The Problem of Sonic Shock Formation

Dissertation

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By

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Abstract

The equations of compressible ideal gas flow in Eulerian coordinates and simplified models of this flow are studied by many mathematicians, physicists and engineers, because of their rich mathematical structure and their close relation to many physics and engineering problems. The Eulerian system of quasilinear PDEs has the feature that smooth solutions do not exist to most problems; one must study weak solutions, which typically contain shocks. While shocks are quite well understood in one space dimension, little is known about the behavior of shocks in multi-dimensional space.

Since Ćanić, Keyfitz and Lieberman’s seminal paper [10], there has been much research on multi-dimensional shocks using self-similar reduction, the free boundary approach and the procedure introduced in [10]. A feature of these reduced systems is that the systems change type from hyperbolic to elliptic in some regions of space. Since quasilinear elliptic equations have smooth solutions, the shock cannot appear inside of the elliptic region. A natural question to ask is whether the shock can form at the sonic line (the curve separating regions in which the system is of different types).

In the first part of the dissertation, we try to give a positive answer to this question. We use the unsteady transonic small disturbance equation (UTSDE) as a model system and choose a configuration where it appears that the shock forms at the sonic line. We follow the procedure of [10] to prove the existence of a solution to this
problem. To implement this procedure, we face some new technical challenges. First, the second order elliptic PDE developed from the UTSDE is degenerate on the sonic line. Second, the oblique derivative condition posed on the shock becomes tangential at the shock formation point. Finally, the first two difficulties happen at the same time at the shock formation point. To overcome those difficulties, we first introduce several cut-off functions and small parameters to regularize the problem. Then we construct several barrier functions to remove the cut-off functions after solving the regularized problem. Finally, we use the Arzela-Ascoli Theorem and a diagonalization argument to obtain the solution to the original problem by sending the small parameters to zero.

In the second part of the dissertation, we study a situation that arises in shock diffraction by a 90 degree corner, using the isentropic Euler equations. In preparing to extend the method in the first part to this problem, we need to resolve a new difficulty that the PDEs in this system become degenerate at “stagnation points”. To gain some understanding of the behavior of the solutions near stagnation points we study the behavior of the solutions to the isentropic Euler equations near stagnation points under the self-similar and axisymmetric assumptions. We find a solution to the self-similar and axisymmetric Euler equations with a stagnation point at the origin with certain asymptotic behavior which gives some insights of how solutions behave near stagnation points.
To my wife Feiran Lei and my parents
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Chapter 1: Introduction

Systems of conservation laws arise naturally from problems related to physics and engineering. Although the existence theory of one-dimensional systems of hyperbolic conservation laws was established by Glimm in 1965 [20], there is no general existence theory for systems of conservation laws in more than one space dimension. In the two-dimensional case, one way to tackle the existence problem is through self-similar reduction. A general framework using this approach was suggested by Čanić and Keyfitz in [5] and [6]. However, unlike one-dimensional systems of conservation laws, the systems after self-similar reduction are usually systems that change type.

In this dissertation, we use self-similar reduction to study two important systems in 2D: the unsteady transonic small disturbance equation (UTSDE)

\[ \begin{cases} 
  u_t + \left( \frac{1}{2} u^2 \right)_x + v_y = 0 \\
  v_x - u_y = 0.
\end{cases} \] (1.1)

and the isentropic Euler system

\[ \begin{cases} 
  \rho_t + (\rho u)_x + (\rho v)_y = 0 \\
  (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0 \\
  (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0.
\end{cases} \] (1.2)

The dissertation is organized in the following way:

In Chapter 1, we first introduce some notation and useful results about systems of conservation laws. Then we provide background and motivation for why we want to study the problem of shock formation at the sonic line.
In Chapter 2, we formulate a problem for the UTSDE in which a shock forms at
the sonic line and we prove the existence of a solution to this problem.

In Chapter 3, first, we formulate the problem of diffraction of a shock wave at a 90
degree corner for the isentropic Euler system. Then we set up the problem in a way
so that the method in Chapter 2 can be applied. Finally, we discuss some difficulties
related to this problem and we show some preliminary results.

1.1 Shock Formation in Systems of Hyperbolic Conservation
Laws

Consider a general system of hyperbolic conservation laws

$$\frac{\partial}{\partial t} u(t, x) + \frac{\partial}{\partial x} f(u(t, x)) = 0, \quad u \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

(1.3)

with Cauchy data

$$u(0, x) = u_0(x)$$

(1.4)

**Definition 1.1.1.** (Strict hyperbolicity) We say (1.3) is **strictly hyperbolic** if for
all values of $u$, $A(u) = Df(u)$ has $n$ real and distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$$

Suppose (1.3) is strictly hyperbolic. If the initial data $u_0(x)$ is continuously differentiable, it is well-known that under some assumptions on the structure of the system, the Cauchy problem has a unique $C^1$ solution on $[0, T) \times \mathbb{R}$ for some $T \leq \infty$. If a singularity develops at $t = T$ then $\limsup_{t \to T} \|\partial_x u(t, \cdot)\|_{L^\infty} = \infty$, i.e. the derivative with
respect to $x$ blows up at the time when a shock forms. We refer to [17] for those classical results. Furthermore, [39] and [29] show that when the smooth solution breaks down, the blow-up is of cusp type (according to the terminology of [1]).

When the classical solution ceases to exist we need to consider weak solutions.

**Definition 1.1.2.** (Weak solutions) A function $u(t,x) \in L^\infty((0,\infty) \times \mathbb{R})$ is a **weak solution** of (1.3) if it satisfies

$$
\int_0^\infty \int_{\mathbb{R}} (u \phi_t + f(u) \phi_x) \, dx \, dt = 0.
$$

for all $\phi \in C^\infty_c((0,\infty) \times \mathbb{R})$.

One important kind of weak solution we will consider is the weak solution with a shock. Suppose $u$ is a weak solution of (1.3), which is $C^1$ in both $\Omega^-$ and $\Omega^+$, where $u$ satisfies (1.3) in the classical sense, and $u$ is discontinuous on $\Sigma = \{(t,x)|x = x(t)\}$. Then $\Sigma$ is called a shock.

![Figure 1.1: Derivation of the Rankine-Hugoniot relation.](image)

Suppose $u$ has one-sided limits $u^-$ and $u^+$ on both sides of $\Sigma$, then we have the following Rankine-Hugoniot condition for $u^-$ and $u^+$. 

3
Theorem 1.1.3. (Rankine-Hugoniot condition) A weak solution \( u \) that is piecewise \( C^1 \) in the neighborhood of the shock \( \Sigma \) must satisfy the Rankine-Hugoniot condition on the shock \( \Sigma = \{(t, x)|x = x(t)\} \):

\[
x'(t)[u] = [f(u)],
\]

where \([u] = u^+ - u^-\) and \([f(u)] = f(u^+) - f(u^-)\).

Proof. We first split the left side of (1.5)

\[
\int \int_{\Omega^-} (u\phi_t + f(u)\phi_x) \, dx \, dt + \int \int_{\Omega^+} (u\phi_t + f(u)\phi_x) \, dx \, dt = 0.
\]  

(1.7)

Apply the divergence theorem to the first term on the left hand side of (1.7) and notice that the test function \( \phi = 0 \) on \( \partial\Omega \), we obtain

\[
\int \int_{\Omega^-} (u\phi_t + f(u)\phi_x) \, dx \, dt = -\int \int_{\Omega^-} (u_t + f(u)_x)\phi \, dx \, dt + \int_{\Sigma} (-x'(t)u^- + f(u^-))\phi \, dS.
\]  

(1.8)

Since \( u \) satisfies (1.3) in \( \Omega^- \) in the classical sense, the first term on the right hand side of (1.8) vanishes. Therefore,

\[
\int \int_{\Omega^-} (u\phi_t + f(u)\phi_x) \, dx \, dt = \int_{\Sigma} (-x'(t)u^- + f(u^-))\phi \, dS.
\]  

(1.9)

Similarly,

\[
\int \int_{\Omega^+} (u\phi_t + f(u)\phi_x) \, dx \, dt = \int_{\Sigma} (x'(t)u^+ - f(u^+))\phi \, dS.
\]  

(1.10)

Apply (1.9) and (1.10) to (1.7), we get

\[
\int_{\Sigma} (x'(t)(u^+ - u^-) - (f(u^+) - f(u^-))\phi \, dS = 0.
\]  

(1.11)

Since \( \phi \) is an arbitrary test function, we then have

\[
x'(t)(u^+ - u^-) - (f(u^+) - f(u^-)) = 0,
\]  

(1.12)

and this finishes the proof. \( \square \)
Given an initial value (1.4), the weak solution of (1.3) defined in definition 1.1.2 is not necessarily unique. In order to achieve uniqueness, we need some admissibility conditions to single out the “physical” solution. There are several different ways to do this. Here, we will present the entropy condition to obtain the unique solution. For other admissibility conditions, please refer to [3].

**Definition 1.1.4.** *(Entropy pair)* A convex function \( \eta(u) \in C^1(\mathbb{R}) \) is called an **entropy** for the system (1.3), with **entropy flux** \( q(u) \), if

\[
D\eta(u) \cdot Df(u) = Dq(u). 
\]

(1.13)

**Remark 1.1.5.** If \( u(t,x) \) a \( C^1 \) solution of (1.3), then

\[
\eta(u)_t + q(u)_x = 0
\]

(1.14)

**Definition 1.1.6.** *(Entropy condition)* A weak solution \( u \) of (1.3) is **entropy admissible** if

\[
\eta(u)_t + q(u)_x \leq 0 
\]

(1.15)

in the distributional sense, for every entropy pair \((\eta, q)\).

### 1.2 Shock Formation in Systems which Change Type

While for systems of strictly hyperbolic PDEs, the formation of a shock can be modelled by a classical 3/2 cusp, systems which change type might exhibit different features.

Before proceeding further, we introduce some definitions.

**Definition 1.2.1.** Consider a system of \( n \) PDEs of the form

\[
A(u,x)u_{x_1} + B(u,x)u_{x_2} = f(u,x). 
\]

(1.16)
It is hyperbolic if \( \det |\mu A + \nu B| = 0 \) has \( n \) real roots \( \mu_i = \mu_i(\nu) \) for all \( \nu \). When \( A \) and \( B \) depend on both \( u \) and \( x \), the system may change type; that is, it may be hyperbolic for some values of \( u \) and \( x \) and not for others. Assume \( A \) and \( B \) are smooth function of \( u \) and \( x \). For a given \( u \), the boundary of the set

\[
\{ x | \text{the system (1.16) is hyperbolic at } x \}
\]

is called the sonic line. More generally, the boundary of the set

\[
\{(u, x) | \text{the system (1.16) is hyperbolic at } (u, x) \}
\]

forms the set of sonic points.

Recent papers of Hunter and Tesdall [21] and [41] consider a problem where a weak plane shock wave of constant strength hits a semi-infinite, rigid screen at normal incidence (see Figure 1.2). Hunter and Tesdall carry out a numerical simulation using the self-similar unsteady transonic small disturbance equation (UTSDE). In the region of interest, the self-similar UTSDE is a system which changes type from hyperbolic to elliptic. According to numerical simulations of Hunter and Tesdall, the shock disappears exactly on the sonic line (see Figure 1.3). At the shock formation point for hyperbolic system of PDEs, the shock strength is proportional to the square root of the distance to the shock formation point. However, in the Hunter-Tesdall problem, the shock strength is not proportional to the square root of the distance to the shock formation point (see Figure 1.4), therefore, we expect the mechanism of the shock formation to be different from the cusp singularity.
Figure 1.2: A weak shock at normal incidence to a screen. Solid lines are shocks; dotted lines are expansion wavefronts.

Figure 1.3: (a) A contour plot of $u$ close to the shock die out point. It is subsonic (elliptic) on the left of the red dash line, and it is supersonic (hyperbolic) on the right of the red dash line. (b) Cross sections of $u - \left[ \frac{x}{t} + \frac{1}{4} \left( \frac{y}{t} \right)^2 \right]$ taken through the shock. It is subsonic (elliptic) when $u - \left[ \frac{x}{t} + \frac{1}{4} \left( \frac{y}{t} \right)^2 \right] > 0$ and supersonic (hyperbolic) when $u - \left[ \frac{x}{t} + \frac{1}{4} \left( \frac{y}{t} \right)^2 \right] < 0$.
Figure 1.4: Strength of the shock VS. Distance to the shock formation point
Chapter 2: Shock Formation at the Sonic Line

In this chapter, we will study Hunter-Tesdall problem analytically. First, we describe this problem in detail and then reformulate this problem into a free boundary problem in Section 2.1. Next, we state our main theorem (the existence of solution to the free boundary problem) in Section 2.3 and then illustrate the method we use to solve this problem in Section 2.4. Finally, we present the details of the proof in Section 2.5, Section 2.6 and Section 2.7.

2.1 Problem Setting

Let us take a close look at the Hunter-Tesdall problem and discuss how to approach this problem analytically. To study the local structure of self-similar solutions of the compressible Euler equations near the point \( S \) where the shock strength approaches zero and the wave changes continuously from a shock into an expansion, they use the method of matched asymptotic expansions to show that the solution near \( S \) can be described by a self-similar solution of the unsteady transonic small disturbance equation (UTSDE):

\[
\begin{align*}
    u_t + \left( \frac{1}{2} u^2 \right)_x + v_y &= 0, \\
    v_x - u_y &= 0.
\end{align*}
\]

(2.1)
with the matching, or initial condition, that

\[ u \sim \alpha \frac{y}{t} \sqrt{\frac{x}{t} + \frac{y^2}{4t^2}} \text{ as } \frac{x}{t} + \frac{y^2}{4t^2} \to -\infty, \]

\[ u = 0 \text{ for } \frac{x}{t} + \frac{y^2}{4t^2} \text{ sufficiently large and positive.} \]

where \( \alpha \) is a parameter that measures the strength of the wave. (In their numerical simulation, they actually used improved matching data, but this will not change the nature of the mathematical problem.)

Because this problem is self-similar, or pseudo-steady, i.e., the solution depends only on \((x/t, y/t)\), we consider (2.1) in self-similar coordinates \( \xi = x/t, \eta = y/t \):

\[
(u - \xi)u_\xi - \eta u_\eta + v_\eta = 0
\]

\[
v_\xi - u_\eta = 0.
\]

Equation (2.3) is hyperbolic (supersonic) when \( u < \xi + \eta^2/4 \), and elliptic (subsonic) when \( u > \xi + \eta^2/4 \).

It is more convenient to work in the \((\rho, \tau)\) coordinate system, with \( \rho = \xi + \eta^2/4 \) and \( \tau = \eta \). In the \((\rho, \tau)\) coordinate system, the self-similar UTSDE becomes

\[
(u - \rho)u_\rho - \frac{\tau}{2} u_\tau + v_\tau = 0
\]

\[
\frac{\tau}{2} u_\rho - v_\rho + u_\tau = 0.
\]

Notice that this system is hyperbolic (supersonic) when \( u - \rho < 0 \) and is elliptic (subsonic) when \( u - \rho > 0 \) in this coordinate system.

The numerical results in [21] and [41] show that the solution has a supersonic (hyperbolic) state \((u_R, v_R) = (0, 0)\) on the right of shock and sonic line, and a subsonic (elliptic) state \((u_L, v_L)\) on the left, and the shock disappears at \((0, \tau_*)\) for some \( \tau_* < 0 \) which depends on the wave strength \( \alpha \). To simplify the problem we move the shock die out point to the origin (see Appendix A).
We are interested in constructing a solution of (2.4) with the configuration shown in Figure 2.1.

In the supersonic region, we already know that \( u = 0 \) and \( v = 0 \), so we are only interested in constructing a solution \( (u, v) \) in the subsonic region. Since the UTSDE is a good approximation of compressible Euler equations near the shock formation point (or shock die out point, they are really the same thing), we consider a compact subsonic region \( \Omega \) (see Figure 2.2) instead of the non-compact subsonic region in Figure 2.1.

We are seeking a solution \( (u, v) \) of (2.4) in \( \Omega \) which connects to the supersonic state \( (u_R, v_R) = (0, 0) \) through the Rankine-Hugoniot conditions on the shock \( \Sigma \) and satisfies \( (u, v) = (0, 0) \) on the sonic line \( \sigma_0 \).
2.1.1 The Second-Order PDE

Eliminating $v$ in the system (2.4) yields the second-order equation

$$Q(u) := \left( (u - \rho) u_{\rho} + \frac{u}{2} \right)_{\rho} + u_{\tau\tau} = 0,$$

(2.5)

or in the non-divergence form

$$Q(u) := (u - \rho) u_{\rho\rho} + u_{\tau\tau} + u_{\rho}^2 - \frac{1}{2} u_{\rho} = 0.$$

(2.6)

The second order PDE (2.5) (or equivalently (2.6)) is elliptic as long as $u - \rho > 0$. However, since (2.5) is quasilinear, ellipticity is not known a priori.

We mention that it is possible to get a second-order PDE from (2.4) using a velocity potential. Introducing $\bar{u} = u - \rho$ and $\bar{v} = v - \frac{7}{2} u$, (2.4) becomes

$$\bar{u} \bar{u}_{\rho} + \bar{v}_{\tau} + \frac{3}{2} \bar{u} + \frac{1}{2} \rho = 0$$

$$\bar{u}_{\tau} - \bar{v}_{\rho} = 0$$

(2.7)
The second equation in (2.7) implies that we can introduce a velocity potential \( \phi \), with \( \bar{u} = \phi_\rho \) and \( \bar{v} = \phi_\tau \); then we obtain a second-order PDE for \( \phi \):

\[
\phi_\rho \phi_{\rho \rho} + \phi_{\tau \tau} + \frac{3}{2} \phi_\rho + \frac{1}{2} \rho = 0
\]  

(2.8)

However, we prefer (2.6) to (2.8), because the coefficients of (2.8) depend on the gradient of the solution \( \phi \) while the coefficients of (2.6) depend only on the solution \( u \).

2.1.2 The Shock Position and the Oblique Derivative Condition for \( u \) along the Shock

The idea of replacing the Rankine-Hugoniot conditions on the shock by an oblique derivative condition on the shock and an ODE for the position of shock was first developed in [10] for the steady transonic small-disturbance equation. Later, it was extended to the UTSDE in [7]. In this section, we will develop the oblique derivative condition for \( u \) and the shock evolution equation for the position of the shock from the Rankine-Hugoniot conditions following the procedure described in [7].

Along the shock \( \Sigma = \{(\rho, \tau) | \rho = \rho(\tau)\} \), the Rankine-Hugoniot conditions must be satisfied (the notion ‘ means \( \frac{d}{d\tau} \)):

\[
\rho'(\tau) = \frac{\left[ \frac{1}{2} u^2 - \rho u \right]}{\left[ v - \frac{1}{2} u^2 \right]} = \frac{\left[ \frac{1}{2} u^2 - v \right]}{\left[ u \right]}.
\]  

(2.9)

Notice that since the shock starts at origin, we must have

\[
\rho(0) = 0.
\]  

(2.10)

Let \( (u, v) \) denote the left state and let \( (u_R, v_R) \) denote the right state. We know that \( (u_R, v_R) = (0, 0) \). From the Rankine-Hugoniot conditions (2.9), we obtain one
equation for \( \rho' \) by eliminating \([\frac{1}{2}u - v]\):

\[
(r')^2 = -\frac{1}{2}u + \rho. \tag{2.11}
\]

Combining \(-\frac{1}{2}u + \rho = (r')^2 \geq 0\) with the requirement that the left state be subsonic, i.e. \(u - \rho > 0\), we must have \(\rho > 0\) for the shock. Therefore,

\[
\rho' = \sqrt{-\frac{1}{2}u + \rho}. \tag{2.12}
\]

A second condition is obtained by eliminating \(\rho'\) from (2.9):

\[
v = \frac{\tau}{2} u - u \sqrt{-\frac{1}{2}u + \rho} \tag{2.13}
\]

We differentiate both sides along \(\Sigma\) (where \(\tau = \rho' \partial_\rho + \partial_\tau = \frac{d}{d\tau}\) along \(\Sigma\)):

\[
v' = \frac{u}{2} + \frac{\tau}{2} u' - u' \sqrt{-\frac{1}{2}u + \rho} - \frac{-\frac{1}{2}u' + \rho'}{2 \sqrt{-\frac{1}{2}u + \rho}} u \tag{2.14}
\]

We express \(u'\) as \(u_\rho \rho' + u_\tau\) and use the differential equation (2.1) to write

\[
v' = v_\rho \rho' + v_\tau = (\frac{\tau}{2} u_\rho + u_\tau) \rho' - ((u - \rho) u_\rho - \frac{\tau}{2} u_\tau). \tag{2.15}
\]

Finally, substituting this for \(u'\) and \(v'\) in (2.14), using (2.12) and collecting terms in \(u_\rho\) and \(u_\tau\), we obtain

\[
N(u) := \beta \cdot \nabla u = 0 \tag{2.16}
\]

where \(\beta = (\beta_1, \beta_2) = ((2 \rho - \frac{7}{4} u) \rho', 2 \rho - \frac{5}{4} u)\).

We can replace the Rankine-Hugoniot conditions (2.9) by (2.10), (2.12) and (2.16).

We notice that \(\beta \cdot \nu = \frac{\frac{1}{2} u \rho'}{\sqrt{1 + (\rho')^2}} \geq 0\) if \(u \geq 0\), where \(\nu = \frac{(-1, \rho')}{\sqrt{1 + (\rho')^2}}\) is a unit inward normal to the subsonic region \(\Omega\) on \(\Sigma\).
2.1.3 The Free Boundary Problem

Now we can replace the problem at the beginning of section 2.1 by the following free boundary problem:

\[
((u - \rho)u_\rho + \frac{u}{2})_\rho + u_{\tau\tau} = 0 \quad \text{in } \Omega,
\]
\[
\beta \cdot \nabla u = 0 \quad \text{on } \Sigma,
\]
\[
u = 0 \quad \text{on } \sigma_0,
\]
\[
u = \varphi \quad \text{on } \sigma,
\]

where, the free boundary \( \Sigma = \{ (\rho, \tau) | \rho = \rho(\tau) \} \) satisfies the following ODE:

\[
\rho'(\tau) = \sqrt{\rho - \frac{1}{2}u},
\]
\[
\rho(0) = 0.
\]

We also pose some special requirements for \( \varphi \):

the function \( \varphi \) is a smooth function defined on an open set that contains \( \partial \Omega / \Sigma \), such that

1. \( \varphi = 0 \) on \( \sigma_0 \),

2. \( \varphi \geq \max\{\frac{8}{7}\rho, \rho\} \) everywhere,

3. when \( \rho > 0 \), \( \varphi \leq 2(1 - \varepsilon_0^2)\rho \) for some small \( \varepsilon_0 > 0 \),

4. when \( \rho < 0 \), \( \varphi \geq (1 - \varepsilon_0)\rho \) for some small \( \varepsilon_0 > 0 \) which will be specified later,

5. \( \varphi - \rho \geq \varepsilon_1 \) in a small neighborhood of the point \( \sigma \cap \{ (\rho, \tau) | \rho = 0, \tau > 0 \} \) for some small \( \varepsilon_1 > 0 \),

6. when \( \rho < 0 \) and close to \( \sigma_0 \), \( \varphi - \rho \leq -C\rho \) for some positive constant \( C \).
Remark 2.1.1. We require that $\varphi$ satisfies those conditions because we need those conditions when we construct barrier functions for the solution in later sections. However, those conditions are not totally artificial, they are satisfied by the numerical results obtained by Hunter and Tesdall and they are also consistent with the asymptotic solution to this problem.

Remark 2.1.2. We write the boundary condition on $\sigma_0$ and the boundary condition on $\sigma$ separately to stress the fact that $\sigma_0$ is a sonic line and it needs to be treated separately sometimes. However, when we do not need to stress this fact, we will just use the condition $u = \varphi$ on $\sigma \cup \sigma_0$ where the definition of $\varphi$ is extended to be 0 on $\sigma_0$.

2.2 Definition of Hölder Norms

In this section we recall the definition of Hölder norms. For a function $u$ defined on an open set $\Omega$ in $\mathbb{R}^2$, first, we define the usual supremum norm and the Hölder seminorms:

$$|u|_{0;\Omega} = \sup_{X \in \Omega} |u(X)| \quad \text{and} \quad [u]_{\alpha;\Omega} = \sup_{X,Y \in \Omega} \frac{|u(X) - u(Y)|}{|X - Y|^\alpha}$$

for $0 < \alpha \leq 1$. We then define the Hölder norm:

$$|u|_{\alpha;\Omega} = |u|_{0;\Omega} + [u]_{\alpha;\Omega} \quad \text{for} \quad 0 < \alpha \leq 1,$$

and, for $a = k + \alpha$ where $k$ is an integer and $0 < \alpha \leq 1$,

$$|u|_{a;\Omega} = \sum_{j<k} |D^j u|_{0;\Omega} + |D^k u|_{a;\Omega}.$$

The notation $C^{k,\alpha}$ or $H_{k+\alpha}$ denotes the space of functions whose $(k + \alpha)$-Hölder norm is finite.
We describe the regularity of $\Sigma = \{(\rho(\tau), \tau) : \tau_{min} < \tau < \tau_{max}\}$ by the regularity of the function $\rho = \rho(\tau)$. We say $\Sigma \in C^{1,\alpha}$ or $\Sigma \in H_{1+\alpha}$ if $\rho(\tau) \in C^{1,\alpha}$, for $0 < \alpha \leq 1$.

The notation $|\Sigma|_{1+\alpha}$ denotes the $H_{1+\alpha}$ norm of $\rho(\tau)$.

Let $V$ denote the set of corners of $\Omega$, $V = \{V_i\}$, and let $d_V(X) = \min_i \{|X - V_i|\}$.

For a subset $S$ of $\partial\Omega$, define

$$\Omega_{\delta;S} = \{X \in \Omega : \text{dist}(X, S) > \delta\}.$$ 

For any $a > 0$ and $a + b \geq 0$, the weighted norms are defined to be

$$|u|^{(b)}_{a;\Omega \cup V^c} = \sup_{\delta > 0} \delta^{a+b} |u|^{(b)}_{a;\Omega_{\delta;S}}.$$ 

The notation $H^{(b)}_{a;\Omega \cup V^c}$ denotes the space of functions $u$ such that $|u|^{(b)}_{a;\Omega \cup V^c}$ is finite. In the following sections, we are interested in the case when $S = V$, so from now on, we will drop the subscript $\Omega \cup V^c$ to make the notation simple.

We note that there are two important properties of these norms

1. If $a' + b \geq 0$ and if $a' < a$, then $|u|^{(b)}_{a'} \leq |u|^{(b)}_a$, for all $u \in H^{(b)}_a$.

2. If $a \geq b > 0$ and if $\{u_m\}$ is a bounded sequence in $H^{(-b)}_a$, then there is a subsequence $\{u_{m_k}\}$ which converges in any $H^{(-b')}_{a'}$ with $0 < b' < b$, $0 < a' < a$, and $a' \geq b'$.

2.3 Main Theorem

In the following sections, we will solve the free boundary problem at the beginning of the chapter.

To state the result precisely, we introduce a set $K$ of possible position for the shock and a cut-off function $g$. The set $K$ and the cut-off function $g$ will be defined in later sections.
Theorem 2.3.1. There exist a \( u \in C^{2,\alpha}(\Omega/\sigma_0 \cup \Sigma) \cap C^{1,\alpha}(\Omega/\{\sigma_0, V_1\}) \cap C^{\gamma}(\Omega/\bar{\sigma}_0) \cap C(\Omega) \) for some \( 0 < \alpha < 1 \) and \( 0 < \gamma < 1 \) and a \( \rho(\tau) \in K \) such that

\[
\begin{align*}
((u - \rho)u_{\rho} + \frac{u}{2})_{\rho} + u_{\tau\tau} &= 0 & \text{in } \Omega, \\
\beta \cdot \nabla u &= 0 & \text{on } \Sigma, \\
u = \varphi & \text{on } \sigma, \\
u = 0 & \text{on } \sigma_0.
\end{align*}
\] (2.19)

where \( \Sigma = \{(\rho, \tau)|\rho = \rho(\tau)\} \), and

\[
\rho'(\tau) = \sqrt{\rho(\tau) - \frac{1}{2} g(u(\rho(\tau), \tau))}, \quad \rho(0) = 0.
\] (2.20)

Remark 2.3.2. To fully solve the free boundary problem in chapter 1 and therefore solve the original transonic shock problem for the UTSDE, we need to remove the cut-off function \( g \). However, owing to some technical difficulties, we are not able to remove \( g \) at this moment.

2.4 Method to Solve the Free Boundary Problem

The method we use to solve the free boundary problem we described in Section 2.1.3 was first developed for solving the steady transonic small disturbance equation in [10]. Then this method was extended to solve free boundary problem with various other systems, including the unsteady transonic small disturbance equation, the nonlinear wave system, the pressure-gradient system and the potential flow. Keyfitz described this approach in detail and discussed difficulties associated with different problems in [26]. In this section, we describe how we can adopt this method to solve our free boundary problem formulated in Section 2.1.3.
2.4.1 A Fixed Point Approach to the Free Boundary Problem

In the free boundary problem (2.17) and (2.18), the shock position is determined by (2.18) and it is unknown a priori because (2.18) involves the solution $u$. We write (2.18) in equivalent integral form

$$
\rho(\tau) = \int_0^\tau \sqrt{\rho(t) - \frac{1}{2} u(\rho(t), t)} \, dt.
$$

(2.21)

Define a mapping $J$

$$
J : \rho \mapsto \tilde{\rho}
$$

(2.22)

where

$$
\tilde{\rho}(\tau) = \int_0^\tau \sqrt{\rho(t) - \frac{1}{2} u(\rho(t), t)} \, dt.
$$

(2.23)

Then the solution to (2.21) can be viewed as a fixed point of the mapping $J$.

This integral form of the mapping $J$ motivates the use of Hölder spaces and the following Schauder Fixed Point Theorem:

**Theorem 2.4.1.** (Corollary 11.2 of [19]) Let $\mathcal{K}$ be a closed, convex subset of a Banach space $\mathcal{B}$, and let $J$ be a continuous mapping from $\mathcal{K}$ into itself such that the image $J\mathcal{K}$ is precompact. Then $J$ has a fixed point.

2.4.2 A First Look at the Fixed Point Procedure

The basic idea to solve the free boundary condition (2.17) and (2.18) is to first fix a $\Sigma \in \mathcal{K}$ for some suitable space $\mathcal{K}$ and solve the fixed boundary problem, and then update the shock position $\Sigma$ through the shock evolution equation (2.18).

The standard procedure consists of three steps:
**Step 1.** Fix an approximate position for the free boundary $\Sigma$, given by $\rho = \rho(\tau)$. Here, $\rho$ belongs to a closed, convex subset $\mathcal{K}$ of the Banach space $H_{1+\alpha_\Sigma}$. The set $\mathcal{K} \subset H_{1+\alpha_\Sigma}$ is defined to satisfy the following conditions:

1. Smoothness: $\rho \in H_{1+\alpha_\Sigma}$ for some $\alpha_\Sigma > 0$ to be specified later.
2. Initial condition: $\rho(0) = 0$ and $\rho'(0) = 0$.
3. Boundedness: $\frac{\alpha_\Sigma}{2} \tau^2 \leq \rho(\tau) \leq \frac{3}{28} \tau^2$.
4. Boundedness of the first derivative: $\epsilon_0 \tau \leq \rho'(\tau) \leq \frac{3}{14} \tau$.

(The bound $\epsilon_0$ is related to the cut-off $g$).

**Step 2.** Solve for $u$ in the fixed boundary problem with Dirichlet boundary condition on $\sigma$ and $\sigma_0$ and oblique derivative condition on $\Sigma$, where $\Sigma \in \mathcal{K}$ is fixed. This step involves solving a quasilinear PDE through linearization and an application of the Leray-Schauder fixed point theorem (Theorem 11.6 in [19]). It relies on the extensive theory for mixed boundary value problems in Lipschitz domains which was developed by Lieberman. We list some references on this subject here: [30], [31], [32], [33] and [34]. We also mention that Lieberman’s new book [35] includes most results on this theory.

**Step 3.** Map $\rho$ to $\tilde{\rho} = J\rho$ by the shock evolution equation (2.18). We show that $J$ maps $\mathcal{K}$ back to $\mathcal{K}$ and $J\mathcal{K}$ is precompact. Therefore, from Theorem 2.4.1 we conclude that there is a fixed point $\rho$ such that $J\rho = \rho$. This $\rho = \rho(\tau)$ together with the solution $u$ to the fixed boundary problem, with the position of $\Sigma$ defined by $\rho = \rho(\tau)$, solve the free boundary problem (2.17) and (2.18).
To solve the original UTSDE (2.4) with the Rankine-Hugoniot condition (2.9), we need to add one more step to the above procedure:

**Step 4.** Solve \( v \) from (2.4) using \( u \) from the above procedure. We also need to check that the \( u, v \) along with \( \rho = \rho(\tau) \) indeed solve (2.4) and (2.9). But this is usually straightforward.

The free boundary approach to problems that involve transonic shocks and the procedure to solve the free boundary problem as described above were first introduced by Čanić, Keyfitz and Lieberman in [10] for the steady transonic small disturbance equation (TSDE). Since [10], these ideas have been extended to solve many different problems in different settings with different PDEs. We discuss these in sequence, beginning with the problems closest to our current work.

1. **Unsteady transonic small disturbance equation (UTSDE):**

   In [7], Čanić, Keyfitz and Kim studied the strong regular reflection problem and obtained a local solution near the reflection point and in [8] they studied the weak regular reflection problem and also obtained a local solution near the reflection point. More detailed study of the problem in [7] was presented in [24] by Jegdić, Keyfitz and Čanić.

   The papers [7] and [24] adapt the method used in [10] for UTSDE. We will follow the procedure in [7] and [24] to prove the existence theorem for our regularized fixed boundary problem. In [8], the elliptic PDE is degenerate on the sonic line, and the authors adapt the method in [16] to construct a local lower barrier to handle the difficulty. In our problem, we have a similar degeneracy on the sonic line. We construct a similar local lower barrier. With this lower barrier and other barriers we
construct, we can show that the PDE is strictly elliptic in any compact subset away from the sonic line.

We also mention related works on other systems:

2. Nonlinear wave system (NLWS):

The Mach reflection problem was studied by Ćanić, Keyfitz and Kim in [9]. The strong regular reflection problem was studied by Jegdić, Keyfitz and Ćanić in [23], and the weak regular reflection problem was studied by Jegdić in [22].

A shock diffraction problem was also studied for NLWS. Shock diffraction by a step was first studied by Kim in [27] with an assumption which is equivalent to the shock not touching the sonic line when it hits the wall. Later, Kim [28] studied the shock diffraction problem again with a rarefaction wave in the configuration. In this case, she found that the shock does coincide with the sonic line at a single point, where the shock touches the wall. In 2014, Chen, Deng and Wei [11] extended Kim’s result in [27] to any convex cornered wedge. In their paper, the assumption which Kim made in [27] was removed. Furthermore, they improved the regularity of the solution and proved their regularity is optimal.

One of the major difficulties in our problem is that at the origin the elliptic PDE becomes degenerate and the oblique derivative condition becomes tangential at the same time. It is worth mentioning that this difficulty was also encountered in both [9] and [28]. However, after careful study of the techniques used in [9] and [28], we find those techniques do not apply to our problem.

3. Pressure-gradient system:
Zheng [44] constructs a global solution to a Riemann problem for the pressure-gradient system which initially has two shock waves and two contact discontinuities. Weak shock reflection was studied by Zheng in [45].

4. Potential flow:

Existence and stability of multidimensional shocks for the Euler equations for steady potential fluids were studied by Chen and Feldman in [12] and [13]. Global solutions of shock reflection by large-angle wedges for potential flow were obtained by Chen and Feldman in [14]. Existence and stability of global solutions of shock diffraction by wedges were studied in [15] by Chen and Wei. We mention that while some parts of the procedure in potential flow problems are the same as the procedure used for the preceding systems, the way the second order PDE is handled is quite different from those papers on other systems. The reason is that for potential flow, the coefficients of the second order terms involve the gradient of the solution, while for other systems, the coefficients only depend on the solution and the independent variables.

2.5 Modified Free Boundary Problem

Since the problem is nonlinear, the ellipticity, the obliqueness and the well-posedness of the shock evolution equation are not known a priori. Hence we introduce a small parameter $\varepsilon$ and cut-off functions $f$, $g$, $h_\varepsilon$ to modify our problem.

To ensure strict ellipticity, we replace the operator $Q$ by

$$Q^{+\varepsilon}u := [(h_\varepsilon(u - \rho) + \varepsilon)u_\rho + \frac{u}{2}]_\rho + u_{\tau\tau}$$

$$= (h_\varepsilon(u - \rho) + \varepsilon)u_{\rho\rho} + u_{\tau\tau} + h'_\varepsilon(u - \rho)u_\rho^2 + (\frac{1}{2} - h'_\varepsilon(u - \rho))u_\rho$$  \hspace{1cm} (2.24)
where, $h_\varepsilon$ is a smooth increasing function such that
\[
h_\varepsilon(t) = \begin{cases} 
s & \text{if } s \geq 0 \\
-\frac{1}{2}\varepsilon & \text{if } s < -\frac{1}{2}\varepsilon
\end{cases}
\]
and $0 \leq h_\varepsilon'(s) \leq 1$.

We also replace our operator $N$ by
\[
N^+ u := \beta_1^+ u_\rho + \beta_2^+ u_\tau
\]
where,
\[
\beta_1^+ = (2\rho - \frac{7}{4}f(u, \rho))\rho' \\
\beta_2^+ = 2\rho - \frac{5}{4}f(u, \rho),
\]
and
\[
f(s, \rho) = \max\{s, a\rho\} \quad \text{for some } 1 < a < \frac{8}{7}.
\]

To ensure the well-posedness of the shock evolution equation, we introduce the cut-off function
\[
g(s, \rho) = \begin{cases} 
2(1 - 2\varepsilon_0)\rho & \text{if } s > 2(1 - 2\varepsilon_0)\rho \\
s & \text{if } s \leq 2(1 - 2\varepsilon_0)\rho
\end{cases}
\]
and replace the shock evolution equation by
\[
\frac{d\rho}{d\tau} = \sqrt{\rho - \frac{1}{2}g(u, \rho)}
\]
To make the notation simple, we will omit the $\rho$ dependence of $g$ and $f$ from now on.

We then consider the following modified free boundary problem:

\[
\begin{align*}
Q^+ u &= 0 & \text{in } \Omega, \\
N^+ u &= 0 & \text{on } \Sigma, \\
\rho &= 0 & \text{on } \sigma_0, \\
\rho &= \varphi & \text{on } \sigma,
\end{align*}
\]
where, the free boundary $\Sigma = \{(\rho, \tau) | \rho = \rho(\tau)\}$ satisfies the following ODE:

\[
\frac{d\rho}{d\tau} = \sqrt{\rho - \frac{1}{2} g(u)}
\]

\[
\rho(0) = 0.
\]

(2.27)

### 2.6 The Fixed Boundary Problem

To make it convenient, we require that $\partial \Omega \cap \{\rho \geq 0\} / \Sigma$ to be a straight line segment where $\tau = \tau_{max}$.

We define $\mathcal{K}$ to consist of functions $\rho(\tau)$ in $H_{1+\alpha\Sigma}([0, \tau_{max}])$ satisfying the following properties:

\[
\begin{align*}
(K1) \quad & \rho(0) = 0 \text{ and } \rho'(0) = 0 \\
(K2) \quad & \frac{\varepsilon_0}{2} \tau^2 \leq \rho(\tau) \leq \frac{3}{28} \tau^2 \\
(K3) \quad & \varepsilon_0 \tau \leq \rho'(\tau) \leq \frac{3}{14} \tau
\end{align*}
\]

Take $\rho(\tau) \in \mathcal{K}$ and let the domain $\Omega$ be bounded by a smooth boundary component $\sigma$, straight line $\sigma_0$ and $\Sigma = \{(\rho, \tau) | \rho = \rho(\tau)\}$ as shown in Figure 2.2. We solve (2.26) in this fixed domain $\Omega$. We notice that the oblique derivative condition on $\Sigma$ becomes tangential at the origin. To resolve this technical difficulty, we consider the following modified problem in the fixed domain $\Omega$:

\[
\begin{align*}
Q^+ u &= 0 \quad \text{in } \Omega, \\
N^+ u &= 0 \quad \text{on } \Sigma / \Sigma(\delta), \\
u &= \frac{8}{7} \rho \quad \text{on } \Sigma(\delta) \quad \text{on } \Sigma(\delta) \quad \text{on } \Sigma(\delta) \quad \text{on } \Sigma(\delta) \quad \text{on } \Sigma(\delta) \\
u &= 0 \quad \text{on } \sigma_0, \\
u &= \phi \quad \text{on } \sigma.
\end{align*}
\]

(2.28)
where $\Sigma(\delta) = \{ (\rho(\tau), \tau) | 0 < \tau \leq \delta \}$ for some small $\delta$ (see Figure 2.3).

For this modified fixed boundary problem (2.28), we compute the obliqueness constant

$$B = \beta \cdot \nu = \frac{\frac{1}{2} f(u)\rho'}{\sqrt{1 + (\rho')^2}} \geq C(\delta) > 0$$

**Remark 2.6.1.** We point out that this is a structure different from [7], [8] and [24].

**Remark 2.6.2.** The solution $u$ to the problem (2.28) depends on the parameters $\varepsilon$ and $\delta$, so we should write $u^{\varepsilon, \delta}$ instead of $u$. However, to make the notation simple, we will use the notation $u$ instead of $u^{\varepsilon, \delta}$ from now on, unless the dependence on $\varepsilon$ and $\delta$ needs to be stated explicitly.
2.6.1 The Linearized Problem

Assume that \( \hat{u} \in H^{(-\gamma_1)}_{1+\alpha_1} \). We define linear operators

\[
Lu := \sum_{i,j} a_{ij}(\hat{u}, \rho, \tau)D_{ij}u + \sum_i b_i(\hat{u}, \rho, \tau)D_i u + \sum_{i,j} c_{ij}(\hat{u}, \rho, \tau)D_i \hat{u} D_j u
\]

\[
= [(h\varepsilon(\hat{u} - \rho) + \varepsilon)u_\rho + \frac{u}{2}]_\rho + u_{\tau\tau}
\]

and

\[
\tilde{N}u := \sum_i \beta_i(\hat{u}, \rho)D_i u
\]

\[
= (2\rho - \frac{7}{4}f(\hat{u}))\rho'u_\rho + (2\rho - \frac{5}{4}f(\hat{u}))u_\tau
\]

Consider the following linear problem:

\[
Lu = 0 \quad \text{in } \Omega,
\]

\[
\tilde{N}u = 0 \quad \text{on } \Sigma/\Sigma(\delta),
\]

\[
u = \frac{8}{7}\rho \quad \text{on } \Sigma(\delta)
\]

\[
u = 0 \quad \text{on } \sigma_0,
\]

\[
u = \varphi \quad \text{on } \sigma,
\]

(2.29)

Let \( \Lambda \) and \( \lambda \) be the largest and smallest eigenvalues of \( A = [a_{ij}] \) respectively. The ellipticity ratio of \( L \) is defined to be the ratio \( \Lambda/\lambda \).

To simplify the notation, we let \( \tilde{\sigma} = \sigma \cap \sigma_0 \cap \Sigma(\delta)/\{V_1, V_2\} \), \( \tilde{\Sigma} = \Sigma/\Sigma(\delta) \), and we define \( \tilde{\varphi} \) so that \( \tilde{\varphi} = \varphi \) on \( \sigma \) and \( \sigma_0 \) and \( \tilde{\varphi} = \frac{8}{7}\rho \) on \( \Sigma(\delta) \). Notice that the regularity of \( \tilde{\varphi} \) is the same as the regularity of \( \varphi \).

**Lemma 2.6.3.** For a small enough \( \delta \), any differentiable solution \( u \) of the problem (2.29) satisfies \( m \leq u \leq M \) in \( \Omega \), where \( M = \max_\sigma \{\varphi\} \) and \( m = \min_\sigma \{\varphi\} \) are independent of \( \varepsilon \) and \( \delta \).
Proof. $L$ is a uniformly elliptic operator, and by the strong maximum principle, the extrema of $u$ are achieved on $\partial \Omega$. Suppose there is an extremum at a point $\Xi_0$ on $\Sigma/\Sigma(\delta)$. Then the tangential derivative along $\Sigma/\Sigma(\delta)$ satisfies $u'(\Xi_0) = 0$, since this value is also an extremum of the function restricted to the boundary. This, combined with $\tilde{N}u = 0$, implies that $\frac{\partial u}{\partial \nu}(\Xi_0) = 0$. However, this violates the Hopf lemma. Hence, the extrema are attained on $\sigma$, $\sigma_0$ or $\Sigma(\delta)$. Therefore, $\min_{\tilde{\sigma}} \{\tilde{\varphi}\} \leq u \leq \max_{\tilde{\sigma}} \{\tilde{\varphi}\}$. Since for small enough $\delta$, $\min_{\sigma} \{\varphi\} < \min_{\Sigma(\delta)} \{\varphi\}$ and $\max_{\sigma} \{\varphi\} > \max_{\Sigma(\delta)} \{\varphi\}$, the estimate becomes $m \leq u \leq M$ in $\Omega$.

Let $V_1 = \tilde{\sigma} \cap \Sigma/\Sigma(\delta)$, $V_2 = \Sigma(\delta) \cap \Sigma/\Sigma(\delta)$ and $V := \{V_1, V_2\}$.

Theorem 2.6.4. Let $\alpha_\Sigma \in (0, 1)$ and $\rho \in K$ be fixed, and let $\varphi$ be any function satisfying the conditions in Section 2.1.3. Suppose that for exponents $0 < \alpha_1 \leq \alpha_\Sigma$ and $0 < \gamma_1 < 1$, $\hat{u} \in H_{1+\alpha_1}^{(-\gamma_1)}$ is a function such that

$$m \leq \hat{u}(\rho, \tau) \leq M, \quad (\rho, \tau) \in \Omega,$$  \hspace{1cm} (2.30)

where $m$ and $M$ are as in Lemma 2.6.3, and there exists a constant $m_0$ such that

$$|b_i(\hat{u}, \rho, \tau) + \sum_j c_{ji}(\hat{u}, \rho, \tau)D_j \hat{u}| \leq m_0 d_\gamma^{\gamma - 1}$$  \hspace{1cm} (2.31)

Then for some $\gamma_0$ determined by the geometry of $\Omega$ and by the operators $L$ and $\tilde{N}$, there exists a unique solution $u \in H_{1+\alpha_\Sigma}^{(-\gamma)}$ of the linear problem (2.29) for any $\gamma \in (0, \gamma_0)$. Furthermore, the solution $u$ satisfies the following two estimates

$$|u|_{1+\alpha_\Sigma}^{(-\gamma)} \leq C(|\tilde{\varphi}|_\gamma + \sup_{i, X \in \Omega} d_{V_i}^{\gamma}(X)|u(X) - u(V_i)|)$$  \hspace{1cm} (2.32)

and

$$|u|_{1+\alpha_\Sigma}^{(-\gamma)} \leq C_1 |\tilde{\varphi}|_\gamma,$$  \hspace{1cm} (2.33)
Proof. We prove Theorem 2.6.4 in three steps.

Step 1. Let \( u \in H^{(-\gamma)}_{1+\alpha} \) be a solution to (2.29), then \( u \in C^2(\Omega) \) by standard elliptic theory. By Lemma 2.6.3, we obtain the \( L^\infty \) estimate:

\[
|u|_{0;\Omega} \leq |\varphi|_0 = |\tilde{\varphi}|_0 \tag{2.34}
\]

Step 2. Now we prove that if \( u \in H^{(-\gamma)}_{1+\alpha} \) is a solution to (2.29), then \( u \) satisfies the estimate (2.32). First we show that

\[
R^{1+\alpha}[Du]_{\alpha;B_R(X_0)\cap\Omega} \leq C(R^\gamma|\tilde{\varphi}|_\gamma + \sup_i |u(X) - u(V_i)|_{0;B_{2R}(X_0)\cap\Omega}) \tag{2.35}
\]

holds in the following three cases

1. \( X_0 \in \tilde{\sigma} \) and \( B_{2R}(X_0) \cap \tilde{\Sigma} = \emptyset \),
2. \( X_0 \in \tilde{\Sigma} \) and \( B_{2R}(X_0) \cap \tilde{\sigma} = \emptyset \), and
3. \( B_{2R}(X_0) \subseteq \Omega \).

In case 1, we are dealing with an elliptic PDE problem with Dirichlet boundary condition, so we can use Theorem 8.33 and the discussion on page 138-139 of [19] to show that

\[
R^{1+\alpha-\gamma}[Du]_{\alpha;B_R(X_0)\cap\Omega} \leq C(|\tilde{\varphi}|_\gamma + \sup_i |u(X) - u(V_i)|_{0;B_{2R}(X_0)\cap\Omega}) \tag{2.36}
\]

where the constant \( C \) depends on \( \alpha_\Sigma \), the domain \( \Omega \) and the norms of the coefficients of the operator \( L \). Since any \( \rho \in \mathcal{K} \) satisfies the boundedness condition \((K_2)\), we
conclude that $C$ does not depend on the choice of $\rho \in K$. The estimate (2.36) together with (2.34) from step 1 gives

$$R^{1+\alpha_{\Sigma}}[Du]_{\alpha_{\Sigma};B_{R}(X_0)\cap\Omega} \leq C|\tilde{\varphi}|_\gamma$$

(2.37)

and this implies (2.35).

In case 2, we deal with $u - u(V_i) i \in 1, 2$ and notice that $u - u(V_i)$ satisfies the same linear PDE as $u$ in $\Omega$ and the same linear oblique derivative condition on $\tilde{\Sigma}$. We can follow the proof of Theorem 8.33 of [19], but use the proof of Theorem 6.26 of [19] to handle the oblique derivative condition to obtain the estimate

$$R^{1+\alpha_{\Sigma}}[Du]_{\alpha_{\Sigma};B_{R}(X_0)\cap\Omega} = R^{1+\alpha_{\Sigma}}[D(u - u(V_i))]_{\alpha_{\Sigma};B_{R}(X_0)\cap\Omega} \leq C|u - u(V_i)|_{0;B_{2R}(X_0)\cap\Omega}$$

(2.38)

where $C$ depends on $\alpha_{\Sigma}$ and the obliqueness constant. Therefore, (2.35) holds for the case 2.

In case 3, we apply theorem 8.32 of [19] to $u - u(V_i)$ to show

$$R^{1+\alpha_{\Sigma}}[Du]_{\alpha_{\Sigma};B_{R}(X_0)\cap\Omega} = R^{1+\alpha_{\Sigma}}[D(u - u(V_i))]_{\alpha_{\Sigma};B_{R}(X_0)\cap\Omega} \leq C|u - u(V_i)|_{0;B_{2R}(X_0)\cap\Omega}$$

(2.39)

where $C$ depends on the $\alpha_{\Sigma}$ norms of the coefficients of $L$, $\alpha_{\Sigma}$ seminorms of the coefficients of $\tilde{N}$, and the size of the domain $\Omega$. Hence, (2.35) holds for the case 3.

Cases 1-3 cover all points $X_0 \in \overline{\Omega}/V$. Let $R = \sin(\theta/4)\frac{d\varphi}{2}$, where $\theta$ stands for the corner angle. We multiply (2.35) by $R^{-\gamma}$ and use the interpolation inequalities (6.8) and (6.9) in [19] to get (2.32).

Next, we derive the estimate (2.33) from (2.32).

First, Theorem 1 of [34] implies that there exist positive constants $\gamma_0$ and $a_0$, depending on the size of the angles at $V$ and the ellipticity ratio for the operator $L$,
such that for any \(0 < \gamma < \gamma_0\) and \(1 < a < 1 + a_0\), the solution \(u\) satisfies
\[
|u|_{a}^{(-\gamma)} \leq C(|\tilde{\varphi}|_{\gamma} + |u|_{0})
\]  
(2.40)

Now, fix \(\gamma\) with \(0 < \gamma < \min\{\gamma_0, 1\}\). Since
\[
|u|_{\gamma; \Omega/V} = |u|^{(-\gamma)} \leq C|u|^{(-\gamma)}
\]
for any \(a > \gamma\), from (2.40) we obtain
\[
|u|_{\gamma; \Omega/V} \leq C(|\tilde{\varphi}|_{\gamma} + |u|_{0}) \leq C|\tilde{\varphi}|_{\gamma}
\]  
(2.41)

where the last inequality follows from the \(L^\infty\) estimate from step 1.

Noticing that for each \(i \in \{1, 2\}\) we have
\[
\sup_{X \in \Omega} d_{V_i}(X)|u(X) - u(V_i)| \leq |u|_{\gamma; \Omega/V},
\]  
(2.42)

(2.33) follows from (2.32) and (2.41).

**Step 3.** In this step we use Theorem 1 of [31] to prove there exists a solution of this linear problem. To use this theorem, we need \(\tilde{\Sigma} \in H^{2+\alpha}_{\Sigma}\), however, we only have \(\tilde{\Sigma} \in H^{1+\alpha}_{\Sigma}\). So we approximate \(\Omega\) by a sequence of domains \(\Omega_k\) using a sequence of sufficiently smooth curves \(\tilde{\Sigma}_k\) in \(\Omega\) that approximate \(\tilde{\Sigma}\). To construct such \(\tilde{\Sigma}_k\) we need the following lemma from [31].

**Lemma 2.6.5.** *Regularized Distance Function* If the boundary component \(\Sigma\) is in \(H^{1+\alpha}_{\Sigma}\), then the domain \(\Omega\) has a regularized distance function \(d(X)\) with the following properties:

1. \(d \in C^{2,\alpha}(\Omega) \cap H^{1+\alpha}_{\Sigma}(\overline{\Omega})\),
2. \(1 \leq \frac{d}{d_{\Sigma}} \leq 2\) where \(d_{\Sigma}\) is the distance to \(\Sigma\),
3. $|D^2d| \leq cd_{\Sigma}^{\alpha - 1}$,

4. $|Dd| \geq d_0$ in $\Omega$ for some positive constant $d_0$, and

5. $|Dd| \leq d_1$ in $\Omega$ for some positive constant $d_1$.

We approximate $\Omega$ and $\tilde{\Sigma}$ by

$$\Omega_k = \{X \in \Omega : d > \frac{1}{k}\}, \quad \tilde{\Sigma}_k = \{X \in \Omega : d = \frac{1}{k}\}$$

Let $u_k$ be the solution to the linear problem (2.29) in $\Omega_k$. Then Lemma 2.6.3 gives the bound $m \leq u_k \leq M$ uniformly in $k$.

In each $\Omega_k$, we apply Theorem 1 from [31] to obtain the solution $u_k$ to the linear problem (2.29). From Step 2, we know $u_k$ satisfies (2.32) and (2.33) uniformly in $k$.

Now, let $k \to \infty$, then $\Omega_k \to \Omega$. Since $u_k$ satisfies (2.33) uniformly in $k$, $\{u_k\}$ is uniformly bounded in $H_{1+\alpha\Sigma}^{(-\gamma)}$. By the Arzela-Ascoli Theorem, $\{u_k\}$ contains a subsequence $\{u_{k_j}\}$ that converges to some $u$ in $H_{1+\alpha'}^{(-\gamma')}$ for some $\gamma' < \gamma$ and $\alpha' < \alpha\Sigma$. Since $u_{k_j}$ is the solution to the linear problem (2.29) in $\Omega_{k_j}$ and $u_{k_j} \to u$ in $H_{1+\alpha'}^{(-\gamma')}$, $u \in H_{1+\alpha'}^{(-\gamma')}$ is a weak solution of linearized problem (2.29) in $\Omega$, and Step 2 gives estimate (2.32) and (2.33) for $u$.

Although this construction gives us only a weak solution, we know that since $\hat{u} \in H_{1+\alpha_1}^{(-\gamma_1)}$, $u \in C^2(\Omega)$. Since $L$ and $\tilde{N}$ are linear operators, we can use Lemma 2.6.3 to conclude that the solution $u$ to the linear problem (2.29) is unique.

\[\square\]

2.6.2 The Modified Nonlinear Problem

To show that the modified nonlinear problem (2.26) with fixed boundary has a solution, we use the Leray-Schauder fixed point theorem (Theorem 11.6) from [19]:
Theorem 2.6.6. Let $T$ be a compact mapping of a Banach space $B$ into itself, and suppose that there exists a constant $C$ such that

$$
\|u\|_B \leq C
$$

for all $u \in B$ and $t \in [0,1]$ satisfying $u = tTu$. Then $T$ has a fixed point.

We define $T : H^{(-\gamma_1)}_{1+\alpha_1} \to H^{(-\gamma_1)}_{1+\alpha_1}$ by letting $u = T\hat{u}$ be the unique weak solution of (2.29) that we constructed in Theorem 2.6.4. Noticing that $T(H^{(-\gamma_1)}_{1+\alpha_1}) \subset H^{(-\gamma)}_{1+\alpha_\Sigma}$, we conclude that the operator $T$ is compact if $\gamma_1 < \gamma$ and $\alpha_1 < \alpha_\Sigma$. We can take $\alpha_1 = \frac{\alpha_\Sigma}{2}$ and $\gamma_1 = \frac{\gamma}{2}$.

To use Theorem 2.6.6 to obtain the fixed point, we only need to show that there exists an $C > 0$ such that

$$
\|u\|_B \leq C
$$

for all $u \in B \equiv H^{(-\gamma_1)}_{1+\alpha_1}$ that solve

\begin{align*}
Q^+u & = 0 \quad \text{in } \Omega, \\
N^+u & = 0 \quad \text{on } \Sigma/\Sigma(\delta), \\
u & = t\frac{8}{7}\rho \quad \text{on } \Sigma(\delta) \tag{2.43} \\
u & = 0 \quad \text{on } \sigma_0, \\
u & = t\varphi \quad \text{on } \sigma.
\end{align*}

Now, we will prove the following estimate:

Theorem 2.6.7. Let $\alpha_\Sigma$, $\gamma$, $\alpha_1$ and $\gamma_1$ be the same as in Theorem 2.6.4. Suppose that $u \in H^{(-\gamma_1)}_{1+\alpha_1}$ is a solution of (2.43). Then,

$$
|u|^{(-\gamma)}_{1+\alpha_*} \leq C, \quad \forall \tau \in [0,1]. \tag{2.44}
$$

where $\alpha_* = \min\{\alpha_\Sigma, \gamma\}$ and $C$ depends on the ellipticity ratio, the obliqueness constant, $|\varphi|_\gamma$ and the geometry of $\Omega$. 

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Proof. We prove this theorem in four steps.

Step 1. $L^\infty$ bound for $u$.

Since $u \in H^{1-\gamma_1}_{1+a_1}$ is a solution of (2.43), $u \in C^2(\Omega)$. By Lemma 2.6.3, we obtain the $L^\infty$ bound

$$|u|_0 \leq t|\varphi|_{0,\sigma} \leq |\varphi|_{0,\sigma}. \quad (2.45)$$

Step 2. Estimate for $\sup_{i, X \in \Omega} d_{V_i}(X)|u(X) - u(V_i)|$.

To derive such an estimate, we need to construct a barrier function for each corner.

We define linear operators $\overline{L}$ and $\overline{N}$ as follows:

$$\overline{L}v := \sum_{i,j} a_{ij}(u, \rho, \tau)D_{ij}v + \sum_i b_i(u, \rho, \tau)D_iv, \quad (2.46)$$

and

$$\overline{N}v := \beta(\rho, u) \cdot \nabla v. \quad (2.47)$$

First, we state a lemma which is a immediate consequence of Lemma 4.1 from [34].

**Lemma 2.6.8. (Miller Corner Barrier)** There exist positive constants $h_0$ and $\gamma_0$ which depend on the opening angles at each $V_i$, the ellipticity ratio of $\overline{L}$, and the obliqueness constant of $\overline{N}$, such that for every fixed $\gamma \in (0, \gamma_0)$ there exist a constant $c_1$ and a function $w_1 \in C^2(\overline{\Omega(h_0)} - \mathbf{V}) \cap C(\overline{\Omega(h_0)})$ such that

$$\overline{L}w_1 \leq 0 \quad \text{in } \Omega(h_0),$$

$$c_1 d_{V_i}^\gamma w_1 \leq d_{V_i}^\gamma \quad \text{in } \Omega(h_0),$$

$$\overline{N}w_1 \leq 0 \quad \text{on } \hat{\Sigma}(h_0),$$

where $\Omega(h_0)$ and $\Sigma(h_0)$ are the subsets of $\Omega$ and $\hat{\Sigma}$ where $d_{V_i} \leq h_0$.
Consider the linear problem

\[ \overline{L}w = 0 \quad \text{in } \Omega, \]
\[ \overline{N}w = 0 \quad \text{on } \Sigma / \Sigma(\delta), \]
\[ w = t\varphi \quad \text{on } \tilde{\sigma}. \]

Since \( u \) is a solution to (2.43),

\[ \overline{L}u = Q^{+\varepsilon}u - h'_\varepsilon(u - \rho)u^2 = -h'_\varepsilon u^2 \leq 0 \quad \text{in } \Omega, \]
\[ \overline{N}u = 0 \quad \text{on } \Sigma / \Sigma(\delta), \]
\[ u = t\tilde{\varphi} \quad \text{on } \tilde{\sigma}. \]

Applying the maximum principle and the Hopf lemma to (2.49) and (2.50), we obtain \( w \leq u \).

To construct an upper bound, we consider another linear problem

\[ \overline{L}W = 0 \quad \text{in } \Omega, \]
\[ \overline{N}W = 0 \quad \text{on } \Sigma / \Sigma(\delta), \]
\[ W = \frac{1}{k}(e^{kt\tilde{\varphi}} - 1) \quad \text{on } \tilde{\sigma}. \]

Let \( w_2 = \frac{1}{k}(e^{ku} - 1) \), then

\[ \overline{L}w_2 = e^{ku}(Q^{+\varepsilon}u + (kh'_\varepsilon(u - \rho) + \varepsilon) - h'_\varepsilon(u - \rho))u^2 + ku^2 \]
\[ \geq 0 \]

for \( k \) big enough.

Hence for \( k \) big enough,

\[ \overline{L}w_2 \geq 0 \quad \text{in } \Omega, \]
\[ \overline{N}w_2 = 0 \quad \text{on } \Sigma / \Sigma(\delta), \]
\[ w_2 = \frac{1}{k}(e^{kt\tilde{\varphi}} - 1) \quad \text{on } \tilde{\sigma}. \]

Again, by applying the maximum principle and the Hopf lemma to (2.51) and (2.52), we get \( w_2 \leq W \). However, by the definition of \( w_2 \), it is clear that \( w_2 \geq u \). Therefore, \( u \leq W \).
Now we use Lemma 2.6.8 to construct a subsolution for (2.49) and a supersolution for (2.51). Notice that to get an estimate for \( \sup_{i,X \in \Omega} d_{V_i}^{-\gamma}(X)|u(X) - u(V_i)| \), we only need to construct an appropriate barrier function for each corner and piece them together, so we can assume \( u = 0 \) at each corner by considering \( u - u(V_i) \) instead of \( u \) in a neighborhood of the corner \( V_i \).

Let \( w_1 \) be as in Lemma 2.6.8, then there exists a big enough \( C_1 \) such that

\[
C_1 w_1 + u(V_i) \geq \frac{1}{k}(e^{kt\tilde{\varphi}} - 1), \quad \text{on } \partial \Omega_i(h_0)/\tilde{\Sigma}_i(h_0),
\]

where \( \Omega_i(h_0) \) and \( \tilde{\Sigma}_i(h_0) \) are the subsets of \( \Omega_i(h_0) \) and \( \tilde{\Sigma}_i \) that are close to \( V_i \), and \( C_1 \) depends on \( |u|_0 \), \( |\tilde{\varphi}|_0 \) and \( h_0 \).

Also, Lemma 2.6.8 gives

\[
\overline{L}(C_1 w_1 + u(V_i)) \leq 0 \quad \text{in } \Omega(h_0),
\]
\[
\overline{N}(C_1 w_1 + u(V_i)) \leq 0 \quad \text{on } \tilde{\Sigma}(h_0).
\]

So by the maximum principle and the Hopf lemma, we get \( C_1 w_1 + u(V_i) \geq W \). Therefore, \( C_1 w_1 + u(V_i) \geq u \), i.e. \( u(X) - u(V_i) \leq C_1 w_1 \leq C_1 d_{V_i}^{-\gamma}(X) \). Hence, \( d^{-\gamma}_{V_i}(X)(u(X) - u(V_i)) \leq C_1 \).

If we apply the same argument again to \( w \), we get \( d^{-\gamma}_{V_i}(X)(u(X) - u(V_i)) \geq -C_2 \). Therefore

\[
\sup_{i,X \in \Omega} d^{-\gamma}_{V_i}(X)|u(X) - u(V_i)| \leq C
\]

for some \( C \) depending on the opening angles at each \( V_i \), on the ellipticity ratio of \( \overline{L} \), on the obliqueness constant of \( \overline{N} \) and on \( |\varphi|_0 \).

**Step 3.** Estimate for \( |u|_{\varepsilon^*} \).

In this step, we show that

\[
|u|_{\varepsilon^*} \leq C
\]
for some $\varepsilon^*$ and $C$ that depend on the geometry of $\Omega$, the ellipticity ratio of $Q^{+,*}$, the obliqueness constant of $N^+$, $\varepsilon$ and $|\varphi|_0$.

First, we use Theorem 2.3 of [36] to establish the Hölder seminorm estimate in a neighborhood of $\Sigma/\Sigma(\delta)$.

To use this theorem, we need to check whether $u$ satisfies the first condition in (2.10) of this theorem which requires that $|\sum_{i,j} a_{ij} D_{ij} u| \leq \lambda (\mu_0 |u_\rho|^2 + \phi_0)$ for some nonnegative constants $\mu_0$ and $\phi_0$.

Notice that any solution $u$ to (2.43) satisfies

$$|(h_\varepsilon(u - \rho) + \varepsilon) u_{pp} + u_{\tau\tau}| \leq |h'_{\varepsilon}(u - \rho)||u_\rho|^2 + \frac{1}{2} - h'_{\varepsilon}(u - \rho)||u_\rho|$$

$$\leq |u_\rho|^2 + \frac{1}{2}|u_\rho|$$

$$\leq \frac{3}{2} |u_\rho|^2 + \frac{1}{2}$$

$$\leq \lambda \left( \frac{3}{2 \min \{1, \varepsilon\}} |u_\rho|^2 + \frac{1}{2 \min \{1, \varepsilon\}} \right),$$

so the condition in Theorem 2.3 of [36] is satisfied. Therefore,

$$[u]_{\alpha_1} \leq C$$

(2.56)

where $\alpha_1$ and $C$ depend on the ellipticity of $Q^{+,*}$, the oblique constant for $N^+$, and $\varepsilon$. Also $C$ depends on $\Omega$.

Second, on $\hat{\sigma} = \sigma \cap \sigma_0 \cap \Sigma(\delta)/\{V_1, V_2\}$,

$$[u]_\gamma \leq [\hat{\varphi}]_\gamma$$

(2.57)

where $\hat{\varphi} = t \frac{\rho}{2} \rho$ on $\Sigma_\delta$, $\hat{\varphi} = 0$ on $\sigma_0$, and $\hat{\varphi} = t \varphi$ on $\sigma$.

Finally, on the corners $V_1$ and $V_2$, the estimate from step 2 implies $[u]_\gamma \leq C$. 

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Let $\alpha_0 = \min\{\alpha_1, \gamma\}$. Combining those three estimates on the boundary, we obtain
\[ \text{osc}_{\partial\Omega \cap B_R(x_0)} u \leq KR^{\alpha_0} \quad \forall x_0 \in \partial\Omega \text{ and } R > 0, \] (2.58)
for some $K > 0$.

With (2.58), Theorem 8.29 of [19] (the Hölder estimate on the boundary implies a global Hölder estimate) implies that there exist $\varepsilon^*$ and $C$ that depend on the ellipticity ratio of $Q^{+,\varepsilon}$, $\varepsilon$, and $\alpha_0$ and $C$ also depends on $|\hat{u}|_{\gamma}$ so that (2.55) holds.

**Step 4.** With the estimate (2.55), we can treat (2.43) as linear PDE with coefficients depending on $u$. Applying Theorem 2.6.4 to (2.43) with $\alpha_\Sigma$ taken to be $\min\{\alpha_\Sigma, \varepsilon^*, \gamma\}$, we get the estimate (2.32) for the problem (2.43):
\[ |u|_{1+\min\{\alpha_\Sigma, \varepsilon^*, \gamma\}}^{(-\gamma)} \leq C(|\tilde{\varphi}|_\gamma + \sup_{i,X \in \Omega} d_{\tilde{\gamma}}^{-\gamma}(X)|u(X) - u(V_i)|). \]
Applying (2.54) in step 2 to the above estimate, we get
\[ |u|_{1+\alpha_\ast}^{(-\gamma)} \leq C \] (2.59)
where $\alpha_\ast = \min\{\alpha_\Sigma, \varepsilon^*, \gamma\}$, $C$ depends on the ellipticity ratio of $Q^{+,\varepsilon}$, the obliqueness constant of $N^+$, $\alpha_\Sigma$, $|\hat{u}|_{\alpha_\Sigma}$, $|\beta^+|_{\alpha_\Sigma}$, $|\tilde{\Sigma}|_{1+\alpha_\Sigma}$, $|\varphi|_\gamma$, and the size of $\Omega$.

We realize that (2.59) implies that $|u|_\gamma \leq C$, so we can replace the $|u|_{\varepsilon^*}$ estimate from step 3 by this $|u|_\gamma$ estimate and repeat step 4 to get rid of the $\varepsilon^*$ dependence in $\alpha_\ast$. \hfill $\square$

Provided $\alpha_\Sigma \leq 2\gamma$, inequality (2.44) implies the bound in Theorem 2.6.6. Hence, $T$ has a fixed point $u$, which in fact lies in $H_{1+\alpha_\ast}^{(-\gamma)}$. This proves the following result:

**Theorem 2.6.9.** Let $\Omega$ be a domain with smooth boundary component $\sigma$ and $\Sigma \in K$.

The modified nonlinear problem (2.28) has a solution $u \in H_{1+\alpha_\ast}^{(-\gamma)}$ where $\gamma$ is determined by the corner angles, and $\alpha_\ast = \min\{\alpha, \gamma\}$. 

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Remark 2.6.10. As was the case in the linear problem, \( u \in C^2(\Omega) \).

### 2.6.3 Removal of the Cut-off Functions \( f \) and \( h_\varepsilon \)

#### Lemma 2.6.11. The solution of (2.28) satisfies \( u \geq \frac{8}{7}\rho \) in \( \overline{\Omega} \) independent of \( \varepsilon \) and \( \delta \).

**Proof.** Let \( \zeta = u - \frac{8}{7}\rho \). We want to prove \( \zeta \geq 0 \).

Since \( u \) is a solution of (2.28),

\[
(h_\varepsilon(u - \rho) + \varepsilon)\zeta_{\rho\rho} + \zeta_{\tau\tau} + \left( \frac{9}{7} h_\varepsilon'(u - \rho) + \frac{1}{2} \right)\zeta_\rho = -\frac{4}{7} - \frac{8}{49} h_\varepsilon'(u - \rho) - h_\varepsilon'(u - \rho)\zeta_\rho^2 < 0.
\]

So by the weak minimum principle, \( \zeta \) does not obtain a minimum in the interior.

If \( \zeta \) obtains a negative minimum at \( \Xi_0 \in \Sigma/\Sigma(\delta) \), then at \( \Xi_0 \), \( u < \frac{8}{7}\rho \), therefore

\[
\beta_1^+ = (2\rho - \frac{7}{4} f(u))\rho' > (2\rho - \frac{7}{4} \frac{8}{7}\rho)\rho' > 0.
\]

Hence,

\[
\beta_1^+ \zeta_\rho + \beta_2^+ \zeta_\rho = -\frac{8}{7} \beta_1^+ < 0
\]

and this contradicts the Hopf’s lemma. Therefore, \( \zeta \) cannot obtain a negative minimum on \( \Sigma/\Sigma(\delta) \).

On the other hand, on the Dirichlet boundary \( \sigma \cup \sigma_0 \cup \Sigma(\delta) \), \( \zeta = u - \frac{8}{7}\rho \geq 0 \).

Hence \( \zeta \geq 0 \) and \( u \geq \frac{8}{7}\rho \) in \( \overline{\Omega} \).

**Lemma 2.6.12.** The solution of (2.28) satisfies \( u \geq (1 - \varepsilon_\varepsilon)\rho \) in \( \overline{\Omega} \cap \{ \rho \leq 0 \} \) independent of \( \varepsilon \) and \( \delta \).
Proof. Let \( \zeta = u - (1 - \varepsilon_e)\rho \). For \( \varepsilon_e > 0 \) small enough so that \((1 - \varepsilon_e) - (1 - \varepsilon_e)^2 > -\frac{1}{2}\), \( \zeta \) satisfies

\[
(h_e(u - \rho) + \varepsilon)\zeta_{\rho\rho} + \zeta_{\tau\tau} + \left(\frac{9}{7}h'_e(u - \rho) + \frac{1}{2}\right)\zeta_{\rho} = -\frac{1}{2} - h'_e(u - \rho)((1 - \varepsilon_e) - (1 - \varepsilon_e)^2) - h'_e(u - \rho)\zeta^2_{\rho} < 0
\]

Applying the weak minimum principle to \( \zeta \) in \( \overline{\Omega} \cap \{\rho \leq 0\} \), we conclude that \( \zeta \) obtains its minimum on the boundary.

By Lemma 2.6.11, \( u \geq \frac{8}{7}\rho \) in \( \overline{\Omega} \). Therefore, \( u \geq \frac{8}{7}\rho = 0 = (1 - \varepsilon_e)\rho \) i.e. \( \zeta \geq 0 \) on \( \overline{\Omega} \cap \{\rho = 0\} \). And \( u = \varphi \geq (1 - \varepsilon_e)\rho \) on \( \sigma \cap \{\rho \leq 0\} \) because of the choice of \( \varphi \).

Therefore, \( \zeta \geq 0 \) and \( u \geq (1 - \varepsilon_e)\rho \) in \( \overline{\Omega} \cap \{\rho \leq 0\} \).

Combining Lemma 2.6.11 and Lemma 2.6.12, we get the following corollary.

**Corollary 2.6.13.** The solution of (2.28) satisfies \( u \geq \rho \) in \( \overline{\Omega} \) independent of \( \varepsilon \) and \( \delta \).

Lemma 2.6.11 implies that \( f \) is actually the identity function, so we can remove \( f \). And Corollary 2.6.13 implies that \( h_e \) is also the identity function, so we can remove \( h_e \) as well. Therefore, we have the following theorem
Theorem 2.6.14. For any fixed domain $\Omega$ with $\rho \in K$ there exists a solution $u^{\varepsilon, \delta} \in H_{1+\alpha_*}^{(\gamma)}$ of

$$((u - \rho + \varepsilon)u_\rho + \frac{u}{2})_\rho + u_{\tau\tau} = 0 \quad \text{in } \Omega,$$

$$\beta \cdot \nabla u = 0 \quad \text{on } \Sigma/\Sigma(\delta),$$

$$u = \frac{8}{7}\rho \quad \text{on } \Sigma(\delta)$$

(2.60)

$$u = 0 \quad \text{on } \sigma_0,$$

$$u = \varphi \quad \text{on } \sigma.$$ (2.61)

2.6.4 The Limiting Solution

Lemma 2.6.11 and Lemma 2.6.12 provide strict ellipticity uniformly in $\varepsilon$ and $\delta$ away from the $\tau$-axis. To obtain strict ellipticity uniformly in $\varepsilon$ and $\delta$ away from $\sigma_0$, we need to construct a local lower barrier independent of $\varepsilon$ and $\delta$ for all points on $T^+ = \overline{\Omega} \cap \{(\rho, \tau)|\rho = 0, \tau > 0\}$.

Lemma 2.6.15. For any $X_1 \in T^+$, there exists a $R > 0$ and a $\delta_1 > 0$ independent of $\delta$ such that

$$u - \rho \geq \phi := \delta_1(R^2 - |X - X_1|^2) \quad \text{in } B_R(X_1) \cap \overline{\Omega}.$$ (2.62)

Proof. We first restrict $R$ so that $0 < R < 1$ and we will choose $R$ more precisely later. Let $w := u - \rho$, and define $Lv := (w + \varepsilon)v_\rho + v_{\tau\tau} + v^2 + \frac{3}{2}v_\rho + \frac{1}{2}$, then

$$Lw = 0.$$ (2.63)

Apply $L$ to $\phi$, we get

$$L\phi = -2\delta_1(w + \varepsilon) - 2\delta_1 + (-2\delta_1(\rho - \rho_1))^2 + \frac{3}{2}(-2\delta_1(\rho - \rho_1)) + \frac{1}{2}$$

$$\geq -2\delta_1(M + \varepsilon) - 2\delta_1 + \frac{3}{2}(-2\delta_1) + \frac{1}{2} > 0$$

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for sufficiently small $\delta_1 > 0$ (notice that $M$ is the upper bound for $w$ in $\Omega$ and it is independent of $\varepsilon$ and $\delta$).

Hence,

$$0 > Lw - L\phi$$

$$= (w + \varepsilon)(w - \phi)_{\rho \rho} + (w - \phi)_{\tau \tau} + (w_{\rho} + \phi_{\rho} + \frac{3}{2})(w - \phi)_{\rho}$$

$$:= \bar{L}(w - \phi).$$

Since $w = u - \rho \geq 0$ in $\Omega$, $\bar{L}$ can be viewed as a linear elliptic operator and we have $\bar{L}(w - \phi) < 0$.

Now we pick the $R$ and prove Lemma 2.6.15 in 2 cases.

**Case 1.** If $X_1 \in T^+ / \partial \Omega$, we take $R > 0$ small enough so that $B_R(X_1) \subset \Omega$.

Since $\bar{L}(w - \phi) < 0$ in $B_R(X_1)$, by the maximum principle $\sup_{B_R(X_1)} \{w - \phi\} = \sup_{\partial B_R(X_1)} \{w - \phi\}$. However, $w \geq 0 = \phi$ on $\partial B_R(X_1)$, so we must have $w \geq \phi$.

**Case 2.** If $X_1 = T^+ \cap \partial \Omega$, we take $R > 0$ small enough so that $B_R(X_1) \cap \Sigma \cup \sigma_0 = \emptyset$.

The proof is similar to case 1, the only difference is that we have to show that $w \geq \phi$ on $\overline{B_R(X_1)} \cap \sigma$. This is true for small enough $R$ and $\delta_1$ because of the choice we make of the boundary data $\varphi$ on $\sigma$.

Now we can prove the theorem about the limiting solution.
Theorem 2.6.16. There exists a limiting solution \( u \in C^{2,\alpha}(\Omega/\sigma_0) \cap C^{1,\alpha}(\Omega/\{\sigma_0, V_1\}) \cap C^\gamma(\Omega/\sigma_0) \cap C(\Omega) \) of \( u^{\varepsilon,\delta} \) for some \( 0 < \alpha < 1 \) and \( 0 < \gamma' < 1 \) such that
\[
\begin{align*}
((u - \rho)u_\rho + \frac{u}{2})_\rho + u_{rr} &= 0 \quad \text{in } \Omega, \\
\beta \cdot \nabla u &= 0 \quad \text{on } \Sigma, \\
u = \varphi \quad &\text{on } \sigma, \\
u = 0 \quad &\text{on } \sigma_0.
\end{align*}
\]
Furthermore, \( u \) satisfies \( u \geq \frac{8}{7}\rho \) in \( \Omega \).

Proof. We organize the proof in 4 parts.

**Part 1.** Convergence of \( u^{\varepsilon,\delta} \) in \( \Omega/\sigma_0 \cup \Sigma \).

Fix \( \Omega_1 \), a compact subset of \( \Omega/\sigma_0 \cup \Sigma \). By Lemma 2.6.11, Lemma 2.6.12 and Lemma 2.6.15, we have a uniform bound on the ellipticity ratio which is independent of \( \varepsilon \) and \( \delta \) in \( \Omega_1 \). Hence, by the same argument as in Theorem 2.6.7, we get \( |u|_{1+\alpha} \leq C \) where \( C \) is independent of \( \varepsilon \) and \( \delta \). With the \( C^{1,\alpha_*} \) estimate, we treat our PDE like a linear PDE, and then use Theorem 6.2 and Lemma 6.5 in [19] to improve the \( C^{1,\alpha_*} \) estimate to get \( |u|_{2+\alpha_*} \leq C \) for a \( C \) independent of \( \varepsilon \) and \( \delta \). By the Arzela-Ascoli theorem, there exists a subsequence of \( \{u^{\varepsilon,\delta}\} \) which is convergent in \( C^{2,\alpha}(\Omega_1) \) for some \( \alpha < \alpha_* \).

Now let \( \Omega_1 \) vary and use a diagonalization argument to obtain a further subsequence of \( \{u^{\varepsilon,\delta}\} \) which converges in \( C^{2,\alpha}_{loc}(\Omega/\sigma_0 \cup \Sigma) \) to a limit \( u \in C^{2,\alpha}(\Omega/\sigma_0 \cup \Sigma) \). And \( u \) obviously satisfies
\[
\begin{align*}
((u - \rho)u_\rho + \frac{u}{2})_\rho + u_{rr} &= 0 \quad \text{in } \Omega, \\
u = \varphi \quad &\text{on } \sigma/V_1.
\end{align*}
\]

**Part 2.** Convergence of \( u^{\varepsilon,\delta} \) up to \( \Sigma \).
For any $X \in \Sigma$ there exists an $r > 0$ such that $B_r(X) \cap \Sigma \subset \Sigma/\Sigma(\delta)$ for all $\delta > 0$ small enough. Fix $r$ small so that $B_r(X) \cap \Sigma \subset \Sigma/\Sigma(\delta)$ for all $\delta > 0$ small enough and $B_r(X) \cap \sigma_0 = \emptyset$. By Lemma 2.6.11, Lemma 2.6.12 and Lemma 2.6.15, we have uniform bounds on the ellipticity ratio and obliqueness constant which are independent of $\varepsilon$ and $\delta$ in $B_r(X) \cap \Omega$. Hence, by the same argument as in Theorem 2.6.7, we get $|u|_{1+\alpha_s} \leq C$ in $B_r(X) \cap \Omega$ where $C$ is independent of $\varepsilon$ and $\delta$. Therefore, $|u|_{1+\alpha} \leq C$ in $B_{\frac{1}{2}r}(X) \cap \Omega$ where $C$ is independent of $\varepsilon$ and $\delta$. By the Arzela-Ascoli theorem, there exists a subsequence of $\{u^{\varepsilon,\delta}\}$ which is convergent in $C^{1,\alpha}(B_{\frac{1}{2}r}(X) \cap \Omega)$ for some $\alpha < \alpha_s$.

Now, take a sequence $\{\delta_n\}$ such that $\delta_n \to 0$ as $n \to \infty$. For each fixed $\delta_n$, for any fixed compact subset of $\Sigma/\Sigma(\delta_n)$, we use a finite number of such $B_{\frac{1}{2}r}(X)$ to cover it. Using a diagonalization argument, we obtain a subsequence of $\{u^{\varepsilon,\delta}\}$ (when we take the subsequence here, we take the subsequence of the subsequence in step 1) which converges in $C^{1,\alpha}$ in a neighborhood of $\Sigma/\Sigma(\delta_n)$ to $u \in C^{1,\alpha}$ in the same neighborhood of $\Sigma/\Sigma(\delta_n)$. Letting $n$ vary, another diagonalization argument will give a further convergent subsequence that converges to $u \in C^{2,\alpha}(\overline{\Omega}/\sigma_0 \cup \Sigma) \cap C^{1,\alpha}(\overline{\Omega}/\{\sigma_0, V_1\})$ which satisfies

$$\beta \cdot \nabla u = 0 \quad \text{on } \Sigma$$

in addition to (2.64).

**Part 3.** Convergence of $u^{\varepsilon,\delta}$ up to $V_1$.

As in Part 2, we can obtain $|u|_\gamma \leq |u|_{1+\alpha_s}^{(\gamma)} \leq C$ independent of $\varepsilon$ and $\delta$. By the Arzela-Ascoli theorem, we can obtain a further subsequence of the subsequence from step 2 which converges in $C^{\gamma'}$ to $u \in C^{\gamma'}$ in a neighborhood of $V_1$ for any positive $\gamma' < \gamma$. 

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Part 4. Continuity of $u$ up to $\sigma_0$.

To prove $u$ is continuous up to $\sigma_0$, we need to construct a barrier function $\phi$ independent of $\varepsilon$ and $\delta$, such that $\phi = 0$ on $\sigma_0$ and $u^{\varepsilon, \delta} - \rho \leq \phi$ in a neighborhood of $\sigma_0$.

Define $w^{\varepsilon, \delta} = u^{\varepsilon, \delta} - \rho$. Because $u^{\varepsilon, \delta}$ satisfies (2.60), $w^{\varepsilon, \delta}$ satisfies

\[(w + \varepsilon)w_{\rho\rho} + w_{rr} + w^2_\rho + \frac{3}{2}w_\rho + \frac{1}{2} = 0. \tag{2.66}\]

To simplify the notation, from now on we will omit the superscript $\varepsilon$ and $\delta$.

We introduce a polar coordinate centered at the origin as shown in figure 2.4.

\[ \text{Figure 2.4: Polar coordinate centered at origin} \]

Now we construct the barrier function $\phi = \phi(\theta)$ such that $\phi \geq 0$, $\phi' \geq 0$, $\phi'' \leq 0$ on $[0, \theta_0]$ and $\phi(0) = 0$, $\phi(\theta_0) = M_0 := \max \limits_{\Omega} w$ for some small $\theta_0$ to be chosen later.
\[(\phi + \varepsilon)\phi_{\rho\rho} + \phi_{\tau\tau} + \phi_{\rho}^2 + \frac{3}{2}\phi_{\rho} + \frac{1}{2}\]
\[= (\phi + \varepsilon)(\frac{1}{r^2} \cos^2 \theta \phi'' - \frac{1}{r^2} \sin \theta \cos \phi') + (\frac{1}{r^2} \sin^2 \theta \phi'' + \frac{1}{r^2} \sin \theta \cos \phi')
+ \frac{1}{r^2} \cos^2 \theta (\phi')^2 - \frac{3}{2} \frac{1}{r} \cos \theta \phi' + \frac{1}{2}
\leq \frac{1}{r^2} ((\phi \phi'' + (\phi')^2) \cos^2 \theta + \theta \phi') + \frac{1}{2}\]

(2.67)

Take \(\phi(\theta) = \sqrt{-a\theta^2 + b\theta}\), where \(a\) and \(b\) are positive constants to be chosen later.

Then
\[\phi \phi'' + (\phi')^2 = -a.\]

Let \(\theta_0 \leq \frac{\pi}{4}\), then
\[(\phi \phi'' + (\phi')^2) \cos^2 \theta + \theta \phi'
\leq -\frac{a}{2} + \frac{1}{2} \sqrt{-a\theta^2 + b\theta}
\leq \frac{1}{2}(M_0 - a)
\leq -\frac{a}{4}\]
for big enough \(a\). Let \(R = \max \{r(X) | X \in \overline{\Omega} \text{ and } \theta(X) \in [0, \frac{\pi}{2}]\}\), then from (2.67) we obtain
\[(\phi + \varepsilon)\phi_{\rho\rho} + \phi_{\tau\tau} + \phi_{\rho}^2 + \frac{3}{2}\phi_{\rho} + \frac{1}{2}\]
\[\leq -\frac{a}{4r^2} + \frac{1}{2}\]
\[\leq -\frac{a}{4R^2} + \frac{1}{2}\]
\[< 0\]
(2.68)

for large enough \(a\). Now pick \(b\) such that \(\phi(\theta_0) = \sqrt{-a\theta_0^2 + b\theta_0} = M_0\).

Subtracting (2.66) from (2.68), we obtain
\[(w + \varepsilon)(\phi - w)_{\rho\rho} + (\phi - w)_{\tau\tau} + (\phi_{\rho} + w_{\rho} + \frac{3}{2})(\phi - w)_{\rho} + \phi_{\rho\rho}(\phi - w) < 0.\]
(2.69)

Let \(\Omega_0 = \{X \in \Omega | \theta \in [0, \theta_0]\}\). First, we want to show that \(\phi - w\) can not achieve a negative local minimum in the interior of \(\Omega_0\). Suppose this is not true, then there
exists $X_0 \in \Omega_0$ where $\phi - w$ achieves a negative local minimum. Therefore, $(\phi - w)(X_0) < 0$ and $(\phi - w)\rho(X_0) = 0$. Also, noticing that $\phi_{\rho\rho} \leq 0$, we obtain from (2.69) that
\[(w + \varepsilon)(\phi - w)_{\rho\rho} + (\phi - w)_{\tau\tau} < 0.\] (2.70)
Since $w \geq 0$, (2.70) contradicts the convexity condition at the local minimum $X_0$. Therefore, $\phi - w$ can not achieve a negative local minimum in the interior of $\Omega_0$.

Now let’s look at $\phi - w$ on $\partial\Omega_0$ in three cases.

1. On $\sigma_0 \cap \partial\Omega_0$, $\phi - w = 0$.

2. On $\{\theta = \theta_0\} \cap \partial\Omega_0$, $\phi = \max_{\Omega} w$, therefore, $\phi - w \geq 0$.

3. On $\sigma \cap \partial\Omega_0$, $w = \varphi - \rho \leq -C\rho$ for some positive constant $C$ and $\phi \sim C_0\sqrt{-\rho}$ for small $\theta_0$. Hence, we can pick $\theta_0$ small enough so that $\phi \geq w$ on $\sigma \cap \partial\Omega_0$.

Therefore, $\phi - w \geq 0$ on $\partial\Omega_0$. Combining this with the fact that $\phi - w$ can not achieve a negative local minimum in the interior of $\Omega_0$, we have proved that $\phi - w \geq 0$ in $\overline{\Omega}_0$. Noticing that we suppressed the superscript $\varepsilon$ and $\delta$ on $w$, we have actually obtained $w^{\varepsilon,\delta} \leq \phi$, i.e., $u^{\varepsilon,\delta} \leq \phi + \rho$ independent of $\varepsilon$ and $\delta$. Since we also proved $u^{\varepsilon,\delta} \geq \rho$ independent of $\varepsilon$ and $\delta$, we have $\rho \leq u^{\varepsilon,\delta} \leq \phi + \rho$ independent of $\varepsilon$ and $\delta$. Therefore, the limiting solution $u$ satisfies $\rho \leq u \leq \phi + \rho$ in $\overline{\Omega}_0$. Since $\rho = \phi + \rho = 0$ on $\sigma_0$, we have proved that $u$ is continuous up to $\sigma_0$ and $u = 0$ on $\sigma_0$.

Finally, the estimate $u \geq \frac{2}{7}\rho$ follows from the estimate for $u^{\varepsilon,\delta}$ in Lemma 2.6.11.

\[\square\]

2.7 The Free Boundary Problem

For $\rho \in K$, we let
\[J\rho = \tilde{\rho}\] (2.71)
be a new approximation to the free boundary defined by
\[ \tilde{\rho} = \int_0^\tau \sqrt{\rho(t) - \frac{1}{2} g(u(\rho(t), t))} dt, \] (2.72)
where \( u \) is the solution of (2.63) we found in theorem 2.6.16 with \( \Sigma = \{ (\rho, \tau) | \rho = \rho(\tau) \} \).

**Proposition 2.7.1.** \( J \) maps \( K \) into \( K \).

**Proof.** Since we have \( u \in C^{1,\alpha}(\Sigma) \) from Theorem 2.6.16 and \( \rho \in H^{1+\alpha}(\Sigma) \), by the definition of \( J\rho = \tilde{\rho} \) in (2.72), we conclude that \( J\rho \in H^{1+\alpha}(\Sigma) \).

Now we verify that \( J\rho \) satisfies conditions (K1)-(K3).

**Part 1.** By the definition (2.72) of \( \tilde{\rho} \), it is obvious that \( \tilde{\rho}(0) = 0 \).

Because \( \rho(0) = 0 \) and \( u(0, 0) = 0 \),
\[ \tilde{\rho}'(0) = \sqrt{\rho(0) - \frac{1}{2} g(u(\rho(0), 0))} = 0. \] (2.73)
Hence \( J\rho \) satisfies (K1).

**Part 2.** Since we have estimate \( u \geq \frac{8}{7} \rho \) and \( \rho \) satisfies (K2) \( \rho \leq \frac{3}{28} \tau^2 \),
\[ \tilde{\rho}'(\tau) = \sqrt{\rho(\tau) - \frac{1}{2} g(u(\rho(\tau), \tau))} \]
\[ \leq \sqrt{\rho(\tau) - \frac{18}{25} \rho(\tau)} \]
\[ \leq \sqrt{\frac{3}{7} \frac{3}{28} \tau^2} \]
\[ \leq \frac{3}{14} \tau. \] (2.74)
Due to the cut-off function \( g \),
\[ \tilde{\rho}'(\tau) = \sqrt{\rho(\tau) - \frac{1}{2} g(u(\rho(\tau), \tau))} \]
\[ \geq \sqrt{\rho(\tau) - \frac{1}{2} 2(1 - 2\varepsilon_0) \rho} \]
\[ \geq \sqrt{2\varepsilon_0 \rho} \]
\[ \geq \sqrt{2\varepsilon_0 \frac{\varepsilon_0}{2} \tau^2} \]
\[ \geq \varepsilon_0 \tau. \] (2.75)
Hence $J\rho$ satisfies (K3).

**Part 3.** Because $J\rho(0) = 0$ and $J\rho$ satisfies (K3), $J\rho$ satisfies (K2) as well.

To sum up, we have proved $J$ maps $\mathcal{K}$ into $\mathcal{K}$.

Since $\mathcal{K}$ is a closed convex subset of the Banach space $H_{1+\alpha\Sigma}$ and $J$ is compact because of the definition of $J$ and Theorem 2.6.16 (the smoothness is increased after map $J$), we can show that there is a fixed point for $J$ using Theorem 2.4.1 (the Schauder fixed point theorem).

Combining this with Theorem 2.6.16, we have proved our main theorem

**Theorem 2.7.2.** There exist a $u \in C^{2,\alpha}((\overline{\Omega}/\overline{\sigma_0} \cup \Sigma)) \cap C^{4,\alpha}((\overline{\Omega}/\overline{\{\sigma_0, V_1\}}) \cap C^{\gamma}((\overline{\Omega}/\overline{\sigma_0}) \cap C(\overline{\Omega})$ for some $0 < \alpha < 1$ and $0 < \gamma < 1$ and a $\rho(\tau) \in \mathcal{K}$ such that

\[
(u - \rho)u_\rho + \frac{u}{2} u_\tau + u_{\tau\tau} = 0 \quad \text{in} \ \Omega, \\
\beta \cdot \nabla u = 0 \quad \text{on} \ \Sigma, \\
u = \varphi \quad \text{on} \ \sigma, \\
u = 0 \quad \text{on} \ \sigma_0.
\]

where $\Sigma = \{(\rho, \tau)|\rho = \rho(\tau)\}$, and

\[
\rho'(\tau) = \sqrt{\rho(\tau) - \frac{1}{2} g(u(\rho(\tau), \tau))}, \quad \rho(0) = 0.
\]
Chapter 3: Diffraction of a Shock Wave at a 90 Degree Corner

3.1 Introduction

Unsteady interactions of shock waves take place when a blast wave impinges upon an aircraft. The shock patterns are often very complex, but can be broken down locally into two basic elements: interaction–reflection at a concave corner between two plane surfaces and diffraction at a convex corner.

The study of the problem concerning shock diffraction at a convex corner dates back to late 40s. Lighthill studied diffraction at small angles by linearising the differential equations of gas dynamics in his papers [37] and [38]. In Lighthill’s papers, he showed that without viscosity the diffraction process is self-similar and he also predicted the distribution of pressure on the wedge. Other authors such as Bargmann [2], and Fletcher, Weimer, and Bleakney [18] have also studied this problem with different approaches. However, rigorous proof of the existence of a solution to this problem has not been completed.

Kim first proved the existence of solution for the nonlinear wave system (NLWS) to this problem at a 90 degree corner in [27]. However, in her paper, she made the assumption that the shock strength does not become 0 at the point where the shock hits the wall (one consequence of this is that the shock does not land on the sonic line
when it hits the wall). Later, Chen, Deng and Xiang proved the same result without this requirement, also for the NLWS, Kim further proved a similar result to her paper [27], but with a rarefaction wave in the configuration to force the shock to be on the sonic line when it hits the wall. With a different model, potential flow, Chen and Xiang obtained the existence of a solution for the problem of shock diffraction by a convex cornered wedge in [15]. All these results provide self-similar solutions to this problem which are consistent with Lighthill’s work. However, because both the NLWS and the potential flow are much simpler than the Euler equations, none of the solutions mentioned above capture the complicated structure inside the subsonic region.

Experiments (for example, see Van Dyke’s book [42] page 148) and numerical simulations for the Euler equations (for example, see Ripley, Lien and Yovanovich’s paper [40]) show that there is more detailed structure inside the subsonic region for the full Euler equations and this cannot be understood by examining simpler models such as the NLWS and the potential flow.

To better understand the diffraction of a shock wave at corner, we look at shock wave diffraction at a 90 degree corner using the isentropic Euler equations

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0
\end{align*}
\]

(3.1)

where the pressure \( p \) satisfies the \( \gamma \) law

\[ p = A \rho^\gamma \]

for some constant \( A > 0 \) and \( \gamma > 1 \).
Notice that in the region where $\rho$, $u$, $v$ are all $C^1$, we can use first equation in (3.1) to eliminate $\rho_t$ in second and third equations to get

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
u_t + uu_x + vu_y + \frac{p}{\rho} &= 0 \\
v_t + uv_x + vv_y + \frac{p}{\rho} &= 0
\end{align*}
\] (3.2)

The method used in Chapter 2 should also work for the isentropic Euler equations. However, due to the complicated structure of the isentropic Euler equations, there are many difficulties in implementing this method to obtain an existence result.

We will first set up the problem in a frame similar to Chapter 2 in Section 3.3. Then we will discuss difficulties in implementing the method mentioned in Chapter 2 in Section 3.4. Finally, we will present some results that could help us to overcome those difficulties in Section 3.5.

### 3.2 Problem Setting

Let us consider a shock moving off a step to the right with left state $(\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0)$ as shown in figure 3.1.

Since both the isentropic Euler equations and the initial condition are scale invariant, we seek a self-similar solution to the isentropic Euler equations.

Let $\xi = \frac{x}{\tau}$ and $\eta = \frac{y}{\tau}$, the isentropic Euler equations in self-similar coordinates $(\xi, \eta)$ become

\[
\begin{align*}
(\rho \tilde{u})_\xi + (\rho \tilde{v})_\eta + 2\rho &= 0 \\
(\rho \tilde{u}^2 + p)_\xi + (\rho \tilde{u} \tilde{v})_\eta + 3\rho \tilde{u} &= 0 \\
(\rho \tilde{v})_\xi + (\rho \tilde{v}^2 + p)_\eta + 3\rho \tilde{v} &= 0
\end{align*}
\] (3.3)

where $(\tilde{u}, \tilde{v}) = (u - \xi, v - \eta)$ is the pseudo-velocity.

The shock diffraction problem in the $\xi - \eta$ plane is shown in Figure 3.2 ($S_0$ and $S_1$ are the sonic lines for state 0 and state 1 respectively).
Figure 3.1: A shock moves off a step to the right.

Figure 3.2: Shock diffraction at a step.

On the shock, the Rankine-Hugoniot relation must hold

\[
\begin{align*}
\frac{d\eta}{d\xi} [\rho \tilde{u}] &= [\rho \tilde{v}] \\
\frac{d\eta}{d\xi} [\rho \tilde{u}^2 + p] &= [\rho \tilde{u} \tilde{v}] \\
\frac{d\eta}{d\xi} [\rho \tilde{u} \tilde{v}] &= [\rho \tilde{v}^2 + p]
\end{align*}
\] (3.4)
Since the shock is vertical before it interacts with the sonic line, the Rankine-Hugoniot condition (3.4) requires that
\[
\begin{align*}
\begin{cases}
[\rho \tilde{u}] = 0 \\
[\rho \tilde{u}^2 + p] = 0 \\
[\rho \tilde{u} \tilde{v}] = 0
\end{cases}
\end{align*}
\] (3.5)

Hence, we must have
\[
u_1 = \sqrt{\frac{[p][\rho]}{\rho_1 \rho_0}}
\] (3.6)
and the shock is at
\[
\xi = \xi_\alpha = \sqrt{\frac{\rho_1 [\rho]}{\rho_0 [p]}},
\] (3.7)
for values of \( \eta \) above the sonic line \( S_1 \).

### 3.3 The Free Boundary Problem

To make the method in Chapter 2 work, we need to set up the problem as a free boundary problem.

#### 3.3.1 The Second Order PDE for \( \rho \) and the Transport Equations for \( (u, v) \)

From (3.3), we can obtain a second order PDE for \( \rho \)
\[
Q(\rho, \tilde{u}, \tilde{v}) := ((\tilde{u}^2 - c^2)\rho_\xi + \tilde{u} \tilde{v} \rho_\eta + \rho \tilde{u})_\xi + ((\tilde{v}^2 - c^2)\rho_\eta + \tilde{u} \tilde{v} \rho_\xi + \rho \tilde{v})_\eta
\]
\[
+ (\tilde{u} \tilde{v}_\eta - \tilde{v} \tilde{u}_\eta) \rho_\xi + (\tilde{v} \tilde{u}_\xi - \tilde{u} \tilde{v}_\xi) \rho_\eta + 2(\tilde{u} \tilde{v}_\xi \eta - \tilde{v} \tilde{u}_\xi \eta) \rho
\] (3.8)
\[
= 0
\]
where \( c = c(\rho) = \sqrt{\rho'(\rho)} \) is the speed of sound. Notice that (3.8) is elliptic if and only if \( \tilde{u}^2 + \tilde{v}^2 < c^2 \). However, unlike the second order PDE for \( u \) in Chapter 2, the coefficients of this PDE depends not only on \( \rho \) but also on \( \tilde{u} \) and \( \tilde{v} \). Therefore, we cannot work with this PDE alone, but we should also provide the transport equations.
for \((\tilde{u}, \tilde{v})\)
\[
\begin{align*}
(\tilde{u}, \tilde{v}) \cdot \nabla \tilde{u} + \tilde{u} + \frac{p}{\rho} &= 0 \\
(\tilde{u}, \tilde{v}) \cdot \nabla \tilde{v} + \tilde{v} + \frac{\rho}{\rho} &= 0
\end{align*}
\] (3.9)

Notice that in (3.8), there is a zeroth order term \(\rho\) and the sign of the coefficient is not known a priori. This will cause trouble when we try to use maximum principle for this second order elliptic PDE. As shown in [25], through some tedious calculation, we can get a new form of \(Q\)
\[
Q(\rho, \tilde{u}, \tilde{v}) := (c^2 - \tilde{u}^2)\rho_{\xi\xi} - 2\tilde{u}\tilde{v}\rho_{\xi\eta} + (c^2 - \tilde{v}^2)\rho_{\eta\eta} + 2cc'(\rho_{\xi}^2 + \rho_{\eta}^2) \\
- 2\rho_{\xi}(\tilde{u}(1 + \tilde{u}_\xi + \tilde{v}_\eta) - \frac{c^2(\tilde{v}_\eta + 1) - \tilde{v}\tilde{v}_\xi}{\tilde{u}^2 + \tilde{v}^2}) \\
- 2\rho_{\eta}(\tilde{v}(1 + \tilde{u}_\xi + \tilde{v}_\eta) - \frac{c^2\tilde{u}\tilde{v}_\eta - \tilde{v}(\tilde{u}_\xi + 1)}{\tilde{u}^2 + \tilde{v}^2})
\] (3.10)
as long as \(\tilde{u}^2 + \tilde{v}^2 \neq 0\).

3.3.2 The Shock Evolution Equation and the Oblique Derivative Condition

We follow the calculation in [25] to get the shock evolution equation and the oblique derivative condition.

We start with the Rankine-Hugoniot equations:
\[
\frac{d\eta}{d\xi} = \frac{[\rho\tilde{v}]}{[\rho\tilde{u}]} = \frac{[\rho\tilde{u}\tilde{v}]}{[\rho\tilde{u}^2 + p]} = \frac{[\rho\tilde{v}^2 + p]}{[\rho\tilde{u}\tilde{v}]}.
\] (3.11)

From now on, \([f] = f - f_0\).

To make the method in Chapter 2 work, we need a shock evolution equation such that \(\frac{d\eta}{d\xi}\) depends only on \(\rho\) and known quantities \(\rho_0, \tilde{u}_0\) and \(\tilde{v}_0\).

In (3.11), we have 3 equations and 5 unknowns. We can eliminate \(\tilde{u}\) and \(\tilde{v}\) to get the desired shock evolution equations.
Setting second and third term equal to each other, after some algebra, we get

\[ \rho_0 \rho \left[ \rho \frac{\partial \bar{v}}{\partial \bar{u}} - \bar{v}_0 \tilde{u} \right] = - \frac{[\rho \tilde{v}]}{[\tilde{u}]} . \]  
(3.12)

Similarly, equating the second term to the fourth term gives

\[ \rho_0 \rho \left[ \tilde{u} \bar{v} - \bar{v}_0 \tilde{u} \right] = \frac{[\rho \tilde{v}]}{[\tilde{v}]} . \]  
(3.13)

Combining (3.12) and (3.13) gives

\[ - \frac{[\rho \tilde{v}]}{[\tilde{u}]} = \frac{[\rho \tilde{v}]}{[\tilde{v}]} . \]  
(3.14)

Now, (3.14) and the first equation in (3.11) together give

\[ \frac{d \eta}{d \xi} = \frac{[\rho \tilde{v}]}{[\rho \tilde{u}]} = \frac{[\tilde{u}]}{[\tilde{v}]} . \]  
(3.15)

The first equation in (3.11) gives

\[ \rho[\tilde{v}] + [\rho] \tilde{v}_0 = \eta' (\rho[\tilde{u}] + [\rho] \tilde{u}_0) , \]  
(3.16)

where \( \eta' = \frac{d \eta}{d \xi} \).

This combining with (3.15) gives

\[ [\tilde{v}] = \frac{\eta' \tilde{u}_0 - \tilde{v}_0}{\rho (1 + (\eta')^2)} [\rho] . \]  
(3.17)

Similarly, setting the first and fourth terms in (3.11) equal, using (3.15) and simplifying the result a little, we obtain

\[ \rho[\tilde{v}]^2 (1 + (\eta')^2) + \rho (\tilde{v}_0 - \eta' \tilde{u}_0)[\tilde{v}] + [p] = 0 . \]  
(3.18)

Replacing \([\tilde{v}]\) in (3.18) using (3.17), we get

\[ (\tilde{u}_0^2 - \tilde{c}^2)(\eta')^2 - 2 \tilde{u}_0 \tilde{v}_0 \eta' + \tilde{v}_0^2 - \tilde{c}^2 = 0 \]  
(3.19)
where $c^2 = \frac{\rho |p|}{\rho_0 |p|}$.

Hence
\[
\eta' = \frac{\tilde{u}_0 \tilde{v}_0 \pm \sqrt{c^2 (\tilde{u}_0^2 + \tilde{v}_0^2 - \bar{c}^2)}}{\tilde{u}_0^2 - \bar{c}^2} = \frac{\tilde{v}_0^2 - \bar{c}^2}{\tilde{u}_0 \tilde{v}_0 \pm \sqrt{c^2 (\tilde{u}_0^2 + \tilde{v}_0^2 - \bar{c}^2)}}.
\] (3.20)

These two shock evolution equations correspond to +and- shock.

Because $\eta' \geq 0$, we can pick the correct shock evolution equation from these two different shock evolution equations (computing $\eta'$ at the point where the shock intersects the $\xi$-axis will give the correct choice):
\[
\eta' = \frac{\tilde{u}_0 \tilde{v}_0 + \sqrt{c^2 (\tilde{u}_0^2 + \tilde{v}_0^2 - \bar{c}^2)}}{\tilde{u}_0^2 - \bar{c}^2} = \frac{\tilde{v}_0^2 - \bar{c}^2}{\tilde{u}_0 \tilde{v}_0 + \sqrt{c^2 (\tilde{u}_0^2 + \tilde{v}_0^2 - \bar{c}^2)}}.
\] (3.21)

To completely decide the position of the shock, we also need an initial condition for (3.21). We can use the position of $\Xi_a$ (see figure 3.3) to serve as our initial condition for the shock evolution equation (3.21).

Figure 3.3: The subsonic domain.
\( \Xi_a \) is the intersection point between the vertical incident shock and the sonic line for state 1. We know that the incident shock is
\[
\xi = \xi_a = \sqrt{\frac{\rho_1}{\rho_0}} \left[ \frac{\rho}{\rho} \right],
\] (3.22)
and the sonic line for state 1 is
\[
(u_1 - \xi)^2 + (0 - \eta)^2 = c^2(\rho_1).
\] (3.23)
Noticing that \( u_1 = \sqrt{\frac{\rho_1}{\rho_1\rho_0}} \), we get \( \Xi_a = (\xi_a, \eta_a) \), where
\[
\xi_a = \sqrt{\frac{\rho_1}{\rho_0}} \left[ \frac{\rho}{\rho} \right],
\]
and
\[
\eta_a = \sqrt{c^2(\rho_1) - \left( \sqrt{\frac{\rho_1}{\rho_1\rho_0}} - \sqrt{\frac{\rho_0}{\rho_0}} \right)^2}.
\]
Hence, we have the initial condition for the shock evolution equation (3.21)
\[
\eta(\xi_a) = \eta_a
\] (3.24)
Before we proceed to find the oblique derivative condition, we derive an equation for \([u]\) from (3.15) and (3.17)
\[
[u] = -\frac{\eta' \eta'\tilde{u}_0 - \tilde{v}_0}{\rho(1 + (\eta')^2)} [\rho].
\] (3.25)

To get the oblique derivative condition for \( \rho \), we notice that taking derivatives of (3.17) and (3.25) gives us two equations for \( \nabla \tilde{u} \) and \( \nabla \tilde{v} \). Those equations involve \( \eta'' \), however, if we take derivative of (3.19) along the shock to get \( \eta'' \) and eliminate \( \eta'' \) in those equations, we get
\[
\begin{align*}
\{ & (1, \eta') \cdot \nabla \tilde{u} = -1 + A\rho + \eta' A\rho, \\
& (1, \eta') \cdot \nabla \tilde{v} = -\eta' + B\rho + \eta' B\rho \}
\end{align*}
\] (3.26)
where
\[
A = \frac{c^2}{\rho_0(\eta' \tilde{u}_0 - \tilde{v}_0)} - \frac{[p] \tilde{u}_0}{\rho_0(\eta' \tilde{u}_0 - \tilde{v}_0)^2} \frac{d\tilde{c}^2}{d\rho} (1 + (\eta')^2)
\]
and
\[
B = -\frac{c^2}{\rho_0(\eta' \tilde{u}_0 - \tilde{v}_0)} + \frac{[p] \tilde{v}_0}{\rho_0(\eta' \tilde{u}_0 - \tilde{v}_0)^2} \frac{d\tilde{c}^2}{d\rho} (1 + (\eta')^2).
\]

With the other two equations for $\nabla \tilde{u}$ and $\nabla \tilde{v}$ in (3.9), we can solve for $\nabla \tilde{u}$ and $\nabla \tilde{v}$. In particular, we get the equation for $\tilde{u}_\xi$ and $\tilde{v}_\eta$

\[
\begin{align*}
\tilde{u}_\xi &= \frac{1}{\tilde{v} - \eta' \tilde{u}} (-\tilde{v} + \tilde{v} A \rho_\xi + \eta' \tilde{v} \rho_\eta + \eta' \frac{c^2}{\rho} \rho_\xi + \eta' \tilde{u}) \\
\tilde{v}_\eta &= \frac{1}{\tilde{v} - \eta' \tilde{u}} (\eta' \tilde{u} B \rho_\xi - \eta' \tilde{u} B \rho_\eta - \tilde{v} - \frac{c^2}{\rho} \rho_\eta)
\end{align*}
\]

Using (3.27) to eliminate $u_\xi$ and $u_\eta$ in the first equation in (3.3), we get the oblique derivative condition for $\rho$ along the shock

\[
(\beta^1, \beta^2) \cdot \nabla \rho = 0
\]

where
\[
\beta^1 = \tilde{u} + \frac{\rho}{\tilde{v} - \eta' \tilde{u}} (\tilde{v} A + \eta' \frac{c^2}{\rho} - \tilde{u} B)
\]
and
\[
\beta^2 = \tilde{v} + \frac{\rho}{\tilde{v} - \eta' \tilde{u}} (\eta' \tilde{v} A - \eta' \tilde{u} B + \frac{c^2}{\rho}).
\]

Notice that $\nu = (-\eta', 1)$ is the inward normal on the shock. To measure the obliqueness, we compute

\[
\beta \cdot \nu = -\frac{1}{\tilde{v} - \eta' \tilde{u}} ((\tilde{v} \eta' + \tilde{u})^2 + (c^2 - \tilde{u}^2 - \tilde{v}^2)(1 + (\eta')^2)).
\]

In section 3.4, we will show that $\tilde{v} - \eta' \tilde{u} > 0$ on shock. Therefore, $\beta \cdot \nu < 0$ if $\tilde{u}^2 + \tilde{v}^2 < c^2$. Notice that $\tilde{u}^2 + \tilde{v}^2 < c^2$ is the same condition to ensure ellipticity of the second order PDE for $\rho$. 

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3.3.3 The Boundary Condition on the Wall

On the wall $\Sigma_w$, we pose the slip boundary conditions

$$\frac{\partial \rho}{\partial \nu} = 0,$$

and

$$\nu \cdot (\tilde{u}, \tilde{v}) = \nu \cdot (u, v) = 0.$$

(3.30) (3.31)

3.3.4 The One Point Dirichlet Boundary Condition at $\Xi_w$

At the point $\Xi_w$ where the shock hits the wall, we use a one-point Dirichlet boundary condition instead of the oblique derivative boundary condition for the shock or the slip boundary condition for the wall.

Physically, the shock will hit the wall perpendicularly. This requires that $\eta'(\xi_w) = 0$ for the shock. Therefore, from the shock evolution equation (3.21), we have

$$\bar{c}^2(\rho) = \tilde{v}_0^2,$$

(3.32)

i.e.

$$\bar{c}^2(\rho) = \eta^2,$$

(3.33)

or

$$\rho = (\bar{c}^2)^{-1}(\eta^2)$$

(3.34)

at $\Xi_w$.

3.3.5 The Dirichlet Boundary Condition on the Sonic Line

The only boundary left is $\sigma$, the sonic line of state 1. The position of $\sigma$ is decided by the equation

$$u_1^2 + v_1^2 = c^2(\rho_1),$$

(3.35)
\[(\xi - u_1)^2 + \eta^2 = c^2(\rho_1). \tag{3.36}\]

On \(\sigma\) we pose the Dirichlet boundary condition

\[\rho = \rho_1, \tag{3.37}\]

and

\[(\tilde{u}, \tilde{v}) = (u_1 - \xi, -\eta). \tag{3.38}\]

### 3.4 The Stagnation Point

From the second order PDE (3.10) for \(\rho\) and the transport equation (3.9) for \((\tilde{u}, \tilde{v})\), we see that those PDEs are degenerate at a point where \((\tilde{u}, \tilde{v}) = (0, 0)\). We call those points stagnation points. Ideally, we would like to avoid stagnation points to avoid technical difficulties. However, we will show that a stagnation point is inevitable in our problem.

First, we show that the vector \((\tilde{u}, \tilde{v})\) on the boundary either points into \(\Omega\) or is tangent to the boundary.

On the wall \(\Sigma_w\), the slip boundary condition requires \((\tilde{u}, \tilde{v})\) to be tangent to \(\Sigma_w\).

On the shock \(\Sigma\), the first equation in the Rankine-Hugoniot condition \(\frac{d\rho}{d\xi} = \frac{\rho \tilde{v}}{\rho u}\) can be written as

\[\rho(-\eta', 1) \cdot (\tilde{u}, \tilde{v}) = \rho_0(-\eta', 1) \cdot (\tilde{u}_0, \tilde{v}_0) = \rho_0(-\eta', 1) \cdot (-\xi, -\eta) \tag{3.39}\]

Notice that \(\mathbf{v} = (-\eta', 1)\) is the inward normal on shock, (3.39) becomes

\[\rho \mathbf{v} \cdot (\tilde{u}, \tilde{v}) = \rho_0 \mathbf{v} \cdot (-\xi, -\eta) > 0. \tag{3.40}\]

Therefore, \((\tilde{u}, \tilde{v})\) points into the \(\Omega\).
On the sonic line $\sigma$, $(\tilde{u}, \tilde{v})$ points towards the center of the sonic circle, i.e., $(u_1, 0)$.

If $(u_1, 0) \in \Omega$, then surely $(\tilde{u}, \tilde{v})$ points into $\Omega$.

If $(u_1, 0) \not\in \Omega$, then $(u_1, 0)$ is on the right hand of the shock $\Sigma$. It is obvious that $(\tilde{u}, \tilde{v})$ points into the $\Omega$ from Figure 3.4.

![Figure 3.4: The pseudo-velocity on the sonic circle.](image)

To sum up, $(\tilde{u}, \tilde{v})$ is parallel to the wall on the wall $\Sigma_w$ and points into $\Omega$ on the other boundaries as shown in Figure 3.5.

Hence, there must be at least one stagnation point either on the wall or in $\Omega$. 
3.5 Axisymmetric and Self-Similar Solutions

Though the stagnation point is very important in our study to this problem, it is very difficult to locate the stagnation points or to conclude whether we have a few discrete stagnation points or a whole region of stagnation points. To understand the behavior of $\rho$ and $\left(\tilde{u}, \tilde{v}\right)$ around the stagnation point, we study special solutions – axisymmetric and self-similar solutions to the isentropic Euler equations with a stagnation point at the origin.

Zheng studied the axisymmetric and self-similar solutions to the 2-D isentropic Euler equations in [43]. He proved existence of solutions with different configurations and categorized those solutions according to the initial condition and the adiabatic exponent $\gamma$. In this section, we will follow Zheng’s idea to reduce the Euler system to a system of ODEs. After that, we focus on studying the behavior of the solution...
near the stagnation point instead of proving the existence of global solutions as in Zheng’s book.

In addition to the self-similar assumption in this chapter, we assume that the solution is also axisymmetric, i.e., the solution \((\rho, u, v)\) has the property

\[
\begin{cases}
\rho(t, R, \theta) = \rho(t, R, 0) \\
(u(t, R, \theta)) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u(t, R, 0) \\ v(t, R, 0) \end{pmatrix}
\end{cases}
\]

where \((R, \theta)\) are polar coordinates in the \(xy\)-plane. We denote \(\bar{u}(t, R) = u(t, R, 0)\) and \(\bar{v}(t, R) = v(t, R, 0)\). We notice that \(\bar{u}\) is the radial velocity and \(\bar{v}\) is the cross-radial velocity, i.e., the tangential velocity.

Because we also assume the solution is self-similar, the solution depends only on \(r = \frac{R}{t}\) and the isentropic Euler equations are reduced to a system of ODEs

\[
\begin{cases}
\rho_r = \frac{\rho v^2 - \bar{u}(r - \bar{u})}{r c^2 - (\bar{u} - \bar{v})^2} \\
\bar{u}_r = \frac{1}{r} \frac{v^2(r - \bar{u}) - \bar{u}c^2}{c^2 - (\bar{u} - \bar{v})^2} \\
\bar{v}_r = \frac{\bar{u}v}{r(r - \bar{u})}
\end{cases}
\]

where \(c^2 = p' = \gamma A \rho^{\gamma-1}\) is the square of the sound speed.

Now we want to study the behavior of \(\rho\) and \((u, v)\) around an isolated stagnation point.

Start with Euler equations (3.1). Since this system is also invariant under Galilean transformation, we can assume the stagnation point is at the origin in the self-similar plane. In this section we still assume that \(p = A \rho^\gamma\). For simplicity, we take \(A = 1\).

In the following subsections, we want to study the behavior of axisymmetric, self-similar solutions that have an isolated stagnation point at the origin, i.e., \((\bar{u}, \bar{v}) \to 0\) as \(r \to 0\).
3.5.1 Necessity of a Vacuum at Stagnation Point

With the axisymmetry and self-similarity assumptions, the second and third equations of (3.2) become

\[
\begin{cases}
(\bar{u} - r)\bar{u}_r - \frac{v^2}{r} + \frac{r\bar{v}}{\rho} = 0 \\
(\bar{u} - r)(\bar{v}_r + \frac{\bar{v}}{r}) + \bar{v} = 0.
\end{cases}
\]  

(3.43)

We integrate the first equation in (3.2) on a disc with radius \(R\)

\[
\iint_{B_R} \rho_t \, dx \, dy + \iint_{B_R} (\rho u)_x + (\rho v)_y \, dx \, dy = 0.
\]  

(3.44)

We apply the divergence theorem to the second term

\[
\iint_{B_R} \rho_t \, dx \, dy + \int_{\partial B_R} \rho (u,v) \cdot \nu \, dS = 0.
\]  

(3.45)

Using the axisymmetry assumption, we get

\[
\int_{0}^{R} \rho_t 2\pi R' \, dR' - \rho \bar{u} R = 0.
\]  

(3.46)

Dividing both sides by \(2\pi t\) and using the self-similarity assumption, the above equation becomes

\[
\int_{0}^{r} r^2 \rho_r(r') \, dr' - \rho \bar{u} r = 0,
\]  

(3.47)

where \(r = \frac{R}{t}\) as before. If we integrate the first term by parts and rearrange the equation, we get

\[
\bar{u} - r = -\frac{2}{r\rho} \int_{0}^{r} r' \rho(r') \, dr'.
\]  

(3.48)

Using this in the second equation in (3.43), we have

\[
(2 \ln (r\bar{v}))' = (\ln \int_{0}^{r} r' \rho(r') \, dr').
\]  

(3.49)

Integrating this and taking exponential on both sides, we obtain

\[
r^2 \bar{v}^2 = B^2 \int_{0}^{r} r' \rho(r') \, dr'.
\]  

(3.50)
where $B$ is an integration constant.

Now we prove that the density must go to 0 at the stagnation point. Suppose this is not true, then $\rho = \rho_* + o(1)$ as $r \to 0^+$ with $\rho_* > 0$. Substituting this for $\rho$ in (3.50), we obtain

$$
\bar{v}^2 = B^2 \int_0^r r' (\rho_* + o(1)) \, dr' \\
= B^2 \frac{1}{2} \rho_* r^2 + o(r^2) \\
= \frac{1}{2} B^2 \rho_* + o(1),
$$

which implies $\bar{v} \to \frac{1}{2} B^2 \rho_* > 0$ as $r \to 0^+$. But this contradicts the fact that we have a stagnation point at the origin. So we must have $\rho \to 0$ as $r \to 0^+$, which means we must have a vacuum at the stagnation point.

We have proved

**Proposition 3.5.1.** If $(\rho, \bar{u}, \bar{v})$ is an axisymmetric and self-similar solution to (3.2) which has a stagnation point $((\bar{u}, \bar{v}) = (0, 0))$ at the origin, then we have a vacuum $(\rho = 0)$ at the origin.

**Remark** Using (3.48) and (3.50) we can prove by a similar argument that if we have an isolated vacuum point at the origin, that point must be a stagnation point.

### 3.5.2 Asymptotic Behavior at the Stagnation Point

From Section 3.5.1, we know that if there is a stagnation point at the origin, there is a solution of (3.42) which satisfies

$$
\rho \to 0, \quad (\bar{u}, \bar{v}) \to (0, 0), \quad \text{as } r \to 0^+
$$

Now we will assume the adiabatic constant $\gamma \in [1, 2)$. We will study the asymptotic behavior of a solution when $(\rho, \bar{u}, \bar{v})$ is close to $(0, 0, 0)$. 
We follow Zheng’s book [43], and define
\[ X = \bar{u} \frac{1}{r}, \quad V_2 = \frac{\bar{v}^2}{c^2}, \quad R_2 = \frac{r^2}{c^2}. \]

We also define a new independent variable \( \tau \) such that
\[
\frac{dr}{d\tau} = r(1 - X)[1 - R_2(X - 1)^2]. \tag{3.53}
\]

Then (3.42) becomes
\[
\begin{align*}
\frac{dX}{d\tau} &= (1 - X)[V_2(1 - X) - 2X + X(1 - X)^2R_2] \\
\frac{dV_2}{d\tau} &= 2V_2\{X - X(1 - X)^2R_2 - \frac{R_2}{2}(1 - X)[V_2 - X(1 - X)R_2]\} \\
\frac{dR_2}{d\tau} &= 2R_2(1 - X)\{1 - (1 - X)^2R_2 - \frac{R_2}{2}[V_2 - X(1 - X)R_2]\} \tag{3.54}
\end{align*}
\]

Noticing that \((X, V_2, R_2) = (0, 0, 0)\) is an equilibrium of (3.54), we consider the asymptotic behavior of (3.54) near \((X, V_2, R_2) = (0, 0, 0)\). We notice that around \((X, V_2, R_2) = (0, 0, 0)\), \(\tau\) behaves like \(\ln r\) as \(r \to 0^+\). Since we are interested in the behavior when \(r \to 0^+\), we should consider the behavior of (3.54) when \(\tau \to -\infty\).

The linearized system of (3.54) around \((X, V_2, R_2) = (0, 0, 0)\) is
\[
\frac{d}{d\tau} \begin{pmatrix} X \\ V_2 \\ R_2 \end{pmatrix} = A \begin{pmatrix} X \\ V_2 \\ R_2 \end{pmatrix} \tag{3.55}
\]
where
\[
A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

The eigenvalues of \(A\) are \(-2, 2\) and \(0\) and the corresponding eigenvectors are \(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\), \(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\).
By the center manifold theorem (refer to [4]), ODE system (3.54) has a center manifold which is tangent to the vector \( \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \). Suppose the center manifold is

\[
\begin{align*}
X &= g(V_2) \\
R_2 &= h(V_2)
\end{align*}
\tag{3.56}
\]
then, it can be computed that \( g(0) = 0 \), \( g'(0) = \frac{1}{2} \) and \( h(0) = h'(0) = 0 \). Using (3.56) in the second equation of (3.54), we get

\[
\frac{dV_2}{d\tau} = (2 - \gamma)V_2^2 + O(V_2^3)
\tag{3.57}
\]
So \( V_2 = -\frac{1}{(2-\gamma)\tau} + o\left(\frac{1}{\tau}\right) \) on the center manifold.

Noticing that the linearized ODE system (3.55) has eigenvalues \(-2\), \(2\) and \(0\), to find the solution which goes to \((0, 0, 0)\), we need to look for the solution on the center-stable manifold \( X = \frac{1}{2} V_2 + \text{small terms} \) (if we only need to get the linear terms for the center-stable manifold, we only need to find the plane that is tangent to \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \)).

So the solution we are seeking has the behavior

\[
\begin{align*}
X &= -\frac{1}{2(2-\gamma)\tau} + o\left(\frac{1}{\tau}\right) \\
V_2 &= -\frac{1}{(2-\gamma)\tau} + o\left(\frac{1}{\tau}\right) \\
R_2 &= O(e^{2\tau})
\end{align*}
\tag{3.58}
\]
as \( \tau \to -\infty \).

Applying the asymptotic behavior (3.58) to (3.53), we obtain

\[
\frac{dr}{d\tau} = r + r \frac{1}{2(2 - \gamma)\tau} + o\left(\frac{1}{\tau}\right).
\tag{3.59}
\]
Hence,

\[
\tau = \ln r - \frac{1}{2(2 - \gamma)} \ln |\ln r| + o(|\ln r|).
\tag{3.60}
\]
Therefore,

\[
\begin{align*}
X &= -\frac{1}{2(2-\gamma) \ln r} + o\left(\frac{1}{\ln r}\right) \\
V_2 &= -\frac{1}{(2-\gamma) \ln r} + o\left(\frac{1}{\ln r}\right) \\
R_2 &= O\left(\frac{r^2}{(-\ln r)^{\frac{1}{2(2-\gamma)}}}\right)
\end{align*}
\]  

(3.61)

as \( r \to 0^+ \).

From the definition of \( X, V_2 \) and \( R_2 \), we obtain

\[
\begin{align*}
\bar{u} &= -\frac{r}{2(2-\gamma) \ln r} + o\left(\frac{r}{\ln r}\right) \\
\bar{v}^2 &= \left(-\frac{1}{(2-\gamma) \ln r} + o\left(\frac{1}{\ln r}\right)\right)c^2
\end{align*}
\]  

(3.62)

as \( r \to 0^+ \).

To get the asymptotic behavior of \( \bar{v} \) and \( \rho \), we first differentiate (3.50) to get

\[
2r\bar{v}^2 + r^2\bar{v}\bar{v}' = B^2 r \rho.
\]  

(3.63)

With the asymptotic behavior (3.61) and (3.62), the correct balance among all 3 terms in (3.63) is that the first term balances with the third term, i.e.

\[
2r\bar{v}^2 \sim B^2 r \rho.
\]  

(3.64)

Use the second equation of (3.62) to eliminate \( \bar{v} \) in (3.64) and notice that \( c^2 = \gamma \rho^{\gamma-1} \); we get

\[
2 \left(-\frac{1}{(2-\gamma) \ln r} + o\left(\frac{1}{\ln r}\right)\right) \gamma \rho^{\gamma-1} \sim B^2 \rho.
\]  

(3.65)

Hence

\[
\rho \sim \frac{b}{(-\ln r)^{\frac{1}{2(2-\gamma)}}}
\]  

(3.66)

where \( b = \left(\frac{2\gamma}{B^2(2-\gamma)}\right)^{\frac{1}{2(2-\gamma)}} > 0 \) is an arbitrary constant.

Using (3.66) to eliminate \( c^2 \) in the second equation of (3.62), we get the asymptotic behavior of \( \bar{v} \)

\[
\bar{v} \sim \frac{\sqrt{\frac{2\gamma}{2-\gamma}b^{\gamma-1}}}{(-\ln r)^{\frac{1}{2(2-\gamma)}}}
\]  

(3.67)
Summing up (3.66), (3.67) and the first equation in (3.62), we get the asymptotic behavior of $\rho$, $\bar{u}$ and $\bar{v}$ as $r \to 0^+$:

$$
\begin{align*}
\rho & \sim \frac{b}{(-\ln r)^{(2-\gamma)}}, \\
\bar{u} & \sim -\frac{r}{2(2-\gamma)\ln r}, \\
\bar{v} & \sim \frac{\sqrt{\frac{2\gamma}{\gamma-1}}}{(-\ln r)^{(2-\gamma)}}
\end{align*}
$$

(3.68)

The behaviors of the velocity field and the density are shown in Figure 3.6 and Figure 3.7.

Figure 3.6: Plot of the velocity field when the adiabatic constant $\gamma = 1.4$. 

Figure 3.7: Plot of the density field when the adiabatic constant $\gamma = 1.4$. 

Figure 3.8: Plot of the density field when the adiabatic constant $\gamma = 1.4$. 

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Figure 3.7: Plot of the density when the adiabatic constant $\gamma = 1.4$. 
Appendix A: Moving Shock Formation Point to the Origin

In [21] and [41], Hunter and Tesdall have \((u_R, v_R) = (0, 0)\) but the shock formation point is not at the origin. Suppose the shock formation point is at \((\rho_*, \tau_*)\).

Now, \((\rho_*, \tau_*)\) is a sonic point, therefore:
\[ u_L(\rho_*, \tau_*) = \rho_* \] \ ...(A.1)

Also, notice that at \((\rho_*, \tau_*)\), the strength of shock is 0, so we must have:
\[ u_L(\rho_*, \tau_*) = u_R(\rho_*, \tau_*) = 0 \] \ ...(A.2)

Combining (A.1) and (A.2), we must have \(u_L(\rho_*, \tau_*) = \rho_* = 0\), i.e. the shock forms at \((0, \tau_*)\) and \(u(0, \tau_*) = 0\).

To move the shock formation point to the origin, we let \(\bar{\tau} = \tau - \tau_*\), then (2.4) becomes:
\[
(u - \rho)u_\rho - \frac{\bar{\tau} + \tau_*}{2}u_\rho + v_\tau = 0
\]
\[\frac{\bar{\tau} + \tau_*}{2}u_\rho - v_\rho + u_\tau = 0, \] \ ...(A.3)

which is slightly different from the UTSDE we had before.

However, eliminating \(v\) in this system yields the second-order equation
\[
Q(u) = ((u - \rho)u_\rho + \frac{u}{2})_\rho + u_{\tau\tau} = 0,
\] \ ...(A.4)

which is the same as our second-order PDE.
Now let us look at the oblique derivative condition and the shock evolution equation.

Along the shock \( \Sigma = \{(\rho, \bar{\tau})|\rho = S(\bar{\tau})\} \), the Rankine-Hugoniot conditions must be satisfied (the notation \( \cdot' \) means \( \frac{d}{d\bar{\tau}} \)):

\[
S'(\bar{\tau}) = \left[ \frac{\frac{1}{2}u^2 - \rho u}{v - \frac{\bar{\tau} + \tau^*}{2}u} \right] - \left[ \frac{\frac{\bar{\tau} + \tau^*}{2}u - v}{u} \right].
\] (A.5)

Notice that in our new coordinate system, \((\rho, \bar{\tau})\), the shock starts at the origin, so we must have

\[ S(0) = 0. \] (A.6)

Let the left state be \((u, v)\) and remember that \((u_R, v_R) = (0, 0)\). From the Rankine-Hugoniot conditions (2.9), we obtain one equation for \( S' \) by eliminating \([\frac{\bar{\tau} + \tau^*}{2}u - v]\):

\[
(S')^2 = -\frac{1}{2}u + \rho.
\] (A.7)

Combining \(-\frac{1}{2}u + \rho = (S')^2 \geq 0\) with the requirement of the left state being subsonic, i.e. \(u - \rho > 0\), we have to have \(\rho > 0\) for the shock. Therefore,

\[
S'(\bar{\tau}) = \sqrt{-\frac{1}{2}u + \rho}.
\] (A.8)

A second condition is obtained by eliminating \(S'\) from (2.9):

\[
v = \frac{\bar{\tau} + \tau^*}{2}u - u\sqrt{-\frac{1}{2}u + \rho}.
\] (A.9)

We differentiate \(v\) along \(\Sigma\) (where \(\cdot' = S'\partial_\rho + \partial_\tau = \frac{d}{d\bar{\tau}}\) along \(\Sigma\)):

\[
v' = \frac{u}{2} + \frac{\bar{\tau} + \tau^*}{2}u' - u'\sqrt{-\frac{1}{2}u + \rho} - \frac{-\frac{1}{2}u' + \rho'}{2\sqrt{-\frac{1}{2}u + \rho}}u.
\] (A.10)

We express \(u'\) as \(u_{\rho}S' + u_{\tau}\) and use the differential equation (A.3) to write

\[
v' = v_{\rho}S' + v_{\tau} = \left(\frac{\bar{\tau} + \tau^*}{2}u_{\rho} + u_{\tau}\right)S' - ((u - \rho)u_{\rho} - \frac{\bar{\tau} + \tau^*}{2}u_{\tau}).
\] (A.11)
Finally, substituting this for $u'$ and $v'$ in (A.10), using (A.8) and collecting terms in $u_\rho$ and $u_\tau$, we obtain

$$N(u) \equiv \beta \cdot \nabla u = 0 \quad (A.12)$$

where $\beta = (\beta^1, \beta^2) = ((2\rho - \frac{7}{4}u)s, 2\rho - \frac{5}{4}u)$.

So we can replace the Rankine-Hugoniot conditions (A.5) by (A.6), (A.8) and (A.12).

**Remark A.0.2.** In our new coordinate system $(\rho, \bar{\tau})$, the shock forms at the origin and $u(0, 0) = 0$. Moreover, (A.4) (A.6), (A.8) and (A.12) are in the same form as the corresponding equations in the $(\rho, \tau)$ coordinate system. Thus, we have moved the shock formation point to the origin, but to keep our notation simple, we still use $(\rho, \tau)$ instead of $(\rho, \bar{\tau})$. 
Bibliography


