) as follows.

(i) We compute \( c_n \) for \( n \leq 11 \) and differentiate the Taylor polynomial of \( G(t) \) twice. We obtain:

\[
P_0(t, \lambda) = \sum_{n=0}^{9} (n + 2)(n + 1)c_{n+2} t^n
\]  

(3.3.7)

(ii) For each coefficient \((n + 2)(n + 1)c_{n+2}\), we expand \((n + 2)(n + 1)c_{n+2}\) as a Taylor series in \( \lambda \) at \( \lambda_0 = 1 + i 4 \). We then approximate \((n + 2)(n + 1)c_{n+2}\) by its 6-th order Taylor polynomial \( c_{T,n} \). After expanding the polynomial, we have

\[
c_{T,n} = \sum_{i=0}^{6} a_{n,i} \lambda^i
\]  

(3.3.8)

so \( P_0(t, \lambda) \) is approximated by

\[
P_1(t, \lambda) = \sum_{n=0}^{9} c_{T,n} t^n = \sum_{n=0}^{9} \sum_{i=0}^{6} a_{n,i} \lambda^i t^n = \sum_{i=0}^{6} \left( \sum_{n=0}^{9} a_{n,i} t^n \right) \lambda^i = \sum_{i=0}^{6} b_i \lambda^i
\]  

(3.3.9)

(iii) Next, we use a Chebyshev polynomial approximation in \( t \) on \([0, 1/4]\) for \( \text{Re}(b_i) \) and \( \text{Im}(b_i) \), with precision \( 2 \cdot 10^{-3} \). Denote the approximations by \( c_{c,i}^{(r)} \) and \( c_{c,i}^{(i)} \), respectively, \( P_1(t, \lambda) \) is approximated by

\[
P_2(t, \lambda) = \sum_{i=0}^{6} (c_{c,i}^{(r)} + i c_{c,i}^{(i)}) \lambda^i = \sum_{i,n} c_{c,n,i} \lambda^i t^n
\]  

(3.3.10)
(iv) Each $c_{n,i}$ is replaced by a rational number $c_{n,i}$ within $e^{-7}$ of it, to allow for rigorous proofs. Finally we have an approximation of $G''(t)$ in the form

$$P_a(t, \lambda) = \sum_{i,n} c_{n,i} \lambda^i t^n$$  \hspace{1cm} (3.3.11)

(v) The approximation $G_a(t)$ of $G(t)$ on $[0, 1/4]$ is obtained by integrating $P_a$ with respect to $t$ twice and the constants of integration are determined by the initial condition:

$$G_a(0) = G(0) = \lambda, \quad G'_a(0) = G'(0) = c_1 = \frac{1}{4} \lambda^2 + \frac{3}{4} \lambda - \frac{1}{2}$$  \hspace{1cm} (3.3.12)

To obtain the approximation $G_a$ of $G$ on $(1/4, 1/2]$ we work with an approximate Taylor polynomial of $G(t)$ at $t = 1/4$ obtained by finding a series solution of (3.3.2) with the initial conditions $G_a(1/4), G'_a(1/4)$. In order to obtain “nicer” expansion coefficients, we allow for a slight discontinuity of $G_a$ at $t = 1/4$.

Similarly, we obtain a piecewise function $H_a$ as an approximation of $H$ on $[1/2, 1]$. See the appendix §3.5.4 for the expressions of $G_a$ and $H_a$.

(II) Estimate of $W(G, H)(1/2)$.

Let $\delta_1(t, \lambda) = G(t, \lambda) - G_a(t, \lambda)$ for $t \in [0, 1/2]$, and $\delta_2(t, \lambda) = H(t, \lambda) - H_a(t, \lambda)$ for $t \in [1/2, 1]$. Both $\delta_1$ and $\delta_2$ are analytic at $t = 1/2$. In a small neighborhood of $t = 1/2$ we have

$$W(G, H) = W(G_a, H_a) + G_a \delta'_2 + \delta_1 H'_a + \delta_1 \delta'_2 - H_a \delta'_1 - \delta_2 G'_a - \delta_2 \delta'_1$$  \hspace{1cm} (3.3.13)

It is not hard to show that

$$|W(G_a, H_a)(1/2)| > 0.46$$  \hspace{1cm} (3.3.14)

(the proof is found in §3.5.2).

We will show that the sum of the absolute values of the other terms on the right hand side of (3.3.13) less than 0.03, implying that $W(G, H)(1/2)$ is not zero.
We detail below the estimates of $\delta(1/2)$ and $\delta'(1/2)$, obtained in two steps.

Denote

$$\epsilon_1 = \mathcal{L}\delta_1 = -\mathcal{L}G_a, \quad t \in (0, 1/4) \cup (1/4, 1/2)$$  \hspace{1cm} (3.3.15)

where $\mathcal{L}$ is the differential operator in (3.3.2). More precisely,

$$\mathcal{L}\delta_1 = t(1-t)\delta''_1 + \left[-\frac{5}{2}t + \lambda(1-t)\right]\delta'_1 + [\alpha + g(t)]\delta_1 = \epsilon_1$$  \hspace{1cm} (3.3.16)

where $\alpha = -\frac{\lambda^2 + 3\lambda}{4} + \frac{1}{2}$, $g(t) = \frac{t(4-t)}{(2-t)^2}$ and $(2-t)^2\epsilon_1$ has an explicit expression as a polynomial.

(i) Estimates for $\delta_1(t)$ and $\delta'_1(t)$ on $[0, 1/4]$. Note that $G_a(0) = G(0)$, $G'_a(0) = G'(0)$, so $\delta_1(t) = t^2\tilde{\delta}_1(t)$, where $\tilde{\delta}_1(t)$ is analytic at $t = 0$. It is also obvious from (3.3.16) that $\epsilon_1(t, \lambda) = t\tilde{\epsilon}_1(t, \lambda)$ where $(2-t)^2\tilde{\epsilon}_1(t)$ is a polynomial as well.

**Lemma 3.3.1.** Let $\lambda \in S_1$.

(i) For all $t \in [0, 1/4]$, we have the following bounds:

$$|\tilde{\epsilon}_1(t)| \leq e_1 = 0.0015$$  \hspace{1cm} (3.3.17)

$$|\alpha| \leq A = \sqrt{\frac{117}{256}} \quad \text{(independent of } t)$$  \hspace{1cm} (3.3.18)

$$1 \leq \frac{4-t}{(2-t)^2} \leq \frac{60}{49}$$  \hspace{1cm} (3.3.19)

(ii) For $t \in (1/4, 1/2]$:

$$\frac{15}{49} \leq \frac{t(4-t)}{(2-t)^2} \leq \frac{7}{9}$$  \hspace{1cm} (3.3.20)

$$|\epsilon_1(t)| \leq e_2 = 2.7 \cdot 10^{-3}$$  \hspace{1cm} (3.3.21)

The proof of 3.3.1 is given in the Appendix §3.5.1.

For $t \in [0, 1/4]$, we have

$$\frac{d}{dt}((1-t)^{5/2}t^\lambda\delta'_1(t)) = (1-t)^{3/2}t^\lambda\tilde{\epsilon}_1(t) - (1-t)^{3/2}t^{\lambda+1}(\alpha + g(t))\tilde{\delta}_1(t)$$  \hspace{1cm} (3.3.22)
Integrating from 0 to $t$, we have

$$
\delta'(t) = (1 - t)^{-5/2}t^{-\lambda} \int_0^t (1 - s)^{3/2}s^{\lambda} \epsilon_1(s) \, ds
- (1 - t)^{-5/2}t^{-\lambda} \int_0^t (1 - s)^{3/2}s^{\lambda+1}(\alpha + g(s)) \tilde{\delta}_1(s) \, ds
$$

(3.3.23)

One more integration gives

$$
\delta_1(t) = \int_0^t (1 - s)^{-5/2}s^{-\lambda} ds \int_0^s (1 - u)^{3/2}u^{\lambda} \epsilon_1(u) \, du
- \int_0^t (1 - s)^{-5/2}s^{-\lambda} ds \int_0^s (1 - u)^{3/2}u^{\lambda+1}(\alpha + g(u)) \tilde{\delta}_1(u) \, du
$$

(3.3.24)

Let $x = \text{Re}\lambda$; (3.3.24) implies

$$
|\delta_1(t)| \leq \int_0^t (1 - s)^{-5/2}s^{-x} ds \int_0^s (1 - u)^{3/2}u^x |\epsilon_1(u)| \, du
+ \int_0^t (1 - s)^{-5/2}s^{-x} ds \int_0^s (1 - u)^{3/2}u^{x+1}(|\alpha| + |g(u)|) |\tilde{\delta}_1(u)| \, du
$$

(3.3.25)

Denote by $D_1$ an uniform bound of $\tilde{\delta}_1$ for $t \in [0, 1/4]$ and $\lambda \in S_1$. From (3.3.25) we have

$$
t^2|\tilde{\delta}_1(t)| \leq e_1 \int_0^t (1 - s)^{-5/2}s^{-x} ds \int_0^s (1 - u)^{3/2} u^x \, du
+ AD_1 \int_0^t (1 - s)^{-5/2}s^{-x} ds \int_0^s (1 - u)^{3/2} u^{x+1} \, du
+ \frac{60}{49} D_1 \int_0^t (1 - s)^{-5/2}s^{-x} ds \int_0^s (1 - u)^{3/2} u^{x+2} \, du
$$

(3.3.26)
and, using the fact that \( \int_0^s (1 - u)^{3/2} u^\xi du \leq \frac{s^{\xi+1}}{\xi + 1} \) for any \( \xi \geq 0 \), we obtain further

\[
t^2 |\tilde{\delta}_1(t)| \leq \frac{e_1}{x + 1} \int_0^t (1 - s)^{-5/2} s ds + \frac{A D_1}{x + 2} \int_0^t (1 - s)^{-5/2} s^2 ds + \frac{60 D_1}{49 x + 3} \int_0^t (1 - s)^{-5/2} s^3 ds \leq e_1 \left( \frac{2}{3} + 3t + 2(1 - t)^{3/2} \right) + \frac{A D_1}{2} \left( \frac{16}{3} - 1 + 3t/2 - 3t^2/8 + (1 - t)^{3/2} \right) + \frac{60 D_1}{49} \left( 32 - 1 + 3t/2 - 3t^2/8 - t^3/16 + (1 - t)^{3/2} \right) \leq e_1 \left( \frac{2}{3} \sqrt{3t^2/2} + \frac{A D_1}{2} \frac{t^3/6\sqrt{3}}{3} \right) + \frac{60 D_1}{49} \left( 32 t^4/12\sqrt{3} \right) \leq \frac{8}{9} e_1 t^2 + \frac{32}{81} A D_1 t^3 + \frac{1280}{3969} D_1 t^4 \]  

(3.3.27)

After canceling \( t^2 \) and taking supremum of \( |\tilde{\delta}_1(t)| \) over \([0, 1/4]\), we have

\[
D_1 \leq \frac{8}{9} e_1 + \left( \frac{32}{81} A \cdot \frac{1}{4} + \frac{1280}{3969} \cdot \frac{1}{4^2} \right) \cdot D_1 \]  

(3.3.28)

Using the bounds of Lemma 3.3.1, we get

\[
D_1 < 0.0015 \]  

(3.3.29)

and consequently

\[
|\delta_1(t)| = t^2 |\tilde{\delta}_1(t)| < \frac{15}{16} \cdot 10^{-4} \]  

(3.3.30)

In a similar way, using (3.3.23) and we obtain

\[
|\delta'_1(t)| < 9 \cdot 10^{-4} \text{ on } [0, 1/4] \]  

(3.3.31)

(ii) Estimate of \( G_a \) on \([1/4, 1/2]\).
For \( t \in (1/4, 1/2] \), we integrate (3.3.22) and obtain the following equations:

\[
\delta'_1(t) = (1 - t)^{-5/2} t^{-\lambda} (3/4)^{5/2} (1/4)^{\lambda} \delta'(1/4 + 0) \\
+ (1 - t)^{-5/2} t^{-\lambda} \int_{1/4}^{t} (1 - s)^{3/2} s^{\lambda-1} \epsilon_1(s) \, ds \\
- (1 - t)^{-5/2} t^{-\lambda} \int_{1/4}^{t} (1 - s)^{3/2} s^{\lambda-1} (\alpha + g(s)) \, \delta_1(s) \, ds \\
(3.3.32)
\]

\[
\delta_1(t) = \delta_1(1/4 + 0) + \int_{1/4}^{t} (1 - s)^{-5/2} s^{-\lambda} (3/4)^{5/2} (1/4)^{\lambda} \delta'_1(1/4 + 0) \, ds \\
+ \int_{1/4}^{t} (1 - s)^{-5/2} s^{-\lambda} \, ds \int_{1/4}^{s} (1 - u)^{3/2} u^{\lambda-1} \epsilon_1(u) \, du \\
- \int_{1/4}^{t} (1 - s)^{-5/2} s^{-\lambda} \, ds \int_{1/4}^{s} (1 - u)^{3/2} u^{\lambda-1} (\alpha + g(u)) \, \delta_1(u) \, du \\
(3.3.33)
\]

Estimating the integrals (3.3.32) and (3.3.33) as in (3.3.26) and (3.3.27), in the end we get

\[
|\delta_1(t)| < 1.6 \cdot 10^{-3} \quad \text{and} \\
|\delta'_1(t)| < 1.3 \cdot 10^{-2} \quad \text{for all } t \in (1/4, 1/2] \\
(3.3.34)
\]

The estimates of \( \delta_2(t) \) and \( \delta'_2(t) \) on \([3/4, 1]\) and then on \([1/2, 3/4]\) are obtained similarly. We obtain:

\[
|\delta_2(1/2)| < 1.3 \cdot 10^{-3} \\
|\delta'_2(1/2)| < 8 \cdot 10^{-3} \\
(3.3.36)
\]

Also, we have the following estimates for the approximating polynomials using the method in §3.5.1:

\[
|G_a(1/2)| < 1.07 \\
|G'_a(1/2)| < 2.63 \\
|H_a(1/2)| < 0.59 \\
|H'_a(1/2)| < 0.74 \\
(3.3.37)
\]
Finally, using (3.3.34), (3.3.36) and (3.3.37) in (3.3.13), we get, at $t = \frac{1}{2}$:

$$|W(G,H)| \geq |W(G_a,H_a)| - (|G_a\delta_2'| + |\delta_1 H'_a| + |\delta_1 \delta'_2| + |H_a\delta_1'| + |\delta'_2 G'_a| + |\delta_2 \delta'_1|) > 0.46 - 0.03 = 0.43 > 0$$

(3.3.38)

Hence $S_1$ contains no eigenvalues.

### 3.3.3 Absence of eigenvalue in $S_2$

Next we show that the rectangle (3.3.4), adjacent to $R_1$, does not contain any eigenvalues either.

**Proposition 3.3.2.** $\lambda$ is not an eigenvalue of (3.3.2) if $\lambda \in S_2 = \{z \in \mathbb{C} | 0 \leq \Re(z) \leq 1/2, 1/2 \leq \Im(z) \leq 4\}$.

**Proof** The idea is similar to the proof in §3.3.2: we look for an approximation $G_a(t)$ of a solution $G(t)$ which is real analytic at $t = 0$, and an approximation $H_a(t)$ of a solution $H(t)$ which is real analytic at $t = 1$ then estimate and compare the Wronskians.

Let the solutions have the following expansions:

$$G(t) = \sum_{n=0}^{\infty} a_n t^n \quad H(t) = \sum_{n=0}^{\infty} b_n (1-t)^n$$

(3.3.39)

where $G(0) = H(1) = 1$.

The coefficients $a_n$ and $b_n$ satisfy the following recurrence relations:

$$a_{n+1} = p_n(\lambda) a_n - \frac{1}{(n+1)(n+\lambda)} \sum_{k=0}^{n-1} \frac{n-k+1}{2^{n-k}} a_k$$

(3.3.40)

$$b_{n+1} = q_n(\lambda) b_n + \frac{1}{(n+1)(n+\frac{3}{2})} \sum_{k=0}^{n-1} 4(-1)^{n-k+1}(n-k+1)b_k$$

(3.3.41)

where

$$p_n = \frac{n^2 + (\lambda + \frac{3}{2})n + \frac{\lambda^2 + 3\lambda}{4} - \frac{1}{2}}{(n+1)(n+\lambda)}$$

(3.3.42)
\[ q_n = \frac{n^2 + (\lambda + \frac{3}{2})n + \frac{\lambda^2 + 3\lambda}{4} - \frac{7}{2}}{(n + 1)(n + \frac{5}{2})} \]  

(3.3.43)

First we prove the following estimates:

**Lemma 3.3.3.** Assume \( \lambda \in S_2 \). There are constants \( J_n \) and \( M_n \) so that

(i) \( |a_n| \leq J_n \), for all \( n \geq 0 \). For \( n \geq 30 \), \( J_n \leq k_1^n \), where \( k_1 = 1.026 \).

(ii) \( |b_n| \leq M_n \), for all \( n \geq 0 \). For \( n \geq 30 \), \( M_n \leq k_2^n \), where \( k_2 = 1.2 \). The values of \( J_n \) and \( M_n \) for \( 0 \leq n \leq 30 \) are given in §3.5.8.

The proof is given in §3.5.3.

Now we obtain estimates of remainders \( G(t) - G_a(t) \) and \( H(t) - H_a(t) \), where \( G_a \) is the type of rational functions discussed in §3.5.5 and \( H_a \) is a polynomial where we choose:

\[ G_a(t) = \sum_{k=0}^{8} a_k t^k, \quad H_a(t) = \sum_{k=0}^{10} b_k (1 - t)^k \]  

(3.3.44)

and consider the Wronskians \( W(G, H) \) and \( W(G_a, H_a) \) at \( t_0 = 0.56 \). With the same notations as in the proof for \( S_1 \), \( \delta_1 = G(t) - G_a(t) \) and \( \delta_2 = H(t) - H_a(t) \), we have (3.3.13) for \( \lambda \in S_2 \). Using the method in §3.5.5 and §3.5.6 we have the following estimates:

\[ |G_a(t_0)| \leq 2.14 \]  

(3.3.45)

\[ |H_a(t_0)| \leq 0.61 \]  

(3.3.46)

\[ |G_a'(t_0)| \leq 5.83 \]  

(3.3.47)

\[ |H_a'(t_0)| \leq 1.32 \]  

(3.3.48)

The lower bound of \( |W(G_a, H_a)(t_0)| \) on \( S_2 \) is proven in §3.5.7.

\[ |W_a(t_0)| = |G_a(t_0)H_a'(t_0) - G_a'(t_0)H_a(t_0)| > 1.06 \]  

(3.3.49)
\[|\delta_1(t_0)| \leq \sum_{i=9}^{\infty} |a_i| t_0^i \leq \sum_{i=9}^{30} J_i t_0^i + \sum_{i=31}^{\infty} (k_i t_0)^i < 0.014 \] (3.3.50)

\[|\delta'_1(t_0)| \leq \sum_{i=9}^{\infty} i |a_i| t_0^{i-1} \leq \sum_{i=9}^{30} i J_i t_0^{i-1} + \sum_{i=31}^{\infty} i k_i t_0^{i-1} < 0.252 \] (3.3.51)

Similarly:

\[|\delta_2(t_0)| < 0.003 \] (3.3.52)

\[|\delta'_2(t_0)| < 0.058 \] (3.3.53)

Combining equations (3.3.45) - (3.3.53) we get

\[|W(G,H)(t_0)| \geq |W(G_a,H_a)(t_0)| - |G_a\delta'_2| |t_0| - |\delta_1 H'_a| |t_0| - |\delta_1 | |t_0| - |H_a\delta'_1| |t_0| - |\delta_2 G'_a| |t_0| - |\delta_2 | |t_0| \\
\geq 1.06 - 2.14 \cdot 0.058 - 0.014 \cdot 1.32 - 0.014 \cdot 0.073 \\
- 0.61 \cdot 0.252 - 0.003 \cdot 5.83 - 0.003 \cdot 0.252 > 0.744 \] (3.3.54)

Hence no \( \lambda \in S_2 \) is an eigenvalue. Proposition 3.3.2 is proved.

### 3.3.4 Absence of eigenvalues in \( S_3 \)

**Proposition 3.3.4.** \( \lambda \) is not an eigenvalue of (3.1.3), (3.1.4) if \( \lambda \in S_3 = \{ z \in \mathbb{C} | 0 \leq \text{Re}(z) \leq 1/2, 4 \leq \text{Im}(z) \leq 10 \} \).

**Proof** The proof follows the idea of [9]. First we introduce some notation. Make a change of variable

\[y(x) = \left( \frac{2}{1 + \rho^2} \right)^{\frac{\rho - 1}{2}} \rho u(\rho), \quad x = \frac{2\rho^2}{1 + \rho^2} \] (3.3.55)

and (3.1.3), (3.1.4) becomes:

\[x^2(1-x)(2-x) \frac{d^2}{dx^2} y(x) + x [1 - (1 + \lambda) x (2 - x)] \frac{d}{dx} y(x) \\
- \frac{1}{4} \left[ \lambda^2 x (1 - x) + 9 x^2 - 17 x + 4 \right] y(x) = 0 \] (3.3.56)
We can see from the Frobenius indices of \( u(\rho) \) (see the §2.1) that the change of variable preserves analyticity of solutions at 0 and 1. So the eigenvalues of \( u(\rho) \) are the same as those of \( y(x) \).

Let \( y \) be the solution of (3.3.56) which is analytic at \( x = 0 \) and satisfies \( y'(0) = 1 \). We write

\[
y(x) = \sum_{n=0}^{\infty} c_n x^n \quad (3.3.57)
\]

If \( \lambda \) is an eigenvalue then the radius of convergence of the series is strictly larger than 1, therefore, it is at least 2. By substituting the series into (3.3.56) we find that \( c_n \) satisfy the recurrence relation:

\[
\begin{align*}
p_2(0)c_2 + p_1(0)c_1 &= 0 \\
p_2(n)c_{n+2} + p_1(n)c_{n+1} + p_0(n)c_n &= 0, \quad n = 1, 2, \ldots
\end{align*}
\]  

(3.3.58)

where \( c_0 = 0 \) and \( c_1 = 1 \) and

\[
\begin{align*}
p_2(n) &= 8n^2 + 28n + 20, \\
p_1(n) &= -12n^2 - (20 + 8\lambda)n - \lambda^2 - 8\lambda + 9, \\
p_0(n) &= 4n^2 + 4\lambda n + \lambda^2 - 9.
\end{align*}
\]  

(3.3.59)

Also, let

\[
A_n = \frac{p_1(n)}{p_2(n)}, \quad B_n = \frac{p_0(n)}{p_2(n)}, \quad r_n = \frac{c_{n+1}}{c_n}.
\]  

(3.3.60)

Then (3.3.58) is rewritten as

\[
\frac{r_n}{A_n + r_{n+1}} = \frac{B_n}{A_n + r_{n+1}} ,
\]  

(3.3.61)

We notice that the recurrence relation (3.3.58) is a second order difference equation of Poincaré type with characteristic roots 1 and \( \frac{1}{2} \), so by Poincaré’s theorem ([10]) either \( c_n \equiv 0 \) for \( n \) large or

\[
\lim_{n \to \infty} r_n \in \left\{ 1, \frac{1}{2} \right\}
\]  

(3.3.62)
Suppose that \( c_n \equiv 0 \) for \( n \) large for some values of \( \lambda \), then there are only finitely many values of \( n \) such that \( c_n \neq 0 \). Assume that \( n = N \) is the largest positive integer such that \( c_n \neq 0 \), then \( c_{N+1} = c_{N+2} = 0 \). Using recurrence relation (3.3.58) and the fact that \( p_0(n) \neq 0 \) for any values of \( \lambda \) with \( \text{Re} \lambda \in [0, 1/2] \) we have \( c_N = 0 \), contradicting the assumption \( c_N \neq 0 \). Hence the situation that \( c_n \equiv 0 \) for \( n \) large will not arise for the values of \( \lambda \) in concern.

Note that \( \lambda \) is an eigenvalue if and only if \( \lim_{n \to \infty} r_n = 1/2 \) (meaning that the radius of convergence of (3.3.57) is 2). In this case \( c_n \) is a minimal solution \(^1\) ([10]) of the recurrence (3.3.58).

For an arbitrary \( \lambda \), let \( \psi_n = \phi_{n+1}/\phi_n \) where \( (\phi_n)_{n \geq 1} \) is a minimal solution of (3.3.58), then \( (\psi_n)_{n \geq 1} \) is unique. By Pincherle’s theorem [10, 50], for each \( n \geq 1 \):

\[
\psi_n = -\frac{B_n}{A_n} - \frac{B_{n+1}}{A_{n+1}} \frac{B_{n+2}}{A_{n+2}} - \cdots \tag{3.3.63}
\]

Therefore we just need to show that for some \( n \geq 1 \) the equation

\[
\frac{c_{n+1}}{c_n} = -\frac{B_n}{A_n} - \frac{B_{n+1}}{A_{n+1}} \frac{B_{n+2}}{A_{n+2}} - \cdots \tag{3.3.64}
\]

does not have any roots in \( S_3 \).

Let \( n_0 = 10 \). Consider the Banach space of sequences

\[
\mathcal{B} := \{(a_n)_{n \geq n_0} : a_n \in \mathbb{C}\} \tag{3.3.65}
\]

with sup norm: \( \|(a_n)_{n \geq n_0}\| := \sup_{n \geq n_0} |a_n| \), and the closed ball in \( \mathcal{B} \)

\[
S = \{(a_n)_{n \geq n_0} \in \mathcal{B} : \|(a_n)_{n \geq n_0}\| \leq 0.6\}. \tag{3.3.66}
\]

\(^1\) A solution \( \Phi(n) \) of a second-order difference equation

\[
\phi(n+2) + p_1(n) \phi(n+1) + p_2(n) \phi(n) = 0
\]

is said to be minimal if

\[
\lim_{n \to \infty} \frac{\Phi(n)}{\Psi(n)} = 0
\]

for any other solution \( \Psi(n) \) of the equation that is not a multiple of \( \Phi(n) \).
Define an operator $\mathcal{N}$ in $S$:

\[
(\mathcal{N}[(a_n)_{n \geq n_0}])_n = -\frac{B_n}{A_n + a_{n+1}}
\] (3.3.67)

**Lemma 3.3.5.** (i) The operator $\mathcal{N}$ preserves $S$ and (ii) $\mathcal{N}$ is contractive on $S$ with a contractivity factor less than 0.8.

**Proof** (i) $\mathcal{N}$ preserves $S$. Let $(r_n)_{n \geq n_0} \in S$, write

\[
(\mathcal{N}[(r_n)_{n \geq n_0}])_n = -\frac{B_n}{A_n + r_{n+1}} = -\frac{B_n/A_n}{1 + r_{n+1}/A_n}
\] (3.3.68)

We obtain estimates of upper bounds of $|B_n/A_n|$ and lower bounds of $|A_n|$ in $S$, which are provided in the formula (3.5.120), (3.5.139) and (3.5.159) in §3.5.9, and from them we know that for $n \geq n_0$,

\[
\left| -\frac{B_n/A_n}{1 + r_{n+1}/A_n} \right| \leq \frac{|B_n/A_n|}{1 - 0.6/|A_n|} \leq 0.6
\] (3.3.69)

(ii) $\mathcal{N}$ is contractive on $S$ with a contraction factor less than 0.8. Similarly, let $(r_n)_{n \geq n_0}, (s_n)_{n \geq n_0} \in S$,

\[
(\mathcal{N}[(r_n)_{n \geq n_0}])_n - (\mathcal{N}[(s_n)_{n \geq n_0}])_n = -\frac{B_n}{A_n + r_{n+1}} + \frac{B_n}{A_n + s_{n+1}}
\]

\[
= \frac{B_n}{(A_n + r_{n+1})(A_n + s_{n+1})} (r_{n+1} - s_{n+1})
\]

\[
= \frac{B_n/A_n^2}{(1 + r_{n+1}/A_n)(1 + s_{n+1}/A_n)} (r_{n+1} - s_{n+1})
\] (3.3.70)

With estimates of upper bounds of $|B_n/A_n^2|$ (see (3.5.159) and (3.5.161)), we are able to show that the contraction factor is less than 0.8, so that proves this part of the lemma.

Hence by contractive mapping theorem, $\mathcal{N}^m(0) \to \mathcal{N}^\infty(0)$ as $m \to \infty$, where $\mathcal{N}^\infty(0)$ is used to denote the unique fixed point in $S$. On the other hand, by Pincherle’s theorem, if $\lambda$ is an eigenvalue then $(r_n)_{n \geq n_0} = (c_{n+1}/c_n)_{n \geq n_0}$ is the fixed point. In fact, both $\prod_{n \geq n_0} (\mathcal{N}^\infty(0))_n$ and $\prod_{n \geq n_0} r_n$ are minimal solutions of the linear
difference equation (3.3.58), so \((\mathcal{N}^\infty(0))_n = r_n\) for each \(n\). In particular, eigenvalues \(\lambda\) are roots of the equation

\[
(\mathcal{N}^\infty(0))_{n_0} = r_{n_0}
\]  
(3.3.71)

We then observe that over the boundary of \(R_3\)

\[
\| (\mathcal{N}(0)) - (\mathcal{N}^2(0)) \| < 0.106
\]  
(3.3.72)

\[
| (\mathcal{N}(0))_{n_0} - r_{n_0} | > 0.595
\]  
(3.3.73)

(see §3.5.10 and §3.5.10 for the proof). So

\[
\| (\mathcal{N}(0)) - (\mathcal{N}^\infty(0)) \| \leq \| \mathcal{N}(0) - \mathcal{N}^2(0) \| (I - \| \mathcal{N} \| )^{-1} < | (\mathcal{N}(0))_{n_0} - r_{n_0} |
\]  
(3.3.74)

Now since over the boundary of \(S_3\) we have

\[
| (\mathcal{N}^\infty(0))_{n_0} - (\mathcal{N}(0))_{n_0} | \\
= | [(\mathcal{N}^\infty(0))_{n_0} - r_{n_0}] - [(\mathcal{N}(0))_{n_0} - r_{n_0}] | \\
< | (\mathcal{N}(0))_{n_0} - r_{n_0} |
\]  
(3.3.75)

By Rouché’s theorem, \((\mathcal{N}^\infty(0))_{n_0} - r_{n_0}\) and \((\mathcal{N}(0))_{n_0} - r_{n_0}\) have the same number of roots in \(S_3\), but the latter is an explicit rational function with no roots in \(S_3\), hence there are no eigenvalues in \(S_3\).

### 3.4 Proof of Theorem 3.1.4

#### 3.4.1 Introduction of notations

Denote

\[
S_4 := \{ \lambda \mid 0 \leq \text{Re}\lambda \leq \frac{1}{2}, \ 10 \leq \text{Im}\lambda \leq 380 \}
\]  
(3.4.1)
With the notations as in §3.3.4, we rewrite (3.3.58) as

\[ r_{n+1} = -A_n - \frac{B_n}{r_n}, \quad (3.4.2) \]

Proving Theorem 3.1.4 is equivalent to proving that for \( \lambda \in S_4 \),

\[ \lim_{n \to \infty} r_n = 1 \quad (3.4.3) \]

The method of obtaining the estimates of \( r_n \) is as follows. We first obtain a “quasi-solution” \( \tilde{r}_n \) of the recurrence (3.4.2) (more precisely for \( r_n = c_{n+1}/c_n \)). That is, a function \( \tilde{r}_n \) which is close enough to \( r_n \). For this purpose the analysis need not be rigorous, since in the sequel we prove that the relative error \( |r_n/\tilde{r}_n - 1| \) is small.

The concrete construction of \( \tilde{r}_n \) is as follows.

For \( n \leq 50 \) we project a suitable branch of \( \log r_n \) on Chebyshev polynomials in \( \text{Re}(\lambda) \) and \( \text{Im}(\lambda) \), of degree at most 6. For \( n > 50 \) we simply define \( \tilde{r}_n \) to be one of the exact solutions of the approximate equation

\[ r_n = -A_n - B_n/r_n \quad (3.4.4) \]

The proof of the estimate of \( |r_n/\tilde{r}_n - 1| \) is based on calculations with polynomials with rational coefficients and is rigorous.

The boundary of \( S_4 \) (see (3.4.1)) is partitioned into 6 line segments,

\[
\begin{align*}
  l_1 &= [10i, 70i] & l_3 &= [10i + 1/2, 70i + 1/2] & l_5 &= [10i, 10i + 1/2] \\
  l_2 &= (70i, 380i] & l_4 &= (70i + 1/2, 380i + 1/2] & l_6 &= [380i, 380i + 1/2] \\

eq (3.4.5)
\end{align*}
\]

Let \( \lambda = ti + s \). We choose the following quasi-solution

\[ r_n^{(1)} = e^{W_n(\lambda)} \quad (1 \leq n \leq 50) \quad (3.4.6) \]
to \( r_n \), where

\[
W_n(\lambda) = \begin{cases} 
\sum_{i=0}^{5} \left( (1 - 2s)a_{n,i}^{(1)} + 2sa_{n,i}^{(2)} \right) T_i \left( \frac{t - 40}{30} \right) : & 10 \leq t \leq 70 \\
\sum_{i=0}^{5} \left( (1 - 2s)b_{n,i}^{(1)} + 2sb_{n,i}^{(2)} \right) T_i \left( \frac{t - 225}{155} \right) : & 70 < t \leq 380
\end{cases}
\]

(3.4.7)

where \( a_{n,i}^{(j)} \) and \( b_{n,i}^{(j)} \) (\( 1 \leq j \leq 2, 1 \leq i \leq 5, 1 \leq n \leq 50 \)) are given in Table 3.5.14 in the Appendix. Define also

\[
r_n^{(2)} = -\frac{A_n}{2} \left( 1 + \sqrt{1 - \frac{4B_n}{A_n^2}} \right)
\]

(3.4.8)

where the branch of square root is the one which is positive on the positive real axis.

We note that \( r_n^{(2)} \) is a solution to the quadratic equation (3.4.4). Also write

\[
\begin{align*}
\delta(1) & = r_n^{(1)} (1 + \delta_n^{(1)}) \\
\delta(2) & = r_n^{(2)} (1 + \delta_n^{(2)})
\end{align*}
\]

(3.4.9)

3.4.2 Proof of the Theorem

We prove Theorem 3.1.4 in four steps.

Lemma 3.4.1. (i)

\[
\left| \delta_1^{(1)}(\lambda) \right| \leq D_1 = \frac{19}{1000} \quad \forall \lambda \in \partial S_4
\]

(3.4.10)

(ii)

\[
\left| \delta_n^{(1)}(\lambda) \right| \leq D_2 = \frac{9}{200} \quad \forall \lambda \in \partial S_4, \ 1 \leq n \leq 50
\]

(3.4.11)

(iii)

\[
\left| \delta_{50}^{(2)}(\lambda) \right| \leq D_3 = \frac{3}{20} \quad \forall \lambda \in \partial S_4
\]

(3.4.12)

(iv)

\[
\left| \delta_n^{(2)}(\lambda) \right| \leq D_3 = \frac{3}{20} \quad \forall \lambda \in \partial S_4, \ n \geq 50
\]

(3.4.13)

Corollary 3.4.2.

\[
\lim_{n \to \infty} r_n = 1 \quad \forall \lambda \in S_4
\]

(3.4.14)
Proof of Lemma 3.4.1 (i)

Direct application of an elementary method described in Section 3.5.11 in the Appendix.

Proof of Lemma 3.4.1 (ii)

By the definitions (3.4.6) and (3.4.9) we have for 1 ≤ n ≤ 50.

\[ r_n^{(1)}(\lambda) = e^{W_n(\lambda)} \left( 1 + \delta_n^{(1)}(\lambda) \right) \quad (3.4.15) \]

Substituting the equation (3.4.15) into (3.4.2) we obtain

\[ \delta_{n+1}^{(1)}(\lambda) = \epsilon_{n+1}^{(1)} + C_n^{(1)} \frac{\delta_n^{(1)}(\lambda)}{1 + \delta_n^{(1)}(\lambda)} \quad (3.4.16) \]

where

\[ \epsilon_{n+1}^{(1)} = -1 - A_n e^{-W_{n+1}} - B_n e^{-W_{n+1}-W_n} \quad (3.4.17) \]

\[ C_n^{(1)} = B_n e^{-W_{n+1}-W_n} \quad (3.4.18) \]

We use the method in Section 3.5.11 to obtain the following upper bounds

\[ |\epsilon_{n+1}^{(1)}| \leq 21/1000 \quad (1 \leq n \leq 49, \lambda \in l_1 \cup l_3 \cup l_5) \quad (3.4.19) \]

\[ |\epsilon_{n+1}^{(1)}| \leq 27/1000 \quad (1 \leq n \leq 49, \lambda \in l_2 \cup l_4 \cup l_6) \quad (3.4.20) \]

\[ |C_n^{(1)}| \leq 1/2 \quad (1 \leq n \leq 49, \lambda \in l_1 \cup l_3 \cup l_5) \quad (3.4.21) \]

\[ |C_n^{(1)}| \leq 1/3 \quad (1 \leq n \leq 49, \lambda \in l_2 \cup l_4 \cup l_6) \quad (3.4.22) \]

Hence if \( \delta_n^{(1)} \leq D_2 = 21/500 \), it is straightforward to check that \( \delta_{n+1}^{(1)} \leq D_2 \) by (3.4.16) and the estimates (3.4.19) – (3.4.22) hold.

Proof of Lemma 3.4.1 (iii)

Define \( \delta^{(3)} \) by

\[ r_{50}^{(2)} = r_{50}^{(1)} \left( 1 + \delta^{(3)} \right) \quad (3.4.23) \]

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By the definition (3.4.9) we have

\[ (1 + \delta_{50}^{(1)}) r_{50}^{(1)} = (1 + \delta_{50}^{(2)}) r_{50}^{(2)} \] \hspace{1cm} (3.4.24)

Substituting (3.4.23) into (3.4.24) we have

\[ \delta_{50}^{(2)} = \frac{\delta_{50}^{(1)} - \delta^{(3)}}{1 + \delta^{(3)}} \] \hspace{1cm} (3.4.25)

Thus

\[ \left| \delta_{50}^{(2)} \right| \leq \frac{\left| \delta_{50}^{(1)} \right| + \left| \delta^{(3)} \right|}{1 - \left| \delta^{(3)} \right|} \] \hspace{1cm} (3.4.26)

By definitions (3.4.6) and (3.4.23) we have

\[ r_{50}^{(2)}(\lambda) = e^{W_{50}(\lambda)} (1 + \delta^{(3)}(\lambda)) \] \hspace{1cm} (3.4.27)

Substituting (3.4.27) into (3.4.4) we get

\[ \delta^{(3)} = \epsilon^{(3)} + C^{(3)} \frac{\delta^{(3)}}{1 + \delta^{(3)}} \] \hspace{1cm} (3.4.28)

where

\[ \epsilon^{(3)} = -1 - A_{50} e^{-W_{50}} - B_{50} e^{-2W_{50}} \]

\[ C^{(3)} = B_{50} e^{-2W_{50}} \] \hspace{1cm} (3.4.29)

Using the method in Section 3.5.11 we obtain

\[ \left| \epsilon^{(3)} \right| < \frac{1}{50} \hspace{1cm} (\lambda \in l_1 \cup l_3 \cup l_5) \] \hspace{1cm} (3.4.30)

\[ \left| C^{(3)} \right| \leq \frac{1}{2} \]

and

\[ \left| \epsilon^{(3)} \right| < \frac{19}{500} \hspace{1cm} (\lambda \in l_2 \cup l_4 \cup l_6) \] \hspace{1cm} (3.4.31)

\[ \left| C^{(3)} \right| \leq \frac{1}{3} \]

Rewriting (3.4.28), we obtain a quadratic equation for \( \delta^{(3)} \)

\[ (\delta^{(3)})^2 + (1 - C^{(3)} - \epsilon^{(3)}) \delta^{(3)} - \epsilon^{(3)} = 0 \] \hspace{1cm} (3.4.32)
Denoting the two roots of the equation above by \( s_1 \) and \( s_2 \) and using the estimates (3.4.30) and (3.4.31) we have

\[
||s_1| - |s_2|| > 19/50 \quad (\lambda \in l_1 \cup l_3 \cup l_5) \quad (3.4.33)
\]
\[
||s_1| - |s_2|| > 49/100 \quad (\lambda \in l_2 \cup l_4 \cup l_6) \quad (3.4.34)
\]

Assume without loss of generality that \( s_1(\lambda) \) is the root with the larger modulus. Then (3.4.33) implies

\[
|s_1| > 19/50
\]
\[
|s_2| = |\epsilon^{(3)}/s_1| < \frac{1/50}{19/50} = \frac{1}{19} \quad (\lambda \in l_1 \cup l_3 \cup l_5) \quad (3.4.35)
\]
\[
|s_1| > 49/100
\]
\[
|s_2| = |\epsilon^{(3)}/s_1| < \frac{19/500}{49/100} = \frac{19}{245} \quad (\lambda \in l_2 \cup l_4 \cup l_6) \quad (3.4.36)
\]

By definition, \( \delta^{(3)} \) is continuous on \( l_1 \cup l_3 \cup l_5 \) and on \( l_2 \cup l_4 \cup l_6 \). In addition, using the definition of \( r_n^{(1)} \) in (3.4.6) and (3.4.7), the definition of \( r_n^{(2)} \) in (3.4.8) and the definition of \( \delta^{(3)} \) in (3.4.23) we can check that

\[
|\delta^{(3)}(10i)| < 1/100 , \quad |\delta^{(3)}(380i)| < 1/25 \quad (3.4.37)
\]

This implies that for all \( \lambda \in \partial S_4 \), \( \delta^{(3)}(\lambda) \) is the root \( s_2 \) of (3.4.32). Lemma 3.4.1 (iii) follows from the estimates (3.4.35), (3.4.36), Lemma 3.4.1 (ii) and (3.4.26).

**Proof of Lemma 3.4.1 (iv)**

Substituting the second equation in (3.4.9) into (3.4.2) and using the fact that \( r_n^{(2)} \) is a solution of the equation (3.4.4) we obtain

\[
\delta^{(2)}_{n+1} = \epsilon^{(2)}_n + C_n^{(2)} \frac{\delta^{(2)}_n}{1 + \delta^{(2)}_n} \quad (3.4.38)
\]
where

\[ \epsilon_n^{(2)} = \frac{r_n^{(2)}}{r_{n+1}^{(2)}} - 1 \]  
(3.4.39)

\[ C_n^{(2)} = \frac{B_n}{r_n^{(2)} r_{n+1}^{(2)}} \]  
(3.4.40)

Denoting

\[ F_n = 1 - \frac{4 B_n}{A_n^2} \]  
(3.4.41)

we get

\[ r_n^{(2)} = -\frac{A_n}{2} \left(1 + \sqrt{F_n}\right) \]  
(3.4.42)

and

\[ \epsilon_n^{(2)} = \frac{A_n \left(1 + \sqrt{F_n}\right)}{A_{n+1} \left(1 + \sqrt{F_{n+1}}\right)} - 1 \]

\[ = \frac{A_n}{A_{n+1}} \left(1 + \frac{F_n}{1 + \sqrt{F_{n+1}}} - 1\right) + \left(\frac{A_n}{A_{n+1}} - 1\right) \]  
(3.4.43)

\[ C_n^{(2)} = \frac{4 B_n}{A_n A_{n+1} \left(1 + \sqrt{F_n}\right) \left(1 + \sqrt{F_{n+1}}\right)} \]  
(3.4.44)

We estimate \( \epsilon_n^{(2)} \) and \( C_n^{(2)} \) by first showing the following inequalities valid for \( \lambda \) on all sides of \( \partial S_1 \) and \( n \geq 50 \), proved in Section 3.5.12

\[ \left| \frac{A_n}{A_{n+1}} - 1 \right| \leq U_1 := \frac{1}{30} \]  
(3.4.45)

\[ |F_n - F_{n+1}| \leq U_2 := \frac{7 \sqrt{2}}{500} \]  
(3.4.46)

\[ \text{Re} \sqrt{F_n} \geq L_1 := \frac{8}{25} \]  
(3.4.47)

\[ \left| \frac{B_n}{A_n A_{n+1}} \right| \leq U_3 := \frac{9}{40} \]  
(3.4.48)
Using (3.4.45) – (3.4.48) in (3.4.43) and (3.4.44) we have

\[ |\epsilon_n^{(2)}| \leq (1 + U_1) \frac{U_2}{(1 + L_1)(2 L_1)} + U_1 < \frac{29}{500} \tag{3.4.49} \]

\[ |C_n^{(2)}| \leq \frac{4U_3}{(1 + L_1)^3} < \frac{517}{1000} \tag{3.4.50} \]

Hence \( |\delta_n^{(2)}| \leq D_3 = \frac{3}{20} \) implies \( |\delta_{n+1}^{(2)}| \leq D_3 \) by (3.4.38), the estimates (3.4.49) and (3.4.50). Also from Lemma 3.4.1 (iii) we know that \( |\delta_{50}^{(2)}| \leq D_3 \). Lemma 3.4.1 (iv) is proved by straightforward induction.

**Proof of Corollary 3.4.2**

First we show that for each \( n > 0 \), \( r_n \) is analytic in \( S_4 \) since we later make use of the maximum principle. We provide below all the details, but conceptually the proof is simple: from the recurrence (3.4.2) one can easily check that if \( r_n \) has a pole, then \( r_{n-1} \) has a zero; we use the argument principle to show that there are no zeros in the strip of interest. Each \( r_n \) is meromorphic since each \( c_n \) is a polynomial by recurrence (3.3.58) and (3.3.59). On the boundary \( \partial S_4 \), the boundedness of the quasi-solutions \( r_n^{(1)} \) and \( r_n^{(2)} \) together with the estimates in Lemma 3.4.1 show that \( r_n \) is analytic for each \( n \) and therefore also nonzero. Assume that \( n = n_0 \) is the smallest index such that \( r_n \) has a zero in \( S_4 \), \( n_0 > 1 \) obviously. If \( n_0 \geq 50 \), \( r_{n_0}^{(2)} \) is manifestly analytic.

Definition of \( \delta^{(2)} \) in (3.4.9) and Lemma 3.4.1 (iv) imply that

\[ |r_{n_0}^{(2)}| < \frac{3}{20} |r_{n_0}^{(2)}| < |r_{n_0}^{(2)}| \quad (\lambda \in \partial S_4) \tag{3.4.51} \]

By Rouché’s Theorem \( r_{n_0} \) has the same number of roots as \( r_{n_0}^{(2)} \) in \( S_4 \), which is zero.

If \( n_0 < 50 \), let \( \phi : [0, 741] \to \partial S_4 \) denote a parameterization of \( \partial S_4 \) by arc length (the perimeter of \( S_4 \) is 741), satisfying \( \phi(0) = \phi(741) = 10i \) and that as \( t \) increases \( \partial S_4 \) is traversed counterclockwise. We observe that the \( W_n \) given in (3.4.7) is continuous in \( S_4 \setminus \{\lambda : \text{Im}\lambda = 70\} \). It is straightforward to check that for each \( n < 50 \) there exists a suitable integer \( k_n \) such that if we redefine \( W_n \) as \( W_n + 2\pi k_n i \) in \( \{\lambda : 70 < \text{Im}\lambda \leq 380\} \)
we obtain the following estimates at the only two discontinuities $70i$ and $70i + 1/2$ of $W_n$ on $\partial S_4$

\[
\begin{align*}
\lim_{t \to 0^+} |W_n(70i + ti) - W_n(70i - ti)| &< 2/250 \\
\lim_{t \to 0^+} |W_n(70i + 1/2 + ti) - W_n(70i + 1/2 - ti)| &< 7/1000
\end{align*}
\]

Thus there exists a function $\tilde{W}_n$ which is continuous on $\partial S_4 \setminus \{10i\}$ and such that both

\[
\left( \tilde{W} \circ \phi \right)(0+) \quad \text{and} \quad \left( \tilde{W} \circ \phi \right)(741-) \quad \text{exist,} \quad \tilde{W}(10i) = \left( \tilde{W} \circ \phi \right)(0+)
\]

and the following holds:

\[
\left| W_n - \tilde{W}_n \right| < 2/250 \quad (\lambda \in \partial S_4)
\]

Consequently, if we let $\tilde{\delta}_n := e^{W_n}/e^{\tilde{W}_n} - 1$ it is straightforward from (3.4.53) that

\[
\left| \tilde{\delta}_n \right| < 13/1000
\]

Next we choose a branch $\bar{W}_n$ of $\log r_n$ on $\partial S_4$, which is continuous on $\partial S_4 \setminus \{10i\}$ with $\bar{W}_n(10i) := \bar{W}_n(\phi(0+))$. Denoting $\tilde{\delta}_n = r_n/e^{\bar{W}_n} - 1$, we have from (3.4.54) and Lemma 3.4.1 (ii) that

\[
\left| \tilde{\delta}_n \right| = \left| e^{W_n-\bar{W}_n} - 1 \right| = \left| (1 + \delta_n^{(1)}) \left( 1 + \tilde{\delta}_n \right) - 1 \right| < 3/50
\]

Therefore $\left| \ln(1 + \tilde{\delta}_n) \right| < 1/10$ where by definition $\text{Im}(\ln z) \in (-\pi, \pi]$. For each $\lambda \in \partial S_4$ we have

\[
W_n - \bar{W}_n = \ln(1 + \tilde{\delta}_n) + 2\pi ki \quad (k \in \mathbb{Z})
\]

In other words, for each $t \in [0, 741)$ we have

\[
\left( W_n \circ \phi \right)(t) - \left( \bar{W}_n \circ \phi \right)(t) \subset \bigcup_{k \in \mathbb{Z}} B(0, 1/10) + 2k\pi i
\]

where $B(0, 1/10) = \{ z : |z| < 1/10 \}$. Since $W_n \circ \phi - \bar{W}_n \circ \phi$ is continuous on $[0, 741)$, its image must be connected, and the right hand side of (3.4.57) is a disjoint union of
open balls, so the image must be contained in only one of them, say $B(0, 1/10)+2k_0\pi i$. Then we have

$$
\left| W_n(\phi(0+)) - W_n(\phi(741-)) \right|
< \left| \left( W_n - \tilde{W}_n \right)(\phi(0+)) - \left( W_n - \tilde{W}_n \right)(\phi(741-)) \right|
+ \left| \tilde{W}_n(\phi(0+)) - \tilde{W}_n(\phi(741-)) \right|
$$

(3.4.58)

$$
< 1/10 \cdot 2 + 2/250 \cdot 2 < 3/10
$$

By argument principle $W_n(\phi(0+)) = W_n(\phi(741-))$ and thus $r_{n_0}$ has no zeros in $S_4$. Therefore each $r_n$ is analytic in $S_4$. We know that for all $\lambda \in \mathbb{C}$ we have

$$
\lim_{n \to \infty} A_n = -\frac{3}{2}
$$

(3.4.59)

$$
\lim_{n \to \infty} B_n = \frac{1}{2}
$$

(3.4.60)

so by the definition (3.4.8),

$$
\lim_{n \to \infty} r_n^{(2)} = 1
$$

(3.4.61)

Also, for each $n \geq 50$, $r_n^{(2)}$ is analytic in $S_4$, thus $\delta_n^{(2)} = r_n/r_n^{(2)} - 1$ is also analytic in $S_4$. Therefore the estimate (3.4.13) in Lemma 3.4.1 (iv) and the maximum modulus principle imply that

$$
|\delta_n^{(2)}| \leq 3/20 \quad (\lambda \in S_4)
$$

(3.4.62)

In addition (3.3.62) implies that either $\lim_{n \to \infty} \delta_n^{(2)} = 0$, which means $r_n \to 1$ or $\lim_{n \to \infty} \delta_n^{(2)} = -\frac{1}{2}$, which means $r_n \to 1/2$. Since (3.4.62) implies $\lim_{n \to \infty} \delta_n^{(2)} \neq -\frac{1}{2}$, Corollary 3.4.2 is proved.
3.5 Appendix

3.5.1 Bounds of polynomials and Proof of Lemma 3.3.1

First we describe the method used to estimate upper bounds of absolute values of polynomials of the form

\[ P(x, \lambda) = \sum_{i,j} a_{i,j} x^i \lambda^j \quad (3.5.1) \]

where \( x \in [-1, 1] \), \( \lambda \in S_1 \subseteq B(\frac{1+i}{4}, \frac{\sqrt{2}}{4}) \), where \( B(\frac{1+i}{4}, \frac{\sqrt{2}}{4}) \) denotes the closed disk centered at \( \frac{1+i}{4} \) with radius \( \frac{\sqrt{2}}{4} \). Reexpand the polynomial \( P(x, \lambda) \) at \( \lambda_0 = \frac{1+i}{4} \).

\[ P(x, \lambda) = \sum_{i,j} \tilde{a}_{i,j} x^i (\lambda - \lambda_0)^j \quad (3.5.2) \]

Then we express each \( x^i \) as a linear combination of Chebyshev polynomials of the first kind.

\[ x^i = \sum_{l=0}^{i} c_l T_l(x) \quad (3.5.3) \]

Rewrite the polynomial again

\[ P(x, \lambda) = \sum_{i,j} \tilde{a}_{i,j} \left( \sum_{l=0}^{i} c_l T_l(x) \right) (\lambda - \lambda_0)^j \quad (3.5.4) \]

\[ = \sum_{i,j} \sum_{l=0}^{i} \tilde{a}_{i,j} c_l T_l(x)(\lambda - \lambda_0)^j \quad (3.5.5) \]

and use the facts that for each \( n \), \( |T_n(x)| \leq 1 \) on \([0, 1]\) and \( |\lambda - \lambda_0| \leq \frac{\sqrt{2}}{4} \) to obtain the following estimate:

\[ |P(x, \lambda)| \leq \sum_{i,j} \sum_{l=0}^{i} |\tilde{a}_{i,j}| |c_l| \left( \frac{\sqrt{2}}{4} \right)^j \quad (3.5.6) \]

**Proof of Lemma 3.3.1** As explained before Lemma 3.3.1,
\[ \epsilon_1(t, \lambda) = L \delta_1 = -L G_a = t \bar{\epsilon}_1(t, \lambda) \]

For \( t \in (0, 1/4) \) and \( \lambda \in S_1 \) we are interested in an upper bound of \( |\bar{\epsilon}_1(t, \lambda)| \). For \( t \in (1/4, 1/2) \) and \( \lambda \in S_1 \) we are interested in an upper bound of \( |\epsilon_1(t, \lambda)| = |L G_a| \).

Let
\[
R_1(t, \lambda) := (2 - t)^2 \bar{\epsilon}_1(t, \lambda) \quad t \in (0, 1/4) \tag{3.5.7}
\]
\[
R_2(t, \lambda) := (2 - t)^2 \epsilon_1(t, \lambda) \quad t \in (1/4, 1/2) \tag{3.5.8}
\]

\( R_1 \) and \( R_2 \) are both polynomials in \( t \) and \( \lambda \). Make a change of variable \( t = (x + 1)/8 \), then an upper bound of \( |R_1((x + 1)/8, \lambda)| \) with \( x \in [-1, 1] \), \( \lambda \in S_1 \) is an upper bound of \( |R_1(t, \lambda)| \) with \( t \in [0, 1/4], \lambda \in S_1 \), which is what we seek. Using the method mentioned in the beginning of the section, we obtain the following:

\[
|R_1((x + 1)/8, \lambda)| \leq 0.00441 \quad x \in [-1, 1] \quad \lambda \in S_1 \tag{3.5.9}
\]

i.e.
\[
|R_1(t, \lambda)| \leq 0.00441 \quad t \in [0, 1/4] \quad \lambda \in S_1 \tag{3.5.10}
\]

Hence

\[
|\bar{\epsilon}_1(t, \lambda)| = \left| \frac{R_1(t, \lambda)}{(2 - t)^2} \right| \leq \frac{0.00441}{(2 - 1/4)^2} = 0.00144 \tag{3.5.11}
\]

For \( R_2(t, \lambda) \), let \( t = (x + 3)/8 \), then we an upper bound of \( |R_2((x + 3)/8, \lambda)| \) with \( x \in [-1, 1] \), \( \lambda \in S_1 \) is what we seek. Apply the method to \( |R_2((x + 3)/8, \lambda)| \) we get

\[
|R_2((x + 3)/8, \lambda)| \leq 0.00595 \quad x \in [-1, 1] \quad \lambda \in S_1 \tag{3.5.12}
\]

i.e.
\[
|R_2(t, \lambda)| \leq 0.00595 \quad t \in [1/4, 1/2] \quad \lambda \in S_1 \tag{3.5.13}
\]

Hence

\[
|\epsilon_1(t, \lambda)| = \left| \frac{R_2(t, \lambda)}{(2 - t)^2} \right| \leq \frac{0.00595}{(2 - 1/2)^2} < 0.00265 \tag{3.5.14}
\]

The other estimates in Lemma 3.3.1 are trivial.
3.5.2 Proof of the estimate 3.3.14

Let $W(\lambda) = W(G, H)(1/2, \lambda)$. $W(\lambda)$ is a polynomial in $\lambda$, the region of interest is $\lambda \in S_1$. First take $W_0 = W(1/2, 1 + i/4)$, which is just a constant with $|W_0| > 0.79$. Then use the method described in the previous section to acquire an upper bound of $|W(\lambda) - W_0|$. Notice that $W(\lambda) - W_0$ is just a special case of the class of polynomial discussed, so we obtain:

$$|W(\lambda) - W_0| \leq 0.33 \quad \lambda \in S_1$$

(3.5.15)

Thus

$$|W(\lambda)| \geq |W_0| - |W(\lambda) - W_0| > 0.79 - 0.33 > 0.46$$

(3.5.16)

Sharper estimates can be achieved at the cost of simplicity of the proof.

3.5.3 Proof of Lemma 3.3.3

For $n \leq 10$ we obtain $J_n$ and $M_n$ using the method in §3.5.5 and §3.5.6. Using (3.3.40) and (3.3.41), we obtain:

$$|a_{n+1}| \leq |p_n(\lambda)||a_n| + \frac{1}{(n + 1)(n + \lambda)} \sum_{k=0}^{n-1} \frac{n - k + 1}{2^{n-k}} |a_k|$$

(3.5.17)

$$|b_{n+1}| \leq |q_n(\lambda)||b_n| + \frac{1}{(n + 1)(n + \frac{5}{2})} \sum_{k=0}^{n-1} 4(n - k + 1) |b_k|$$

(3.5.18)

which give upper bounds $J_n$ of $|a_n|$ and $M_n$ of $|b_n|$ for $11 \leq n \leq 30, \lambda \in S_2$. The details are as follows.

We first estimate the difference between $p_n(\lambda)$ and its leading behavior for $n \geq 30$:

$$e_n(\lambda) := p_n(\lambda) - \left(1 + \frac{1}{2n}\right) = \frac{1}{4} \frac{\lambda^2 n - 3 \lambda n - 2 \lambda - 4 n}{(n + 1)(n + \lambda)}$$

(3.5.19)
With the notation
\[
\lambda = r + is \quad (0 \leq r \leq 1/2, 1/2 \leq s \leq 4)
\]
\[
z = 1/n \quad (0 < z \leq 1/30)
\]
we have
\[
\text{Re}(e_n) = z^2\left( r^3z - 2r^2z^2 + rs^2z - 2s^2z^2 - 3r^2z - 3s^2z + r^2 - 6rz \\
- s^2 - 3r - 4 \right) / \left[ 4 (1 + z) \left( r^2z^2 + s^2z^2 + 2rz + 1 \right) \right]
\leq \frac{z^2(r^3z + rs^2z + r^2 - 4)}{4(1 + z)(r^2z^2 + s^2z^2 + 2rz + 1)}
\leq \frac{z^2(0.5^3/30 + (0.5)4^2/30 + 0.5^2 - 4)}{4(1 + z)(r^2z^2 + s^2z^2 + 2rz + 1)} < 0
\]
and
\[
\text{Im}(e_n) = \frac{1}{4} \frac{z^2 s (r^2z + s^2z + 2r + 2z - 3)}{(1 + z)(r^2z^2 + s^2z^2 + 2rz + 1)}
\geq -\frac{4187}{648000}
\]
Hence
\[
1/100 < \text{Re}(e_n) < 0
\]
\[
0 > \text{Im}(e_n) > -\frac{3z^2}{z^2(r^2 + s^2) + 1} \geq -\frac{3}{r^2 + s^2 + 1/z^2} \geq -\frac{3}{1/4 + n^2}
\]
for \(\lambda \in S_2 \) and \(n \geq 30\). So
\[
|p_n|^2 = \left( 1 + \frac{1}{2n} + \text{Re}(e_n) \right)^2 + \left( \text{Im}(e_n) \right)^2
\leq \left( 1 + \frac{1}{2n} \right)^2 + \left( \frac{3}{1/4 + n^2} \right)^2
\]
(3.5.27)
Let
\[ P_n = \left( \left( 1 + \frac{1}{2n} \right)^2 + \left( \frac{3}{1/4 + n^2} \right)^2 \right)^{1/2} \]  \hspace{1cm} (3.5.28)
then \(|p_n| \leq P_{30}\) for \(n \geq 30\). For \(10 \leq n \leq 29\), we obtain \(P_n\) using the method in §3.5.5, see §3.5.8 for these values of \(P_n\). From (3.5.17) we have:
\[ |a_{n+1}| \leq P_n |a_n| + \frac{1}{(n+1)|n+i/2|} \sum_{k=0}^{n-1} \frac{n-k+1}{2^{n-k}} |a_k| \quad (n \geq 10) \]  \hspace{1cm} (3.5.29)
For \(10 \leq n \leq 29\) if we choose \(J_{n+1}\) such that
\[ J_{n+1} > P_n J_n + \frac{1}{(n+1)|n+i/2|} \sum_{k=0}^{n-1} \frac{n-k+1}{2^{n-k}} J_k \]  \hspace{1cm} (3.5.30)
then
\[ |a_{n+1}| < J_{n+1} \quad (10 \leq n \leq 29) \]  \hspace{1cm} (3.5.31)
We use induction to take care of larger \(n\). Explicit calculation shows that for our choice of \(k_1\) and \(J_n\), \(J_{30} < k_1^{30}\) (see §3.5.8). For \(n > 30\), assuming for \(30 \leq m \leq n\),
\(|a_m| \leq k_1^n\), we have:

\[
|a_{n+1}| < P_n|a_n| + \frac{1}{(n+1)n} \sum_{i=0}^{n-1} \frac{n-i+1}{2^{n-i}} |a_i|
\]

\[
< P_n k_1^n + \frac{1}{(n+1)n} \sum_{i=0}^{n-1} \frac{n-i+1}{2^{n-i}} k_i^n + \frac{1}{(n+1)n} \sum_{i=0}^{n-1} \frac{n-i+1}{2^{n-i}}(|a_i| - k_i^n)
\]

\[
< P_{30} k_1^n + \frac{4(2k_1)^n k_1 - 2nk_1 - (2k_1)^n - 4k_1 + n + 1}{2^n(2k_1 - 1)^2(n+1)n} + \frac{1}{(n+1)n} \sum_{i=0}^{30} \frac{n-i+1}{2^{n-i}}(J_i - k_i^n)
\]

\[
< P_{30} k_1^n + k_1^n \frac{4k_1 - 1}{(2k_1 - 1)^2(n+1)n} + \frac{(1 - 2k_1)n + 1 - 4k_1}{2^n(n+1)(2k_1 - 1)^2} + \frac{1}{(n+1)n}(\tilde{V}_1 n + \tilde{V}_0)
\]

\[
< \left( P_{30} + \frac{4k_1 - 1}{(2k_1 - 1)^2(n+1)n} \right) k_1^n + \frac{(-1)_{2k_1-1} + \tilde{V}_1}{2^n(n+1)} n + \frac{1 - 4k_1}{(2k_1 - 1)^2} + \tilde{V}_0
\]

\[
< (1.01668 + 0.00301) k_1^n
\]

\[
< 1.0197 k_1^n < k_1^{n+1} \quad (3.5.32)
\]

where \(\tilde{V}_1 = -196921202\) and \(\tilde{V}_0 = 5563721416\) are chosen to satisfy

\[
\sum_{i=0}^{30} \frac{n-i+1}{2^{n-i}}(J_i - k_i^n) < \tilde{V}_1 n + \tilde{V}_0 \quad (3.5.33)
\]

and explicit calculation shows that

\[
\left( \frac{-1}{2k_1 - 1} + \tilde{V}_1 \right) n + \frac{1 - 4k_1}{(2k_1 - 1)^2} + \tilde{V}_0 < 0 \quad (n \geq 30)
\]  

(3.5.34)

So Lemma 3.3.3 (i) is proved.

Part (ii) is similar. It is straightforward to check that for each \(n \geq 2\), \(|q_n|\) is monotonic on each line segment of \(\partial S_2\) and the maximum is attained at \(\lambda = 4i+1/2\). Let

\[
Q_n := |q_n(4i + 1/2)| \quad (n \geq 10)
\]  

(3.5.35)
Then \(|q_n(\lambda)| \leq Q_n\) for all \(\lambda \in S_2\) and \(n \geq 10\). It is also straightforward to check that \(|Q_n| < 1\) for all \(n \geq 2\).

From (3.5.18) and \(M_n\) \((n \leq 10)\) we choose \(M_n\) recursively for \(11 \leq n \leq 30\) such that

\[
M_{n+1} > Q_n M_n + \frac{1}{(n+1)(n+\frac{5}{2})} \sum_{k=0}^{n-1} 4(n-k+1)M_k
\]

so

\[
|b_{n+1}| \leq |q_n(\lambda)||b_n| + \frac{1}{(n+1)(n+\frac{5}{2})} \sum_{k=0}^{n-1} 4(n-k+1)|b_k| < M_{n+1}
\]

It is easy to check (see §3.5.8) that \(M_{30} < k_{30}^n\). For \(n > 30\), assuming for all \(30 \leq m \leq n\), \(|b_m| \leq k_m^n\), we have:

\[
|b_{n+1}| < |b_n| + \frac{4}{(n+1)(n+\frac{5}{2})} \sum_{i=0}^{n-1} (n-i+1)|b_i|
\]

\[
< k_2^n + \frac{4}{(n+1)(n+\frac{5}{2})} \sum_{i=0}^{n-1} (n-i+1)k_i^2
\]

\[
+ \frac{4}{(n+1)(n+\frac{5}{2})} \sum_{i=0}^{30} (n-i+1)(M_i - k_i^2)
\]

\[
< k_2^n + \frac{4}{(n+1)(n+\frac{5}{2})} \left[ k_2^n \frac{2k_2 - 1}{(k_2 - 1)^2} + \frac{n(1-k_2) + 1 - 2k_2}{(k_2 - 1)^2} + \bar{W}_1 n + \bar{W}_0 \right]
\]

\[
< k_2^n \left( 1 + \frac{4(2k_2 - 1)}{(k_2 - 1)^2(n+1)(n+\frac{5}{2})} \right)
\]

\[
+ k_2^n \left( 4k_2^{-n} \frac{(\bar{W}_1 - \frac{1}{k_2-1})n + \left( \frac{1-2k_2}{(k_2-1)^2} + \bar{W}_0 \right)}{(n+1)(n+\frac{5}{2})} \right)
\]

\[
< k_2^n \left( 1 + \frac{4 \cdot 1.4}{0.2^2 \cdot 30 \cdot 32.5} + \frac{4}{1.2^{30} \cdot 2.89} \right)
\]

\[
< k_2^n (1 + 0.144 + 0.049)
\]

\[
< k_2^{n+1}
\]

where \(\bar{W}_1 = 271\) and \(\bar{W}_0 = -5622\) are chosen such that

\[
\sum_{i=0}^{30} (n-i+1)(M_i - k_i^2) < \bar{W}_1 n + \bar{W}_0
\]
and explicit calculation shows that for $n \geq 30$

\[
\frac{(\tilde{W}_1 - \frac{1}{k_2-1})n + \left(\frac{1-2k_2}{(k_2-1)^2} + \tilde{W}_0\right)}{(n+1)(n+5/2)} = \frac{266n - 5657}{(n+1)(n+5/2)} \leq 2.89
\]  \quad (3.5.40)

ending the proof of Lemma 3.3.3.
3.5.4 Approximations $G_a$ and $H_a$

For $t \in [0, 1/4)$:

$$G_a(t) := \left( \frac{106}{111} \lambda^4 - \frac{89}{86} + \frac{52}{113} t \lambda^5 - \frac{82}{89} \lambda^3 + \frac{61}{36} \lambda^2 \right) t^6 + \left( \frac{18}{181} \lambda^3 - \frac{79}{214} \lambda^5 \right) t^5$$

$$- \frac{11}{75} - \frac{26}{259} \lambda^3 - \frac{59}{362} \lambda^6 - \frac{19}{88} \lambda - \frac{29}{93} t \lambda^4 + \frac{44}{205} \lambda^2 - \frac{28}{283} \lambda^4 + \frac{58}{1195} \lambda^5 \right) t^5$$

$$+ \left( - \frac{58}{161} \lambda^3 - \frac{2}{29} \lambda^6 - \frac{35}{96} \lambda^4 + \frac{35}{243} t \lambda^5 + \frac{17}{749} \lambda \right) t^4 + \left( - \frac{41}{90} - \frac{31}{257} \lambda^4 + \frac{22}{75} \lambda^4 + \frac{78}{127} \lambda^2 \right) t^3$$

$$+ \frac{271}{7045} \lambda^3 + \frac{45}{314} \lambda^6 - \frac{33}{230} \lambda^5 - \frac{87}{293} \lambda^3 + \frac{6}{713} \lambda^6 - \frac{3}{1081} \lambda^2 - \frac{7}{111} \lambda^6 \right) t^2$$

$$+ \frac{43}{519} \lambda \right) t + \lambda$$

$$+ \frac{3}{4} \lambda - \frac{1}{2} \right) t + \lambda$$
For $t \in (1/4, 1/2)$

\[ G_a(t) := \left( -\frac{121}{18} i\lambda^4 - \frac{421}{77} \lambda - \frac{183}{23} \lambda^5 - \frac{1045}{58} - 17 \lambda^3 \right) t^7 + \left( \frac{361}{27} i\lambda^4 \right. \\
- \frac{38}{17} i\lambda^6 + \frac{76}{7} \lambda + \frac{657}{122} i\lambda^5 + \frac{287}{15} \lambda^2 + \frac{236}{21} \lambda^4 + \frac{491}{31} \lambda^5 + \frac{250}{7} + \frac{175}{129} i\lambda^3 \\
+ \frac{72}{163} \lambda^6 + \frac{1387}{41} i\lambda^3 \bigg) t^6 + \left( -\frac{313}{21} \lambda^5 - \frac{29}{330} i\lambda^2 - \frac{327}{26} i\lambda^4 - \frac{571}{31} \lambda^4 \\
- \frac{1305}{41} \lambda^3 - \frac{193}{19} \lambda + \frac{216}{59} i\lambda^6 - \frac{218}{7} \lambda^2 - \frac{238}{27} i\lambda^5 - \frac{1178}{35} - \frac{62}{85} \lambda^6 \\
- \frac{71}{32} i\lambda^3 \bigg) t^5 + \left( \frac{153}{22} i\lambda^5 + \frac{7/4 i\lambda^3 + 335}{64} \lambda + \frac{423}{25} + \frac{145}{9} \lambda^3 + \frac{1381}{56} \lambda^2 \\
+ \frac{76}{677} i\lambda^2 + \frac{151}{20} \lambda^5 + \frac{116}{201} \lambda^6 + \frac{247}{17} \lambda^4 + \frac{312}{49} i\lambda^4 - \frac{26}{9} i\lambda^6 \bigg) t^4 + \left( -\frac{181}{33} \lambda^3 \\
- \frac{93}{43} i\lambda^4 - \frac{113}{72} \lambda - \frac{136}{53} \lambda^5 + \frac{65}{59} i\lambda^6 - \frac{61}{839} i\lambda^2 - \frac{220}{83} i\lambda^5 - \frac{39}{176} \lambda^6 - \frac{504}{85} \\
- \frac{172}{31} \lambda^4 - \frac{5}{3333} i\lambda - \frac{324}{35} \lambda^2 - \frac{2543}{3815} i\lambda^3 \bigg) t^3 + \left( \frac{4}{223} i\lambda^2 + \frac{530}{187} \lambda^2 + \frac{67}{107} \lambda \\
+ \frac{63}{83} i\lambda^5 + \frac{17}{89} \lambda^3 + \frac{34}{45} \lambda^3 + \frac{71}{132} + \frac{1}{9639} i - \frac{35}{111} i\lambda^6 + \frac{264}{167} \lambda^4 + \frac{14}{39} \lambda^5 \\
+ \frac{8}{127} \lambda^6 + \frac{56}{185} i\lambda^4 + \frac{1}{1180} i\lambda \bigg) t^2 + \left( \frac{32}{953} i\lambda^6 - \frac{18}{511} \lambda^2 - \frac{2}{7333} i\lambda - \frac{23}{222} \lambda^3 \\
- \frac{17}{837} i\lambda^3 - \frac{2}{637} i\lambda^2 - \frac{133}{787} \lambda^4 - \frac{5}{739} \lambda^6 - \frac{9}{220} i\lambda^4 - \frac{12}{247} \lambda^5 - \frac{53}{656} i\lambda^5 + \frac{38}{53} \lambda \\
- \frac{92}{151} - \frac{1}{19278} i \bigg) t - \frac{673}{672} \lambda + \frac{3}{1631} i\lambda^4 + \frac{8}{1717} \lambda^3 - \frac{5}{2888} i\lambda^6 + \frac{1}{955} i\lambda^3 \\
+ \frac{1}{44699} i\lambda + \frac{1}{240} i\lambda^5 + \frac{15}{3052} + \frac{1}{154231} i + \frac{1}{5265} i\lambda^2 + \frac{11}{1262} \lambda^4 + \frac{3}{1372} \lambda^5 \\
+ \frac{2}{5759} \lambda^6 + \frac{110}{7481} \lambda^2 \right)
For $t \in (1/2, 3/4)$, let $s = 1 - t$

$$H_a(1 - s) := \left( \frac{64}{67} i\lambda^5 - \frac{95}{158} i\lambda^6 - \frac{205}{461} i\lambda^4 - \frac{17}{13} \right) s^6 + \left( \frac{148}{61} + \frac{19}{305} i\lambda^4 + \frac{10}{199} \right) \lambda s^5 + \left( \frac{182}{109} i\lambda^5 \right) \lambda^2 s^4 + \left( \frac{9}{134} i\lambda^6 + \frac{61}{130} i\lambda^6 - \frac{33}{236} + \frac{60}{193} \lambda - \frac{24}{593} \lambda^2 - \frac{119}{160} i\lambda^5 \right) \lambda^3 s^3 + \left( -\frac{233}{599} \lambda - \frac{83}{975} i\lambda^4 - \frac{15}{697} \lambda^6 \right) \lambda^4 s^2 + \left( \frac{193}{643} \lambda \right) \lambda^5 s + \left( -\frac{5}{2957} \lambda^3 - \frac{79}{58} + \frac{9}{92} \lambda^2 + \frac{11}{733} i\lambda^6 + \frac{5}{1156} i\lambda^6 - \frac{75}{3151} i\lambda^5 + \frac{4}{1967} \lambda^4 + \frac{5}{453} i\lambda^4 \right) \lambda^6 s - \left( \frac{4}{6811} i\lambda^4 - \frac{1}{24270} \lambda - \frac{5}{6274} i\lambda^6 + \frac{10}{7901} i\lambda^5 - \frac{2}{7433} \lambda^5 \right) \lambda^7 - \frac{1}{4713} \lambda^6 + \frac{498}{499} + \frac{1}{8478} \lambda^3 + \frac{1}{7548} \lambda^2 - \frac{1}{8040} \lambda^4$$

For $t \in (3/4, 1]$, let $s = 1 - t$,

$$H_a(1 - s) := \left( \frac{13}{76} \lambda - \frac{201}{286} \right) s^5 + \left( \frac{11}{2267} \lambda^5 + \frac{13}{1097} \lambda^4 + \frac{21}{1000} \lambda^3 - \frac{17}{3153} \lambda^2 \right) s^4 + \left( \frac{1}{1267} \lambda^5 + \frac{11}{1538} \lambda^4 + \frac{5}{1326} \lambda^3 + \frac{7}{953} \lambda^2 + \frac{85}{246} \lambda \right) s^3 + \left( \frac{1}{6961} \lambda^6 + \frac{1}{37023} \lambda^5 + \frac{4}{1103} \lambda^4 + \frac{60}{1679} \lambda^3 + \frac{17}{1590} \lambda^2 - \frac{152}{387} \lambda \right) s^2 + \left( \frac{98}{73} \right) s + 1$$
3.5.5 Estimate of modulus of some rational functions on the boundary of a rectangle.

A description of the method

Here we describe a method to obtain an upper bound and a lower bound of $|F(\lambda)|$ on the boundary of a rectangle

$$R := \{ z : \text{Re}(z) \in [0, 1/2], \text{Im}(z) \in [a, b] \}$$

where

$$F(\lambda) = P(\lambda) + \sum_{j=0}^{n-1} \frac{\tilde{a}_j}{\lambda - s_j}$$

(3.5.41)

where $P(\lambda)$ is a polynomial of degree $m$, $(m \leq 12)$, $s_j$ are distinct complex numbers outside of $R$, and $\tilde{a}_j$’s are constants. The estimates of $F(\lambda)$ are obtained in 4 steps.

(i) Define a partition $\mathcal{P}$ on the boundary $\partial S_2$, given by $\{d_i\}_{0 \leq i \leq N-1}$, satisfying $d_0 \in \{a, a+i, b, b+i\} \subseteq \mathcal{P}$

(3.5.42)

d_i’s are ordered counterclockwise. Let $d_N = d_0$. For each $i \in \{1, 2, ..., N\}$, denote the disk that has $[d_{i-1}, d_i]$ as a diameter by $D_i$, the center of $D_i$ by $c_i$. Also let $r_i = \frac{d_i - d_{i-1}}{2}$. Since all vertices of $S_2$ are in $\mathcal{P}$, each $[d_{i-1}, d_i]$ either lies on a horizontal side or a vertical side of $\partial S_2$. In the following steps we will find an upper bound and a lower bound of $|F(\lambda)|$ on each $[d_{i-1}, d_i]$.

Now fix $i \in \{1, 2, ..., N\}$.

(ii) Make a change of variable

$$\lambda_i(x) = c_i + r_i x$$

(3.5.43)
λ_i is a bijection that maps the unit disk \( \{ |x| \leq 1 \} \) to \( D_i \), and the image of \([-1, 1]\) under \( \lambda_i \) is \([d_{i-1}, d_i]\). Consider the Taylor expansion of \( F(\lambda_i(x)) \):

\[
F(\lambda_i(x)) = P(c_i + r_ix) + \sum_{j=0}^{n-1} \frac{\tilde{a}_j}{c_i + r_ix - s_j}
\]

\[
= P(c_i + r_ix) + \sum_{j=0}^{n-1} \frac{\tilde{a}_j}{c_i - s_j} \left( \frac{r_ix}{c_i - s_j} \right)^k + \sum_{j=0}^{n-1} \frac{\tilde{a}_j}{c_i - s_j} \left( \frac{-r_ix}{c_i - s_j} \right)^{m_0 + 1}
\]

(3.5.44)

Let

\[
\tilde{F}_{i,1}(x) = P(c_i + r_ix) + \sum_{j=0}^{n-1} \frac{\tilde{a}_j}{c_i - s_j} \sum_{k=0}^{m_0} \left( -\frac{r_ix}{c_i - s_j} \right)^k = \sum_{k=0}^{M} \tilde{b}_{k,i} x^k
\]

(3.5.45)

\[
E_{i,1} > \sum_{j=0}^{n-1} \left| \frac{\tilde{a}_j}{c_i - s_j} \right| \left| \frac{r_ix}{c_i - s_j} \right|^{m_0 + 1}
\]

(3.5.46)

where \( M = \max(m, m_0) \), then

\[
|F(\lambda_i(x)) - \tilde{F}_{i,1}(x)| < E_{i,1} \quad (|x| \leq 1)
\]

(3.5.47)

Repaing each coefficient \( \tilde{b}_{k,i} \) by a rational number \( \tilde{c}_{k,i} \) within \( e^{-9} \) of it we obtain

\[
F_{i,1}(x) = \sum_{k=0}^{M} \tilde{c}_{k,i} x^k
\]

(3.5.48)

Let

\[
E_{i,2} > \sum_{k=0}^{M} |\tilde{b}_{k,i} - \tilde{c}_{k,i}|
\]

(3.5.49)

Then

\[
|F_{i,1}(x) - \tilde{F}_{i,1}(x)| < E_{i,2} \quad (|x| \leq 1)
\]

(3.5.50)

70
(iii) Consider $x \in [-1, 1]$. We estimate $|F_{i,1}(x)|^2$ using Chebyshev polynomials of the first kind.

$$|F_{i,1}(x)|^2 = [\text{Re}(F_{i,1}(x))]^2 + [\text{Im}(F_{i,1}(x))]^2$$

(3.5.51)

$|F_{i,1}(x)|^2$ is a polynomial of degree $2M$. Express each power of $x$ by a linear combination of Chebyshev polynomials $T_k(x)$ then we have a nonnegative polynomial:

$$|F_{i,1}(x)|^2 = \sum_{k=0}^{2M} \tilde{e}_{k,i} T_k(x)$$

(3.5.52)

For $k = 0, 1, 2, 3$ let $\tilde{f}_{k,i}$ be a rational number within $e^{-9}$ of it and let

$$A_i(x) = \sum_{k=0}^{3} \tilde{f}_{k,i} T_k(x)$$

(3.5.53)

$$E_{i,3} > \sum_{k=0}^{3} |\tilde{e}_{k,i} - \tilde{f}_{k,i}| + \sum_{k=4}^{2M} |\tilde{e}_{k,i}|$$

(3.5.54)

Then

$$||F_{i,1}(x)|^2 - A_i(x)| < E_{i,3} \quad x \in [-1, 1]$$

(3.5.55)

Now $A_i(x)$ is a cubic polynomial, so we can calculate its maximum $U_{i,1}$ and minimum $L_{i,1}$ on $[-1, 1]$,

$$L_{i,1} \leq A_i(x) \leq U_{i,1}$$

(3.5.56)

In cases we deal with in this paper, we can always choose $E_{i,l}$ ($l = 1, 2, 3$), $U_i$ and $L_i$ such that

$$\sqrt{L_{i,1} - E_{i,3}} >> E_{i,1} + E_{i,2}$$

so from (3.5.55) and (3.5.56) we get:

$$\sqrt{L_{i,1} - E_{i,3}} \leq |F_{i,1}(x)| \leq \sqrt{U_{i,1} + E_{i,3}}$$

(3.5.57)
Furthermore, from (3.5.47) and (3.5.50) we get:

\[ |F_{i,1}(x) - F(\lambda_i(x))| \leq E_{i,1} + E_{i,2} \] (3.5.58)

Thus if for each \( i \) we choose

\[ U_i > \sqrt{U_{i,1} + E_{i,3} + E_{i,1} + E_{i,2}} \] (3.5.59)
\[ L_i < \sqrt{L_{i,1} - E_{i,3} - E_{i,1} - E_{i,2}} \] (3.5.60)

then

\[ L_i < |F(\lambda_i(x))| < U_i, \quad x \in [-1, 1] \] (3.5.61)

i.e.

\[ L_i < |F(\lambda)| < U_i, \quad \lambda \in [d_{i-1}, d_i] \] (3.5.62)

Finally let

\[ U = \max_{1 \leq i \leq N} U_i \] (3.5.63)
\[ L = \min_{1 \leq i \leq N} L_i \] (3.5.64)

Then

\[ L < |F(\lambda)| < U \quad (\lambda \in \partial R) \] (3.5.65)

**Derivation of \( J_n, P_n, \) upper bounds of \( |G_a(t_0)| \) and \( |G_a'(t_0)| \) on \( S_2 \) and a lower bound of \( |W_a(t_0)| \) on \( \partial S_2. \)**

Each one of \( a_n, \) (1 \( \leq n \leq 10), G_a(t_0), \) \( G_a'(t_0) \) and \( W_a(t_0) \) is a rational function of the form in (3.5.41) with \( R = S_2. \) We use the method described in §3.5.5 to obtain upper bounds of \( |a_n|, \) (1 \( \leq n \leq 10), |G_a(t_0)|, |G_a'(t_0)| \) over \( \partial S_2 \) and a lower bound of \( |W_a(t_0)| \) on \( \partial S_2. \)
Two partitions are used

\[ \mathcal{P}_1 = \{ d_i^{(1)} \}_{0 \leq i \leq 8} \]  
(3.5.66)

\[ \mathcal{P}_2 = \{ d_i^{(2)} \}_{0 \leq i \leq 8} \]  
(3.5.67)

where

\[ d_0^{(1)} = \frac{1}{2} + 4i, \quad d_1^{(1)} = \frac{1}{4} + 4i, \quad d_2^{(1)} = 4i, \quad d_3^{(1)} = \frac{17}{6}i, \quad d_4^{(1)} = \frac{5}{3}i, \]

\[ d_5^{(1)} = \frac{1}{2}i, \quad d_6^{(1)} = \frac{1}{2} + \frac{1}{2}i, \quad d_7^{(1)} = \frac{1}{2} + \frac{17}{8}i, \quad d_8^{(1)} = \frac{1}{2} + \frac{15}{4}i, \]

and \( d_i^{(1)} + d_i^{(2)} = \frac{9}{2}i + \frac{1}{2} \) for each \( i \).

To obtain upper bounds of \( |a_n| \), \( (1 \leq n \leq 6) \) we used partition \( \mathcal{P}_1 \). To obtain upper bounds of \( |a_n| \), \( (7 \leq n \leq 10) \), \( |p_n| \) \( (11 \leq n \leq 29) \), \( |G_a(t_0)| \) and \( |G'_a(t_0)| \) we used partition \( \mathcal{P}_2 \).

In all these calculations we chose to keep 11 terms in the Taylor expansions in Step (i), i.e. \( m_0 = 10 \).

Note that each one of \( a_n \)’s, \( G_a(t_0) \) and \( G'_a(t_0) \) is analytic in \( S_2 \), so an upper bound of its modulus on the boundary \( \partial S_2 \) is also an upper bound on whole \( S_2 \).

To obtain a lower bound of \( W_a(t_0) \) on \( S_2 \) we added a couple of steps. First approximate \( W_a \) by a rational function \( F(\lambda) \) which can be estimated by the method in §3.5.5:

\[ W_a(t_0) = F(\lambda) + \lambda^{13} \sum_{k=0}^{15} \tilde{g}_i \lambda^i \]  
(3.5.68)

where \( F(\lambda) \) is a rational function as described in §3.5.5, and fortunately by explicit calculation

\[ |\lambda|^{13} \sum_{k=0}^{15} |\tilde{g}_i| |\lambda|^i < 0.079 \quad (|\lambda| \leq |4i + 1/2|) \]  
(3.5.69)

Use the method in §3.5.5 and choose the partition to be \( \mathcal{P}_2 \), \( m_0 = 10 \), we obtain:

\[ |F(\lambda)| > 1.14 \quad (\lambda \in \partial S_2) \]  
(3.5.70)
Thus

\[ |W_\alpha(t_0)| > 1.14 - 0.08 = 1.06 \quad (\lambda \in \partial S_2) \quad (3.5.71) \]

Finally, to show a lower bound of \( |W_\alpha(t_0)| \) on the boundary \( \partial S_2 \) is also a lower bound on \( S_2 \) it suffices to show that \( W_\alpha(t_0) \) has no roots in \( S_2 \), see §3.5.7 for the proof.

### 3.5.6 Estimate of modulus of some polynomials on the boundary of \( S_2 \)

#### A description of the method

In this subsection we describe the method used to estimate \( |P(\lambda)| \) on the boundary of \( S_2 \), where \( P(\lambda) \) can be any one of \( b_n \) (1 \( \leq n \leq 10 \)), \( H_\alpha(t_0) \) or \( H'_\alpha(t_0) \). Denote the degree of \( P(\lambda) \) by \( m \). The method is similar to the one in §3.5.5. Slight modifications are made. (i) Let \( \mathcal{P}, \{d_i\}_{0 \leq i \leq N} \) and \( \{c_i\}_{1 \leq i \leq N} \) all be the same as in §3.5.5

Fix \( i \in \{1, 2, ..., N\} \).

(ii) Make a change of variable (3.5.43), then

\[
P(\lambda_i(x)) = \sum_{k=0}^{m} \tilde{a}_{k,i} x^k
\]

(3.5.72)

For each \( k \) express \( x^k \) by a linear combination of Chebyshev polynomials \( T_k(x) \) then we have

\[
P(\lambda_i(x)) = \sum_{k=0}^{m} \tilde{b}_{k,i} T_k(x)
\]

(3.5.73)

Let

\[
P_{i,1}(x) = \sum_{k=0}^{5} \tilde{b}_{k,i} T_k(x)
\]

(3.5.74)

\[
E_{i,1} > \sum_{k=6}^{m} |\tilde{b}_{k,i}|
\]

(3.5.75)

Then

\[
|P(\lambda_i(x)) - P_{i,1}(x)| < E_{i,1} \quad x \in [-1, 1]
\]

(3.5.76)
(iii) Estimate the polynomial \(|P_{i,1}(x)|^2\) which is of degree at most 10.

\[
|P_{i,1}(x)|^2 = \sum_{i=0}^{10} \tilde{c}_{k,i} x^k
\]  

(3.5.77)

Express each power of \(x\) by a linear combination of Chebyshev polynomials \(T_k(x)\) then we have

\[
|P_{i,1}(x)|^2 = \sum_{i=0}^{10} \tilde{d}_{k,i} T_k(x) \]  

(3.5.78)

Let

\[
A_i(x) = \sum_{k=0}^{3} \tilde{e}_{k,i} T_k(x) \]  

(3.5.79)

\[
|E_{i,2}| > \sum_{k=0}^{3} \left| \tilde{d}_{k,i} - \tilde{e}_{k,i} \right| + \sum_{k=1}^{10} |\tilde{d}_{k,i}| \]  

(3.5.80)

where each \(\tilde{e}_{k,i}\) is a rational number within \(e^{-7}\) of \(\tilde{d}_{k,i}\), then

\[
||P_{i,1}(x)|^2 - A_i(x)| < E_{i,2} \quad x \in [-1,1]
\]  

(3.5.81)

Now we can determine the maximum \(U_{i,1}\) and the minimum \(L_{i,1}\) of the cubic polynomial \(A_i\) on \([-1,1]\). We obtain

\[
L_{i,1} \leq A_i(x) \leq U_{i,1} \quad x \in [-1,1]
\]  

(3.5.82)

From (3.5.81) and (3.5.82) we get

\[
\sqrt{L_{i,1} - E_{i,2}} \leq |P_{i,1}(x)| \leq \sqrt{U_{i,1} + E_{i,2}}
\]  

(3.5.83)

and in view of (3.5.76) if we choose for each \(i\):

\[
U_i > \sqrt{U_{i,1} + E_{i,2} + E_{i,1}}
\]  

(3.5.84)

\[
L_i < \sqrt{L_{i,1} - E_{i,2} - E_{i,1}}
\]  

(3.5.85)

then

\[
L_i < |P(\lambda_i(x))| < U_i \quad x \in [-1,1]
\]  

(3.5.86)
\[ L_i < |P(\lambda)| < U_i \quad \lambda \in [d_{i-1}, d_i] \]  

(3.5.87)

Let

\[ U = \max_{1 \leq i \leq N} U_i, \quad L = \min_{1 \leq i \leq N} L_i \]  

(3.5.88)

Then

\[ L < |P(\lambda)| < U, \quad \lambda \in \partial S_2 \]  

(3.5.89)

**Derivation of \( M_n \), upper bounds of \( H_a(t_0) \) and \( H'_a(t_0) \)**

Each one of \( b_n \), \( H_a(t_0) \) and \( H'_a(t_0) \) is a polynomial. To obtain the upper bounds of their moduli, 2 partitions are used

\[ \mathcal{P}_3 = \{d_i^{(3)}\}_{0 \leq i \leq 6} \]  

(3.5.90)

\[ \mathcal{P}_4 = \{d_i^{(4)}\}_{0 \leq i \leq 6} \]  

(3.5.91)

where

\[ d_0^{(3)} = \frac{i}{2}, \quad d_1^{(3)} = \frac{i}{2} + \frac{1}{2}, \quad d_2^{(3)} = \frac{7i}{2} + \frac{1}{2} \]

\[ d_3^{(3)} = \frac{1}{2} + 4i, \quad d_4^{(3)} = \frac{1}{4} + 4i, \quad d_5^{(3)} = 4i, \quad d_6^{(3)} = \frac{7}{2}i, \]

and \( d_i^{(3)} + d_i^{(4)} = \frac{9}{2}i + \frac{1}{2} \) for each \( i \).

To obtain upper bounds of \( |b_n| \) and \( |H'_a(t_0)| \), (1 \leq n \leq 10) we used partition \( \mathcal{P}_3 \).

To obtain upper bounds of \( |H_a(t_0)| \) we used partition \( \mathcal{P}_4 \).

**3.5.7 Estimate of \( W(Ga, Ha)(t_0) \) in \( S_2 \)**

Equally divide \( S_2 \) into 7 squares:

\[ S_{2,k} := \{ z : 0 \leq \text{Re}(z) \leq \frac{1}{2}, \text{Im}(C_k) - \frac{1}{4} \leq \text{Im}(z) \leq \text{Im}(C_k) + \frac{1}{4} \} \]  

(3.5.92)
where for each $k \in \{1, 2, \ldots, 7\}$

$$C_k = \frac{3i}{4} + \frac{1}{4} + (k - 1) \frac{i}{2}$$

is the center of the square $S_{2,k}$. $W_a(t_0) := W(G_a, H_a)(t_0)$ is of the form

$$W_a(t_0) = \sum_{j=0}^{7} \frac{\tilde{a}_j}{\lambda + j} + \sum_{k=0}^{12} \tilde{b}_k \lambda^k + \lambda^{13} \sum_{k=0}^{15} \tilde{g}_k \lambda^k$$

(3.5.93)

In view of (3.5.69), if we let

$$W_1(\lambda) = \sum_{j=0}^{7} \frac{\tilde{a}_j}{\lambda + j} + \sum_{k=0}^{12} \tilde{b}_k \lambda^k$$

(3.5.94)

then

$$|W_a(t_0) - W_1(\lambda)| < 0.079 \quad (\lambda \in S_2)$$

(3.5.95)

For each $i$, let

$$\bar{A}_i = W_1(C_i)$$

(3.5.96)

then for $\lambda \in S_{2,i}$:

$$|W_1(\lambda) - \bar{A}_i| = \left| \sum_{j=0}^{7} \frac{\tilde{a}_j}{\lambda + j} - \sum_{j=0}^{7} \frac{\tilde{a}_j}{C_i + j} + \sum_{k=0}^{12} \tilde{b}_j \lambda^k - \sum_{k=0}^{12} \tilde{b}_k C_i^k \right|$$

$$\leq \sum_{j=0}^{7} \frac{|\tilde{a}_j| |C_i - \lambda|}{|\lambda + j| |C_i + j|} + \sum_{k=0}^{12} \left| \tilde{b}_k - C_i^k \right|$$

(3.5.97)

For $\lambda \in S_{2,i}$,

$$|C_i - \lambda| \leq \frac{\sqrt{2}}{4};$$

(3.5.98)

$$|\lambda + j| \geq \left| C_i - \frac{1}{4} - \frac{i}{4} + j \right|$$

(3.5.99)

so

$$\sum_{j=0}^{7} \frac{|\tilde{a}_j| |C_i - \lambda|}{|\lambda + j| |C_i + j|} \leq \sum_{j=0}^{7} \frac{|\tilde{a}_j| \sqrt{2}/4}{|C_i - \frac{1}{4} - \frac{i}{4} + j| |C_i + j|} := R_{i,1}$$

(3.5.100)
On the other hand
\[ \sum_{k=0}^{12} (\tilde{b}_k - C_i^k) = \sum_{k=1}^{12} \tilde{c}_{k,i} (\lambda - C_i)^k \] (3.5.101)
so by explicit calculation,
\[ \left| \sum_{k=0}^{12} (\tilde{b}_k - C_i^k) \right| \leq \sum_{k=1}^{12} |\tilde{c}_{k,i}| |\lambda - C_i|^k \leq \sum_{k=1}^{12} |\tilde{c}_{k,i}| \left( \frac{\sqrt{2}}{4} \right)^k =: R_{i,2} \] (3.5.102)

Now choose \( E_i \) such that
\[ E_i > R_{i,1} + R_{i,2} \] (3.5.103)

Combine (3.5.97), (3.5.100), (3.5.102) and (3.5.103) we have
\[ |W_1(\lambda) - \bar{A}_i| < E_i \quad (\lambda \in S_{2,i}) \] (3.5.104)

Combine (3.5.104) with (3.5.95) we obtain:
\[ |W_a(t_0)| > \bar{A}_i - (E_i + 0.079) \] (3.5.105)

From straightforward calculation we have the following:
\[ |\bar{A}_1| > 1.95, \quad |\bar{A}_2| > 1.80, \quad |\bar{A}_3| > 1.74, \quad |\bar{A}_4| > 1.68, \]
\[ |\bar{A}_5| > 1.61, \quad |\bar{A}_6| > 1.54, \quad |\bar{A}_7| > 1.48, \] (3.5.106)

According to calculations for \( R_{i,1} \) and \( R_{i,2} \), we can choose valid \( E_i \) as follows:
\[ |E_1| = 1.28, \quad |E_2| > 0.77, \quad |E_3| = 0.67, \quad |E_4| = 0.65, \]
\[ |E_5| = 0.66, \quad |E_6| = 0.67, \quad |E_7| = 0.65, \] (3.5.107)

Hence from (3.5.105), (3.5.106) and (3.5.107) we have
\[ |W_a(t_0)| > \min_{1 \leq i \leq 7} \{ \bar{A}_i - (E_i + 0.079) \} > 0.59 \] (3.5.108)

This crude estimate is enough for the proof of Proposition 3.3.2, but we are able to obtain a better estimate once we know that that \( W_a(t_0) \) has no roots in \( S_2 \) and thus the minimum of \( |W_a(t_0)| \) on \( S_2 \) is attained on the boundary. The method in §3.5.5 provides us with a sharper result (3.5.71).
### 3.5.8 Values of \(J_n\), \(P_n\) and \(M_n\)

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### 3.5.9 Estimates of $|B_n/A_n|$, $|B_n/A_n|$ and $|B_n/A_n^2|$.

**Upper bound of $|B_n/A_n|$**

Let $\lambda = x + iy$, then

$$F_1(x, y, n) := \frac{B_n(x + iy)}{A_n(x + iy)} \quad (3.5.109)$$

is a rational function in $x$, $y$ and $n$. Let $z = 1/n$, then

$$M_1(x, y, z) := |F_1(x, y, 1/z)|^2 = \frac{P_1(x, y, z)}{Q_1(x, y, z)} \quad (3.5.110)$$
is a real valued rational function, where

\[
P_1(x, y, z) = \left( x^2 z^2 + z^2 y^2 - 6xz^2 + 4xz + 9z^2 - 12z + 4 \right) \left( x^2 z^2 + z^2 y^2 + 6xz^2 + 4xz + 9z^2 + 12z + 4 \right)
\]

\[
Q_1(x, y, z) = x^4 z^4 + 2x^2 y^2 z^4 + y^4 z^4 + 16x^3 z^4 + 16 xy^2 z^4 + 16 x^3 z^3 + 46 x^2 z^4 + 16 xy^2 z^3 + 82 y^2 z^4 + 168 x^2 z^3 - 144 xz^4 + 88 y^2 z^3 + 88 x^2 z^2 + 176 xz^3 + 40 z^2 y^2 + 81 z^4 + 512 xz^2 - 360 z^3 + 192 xz + 184 z^2 + 480 z + 144
\]

(3.5.111) (3.5.112)

Take partial derivative of \( M_1(x, y, z) \) with respect to \( x \):

\[
\frac{\partial}{\partial x} M_1(x, y, z) = z \frac{P_2(x, y, z)}{Q_2(x, y, z)}
\]

(3.5.113)
where

\[ P_2(x, y, z) = 1536 + (4096 x + 7168) z + (4480 x^2 + 640 y^2 + 17664 x \\
+ 6528) z^2 + (2560 x^3 + 1024 xy^2 + 17664 x^2 + 2304 y^2 + 14848 x \\
- 6912) z^3 + (800 x^4 + 576 x^2 y^2 + 32 y^4 + 9088 x^3 + 3456 xy^2 \\
+ 13504 x^2 - 2496 y^2 - 10368 x - 13536) z^4 + (128 x^5 + 128 x^3 y^2 \\
+ 2496 x^4 + 1792 x^2 y^2 + 64 y^4 + 6144 x^3 - 384 xy^2 - 5760 x^2 \\
- 16128 y^2 - 10368 x - 25920) z^5 + (8 x^6 + 8 x^4 y^2 - 8 x^2 y^4 - 8 y^6 \\
+ 336 x^5 + 352 x^3 y^2 + 16 xy^4 + 1400 x^4 + 944 x^2 y^2 + 120 y^4 \\
- 1440 x^3 - 10656 xy^2 - 1800 x^2 - 9720 y^2 - 14256 x - 20088) z^6 \\
+ (16 x^6 + 16 x^4 y^2 - 16 x^2 y^4 - 16 y^6 + 128 x^5 + 256 x^3 y^2 + 128 xy^4 \\
- 144 x^4 - 1440 x^2 y^2 - 144 y^4 - 4608 xy^2 - 1296 x^2 + 1296 y^2 \\
- 10368 x + 11664) z^7 \]  

(3.5.114)

\[ Q_2(x, y, z) = (x^4 z^4 + 2 x^2 y^2 z^4 + y^4 z^4 + 16 x^3 z^4 + 16 xy^2 z^4 \\
+ 16 x^3 z^3 + 46 x^2 z^4 + 16 xy^2 z^3 + 82 y^2 z^4 + 168 x^2 z^3 - 144 xz^4 \\
+ 88 y^2 z^3 + 88 x^2 z^2 + 176 xz^3 + 40 z^2 y^2 + 81 z^4 + 512 xz^2 - 360 z^3 \\
+ 192 xz + 184 z^2 + 480 z + 144)^2 \]  

(3.5.115)

It is straightforward to check that for \( x \in [0, 1/2], \ y \in [4, 10] \) and \( z \in [0, 1/10] \), both \( P_2(x, y, z) \) and \( Q_2(x, y, z) \) are positive, so

\[ \frac{\partial}{\partial x} M_1(x, y, z) \geq 0 \]  

(3.5.116)

so

\[ M_1(x, y, z) \leq M_1(1/2, y, z) \]  

(3.5.117)

Use similar method we also have:

\[ \frac{\partial}{\partial y} M_1(x, y, z) \geq 0 \]  

(3.5.118)
To summarize, for any fixed \( z \in [0, 1/10] \) the maximum of \( M_1 \) is attained at \( x = 1/2, y = 10 \). In other words, given fixed \( n \geq 10 \), the maximum of \(|B_n(\lambda)/A_n(\lambda)|\) on \( S_3 \) is attained at \( \lambda = 1/2 + 10i \). Let \( Z_n^{(1)} \) denote the maximum of \(|B_n(\lambda)/A_n(\lambda)|\) on \( S_3 \). Then

\[
|B_n(\lambda)/A_n(\lambda)| \leq Z_n^{(1)}
\]

\[
= \sqrt{M_1(1/2, 10, 1/n)}
\]

\[
= \left| \frac{B_n(1/2 + 10i)}{A_n(1/2 + 10i)} \right|
\]

\[
= \left( \frac{(16 n^2 - 40 n + 425) (16 n^2 + 56 n + 449)}{2304 n^4 + 9216 n^3 + 71392 n^2 + 149952 n + 305161} \right)^{1/2}
\]

If we take derivative of \( M_1(1/2, 10, z) \) with respect to \( z \), then

\[
\frac{\partial}{\partial z} M_1(1/2, 10, z) = \frac{P_3(z)}{Q_3(z)}
\]

where

\[
P_3(z) = 26832450160 z^6 + 20079135232 z^5 + 3697458944 z^4
\]

\[+ 1477025792 z^3 + 15159296 z^2 + 17563648 z - 1769472
\]

\[
Q_3(z) = (305161 z^4 + 149952 z^3 + 71392 z^2 + 9216 z + 2304)^2
\]

It is obvious that \( Q_3 \) is positive and \( P_3 \) is increasing on \([0, 1/10]\). We notice that

\[
P_3(1/15) > 0
\]

\[
P_3(1/16) < 0
\]

Hence for \( z \in [0, 1/16] \), \( M_1 \) decreases as \( z \) increases. In other words, for \( n \geq 16 \), \( Z_n^{(1)} \) increases as \( n \) increases. Since

\[
\frac{B_n}{A_n} = \frac{\lambda^2 + 4 \lambda n + 4 n^2 - 9}{-12 n^2 - (20 + 8 \lambda) n - \lambda^2 - 8 \lambda + 9}
\]

we can see that for each \( \lambda \), as \( n \to \infty \), \(|B_n/A_n| \to 1/3\), so \( Z_n^{(1)} \uparrow 1/3 \) as \( n \to \infty \).
Lower bound of $|A_n|$

The study of $A_n$ is similar.

$$A_n(\lambda) = \frac{-12n^2 - (20 + 8\lambda)n - \lambda^2 - 8\lambda + 9}{8n^2 + 28n + 20} \quad (3.5.127)$$

Let $\lambda = x + iy$, then

$$F_2(x, y, n) := A_n(x + iy) \quad (3.5.128)$$

is a rational function in $x$, $y$ and $n$. Let $z = 1/n$, then

$$M_2(x, y, z) := |F_2(x, y, 1/z)|^2 = \frac{P_4(x, y, z)}{Q_4(x, y, z)} \quad (3.5.129)$$

is a real valued rational function, where

$$P_4(x, y, z) = x^4z^4 + 2x^2y^2z^4 + y^4z^4 + 16x^3z^4 + 16xy^2z^4 + 16x^3z^3$$

$$+ 46x^2z^4 + 16xy^2z^3 + 82y^2z^4 + 168x^2z^3 - 144xz^4 + 88y^2z^3$$

$$+ 88x^2z^2 + 176xz^3 + 40z^2y^2 + 81z^4 + 512xz^2 - 360z^3 + 192xz$$

$$+ 184z^2 + 480z + 144 \quad (3.5.130)$$

$$Q_4(x, y, z) = 16(z + 1)^2(5z + 2)^2 \quad (3.5.131)$$

Take partial derivative of $M_2(x, y, z)$ with respect to $x$:

$$\frac{\partial}{\partial x}M_2(x, y, z) = z\frac{P_6(x, y, z)}{Q_5(x, y, z)} \quad (3.5.132)$$

where

$$P_6(x, y, z) = (xz + 4z + 4)(x^2z^2 + z^2y^2 + 8xz^2 + 8xz - 9z^2$$

$$+ 20z + 12) \quad (3.5.133)$$

$$Q_5(x, y, z) = 4(z + 1)^2(5z + 2)^2 \quad (3.5.134)$$

By inspection $P_6$ and $Q_6$ are both positive, so

$$\frac{\partial}{\partial x}M_2(x, y, z) \geq 0 \quad (3.5.135)$$
\[ M_2(x, y, z) \geq M_2(0, y, z) \] (3.5.136)

Now consider the partial derivative of \( M_2(0, y, z) \) with respect to \( y \):
\[
\frac{\partial}{\partial y} M_2(0, y, z) = \frac{4 y^3 z^4 + 164 y z^4 + 176 y z^3 + 80 z^2 y}{16 (z + 1)^2 (5 z + 2)^2} \] (3.5.137)

It is obviously nonnegative for \( y \geq 0 \) and \( z \geq 0 \), so
\[ M_2(0, y, z) \geq M_2(0, 4, z) \] (3.5.138)

So given any \( z \in [0, 1/10] \) the minimum of \( M_2(x, y, z) \) is attained at \( x = 0, y = 4 \).
Equivalently, for each \( n \geq 10 \), the minimum of \( |A_n| \) on \( S_3 \) is attained at \( \lambda = 4i \). Let \( Z_n^{(2)} \) denote the minimum of \( |A_n(\lambda)| \) on \( S_3 \). Then
\[
|A_n(\lambda)| \geq Z_n^{(2)} \\
= \sqrt{M_2(0, 4, 1/n)} \\
= |A_n(4i)| \\
= \left( \frac{144 n^4 + 480 n^3 + 824 n^2 + 1048 n + 1649}{16 (1 + n)^2 (5 + 2 n)^2} \right)^{1/2} \] (3.5.139)

Notice that
\[
\frac{\partial}{\partial z} M_2(0, 4, z) = \frac{8923 z^4 + 6144 z^3 - 456 z^2 - 1472 z - 528}{8 (z + 1)^3 (5 z + 2)^3} \] (3.5.140)

By inspection the derivative is negative for \( z \in [0, 1/10] \), so for \( n \geq 10 \), \( Z_n^{(2)} \) is monotonically increasing. Since for each \( n \), \( \lim_{n \to \infty} |A_n| = 3/2 \), \( Z_n^{(2)} \uparrow 3/2 \) as \( n \to \infty \).

**Upper bound of** \( |B_n/A_n^2| \):
\[
\frac{B_n(\lambda)}{A_n^2(\lambda)} = \frac{(\lambda^2 + 4 \lambda n + 4 n^2 - 9) (8 n^2 + 28 n + 20)}{(-12 n^2 - (20 + 8 \lambda) n - \lambda^2 - 8 \lambda + 9)^2} \] (3.5.141)

Let \( \lambda = x + iy \), then
\[
F_3(x, y, n) := \frac{B_n(x + iy)}{A_n^2(x + iy)} \] (3.5.142)

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is a rational function in $x$, $y$ and $n$. Let $z = 1/n$, then

$$M_3(x, y, z) := |F_3(x, y, 1/z)|^2 \quad (3.5.143)$$

is a real valued rational function. Consider the partial derivative of $M_3(x, y, z)$ with respect to $x$.

$$\frac{\partial}{\partial x} M_3(x, y, z) = 64 z (z + 1)^2 (5 z + 2)^2 \frac{P_6(x, y, z)}{Q_6(x, y, z)} \quad (3.5.144)$$

where

$$P_6(x, y, z) = \sum_{i=0}^{7} f_i(x, y) z^i$$

$$Q_6(x, y, z) = (x^4 z^4 + 2 x^2 y^2 z^4 + y^4 z^4 + 16 x^3 z^4 + 16 x y^2 z^4 + 16 x^3 z^3$$

$$+ 46 x^2 z^4 + 16 x y^2 z^3 + 82 y^2 z^4 + 168 x^2 z^3 - 144 x z^4 + 88 y^2 z^3$$

$$+ 88 x^2 z^2 + 176 x z^3 + 40 z^2 y^2 + 81 z^4 + 512 x z^2 - 360 z^3$$

$$+ 192 x z + 184 z^2 + 480 z + 144)^3 \quad (3.5.145)$$
where

\[ f_0(x, y) = -384 \]  
\[ f_1(x, y) = -1216 x - 256 \]  
\[ f_2(x, y) = -1632 x^2 - 288 y^2 - 1024 x + 4384 \]  
\[ f_3(x, y) = -1200 x^3 - 624 xy^2 - 1536 x^2 - 512 y^2 + 8560 x + 8064 \]  
\[ f_4(x, y) = -520 x^4 - 528 x^2 y^2 - 72 y^4 - 1152 x^3 - 960 xy^2 + 6480 x^2 \]  
\[ - 1552 y^2 + 13824 x - 4104 \]  
\[ f_5(x, y) = -132 x^5 - 216 x^3 y^2 - 84 xy^4 - 464 x^4 - 672 x^2 y^2 - 144 y^4 \]  
\[ + 2360 x^3 - 1064 xy^2 + 8640 x^2 - 5760 y^2 - 1332 x - 19440 \]  
\[ f_6(x, y) = -18 x^6 - 42 x^4 y^2 - 30 x^2 y^4 - 6 y^6 - 96 x^5 - 208 x^3 y^2 \]  
\[ - 112 xy^4 + 410 x^4 - 108 x^2 y^2 - 86 y^4 + 2304 x^3 - 3600 xy^2 \]  
\[ + 1026 x^2 - 3546 y^2 - 12960 x - 8586 \]  
\[ f_7(x, y) = -x^7 - 3 x^5 y^2 - 3 x^3 y^4 - x y^6 - 8 x^6 - 24 x^4 y^2 - 24 x^2 y^4 8 y^6 \]  
\[ + 27 x^5 + 18 x^3 y^2 - 9 xy^4 + 216 x^4 - 432 x^2 y^2 - 72 y^4 \]  
\[ + 333 x^3 - 1647 xy^2 - 1944 x^2 + 648 y^2 - 4455 x + 5832 \]  

It is obvious that \( Q_6(x, y, z) > 0 \), \( f_1(x, y) \leq 0 \) and \( f_2(x, y) \leq 0 \) for \( x \in [0, 1/2] \), \( y \in [4, 10] \) and \( z \in [-0, 1/10] \). By taking absolute value of every term and use the condition that \(|x| \leq 1/2\) and \(|y| \leq 10\) we obtain crude estimates for upper bounds of \(|f_i(x, y)|\). Thus

\[
P_6(x, y, z) \leq \sum_{i=0}^{7} f_i(x, y) z^i
\]
\[
\leq -384 + \sum_{i=3}^{7} |f_i(x, y)|(1/10)^i
\]
\[
< -384 + 96 + 95 + 26 + 9 < 0
\]  

(3.5.154)
We get
\[ \frac{\partial}{\partial x} M_3(x, y, z) \leq 0 \tag{3.5.155} \]

\[ M_3(x, y, z) \leq M_3(0, y, z) \tag{3.5.156} \]

From §3.5.9 and §3.5.9 we see that given \( n \) both \( |B_n(iy)/A_n(iy)| \) and \( |A_n(iy)| \) are monotonically increasing for \( y \in [4, 10] \), so for \( \lambda \in [j, j + 1] \), \( j \in \{4, 5, \ldots, 9\} \) we have
\[ \frac{|B_n(\lambda)|}{|A_n(\lambda)|} = \frac{|B_n(\lambda)|}{|A_n(\lambda)|} \leq \frac{M_1(0, j + 1, 1/n)}{M_2(0, j, 1/n)} \tag{3.5.157} \]

For \( j \in \{4, 5, \ldots, 9\} \) let
\[ M_{4j}(z) = \frac{M_1(0, j + 1, z)}{M_2(0, j, z)} \tag{3.5.158} \]

\[ Z^{(3)}(n) = \max_{j=4,5,\ldots,9} \left\{ \left( M_{4j}(z) \right)^{1/2} \right\} \tag{3.5.159} \]

then for all \( \lambda \in [4i, 10i] \)
\[ \frac{|B_n(\lambda)|}{|A_n(\lambda)|} \leq Z^{(3)}(n)^2 \tag{3.5.160} \]

In view of (3.5.156), for each \( n \) and \( \lambda \in S_3 \):
\[ \frac{|B_n(\lambda)|}{|A_n(\lambda)|} \leq Z^{(3)}(n) \tag{3.5.161} \]
The expressions of $M_{4,j}$ are:

$$M_{4,4}(z) = 16 (34 z^2 + 12 z + 4) (34 z^2 - 12 z + 4) (z + 1)^2 (5 z + 2)^2 / \left[ (2756 z^4 + 1840 z^3 + 1184 z^2 + 480 z + 144) (1649 z^4 + 1048 z^3 + 824 z^2 + 480 z + 144) + 1184 z^2 + 480 z + 144 \right]$$

$$M_{4,5}(z) = 16 (45 z^2 + 12 z + 4) (45 z^2 - 12 z + 4) (z + 1)^2 (5 z + 2)^2 / \left[ (4329 z^4 + 2808 z^3 + 1624 z^2 + 480 z + 144) (2756 z^4 + 1840 z^3 + 1184 z^2 + 480 z + 144) + 1184 z^2 + 480 z + 144 \right]$$

$$M_{4,6}(z) = 16 (73 z^2 + 12 z + 4) (73 z^2 - 12 z + 4) (z + 1)^2 (5 z + 2)^2 / \left[ (9425 z^4 + 5272 z^3 + 2744 z^2 + 480 z + 144) (6500 z^4 + 3952 z^3 + 2144 z^2 + 480 z + 144) + 2144 z^2 + 480 z + 144 \right]$$

$$M_{4,7}(z) = 16 (73 z^2 + 12 z + 4) (73 z^2 - 12 z + 4) (z + 1)^2 (5 z + 2)^2 / \left[ (9425 z^4 + 5272 z^3 + 2744 z^2 + 480 z + 144) (6500 z^4 + 3952 z^3 + 2144 z^2 + 480 z + 144) + 2144 z^2 + 480 z + 144 \right]$$

$$M_{4,8}(z) = 16 (90 z^2 + 12 z + 4) (90 z^2 - 12 z + 4) (z + 1)^2 (5 z + 2)^2 / \left[ (13284 z^4 + 6768 z^3 + 3424 z^2 + 480 z + 144) (9425 z^4 + 5272 z^3 + 2744 z^2 + 480 z + 144) + 2744 z^2 + 480 z + 144 \right]$$

$$M_{4,9}(z) = 16 (109 z^2 + 12 z + 4) (109 z^2 - 12 z + 4) (z + 1)^2 (5 z + 2)^2 / \left[ (18281 z^4 + 8440 z^3 + 4184 z^2 + 480 z + 144) (13284 z^4 + 6768 z^3 + 3424 z^2 + 480 z + 144) + 3424 z^2 + 480 z + 144 \right]$$

(3.5.162)

It is straightforward to check that each $M_{4,j}(z)$ has a positive derivative with respect to $z$ on $[0, 1/10]$. Hence as $n$ increases, $Z_n^{(3)}$ decreases monotonically.
3.5.10 Proof of (3.3.72) and (3.3.73)

Proof of (3.3.72)

\[
\left| \left[ (N(0)) - (N^2(0)) \right] \right|_n = \left| \frac{B_n}{A_n} - \left( \frac{B_n}{A_n - \frac{B_{n+1}}{A_{n+1}}} \right) \right|
\]

\[
= \left| \frac{B_n}{A_n} \right| \left| \frac{B_{n+1}}{A_{n+1}} \right| \left| A_n - \frac{B_{n+1}}{A_{n+1}} \right|
\]

\[
\leq \frac{Z_n^{(1)} Z_{n+1}^{(1)}}{Z_n^{(2)} - Z_{n+1}^{(1)}}
\]

(3.5.163)

For 10 \leq n \leq 20 we can check directly from (3.5.120) and (3.5.139) that the value is less than 0.106. For n > 20 we have

\[
\frac{Z_n^{(1)} Z_{n+1}^{(1)}}{Z_n^{(2)} - Z_{n+1}^{(1)}} \leq \frac{(1/3)^2}{Z_{21}^{(2)} - 1/3} < 0.106
\]

(3.5.164)

Proof of (3.3.73)

Note that we can write

\[
[N(0)]_{n_0} = [N(0)]_{10}
\]

\[
= 1 - \frac{4}{545} \left( \frac{-139 + 6 \sqrt{545}}{\lambda + 44 - \sqrt{545}} \right) \sqrt{545} - \frac{4}{545} \left( \frac{139 + 6 \sqrt{545}}{\lambda + 44 + \sqrt{545}} \right) \sqrt{545}
\]

(3.5.165)

so we have unique numbers \( \tilde{e}_i, (i = 18, 19) \), and \( s_{18} = -44 + \sqrt{545} \), \( s_{19} = -44 - \sqrt{545} \), such that

\[
[N(0)]_{10} = 1 + \sum_{i=18}^{19} \frac{\tilde{e}_i}{\lambda - s_i}
\]

(3.5.166)

To simplify calculations, we approximate \([N(0)]_{10}\) by \(Q(\lambda)\), where

\[
Q(\lambda) = 1 + \frac{\tilde{d}_{18}}{\lambda - \tilde{s}_{18}} + \frac{\tilde{d}_{19}}{\lambda - \tilde{s}_{19}}
\]

(3.5.167)

where

\[
\tilde{s}_{18} = -1735/84, \quad \tilde{s}_{19} = -1953/29,
\]

\[
\tilde{d}_{18} = -38/207, \quad \tilde{d}_{19} = -2343/49
\]

(3.5.168)
By straightforward calculation:

\[
|\mathcal{N}(0)_{10} - Q(\lambda)| \leq \sum_{i=18}^{19} \left| \frac{\bar{e}_i}{\lambda - s_i} - \frac{\bar{d}_i}{\lambda - \tilde{s}_i} \right|
\]

\[
= \sum_{i=18}^{19} \frac{\lambda |\bar{e}_i - \bar{d}_i| + |\bar{e}_i \tilde{s}_i - \bar{d}_i s_i|}{|\lambda - s_i| |\lambda - \tilde{s}_i|}
\]

\[
< \frac{1}{312500} \tag{3.5.169}
\]

\[c_{10}\] is a polynomial of degree 18 and it has 18 distinct real roots \(\{s_j\}_{0 \leq i \leq 17}\) satisfying

\[1 = s_0 > -0.54 > s_1 > s_2 > ... > s_{17} \tag{3.5.170}\]

\[c_{11}\] is a polynomial of degree 20 and it has 20 distinct real roots \(\{t_j\}_{0 \leq i \leq 19}\) satisfying

\[1 = t_0 > -0.54 > t_1 > t_2 > ... > t_{19} \tag{3.5.171}\]

The only common root of \(c_{10}\) and \(c_{11}\) is \(\lambda = 1\). We can write:

\[c_{10} = l_1 \prod_{i=0}^{17} (\lambda - s_i) \tag{3.5.172}\]

\[c_{11} = l_2 \prod_{i=0}^{19} (\lambda - t_i) \tag{3.5.173}\]

where \(l_1\) is the leading coefficient of \(c_{10}\) and \(l_2\) is the leading coefficient of \(c_{11}\), so

\[r_{10} = \frac{c_{11}}{c_{10}} = P_1(\lambda) + \sum_{i=1}^{17} \frac{\tilde{a}_i}{\lambda - s_i} \tag{3.5.174}\]

where \(\tilde{a}_i\)'s are constants and \(P_1(\lambda)\) is a quadratic polynomial.

Now we approximate \(c_{10}\) by \(\tilde{c}_{10}\)

\[\tilde{c}_{10} = l_1 \prod_{i=0}^{17} (\lambda - \tilde{s}_i) \tag{3.5.175}\]

where for each \(1 \leq i \leq 17\), \(\tilde{s}_i\) is a rational number within \(e^{-9}\) of \(s_i\), and \(\tilde{s}_0 = s_0 = 1\).

Now we crudely estimate \(c_{11}\), \(c_{10} - \tilde{c}_{10}\) and \(\tilde{c}_{10}\). Rewrite

\[c_{11} = \sum_{j=0}^{20} \tilde{b}_j (\lambda - 7 i - 1/4)^j \tag{3.5.176}\]
then
\[ |c_{11}| \leq \sum_{j=0}^{20} |\tilde{b}_j| \left( \frac{\sqrt{145}}{4} \right)^j < 37209/50 \quad (\lambda \in S_3) \] (3.5.177)

Rewrite
\[ c_{10} - \tilde{c}_{10} = \sum_{j=0}^{18} \tilde{c}_j (\lambda - 7 i - 1/4)^j \] (3.5.178)

then
\[ |c_{10} - \tilde{c}_{10}| \leq \sum_{j=0}^{18} |\tilde{c}_j| \left( \frac{\sqrt{145}}{4} \right)^j < 3/250000 \quad (\lambda \in S_3) \] (3.5.179)

By definition
\[ \tilde{c}_{10} = l_1 \prod_{i=0}^{17} (\lambda - \tilde{s}_i) = l_1 (\lambda - 1) \prod_{i=1}^{17} (\lambda - \tilde{s}_i) \] (3.5.180)

Since for each \( 1 \leq i \leq 17, \tilde{s}_i < 0 \), we have
\[ |\tilde{c}_{10}| > |l_1| \cdot 4 i - 1/2 \prod_{i=1}^{17} |4 i - \tilde{s}_i| > 786187/1000000 \] (3.5.181)

Hence from (3.5.177), (3.5.179) and (3.5.181) we have
\[ \frac{|c_{11} - c_{11}|}{|c_{10} - \tilde{c}_{10}|} = \frac{|c_{11}| |\tilde{c}_{10} - c_{10}|}{|c_{10} c_{10}|} \]
\[ = \frac{37209}{\left( \frac{786187}{1000000} \right)^3} \frac{3}{250000} \left( \frac{786187}{1000000} \right) < 0.0145 \] (3.5.182)

Now we have explicit numbers \( \tilde{c}_i \)'s and a quadratic polynomial \( \tilde{P}_1(\lambda) \) such that
\[ \frac{c_{11}}{\tilde{c}_{10}} = \tilde{P}_1(\lambda) + \sum_{i=1}^{17} \frac{-\tilde{c}_i}{\lambda - \tilde{s}_i} \] (3.5.183)

To simplify the calculation, replace each \( \tilde{c}_i \) by a rational number \( \tilde{d}_i \) with \( e^{-7} \) of it and each coefficient in \( \tilde{P}_1(\lambda) \) by a rational number within \( e^{-7} \) of it we obtain:
\[ \tilde{r}_{10} := \tilde{P}_1(\lambda) + \sum_{i=1}^{17} \frac{-\tilde{d}_i}{\lambda - \tilde{s}_i} \] (3.5.184)

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By straightforward calculation
\[
\left| \tilde{r}_{10} - \frac{c_{11}}{c_{10}} \right| \leq \left| \tilde{P}_1(\lambda) - \tilde{P}_1(\lambda) \right| + \sum_{i=1}^{17} \frac{|\tilde{d}_i - \tilde{e}_i|}{|4i - \tilde{s}_i|} < 3 \cdot 10^{-7} \quad (3.5.185)
\]

Finally, consider
\[
F(\lambda) := Q(\lambda) - \tilde{r}_{10} = 1 - \tilde{P}_1(\lambda) + \sum_{i=1}^{19} \frac{\tilde{d}_i}{\lambda - \tilde{s}_i} \quad (3.5.186)
\]
(see (3.5.189) for explicit expression of \( F(\lambda) \)). Use the method in §3.5.5 with partition
\( \mathcal{P} = \{4i, 4i + 1/2, 7i + 1/2, 10i + 1/2, 7i\} \) and \( m_0 = 2 \), we have
\[
|F(\lambda)| > 0.6098 \quad (3.5.187)
\]

In view of (3.5.169), (3.5.182) (3.5.185) and (3.5.187) we have:
\[
\left| [N(0)]_{10} - \frac{c_{11}}{c_{10}} \right|
\geq |Q(\lambda) - \tilde{r}_{10}| - |[N(0)]_{10} - Q(\lambda)| - \left| \frac{c_{11}}{c_{10}} - \frac{c_{11}}{c_{10}} \right| - |\tilde{r}_{10} - \frac{c_{11}}{c_{10}}|
\]
\[
> 0.595 \quad (3.5.188)
\]

Thus we have (3.7.33). The expression of \( F(\lambda) \):
\[
F(\lambda) = \frac{1}{920} \lambda^2 + \frac{2}{23} \lambda + \frac{387}{920} - \frac{38}{207} \left( \lambda + \frac{1735}{84} \right)^{-1} - \frac{2343}{49} \left( \lambda + \frac{1953}{29} \right)^{-1}
- \frac{1704}{101} \left( \lambda + \frac{108406}{1283} \right)^{-1} - \frac{4297}{537} \left( \lambda + \frac{73708}{1189} \right)^{-1} - \frac{2765}{768} \left( \lambda + \frac{59605}{1299} \right)^{-1}
- \frac{1588}{1195} \left( \lambda + \frac{53402}{1589} \right)^{-1} - \frac{763}{3245} \left( \lambda + \frac{31328}{1279} \right)^{-1} - \frac{53}{10426} \left( \lambda + \frac{19885}{1029} \right)^{-1}
+ \frac{23}{2173} \left( \lambda + \frac{52237}{3423} \right)^{-1} + \frac{367}{7339} \left( \lambda + \frac{21932}{1491} \right)^{-1} + \frac{160}{4019} \left( \lambda + \frac{38219}{3029} \right)^{-1}
+ \frac{71}{9409} \left( \lambda + \frac{1316}{117} \right)^{-1} + \frac{358}{3591} \left( \lambda + \frac{28039}{2844} \right)^{-1} + \frac{717}{10073} \left( \lambda + \frac{11626}{1435} \right)^{-1}
+ \frac{274}{14543} \left( \lambda + \frac{36794}{5013} \right)^{-1} + \frac{280}{3639} \left( \lambda + \frac{121869}{21650} \right)^{-1} + \frac{179}{3360} \left( \lambda + \frac{15715}{4138} \right)^{-1}
+ \frac{159}{8224} \left( \lambda + \frac{19660}{9407} \right)^{-1} + \frac{43}{17095} \left( \lambda + \frac{2595}{4718} \right)^{-1}
\]
(3.5.189)
3.5.11 The method to prove (3.4.10), (3.4.19) – (3.4.22), and (3.4.30) – (3.4.31)

The functions we estimate on ∂R₂ are of two types:

Group (I)

\[ \delta^{(1)}_1 = r_1 e^{-W_1} - 1 \]  
\[ \epsilon^{(1)}_n = -1 - A_{n-1} e^{-W_n} - B_{n-1} e^{-W_n - W_{n-1}} \quad (2 \leq n \leq 50) \]  
\[ \epsilon^{(3)} = -1 - A_{50} e^{-W_{50}} - B_{50} e^{-2W_{50}} \]

Group (II)

\[ C^{(1)}_n = B_n e^{-W_{n+1} - W_n} \]  
\[ C^{(3)} = B_{50} e^{-2W_{50}} \]

We obtain estimates for each function on each one of the following subintervals of \( l_i \), \( (1 \leq i \leq 6) \):

\[ l_{1,1} = [10i, 18i] \quad l_{1,2} = [18i, 30i] \quad l_{1,3} = [30i, 48i] \]  
\[ l_{1,4} = [48i, 70i] \]  
\[ l_{3,j} = l_{1,j} + 1/2 \quad (j = 1, 2, 3, 4) \]  
\[ l_{2,1} = [70i, 110i] \quad l_{2,2} = [110i, 175i] \quad l_{2,3} = [175i, 275i] \]  
\[ l_{2,4} = [275i, 380i] \]  
\[ l_{4,j} = l_{2,j} + 1/2 \quad (j = 1, 2, 3, 4) \]  
\[ l_{5,1} = l_5 = [10i, 10i + 1/2] \]  
\[ l_{6,1} = l_6 = [380i, 380i + 1/2] \]

For each function in Group (II), the estimate follows easily from intermediate results obtained in the process of estimating the corresponding function in Group (I). See (3.5.223). We shall describe the method we use to estimate a function \( \tilde{G}(\lambda) \) of Group (I) on an interval \([a_0, b_0]\) in detail.
We see that $\tilde{G}(\lambda)$ is of the form

$$\tilde{G}(\lambda) = \tilde{q}_1 e^{\tilde{f}_1} + \tilde{q}_2 e^{\tilde{f}_2} - 1$$  \hspace{1cm} (3.5.201)

where $\tilde{q}_1$ and $\tilde{q}_2$ are polynomials of degree 2 in $\lambda$ and $\tilde{f}_1$ and $\tilde{f}_2$ are polynomials of degree at most 5 in $\text{Im}(\lambda)$ and of degree 1 in $\text{Re}(\lambda)$. First let $\phi$ be a polynomial of degree 1 which maps $[-1,1]$ bijectively to $[a,b]$ and let

$$G := \tilde{G} \circ \phi \quad q_i := \tilde{q}_i \circ \phi \quad f_i := \tilde{f}_i \quad (i = 1, 2)$$  \hspace{1cm} (3.5.202)

The problem thus transforms into finding an upper bound for the function $|G|$ of a single real variable $x$

$$|G| = |q_1 e^{f_1} + q_2 e^{f_2} - 1| \quad x \in [-1,1]$$  \hspace{1cm} (3.5.203)

where $q_{1,2}$ are polynomials of degree 2 in $x$ and $f_{1,2}$ are polynomials of degree 5 or 1 in $x$. The estimate of $|G|$ is obtained in 3 steps: Step 1: For each $i$, express $f_i$ as a linear combination of Chebyshev polynomials $T_j(x)$, $0 \leq j \leq 5$.

$$f_i(x) = \sum_{j=0}^{5} g_{i,j} T_j(x)$$  \hspace{1cm} (3.5.204)

Denote

$$h_i(x) = \sum_{j=1}^{2} g_{i,j} T_j(x) \quad \tilde{r}_i(x) = \sum_{j=3}^{5} g_{i,j} T_j(x) \quad \tilde{r}_i = \sum_{j=3}^{5} |g_{i,j}|$$  \hspace{1cm} (3.5.205)

then

$$f_i = g_{i,0} + h_i + \tilde{r}_i$$  \hspace{1cm} (3.5.206)

Step 2: For each $i$, approximate $e^{f_i}$ by a polynomial

$$e^{f_i} = e^{g_{i,0} + h_i + \tilde{r}_i}$$

$$= e^{g_{i,0}} \left( \sum_{k=0}^{3} \frac{(h_i)^{k}}{k!} + \int_{0}^{h_i} \int_{0}^{s_1} \int_{0}^{s_2} \int_{0}^{s_3} e^{s_4} ds_4 ds_3 ds_2 ds_1 \right) \left( 1 + \int_{0}^{\tilde{r}_i} e^{s} ds \right)$$

$$= H_i + R_{i,1} + R_{i,2}$$  \hspace{1cm} (3.5.207)
where

\[ H_i = e^{q_i,0} \left( 1 + h_i + \frac{h_i^2}{2} + \frac{h_i^3}{6} \right) \] (3.5.208)

\[ R_{i,1} = e^{q_i,0} e^{h_i} \int_{0}^{\bar{r}_i} e^s ds \] (3.5.209)

\[ R_{i,2} = e^{q_i,0} \left( \int_{0}^{h_i} \int_{0}^{s_1} \int_{0}^{s_2} \int_{0}^{s_3} e^{s_4} ds_4 ds_3 ds_2 ds_1 \right) \] (3.5.210)

Step 3: We have

\[ |G(x)| \leq |H(x)| + |q_1 R_{1,1}| + |q_1 R_{1,2}| + |q_2 R_{2,1}| + |q_2 R_{2,2}| \] (3.5.211)

where

\[ H(x) = q_1(x)H_1(x) + q_2(x)H_2(x) - 1 \] (3.5.212)

is a polynomial of degree 8.

To estimate \( H(x) \) we simply express \( H(x) \) as a linear combination of Chebyshev polynomials \( T_l(x) \), \( 0 \leq l \leq 8 \).

\[ |H(x)| = \left| \sum_{l=0}^{8} \tilde{h}_l T_l(x) \right| \leq \sum_{l=0}^{8} |\tilde{h}_l| \] (3.5.213)

since

\[ \sup_{x \in [-1,1]} |T_l(x)| \leq 1 \quad \forall l \in \mathbb{N} \] (3.5.214)

For \( i \in \{1, 2\} \), to estimate \( q_i R_{i,1} \) we first notice that by (3.5.205) and (3.5.214) we have

\[ |\tilde{r}_i(x)| \leq \bar{r}_i \] (3.5.215)

We choose \( E_{i,1} \) such that

\[ E_{i,1} \geq \|q_i e^{h_i} \| = \sup_{x \in [-1,1]} |q_i(x) e^{h_i(x)}| \] (3.5.216)
$E_{i,1}$ is chosen rigorously in the following way. For each $i \in \{1, 2\}$, denote $Q_i = |q_i|^2$ and consider the function

$$S(x) := |q_i(x)e^{h_i(x)}|^2 = Q_i(x)e^{2\text{Re}h_i(x)} \quad (3.5.217)$$

We approximate $S(x)$ by approximating first its derivative:

$$S'(x) = e^{2\text{Re}h_i(x)} (2\text{Re}(h_i'(x)) Q_i(x) + Q_i'(x)) \quad (3.5.218)$$

Let $\tilde{\epsilon}$ be small enough. We construct a polynomial $\tilde{P}(x)$ with real coefficients and known roots $r_i$ such that explicitly

$$|\tilde{P}(x) - (2\text{Re}(h_i'(x)) Q_i(x) + Q_i'(x))| < \tilde{\epsilon} \quad x \in [-1, 1] \quad (3.5.219)$$

By construction, the maximum value of the approximating function

$$\tilde{S}(x) := S(0) + \int_0^x e^{2\text{Re}h_i(s)} \tilde{P}(s) ds \quad (3.5.220)$$

on $[-1, 1]$ is known exactly. Also

$$|S(x) - \tilde{S}(x)| \leq \left| \int_0^x e^{2\text{Re}h_i(s)} \left[ \tilde{P}(s) - (2\text{Re}(h_i'(s)) Q_i(s) + Q_i'(s)) \right] ds \right| \leq \exp \left( \sup_{x \in [-1, 1]} 2\text{Re}(h_i(x)) \right) \tilde{\epsilon} \quad (3.5.221)$$

Thus

$$|q_i(x)e^{h_i(x)}| \leq \sqrt{\tilde{S}(x)}$$

$$\leq \left( \sup_{-1 \leq x \leq 1} \tilde{S}(x) + \exp \left( \sup_{x \in [-1, 1]} 2\text{Re}(h_i(x)) \right) \tilde{\epsilon} \right)^{\frac{1}{2}} := E_{i,1} \quad (3.5.222)$$

In particular for $i = 2$ we obtain upper bounds for the corresponding functions in Group (II):

$$|q_i(x)e^{h_i(x)}| \leq |q_i(x)e^{h_i(x)}| |\exp(g_{i,0})| \exp(\tau_i) \leq E_{i,1} |\exp(g_{i,0})| \exp(\tau_i) \quad (3.5.223)$$
Now let us get back to estimating \( q_i R_{i,1} \),

\[
|q_i R_{i,1}| < |e^{g_{i,0}}| E_{i,1} e^{\pi i} \tag{3.5.224}
\]

To estimate \( q_i R_{i,2} \) we write

\[
q_i R_{i,2} = e^{g_{i,0}} q_i \left( \int_0^{h_i} \int_0^{s_1} \cdots \int_0^{s_4} e^{s_5} ds_5 \cdots ds_1 \right) + e^{g_{i,0}} q_i \frac{h_i^4}{24} \tag{3.5.225}
\]

and let

\[
E_{i,3} := |e^{g_{i,0}}| \|q_i\| \|h_i\|^{1.5} \exp \left( \sup_{x \in [-1,1]} \text{Re}(h_i(x)) \right) \tag{3.5.226}
\]

\[
e^{g_{i,0}} \frac{h_i^4}{24} = \sum_j \tilde{t}_j T_j(x) \tag{3.5.227}
\]

\[
E_{i,4} := \sum_j |\tilde{t}_j| \tag{3.5.228}
\]

then

\[
|q_i R_{i,2}| \leq E_{i,3} + E_{i,4} \tag{3.5.229}
\]

and

\[
|G(x)| \leq \sum_{i=0}^{8} |\tilde{h}_i| + \sum_{i=1}^{2} \left( |e^{g_{i,0}}| \|E_{i,1} e^{\pi i} + E_{i,3} + E_{i,4}\right) \tag{3.5.230}
\]

implying an upper bound of \( |\hat{G}| \) on \([a,b]\) in terms of \( \tilde{h}_i, g_{i,0}, \pi_i \) and \( E_{i,j} \) (0 \( \leq l \leq 8, i = 1, 2, j = 1, 3, 4 \)).

3.5.12 Proof of the estimates in Lemma 3.4.1 (iv)

\( U_1 = 1/30 \)

On \([10i, 380i]\), write \( \lambda = ti \). Then \( t \) is real and positive. We have

\[
\frac{|A_n(t i)|}{|A_{n+1}(t i)| - 1|^2} = \frac{P_1(n, t)}{Q_1(n, t)} \tag{3.5.231}
\]

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where $P_1, Q_1$ are real-valued polynomials in $n$ and $t$, where
\begin{align*}
P_1(n, t) &= 256 n^4 t^2 + 16 n^2 t^4 + 1888 n^3 t^2 + 72 n t^4 + 1936 n^4 \\
&+ 5720 n^2 t^2 + 81 t^4 + 17600 n^3 + 8600 n t^2 + 61208 n^2 \\
&+ 5362 t^2 + 96400 n + 58081 \tag{3.5.232}
\end{align*}
\begin{align*}
Q_1(n, t) &= (2n + 5)^2 (n + 1)^2 (144 n^4 + 40 n^2 t^2 + t^4 + 1056 n^3 \\
&+ 168 n t^2 + 2488 n^2 + 210 t^2 + 2024 n + 529)
\end{align*}

Now, $P_1/Q_1$ is increasing in $t$ since,
\begin{equation}
\frac{\partial}{\partial t} \left| \frac{A_n(t i)}{A_{n+1}(t i)} - 1 \right|^2 = \frac{\partial}{\partial t} \frac{P_1(n, t)}{Q_1(n, t)} = \frac{P_2(n, t)}{Q_2(n, t)} \tag{3.5.233}
\end{equation}

where, after simplification $P_2$ and $Q_2$ are manifestly positive polynomials whose explicit forms are not relevant. Thus
\begin{align*}
\frac{P_1(n, t)}{Q_1(n, t)} &\leq \frac{P_1(n, 380)}{Q_1(n, 380)} = (36968336 n^4 + 272644800 n^3 + 33447789208 n^2 \\
&+ 1502539856400 n + 1689734490881)/[(144 n^4 + 1056 n^3 + 5778488 n^2 \\
&+ 24261224 n + 20881684529)(2n + 5)^2 (n + 1)^2] \tag{3.5.234}
\end{align*}

The right side of (3.5.234) is decreasing in $n$. Indeed
\begin{equation}
\frac{d}{dn} \frac{P_1(n, 380)}{Q_1(n, 380)} = -\frac{P_3(n)}{Q_3(n)} \tag{3.5.235}
\end{equation}

where $P_3$ and $Q_3$ are manifestly positive for positive values of $n$. Hence for $n \geq 50$ and $10 \leq t \leq 380$ we have
\begin{equation}
\left| \frac{A_n(t i)}{A_{n+1}(t i)} - 1 \right|^2 \leq \frac{P_1(n, t)}{Q_1(n, t)} \leq \frac{P_1(50, 380)}{Q_1(50, 380)} < U_1^2 \tag{3.5.236}
\end{equation}

Similarly, on $[10 i + 1/2, 380 i + 1/2]$, writing $\lambda = t i + 1/2$, $|A_n/A_{n+1} - 1|^2$ is monotonically increasing in $t$ and thus
\begin{equation}
\left| \frac{A_n(t i + 1/2)}{A_{n+1}(t i + 1/2)} - 1 \right|^2 \leq \left| \frac{A_n(380 i + 1/2)}{A_{n+1}(380 i + 1/2)} - 1 \right|^2 \tag{3.5.237}
\end{equation}
The right hand side of the equation (3.5.237) is monotonically decreasing in \( n \), so
\[
\left| \frac{A_n(t+i+1/2)}{A_{n+1}(t+i+1/2)} - 1 \right|^2 \leq \frac{A_{50}(380i+1/2)}{A_{51}(380i+1/2)} - 1 \right|^2 \leq U_1^2 \tag{3.5.238}
\]
On \([10i, 10i+1/2] \), write \( \lambda = t + 10i \), \(|A_n/A_{n+1} - 1|^2 \) is monotonically decreasing in \( t \) and thus
\[
\left| \frac{A_n(t+10i)}{A_{n+1}(t+10i)} - 1 \right|^2 \leq \left| \frac{A_{50}(10i)}{A_{51}(10i)} - 1 \right|^2 \leq U_1^2 \tag{3.5.239}
\]
The right hand side of the equation (3.5.239) is monotonically decreasing in \( n \), so
\[
\left| \frac{A_n(t+10i)}{A_{n+1}(t+10i)} - 1 \right|^2 \leq \left| \frac{A_{50}(10i)}{A_{51}(10i)} - 1 \right|^2 \leq U_1^2 \tag{3.5.240}
\]
Finally, on \([380i, 380i+1/2] \), write \( \lambda = t + 380i \), \(|A_n/A_{n+1} - 1|^2 \) is monotonically decreasing in \( t \) and thus
\[
\left| \frac{A_n(t+380i)}{A_{n+1}(t+380i)} - 1 \right|^2 \leq \left| \frac{A_{50}(380i)}{A_{51}(380i)} - 1 \right|^2 \leq U_1^2 \tag{3.5.241}
\]
The right hand side of the equation (3.5.241) is monotonically decreasing in \( n \), so
\[
\left| \frac{A_n(t+380i)}{A_{n+1}(t+380i)} - 1 \right|^2 \leq \left| \frac{A_{50}(380i)}{A_{51}(380i)} - 1 \right|^2 \leq U_1^2 \tag{3.5.242}
\]
\[U_2 = 7\sqrt{2}/500 \]
We now obtain an upper bound of \(|F_n - F_{n+1}| \) for \( n \geq 50 \) and \( \lambda \in \partial R_2 \). Let \( N = n - 50 \). Consider \( \lambda \in [10i, 380i] \), write \( \lambda = ti \), and denote \( \Delta_N = F_{N+50} - F_{N+51} = R(N,t) + iS(N,t) \), where
\[
R(N,t) = \frac{P_4(N,t)}{Q_4(N,t)} \quad t \in [10, 380], N \geq 0 \tag{3.5.243}
\]
\[
S(N,t) = \frac{P_5(N,t)}{Q_4(N,t)} \quad t \in [10, 380], N \geq 0 \tag{3.5.244}
\]
where \( P_4, Q_4 \) and \( P_5 \) are all real polynomials in \( N \) and \( t \). Explicitly
\[
Q_4(N,t) = (144N^4 + 40N^2t^2 + t^4 + 29280N^3 + 4088t^2N + 2232184N^2 \\
+ 104482t^2 + 75618040N + 960442081)^2 \left(144N^4 + 40N^2t^2 + t^4 + 29856N^3 \\
+ 4168t^2N + 2320888N^2 + 108610t^2 + 80170824N + 1038321729 \right)^2
\tag{3.5.245}
is positive. We shall obtain bounds for \( R(N, t) \) and \( S(N, t) \) separately. Let \( U_{2,R} = 7/500 \). Since \( Q_4 \) is obviously positive, proving \( |R(N, t)| \leq U_{2,R} \) and \( |S(N, t)| \leq U_{2,I} \) is equivalent to proving the following inequalities

\[
- U_{2,R}Q_4(N, t) \leq P_4(N, t) \leq U_{2,R}Q_4(N, t) \leq 0 \quad (3.5.246)
\]

\[
- U_{2,I}Q_4(N, t) \leq P_5(N, t) \leq U_{2,I}Q_4(N, t) \leq 0 \quad (3.5.247)
\]

To prove (3.5.246) and (3.5.247) we write

\[
T_1(N, t) = P_4(N, t) - U_{2,R}Q_4(N, t) = \sum_{i=0}^{16} w_i^{(1)}(t)N^i \quad (3.5.248)
\]

\[
T_2(N, t) = P_4(N, t) + U_{2,R}Q_4(N, t) = \sum_{i=0}^{16} w_i^{(2)}(t)N^i \quad (3.5.249)
\]

\[
T_3(N, t) = P_5(N, t) - U_{2,I}Q_4(N, t) = \sum_{i=0}^{16} w_i^{(3)}(t)N^i \quad (3.5.250)
\]

\[
T_4(N, t) = P_5(N, t) + U_{2,I}Q_4(N, t) = \sum_{i=0}^{16} w_i^{(4)}(t)N^i \quad (3.5.251)
\]

By our choice of \( U_{2,R} \) and \( U_{2,I} \), we can show the following,

\[
w_i^{(j)}(t) \leq 0 \quad (j = 1, 3, 0 \leq i \leq 16, t \in [10, 380]) \quad (3.5.252)
\]

\[
w_i^{(j)}(t) \geq 0 \quad (j = 2, 4, 0 \leq i \leq 16, t \in [10, 380]) \quad (3.5.253)
\]

and consequently (3.5.246) and (3.5.247). The details are in Section 3.5.13 in the Appendix.

Use the same method and still let \( U_{2,R} = 7/500 \) be an upper bound of \( |\text{Re}(\Delta_n)| \) and \( U_{2,I} = 7/500 \) as an upper bound of \( |\text{Im}(\Delta_n)| \), then we are able to show that on the other three sides of \( \partial R_2 \)

\[
|F_n - F_{n+1}| \leq U_2 \quad (3.5.254)
\]
\[ L_1 = 8/25 \]

We would like to show now for \( n \geq 50 \) and \( \lambda \in \partial R_2 \)

\[ \text{Re} \sqrt{F_n} > L_1 := 8/25 \quad (3.5.255) \]

Let \( u = \text{Re} \sqrt{F_n} \) and \( v = \text{Im} \sqrt{F_n} \) we have

\[ \text{Re}(F_n) = u^2 - v^2 \quad (3.5.256) \]
\[ \text{Im}(F_n) = 2uv \quad (3.5.257) \]

We will show that \( \text{Re}(F_n) \geq L_1^2 \) and \( \text{Im}(F_n) > 0 \). Since we chose the branch of square root to be the one that is positive on the positive real axis, we have

\[ u > \sqrt{u^2 - v^2} \geq L_1 \]

For \( \lambda \in [10i, 380i] \), let \( \lambda = ti \) we can check straightforwardly that for all non-negative \( t \) and \( n \geq 50 \)

\[ \frac{\partial}{\partial t} \text{Re} (F_n(t)) > 0 \quad (3.5.258) \]
\[ \text{Im} (F_n(t)) > 0 \quad (3.5.259) \]

So we have

\[ \text{Re} (F_n(t)) \geq \text{Re} (F_n(0)) = \frac{16 n^4 + 32 n^3 + 152 n^2 + 648 n + 801}{(12 n^2 + 20 n - 9)^2} \quad (3.5.260) \]

Then we can check that

\[ \frac{d}{dn} \text{Re} (F_n(0)) > 0 \quad (3.5.261) \]

and thus

\[ \text{Re} (F_n(t)) \geq \text{Re} (F_n(0)) \geq \text{Re} (F_{50}(0)) > L_1^2 \quad (3.5.262) \]
For $\lambda \in [10i + 1/2, 380i + 1/2]$, let $\lambda = t + 1/2$ we can check straightforwardly that for all nonnegative $t$ and $n \geq 50$,

\[
\frac{\partial}{\partial t} \Re(F_n(t + 1/2)) > 0 
\] (3.5.263)

\[
\Im(F_n(t + 1/2)) > 0
\] (3.5.264)

So we have

\[
\Re(F_n(t + 1/2)) \geq \Re(F_n(1/2))
\] (3.5.265)

\[
= \frac{256 n^4 + 1024 n^3 + 3168 n^2 + 9472 n + 11561}{(48 n^2 + 96 n - 19)^2}
\] (3.5.266)

Then we can check that

\[
\frac{d}{dn} \Re(F_n(1/2)) < 0
\] (3.5.267)

and thus

\[
\Re(F_n(t + 1/2)) \geq \Re(F_n(1/2)) \geq \lim_{n \to \infty} \Re(F_n(1/2)) = 1/9 > L_1^2
\] (3.5.268)

For $\lambda \in [10i, 10i + 1/2]$, let $\lambda = t + 10i$ we can check straightforwardly that for all nonnegative $t$ and $n \geq 50$,

\[
\frac{\partial}{\partial t} \Re(F_n(10i + t)) > 0
\] (3.5.269)

\[
\Im(F_n(10i + t)) > 0
\] (3.5.270)

Thus, we have

\[
\Re(F_n(10i + t)) \geq \Re(F_n(10i))
\] (3.5.271)

Then by (3.5.258) and (3.5.261) we obtain

\[
\Re(F_n(10i)) \geq \Re(F_n(0)) \geq \Re(F_{50}(0)) > L_1^2
\] (3.5.272)
For $\lambda \in [380i, 380i + 1/2]$, let $\lambda = t + 380i$ and write

$$\text{Re}(F_n) = \frac{P_6(n,t)}{Q_6(n,t)}$$

(3.5.273)

where

$$Q_6(n,t) = (144n^4 + 192n^3 + 88nt^2 + 16nt^3 + 480n^3 + 512nt)$$

$$+ 168nt^2 + 16t^3 + t^4 + 5776184n^2 + 2310576nt + 288846t^2$$

(3.5.274)

$$+ 12706840n + 2310256t + 20863200881)^2$$

Defining $N = n - 50$ then we can check directly that

$$T_5(N, t) = P_6(N + 50, t) - L_1^2Q_6(N + 50, t)$$

(3.5.275)

is positive for all nonnegative $N$ and $t$. Hence $\text{Re}(F_n) > L_1^2$ for $\lambda \in [380i, 380i + 1/2]$ and all $n \geq 50$.

$U_3 = 9/40$

We show that $U_3 = 9/40$ is an upper bound for $|B_n/(A_nA_{n+1})|$ for $\lambda \in \partial R_2$. Denote

$$V_n(\lambda) = \left| \frac{B_n(\lambda)}{A_n(\lambda)A_{n+1}(\lambda)} \right|^2$$

(3.5.276)

For $\lambda \in [10i, 380i]$, let $\lambda = ti$ then

$$V_n(ti) = \frac{P_7(n,t)}{Q_7(n,t)}$$

(3.5.277)

where

$$Q_7(n,t) = (144n^4 + 40n^2t^2 + t^4 + 1056n^3 + 168nt^2 + 2488n^2$$

$$+ 210t^2 + 2024n + 529)(144n^4 + 40n^2t^2 + t^4 + 480n^3 + 88nt^2$$

(3.5.278)

$$+ 184n^2 + 82t^2 - 360n + 81)$$

which is manifestly positive for all $t \in [10, 380]$ and $n \geq 50$. Next we just check that for all nonnegative $N$ and $t$

$$P_7(N + 50, t) - U_3^2Q_7(N + 50, t) \leq 0$$

(3.5.279)
which is also obvious.

For $\lambda \in [10i + 1/2, 380i + 1/2]$, let $\lambda = t i + 1/2$. Then

$$V_n(ti + 1/2) = \frac{P_8(n, t)}{Q_8(n, t)} \quad (3.5.280)$$

where

$$Q_8(n, t) = (2304 n^4 + 640 n^2 t^2 + 16 t^4 + 18432 n^3 + 2816 nt^2$$

$$+ 48864 n^2 + 3624 t^2 + 48000 n + 15625)(2304 n^4 + 640 n^2 t^2$$

$$+ 16 t^4 + 9216 n^3 + 1536 nt^2 + 7392 n^2 + 1448 t^2 - 3648 n + 361) \quad (3.5.281)$$

which is manifestly positive for all $t \in [10, 380]$ and $n \geq 50$. Next we check that for all nonnegative $N$ and $t$

$$P_8(N + 50, t) - U_3^2 Q_8(N + 50, t) \leq 0 \quad (3.5.282)$$

For $\lambda \in [10i, 10i + 1/2]$ and $\lambda \in [380i, 380i + 1/2]$, using a similar method we obtain that for $t \in [0, 1/2]$ and $n \geq 50$

$$V_n(10i + t) \leq U_3^2 \quad (3.5.283)$$

$$V_n(380i + t) \leq U_3^2 \quad (3.5.284)$$

### 3.5.13 Proof of (3.5.252) and (3.5.253)

We give the proof for $\lambda \in [10i, 380i]$. The proof is similar for $\lambda \in [10i + 1/2, 380i + 1/2]$. On the other two sides of $\partial R_2$ the sign of $w_i^{(j)}(t)$ is clear by inspection.

$t \in [10, 60]$

For each $0 \leq i \leq 16$ and $j \in \{1, 2, 3, 4\}$, assume that $w_i^{(j)}(t)$ is a polynomial of degree $\alpha$. If $\alpha \leq 3$ it is trivial to calculate the maximum or minimum of $w_i^{(j)}(t)$ on $[10, 60]$. If $\alpha > 3$, we express $w_i^{(j)}(t)$ as a linear combination of Chebyshev polynomials:

$$w_i^{(j)}(t) = \sum_{l=0}^{\alpha} \tilde{a}_l T_l \left( \frac{t - 35}{25} \right) \quad (3.5.285)$$
Using (3.5.214) we have lower bounds and upper bounds for \( w_i^{(j)}(t) \)

\[
\inf_{t \in [10, 60]} \left\{ \sum_{l=0}^{3} \tilde{a}_l T_l \left( \frac{t - 35}{25} \right) \right\} - \sum_{l=4}^{\alpha} |\tilde{a}_l| \leq w_i^{(j)}(t)
\]

\[
\leq \sup_{t \in [10, 60]} \left\{ \sum_{l=0}^{3} \tilde{a}_l T_l \left( \frac{t - 35}{25} \right) \right\} + \sum_{l=4}^{\alpha} |\tilde{a}_l|
\]

(3.5.286)

We can check that for \( j = 1, 3 \) the upper bound of \( w_i^{(j)}(t) \) given by (3.5.286) is negative and for \( j = 2, 4 \) the lower bound of \( w_i^{(j)}(t) \) given by (3.5.286) is positive.

\[ t \in [60, 380] \]

For each \( j \in \{1, 2, 3, 4\} \), assume that \( K(t) = w_i^{(j)}(t) \) is a polynomial of degree \( \alpha \), then \( K(t) \) has the property that the \( \beta \)-th derivative of \( K(t) \) is monotonic and does not change sign on \([60, 380]\) for each \( 0 \leq \beta \leq \alpha \). We can show this by induction.

It is obvious that the \( \alpha \)-th derivative of \( K(t) \), which is a constant, does not change sign on any interval. Since the \( \beta \)-th derivative \( K^{(\beta)}(t) \) does not change sign on \([60, 380]\) we have for each \( \beta \)

\[
\frac{d^\beta K(60)}{dt^\beta} \frac{d^\beta K(380)}{dt^\beta} \geq 0
\]

(3.5.287)

Assume that \( j \leq 0 \). Suppose the \( (\alpha - j) \)-th derivative does not change sign on \([60, 380]\) and (3.5.287) holds for \( \beta = \alpha - j - 1 \). Then the \( (\alpha - j - 1) \)-th derivative is monotonic and does not change sign on \([60, 380]\). By induction on \( j \) we see that \( K(t) \) is monotonic and does not change sign on \([60, 380]\). Now it is trivial to check that

\[
w_i^{(j)}(60) < 0 \quad w_i^{(j)}(380) < 0 \quad (j = 1, 3)
\]

(3.5.288)

\[
w_i^{(j)}(60) > 0 \quad w_i^{(j)}(380) > 0 \quad (j = 2, 4)
\]

(3.5.289)

Hence each \( w_i^{(j)}(t) \) does not change sign on the interval \([60, 380]\) and (3.5.252) and (3.5.253) are proved.
### 3.5.14 Table of coefficients $a_{n,i}^{(1,2)}$ and $b_{n,i}^{(1,2)}$

Note: In the table below, a pair of the form “A, B” is used to denote the complex number $A + Bi$.

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Chapter 4

TRONQUÉE SOLUTIONS OF THE THIRD AND FOURTH PAINLEVÉ EQUATIONS

4.1 Introduction

The third Painlevé equation $P_{III}$ and the fourth Painlevé equation $P_{IV}$ are:

\begin{align*}
P_{III} : & \quad \frac{d^2 y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{1}{x} (\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \quad (4.1.1) \\
P_{IV} : & \quad \frac{d^2 y}{dx^2} = \frac{1}{2y} \left( \frac{dy}{dx} \right)^2 + \frac{3}{2} y^3 + 4xy^2 + 2 (x^2 - \alpha) y + \frac{\beta}{y} \quad (4.1.2)
\end{align*}

where $\alpha, \beta, \gamma$ and $\delta$ are arbitrary complex numbers. By Bäcklund transformations (see [4]) $P_{III}$ can be reduced to:

\begin{align*}
P_{III}^{(i)} : & \quad \frac{d^2 y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{1}{x} (\alpha y^2 + \beta) + y^3 - \frac{1}{y} \quad (4.1.3) \\
P_{III}^{(ii)} : & \quad \frac{d^2 y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{1}{x} (y^2 + \beta) - \frac{1}{y} \quad (4.1.4)
\end{align*}

In this paper we study tronquée solutions of (4.1.3), (4.1.4) and (4.1.2) by first transforming each of them into second order differential equations of the following form:

\[ h''(w) - h(w) + \frac{1}{w} \left[ (\beta_2 - \beta_1) h(w) + (\beta_2 + \beta_1) h'(w) \right] = g(w, h, h') \quad (4.1.5) \]
where $\beta_1$ and $\beta_2$ are constants and $g(w, h, h')$ is analytic at $(\infty, 0, 0)$. Next we make the substitution

$$
\begin{bmatrix}
  h(w) \\
  h'(w)
\end{bmatrix} =
\begin{bmatrix}
  1 - \frac{\beta_1}{2w} & 1 + \frac{\beta_2}{2w} \\
  -1 - \frac{\beta_1}{2w} & 1 - \frac{\beta_2}{2w}
\end{bmatrix} u(w)
$$

Then $u$ is a solution to the following normalized (see [2]) 2-dimensional differential system:

$$
u' + \left(\hat{\Lambda} + \frac{\hat{B}}{w}\right) u = g(w, u)$$

where

$$
\hat{\Lambda} =
\begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix} \quad \hat{B} =
\begin{bmatrix}
  \beta_1 & 0 \\
  0 & \beta_2
\end{bmatrix}
$$

and $g(w, u)$ is analytic at $(\infty, 0)$ with $g(w, u) = O(w^{-2}) + O(|u|^2)$ as $w \to \infty$ and $u \to 0$.

We obtain information about the tronquée and tritronquée solutions of the normalized system (4.1.7) such as their existence, regions of analyticity and asymptotic position of poles. Our method uses results in [5] and [2], through which we obtain corresponding results regarding tronquée and tritronquée solutions of $P_{III}$ and $P_{IV}$. See also [7] in which a similar approach is used to study the tronquée solutions of the first Painlevé equation.

### 4.2 Tronquée solutions of (4.1.5)

#### 4.2.1 Formal solutions and tronquée solutions of (4.1.5)
In Proposition 4.2.1 and Theorem 4.2.2 we present formal and actual solutions on the right half \( w \)-plane \( S_1 := \{ w : \arg(w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\} \). Then through a simple symmetry transformation we obtain solutions in the left half plane \( S_2 := \{ w : \arg(w) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\} \). We start with the formal expansions of the solutions. Assume that \( d \) is a ray of the form \( e^{i\phi} \mathbb{R}^+ \) with \( \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \). We have the following results on transseries solutions, formal expansions in powers of \( 1/w \) and \( e^{-w} \) (see [2]), of (4.1.5) valid on \( d \) and, moreover, in the sector \( S_1 \):

**Proposition 4.2.1.** Assume that \( d \) is a ray of the form \( e^{i\phi} \mathbb{R}^+ \) with \( \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \). Then (i) the one-parameter family of transseries solutions of (4.1.5) satisfying \( h(w) \to 0 \) as \( |w| \to \infty \) on \( d \) are:

\[
\tilde{h}(w) = \tilde{h}_0(w) + \sum_{k=1}^{\infty} C^k e^{-kw} w^{-\beta_1 k} \tilde{s}_k(w) \tag{4.2.1}
\]

where for each \( k \geq 1 \),

\[
\tilde{s}_k(w) = \sum_{j=0}^{\infty} \frac{s_{k,j}}{w^j} \tag{4.2.2}
\]

is a formal power series in \( w^{-1} \). (ii) The formal power series in \( w^{-1} \)

\[
\tilde{h}_0(w) = \sum_{j=2}^{\infty} \frac{h_{0,j}}{w^j} \tag{4.2.3}
\]

is the unique formal power series solution of (4.1.5).

The results in [2] provides us with the relation between these transseries solutions and actual solutions.

In the following, we denote by \( \mathcal{L}_\phi \) the Laplace transform

\[
f \mapsto \int_0^{\infty} e^{i\phi p} f(p) e^{-xp} dp \tag{4.2.4}
\]

where \( \phi \in \mathbb{R} \). See also [5] (p. 8) for the notation.
Theorem 4.2.2. Let $d, \tilde{h}_0(w)$ and $\tilde{s}_k(w)$ be as in Proposition 4.2.1. Let $h(w)$ be a solution to (4.1.5) on $d$ for $|w|$ large enough satisfying

$$h(w) \to 0 \quad w \in d, \quad |w| \to \infty \quad (4.2.5)$$

Then:

(i) There is a unique pair of constants $(C_+, C_-)$ associated with $h(w)$, and $h(w)$ has the following representations:

$$h(w) = \mathcal{L}_\phi H_0(w) + \sum_{k=1}^{\infty} C_+^k e^{-kw} w^{M_1} \mathcal{L}_\phi H_k(w),$$

$$- \phi = \arg(w) \in \left(0, \frac{\pi}{2}\right) \quad (4.2.6)$$

$$h(w) = \mathcal{L}_\phi H_0(w) + \sum_{k=1}^{\infty} C_-^k e^{-kw} w^{M_1} \mathcal{L}_\phi H_k(w),$$

$$- \phi = \arg(w) \in \left(-\frac{\pi}{2}, 0\right) \quad (4.2.7)$$

where

$$M_1 = \lfloor \text{Re}(-\beta_1) \rfloor + 1,$$

$$H_0 = \mathcal{B} \tilde{h}_0,$$

$$H_k = \mathcal{B} \tilde{h}_k = \mathcal{B} \left( w^{-k\beta_1} - k^{M_1} \tilde{s}_k \right) \quad (k = 1, 2, \ldots)$$

where each $H_k$ is analytic on the Riemann surface of $\mathbb{C} \setminus (\mathbb{Z}^+ \cup \mathbb{Z}^-)$, and the branch cut for each $H_k$, $(k \geq 1)$ is chosen to be $(-\infty, 0]$.

(ii) There exists $\epsilon_0 > 0$ such that for each $0 < \epsilon \leq \epsilon_0$ there exist $\delta_\epsilon > 0$, $R_\epsilon > 0$ such that $h(w)$ can be analytically continued to (at least) the following region:

$$S_{an, \epsilon}(h(w)) = S_\epsilon^+ \cup S_\epsilon^- \quad (4.2.9)$$

where

$$S_\epsilon^- = \{ w : |w| > R_\epsilon, \arg(w) \in \left[-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} - \epsilon\right] \text{ and } |C_- e^{-w} w^{-\beta_1}| < \delta_\epsilon^{-1} \}$$

$$S_\epsilon^+ = \{ w : |w| > R_\epsilon, \arg(w) \in \left[-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} + \epsilon\right] \text{ and } |C_+ e^{-w} w^{-\beta_1}| < \delta_\epsilon^{-1} \}$$

(4.2.10)
Consequently, \( h(w) \) is analytic (at least) in

\[
S_{an}(h) = \bigcup_{0<\epsilon \leq \epsilon_0} (S^-_i \cup S^+_i) \tag{4.2.11}
\]

(iii) \( h(w) \sim h_0(w) \) in \( S_1 \).

**Note 4.2.3.** (i) It is straightforward to check that if \( \text{Re}(\beta_1) > 0 \), \( S_{an} \) contains all but a compact subset of \( i\mathbb{R} \). In other words there exists \( R_0 > 0 \) such that \( h(w) \) is analytic in the closure of \( S_1 \setminus D_{R_0} \), where \( S_1 \) is the open right half plane and \( D_{R_0} = \{|w| < R_0\} \) is the open disk centered at origin with radius \( R_0 \). (ii) On the other hand if \( \text{Re}(\beta_1) < 0 \), \( S^c_{an} \) contains all but a compact subset of \( i\mathbb{R} \). We point out that in particular, the solution is not analytic in \( S_1 \setminus D_{R_0} \) for any \( R_0 > 0 \) as claimed in [8], singularities of the tronquée solutions exist for large \( w \) in \( S_1 \) as seen in Theorem 4.2.4, 4.2.5, 4.3.2, 4.3.3, 4.3.9 and 4.3.10.

**Theorem 4.2.4** (Asymptotic position of singularities). Let \( h, C_+ \) and \( C_- \) be as in Theorem 4.2.2.

(i) Assume \( C_+ \neq 0 \). Denote

\[
\xi_+(w) = C_+ w^{-\beta_1} e^{-w} \tag{4.2.12}
\]

Then

\[
h(w) \sim \sum_{m=0}^{\infty} \frac{F_m(\xi_+(w))}{w^m} \quad (|w| \to \infty, w \in D_{R}^+) \tag{4.2.13}
\]

where \( F_m \) (\( m \geq 0 \)) are analytic at \( \xi = 0 \) and

\[
D_{w}^+ = \left\{|w| > R : \arg w \in \left(-\frac{\pi}{2} + \delta, -\frac{\pi}{2} + \delta\right), \right.
\]

\[
\text{dist} (\xi_+(w), \Xi) > \epsilon, |\xi(w)| < \epsilon^{-1} \right\} \tag{4.2.14}
\]

for any \( \delta, \epsilon > 0 \) small enough and \( R \) large enough, and where \( \Xi \) is the set of singularities of \( F_0(\xi) \). \( F_0(\xi) \) satisfies

\[
F_0(0) = 0, \quad F_0'(0) = 1 \tag{4.2.15}
\]
(ii) Assume $C_+ \neq 0$, and $\xi_s \in \Xi$ is a singularity of $F_0$. Then the singular points of $h, w^+_n$, near the boundary $\{x : \arg(w) = \pi/2\}$ of the sector of analyticity are given asymptotically by

$$w^+_n = 2n\pi i - \beta_1 \ln(2n\pi i) + \ln(C_+) - \ln(\xi_s) + o(1)$$

as $n \to \infty$.

(iii) Assume $C_- \neq 0$. Denote

$$\xi_-(w) = C_- w^{-\beta_1} e^{-w}$$

Then

$$h(w) \sim \sum_{m=0}^{\infty} \frac{F_m(\xi_-(w))}{w^m} \quad (|w| \to \infty, w \in D_w^-)$$

where

$$D_w^- = \left\{ |w| > R : \arg w \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} - \delta \right), \right.\right.
\dist(\xi_-(w), \Xi) > \epsilon, |\xi(w)| < \epsilon^{-1}\left.\right\}$$

for any $\delta, \epsilon > 0$ small enough and $R$ large enough, and where $F_m (m \geq 0)$ and $\Xi$ are as described in (i).

(iv) Assume $C_- \neq 0$, and $\xi_s \in \Xi$ is a singularity of $F_0$. Then the singular points of $h, w^-_n$, near the boundary $\{x : \arg(w) = -\pi/2\}$ of the sector of analyticity are given asymptotically by

$$w^-_n = -2n\pi i - \beta_1 \ln(-2n\pi i) + \ln(C_-) - \ln(\xi_s) + o(1)$$

as $n \to \infty$.

The expression of $F_0$ (see (4.3.11), (4.3.34), (4.3.55) and (4.3.67)) is obtained explicitly in each case where asymptotic position of singularities is presented.
4.2.2 Tritronquée solutions of (4.1.5)

The information on formal and actual tronquée solutions of (4.1.5) in the left half plane \( S_2 := \{ w : \text{arg}(w) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \} \) is obtained by means of a simple transformation

\[
h(w) = \hat{h}(-w), \quad \tilde{w} = -w
\]  

(4.2.21)

(4.1.5) is rewritten as

\[
\hat{h}''(\tilde{w}) - \frac{1}{\tilde{w}} \left[ (\beta_1 - \beta_2) \hat{h}(\tilde{w}) + (\beta_1 + \beta_2) \hat{h}'(\tilde{w}) \right] = g(-\tilde{w}, \hat{h}, -\hat{h}')
\]

(4.2.22)

which is of the form (4.1.5) with \( \beta_1 \) and \( \beta_2 \) exchanged, and thus all results in Proposition 4.2.1 - Theorem 4.2.4 apply. Without repeating all of the results, we introduce some notations needed for describing the tritronquée solutions of (4.1.5).

The small transseries solutions of (4.2.22) in the right half \( \tilde{w} \)-plane is:

\[
\tilde{h}_1(\tilde{w}) = \tilde{h}_0(-\tilde{w}) + \sum_{k=1}^{\infty} \hat{C}_k e^{-k\tilde{w}} \tilde{w}^{\beta_2 k} \tilde{t}_k(\tilde{w})
\]

(4.2.23)

where for each \( k \geq 1 \), \( \tilde{t}_k(\tilde{w}) \) is a formal power series in \( \tilde{w}^{-1} \).

Assume that \( \hat{h}(\tilde{w}) \) is an actual solution to (4.2.22) on \( d = e^{i\theta \mathbb{R}^+} \) with \( \cos \theta > 0 \), such that \( \hat{h}(\tilde{w}) = o(1) \) as \( |\tilde{w}| \to \infty \). Then there exists a unique pair of constants \((\hat{C}_+, \hat{C}_-)\) such that

\[
\hat{h}(\tilde{w}) = \begin{cases} 
\mathcal{L}_{\phi} \hat{H}_0(\tilde{w}) + \sum_{k=1}^{\infty} \hat{C}_k e^{-k\tilde{w}} \tilde{w}^{\beta_2 k} \mathcal{L}_\phi \hat{H}_k(\tilde{w}), \\
\quad \phantom{=} \quad -\phi = \text{arg}(\tilde{w}) \in \left(0, \frac{\pi}{2}\right); \\
\mathcal{L}_{\phi} \hat{H}_0(\tilde{w}) + \sum_{k=1}^{\infty} \hat{C}_k e^{-k\tilde{w}} \tilde{w}^{\beta_2 k} \mathcal{L}_\phi \hat{H}_k(\tilde{w}), \\
\quad \phantom{=} \quad -\phi = \text{arg}(\tilde{w}) \in \left(-\frac{\pi}{2}, 0\right).
\end{cases}
\]

(4.2.24)
where

\[ M_2 = \left\lfloor \text{Re}(-\beta_2) \right\rfloor + 1, \]

\[ \hat{H}_0(p) = -H_0(-p), \]  \hspace{1cm} (4.2.25)

\[ \hat{H}_k = B \left( w^{-k\beta_2-kM_2} \tilde{t}_k \right) \quad (k \geq 1) \]

where each \( \hat{H}_k \) is analytic in the Riemann surface of \( \mathbb{C} \setminus (\mathbb{Z}^+ \cup \mathbb{Z}^-) \), and the branch cut for each \( \hat{H}_k \), \((k \geq 1)\) is chosen to be \((-\infty, 0]\). Note that the second equation in (4.2.25) holds because the power series solution of (4.2.22) must be \( \tilde{h}_0(-\tilde{w}) \). By the definition of the Borel transform (see \( \S 4.5 \)) we have \( \hat{H}_0(p) = -H_0(-p) \).

By Theorem 4.2.2(ii), \( \hat{h} \) is analytic at least on

\[ \hat{S}_{an}(h) := -S_{an}(\hat{h}) \]  \hspace{1cm} (4.2.26)

where \( S_{an}(\hat{h}) \) is given by (4.2.9) – (4.2.11) with \( \beta_1 \) replaced by \( \beta_2 \). Denote \( \hat{\xi}_\pm = \hat{C}_\pm \tilde{w}^{-\beta_2} e^{-\tilde{w}} \) as in (4.2.12) and (4.2.17). By Theorem 4.2.4 if \( \hat{C}_+ \neq 0 \) then

\[ \hat{h}(\tilde{w}) \sim \sum_{m=0}^{\infty} \frac{\hat{F}_m(\hat{\xi}_+(\tilde{w}))}{\tilde{w}^m} \quad (|\tilde{w}| \to \infty, \tilde{w} \in D^+_\tilde{w}) \]  \hspace{1cm} (4.2.27)

where \( \hat{F}_m \) are analytic at \( \xi = 0 \). If \( \hat{C}_- \neq 0 \) then

\[ \hat{h}(\tilde{w}) \sim \sum_{m=0}^{\infty} \frac{\hat{F}_m(\hat{\xi}_-(\tilde{w}))}{\tilde{w}^m} \quad (|\tilde{w}| \to \infty, \tilde{w} \in D^-_\tilde{w}) \]  \hspace{1cm} (4.2.28)

where \( D^+_\tilde{w} \) are defined by (4.2.14) and (4.2.19) with \( \xi_\pm \) replaced by \( \hat{\xi}_\pm \) respectively and \( \Xi \) replaced by \( \hat{\Xi} \) which is defined to be the set of singularities of \( \hat{F}_0 \).

Tritronquée solutions are special cases of tronquée solutions with \( C_+ = 0 \) or \( C_- = 0 \). Denote

\[ h^+(w) = \mathcal{L}_\phi H_0(w) \quad -\phi = \text{arg}(w) \in (0, \pi) \]  \hspace{1cm} (4.2.29)

\[ h^-(w) = \mathcal{L}_\phi H_0(w) \quad -\phi = \text{arg}(w) \in (-\pi, 0) \]  \hspace{1cm} (4.2.30)

\[ \hat{h}^+(w) = \mathcal{L}_\phi \hat{H}_0(w) \quad -\phi = \text{arg}(w) \in (0, \pi) \]  \hspace{1cm} (4.2.31)

\[ \hat{h}^-(w) = \mathcal{L}_\phi \hat{H}_0(w) \quad -\phi = \text{arg}(w) \in (-\pi, 0) \]  \hspace{1cm} (4.2.32)
Corollary 4.2.5. Assume $\phi \in \left(0, \frac{\pi}{2}\right)$. Let $C_{1t}^t, C_{2t}^t, C_{3t}^t, C_{4t}^t$ be the constants in the transseries of $h^\pm$ and $\hat{h}^\pm$, namely:

\[
h^+(w) = L_{-\phi}H_0(w) + \sum_{k=1}^{\infty} (C_{1t}^t)^k e^{-kw}w^{M_{t1}k}L_{\phi}H_k(w) \tag{4.2.33}
\]

\[
h^-(w) = L_{\phi}H_0(w) + \sum_{k=1}^{\infty} (C_{2t}^t)^k e^{-kw}w^{M_{t2}k}L_{-\phi}H_k(w) \tag{4.2.34}
\]

\[
\hat{h}^+(w) = L_{-\phi}\hat{H}_0(w) + \sum_{k=1}^{\infty} (C_{3t}^t)^k e^{-kw}w^{M_{t2}\hat{k}}L_{\phi}\hat{H}_k(w) \tag{4.2.35}
\]

\[
\hat{h}^-(w) = L_{\phi}\hat{H}_0(w) + \sum_{k=1}^{\infty} (C_{4t}^t)^k e^{-kw}w^{M_{t2}\hat{k}}L_{-\phi}\hat{H}_k(w) \tag{4.2.36}
\]

(i) We have

\[
h^+(w) = \hat{h}^-(w) \tag{4.2.37}
\]

\[
h^-(w) = \hat{h}^+(w) \tag{4.2.38}
\]

A consequence of Theorem 4.2.2 (ii) is that for any $\delta > 0$ there exists $R > 0$ such that $h^+$ is analytic in the sector

\[
T_{\delta,R}^+ := \left\{ w : |w| > R, \arg(w) \in \left(-\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta\right) \right\}. \tag{4.2.39}
\]

$h^-$ is analytic in the sector

\[
T_{\delta,R}^- := \left\{ w : |w| > R, \arg(w) \in \left(-\frac{3\pi}{2} + \delta, \frac{\pi}{2} - \delta\right) \right\}. \tag{4.2.40}
\]

(ii) Assume $\xi_s \in \Xi$ is a singularity of $F_0$ (see Theorem 4.2.4 (ii)) and $\hat{\xi}_s \in \hat{\Xi}$ is a singularity of $\hat{F}_0$. Then the singular points of $h^+$, $w_{1,n}^-$ near the boundary $\left\{ w : \arg w = -\frac{\pi}{2} \right\}$ and $w_{1,n}^+$ near the boundary $\left\{ w : \arg w = \frac{3\pi}{2} \right\}$, are given asymptotically by

\[
w_{1,n}^- = -2n\pi i - \beta_1 \ln(-2n\pi i) + \ln(C_1^t) - \ln(\xi) + o(1) \tag{4.2.41}
\]

\[
w_{1,n}^+ = -2n\pi i + \beta_2 \ln(2n\pi i) - \ln(C_4^t) + \ln(\hat{\xi}) + o(1) \tag{4.2.42}
\]
as $n \to \infty$. The singular points of $h^-, w_{2,n}^-$ near the boundary \( \{ w : \arg w = -\frac{3\pi}{2} \} \)
and $w_{2,n}^+$ near the boundary \( \{ w : \arg w = \frac{\pi}{2} \} \), are given asymptotically by

\[
\begin{align*}
w_{2,n}^- &= 2n\pi i + \beta_2 \ln(-2n\pi i) - \ln(C_3^1) + \ln(\xi_s) + o(1) \\
w_{2,n}^+ &= 2n\pi i - \beta_1 \ln(2n\pi i) + \ln(C_2^2) - \ln(\xi_s) + o(1)
\end{align*}
\tag{4.2.43}
\tag{4.2.44}
\]

4.3 Normalizations and Tronquée solutions of $P_{\text{III}}$ and $P_{\text{IV}}$

4.3.1 Tronquée solutions of $P_{\text{III}}^{(i)}$

If $y(x)$ is a solution of (4.1.3) which is asymptotic to a formal power series on a ray $d$ which is not an antistokes line (lines on which $\arg w = \pm \frac{\pi}{2}$ where $w$ is the independent variable in the normalized equation), then by dominant balance we have

\[
y(x) \sim l(x) \quad (|x| \to \infty \quad x \in d)
\tag{4.3.1}
\]

where

\[
l(x) = A - \left( \frac{\alpha + A^2 \beta}{4} \right) \frac{1}{x} \tag{4.3.2}
\]

for some $A$ satisfying $A^4 = 1$. Fix some $A$ satisfying $A^4 = 1$ and make the change of variables

\[
w = 2Ax, \quad y(x) = h(w) + l \left( \frac{w}{2A} \right) \tag{4.3.3}
\]

Then the equation (4.1.3) is transformed into an equation for $h$ of the form (4.1.5) with

\[
\beta_1 = \frac{1}{2} + \frac{\alpha}{4} - \frac{A^2 \beta}{4}, \quad \beta_2 = \frac{1}{2} - \frac{\alpha}{4} + \frac{A^2 \beta}{4} \tag{4.3.4}
\]

Results in §4.2 apply. Let the notations be the same as in §4.2.
Theorem 4.3.1. (i) There is a unique formal power series solution

$$\tilde{y}_0(x) = \sum_{k=0}^{\infty} \frac{y_{0,k}}{x^k}$$

(4.3.5)

to (4.1.3), where

$$y_{0,0} = A, \quad y_{0,1} = -\frac{\alpha + A^2 \beta}{4}$$

(4.3.6)

(ii) There is a one-parameter family \( F_{A,1} \) of tronquée solutions of (4.1.3) in \( A^{-1}S_1 \) with representations

$$y(x) = \begin{cases} 
  l(x) + h^+(2Ax) + \sum_{k=1}^{\infty} C^k_+ e^{-2Akx}(2Ax)^k \mathcal{L}_\phi H_k(2Ax), \\
  -\phi \in \left(0, \frac{\pi}{2}\right) 

de, -\phi \in \left(-\frac{\pi}{2}, 0\right) 

de, \quad & -\phi \in \left(0, \frac{\pi}{2}\right); 

de, \quad & -\phi \in \left(-\frac{\pi}{2}, 0\right). 
\end{cases}$$

(4.3.7)

(iii) There is a one-parameter family \( F_{A,2} \) of tronquée solutions of (4.1.3) in \( A^{-1}S_2 \) with representations

$$y(x) = \begin{cases} 
  l(x) + h^-(2Ax) + \sum_{k=1}^{\infty} C^k_- e^{-2Akx}(2Ax)^k \mathcal{L}_\phi \hat{H}_k(2Ax), \\
  -\phi \in \left(0, \frac{\pi}{2}\right) 

de, -\phi \in \left(-\frac{\pi}{2}, 0\right) 

de, \quad & -\phi \in \left(0, \frac{\pi}{2}\right); 

de, \quad & -\phi \in \left(-\frac{\pi}{2}, 0\right). 
\end{cases}$$

(4.3.8)

(iv) For each tronquée solution in (ii) or (iii) we have

$$y(x) \sim \tilde{y}_0(x) \quad x \in d = A^{-1}e^{i\theta}\mathbb{R}^+, \quad |x| \to \infty$$

(4.3.9)

and the solution is analytic at least in \((2A)^{-1}S_{an}\) if \(\cos \theta > 0\), in \((2A)^{-1}\hat{S}_{an}\) if \(\cos \theta < 0\). \(S_{an}\) and \(\hat{S}_{an}\) are as defined in Theorem 4.2.2 and §4.2.2.
From Theorem 4.2.4 we obtain information about the singularities of \( y \). Assume that \( y \) is a tronquée solution with representation (4.3.7) or (4.3.8). Let \( \xi_+ = C_+ e^{-w} w^{-\beta_1} \), \( \xi_- = C_- e^{-w} w^{-\beta_2} \), \( F_m \) and \( \hat{F}_m \) be as in \( \S 4.2 \). Then the equation satisfied by \( F_0 \) is

\[
\xi^2 \frac{d^2}{d\xi^2} F_0 (\xi) + \xi \frac{d}{d\xi} F_0 (\xi) - \frac{\xi^2 \left( \frac{d}{d\xi} F_0 (\xi) \right)^2}{A + F_0 (\xi)} - \frac{(A + F_0 (\xi))^3}{4A^2} + \frac{1}{4A^2 (A + F_0 (\xi))} = 0
\]

(4.3.10)

The equation of \( \hat{F}_0 \) is the same as (4.3.10). The solution satisfying (4.2.15) is

\[
F_0 (\xi) = \frac{2A \xi}{2A - \xi}
\]

(4.3.11)

**Theorem 4.3.2.** (i) Assume \( y(x) \in \mathcal{F}_{A,1} \) is given by the representation (4.3.7). If \( C_+ \neq 0 \), then the singular points of \( y \), \( x_n^- \), near the boundary

\[
\{ x : \arg(2Ax) = \pi/2 \}
\]

of the sector of analyticity are given asymptotically by

\[
(2A)x_n^+ = 2n\pi i - \beta_1 \ln(2n\pi i) + \ln(C_+) - \ln(2A) + o(1) \quad (n \to \infty)
\]

(4.3.12)

If \( C_- \neq 0 \), then the singular points of \( y \), \( x_n^- \), near the boundary

\[
\{ x : \arg(2Ax) = -\pi/2 \}
\]

of the sector of analyticity are given asymptotically by

\[
(2A)x_n^- = -2n\pi i - \beta_1 \ln(-2n\pi i) + \ln(C_-) - \ln(2A) + o(1) \quad (n \to \infty)
\]

(4.3.13)

(ii) Assume \( y(x) \in \mathcal{F}_{A,2} \) is given by the representation (4.3.8). If \( \hat{C}_+ \neq 0 \), then the singular points of \( y \), \( \tilde{x}_n^+ \), near the boundary

\[
\{ \tilde{x} : \arg(-2A\tilde{x}) = \pi/2 \}
\]
of the sector of analyticity are given asymptotically by

$(-2A)x_n^+ = 2n\pi i - \beta_2 \ln(2n\pi i) + \ln(\hat{C}_+) - \ln(2A) + o(1) \quad (n \to \infty)$  \hspace{1cm} (4.3.14)

If $\hat{C}_- \neq 0$, then the singular points of $y, \bar{x}_n^-$, near the boundary

$\{\bar{x} : \arg(-2A\bar{x}) = -\pi/2\}$

of the sector of analyticity are given asymptotically by

$(-2A)x_n^- = -2n\pi i - \beta_2 \ln(-2n\pi i) + \ln(\hat{C}_-) - \ln(2A) + o(1) \quad (n \to \infty)$

(4.3.15)

From Theorem 4.2.4 we obtain the following results about tritronquée solutions of

(4.1.3):

**Theorem 4.3.3.** (4.1.3) has two tritronquée solutions $y^+(x)$ and $y^-(x)$ given by

$y^+(x) = l(x) + h^+(2Ax)$  \hspace{1cm} (4.3.16)

$y^-(x) = l(x) + h^-(2Ax)$  \hspace{1cm} (4.3.17)

Let $C_j^t$, $1 \leq j \leq 4$ be as in (4.2.33). Then:

(i) $\mathcal{F}_{A,1} \cap \mathcal{F}_{A,2} = \{y^+, y^-\}$.

(ii) For each $\delta > 0$ there exists $R > 0$ such that $y^+(x)$ is analytic in $A^{-1}T_{\delta,R}^+$, and

$y^+$ is asymptotic to $y_0(x)$ in the sector

$\bigcup_{\frac{\pi}{2} < \phi < \frac{3\pi}{2}} (A^{-1}e^{i\phi}\mathbb{R}^+)$

The singular points of $y^+(x)$, $x_{1,n}^\pm$, near the boundary of the sector of analyticity

are given asymptotically by

$(2A)x_{1,n}^- = -2n\pi i - \beta_1 \ln(-2n\pi i) + \ln\left(\frac{C_1^t}{2A}\right) + o(1) \quad (n \to \infty)$  \hspace{1cm} (4.3.18)

$(2A)x_{1,n}^+ = -2n\pi i + \beta_2 \ln(2n\pi i) - \ln\left(\frac{C_4^t}{2A}\right) + o(1) \quad (n \to \infty)$  \hspace{1cm} (4.3.19)
(iii) For each $\delta > 0$ there exists $R > 0$ such that $y^-(x)$ is analytic in $A^{-1}T_{\delta R}$, and $y^-$ is asymptotic to $y_0(x)$ in the sector

$$\bigcup_{-\frac{3\pi}{2} < \phi < \frac{\pi}{2}} (A^{-1}e^{i\phi}R^+)$$

The singular points of $y^-(x)$, $x_{2,n}^\pm$, near the boundary of the sector of analyticity are given asymptotically by

$$(2A)x_{2,n}^- = 2n\pi i + \beta_2 \ln(-2n\pi i) - \ln(C_3^t) + \ln(2A) + o(1) \quad (n \to \infty) \quad (4.3.20)$$

$$(2A)x_{2,n}^+ = 2n\pi i - \beta_1 \ln(2n\pi i) + \ln(C_3^t) - \ln(2A) + o(1) \quad (n \to \infty) \quad (4.3.21)$$

### 4.3.2 Tronquée solutions of $P_{iii}^{(ii)}$

If $y(x)$ is a solution of (4.1.4) which is asymptotic to a formal power series on a ray $d$ which is not an antistokes line, then by dominant balance we have

$$y(x) \sim l(x) \quad (|x| \to \infty \quad x \in d) \quad (4.3.22)$$

where

$$l(x) = A x^{1/3} - \frac{\beta}{3A x^{1/3}} \quad (4.3.23)$$

for some $A$ satisfying $A^3 = 1$. Fix an $A$ satisfying $A^3 = 1$ and make the change of variables

$$w = (27A/4)^{1/2}x^{2/3}, \quad y(x) = x^{1/3}h(w) + l(x) \quad (4.3.24)$$

Then the equation (4.1.4) is transformed into an equation for $h$ of the form (4.1.5) with

$$\beta_1 = \beta_2 = \frac{1}{2} \quad (4.3.25)$$
Let the notations be the same as in §4.2. In view of the transformation (4.3.24), we denote

\[ S_R^{(0)} := \{ x : |x| \geq R, \arg(x) \in \left[ -\frac{3\pi}{4}, -\frac{3\arg A}{4}, \frac{3\pi}{4}, \frac{3\arg A}{4} \right] \} \]

\[ S_R^{(1)} := \{ x : |x| \geq R, \arg(x) \in \left[ \frac{3\pi}{4}, \frac{3\arg A}{4}, \frac{9\pi}{4}, \frac{3\arg A}{4} \right] \} \]

\[ S_R^{(2)} := \{ x : |x| \geq R, \arg(x) \in \left[ \frac{9\pi}{4}, \frac{3\arg A}{4}, \frac{15\pi}{4}, \frac{3\arg A}{4} \right] \} \]

\[ S_R^{(3)} := \{ x : |x| \geq R, \arg(x) \in \left[ \frac{15\pi}{4}, \frac{3\arg A}{4}, \frac{21\pi}{4}, \frac{3\arg A}{4} \right] \} \]

(4.3.26)

We notice that for \( j \in \{0, 2\} \), \( S_R^{(j)} \) is mapped under the transformation (4.3.24) bijectively to the closed sector \( S_1 \setminus \mathbb{D}_{R_0} \) in the \( w \)-plane, where \( R_0 = R^{2/3} \) (see also Note 4.2.3); for \( j \in \{1, 3\} \), \( S_R^{(j)} \) is mapped bijectively to the closed sector \( S_2 \setminus \mathbb{D}_{R_0} \) in the \( w \)-plane.

**Theorem 4.3.4.** (i) There is a unique formal power series solution

\[ \tilde{y}_0(x) = x^{1/3} \sum_{k=0}^{\infty} \frac{y_{0,k}}{x^{2k/3}} \]

(4.3.27)

to (4.1.4), where

\[ y_{0,0} = A, \quad y_{0,1} = -\frac{\beta}{3A} \]

(4.3.28)

(ii) For each \( j \in \{0, 1, 2, 3\} \), there is a one-parameter family \( \mathcal{F}_{A,j} \) of tronquée solutions of (4.1.4) in \( S_R^{(j)} \) where

\[ y(x) = l(x) + x^{1/3} h(w), \quad w = K x^{2/3}, \quad K = (27A/4)^{1/2} \]

(4.3.29)

If \( j \) is even, then \( h(w) \) has the representations

\[ h(w) = \begin{cases} 
  h^+(w) + \sum_{k=1}^{\infty} C^k_+ e^{-kw} \mathcal{L}_\phi H_k(w) & -\phi \in \left( 0, \frac{\pi}{2} \right] \\
  h^-(w) + \sum_{k=1}^{\infty} C^k_- e^{-kw} \mathcal{L}_\phi H_k(w) & -\phi \in \left[ -\frac{\pi}{2}, 0 \right) 
\end{cases} \]

(4.3.30)
If \( j \) is odd, then \( h(w) \) has the representations

\[
\begin{align*}
   h^-(w) &= \sum_{k=1}^{\infty} \hat{C}_k e^{kL} \hat{H}_k(-w) & -\phi &\in \left(0, \frac{\pi}{2}\right) \\
   h^+(w) &= \sum_{k=1}^{\infty} \hat{C}_k e^{kL} \hat{H}_k(-w) & -\phi &\in \left[-\frac{\pi}{2}, 0\right)
\end{align*}
\]

(4.3.31)

(iii) Let \( y(x) \) be a tronquée solution in \( F_{A,j} \). If \( j \) is even, the region of analyticity contains the corresponding branch of \( (K^{-1}S_{an}(h))^{2/3} \), which contains \( S_{R}^{(j)} \) for \( R \) large enough. If \( j \) is odd, the region of analyticity contains the corresponding branch of \( (K^{-1}S_{an}(h))^{2/3} \), which contains \( S_{R}^{(j)} \) for \( R \) large enough, and

\[
y(x) \sim \tilde{y}_0(x) \quad x \in d, \quad |x| \to \infty
\]

(4.3.32)

where \( d \) is a ray whose infinite part is contained in the interior of \( S_{R}^{(j)} \).

Assume that \( y(x) \) is a tronquée solution to (4.1.4) and \( h \) is defined by (4.3.24). Then \( h \) has the representation (4.3.30) or (4.3.31). From Theorem 4.2.4 we obtain information about singularities of \( h \). Let \( \xi_+ = C_+ e^{-w^{-1/2}} \), \( \xi_- = C_- e^{-w^{-1/2}} \), \( F_0 \) and \( \hat{F}_0 \) be as in §4.2. Then the equation satisfied by \( F_0 \) and \( \hat{F}_0 \) is the same:

\[
\begin{align*}
   \xi^2 \frac{d^2}{d\xi^2} F_0(\xi) + \left( \frac{d}{d\xi} F_0(\xi) \right) \xi - \frac{\left( \frac{d}{d\xi} F_0(\xi) \right)^2}{A + F_0(\xi)} &\xi^2 \\
   - \frac{(A + F_0(\xi))^2}{3A} + \frac{1}{3A (A + F_0(\xi))} &= 0
\end{align*}
\]

(4.3.33)

The solution satisfying (4.2.15) is

\[
F_0(\xi) = \frac{36A^2 \xi}{(6A - \xi)^2}
\]

(4.3.34)

**Theorem 4.3.5.** (i) If \( j \in \{0, 2\} \), then \( h \) has representation (4.3.30) for a unique pair of constants \((C_+, C_-)\). If \( C_+ \neq 0 \), then the singular points of \( h \), \( w_n^+ \), near the boundary \( \{w : \arg w = \pi/2\} \) of the sector of analyticity are given asymptotically by

\[
w_n^+ = 2n\pi i - \frac{\ln(2n\pi i)}{2} + \ln(C_+) - \ln(6A) + o(1) \quad (n \to \infty)
\]

(4.3.35)
If $C_- \neq 0$, then the singular points of $h$, $w_\mp_n$, near the boundary $\{w : \arg w = -\pi/2\}$ of the sector of analyticity are given asymptotically by

$$w_\mp_n = -2n\pi i - \frac{\ln(-2n\pi i)}{2} + \ln(C_-) - \ln(6A) + o(1) \quad (n \to \infty) \quad (4.3.36)$$

(ii) If $j \in \{1, 3\}$, then $h$ has representation (4.3.31) for a unique pair of constants $(\hat{C}_+, \hat{C}_-)$. If $\hat{C}_+ \neq 0$, then the singular points of $h$, $\tilde{w}_n^+$, near the boundary $\{\tilde{w} : \arg \tilde{w} = -\pi/2\}$ of the sector of analyticity are given asymptotically by

$$\tilde{w}_n^+ = -2n\pi i + \frac{\ln(2n\pi i)}{2} - \ln(\hat{C}_+) + \ln(6A) + o(1) \quad (n \to \infty) \quad (4.3.37)$$

If $\hat{C}_- \neq 0$, then the singular points of $h$, $\tilde{w}_n^-$, near the boundary $\{\tilde{w} : \arg \tilde{w} = \pi/2\}$ of the sector of analyticity are given asymptotically by

$$\tilde{w}_n^- = 2n\pi i + \frac{\ln(-2n\pi i)}{2} - \ln(\hat{C}_-) + \ln(6A) + o(1) \quad (n \to \infty) \quad (4.3.38)$$

**Theorem 4.3.6.** (i) For each $j \in \{0, 2\}$ we have a tritronquée solution $y_j^+$ analytic in $S_R^{(j)} \cup S_R^{(j+1)}$ for $R$ large enough, given by

$$y_j^+(x) = l(x) + h^+(w) \quad (4.3.39)$$

(ii) For each $j \in \{1, 3\}$ we have a tritronquée solution $y_j^-$ analytic in $S_R^{(j)} \cup S_R^{(j+1)}$, where $S_R^{(4)} = S_R^{(0)}$ and $R$ is large enough, given by

$$y_j^-(x) = l(x) + h^-(w) \quad (4.3.40)$$

Let $C_j^t$, $1 \leq j \leq 4$ be as in (4.2.33). Then:

(iii) The singular points of $h^+(w)$, $w_{1,n}^\pm$, near the boundary of the sector of analyticity are given asymptotically by

$$w_{1,n}^- = -2n\pi i - \frac{\ln(-2n\pi i)}{2} + \ln(C_1^t) - \ln(6A) + o(1) \quad (4.3.41)$$

$$w_{1,n}^+ = -2n\pi i + \frac{\ln(2n\pi i)}{2} - \ln(C_4^t) + \ln(6A) + o(1)$$
(iv) The singular points of \( h^{-}(w) \), \( w_{2,n}^{\pm} \), near the boundary of the sector of analyticity are given asymptotically by

\[
\begin{align*}
    w_{2,n}^{-} &= 2n\pi i + \frac{\ln(-2n\pi i)}{2} - \ln(C_3^t) + \ln(6A) + o(1) \\
    w_{2,n}^{+} &= 2n\pi i - \frac{\ln(2n\pi i)}{2} + \ln(C_2^t) - \ln(6A) + o(1)
\end{align*}
\] (4.3.42)

4.3.3 Tronquée solutions of \( P_{IV} \)

By dominant balance we have four possibilities for the leading behavior of \( P_{IV} \). We shall study them one by one.

\[ y(x) \sim l(x) \quad |x| \to \infty \quad x \in d \] (4.3.43)

Case 1

\[ l(x) = \frac{2x}{3} + \frac{\alpha}{x} \] (4.3.44)

make the change of variables

\[ x = \left( \sqrt{3} i w \right)^{1/2}, \quad y(x) = xh(w) + l(x) \] (4.3.45)

Then the equation (4.1.2) is transformed into an equation for \( h \) of the form (4.1.5) with

\[ \beta_1 = \beta_2 = \frac{1}{2} \] (4.3.46)

Let the notations be the same as in §4.2. In view of the transformation (4.3.45), we denote

\[ S_R^{(0)} := \left\{ x : |x| \geq R, \ arg(x) \in \left[ 0, \frac{\pi}{2} \right] \right\} \]
\[ S_R^{(1)} := \left\{ x : |x| \geq R, \ arg(x) \in \left[ \frac{\pi}{2}, \pi \right] \right\} \]
\[ S_R^{(2)} := \left\{ x : |x| \geq R, \ arg(x) \in \left[ \pi, \frac{3\pi}{2} \right] \right\} \]
\[ S_R^{(3)} := \left\{ x : |x| \geq R, \ arg(x) \in \left[ \frac{3\pi}{2}, 2\pi \right] \right\} \] (4.3.47)
We notice that for \( j \in \{0,2\} \), \( S_{R}^{(j)} \) is mapped under the transformation (4.3.24) bijectively to the closed sector \( S_{1} \setminus \mathbb{D}_{R^2} \) in the \( w \)-plane, (see also Note 4.2.3); for \( j \in \{1,3\} \), \( S_{R}^{(j)} \) is mapped bijectively to the closed sector \( S_{2} \setminus \mathbb{D}_{R^2} \) in the \( w \)-plane.

**Theorem 4.3.7.** (i) There is a formal power series solution of (4.1.2):

\[
\tilde{y}_0(x) = x \sum_{k=0}^{\infty} \frac{y_{0,k}}{x^2}
\]

where

\[
y_{0,0} = -\frac{2}{3}, \quad y_{0,1} = \alpha
\]

(ii) For each \( j \in \{0,1,2,3\} \), there is a one-parameter family \( \mathcal{F}_{A,j} \) of tronquée solutions of (4.1.2) in \( S_{R}^{(j)} \) where

\[
y(x) = l(x) + x h(w), \quad w = \frac{x^2}{\sqrt{3}i}
\]

If \( j \) is even, then \( h(w) \) has the representations

\[
h(w) = \begin{cases}
    h^+(w) + \sum_{k=1}^{\infty} C^k_+ e^{-kw} \mathcal{L}_\phi H_k(w) & -\phi \in \left(0, \frac{\pi}{2}\right) \\
    h^-(w) + \sum_{k=1}^{\infty} C^k_- e^{-kw} \mathcal{L}_\phi H_k(w) & -\phi \in \left[-\frac{\pi}{2}, 0\right)
\end{cases}
\]

If \( j \) is odd, then \( h(w) \) has the representations

\[
h(w) = \begin{cases}
    h^-(w) + \sum_{k=1}^{\infty} \hat{C}^k_+ e^{kw} \mathcal{L}_\phi \hat{H}_k(-w) & -\phi \in \left(0, \frac{\pi}{2}\right] \\
    h^+(w) + \sum_{k=1}^{\infty} \hat{C}^k_- e^{kw} \mathcal{L}_\phi \hat{H}_k(-w) & -\phi \in \left[-\frac{\pi}{2}, 0\right)
\end{cases}
\]

(iii) Let \( y(x) \) be a tronquée solution in \( \mathcal{F}_{A,j} \). If \( j \) is even, the region of analyticity contains the corresponding branch of \( \left(\sqrt{3}iS_{an}(h)\right)^{1/2} \), which contains \( S_{R}^{(j)} \) for \( R \) large enough. If \( j \) is odd, the region of analyticity contains the corresponding branch of \( \left(\sqrt{3}iS_{an}(h)\right)^{1/2} \), which contains \( S_{R}^{(j)} \) for \( R \) large enough, and

\[
y(x) \sim \tilde{y}_0(x) \quad x \in d, \quad |x| \to \infty
\]
where \( d \) is a ray whose infinite part is contained in the interior of \( S^{(i)}_R \).

Assume that \( y(x) \) is a tronquée solution to (4.1.2) satisfying \( y(x) \sim -\frac{2x}{3} \) and \( h \) is defined by (4.3.45). Then \( h \) has representation (4.3.51) or (4.3.52). From Theorem 4.2.4 we obtain information about singularities of \( h \). Let \( \xi_+ = C_+e^{-w}w^{-1/2} \), \( \xi_- = C_-e^{-w}w^{-1/2} \), \( F_m \) and \( \hat{F}_m \) be as in \( \S 4.2 \). Then the equation satisfied by \( F_0 \) and \( \hat{F}_0 \) is the same

\[
\xi^2 \frac{d^2}{d\xi^2} F_0(\xi) + \xi \frac{d}{d\xi} F_0(\xi) - \frac{3\xi^2}{2} \left( \frac{d}{d\xi} F_0(\xi) \right)^2 + \frac{3F_0(\xi) - 2}{24} + \frac{(3F_0(\xi) - 2)^3}{3} + \frac{3F_0(\xi) - 2}{2} = 0
\]

(4.3.54)

The solution satisfying (4.2.15) is

\[
F_0(\xi) = \frac{4\xi}{\xi^2 + 2\xi + 4}
\]

(4.3.55)

with simple poles at \( \xi_s^{(1)} = -1 - \sqrt{3}i \) and \( \xi_s^{(2)} = -1 + \sqrt{3}i \). Hence the statements in Theorem 4.3.5 and Theorem 4.3.6 hold true for function \( h \), with \( S^{(i)}_R \) as defined in (4.3.47) and 6A in the formula replaced by \( \xi_s^{(i)} \), \( i = 1 \) or \( i = 2 \).

Case 2

\[
l(x) = -2x - \frac{\alpha}{x}
\]

(4.3.56)

make the change of variables

\[
x = (w)^{1/2}, \quad y(x) = xh(w) + l(x)
\]

(4.3.57)

Then the equation (4.1.2) is transformed into an equation for \( h \) of the form (4.1.5) with

\[
\beta_1 = \alpha + \frac{1}{2}
\]

(4.3.58)

\[
\beta_2 = -\alpha + \frac{1}{2}
\]
Let the notations be the same as in §4.2. In view of the transformation (4.3.45), we denote

\[ S^{(0)} := \left\{ x : \arg(x) \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \right\}, \quad S^{(1)} := \left\{ x : \arg(x) \in \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \right\} \]

\[ S^{(2)} := \left\{ x : \arg(x) \in \left( \frac{3\pi}{4}, \frac{5\pi}{4} \right) \right\}, \quad S^{(3)} := \left\{ x : \arg(x) \in \left( \frac{5\pi}{4}, \frac{7\pi}{4} \right) \right\} \]

\[(4.3.59)\]

For \( j \in \{0, 2\} \), \( S^{(j)} \) is mapped under the transformation (4.3.24) bijectively to the right half \( w \)-plane \( S_1 \); for \( j \in \{1, 3\} \), \( S^{(j)} \) is mapped bijectively to the sector to the left half \( w \)-plane \( S_2 \).

**Theorem 4.3.8.** (i) There is a formal power series solution of (4.1.2):

\[ \tilde{y}_0(x) = x \sum_{k=0}^{\infty} \frac{y_{0,k}}{x^2} \]

(4.3.60)

where

\[ y_{0,0} = -2, \quad y_{0,1} = -\alpha \]

(4.3.61)

(ii) For each \( j \in \{0, 1, 2, 3\} \), there is a one-parameter family \( \mathcal{F}_{A,j} \) of tronquée solutions of (4.1.2) in \( S^{(j)} \) where

\[ y(x) = l(x) + xh(w), \quad w = x^2 \]

(4.3.62)

If \( j \) is even, then \( h(w) \) has the representations

\[ h(w) = \begin{cases} 
  h^+(w) + \sum_{k=1}^{\infty} \hat{C}_+ e^{-k \phi} w^{kM_1} \mathcal{L}_\phi H_k(w) & \phi \in \left( 0, \frac{\pi}{2} \right) \\
  h^-(w) + \sum_{k=1}^{\infty} \hat{C}_- e^{-k \phi} w^{kM_1} \mathcal{L}_\phi H_k(w) & \phi \in \left( -\frac{\pi}{2}, 0 \right) 
\end{cases} \]

(4.3.63)

If \( j \) is odd, then \( h(w) \) has the representations

\[ h(w) = \begin{cases} 
  h^-(w) + \sum_{k=1}^{\infty} \hat{C}_+ e^{k \phi} (-w)^{kM_2} \mathcal{L}_\phi \hat{H}_k(-w) & \phi \in \left( 0, \frac{\pi}{2} \right) \\
  h^+(w) + \sum_{k=1}^{\infty} \hat{C}_- e^{k \phi} (-w)^{kM_2} \mathcal{L}_\phi \hat{H}_k(-w) & \phi \in \left( -\frac{\pi}{2}, 0 \right) 
\end{cases} \]

(4.3.64)
(iii) Let \( y(x) \) be a tronquée solution in \( F_{A,j} \). If \( j \) is even, then the region of analyticity contains the corresponding branch of \((S_{an}(h))^{1/2}\). If \( j \) is odd, then the region of analyticity contains the corresponding branch of \( \left( \tilde{S}_{an}(h) \right)^{1/2} \), and

\[
y(x) \sim \tilde{y}_0(x) \quad x \in d \subset S^{(j)}, \ |x| \to \infty
\]  

(4.3.65)

Assume that \( y(x) \) is a tronquée solution to (4.1.2) satisfying \( y(x) \sim -2x \) and \( h \) is defined by (4.3.57). Then \( h \) has representation (4.3.63) or (4.3.64). From Theorem 4.2.4 we obtain information about singularities of \( h \). Let \( F_m \) and \( \hat{F}_m \) be as in §4.2.

Then the equation satisfied by \( F_0 \) and \( \hat{F}_0 \) is the same

\[
\xi^2 \frac{d^2}{d\xi^2} F_0(\xi) + \xi \frac{d}{d\xi} F_0(\xi) - \frac{\xi^2 \left( \frac{d}{d\xi} F_0(\xi) \right)^2}{2(F_0(\xi) - 2)}
\]

\[
- \frac{3}{8} \frac{(F_0(\xi) - 2)^3}{(F_0(\xi) - 2)^2} - \frac{F_0(\xi) - 2}{2} = 0
\]

(4.3.66)

The solution satisfying (4.2.15) is

\[
F_0(\xi) = \frac{2\xi}{\xi + 2}
\]

(4.3.67)

with a simple pole at \( \xi_s = -2 \).

**Theorem 4.3.9.** (i) If \( j \in \{0, 2\} \), then \( h \) has representation (4.3.63) for a unique pair of constants \((C_+, C_-)\). If \( C_+ \neq 0 \), then the singular points of \( h \), at \( w_n^+ \), near the boundary \( \{w : \arg w = \pi/2\} \) of the sector of analyticity are given asymptotically by

\[
w_n^+ = 2n\pi i - (\alpha + 1/2) \ln(2n\pi i) + \ln(C_+) - \ln(-2) + o(1) \quad (n \to \infty)
\]

(4.3.68)

If \( C_- \neq 0 \), then the singular points of \( h \), \( w_n^- \), near the boundary \( \{w : \arg w = -\pi/2\} \) of the sector of analyticity are given asymptotically by

\[
w_n^- = -2n\pi i - (\alpha + 1/2) \ln(-2n\pi i) + \ln(C_-) - \ln(-2) + o(1) \quad (n \to \infty)
\]

(4.3.69)
(ii) If \( j \in \{1, 3\} \), then \( h \) has representation (4.3.64) for a unique pair of constants \((\hat{C}_+, \hat{C}_-)\). If \( \hat{C}_+ \neq 0 \), then the singular points of \( h, \tilde{w}_n^+ \), near the boundary \( \{ \tilde{w} : \arg \tilde{w} = -\pi/2 \} \) of the sector of analyticity are given asymptotically by

\[
\tilde{w}_n^+ = -2n\pi i + (-\alpha + 1/2) \ln(2n\pi i) - \ln(\hat{C}_+) + \ln(-2) + o(1) \quad (n \to \infty)
\] (4.3.70)

If \( \hat{C}_- \neq 0 \), then the singular points of \( h, \tilde{w}_n^- \), near the boundary \( \{ \tilde{w} : \arg \tilde{w} = \pi/2 \} \) of the sector of analyticity are given asymptotically by

\[
\tilde{w}_n^- = 2n\pi i + (-\alpha + 1/2) \ln(-2n\pi i) - \ln(\hat{C}_-) + \ln(-2) + o(1) \quad (n \to \infty)
\] (4.3.71)

Denote

\[
T_{\delta,R}^{(0)} := \left\{ w : |w| > R, \arg(w) \in \left[ -\frac{\pi}{4} + \delta, \frac{3\pi}{4} - \delta \right] \right\}
\]
\[
T_{\delta,R}^{(1)} := \left\{ w : |w| > R, \arg(w) \in \left[ \frac{\pi}{4} + \delta, \frac{5\pi}{4} - \delta \right] \right\}
\]
\[
T_{\delta,R}^{(2)} := \left\{ w : |w| > R, \arg(w) \in \left[ \frac{3\pi}{4} + \delta, \frac{7\pi}{4} - \delta \right] \right\}
\]
\[
T_{\delta,R}^{(3)} := \left\{ w : |w| > R, \arg(w) \in \left[ -\frac{3\pi}{4} + \delta, \frac{\pi}{4} - \delta \right] \right\}.
\]

**Theorem 4.3.10.** (i) Let \( j \in \{0, 2\} \). For each \( \delta > 0 \) there exists \( R \) large enough such that we have a tritronquée solution \( y_j^+ \) analytic in \( T_{\delta,R}^{(j)} \) given by

\[
y_j^+(x) = -2x + \frac{\alpha}{x} + h^+(x^2)
\] (4.3.73)

(ii) Let \( j \in \{1, 3\} \). For each \( \delta > 0 \) there exists \( R \) large enough such that we have a tritronquée solution \( y_j^- \) analytic in \( T_{\delta,R}^{(j)} \) given by

\[
y_j^-(x) = -2x + \frac{\alpha}{x} + h^-(x^2)
\] (4.3.74)

Let \( C_j^t, 1 \leq j \leq 4 \) be as in (4.2.33). Then:
(iii) The singular points of $h^+(w)$, $w^\pm_{1,n}$, near the boundary of the sector of analyticity are given asymptotically by

$$w^-_{1,n} = -2n\pi i - (\alpha + 1/2)\ln(-2n\pi i) - \ln(-2) + o(1)$$

(4.3.75)

$$w^+_{1,n} = -2n\pi i + (-\alpha + 1/2)\ln(2n\pi i) + \ln(-2) + o(1)$$

(4.3.76)

(iv) The singular points of $h^-(w)$, $w^\pm_{2,n}$, near the boundary of the sector of analyticity are given asymptotically by

$$w^-_{2,n} = 2n\pi i + (-\alpha + 1/2)\ln(-2n\pi i) - \ln(-2) + o(1)$$

(4.3.77)

$$w^+_{2,n} = 2n\pi i - (\alpha + 1/2)\ln(2n\pi i) + \ln(-2) + o(1)$$

(4.3.78)

Case 3

$$l(x) = \frac{A}{x} + \frac{\alpha A + \beta}{2x^3}$$

(4.3.79)

Make the change of variables

$$x = (w)^{1/2}, \quad y(x) = x^{-1}h(w) + l(x)$$

(4.3.80)

Then the equation (4.1.2) is transformed into an equation for $h$ of the form (4.1.5) with

$$\beta_1 = -\frac{\alpha}{2} + \frac{3A}{2}$$

(4.3.81)

$$\beta_2 = \frac{\alpha}{2} - \frac{3A}{2}$$

Let the notation be as in §4.2, $S^{(i)}$ be as in (4.3.59) and $T^{(j)}_{k,R}$ be as in (4.3.72).

**Theorem 4.3.11.** (i) There is a formal power series solution of (4.1.2):

$$\tilde{y}_0(x) = \frac{1}{x} \sum_{k=0}^{\infty} \frac{y_{0,k}}{x^2}$$

(4.3.82)

where

$$y_{0,0} = A, \quad y_{0,1} = \frac{\alpha A + \beta}{2}$$

(4.3.83)
(ii) For each \( j \in \{0, 1, 2, 3\} \), there is a one-parameter family \( \mathcal{F}_{A,j} \) of tronquée solutions of (4.1.2) in \( S^{(j)} \) where

\[
y(x) = l(x) + x^{-1}h(w), \quad w = x^2
\]

If \( j \) is even, then \( h(w) \) has the representations

\[
h(w) = \begin{cases} 
h^+(w) + \sum_{k=1}^{\infty} C^+_ke^{-kM_1L_\phi H_k(w)} & -\phi \in \left(0, \frac{\pi}{2}\right) \\
h^-(w) + \sum_{k=1}^{\infty} C^-_ke^{-kM_1L_\phi H_k(w)} & -\phi \in \left(-\frac{\pi}{2}, 0\right) 
\end{cases}
\]

(4.3.85)

If \( j \) is odd, then \( h(w) \) has the representations

\[
h(w) = \begin{cases} 
h^-(w) + \sum_{k=1}^{\infty} \hat{C}^+_k e^{kM_2L_\phi \hat{H}_k(-w)} & -\phi \in \left(0, \frac{\pi}{2}\right) \\
h^+(w) + \sum_{k=1}^{\infty} \hat{C}^-_k e^{kM_2L_\phi \hat{H}_k(-w)} & -\phi \in \left(-\frac{\pi}{2}, 0\right) 
\end{cases}
\]

(4.3.86)

(iii) Let \( y(x) \) be a tronquée solution in \( \mathcal{F}_{A,j} \). If \( j \) is even, then the region of analyticity contains the corresponding branch of \( (S_{an}(h))^{1/2} \). If \( j \) is odd, then the region of analyticity contains the corresponding branch of \( \left(\hat{S}_{an}(h)\right)^{1/2} \), and

\[
y(x) \sim \tilde{y}_0(x) \quad x \in d \subset S^{(j)}, \quad |x| \to \infty
\]

(4.3.87)

Theorem 4.3.12. (i) Let \( j \in \{0, 2\} \). For each \( \delta > 0 \) there exists \( R \) large enough such that we have a tritronquée solution \( y_j^+ \) analytic in \( T_{\delta,R}^{(j)} \) given by

\[
y_j^+(x) = \frac{A}{x} + \frac{\alpha A + \beta}{2x^3} + h^+(x^2)
\]

(4.3.88)

(ii) Let \( j \in \{1, 3\} \). For each \( \delta > 0 \) there exists \( R \) large enough such that we have a tritronquée solution \( y_j^- \) analytic in \( T_{\delta,R}^{(j)} \) given by

\[
y_j^-(x) = \frac{A}{x} + \frac{\alpha A + \beta}{2x^3} + h^-(x^2)
\]

(4.3.89)

Note 4.3.13. In this case, the corresponding \( F_0 \) and \( \hat{F}_0 \) turn out to be \( \xi \), which yield not singularities for \( h \). However, it does not imply that the poles are nonexistent. More research need to be done for this case.
4.4 Proofs and Further results

4.4.1 Proof of Proposition 4.2.1

Let \( h \) and \( u \) be as defined in §4.1. We have a system of differential equation (4.1.7) for \( u \). It is known (see [2], [5] or [3]) that it admits transseries solutions (i.e. formal exponential power series solutions) of the form

\[
\tilde{u}(w) = \tilde{u}_0(w) + \sum_{k=1}^{\infty} C^k e^{-k w} w^{-\beta k} \tilde{u}_k(w) \tag{4.4.1}
\]

where \( \tilde{u}_0(w) \) and \( \tilde{u}_k(w) \) are formal power series in \( w^{-1} \), namely

\[
\tilde{u}_k(w) = \sum_{r=0}^{\infty} \frac{u_{k,r}}{w^r}, \quad (k \geq 1) \tag{4.4.2}
\]

\[
\tilde{u}_0(w) = \sum_{r=2}^{\infty} \frac{u_{0,r}}{w^r} \tag{4.4.3}
\]

Also, \( \tilde{u}_0(w) \) is the unique power series solution of (4.1.7). The coefficients in the series \( \tilde{u}_k \) can be determined by substitution of the formal exponential power series \( \tilde{u}(w) \) into (4.4.1) and identification of each coefficient of \( e^{-k w} \). Proposition 4.2.1 is then obtained through (4.1.6). Furthermore,

\[
\tilde{h}_0(w) = r_1 \cdot \tilde{u}_0(w) \\
\tilde{s}_k(w) = r_1 \cdot \tilde{u}_k(w) \tag{4.4.4}
\]

\[
r_1 = \begin{bmatrix} 1 - \frac{\beta_1}{2w} & 1 + \frac{\beta_2}{2w} \end{bmatrix}
\]

4.4.2 Proof of Theorem 4.2.2

Let \( d = e^{i\theta} \mathbb{R}^+ \) with \( \cos \theta > 0 \), and let \( u \) be a solution to (4.1.7) on \( d \) for \( w \) large enough, satisfying

\[
u(w) \to 0 \quad (w \in d, |w| \to \infty) \tag{4.4.5}
\]

Theorem 3 in [3], Theorem 16, Lemma 17 and Theorem 19 in [2] imply the following results:
Proposition 4.4.1. (i) For any $d' = e^{i\theta'} \mathbb{R}^+$ where $\cos \theta' > 0$, the solution $u(w)$ is analytic on $d'$ for $w$ large enough and $u \sim \tilde{u}_0(w)$ on $d'$. (ii) Given $\phi \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, there exists a unique constant $C(\phi)$ such that $u$ has the following representation:

$$u(w) = \mathcal{L}_\phi U_0(w) + \sum_{k=1}^{\infty} (C(\phi))^k e^{-kw} w^{kM_1} \mathcal{L}_\phi U_k(w)$$

(4.4.6)

where

$$U_0 = B \tilde{u}_0,$$

$$U_k = B \left(w^{-k\beta_1-kM_1} \tilde{u}_k\right) \quad (k = 1, 2, \ldots)$$

(4.4.7)

where each $U_k$ is analytic in the Riemann surface of $C \setminus (Z^+ \cup Z^-)$, and the branch cut for each $U_k$, $(k \geq 1)$ is chosen to be $(-\infty, 0]$. The function $C(\phi)$ is constant on $\left(-\frac{\pi}{2}, 0\right)$ and also constant on $\left(0, \frac{\pi}{2}\right)$. (iii) Let $\epsilon$ be small. There exist $\delta, R > 0$ such that $u(w)$ is analytic on

$$S_{\alpha,\epsilon}(u(w)) = S^+_\epsilon \cup S^-_\epsilon$$

(4.4.8)

where $S^\pm$ is as defined in (4.2.10).

We now return to the proof of Theorem 4.2.2. Assume that $h(w)$ is a solution of (4.1.5) on $d = e^{i\phi} \mathbb{R}^+$ with $\cos \phi > 0$ for $|w| > w_0$, where $w_0 > 0$ is large enough. Without loss of generality we may assume that $w_0 > \frac{\sqrt{\beta_1 \beta_2}}{2}$. Thus the vector function $u(w)$ defined by

$$u(w) = \begin{bmatrix} 1 - \frac{\beta_1}{2w} & 1 + \frac{\beta_2}{2w} \\ -1 - \frac{\beta_1}{2w} & 1 - \frac{\beta_2}{2w} \end{bmatrix}^{-1} \begin{bmatrix} h(w) \\ h'(w) \end{bmatrix}$$

(4.4.9)

is a solution of the differential system (4.1.7), and $h(w) = r_1 \cdot u(w)$.

Next we use the basic properties (see Lemma 4.5.1 and 4.5.2) of the operators $B$ and $\mathcal{L}_\phi$ and obtain the following:

$$\mathcal{L}_\phi B (r_1 \cdot \tilde{u}_0) = r_1 \cdot \mathcal{L}_\phi B (\tilde{u}_0) = r_1 \cdot \mathcal{L}_\phi U_0$$

$$\mathcal{L}_\phi B (w^{-k\beta_1-kM_1} r_1 \cdot \tilde{u}_k) = r_1 \cdot \mathcal{L}_\phi B (w^{-k\beta_1-kM_1} \tilde{u}_k) = r_1 \cdot \mathcal{L}_\phi U_k$$

(4.4.10)

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By Proposition 4.4.1 (i), given a ray $d'$ in the right half $w$-plane, $h(w) = r_1 \cdot u(w)$ is analytic on $d'$ for $|w|$ large enough and is asymptotic to $\tilde{h}_0(w) = r_1 \cdot \tilde{u}_0(w)$ on $d'$. From the representations (4.4.6), (4.4.7) in Proposition 4.4.1 (ii) of $u(w)$ and (4.4.10) we obtain the representations for $h(w) = r_1 \cdot u(w)$ as in (4.2.6) and (4.2.7). For $|w|$ large enough, $h(w)$ is analytic where $u(w)$ is analytic, hence Proposition 4.4.1 (iii) implies Theorem 4.2.2 (ii). Thus Theorem 4.2.2 is proved.

4.4.3 Proof of Theorem 4.2.4

Let $h(w)$ and $u(w)$ be as in the proof of Theorem 4.2.2. $h(w)$ has representations (4.2.6) and (4.2.7). We will consider the case $C_+ \neq 0$ and prove (i) and (ii). The statements (iii) and (iv) about the case $C_- \neq 0$ follow by symmetry.

By Theorem 1 in [2] there exists $\delta_1 > 0$ such that for $|\xi| < \delta_1$ the power series

$$G_m(\xi) = \sum_{k=0}^{\infty} \xi^k u_{k,m}, \quad m = 0, 1, 2... \tag{4.4.11}$$

converge, where $G_0$ satisfies

$$G_0 = 0, \quad G'_0 = e_1 \tag{4.4.12}$$

Furthermore,

$$u(w) \sim \sum_{m=0}^{\infty} w^{-m} G_m(\xi_+(w)) \quad (|w| \to \infty) \tag{4.4.13}$$

holds uniformly in

$$S_{\delta_1} = \{ w : \arg(w) \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} + \delta\right), |\xi_+(w)| < \delta_1 \}$$

By Theorem 2 in [2], for $R$ large enough and $\delta, \epsilon$ small enough, $u(w)$ is analytic in $D^+_w$ (see (4.2.14)). Also, the asymptotic representation (4.4.13) holds in $D^+_w$. Moreover, if $G_0$ has an isolated singularity at $\xi_s$, then $u(w)$ is singular at a distance at most $o(1)$ of $w^+_n$ given in (4.2.16), as $w^+_n \to \infty$. Since $h(w) = r_1 \cdot u(w)$, Theorem 4.2.4 (i) follows from the results cited.
Assume $|w| > \sqrt{\beta_1\beta_2}/2$. Both (4.1.6) and (4.4.9) hold. While (4.1.6) implies that $h$ is analytic at least where $u$ is analytic, (4.4.9) implies that $h$ is singular where $u$ is singular. Thus the asymptotic position of singularities, i.e. poles of $h(w)$ is the same as that of $u(w)$, which is presented in equation (4.2.16). Thus Theorem 4.2.4 (ii) is proved.

4.4.4 Proof of Corollary 4.2.5

Let the notations be the same as in §4.2.2. First we point out some properties of $U_0(p)$. See also [5].

We apply formal inverse Laplace transform to the system (4.1.7). To be precise, assume the analytic function $g(w, u)$ has the Taylor expansion at $(\infty, 0)$ as follows

$$g(w, u) = \sum_{m \geq 0; |l| \geq 0} g_{m,l} w^{-m} u^l \quad (|w^{-1}| < \xi_0, |u| < \xi_0)$$

(4.4.14)

Note that by assumption $g_{m,l} = 0$ if $|l| \leq 1$ and $m \leq 1$. Denote $U = \mathcal{L}^{-1}u$. Then the formal inverse Laplace transform of the differential system (4.1.7) is the system of convolution equations

$$-p U(p) = -\left[\hat{\Lambda} U(p) + \hat{B} \int_0^p U(s)ds\right] + \mathcal{N}(U)(p)$$

(4.4.15)

where

$$\mathcal{N}(U)(p) = \sum_{m=2}^{\infty} \frac{g_{m,0}}{(m-1)!} p^{m-1} + \sum_{|l| \geq 2} g_{0,l} U^*l + \sum_{|l| \geq 1} \left( \sum_{m=1}^{\infty} \frac{g_{m,1}}{(m-1)!} p^{m-1} \right) * U^*l$$

(4.4.16)

Let $v(p) = (v_1(p), \ldots, v_n(p))$ be an $n$-dimensional complex vector function, $f(p)$ be a locally integrable complex function and $l = (l_1, \ldots, l_n)$ be an $n$-dimensional multi-index. Then

$$v^*l := v_1^{*l_1} * v_2^{*l_2} * \ldots * v_n^{*l_n}$$

(4.4.17)

$$(v * f)(p) \in \mathbb{C}^n, \quad (v * f)_j = v_j * f \quad (j = 1, \ldots, n)$$
We gather the following facts about $U_0$.

**Proposition 4.4.2.** (i) Let $K \in \mathcal{O}$ be a closed set such that for every point $p \in K$, the line segment connecting the origin and $p$ is contained in $K$. Then $U_0$ is the unique solution to (4.4.15) in $K$. (ii) $U_0 = \mathcal{B} \hat{u}_0$. $U_0$ is analytic in the domain $\mathcal{O} = \mathbb{C} \setminus \left( (\infty, -1] \cup [1, \infty) \right)$, and is Laplace transformable along any ray $e^{i\phi} \mathbb{R}^+$ contained in $\mathcal{O}$. $L_\phi U_0$ is a solution of (4.4.15) for each $\phi$ such that $|\cos(\phi)| < 1$. (iii) Let $K$ be as in (i). There exists $b_K > 0$ large enough such that

$$\sup_{p \in K} \int_{[0, p]} |U_0(s)| e^{-b_K |s|} |ds| < \infty$$

(4.4.18)

Proposition 4.4.2 (i) and (ii) come from Proposition 6 in [5]. Although (iii) is not stated explicitly in [5], it can be easily obtained by the same approach used to prove Proposition 6. Let $K$ be as in (i). Consider the Banach space

$$L_{ray}(K) := \{ f : f \text{ is locally integrable on } [0, p] \text{ for each } p \in K \}$$

(4.4.19)

equipped with the norm $\| \cdot \|_{b,K}$ defined by

$$\|f\|_{b,K} := \sup_{p \in K} \int_{[0, p]} \|f(s)\| e^{-b|s|} |ds|$$

(4.4.20)

where $\|f(s)\| = \max\{|f_1(s)|, |f_2(s)|\}$. We can show that for $b$ large enough, the operator

$$N_1 := U(p) \mapsto \left( \hat{\Lambda} - pI \right)^{-1} \left( -\hat{B} \int_0^p U(s)ds + N(U)(p) \right)$$

(4.4.21)

is contractive in the closed ball $S := \{ f \in L_{ray}(K) : \|f\|_{b,K} \leq \delta \}$ of $L_{ray}(K)$ if $\delta$ is small enough. By contractive mapping theorem there is a unique solution of $N_1 U = U$ in $S$, namely $U_0$ by uniqueness of the solution. Using integration by parts and (iii) we have the following:
Corollary 4.4.3. (i) If \( \phi \in (0, \pi) \) or \( \phi \in (-\pi, 0) \), \( \mathcal{L}_\phi U_0(w) \) is analytic (at least) in the region

\[
\mathcal{A}_\phi := \{ w : |w| \cos (\phi + \arg(w)) > b \}
\]

(4.4.22)

where \( b = b_K \) is as in Proposition 4.4.2 (iii) with \( K = e^{i\phi} \mathbb{R}^+ \). (ii) If \( 0 < \phi_1 < \phi_2 < \pi \) or \( 0 < -\phi_1 < -\phi_2 < \pi \), then \( \mathcal{L}_{\phi_1} U_0 \) and \( \mathcal{L}_{\phi_2} U_0 \) are analytic continuations of each other.

Since \( H_0 = \mathcal{B} \tilde{h}_0 \), by (4.4.4) and Lemma 4.5.1 we have

\[
H_0(p) = (\mathcal{B} u_{0,1})(p) + (\mathcal{B} u_{0,2})(p) - \frac{\beta_1}{2} [1 * (\mathcal{B} u_{0,1})](p) + \frac{\beta_2}{2} [1 * (\mathcal{B} u_{0,2})](p)
\]

\[
= U_{0,1}(p) + U_{0,2}(p) - \frac{\beta_1}{2} (1 * U_{0,1})(p) + \frac{\beta_2}{2} (1 * U_{0,2})(p)
\]

(4.4.23)

where \( u_{0,i} \) \((i = 1, 2)\) is the \( i \)-th component of the vector function \( u_0 \) and \( U_{0,i} \) \((i = 1, 2)\) is the \( i \)-th component of \( U_0 \). It is clear from (4.4.23) that Proposition 4.4.2 (iii) and Corollary 4.4.3 hold with \( U_0 \) replaced by \( H_0 \). Merely by \( \tilde{H}_0(p) = -H_0(-p) \) and Corollary 4.4.3 (ii) with \( U_0 \) replaced by \( H_0 \) we obtain Corollary 4.2.5 (i). Moreover, both \( h^+ \) and \( h^- \) are special cases of tronquée solutions, thus Theorem 4.2.2 and Theorem 4.2.4 apply. \( h^+ \) is analytic at least on \( S_{an}(h^+) \cup (-S_{an}(\tilde{h}^+)) \) and \( h^- \) is analytic at least on \( S_{an}(h^-) \cup (-S_{an}(\tilde{h}^-)) \). We also obtain the asymptotic position of singularities of the tritronquée solutions as in Corollary 4.2.5(ii).

4.4.5 Proof of the results in §4.3

Once we have the normalizations in the form of (4.1.5) of the equations (4.1.3), (4.1.4) and (4.1.2), the results in §4.3 follow from the results in §4.2. Here we present the details of finding solutions to (4.3.10), (4.3.33), (4.3.54) and (4.3.66) satisfying (4.2.15).
Solving (4.3.10)

Make the substitution $Q(s) = A + F_0(e^s)$ then (4.3.10) transforms into

$$
\frac{d^2}{ds^2} Q(s) - \left( \frac{d}{ds} Q(s) \right)^2 - \left( \frac{1}{Q(s)} \right)^3 + \frac{1}{4A^2} \frac{1}{Q(s)} = 0 \quad (4.4.24)
$$

Multiplying both sides by $1/Q(s)$ we obtain

$$
\frac{d}{ds} \left( \frac{Q'(s)}{Q(s)} \right) = \frac{1}{4A^2} \left( Q^2(s) - \frac{1}{Q^2(s)} + C_1 \right) \quad (4.4.25)
$$

Multiplying both sides by $2Q'(s)/Q(s)$ and integrating with respect to $s$ we have

$$
\left( \frac{Q'(s)}{Q(s)} \right)^2 = \frac{1}{4A^2} \left( Q^4(s) + C_1 Q^2(s) + 1 \right) \quad (4.4.26)
$$

i.e.

$$
(Q'(s))^2 = \frac{1}{4A^2} \left( Q^4(s) + C_1 Q^2(s) + 1 \right) \quad (4.4.26)
$$

By a linear transformation $Q(s) = \tilde{Q}(s)/(2A)$ (4.4.26) is reduced to the Jacobi normal form which is solved by Jacobi elliptic functions unless $C_1 \in \{-2, 2\}$. Since $Q(s) = A + F_0(e^s)$ and $F_0(0) = 0$ (see (4.2.15)), the solution we look for cannot be an elliptic function. Moreover, as $\text{Re}(s) \to -\infty$, $Q(s) \to A$ implies $Q'(s) \to 0$, so (4.4.26) needs to be of the form

$$
(Q'(s))^2 = \frac{1}{4A^2} \left( Q^2(s) - A^2 \right)^2 \quad (4.4.27)
$$

the solution satisfying $Q(s) \to A$ as $\text{Re}(s) \to -\infty$ is

$$
Q(s) = A \cdot \frac{C_2 - e^s}{C_2 + e^s} \quad (C_2 \neq 0) \quad (4.4.28)
$$

Thus the solution to (4.3.10) is

$$
F_0(\xi) = -\frac{2A\xi}{C_2 + \xi} \quad (4.4.29)
$$

Hence the solution to (4.3.10) satisfying (4.2.15) is

$$
F_0(\xi) = \frac{2A\xi}{2A - \xi} \quad (4.4.30)
$$

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Solving (4.3.33)

Make the substitution \( Q(s) = A + F_0(e^s) \) then (4.3.33) transforms into

\[
\frac{d^2}{ds^2}Q(s) - \left( \frac{d}{ds}Q(s) \right)^2 - \frac{(Q(s))^2}{3A} + \frac{1}{3AQ(s)} = 0 \quad (4.4.31)
\]

Multiplying both sides by \( 1/Q(s) \) we obtain

\[
\frac{d}{ds} \left( \frac{Q'(s)}{Q(s)} \right) = \frac{1}{3A} \left( Q(s) - \frac{1}{Q^2(s)} \right) \quad (4.4.32)
\]

Multiplying both sides by \( 2Q'(s)/Q(s) \) and integrating with respect to \( s \) we have

\[
\left( \frac{Q'(s)}{Q(s)} \right)^2 = \frac{1}{3A} \left( 2Q^3(s) + C_1Q^2(s) + 1 \right) \quad (4.4.33)
\]

\[
i.e. \quad (Q'(s))^2 = \frac{1}{3A} \left( 2Q^3(s) + C_1Q^2(s) + 1 \right)
\]

Notice that if the equation \( 2x^3 + C_1x^2 + 1 = 0 \) has three distinct roots then (4.4.33) is known to have Weierstrass \( \wp \)-functions as general solutions, in which case the corresponding \( F_0(\xi) = Q(\ln \xi) - A \) fails to satisfy the condition (4.2.15). Hence \( C_1 \) must be such that the equation \( 2x^3 + C_1x^2 + 1 = 0 \) has a multiple root. Denote the multiple root by \( r_1 \). Then

\[
2x^3 + C_1x^2 + 1 = 2(x - r_1)^2(x - r_2) \quad (4.4.34)
\]

Then we obtain

\[
r_1 = -2r_2, \quad r_1^3 = 1, \quad C_1 = -3r_1 \quad (4.4.35)
\]

Since \( Q(s) \to A \) as \( \text{Re}(s) \to -\infty, \) \( Q'(s) \to 0 \). Hence \( r_1 = A \) or \( r_2 = A \). We knew from the normalization (see (4.3.23)) that \( A^3 = 1 \). Thus \( r_1 = A, \) \( r_2 = -A/2, \) and \( C_1 = -3A \). Hence (4.4.33) is of the form

\[
(Q'(s))^2 = \frac{2}{3A} (Q(s) - A)^2 \left( Q(s) + \frac{A}{2} \right) \quad (4.4.36)
\]

The solution to (4.4.36) is

\[
Q(s) = -\frac{A}{2} + \frac{3A}{2} \left( \frac{C_2 - e^s}{C_2 + e^s} \right)^2 \quad (4.4.37)
\]
Hence the solution of (4.3.33) satisfying (4.2.15) is

$$F_0(\xi) = \frac{36A^2\xi}{(6A - \xi)^2}$$

Solving (4.3.54)

Make the substitution $Q(s) = F_0(e^s) - 2/3$ then (4.3.54) transforms into

$$\frac{d^2}{ds^2}Q(s) - \left(\frac{\frac{d}{ds}Q(s)}{2Q(s)}\right)^2 + \frac{9Q^3(s)}{8} + 3Q^2(s) + \frac{3Q(s)}{2} = 0$$

(4.4.39)

Multiplying both sides by $2Q'(s)/Q(s)$ and we have

$$\frac{d}{ds} \left[ \frac{(Q'(s))^2}{Q(s)} \right] = \frac{d}{ds} \left( -\frac{3}{4}Q^3(s) - 3Q^2(s) - 3Q(s) \right)$$

(4.4.40)

Integrating with respect to $s$ we have

$$(Q'(s))^2 = -\frac{3}{4}Q^4(s) - 3Q^3(s) - 3Q^2(s) + C_1Q(s)$$

(4.4.41)

Letting $\text{Re}(s) \to -\infty$ we have $Q(s) \to -2/3$ and $Q'(s) \to 0$. Thus $C_1 = -8/9$ and the equation (4.4.47) is of the form

$$(Q'(s))^2 = -\frac{3}{4}Q(s) \left( Q(s) + \frac{8}{3} \right) \left( Q(s) + \frac{2}{3} \right)^2$$

(4.4.42)

This is a separable differential equation with general solutions

$$Q(s) = -\frac{2}{3} e^{2s} - C_2 e^{s}$$

(4.4.43)

Hence the solution of (4.3.54) satisfying (4.2.15) is

$$F_0(\xi) = \frac{4\xi}{\xi^2 + 2\xi + 4}$$

(4.4.44)

Solving (4.3.66)

Make the substitution $Q(s) = F_0(e^s) - 2$ then (4.3.54) transforms into

$$\frac{d^2}{ds^2}Q(s) - \left(\frac{\frac{d}{ds}Q(s)}{2Q(s)}\right)^2 - \frac{3Q^3(s)}{8} - Q^2(s) - \frac{Q(s)}{2} = 0$$

(4.4.45)
Multiplying both sides by \(2Q'(s)/Q(s)\) and we have

\[
\frac{d}{ds} \left[ \frac{(Q'(s))^2}{Q(s)} \right] = \frac{d}{ds} \left( \frac{1}{4} Q^3(s) + Q^2(s) + Q(s) \right)
\]

(4.4.46)

Integrating with respect to \(s\) we have

\[(Q'(s))^2 = \frac{1}{4} Q^4(s) + Q^3(s) + Q^2(s) + C_1 Q(s)\]

(4.4.47)

Letting \(\text{Re}(s) \to -\infty\) we have \(Q(s) \to -2\) and \(Q'(s) \to 0\). Thus \(C_1 = 0\) and the equation (4.4.47) is of the form

\[(Q'(s))^2 = \frac{1}{4} Q^2(s) (Q(s) + 2)^2\]

(4.4.48)

This differential equation has general solutions

\[Q(s) = -\frac{2 C_2}{C_2 + e^s}\]

(4.4.49)

Hence the solution of (4.3.66) satisfying (4.2.15) is

\[F_0(\xi) = \frac{2 \xi}{\xi + 2}\]

(4.4.50)

4.5 Appendix

4.5.1 Borel transform

Assume that we have a formal series

\[\tilde{f}(w) = \sum_{n=0}^{\infty} a_n w^{-r-n} \quad (\text{Re}(r) > 0)\]

(4.5.1)

where the series \(\sum_{n=0}^{\infty} a_n x^n\) has a positive radius of convergence. Then the Borel transform of \(\tilde{f}\) is defined to be the formal power series

\[(B \tilde{f})(p) := \sum_{n=0}^{\infty} \frac{a_n p^{n+r-1}}{\Gamma(n+r)}\]

(4.5.2)
Lemma 4.5.1. Assume that we have two formal series \( \tilde{f} \) and \( \tilde{g} \),

\[
\tilde{f}(w) = \sum_{n=0}^{\infty} a_n w^{-r-n} \quad (\text{Re}(r) > 0) \\
\tilde{g}(w) = \sum_{n=0}^{\infty} b_n w^{-r-s} \quad (\text{Re}(s) > 0)
\]

(4.5.3)

where both series \( \sum_{n=0}^{\infty} a_n x^n \) and \( \sum_{n=0}^{\infty} b_n x^n \) have positive radii of convergence. Then

\[
\mathcal{B}(f \cdot g)(p) = (\mathcal{B}f \ast \mathcal{B}g)(p) = p^{r+s-1} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) \frac{p^n}{\Gamma(n+r+s)}
\]

(4.5.4)

where

\[
(\mathcal{B}f) \ast (\mathcal{B}g)(p) := \int_{0}^{p} (\mathcal{B}f)(t) (\mathcal{B}g)(p-t) dt
\]

(4.5.5)

Lemma 4.5.2. Assume that the function \( f \) is integrable over the ray \( e^{i\phi} \mathbb{R}^+ \), namely

\[
\int_{0}^{\infty} |f(p)| |dp| < \infty
\]

(4.5.6)

Then for \( \text{Re}(we^{i\phi}) > 0 \)

\[
\mathcal{L}_\phi (1 \ast f)(w) = \frac{1}{w} \mathcal{L}_\phi (f)(w)
\]

(4.5.7)
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