Modeling and Nonlinear Control of Highly Maneuverable Bio-Inspired Flapping-Wing Micro Air Vehicles

Dissertation

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By

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Abstract

Over the past decade, the promise of achieving the level of maneuverability exhibited in insect flight has prompted the research community to develop bio-inspired flapping-wing micro air vehicles (FW-MAVs). Flying insects employ their wings to produce lift to perform complex maneuvers. Mimicking insect capabilities could enable FW-MAVs to perform missions in tight spaces and cluttered environments, otherwise unattainable by fixed- or rotary-wing UAVs. The inherent mechanism of flapping-wing flight requires periodically-varying actuation, requiring the use of averaging methods for analysis and design of controllers for flapping-wing MAVs.

The main objective of this research is establishing a rigorous theoretical framework from a control theory point of view that combines averaging theory and robust nonlinear control theory towards the design of flight controllers for general models of FW-MAVs. The point of departure of this work is the adoption of Kane’s method to obtain equations of motion for multi-actuated, multi-body flapping-wing MAVs.

The first contribution of the present work is the formulation of a framework which investigates the effect of multiple actuation, including the presence of a movable appendage (abdomen), on vehicle controllability. The resulting formulation establishes a mathematically precise framework which lays the groundwork for the development of theoretically sound control design strategies.
The second contribution is the development of a novel wingbeat forcing function that ensures continuity of the wing motion across consecutive flapping cycles. The proposed wingbeat is differentiable, does not require manipulations of the waveform within each period, and is amenable to a rigorous application of averaging theory. A noticeable feature of the proposed method is its capability of generating sufficient aerodynamic forces and moments using only two physical actuators, thus satisfying the desired feature of minimal-actuation. Using the smallest possible number of actuators in controlling a FW-MAV remains a highly desirable feature due to the constraints posed by extreme miniaturization.

The framework presented in this investigation encompasses various actuation mechanisms. This analysis is particularly important as the choice of an input from a set of physical actuators must be carefully vetted against the constraints imposed by stringent size and weight constraints for such vehicles. As a consequence, it is crucial that candidate control inputs satisfy minimal-actuation requirement, alongside achieving robust stable maneuvers for FW-MAV. The effectiveness of multiple wing kinematic parameters were evaluated within the context of the developed theoretical framework.

A nonlinear controller was developed for the control of the longitudinal model of a minimally-actuated FW-MAV model using the proposed wingbeat function. The flight controller exhibits an inner-outer loop architecture, where inner-loop controller stabilizes the vehicle pitching dynamics by generating appropriate wing-bias command in the wingbeat function. The controlled pitch dynamics is employed as a virtual control input to regulate the longitudinal and vertical dynamics, while the
output of the outer-loop is used as a reference input for the inner-loop. The performance of the developed controller was tested and validated in various case studies. Simulation results illustrated that even in the presence of sizable modeling uncertainties, controller is capable of letting the vehicle model to track satisfactory the reference trajectories.
Acknowledgments

All praises and thanks are due to Allah, Almighty God, and peace be upon His messenger Mohammed, who said: “Whoever is not thankful to people then he cannot be thankful to Allah, either”.

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NOMENCLATURE

\[ C_D(\cdot) \] = wing drag coefficients
\[ C_L(\cdot) \] = right and left wing lift coefficients
\[ D_{uRW}(t), D_{uLW}(t) \] = drag forces for right and left wings
\[ = \text{during upstroke} \]
\[ D_{dRW}(t), D_{dLW}(t) \] = drag forces for right and left wings
\[ = \text{during downstroke} \]
\[ L_{uRW}(t), L_{uLW}(t) \] = lift forces for right and left wings
\[ = \text{during upstroke} \]
\[ L_{dRW}(t), L_{dLW}(t) \] = lift forces for right and left wings
\[ = \text{during downstroke} \]
\[ \textbf{F}_{uRW}^B(t), \textbf{F}_{dRW}^B(t) \] = right wing upstroke and downstroke aerodynamic force vectors in the body frame
\[ \textbf{F}_{uLW}^B(t), \textbf{F}_{dLW}^B(t) \] = left wing upstroke and downstroke aerodynamic force vectors in the body frame
\[ \textbf{M}_{uRW}^B(t), \textbf{M}_{dRW}^B(t) \] = right wing upstroke and downstroke aerodynamic moment vectors in the body frame
\[ \textbf{M}_{uLW}^B(t), \textbf{M}_{dLW}^B(t) \] = left wing upstroke and downstroke aerodynamic moment vectors in the body frame
$\bar{F}_{xRW}, \bar{F}_{xLW} = $ right and left cycle-averaged $x$-body axis forces

$\bar{F}_{yRW}, \bar{F}_{yLW} = $ right and left cycle-averaged $y$-body axis forces

$\bar{F}_{zRW}, \bar{F}_{zLW} = $ right and left cycle-averaged $z$-body axis forces

$\bar{M}_{xRW}, \bar{M}_{xLW} = $ right and left cycle-averaged moments about the $x$-body axis

$\bar{M}_{yRW}, \bar{M}_{yLW} = $ right and left cycle-averaged moments about the $y$-body axis

$\bar{M}_{zRW}, \bar{M}_{zLW} = $ right and left cycle-averaged moments about the $z$-body axis

$[x_w, y_w, 0]^T = $ location of wing center of pressure in local wing planform frame

$I_€ = $ area moment of inertia of wing planform about wing root

$\alpha = $ angular displacement of the planform about the passive rotation hinge joint, which is equivalent to wing angle-of-attack in still air

$\rho = $ atmospheric density

$[p_x, p_y, p_z]^T = $ location of wing root hinge with respect to the vehicle center of gravity wing planform frame

$\omega_0 = $ constant carrier frequency

$\mu_{uRW}, \mu_{uLW} = $ wingbeat forcing functions for right and left wings during upstroke

$\mu_{dRW}, \mu_{dLW} = $ wingbeat forcing functions for right and left wings during downstroke

$\delta_{RW}, \delta_{LW} = $ phase shifts for right and left wings

$b_{RW}, b_{LW} = $ right and left wing-bias
Chapter 1: Introduction

The ability of flying insects to hover and fly backwards has drawn the attention of scientists for a long time. Considerable studies have been spent in the quest for understanding the aerodynamics of insect flight and answer the question of how such tiny wings provide lift to much heavier bodies. The astonishing maneuverability of flying insects has inspired engineers to develop biologically-inspired flapping wing micro air vehicles (FW-MAV). Recent advances in miniaturization technologies [91] [40] had a dramatic impact on the growing interest in designing insect-size FW-MAVs that are more maneuverable than existing conventional MAVs. One of the motivations for developing FW-MAV is their envisioned capabilities of performing a wide range of missions in urban environments where the involvement of a human might be dangerous or unattainable. For example, FW-MAVs can be equipped with sensors in order to probe biological, chemical or radiation contaminated areas. Consequently, FW-MAVs must be able to perform agile autonomous flight with high maneuverability in order to accomplish such missions in cluttered spaces.

In spite of several remarkable achievements in investigating insects’ flight and understanding the mechanisms behind the aerodynamics of flapping wings for implementation on micro air vehicles [52] [15] [88], there is still a need for a general theoretical framework that can be used as a guideline for analyzing systems-theoretic
properties and developing potential control strategies of FW-MAVs. As a matter of fact, the development of control scheme that achieves successful stabilization and maneuverability for an insect-size FW-MAV is a particularly challenging problem due to the inherent time-variability associated with flapping wing. This dissertation addresses some of the fundamental aspects of bio-inspired FW-MAVs from the point of view of control theory. Emphasis is given to some of the most significant challenges in control and stability analysis of FW-MAV systems. In particular, this dissertation is concerned with establishing a rigorous framework to develop robust nonlinear flight controllers for generic model of a highly maneuverable FW-MAV based on averaging theory and stabilization by bounded feedback. A second goal of this work is to develop a mathematical model of FW-MAV dynamics amenable to investigating the effect of multiple actuation on vehicle controllability. The control solutions that have been proposed in the literature, although sensible from an engineering point of view, suffer from a general lack of methodological soundness, insofar as the correct use of certain analysis techniques - in particular averaging theory - is concerned. In addition, most of the control solutions are substantially heuristic in nature, and a rigorous stability analysis is missing. In particular, the majority of the flight control systems available in the literature employ the concept of averaging incorrectly, in the sense that the assumptions for application of the method are not satisfied. Specifically, although the models that results from these control policies are hybrid systems, they are not being analyzed with the relevant tools. On the other hand, there is still no general framework to incorporate different forms of actuation mechanisms in a comprehensive control design framework. In addition, most of the developed control designs that have been proposed in the field are of local validity, since these designs are based on
the linear approximation of the dynamics around the operating condition at hover. The goal of this research, from a control standpoint, is to establish a rigorous theoretical framework for the analysis and control designs for FW-MAV that involves using a novel wingbeat function that is continuously differentiable function of kinematics control parameters. Thus, averaging methods can be applied rigorously both as design and as analysis tools. Furthermore, this research seeks to develop solutions for controlling 3-DOF vehicle motion using a minimal actuation, with a largest possible domain of validity. This is considered to be a challenging task as the ensuing vehicle dynamics are under-actuated.

All existing micro air vehicles - (MAV) - [40] [46] [11] are still far from matching the superior maneuverability exhibited by natural fliers. Conventional fixed-wing MAVs may be impractical for indoor applications for various reasons. For instance, surveying confined spaces entails slow flight maneuver, whereas fixed-wing MAVs need to fly at certain speeds to generate sufficient lift and sustain the vehicle weight. On the contrary, rotary-wing MAVs offers more beneficial features over fixed-wing counterparts in terms of hovering and maneuvering capabilities. Nonetheless, rotary-wing MAV demand a considerable amount of power consumption at small scale. This is due to the fact that reducing rotor diameter entails increasing the propeller rotations in order to sustain sufficient lift to hover. On the other hand, biological studies [24] [30] have corroborated that aerodynamic lift generated by flapping wing insects is extremely power-efficient, particularly for an insect-size vehicle. Biologically-inspired flapping wing MAVs bear the promise of significant advancements in terms of performance and maneuverability in a similar manner as insect flight.
It is of a great importance to understand the aerodynamics of insect flapping flight in order to efficiently exploit flapping-wing mechanisms to achieving fully controlled maneuvers for FW-MAVs. Flying insects achieve remarkable maneuverability by moving their wings forward and backward in a stroke plane at a relatively high angle of attack [64] [65] [22] [71]. In order to understand the complex motion of a flapping wing, the entire wingbeat can be broken down into four phases, namely, downstroke, upstroke, pronation, and supination. The downstroke and upstroke defines the forward and backward motion of the wing. The downstroke motion is characterized by the wing motion from dorsal side of the insect’s body towards its ventral side. At the end of each downstroke and upstroke, the wing twists rapidly and exhibits a rotation about its radial axis. The twisting movements are termed supination and pronation, respectively. The wing leading edge will be pointing forward (pronation) during downstroke, and will be pointing backward (supination) during upstroke [64].

The flapping kinematics describe the wing motion pattern relative to the insect body. In order to parameterize the wing kinematics, the wing motion relative to an insect’s body can be assigned three degrees of freedom. The reciprocating flapping motion of each wing identifies the stroke plane. The wing angle of attack with respect to the stroke plane characterizes the second degree of freedom of the flapping wing. The deviation of wing motion from the stroke plane defines the third degree of freedom such that the wing tip follows a figure-8 pattern. The steering maneuvers performed by the insect are controlled by subtle changes in wing kinematics to produce appropriate aerodynamic forces and moments [64]. Ellington [21] observed that insects stabilize their flight maneuvers by modifying three wing kinematic parameters, namely, flapping amplitude, wing bias and wing angle of attack.
A primary challenge in designing control strategies for a FW-MAV is to select the relevant parameters of the wing kinematics that can generate desired maneuvers. Several studies [30],[64] have been conducted to elucidate the mechanisms used by wing motion of an insect during complex maneuvers. This, in turn, revealed the wing kinematics that alter the pertinent aerodynamic forces and moments, which control the flight dynamics of insects [16][32]. A theoretical framework can be developed to assess the significance of manipulating wing kinematic parameters on flapping insect maneuverability. Muijres et al. [53] have used high-speed cameras to investigate the remarkable subtle changes in wings and body motions of a fly that is evading an approaching threat. In [53], the authors observed that flies employ a coordinate turn strategy, in the same manner as helicopter and aircraft. The escape maneuvers of flies were associated with simultaneous roll and pitch of their bodies to produce these rapid maneuvers. The authors of [53] have noticed that flies control their position by adjusting their orientation such that the horizontal thrust component points towards desired direction of motion, analogous to helicopter maneuvers.

Experimental identification of wing kinematic parameters have been carried out in order to model the most significant effects of wingbeat patterns on a variety of maneuvers. These studies can give indications on how to design FW-MAVs and how to improve their performance, as well as aiding the processes of control analysis and design in finding a relationship between kinematic parameters and the associated aerodynamic forces and moments. Numerous experiments have fostered the successful design of various FW-MAV configurations by correlating the aerodynamic forces to the wing kinematics. Sane and Dickinson [64] [65] were among the first to identify aerodynamic forces and moments associated with the wing motion and show which
kinematic parameters can generate specific maneuvers. In [64]-[65], it is shown that small changes to the kinematic parameters, such as wing angle of attack, flapping wing amplitude, and timing of the wing rotation have significant effects in the aerodynamic forces and moments. Subsequent studies have resulted in an improved understanding of the aerodynamics of insect flight and its relation with relevant kinematic parameters. Fry et al. [30] have shown that insect flight is similar to helicopter flight where the translational motion is controlled by changing the vehicle’s attitude. Specifically, the stroke plane angle is manipulated to control the pitch moment and the thrust. Most insects have been observed hovering with a nearly horizontal stroke plane [70]. Moreover, it has been noticed that yaw moment is generated by asymmetric changes in the flapping amplitude [70]. This reveals an important device used in insect flight, namely that asymmetric changes in the wing kinematics can control aerodynamic forces and moments. Flying insects modulate their altitude by either varying flapping amplitude or flapping frequency [81] [16] [70] [44] [1]. Rolling moment can be created by modifying the stroke plane inclination of the insect wing [16] or by applying a different flapping amplitude [28] to their right and left wings. Pitching moment can be generated by manipulating center of lift position with respect to the vehicle center of gravity [23] [62]. Modifications of the wing-bias parameter has a significant role on the pitching moment since it basically shifts the mean stroke angle towards the fore or aft-body [27][30]. If the mid-point of the flapping motion moves forward, the resulting pitching moment cause a nose-up motion and vice versa. Yawing moment can be generated by varying the stroke plane inclination [30] or by creating different drag forces between the right and left wings [19]. Previous studies [86, 87, 33, 48, 55, 77] have emphasized the importance of applying asymmetric wing angle of attack in the
upstroke and downstroke. Yawing moment can be produced by modifying the wing angle of attack [23]. More recently, it has been pointed out that translational forces can be generated by applying asymmetric change in the wing angle of attack during downstroke and upstroke [76] [3] [30] [61]. In addition, in [20], [32] and [58] have studies investigated the role of the insect’s abdomen in flight stability and concluded that abdomen movements have a significant impact on the pitching stability. The most significant wingbeat kinematic parameters that have a great influence in producing effective aerodynamic forces and moments are summarized in Table 1.1.

Several studies [6] [17] [56] [77] [78] revealed that open-loop dynamics of flapping wing MAV are unstable near hovering conditions, which emphasizes the need for active control enhanced with continuous feedback. Experimental studies [79] pointed out that insects actively stabilize their flight maneuvers by continuous wing kinematic modifications to maintain a desired body attitude. Candidate control design methodologies must take into consideration stringent size and weight constraints which dictate minimizing the number of actuators. Furthermore, control design strategies are required to provide robustness with respect to modeling uncertainties as well as to

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<tr>
<th>Wing Kinematic Parameter</th>
<th>$F_x$</th>
<th>$F_y$</th>
<th>$F_z$</th>
<th>$M_x$</th>
<th>$M_y$</th>
<th>$M_z$</th>
</tr>
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<tr>
<td>Flapping Frequency</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wing Angle of Attack</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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<td>Stroke Plane Angle</td>
<td>✓</td>
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<tr>
<td>Wing Bias</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Abdomen</td>
<td></td>
<td></td>
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<td>✓</td>
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</tbody>
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Table 1.1: Summary Table of the Control Effectiveness of Wing Kinematic Parameters
account for constraints and saturations of the physical actuators. Hence, a robust control strategy must effectively combine bounded feedback with the desired feature of low computational cost. On the other hand, the majority of the proposed control designs developed in the literature are linear control laws based on linearized dynamics about the hover equilibrium point. Linear control designs are characterized by a local domain of convergence and cannot arguably provide high-performance control over a wide range of operating conditions. It is important to point out that flapping flight involve periodically-varying actuation wherein aerodynamic forces and moments are highly time-varying and periodic. Averaging techniques can be applied to periodically forced systems to remove time dependency of the original periodic system. This has motivated researchers to employ averaging methods in analysis and design of controllers for flapping wing MAV [12] [13]. For the application of averaging techniques, the system dynamics must be slower than the periodic excitation of forcing function. Hence, such systems must naturally exhibit a two-time scale dynamics in our case, a fast time scale due to high flapping frequency, and a slow time scale where the rigid-body motion of the vehicle evolves. This time scale separation allows aerodynamic forces and moments to be averaged over a wingbeat cycle, thereby leading to a drastic model simplification. However, resorting to averaging techniques must be exercised with caution, since the conditions for the applicability of the averaging method need to be satisfied. As a result, any flight control scheme that is based on averaging must be carefully vetted against the conditions that are necessary for an averaged system to be a suitable representation of the original dynamics.

As mentioned, a variety of studies have found that flapping-wing flight is inherently unstable [82] [62] [76]. While this instability provides enhanced responsiveness
to commands and increased agility, it comes at the expenses of requiring constant control actions to ensure stable flight [74] [10]. The survey papers [80] [57] provide a review of the current state of the art in FW-MAV modeling and of outstanding issues in control design. Flapping flight is an open challenge area of research and the field is still lacking a complete model that is convenient in a theoretical framework from control point of view. Considerable research on control design focuses linearized models of FW-MAV around a specific operating point. The majority of the proposed controllers are, therefore, linear control, where stability may be restricted to a small domain of attraction around the operating point. Accordingly, nonlinear control schemes may provide a better alternative in the quest for larger domains of attraction. FW-MAVs are, in general, under-actuated systems. This leads to systems that exhibit non-minimum phase behaviors [69] [67]. Similar to helicopters [35] [51], translational motion in insect flight is achieved by tilting the thrust vector, which is perpendicular to the stroke plane, towards the desired direction of motion [53]. The vertical component of the tilted thrust vector still provides lift to support vehicle’s weight, while the horizontal component will accelerate the vehicle into the desired direction. FW-MAV dynamics has many features in common with Vertical Takeoff and Landing (VTOL) aircraft, whose models are known to be unstable and non-minimum phase [27] [34].

Deng et al. [13] emphasized the similarities between insect and helicopter flights, from a control design standpoint, in that a forward thrust is produced by pitching the vehicle body forward while side way motions are obtained through rolling moment. The authors in [13] claimed that four control variables are required to control a flapping wing MAV wherein four control inputs are sufficient to stabilize helicopter’s
flight. The same authors proposed in [12] a method of manipulating aerodynamic forces and moments through two physical actuators per wing to utilize the proposed wing kinematics, namely, flapping amplitude and wing angle of attack. The number of actuators is a critical issue in designing a flapping wing MAV due to size and weight limitations. All three aerodynamic moments involve non-negligible couplings where pitching moment is always associated with forward thrust while rolling and yawing moments are associated with sideways force. The authors in [12] used linear quadratic regulator method to design a flight controller and demonstrated in simulations that hovering flight can be stabilized with a bounded error.

Khan et al. [42] developed a longitudinal flight controller for a flapping wing MAV using a differential flatness based control technique. The authors demonstrated that the longitudinal dynamics of the flapping wing MAV system is differentially flat with respect to the Huygens center of oscillation selected as the flat output. In [42], a differential flatness based trajectory tracking controller has been developed and applied to the averaged system. Simulations show that the averaged system trajectories converge to the trajectories defined by the flat outputs dynamics with a bounded error. However, this does not guarantee that the trajectories of the original system will converge to the averaged ones since the averaging theorem conditions were not satisfied in that work.

Doman et al.[19] proposed a technique to generate horizontal thrust and yaw moment by manipulating the duration of the downstroke and upstroke in each wingbeat cycle. The seminal work [19] employs the so-called split-cycle parameters to generate asymmetric velocities for the downstroke and upstroke. According to [19], the main motivation for introducing the split-cycle parameters is to produce non-zero cycle
averaged wing drag which is necessary to generate forward thrust as well as yawing moment. The wing angle of attack is kept constant throughout both the upstroke and downstroke and changes instantaneously at the end of each stroke. The proposed split-cycle method parameterizes the wing kinematics by manipulating the fundamental frequency and split-cycle parameter of each wing. However, the split-cycle method can only provide vertical and horizontal body forces as well as rolling and yawing moments. In [19] the use of a bob-weight is proposed to generate pitching moments by shifting the vehicle center of gravity. This dictates that a flapping wing MAV must have three physical actuators, one per each wing and one to control the bob-weight location. The authors of [19], then, applied the averaging technique to the instantaneous aerodynamic forces and moments and developed a mapping between the control variables and the averaged forces and moments. A linear control design for the linearized dynamics in the vicinity of hover was developed in [19]. However, simulations of the designed controller resulted in excessive shift of the bob-weight location. This suggests the impracticality of the bob-weight actuator due to the limited size of the vehicle.

Oppenheimer et al. [54] introduced a different wing kinematic parameterization to eliminate the need of the bob-weight actuator in their earlier work [19]. The work [54] modifies the split-cycle method by introducing a wing bias parameter that can offset the mean position of the wings motion, which results in direct control of the pitching moment. The proposed parameterization involves three wing kinematic parameters per each wing where fundamental frequency and split-cycle parameter provide the same four degrees of freedom as in their previous work, while wing bias parameter will replace the bob-weight to provide pitching moment. The authors have not proved
that the averaged system is exponentially stable which is essential for the original system to be stable by the averaging theory statement. Moreover, the proposed wing kinematic parametrization by the authors in [54] encounters discontinuities across consecutive flapping cycles which might introduce complex dynamics that are not physically realizable and render the application of averaging techniques problematic. The other limitations of their method is that the computation associated with the averaged forces and moments may be cumbersome and their approach tends to be *ad hoc* and does not rely on a rigorous framework.

Vernekar and Serrani [89] proposed a new wingbeat control strategy that manipulates the split-amplitude and the wingbeat frequency. In [89], the authors proposed a new kinematic parameterization, similar to the split-cycle parameter, which employs asymmetric amplitude rather than asymmetric frequency within the same wingbeat cycle. The proposed approach exploits asymmetric amplitudes between upstroke and downstroke of each wingbeat cycle while the wingbeat frequency exhibits phase modulation. By applying averaging technique, the authors of [89] computed the mapping between the proposed wing kinematic parameters and the averaged forces and moments. The ensuing linearized model is controllable and linear quadratic integral controller has been adopted in that study. Simulations show that the proposed controller is robust to a certain degree with respect to parametric uncertainties. However, the proposed modulation of the flapping amplitude might be difficult to realize physically. Bhatia *et al.*[5] have developed a linear controller to stabilize a hovering flight of FW-MAV using several sets of wing kinematic parameters under gust disturbances. The authors argued in [5] that flapping wing amplitude and wing-bias offers the best performance in terms of tolerating gust disturbances. Faruque and Humbert [27] [28]
analyzed fruit fly hovering stability for longitudinal and lateral dynamics. In addition, the same authors presented in [33] an analysis to evaluate the effect of stroke plane angle, wing-bias, and variable wing angle of attack on the FW-MAV stability. The authors concluded that modulating wing-bias and wing angle of attack are the most effective set of wing kinematic parameters.

Rifai et al.[60] adopted similar wing kinematics parameterization strategy to that proposed by Schenato et al. in [66]. However, Rifai et al.[60] introduced an actuated mass to control pitching moment rather than wing-bias proposed in [66]. Rifai et al.[60] assume that the equations of motion exhibit a cascade structure, where the translational dynamics depends on the rotational dynamics, whereas the rotational dynamics are independent on the translational ones. However, such assumption does not hold since strong cross coupling exist in vehicles of this kind, and longitudinal forces and moments can produce significant disturbances in the lateral dynamics [40].

Lee et al.[43] developed a linear control design to stabilize the longitudinal dynamics of FW-MAV through varying wingbeat frequency and stroke plane angle.

A robust nonlinear control design was developed by Serrani et al.[69] to the longitudinal dynamics of a flapping wing MAV. In [69], the authors established a rigorous methodology for controlling a minimally-actuated flapping wing MAV in which averaging techniques and stabilization by bounded state feedback can be effectively combined. The control strategy employs wingbeat frequency and stroke plane angle as wing kinematic parameters. The overall system was decomposed into subsystems with different time scales, where the vertical and horizontal dynamics represent the slower dynamics while the controlled pitch dynamics represent the fastest dynamics.
The proposed controller in [69] stabilizes the pitch angle oscillations while directly controlling the vertical and horizontal positions in the vicinity of hover.

The literature currently is lacking a theoretical justification of the averaging method for stability analysis where a rigorous two-time scale analysis is still missing. Averaging techniques constitute a powerful tool for analysis of periodically-forced systems. The approaches found in the literature for analyzing and designing control strategies for flapping-wing MAVs are sketchy at best, as the conditions for the applicability of averaging theorem are usually not satisfied, mostly because the wingbeat functions proposed in the literature are discontinuous or not differentiable. The aim of the present work is exactly that of overcoming all the mentioned limitation by proposing a rigorous theoretical framework for the analysis and control design of FW-MAV.

This investigation presents first a comprehensive control-oriented modeling for FW-MAV from a control standpoint, where all possible forms of actuation are considered. A novel wingbeat forcing function is proposed that ensures continuity of wingbeat motion across consecutive flapping cycles. The proposed wingbeat function is amenable to averaging techniques due to the fact that is continuously differentiable, with continuous and bounded derivatives of its control parameters. The novel wingbeat forcing function is used to develop a continuous-feedback for a minimally-actuated FW-MAV in the longitudinal plane. This control is based on two control inputs, provided by the proposed wingbeat function, which is the minimum number of the control inputs required for the vehicle in order to be capable of hovering and maneuvering in the longitudinal plane. It is shown that the longitudinal model of FW-MAV flight dynamics can be controlled with only the novel wingbeat function.
The controller developed in this work provides stable tracking of reference trajectories in the large and achieves boundedness of all trajectories of the system robustly with respect to parameter model uncertainty. We believe that this is the first result in the literature that combines a theoretically sound methodology based on averaging with the design of a controller that achieves a semiglobal domain of attraction with bounded effort, and is proved to be robust with respect to modeling uncertainty.

The thesis is organized as follows: the multi-body equations of motion for FW-MAV, including the tail dynamics, are developed in Chapter 2. In Chapter 3, the control-oriented modeling of the vehicle, including various actuation mechanisms is developed and the novel wingbeat forcing function is introduced. In Chapter 4, a robust nonlinear control for a minimally-actuated FW-MAV in the longitudinal plane is developed. Conclusion and future directions of research in this area are discussed in Chapter 5.
In this chapter, a nonlinear model for a prototypical bio-inspired FW-MAV is developed from first principles following the results already available in the literature. This model will be used as a basis for the development of a control-design model (performed in the next chapter) and shall serve as a simulation model for control validation. Our work follows very closely the one of Bolender [7], where Kane’s equations [38] were employed to derive the equations of motion of a multi-body FW-MAV. In addition, in this chapter, the longitudinal dynamics of the system are also derived using Euler-Lagrange's method for the purpose of model validation and verification of the longitudinal dynamics developed using Kane’s approach.

The vehicle considered in this work is similar to the one given in Bolender [7]. The vehicle configuration consists of two articulated rigid bodies, namely a main body (thorax) and a tail (abdomen), and a pair of massless wings. The general structure of the FW-MAV is shown in Figure 2.1. The main body of the FW-MAV system, denoted by the letter B, and the tail, denoted by the letter T, are connected by a fixed hinge joint. The wings are attached to the thorax segment of the vehicle. The wing inertia is neglected as wing mass comprise less than 1% of the total vehicle weight. The aerodynamic forces and moments are assumed to be produced by the
wings only. Figure 2.2 shows the wings motion within the stroke plane. The tail is used primarily for pitch control of the vehicle attitude. A minor contributions of this investigation is the incorporation of the tail dynamics of the vehicle, which was not given in [7]. A detailed description of the vehicle is given in [7]. Here, we give a summary of the main features of the vehicle dynamics, and provide an analysis of the control forces which are produced by the chosen actuation methods.

**Coordinate Frames of Reference**

Analysis and modeling of the vehicle dynamics require the use of several coordinate frames. Figure 2.1 shows the coordinate frames of the FW-MAV used for modeling purpose, as given originally in [7]. The abdomen/tail section $T$ has a single rotational degree of freedom and rotates relative to the body $B$. 

![Figure 2.1: Schematics of a multi-body flapping wing MAV [6]](image-url)
The downstroke can occur. The schematic in Fig. 2.1 illustrates the evolution of aerodynamic force throughout the stroke cycle. In particular, samples labeled 2 to 4 represent the transition from downstroke to upstroke, i.e. supination.

In presence of both translational and rotational movements of the wings, conventional steady-state aerodynamic theory is unable to explain how flying insects manage to remain in the air, let alone their remarkable capabilities such as hovering, flying sideways and in some cases, even landing upside down. This has prompted the search for unsteady mechanisms that might explain the high forces produced by flapping wings.

It is customary in mechanical and aerospace engineering to develop the equations of motion with respect to a body-fixed frame, due to the fact that the ensuing computations and analysis are considerably simplified. Hence, one needs to introduce several reference frames in order to precisely describe the motion of the main body of the FW-MAV relative to an inertial coordinate frame $\mathcal{N}$ [7]. The body coordinate frame, denoted by $\mathcal{B}$, is a body-fixed frame attached to the center of mass of main body $\mathcal{B}$ as depicted in Figure 2.3. Let an inertial (fixed) coordinate frame $\mathcal{N}$ be fixed in Euclidean space, and let $(x, y, z)$ be the coordinates of the main body $\mathcal{B}$ center of mass in the frame $\mathcal{N}$. The orientation of the inertial frame $\mathcal{N}$ and the body-fixed frame $\mathcal{B}$ are defined to be consisted with the standard aircraft coordinate system [75] [9]. The origin of the body-fixed coordinate frame $\mathcal{B}$ is located at the center of mass of body $\mathcal{B}$. As shown in Figure 2.3, the body-fixed coordinate frame $\mathcal{B}$ is defined such that the $x$-axis is positive forward along the longitudinal axis of the main body $\mathcal{B}$. The $y$-axis is positive out of the right side of the main body $\mathcal{B}$, and the $z$-axis is positive downward to create a right handed coordinate system. The unit vectors of
Figure 2.3: The FWMAV system consists of main body $\mathbf{B}$, where the wings are attached to the system, and the tail/abdomen $\mathbf{T}$.

The body frame $\mathbf{B}$ are defined by $\mathbf{B} = (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$, where the plane spanned by the unit vectors $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_3)$ is assumed to be the plane of symmetry for the main body $\mathbf{B}$.

In addition to the main body coordinate frame $\mathbf{B}$, a reference coordinate frame $\mathbf{T}$ is attached to the tail section $\mathbf{T}$ of the vehicle with its origin at the center of mass of the tail segment. As shown in Figure 2.3, the coordinate triad for the tail reference frame $\mathbf{T}$ is aligned such that the $x$-axis is pointing forward, the $y$-axis is positive out of the right side of tail body, and the $z$-axis is positive downward, where the $y$-axis creates a right handed coordinate system with respect to the $x$-$z$ plane. The unit vectors of the tail coordinate frame $\mathbf{T}$ are denoted by $\mathbf{T} = (\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2, \hat{\mathbf{t}}_3)$. The plane formed by the unit vectors $(\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_3)$ is assumed to be the plane of symmetry for the tail section $\mathbf{T}$. 

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As mentioned, the main body $B$ and the tail $T$ are the two articulated rigid bodies that comprise the fuselage of the multi-body FW-MAV system. The tail section $T$ is attached to the main body $B$ through a fixed hinge joint and allowed only a single degree of freedom to rotate about $\hat{\mathbf{b}}_2$ of the $B$ frame through the fixed hinge joint.

2.1 Modeling of the 6-DOF FW-MAV using Kane’s Method

When dealing with a multi-body system that is comprised of several connected rigid bodies, Kane’s method is known to be advantageous over Lagrange’s method for developing the equations of motion of the overall system. The advantage of Kane’s approach is that once the kinematics of the main body in the system are developed, the kinematics of other bodies in the system can be easily incorporated [7]. On the other hand, developing the flight dynamics of a multi-body vehicle with Lagrange’s Equations can be very complex and lengthy as a result of the fact that the generalized velocities are assigned to be the time derivative of the generalized coordinates. Moreover, Lagrange’s equations describe the dynamical system with respect to an inertial reference frame which may be undesirable, particularly where describing computing the flight dynamics of an aircraft. This is due to the fact that the aircraft lift and drag forces produced by its wings, and hence the aerodynamic forces and moments, are best defined in terms of the velocity components in the body reference frame [39]. Kane’s method for formulating dynamical equations, on the other hand, is best suited in modeling complex dynamical systems and developing the equations of motion of a multi-body systems. Kane’s method applies the concept of the generalized speeds which are defined as functions of the generalized coordinates and their time derivatives that describe the configuration space of a multi-body system. Furthermore,
Kane’s equations has the advantage that it can be developed using the body generalized speeds rather than the inertial speeds. Hence, one can work with quantities that are resolved in more natural reference frame. It is important to note that at least one generalized speed is required for each degree of freedom given to the dynamical system. In order to use Kane’s equations, one needs to compute the linear and angular accelerations of each body in the same set of coordinate basis vectors. Moreover, the other requirement needed for developing Kane’s Equations for a system consisting of two articulated bodies, is that the linear and angular accelerations of the second body (tail T) must be written in the same coordinate frame of the main body B [7].

2.1.1 Introduction to Kane’s Equations

Kane’s equations describe the motion of a dynamical system by formulating equations for each degree of freedom given to the system using the associated generalized coordinates. To introduce Kane’s method, a brief description of the fundamental basis behind Kane’s formulations [7] [37] [38] [39] will be presented.

Kane’s equations exploit Jourdain’s Principle to develop a system equations of motion. This principle can be described for a system consisting of N particles and given p degrees of freedom as:

$$\sum_{r}^{p} \left( \sum_{i}^{N} (m_{i} \ddot{\mathbf{r}}_{i} - \mathbf{F}_{i}) \cdot \frac{\partial \mathbf{r}_{i}}{\partial u_{r}} \right) \delta u_{r} = 0$$ (2.1)

where \( \mathbf{r}_{i} \) is the inertial position vector locating the \( i^{th} \) particle, with mass \( m_{i} \), \( \mathbf{F}_{i} \) represents the total force on the particle \( i \), \( u_{r} \) is the generalized speed of the \( i^{th} \) degree of freedom, and \( \frac{\partial \mathbf{r}_{i}}{\partial u_{r}} \) is the \( i^{th} \) partial velocity of \( i \) associated with the generalized speed \( u_{r} \). Jourdain’s Principle can be simplified for a system of N particles with p degrees
of freedom as
\[ \sum_{i}^{N} (m_i \ddot{r}_i - F_i) \cdot \frac{\partial \dot{r}_i}{\partial u_r} = 0, \quad r = 1, \cdots, p \] (2.2)

Kane’s equations, which are associated with each degree of freedom that is given to the system, can be formulated by summing up the generalized inertial forces, \( F_r^* \), and the generalized active forces, \( F_r \), and equate them to zero. This can be simply expressed in the following form
\[ F_r + F_r^* = 0, \quad r = 1, \cdots, p \] (2.3)
where \( F_r \) is the \( r^{th} \) generalized active forces, \( F_r^* \) is the \( r^{th} \) generalized inertial forces, and \( p \) is the number of degrees of freedom given to the system.

The generalized inertial forces, that are associated with each degree of freedom are found as
\[ F_r^* = \sum_{i=1}^{N} \frac{\partial \dot{r}_i}{\partial u_r} \cdot F_i^* + \sum_{j=1}^{N_{rb}} \frac{\partial \omega_i}{\partial u_r} \cdot M_i^*, \quad r = 1, \cdots, p \] (2.4)
where \( \frac{\partial \dot{r}_i}{\partial u_r}, \frac{\partial \omega_i}{\partial u_r} \) are the partial velocity and the partial angular velocity associated with generalized speeds, \( u_r \), respectively. The first term of the generalized inertial force is found in terms of the total sum of number of particles, \( N_p \), and number of rigid bodies, \( N_{rb} \), of the system under consideration, that is, \( N = N_p + N_{rb} \).

The generalized inertial forces, \( F_r^* \), and the generalized inertial moment, \( M_i^* \), in the presence of translational and rotational motion, can be found using the following expressions
\[ F_r^* = -m_i \ddot{r}_i \] (2.5)
\[ M_i^* = -I_i \dot{\omega}_i - \omega_i \times I_i \omega_i - r_i/O \times m_i \ddot{r}_i \]
where \( r_i/O \) is the position vector that locates the force action point relative to the center of mass of the \( i^{th} \) rigid body, of the multi-body system, \( \ddot{r}_i \) is the linear acceleration of the center of mass of the \( i^{th} \) rigid body, \( I_i \) is the moment of inertia about
the center of mass, \( m_i \) is the mass of the \( i^{th} \) rigid body, with angular velocity and angular acceleration, respectively, given by \( \omega_i, \dot{\omega}_i \). The last term in the generalized inertial moment, \((\mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i)\) includes the coupling contribution between the two rigid bodies in the FW-MAV system.

The generalized active forces that are associated with \( p \) degrees of freedom are expressed by the relation

\[
F_r = \sum_{i=1}^{N} \frac{\partial \mathbf{r}_i}{\partial u_r} \cdot F_i + \sum_{j=1}^{N_{rb}} \frac{\partial \omega_i}{\partial u_r} \cdot M_i, \quad r = 1, \cdots, p
\]

(2.6)

where \( F_i \) is the external forces acting on the system and \( M_i \) are the external moments applied to the system. The generalized active forces include the effect of the gravitational forces acting on each particle and rigid bodies of the system.

Once the generalized inertial and active forces have been computed, the system equations of motion can be formed by formulating Kane’s equations, which can be further simplified to the following familiar form [7]

\[
\mathbf{M} (t, \mathbf{q}) \ddot{\mathbf{u}} (t) + \mathbf{f} (\mathbf{u}, \mathbf{q}, t) + F_r (t) = 0
\]

(2.7)

where the matrix \( \mathbf{M} \) is the generalized inertia matrix.

### 2.1.2 Generalized Speeds

The first step in formulating the equations of motion for any dynamical system is to define the configuration space of the system using generalized coordinates. The generalized coordinates describe the position and orientation of each body in a dynamical system relative to each other or even to an inertial frame. In order to embed any configuration constraints of a multi-body system, Kane’s approach introduces the concept of the generalized speed. The generalized speeds are defined as linear combinations of the time derivatives of the generalized coordinates, leading to equations
of the following form [38]

\[ u_r = \sum_{i=1}^{p} Y_{rs} \dot{q}_s + Z_r, \quad r = 1, \ldots, p \]  

(2.8)

where \( Y_{rs} \) and \( Z_r \) are functions of the generalized coordinates. The functions \( Y_{rs} \) and \( Z_r \) should be chosen appropriately such that mapping between the generalized speeds and the derivative of the generalized coordinates is invertible. This invertibility is necessary in order to recover the time derivative of the generalized coordinates.

It is important to choose the generalized speeds as quantities in the dynamical system that are convenient for the analysis. In developing aircraft flight dynamics, the natural choice in defining the generalized speeds would be the linear velocity components and the angular velocity components of the aircraft fixed-body reference frame [37]. These generalized speeds can be correlated to the generalized coordinates of the Cartesian coordinates and the Euler angles, for the position and the orientation of an aircraft, with an invertible transformation. The angular velocity components in terms of the body reference frame \((P, Q, R)\) can be related to Euler angles and their time derivatives by the following invertible transformation [38]:

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}
= \begin{bmatrix}
P \\
  Q \\
  R
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & -\sin \phi \\
  0 & \cos \phi & \cos \theta \sin \phi \\
  0 & -\sin \phi & \cos \theta \cos \phi
\end{bmatrix}
\begin{bmatrix}
  \dot{\phi} \\
  \dot{\theta} \\
  \dot{\psi}
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & -\sin \phi \\
  0 & \cos \phi & \cos \theta \sin \phi \\
  0 & -\sin \phi & \cos \theta \cos \phi
\end{bmatrix}
\begin{bmatrix}
  \dot{q}_1 \\
  \dot{q}_2 \\
  \dot{q}_3
\end{bmatrix}
\]  

(2.9)

The time derivatives of the generalized coordinates are obtained as function of the generalized speeds as follows:

\[
\begin{bmatrix}
  \dot{q}_1 \\
  \dot{q}_2 \\
  \dot{q}_3
\end{bmatrix}
= \begin{bmatrix}
  1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
  0 & \cos \phi & -\sin \phi \\
  0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}
\]  

(2.10)

The generalized speeds that will used to derive Kane’s equations will be the traditional aircraft body axis velocity, \((U, V, W)\) which defines the translational velocity.
components in the main body frame, and \((P, Q, QT, R)\) which are the angular velocity components in the main body frame as well. The generalized speed \(QT\) is added in order to describe the angular deflection of the tail \(T\). Hence, the generalized speeds of the FW-MAV multi-body system are defined as \(u_r = \{U, V, W, P, Q, QT, R\}\).

2.1.3 Development of FW-MAV Kinematics

Kane’s method requires that the linear and angular accelerations of the system must be formulated \(a \text{ priori}\) for each body. In addition, the kinematics of each body must be expressed in the same set of coordinate frame. In order to track the velocity of the main body \(B\) relative to the inertial frame \(N\), one needs to define the position vector of body \(B\), expressed in the coordinate frame \(B\), with respect to the inertial frame \(N\) as follows

\[
r_B = x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3
\]  

(2.11)

To compute the velocity of a position vector of a rigid body defined in a rotating coordinate (body) frame relative to a fixed coordinate (inertial) frame, one needs to apply the \(Transport\ \text{Theorem}\), which allows one to compute the time derivative of a position vector of a rigid body relative to the inertial frame \(N\), where the position vector \(r_B\) is defined in the (rotating) body frame \(B\).

In order to determine the velocity of body \(B\) relative to the inertial frame \(N\), the rate of change of \(r_B\) relative to the inertial frame \(N\) is determined. Using the chain rule of differentiation, the time derivative of the vector \(r_B\) can be found as follows

\[
\left\{\frac{dr_B}{dt}\right\}_N = N\dot{r}_B = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 + \dot{z}\hat{b}_3 + x\ddot{b}_1 + y\ddot{b}_2 + z\ddot{b}_3
\]  

(2.12)
The first three terms are the time derivative of \( \mathbf{r}_B \) measured in \( \mathcal{B} \), that is,

\[
\left\{ \frac{d \mathbf{r}_B}{dt} \right\}_B = \dot{x} \hat{b}_1 + \dot{y} \hat{b}_2 + \dot{z} \hat{b}_3 \quad (2.13)
\]

On the other hand, \((\dot{\hat{b}}_1, \dot{\hat{b}}_2, \dot{\hat{b}}_3)\) are the time derivatives of the body frame \( \mathcal{B} \) unit vectors relative to the inertial frame \( \mathcal{N} \). For any unit vector that is fixed in the body frame \( \mathcal{B} \) while it is rotating relative to the inertial frame \( \mathcal{N} \), its rate of change can be described as

\[
\left\{ \frac{d \hat{b}_1}{dt} \right\}_N = \mathcal{N} \omega^B_B \times \hat{b}_1 \quad (2.14)
\]

The superscript, \( \mathcal{N} \), to the left of \( \mathcal{N} \omega^B_B \) denotes the frame in which the body \( \mathcal{B} \) is rotating, while the superscript, \( \mathcal{B} \), to the right of \( \mathcal{N} \omega^B_B \) is the frame where the unit vector is fixed and not rotating. The last three terms define the rotation of body \( \mathcal{B} \) as seen by an observer from the inertial frame \( \mathcal{N} \):

\[
\left\{ \frac{d \mathbf{r}_B}{dt} \right\}_N = \left\{ \frac{d \mathbf{r}_B}{dt} \right\}_B + \mathcal{N} \omega^B_B \times \mathbf{r}_B \quad (2.15)
\]

where \( \mathcal{N} \omega^B_B \) is the angular velocity of body \( \mathcal{B} \) defined in the coordinate frame \( \mathcal{B} \) as measured by an observer in inertial frame \( \mathcal{N} \). The angular velocity vector, expressed in the body frame \( \mathcal{B} \), is given by

\[
\mathcal{N} \omega^B_B = P \hat{b}_1 + Q \hat{b}_2 + R \hat{b}_3 \quad (2.16)
\]

**Body B Kinematics**

The position of the center of mass of the body \( \mathcal{B} \) relative to the inertial frame \( \mathcal{N} \) is described using the Cartesian coordinates given by the position vector \( \mathbf{r}_B = (x, y, z)^T \in \mathbb{R}^3 \). The translational velocity of the body \( \mathcal{B} \) relative to the inertial frame \( \mathcal{N} \) is given by the velocity vector \( \mathcal{N} \dot{\mathbf{r}}_B = (U, V, W)^T \in \mathbb{R}^3 \) expressed in the body frame \( \mathcal{B} \). Likewise, the angular velocity of the main body with respect to the inertial
frame $\mathcal{N}$ is defined by the vector $^\mathcal{N}\omega_B^B = (P, Q, R)^T \in \mathbb{R}^3$ expressed in the main body frame $\mathcal{B}$. Let the rotation matrix that maps vectors from the body frame $\mathcal{B}$ to the inertial frame $\mathcal{N}$ be $^\mathcal{N}R_B^B \in SO(3)$. The rotation matrix $^\mathcal{N}R_B^B$ is parameterized in terms of three Euler angles. The translational and angular velocities of the center of mass of main body $\mathcal{B}$ is defined in the body frame $\mathcal{B}$ with respect to the inertial frame $\mathcal{N}$ as follows

$$^\mathcal{N}\dot{r}_B = U\hat{b}_1 + V\hat{b}_2 + W\hat{b}_3$$

$$^\mathcal{N}\omega_B^B = P\hat{b}_1 + Q\hat{b}_2 + R\hat{b}_3$$

The attitude of the body $\mathcal{B}$ is described with respect to the inertial frame $\mathcal{N}$ by the Euler angles $\Psi$ (yaw), $\Theta$ (pitch), and $\Phi$ (roll). The orientation of coordinate frame $\mathcal{B}$ is related to the inertial frame $\mathcal{N}$ by 3-2-1 rotation sequence, defined by the following transformation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_\mathcal{N} = ^\mathcal{N}R_B^B \begin{bmatrix} U \\ V \\ W \end{bmatrix}_B$$

Applying the Transport Theorem, the velocity of body $\mathcal{B}$ relative to the inertial frame $\mathcal{N}$ expressed in $\mathcal{B}$ frame, is found as:

$$^\mathcal{N}\dot{r}_B = \frac{\delta r_B}{\delta t} + ^\mathcal{N}\omega_B^B \times r_B$$

$$= (\dot{x} + Qz - Ry)\hat{b}_1 + (\dot{y} - Pz + Rx)\hat{b}_2 + (\dot{z} + Py - Qx)\hat{b}_3$$

$$= U\hat{b}_1 + V\hat{b}_2 + W\hat{b}_3$$

**Tail Section T Kinematics**

The main body $\mathcal{B}$ and the tail segment $\mathcal{T}$ rotate about a fixed hinge joint that connects the two rigid bodies. The tail segment $\mathcal{T}$ is given a single degree of freedom as it rotates relatively to the main body $\mathcal{B}$ about the $\hat{b}_2$ axis. One of the motivations for an actuated tail $\mathcal{T}$ is to have an independent control authority over the pitch attitude of the body $\mathcal{B}$ obtained by changing the vehicle center of mass. The pitch
angle of the body \( \mathbf{B} \) (denoted by \( \Theta \)) is determined by the rotation of the longitudinal axis \( \mathbf{b}_1 \) with respect to the horizontal, while the rotation of the tail segment \( \mathbf{T} \) relative to the body \( \mathbf{B} \), is defined by the angle \( \Theta_T \), as shown in Figure 2.4. The motion of the tail relative to \( \mathbf{B} \) is described by the angle velocity \( Q_T \) of the tail about the axes \( \mathbf{t}_2 = \mathbf{b}_2 \). It is assumed that the body axes fixed in \( \mathbf{B} \) and \( \mathbf{T} \) are the principal axes of inertia.

Since the origin of the tail coordinate frame \( \mathcal{T} \) is located at the center of mass of the tail, the position vector of the tail relative to the center of mass of main body is given by

\[
r_{T/B} = r_{T/H} + r_{H/B}
\]
where the subscript $H$ denotes the hinge point location where the tail $T$ is attached to the body $B$. It is important to note that the location of the hinge point does not coincide with the center of mass of $B$. As shown in Figure 2.4, the center of mass of $B$ is located at a distance $\ell_b$ from the hinge joint on the $\mathbf{b}_1$ axis, while the center of mass of $T$ is located at a distance $\ell_t$ from the hinge joint on the $\mathbf{t}_1$ axis. Hence, the components of the vector $r_{T/B}$ are constant where $r_{H/B} = -\ell_b \mathbf{b}_1$ and $r_{T/H} = -\ell_t \mathbf{t}_1$.

Since the position vector, $r_{T/H}$, from the tail $T$ to the hinge $H$ is expressed in the tail frame $T$, a rotation matrix is needed to resolve its coordinates in the frame $B$ as follows

$$r_{T/B} = r_{T/H} + r_{H/B}$$

$$= \begin{bmatrix} \cos \Theta_T & 0 & -\sin \Theta_T \\ 0 & 1 & 0 \\ \sin \Theta_T & 0 & \cos \Theta_T \end{bmatrix} \begin{bmatrix} -\ell_t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\ell_b \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\ell_b - \ell_t \cos \Theta_T \\ 0 \\ \ell_t \sin \Theta_T \end{bmatrix}$$

The tail $T$ is allowed to rotate about $\mathbf{b}_2$ only, hence, the angular velocity of $T$ relative to the inertial frame $N$ is

$$\dot{N} \omega_T^T = \dot{N} \omega_B^B + \dot{N} \omega_{T/B}$$

$$\dot{N} \omega_T^T = P \dot{\mathbf{b}}_1 + (Q + Q_T) \mathbf{b}_2 + R \mathbf{b}_3$$

where $\dot{N} \omega_{T/B} = Q_T \dot{\mathbf{b}}_1 = Q_T \dot{\mathbf{t}}_2$ and $Q_T = d\Theta_T/dt$.

Since the velocity of the center of mass of body $B$ has been defined, the velocity of the tail $T$ is obtained by considering the motion relative to the main body. The velocity of the tail written in the main body frame $B$, relative to the inertial frame $N$ as follows

$$\dot{N} \mathbf{i}_T = \dot{N} \mathbf{i}_B + \dot{N} \omega_B^B \times r_{H/B} + \dot{N} \omega_T^T \times r_{T/H}$$

$$= \begin{bmatrix} U + \ell_t \sin \Theta_T (Q + Q_T) \\ V - \ell_b R - \ell_t \cos \Theta_T R - \ell_t \sin \Theta_T P \\ W + \ell_b Q + (Q + Q_T) \ell_t \cos \Theta_T \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$

(2.23)
The velocity of the hinge point $H$ depends on the velocity of the main body center of mass $\dot{r}_B$, as shown by the following expression

$$N\dot{r}_{H/B} = N\dot{r}_B + N\omega_B \times r_{H/B}$$

$$= \begin{bmatrix} U \\ V - \ell_b R \\ W + \ell_b Q \end{bmatrix} [\hat{b}_1 \hat{b}_2 \hat{b}_3] \quad (2.24)$$

Note that the velocity vector $N\dot{r}_B$ is defined at the main body center of mass while the velocity vector $N\dot{r}_{H/B}$ is defined at the hinge point which is at a negative $\ell_b$ distance from the main body center of mass.

**Development of Linear and Angular Accelerations**

Once the linear and angular velocities of both bodies have been computed, the linear and angular accelerations of body $B$ and tail $T$ can be found. The linear and angular accelerations of body $B$ and tail $T$ are obtained relative to the inertial frame $N$ by applying the *Transport Theorem* as done previously. The linear accelerations of the body $B$, the hinge point $H$, and the tail $T$ are expressed as follows

$$N\ddot{r}_{H/B} = \begin{bmatrix} \dot{U} + QW - RV + \ell_b(Q^2 + R^2) \\ \dot{V} + RU - PW - \ell_b \dot{R} - \ell_b PQ \\ \dot{W} + PV - QU + \ell_b \dot{Q} - \ell_b PR \end{bmatrix} [\hat{b}_1 \hat{b}_2 \hat{b}_3]$$

$$N\ddot{r}_T = A_T [\hat{b}_1 \hat{b}_2 \hat{b}_3]$$

where

$$A_T = \begin{bmatrix} \dot{U} + QW - RV + \ell_t \sin \theta_T(\dot{Q} + \dot{Q}_T) + \ell_b Q^2 + \ell_t \cos \theta_T(Q + Q_T)^2 + \ldots + \ell_b R^2 + \ell_t \cos \theta_T R^2 + \ell_t \sin \theta_T PR \\ \dot{V} + RU - PW - (\ell_t - \ell_t \cos \theta_T) \dot{R} + (\ell_t \sin \theta_T) \dot{P} - \ell_t PQ + \ldots - \ell_t \cos \theta_T(Q + Q_T)^2 + \ell_t \sin \theta_T(RQ + 2RQ_T) \\ \dot{W} + PV - QU + \ell_b \dot{Q} + \ell_t \cos \theta_T(\dot{Q} + \dot{Q}_T) - \ell_b PR - \ell_t \cos \theta_T PR + \ldots - \ell_t \sin \theta_T P^2 - \ell_t \sin \theta_T(Q + Q_T)^2 \end{bmatrix}$$

The angular accelerations of $B$ and $T$ read as follows

$$N\ddot{\omega}_B = \dot{P}\hat{b}_1 + \dot{Q}\hat{b}_2 + \dot{R}\hat{b}_3$$

$$N\ddot{\omega}_T = (\dot{P} - RQ_T)\hat{b}_1 + (\dot{Q} + \dot{Q}_T)\hat{b}_2 + (\dot{R} + PQ_T)\hat{b}_3$$

(2.26)
2.1.4 Generalized Inertial Forces

Kane’s equations are formulated by combining the generalized inertial forces and the generalized active forces. The generalized inertial forces involve the contribution of quantities that depend on the time rate of change of momentum of the system. Since the linear and angular accelerations of both bodies of the system have been computed, the generalized inertial forces of both bodies are found using Equation (2.4) for each generalized speed, \( u_r \). The generalized inertial forces for \( B \) and \( T \) can be computed by

\[
\mathbf{F}^*_r = \frac{\partial \omega^*_B}{\partial u_r} \cdot (\mathbf{M}^*_B - \mathbf{r}_{H/B} \times m_b \ddot{\mathbf{r}}_{H/B}) + \frac{\partial \dot{r}^*_B}{\partial u_r} \cdot (\mathbf{F}^*_B) \\
+ \frac{\partial \omega^*_T}{\partial u_r} \cdot (\mathbf{M}^*_T - \mathbf{r}_{T} \times m_t \ddot{\mathbf{r}}_T - \mathbf{r}_{T/H} \times m_t \ddot{\mathbf{r}}_T) + \frac{\partial \dot{r}^*_T}{\partial u_r} \cdot (\mathbf{F}^*_T) \tag{2.27}
\]

where \( \mathbf{F}^*_B \) and \( \mathbf{F}^*_T \) are the inertial forces for body \( B \) and tail \( T \), while \( \mathbf{M}^*_B \) and \( \mathbf{M}^*_T \) are the inertial moments for body \( B \) and tail \( T \). The coupling contribution of the rotational dynamics of each body about the hinge joint is computed using the cross product terms. The resulting expressions of the equations of motion in the Euclidean space take on a cumbersome form that will not be reported here. The reader is referred to [7] for the complete expressions. In what follows, however, we will develop explicitly the equations of motion in the longitudinal plane, and verify their correctness by comparison with the equations obtained adopting a Lagrangian formulation.

For the longitudinal plane, it is assumed that the generalized forces directed along the body axes \( \hat{\mathbf{b}}_1 \) and \( \hat{\mathbf{b}}_3 \) only, whereas the generalized torques act only along the \( \hat{\mathbf{b}}_2 \) axis, consequently.
The inertial forces for $B$ and $T$ are given as
\[
F^*_B = -m_b \ddot{r}_{H/B} = -m_b \begin{bmatrix} \dot{U} + QW + \ell_b \dot{Q}^2 \\ 0 \\ \dot{W} - QU + \ell_b \dot{Q} \end{bmatrix}
\]
\[
F^*_T = -m_t \ddot{r}_T = -m_t \begin{bmatrix} \dot{U} + QW + \ell_b \dot{Q}^2 + \ell_t \sin \theta_T (\dot{Q} + \dot{Q}_T) + \ell_t \cos \theta_T (Q + Q_T)^2 \\ 0 \\ \dot{W} - QU + \ell_b \dot{Q} + \ell_t \cos \theta_T (\dot{Q} + \dot{Q}_T) - \ell_t \sin \theta_T (Q + Q_T)^2 \end{bmatrix}
\]
(2.28)
where $m_b$ and $m_t$ are the moments of inertia of the body $B$ and tail $T$, respectively.

The inertial torques for $B$ and $T$ are given as
\[
M^*_B = -I^B_B{\omega}^E_B = -I^B_B{\omega}^E_B \times I^B_B{\omega}^E_B = - \begin{bmatrix} 0 \\ I^B_{yy} \dot{Q} \\ 0 \end{bmatrix}
\]
(2.29)
\[
M^*_T = -I^B_T{\omega}^E_T = -I^B_T{\omega}^E_T \times I^B_T{\omega}^E_T = - \begin{bmatrix} 0 \\ (\dot{Q} + \dot{Q}_T) I^T_{yy} \\ 0 \end{bmatrix}
\]
where $I^B_{yy}$ and $I^T_{yy}$ are the masses of the body $B$ and tail $T$, respectively.

The coupling of the rotational dynamics for $B$ relative to the hinge point $H$ is given as follows
\[
r_{H/B} \times m_b \ddot{r}_{H/B} = \begin{bmatrix} -l_b \\ 0 \\ 0 \end{bmatrix} \times m_b \ddot{r}_{H/B} = m_b \begin{bmatrix} \ell_b (\dot{W} - QU + \ell_b \dot{Q} - \ell_b PR) \\ 0 \\ 0 \end{bmatrix}
\]
(2.30)
The coupling of the rotational dynamics for $T$ relative to $B$ is given as follows
\[
r_{T/B} \times m_t \ddot{r}_T = \begin{bmatrix} -\ell_b - \ell_t \cos \Theta_T \\ 0 \\ \ell_t \sin \Theta_T \end{bmatrix} \times m_t \ddot{r}_T
\]\[
= \begin{bmatrix} m_t (l_t \sin \theta_T) [\dot{U} + QW + l_t \sin \theta_T (\dot{Q} + \dot{Q}_T) + l_b \dot{Q}^2 + \cdots] \\ + l_t \cos \theta_T (Q + Q_T)^2 - m_t (-l_b - l_t \cos \theta_T) [\dot{W} - QU + l_b \dot{Q} + \cdots] \\ + l_t \cos \theta_T (\dot{Q} + \dot{Q}_T) - l_t \sin \theta_T (Q + Q_T)^2 \end{bmatrix}
\]
(2.31)
The coupling of the rotational dynamics for $T$ relative to the hinge joint for $H$ is given as follows

$$
\mathbf{r}_{T/H} \times m_t \ddot{\mathbf{r}}_T = \begin{bmatrix}
-\ell_t \cos \Theta_T \\
0 \\
\ell_t \sin \Theta_T
\end{bmatrix} \times m_t \ddot{\mathbf{r}}_T
$$

$$
= \begin{bmatrix}
m_t (\ell_t \sin \theta_T) [\dot{U} + QW + \ell_t \sin \theta_T (\dot{Q} + \dot{Q}_T) + l_b Q^2 + \cdots] \\
+ \ell_t \cos \theta_T (Q + Q_T)^2 - m_t (-\ell_t \cos \theta_T) [\dot{W} - QU + l_b \dot{Q} + \cdots] \\
+ \ell_t \cos \theta_T (\dot{Q} + \dot{Q}_T) - \ell_t \sin \theta_T (Q + Q_T)^2
\end{bmatrix}
$$

(2.32)

### 2.1.5 Generalized Active Forces

The generalized active forces involve all external forces that are applied to the system as well as the gravitational forces. These external forces include the aerodynamic forces and moments produced by the flapping wings and the force due to gravity of each body in the system. The generalized active forces of Kane’s Equations have the following form

$$
\mathbf{F}_r = \frac{\partial \omega_B}{\partial \mathbf{u}_r} (\mathbf{M}_B + \mathbf{r}_{H/B} \times m_b \mathbf{g}^B) + \frac{\partial \dot{\mathbf{r}}_B}{\partial \mathbf{u}_r} (\mathbf{F}_B + (m_b + m_t) \mathbf{g}^B) + \frac{\partial \overset{\cdot}{\mathbf{r}}_T}{\partial \mathbf{u}_r} (\mathbf{F}_T)
$$

(2.33)

where $\mathbf{g}^B$ is acceleration vector due to the gravity, written in the main body frame $\mathbf{B}$, which has the following form

$$
\mathbf{g}^B = g \begin{bmatrix}
-\sin \Theta \\
\cos \Theta \sin \Phi \\
\cos \Theta \cos \Phi
\end{bmatrix}
$$

(2.34)

Hence, the tail weight can be expressed in the main body frame using this gravity vector. The effect of the tail weight on the main body moment is computed by the term $\mathbf{r}_{T/B} \times m_t \mathbf{g}^B$. The aerodynamic forces and moments from the flapping wings
are $F_B$ and $M_B$, respectively. The moment induced about body $B$ due to the weight of FW-MAV is computed as follow

$$r_{H/B} \times m_b g_B = \begin{bmatrix} -l_b \\ 0 \\ 0 \end{bmatrix} \times m_b g_B = g \begin{bmatrix} -\sin \Theta \\ \cos \Theta \sin \Phi \\ \cos \Theta \cos \Phi \end{bmatrix} = m_b g \begin{bmatrix} 0 \\ l_b \cos \Theta \cos \Phi \\ -l_b \cos \Theta \cos \Phi \end{bmatrix}$$

(2.35)

and

$$r_{T/B} \times m_t g_B = \begin{bmatrix} -l_b - l_t \cos \Theta_T \\ 0 \\ l_t \sin \Theta_T \end{bmatrix} \times m_t g \begin{bmatrix} -\sin \Theta \\ \cos \Theta \sin \Phi \\ \cos \Theta \cos \Phi \end{bmatrix} = m_t g \begin{bmatrix} -l_t \sin \Theta_T \cos \Theta \sin \Phi \\ l_t \cos \Theta \cos \Theta_T \cos \Phi - l_t \sin \Theta \sin \Theta_T + l_b \cos \Theta \cos \Phi \\ -(l_b + l_t \cos \Theta_T) \cos \Theta \sin \Phi \end{bmatrix}$$

(2.36)

### 2.1.6 Longitudinal Equations of Motion using Kane’s Method

Since the generalized inertial forces and the generalized active forces have been computed, one can formulate Kane’s Equations using Equation (2.3). The generalized inertial and active forces can be combined to form Kane’s equations of motion. Recall that, from Equation (2.7), that Kane’s equations can take the following form

$$M(t, q) \dot{u}(t) + f(u, q, t) + F_r(t) = 0$$

(2.37)

The longitudinal equations of motion using Kane’s method can be arranged into matrix form as

$$M_{long} \begin{bmatrix} \dot{U} \\ \dot{W} \\ \dot{Q} \end{bmatrix} + f_{long}(u, q, t) + \begin{bmatrix} F_x - (m_b + m_t)g \sin \Theta \\ F_z + (m_b + m_t)g \cos \Theta \\ \tau + m_b g l_b \cos \Theta + m_t g (l_t \cos(\Theta + \Theta_T) + l_b \cos \Theta) \\ -\tau + m_t g l_t \cos(\Theta + \Theta_T) \end{bmatrix} = 0$$

(2.38)

where $M_{long}$ is the inertia matrix and given by

$$M_{long} = \begin{bmatrix} -m_b - m_t & 0 & M_{1,3} & M_{1,4} \\ 0 & -m_b - m_t & M_{2,3} & M_{2,4} \\ M_{1,3} & M_{2,3} & M_{3,3} & M_{3,4} \\ M_{1,4} & M_{2,4} & M_{3,4} & -m_t l_t^2 - I_{T_{yy}} \end{bmatrix}$$

(2.39)
\[ M_{1,3} = -m_t \ell_t \sin \Theta_T \]
\[ M_{1,4} = -m_t \ell_t \sin \Theta_T \]
\[ M_{2,3} = -m_b \ell_b - m_t (\ell_b + \ell_t \cos \Theta_T) \]
\[ M_{2,4} = -m_t \ell_t \cos \Theta_T \]
\[ M_{3,4} = -I_{yy} - m_t (\ell_t^2 + \ell_b \ell_t \cos \Theta_T) \]
\[ M_{3,3} = -m_b \ell_b^2 - m_t (\ell_b^2 + \ell_t^2 + 2 \ell_b \ell_t \cos \Theta_T) - I_{B_{yy}} - I_{T_{yy}} \]

\[
\mathbf{f}_{\text{long}}(\mathbf{u}, \mathbf{q}, t) = 
\begin{bmatrix}
-(m_b + m_t)QW - (m_b + m_t)\ell_b Q^2 - m_t \ell_t \cos \Theta_T (Q + Q_T)^2 \\
(m_b + m_t)QU + m_t \ell_t \sin \Theta_T (Q + Q_T)^2 \\
m_t QU (\ell_b + \ell_t \cos \Theta_T) + \cdots \\
+ m_t \ell_t \ell_b \sin \Theta_T Q_T (Q_T + 2Q) - m_t \ell_t \sin \Theta_T QW \\
-m_t \ell_b \ell_t \sin \Theta_T Q^2
\end{bmatrix}
\]

The external forces \((F_x, F_z)\) are the aerodynamic forces produced by the flapping wings and \(\tau\) is the torque applied at the joint connecting the main body and the tail. Conservation of the angular momentum entails that applying a torque at the first body, in the absence of external forces, will be balanced by an equal and opposite torque on the second body of the FW-MAV system.

### 2.2 Longitudinal Dynamics using Lagrange’s Method

The longitudinal dynamics of FW-MAV system were also derived using Lagrange’s Equations, mainly for the purpose of verifying the correctness of the expressions previously determined using Kane’s method.

Lagrange’s approach exploit the concept of the system energy. The Lagrangian formulation employs the system kinetic and potential energies in the derivation of the equations of motion with respect to the inertial frame. The Lagrangian \(\mathcal{L}\) is defined as the difference between the total kinetic energy and the total potential energy of
the system \[72\]

\[ \mathcal{L}(q_i, \dot{q}_i) = K(q_i, \dot{q}_i) - V(q_i) \] (2.42)

where \( K(q_i, \dot{q}_i) \) is the kinetic energy and \( V(q_i) \) the potential energy. Lagrange's equations of motion are given by

\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \quad i = 1, \ldots, N \] (2.43)

where \( \tau_i \) are the generalized forces acting on the generalized coordinates \( q_i \) for the system and \( N \) is the number of degrees of freedom. The total kinetic energy of the system can be written as the sum of translational and rotational kinetic energies of each segment of the system as follows

\[ K(q_i, \dot{q}_i) = K_t(q_i, \dot{q}_i) + K_r(q_i, \dot{q}_i) \] (2.44)

where \( K_t(q_i, \dot{q}_i) \) represents translational kinetic energy and \( K_r(q_i, \dot{q}_i) \) represents the rotational kinetic energy.

### 2.2.1 Generalized Coordinates

A set of generalized coordinates were selected to describe the position and orientation of both rigid bodies in the system. The most convenient set of generalized coordinates are \((x, z, \Theta, \Theta_T)\). The generalized coordinates \((x, z)\) represents the horizontal and vertical displacements, respectively, of the center of mass of body \( B \) with respect to the inertial frame. Recall from the previous section that \( \Theta \) represents the angular rotation of body \( B \) longitudinal axis relative to the horizontal, and \( \Theta_T \) represents the angular rotation of tail \( T \) longitudinal axis relative to the main body \( B \), as depicted in Figure 2.4. Counterclockwise rotation of the main body angle \( \Theta \) relative
to the horizontal is defined as positive rotation. Let $q$ be the vector of the generalized coordinates, as follows

$$ q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} x \\ z \\ \Theta \\ \Theta_T \end{bmatrix} $$ (2.45)

The configuration space of the selected generalized coordinates characterize the location of the center of mass of body $B$ with respect to the inertial frame as well as the angular deflections of longitudinal axis of the body $B$ and the tail $T$. The position vectors of the main body/thorax and the abdomen/tail with respect to the inertial frame are

$$ r_b = \begin{bmatrix} x - l_b \cos \Theta \\ z + l_b \sin \Theta \end{bmatrix} \hat{n}_1 \\ \hat{n}_3 $$

$$ r_t = \begin{bmatrix} x - l_b \cos \Theta - l_t \cos(\Theta + \Theta_T) \\ z + l_b \sin \Theta + l_t \sin(\Theta + \Theta_T) \end{bmatrix} \hat{n}_1 \\ \hat{n}_3 $$ (2.46)

The velocity vectors of the main body/thorax and the abdomen/tail are

$$ \dot{r}_b = \begin{bmatrix} \dot{x} + l_b Q \sin \Theta \\ \dot{z} + l_b Q \cos \Theta \end{bmatrix} \hat{n}_1 \\ \hat{n}_3 $$

$$ \dot{r}_t = \begin{bmatrix} \dot{x} + l_b Q \sin \Theta + l_t \sin(\Theta + \Theta_T)(Q + Q_T) \\ \dot{z} + l_b Q \cos \Theta + l_t \cos(\Theta + \Theta_T)(Q + Q_T) \end{bmatrix} \hat{n}_1 \\ \hat{n}_3 $$ (2.47)

where $Q = \dot{\Theta}$ and $Q_T = \dot{\Theta}_T$.

### 2.2.2 Kinetic and Potential Energy

The translational kinetic energy can be found by

$$ K_t = \frac{1}{2} m_b \dot{r}_b^T \dot{r}_b + \frac{1}{2} m_t \dot{r}_t^T \dot{r}_t $$

$$ = \frac{1}{2} m_b (\dot{x}^2 + \dot{z}^2 + l_b^2 Q^2 + 2 \dot{x} l_b Q \sin \Theta + 2 \dot{z} l_b Q \cos \Theta) $$

$$ + \frac{1}{2} m_t (\dot{x}^2 + \dot{z}^2 + 2 \dot{x} l_b Q \sin \Theta + 2 \dot{x} l_t Q \sin(\Theta + \Theta_T) + 2 \dot{z} l_t Q T \sin(\Theta + \Theta_T)) $$

$$ + 2 \dot{z} l_b Q \cos \Theta + 2 \dot{z} l_t Q \cos(\Theta + \Theta_T) + 2 \dot{z} l_t Q T \cos(\Theta + \Theta_T) + l_b^2 Q^2 $$

$$ + l_t^2 Q^2 + l_t^2 Q_T^2 + 2 l_t^2 Q Q_T + 2 l_b l_t Q^2 \cos \Theta_T + 2 l_b l_t Q Q_T \cos \Theta_T) $$ (2.48)
The rotational kinetic energy can be found as

\[ K_r = \frac{1}{2} I_B Q^2 + \frac{1}{2} I_T (Q + Q_T)^2 \]  

(2.49)

One can apply Equation (2.43) to the translational kinetic energy. The resultant terms for each generalized coordinate read as follows:

\[
\frac{d}{dt} \frac{\partial K_t}{\partial \dot{x}} - \frac{\partial K_t}{\partial x} = m_b [\ddot{x} + l_b \dot{Q} \sin \Theta + l_b Q^2 \cos \Theta] + m_t [\ddot{x} + l_b \dot{Q} \sin \Theta + l_b Q^2 \cos \Theta] \\
+ l_t \dot{Q} \sin(\Theta + \Theta_T) + l_t Q_T \cos(\Theta + \Theta_T)
\]

\[
\frac{d}{dt} \frac{\partial K_t}{\partial \dot{z}} - \frac{\partial K_t}{\partial z} = m_b [\ddot{z} + l_b \dot{Q} \cos \Theta - l_b Q^2 \sin \Theta] + m_t [\ddot{z} + l_b \dot{Q} \cos \Theta - l_b Q^2 \sin \Theta] \\
+ l_t \dot{Q} \cos(\Theta + \Theta_T) - l_t Q_T \sin(\Theta + \Theta_T)
\]

\[
\frac{d}{dt} \frac{\partial K_t}{\partial Q} - \frac{\partial K_t}{\partial \dot{Q}} = m_b [l_b^2 \dot{Q} + l_b \ddot{x} \sin \Theta + l_b \ddot{z} \cos \Theta] + m_t [l_b \ddot{x} \sin \Theta + l_b \ddot{z} \cos(\Theta + \Theta_T)] \\
+ l_b \dot{Q} \sin \Theta + l_t \dot{Q} \cos(\Theta + \Theta_T) + 2l_b l_t \dot{Q} \cos \Theta_T - 2l_b l_t Q_T \sin \Theta_T \\
+ l_b l_t \dot{Q} T \cos \Theta_T - l_b l_t Q_T \sin \Theta_T + l_b^2 \dot{Q} + l_t^2 \dot{Q} + l_t^2 \dot{Q}_T
\]

\[
\frac{d}{dt} \frac{\partial K_t}{\partial Q_T} - \frac{\partial K_t}{\partial \dot{Q}_T} = m_t [l_b \dot{x} \sin(\Theta + \Theta_T) + l_b \dot{z} \cos(\Theta + \Theta_T) + l_b l_t \dot{Q} \cos \Theta_T] \\
+ l_t^2 \dot{Q} + l_t^2 \dot{Q}_T + l_b l_t Q_T \sin \Theta_T
\]  

(2.50)

Carrying out the computation of the Lagrangian for the rotational kinetic energy, one obtains

\[
\frac{d}{dt} \frac{\partial K_r}{\partial \dot{x}} - \frac{\partial K_r}{\partial x} = 0 \\
\frac{d}{dt} \frac{\partial K_r}{\partial \dot{z}} - \frac{\partial K_r}{\partial z} = 0 \\
\frac{d}{dt} \frac{\partial K_r}{\partial Q} - \frac{\partial K_r}{\partial \dot{Q}} = I_B \dot{Q} + I_T (\dot{Q} + \dot{Q}_T) \\
\frac{d}{dt} \frac{\partial K_r}{\partial Q_T} - \frac{\partial K_r}{\partial \dot{Q}_T} = I_T (\dot{Q} + \dot{Q}_T)
\]  

(2.51)
The total potential energy of the system is due only to gravity and is given by

\[ V = m_{b}g^{B}r_{b} + m_{t}g^{B}r_{t} \]  

(2.52)

Similarly, the following can be computed for the total potential energy,

\[ \frac{\partial V}{\partial x} = -(m_{b} + m_{t})g \sin \Theta \]

\[ \frac{\partial V}{\partial z} = (m_{b} + m_{t})g \cos \Theta \]

\[ \frac{\partial V}{\partial \Theta} = m_{b}g \ell_{b} \cos \Theta + m_{t}g(\ell_{b} \cos \Theta + \ell_{t} \cos(\Theta + \Theta_{T})) \]

\[ \frac{\partial V}{\partial \Theta_{T}} = m_{t}g \ell_{t} \cos(\Theta + \Theta_{T}) \]  

(2.53)

### 2.2.3 Lagrangian Longitudinal Equations of Motion

After the terms of the Lagrangian equations of motion have been computed, one can combine the total kinetic and potential energies of the Lagrangian function and obtain the longitudinal equations of motion for the system. The nonlinear equations of motion of the FW-MAV system can be expressed in the following compact form:

\[ M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \Gamma \]  

(2.54)

where \( M(q) \) is the \textit{inertia matrix}, \( C(q, \dot{q}) \) is the \textit{matrix of centripetal and Coriolis terms}, \( G(q) \) includes the \textit{gravitational forces} and \( \Gamma \) is the external forces applied to the generalized coordinates. The inertia matrix \( M(q) \) is given by

\[
M(q) = \begin{bmatrix}
m_{b} + m_{t} & 0 & M_{1,3} & M_{1,4} \\
0 & m_{b} + m_{t} & M_{2,3} & M_{2,4} \\
M_{1,3} & M_{2,3} & m_{b}l_{b}^2 + m_{t}(l_{b}^2 + l_{t}^2 + 2l_{b}l_{t} \cos \theta_{t}) + I_{b} + I_{t} & M_{3,4} \\
M_{1,4} & M_{2,4} & M_{3,4} & m_{t}l_{t}^2 + I_{t}
\end{bmatrix}
\]  

(2.55)
where

\[ M_{1,3} = (m_b + m_t)l_b \sin \theta_t + m_t l_t \sin (\theta_b + \theta_t) \]

\[ M_{1,4} = m_t l_t \sin (\theta_b + \theta_t) \]

\[ M_{2,3} = (m_b + m_t)l_b \cos \theta_t + m_t l_t \cos (\theta_b + \theta_t) \]

\[ M_{2,4} = m_t l_t \cos (\theta_b + \theta_t) \]

\[ M_{3,4} = m_t l_t^2 + m_t l_bl_t \cos \theta_t + I_t \]

The centripetal and Coriolis matrix \( G(q) \) reads as

\[
C(q, \dot{q}) = \begin{bmatrix}
(m_b + m_t)l_b \cos \theta_b \dot{\theta}_b^2 + m_t l_t \cos (\theta_b + \theta_t)(\dot{\theta}_b + \dot{\theta}_t)^2 \\
-(m_b + m_t)l_b \sin \theta_b \dot{\theta}_b^2 - m_t l_t \sin (\theta_b + \theta_t)(\dot{\theta}_b + \dot{\theta}_t)^2 \\
-m_t l_bl_t \sin \theta_t (\dot{\theta}_t^2 + 2\dot{\theta}_b \dot{\theta}_t) \\
m_t l_bl_t \sin \theta_t \dot{\theta}_b^2
\end{bmatrix}
\] (2.57)

The gravitational forces matrix \( G(q) \) is

\[
G(q) = \begin{bmatrix}
(m_b + m_t)g \sin \theta_b \\
-(m_b + m_t)g \cos \theta_b \\
-m_b gl_b \cos \theta_b - m_t g(l_b \cos \theta_b + l_t \cos (\theta_b + \theta_t)) \\
-m_t gl_t \cos (\theta_b + \theta_t)
\end{bmatrix}
\] (2.58)

The external forces for the FW-MAV are generated by the wings motion and can be represented by

\[
\Gamma = \begin{bmatrix} F_x \\ F_z \\ \tau \\ -\tau \end{bmatrix}^T
\] (2.59)

where \( F_x, F_z \) denotes the aerodynamic forces in the horizontal and vertical axes, respectively, and \( \tau \) is the torque applied at the joint connecting the main body and the tail.

### 2.3 Kane’s and Lagrange’s FW-MAV Longitudinal Dynamics

Kane’s equations of motion have been expressed in the main body coordinate frame \( \mathcal{B} \), while the equations of motion that were developed with Lagrangian method are expressed in the inertial coordinate frame \( \mathcal{N} \). Hence, one needs to change coordinates in order to relate Lagrange’s longitudinal model to Kane’s longitudinal model.
Moreover, one can notice that Kane’s equations use the time derivatives of the generalized speeds \((\dot{U}, \dot{W}, \dot{Q}, \dot{Q}_T)\) while Lagrange equations use the second derivatives of the generalized coordinates \((\ddot{x}, \ddot{z}, \dot{Q}, \dot{Q}_T)\). To transform the equations from the inertial frame \(N\) to the main body coordinate frame \(B\), the following kinematics relation

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
U \\
W
\end{bmatrix}
\] (2.60)

Using this kinematics relation, one obtains the following

\[
\begin{bmatrix}
\ddot{x} \\
\ddot{z}
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
\dot{U} \\
\dot{W}
\end{bmatrix} +
\begin{bmatrix}
-sin \Theta & \cos \Theta \\
-cos \Theta & -sin \Theta
\end{bmatrix}
\begin{bmatrix}
U \\
W
\end{bmatrix} \dot{Q} \tag{2.61}
\]

Since the relation matrix is nonsingular, one obtains

\[
\begin{bmatrix}
\cos \Theta & -\sin \Theta \\
\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\ddot{z}
\end{bmatrix} =
\begin{bmatrix}
\dot{U} + QW \\
\dot{W} - QU
\end{bmatrix} \tag{2.62}
\]

In order to apply this transformation, one can re-write the Lagrangian equations of motion in the following form

\[
(m_b + m_t) \left( \begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} + \begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix} \begin{bmatrix}
0 \\
\ell_b \dot{Q}
\end{bmatrix} + \begin{bmatrix}
-sin \Theta & \cos \Theta \\
-cos \Theta & -sin \Theta
\end{bmatrix} \begin{bmatrix}
0 \\
\ell_b Q^2
\end{bmatrix} \right) + \cdots
\]

\[
+ m_t \ell_t \left( \begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix} \begin{bmatrix}
\sin \Theta_T \\
\cos \Theta_T
\end{bmatrix} (\dot{Q} + \dot{Q}_T) \right) + \cdots
\]

\[
+ m_t \ell_t \left( \begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix} \begin{bmatrix}
\cos \Theta_T \\
-sin \Theta_T
\end{bmatrix} \right) \left( Q + Q_T \right)^2 = \begin{bmatrix}
F_x \\
F_z
\end{bmatrix} \tag{2.63}
\]

where \(F_x\) and \(F_z\) are forward and vertical forces, respectively, while the torque applied at the hinge joint reads as follows

\[
(m_b + m_t) \left( \begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix} \begin{bmatrix}
0 \\
\ell_b
\end{bmatrix} \dot{x} \right) + m_t \ell_t \left( \begin{bmatrix}
\cos \Theta & \sin \Theta \\
-sin \Theta & \cos \Theta
\end{bmatrix} \begin{bmatrix}
\sin \Theta_T \\
\cos \Theta_T
\end{bmatrix} \dot{x} \right) +
\]

\[
+ [m_b \ell_b^2 + m_t (\ell_b^2 + \ell_t^2 + 2 \ell_b \ell_t \cos \Theta_T)] + I_B + I_T \dot{Q}_T + \left[ m_t \ell_t^2 + m_t \ell_b \ell_t \cos \Theta_T + I_T \right] \dot{Q}_T +
\]

\[- m_t \ell_b \ell_t \sin \Theta_T (Q_T^2 + 2 Q Q_T) = \tau \tag{2.64}
\]
By using the kinematics relation Equation (2.62), one obtains the following

\[
\begin{align*}
(m_b + m_t) \left( \begin{bmatrix} \dot{U} + QW \\ \dot{W} - QU \end{bmatrix} &+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \ell_b \dot{Q} \end{bmatrix} \right) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \ell_b Q^2 \end{bmatrix} \right) + \cdots \\
+m_t \ell_t \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \Theta_T \\ \cos \Theta_T \end{bmatrix} \begin{bmatrix} \dot{Q} + \dot{Q}_T \\ \dot{Q} + \dot{Q}_T \end{bmatrix} \right) &+ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos \Theta_T \\ -\sin \Theta_T \end{bmatrix} \begin{bmatrix} Q + Q_T \end{bmatrix}^2 \right) = \begin{bmatrix} F_x \\ F_z \end{bmatrix}
\end{align*}
\]

(2.66)

\[
\begin{align*}
\left( m_b + m_t \right) \left[ \begin{bmatrix} \ell_b \sin \theta_T (\dot{Q} + \dot{Q}_T) + \ell_t \cos \theta_T (Q + Q_T) \end{bmatrix} \right] + \left( m_t \ell_t \left[ \begin{bmatrix} \sin \Theta_T \\ \cos \Theta_T \end{bmatrix} \begin{bmatrix} \dot{U} + QW \\ \dot{W} - QU \end{bmatrix} \right) + \\
m_t \ell_t \left[ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \ell_b Q^2 \end{bmatrix} \right] \right) + \cdots \\
+m_t \ell_t \left[ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos \Theta_T \\ -\sin \Theta_T \end{bmatrix} \begin{bmatrix} Q + Q_T \end{bmatrix}^2 \right) = \begin{bmatrix} F_x \\ F_z \end{bmatrix}
\end{align*}
\]

(2.67)

\[
\begin{align*}
m_t \ell_t \left[ \begin{bmatrix} \sin \Theta_T \\ \cos \Theta_T \end{bmatrix} \begin{bmatrix} \dot{U} + QW \\ \dot{W} - QU \end{bmatrix} \right] + \left[m_t \ell_t^2 + m_t \ell_b \ell_t \cos \Theta_T + I_T \right] \dot{Q} + \\
m_t \ell_t \left[ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \ell_b Q^2 \end{bmatrix} \right] \right) \]
\end{align*}
\]

(2.68)

The resulting equations of motion are given by

\[
\begin{align*}
(m_b + m_t)[\dot{U} + QW + \ell_b Q^2] + m_t[\ell_t \sin \theta_T (\dot{Q} + \dot{Q}_T) + \ell_t \cos \theta_T (Q + Q_T)^2] &= F_x \\
(m_b + m_t)[W - QU + \ell_b \dot{Q}] + m_t[\ell_t \cos \theta_T (\dot{Q} + \dot{Q}_T) - \ell_t \sin \theta_T (Q + Q_T)^2] &= F_z
\end{align*}
\]

(2.69)
\[
(m_b + m_t)\ell_b(\dot{W} - QU) + m_t\ell_t \sin \theta_T (\dot{U} + QW) + m_t\ell_t \cos \theta_T (\dot{W} - QU) + \cdots
\]
\[
+ [m_b\ell_b^2 + m_t(\ell_b^2 + \ell_t^2 + 2\ell_b\ell_t \cos \theta_T) + I_{Byy} + I_{Tyy}]\dot{Q} + \cdots
\]
\[
+ [m_t\ell_t^2 + m_t\ell_b\ell_t \cos \theta_T + I_{Tyy}]\dot{Q}_T + m_t\ell_b\ell_t \sin \theta_T (Q_T^2 + 2QQ_T) = \tau
\]
\[
m_t\ell_t \sin \theta_t (\dot{U} + QW) + m_t\ell_t \cos \theta_t (\dot{W} - QU) + \cdots
\]
\[
+ [m_t\ell_t^2 + m_t\ell_b\ell_t \cos \theta_t + I_t]\dot{Q} + (m_t\ell_t^2 + I_t)\dot{Q}_T + m_t\ell_b\ell_t \sin \theta_t Q^2 = -\tau
\]
(2.70)

Thus, FW-MAV longitudinal equations of motion developed using Lagrangian method match the ones developed earlier using Kane’s method.
Chapter 3: Control-Oriented Model for the FW-MAV

3.1 Introduction

Assessment of relevant features of wing motion to be replicated by a control mechanism is critical in achieving highly maneuverable flight of FW-MAV. In the literature, various investigations were conducted to provide interpretation of the effect of changes in wing motion that generate aerodynamic forces and moments in flying insects. The unique features of a set of wing kinematic parameters are the focus of this chapter. The effect of each control variable on the vehicle dynamics is examined in order to enable the development of control strategies that can in principle let the vehicle achieve a desired level of performance. Focus is given to the characterization of the effect of all possible kinematics parameters to the actuation of the vehicle model presented in the previous chapter. The methodology we follow is borrowed from Doman and Oppenheimer [19] [54], in that an averaged model of the vehicle dynamics is derived, which shall serve as a control-design model. Differently from [19] and [54], however, we employ a novel wingbeat function which depends on two parameters, namely, a phase-shift and a wing-bias. The novel wingbeat function has the important property of being periodic and continuously differentiable with respect to the control parameters. This important feature allows a rigorous application of the averaging theorem,
allowing the use of robust nonlinear control techniques based on bounded feedback. This leads to the design of a robust controller for the longitudinal model, developed in the next chapter. It is worth noticing that the proposed method yields a controller for the longitudinal model that employs only one physical actuator for each wing, hence a minimal actuator suite. As a matter of fact, we believe that our work fills an important gap in the literature on control of FW-MAVs, where the use of averaged model is predominantly. However, none of the existing results have been proved to satisfy the assumptions required by application of the averaging theorem. In particular, the method in [19] [54] leads to a discontinuous wingbeat functions, whose use in the context of averaging is problematic from a mathematical standpoint. The wingbeat function proposed in this chapter removes all these difficulties and leads to a mathematically sound model for averaging purposes.

As far as the control-oriented modeling, presented in this chapter, is concerned, the procedure to obtain the averaged model is fairly standard and - as mentioned - follows closely the work of [19] [54]. The methodology proceeds by evaluating the aerodynamic forces and moments produced by the wings, performing upstroke and downstroke flapping motion, and applying the averaging procedure. A distinctive feature of our work - apart from the aforementioned wingbeat function and its role in the averaging - is the evaluation of all possible actuation configurations resulting from all possible contributions of the available wing kinematics parameters used as control inputs, namely, the two wingbeat parameters just mentioned, the stroke-plane angle and the wing angle of attack. It will be shown that, with the considered wing kinematics parameters, full actuation of the vehicle is not possible, and the maximum number of degrees of freedom of the main body that can be directly actuated is
five. This means that the system dynamics is under-actuated from a mechanical control system perspective, but redundancy exists in the input space when all control parameters are employed. This redundancy may be used for optimization of a given control policy, although this is outside the scope of this work.

3.2 Instantaneous Blade-Element Aerodynamic Model of the FW-MAV

In this section, we derive the expressions for the instantaneous aerodynamic forces and moments in the body frame. These instantaneous aerodynamic forces and moments are later used to calculate the cycle averaged aerodynamic forces and moments, and thus the development of the control oriented cycle averaged model of the FW-MAV. The key assumptions made in the formulation of this analytical control-oriented model are given in [7] [6]. An important feature of this control-oriented model is the ability to analytically compute the cycle-averaged control derivatives for use in the control law, which would be computationally very complex using a higher-fidelity model based on computational fluid dynamics or finite element analysis.

3.2.1 Instantaneous Aerodynamic Forces in the Body Frame

The instantaneous aerodynamic forces are derived using blade-element theory, for a pair of wings that have two degrees of freedom, namely, angular displacement, $\mu(t)$, about the wing root in the stroke plane, and angular displacement about the passive rotation hinge joint, which is equivalent to wing angle-of-attack, $\alpha$, in still air [7]. The lift and drag, produced by each wing can be expressed as a product of time invariant parameters and time varying functions [6]

$$L = k_L\ddot{\mu}^2(t) \quad D = k_D\ddot{\mu}^2(t)$$

(3.1)
where

\[ k_L \triangleq \frac{\rho}{2} C_L(\alpha) I_A \quad k_D \triangleq \frac{\rho}{2} C_D(\alpha) I_A \] (3.2)

and \( I_A \) is the area moment of inertia of the wing about the axis of the root-hinge, \( \rho \) is the atmospheric density, and \( C_L(\alpha) \) and \( C_D(\alpha) \) are the lift and drag coefficients.

The following expressions for lift and drag coefficients, as functions of the wing angle of attack, defined using Berman and Wang [4] empirical expressions for bumblebee, are used in this work:

\[ C_L(\alpha) = C_T \sin(2\alpha) \] (3.3)

\[ C_D(\alpha) = C_D(0) \cos^2(\alpha) + C_D(\pi/2) \sin^2(\alpha) \]

The nominal values of these parameters adopted in this work are \( C_T = 1.341, C_D(0) = 0, C_D(\pi/2) = 2.93 \) and \( \alpha \) is in degrees. Figure 3.1 shows the lift and drag coefficients obtained using Berman and Wang model [4]. One can notice that the lift coefficient is maximized when the wing angles of attack is approximately 45°. The instantaneous values of lift and drag on each wing are transformed into the body-axis coordinate frame. To resolve the lift and drag into the body axes, it is recognized that the lift acts normal to the stroke plane, while drag lies within the stroke plane and is in the direction opposite to the tangential velocity of the wing. The aerodynamic force vectors associated with the right and left wings for each stroke in the body frame can be written as [6]

\[
F_{uRW}^B = \begin{bmatrix}
L_{RW} \sin \lambda_{RW} + D_{RW} \cos \lambda_{RW} \cos \mu_{RW} \\
- D_{RW} \sin \mu_{RW} \\
- L_{RW} \cos \lambda_{RW} + D_{RW} \cos \lambda_{RW} \sin \lambda_{RW}
\end{bmatrix}
\]

\[
F_{dRW}^B = \begin{bmatrix}
L_{RW} \sin \lambda_{RW} - D_{RW} \cos \lambda_{RW} \cos \mu_{RW} \\
D_{RW} \sin \mu_{RW} \\
- L_{RW} \cos \lambda_{RW} - D_{RW} \cos \lambda_{RW} \sin \lambda_{RW}
\end{bmatrix}
\] (3.4)
Figure 3.1: Lift and drag coefficients for different values of the wing angle of attack.

\[
\begin{align*}
\mathbf{F}_{BuLW}^B &= \begin{bmatrix}
L_{LW} \sin \lambda_{LW} + D_{LW} \cos \lambda_{LW} \cos \mu_{LW} \\
D_{LW} \sin \mu_{LW} \\
-L_{LW} \cos \lambda_{LW} + D_{LW} \cos \mu_{LW} \sin \lambda_{LW}
\end{bmatrix} \\
\mathbf{F}_{dLW}^B &= \begin{bmatrix}
L_{LW} \sin \lambda_{LW} - D_{LW} \cos \lambda_{LW} \cos \mu_{LW} \\
-D_{LW} \sin \mu_{LW} \\
-L_{LW} \cos \lambda_{LW} - D_{LW} \cos \mu_{LW} \sin \lambda_{LW}
\end{bmatrix}
\end{align*}
\]

(3.5)

where the subscripts \( u \) and \( d \) represent upstroke and downstroke, respectively, while \( RW \) and \( LW \) represent right and left wing, respectively.

### 3.2.2 Instantaneous Aerodynamic Moments in the Body Frame

In order to calculate the aerodynamic moment produced by wings motion that is applied to the vehicle center of mass, the center of pressure vector for each wing in the body frame must be defined. The position vector that defines each wing center
of pressure to the vehicle center of mass can be defined as [7] [6]

\[ \mathbf{r}_{R/B} = \mathbf{r}_{R/A} + \mathbf{r}_{A/B} \]  

(3.6)

where \( \mathbf{r}_{A/B} \) denotes the wing attachment point relative to the vehicle center of mass, and \( \mathbf{r}_{R/A} \) locates the wing center of pressure relative to the wing attachment point. The vectors \( \mathbf{r}_{A/B} \) and \( \mathbf{r}_{R/A} \) can be expressed as [7]

\[ \mathbf{r}_{A/B} = p_x \mathbf{b}_1 + p_y \mathbf{b}_2 + p_z \mathbf{b}_3 \]  

\[ \mathbf{r}_{R/A} = x_w \mathbf{r}_1 + y_w \mathbf{r}_2 + z_w \mathbf{r}_3 \]  

(3.7)

Figure 3.2 illustrates the vehicle parameters that are required to locate the wing center of pressure. The letters \( \mathbf{C} \) and \( \mathbf{H} \) in Figure 3.2 represent the locations of the center of pressure and the wing root attachment hinge point, respectively. The parameter \( z_w \) is the wing thickness which is assumed to be negligible, and \( p_y \) is the distance between the wing attachment hinge point and the vehicle center of mass in the \( y \)-body axis direction.

The aerodynamic moment that is applied to the vehicle center of mass can be calculated as follows

\[ \mathbf{M} = \mathbf{r}_{R/B} \times \mathbf{F} \]  

(3.8)
Figure 3.2: Lateral view of the vehicle illustrating the design parameters that define each wing center of pressure

The instantaneous aerodynamic moments of right and left wings for each stroke can be written as [7]

\[
\begin{align*}
M_{uRW}^B &= \begin{bmatrix}
-L_{uRW} \cos \lambda (x_w \cos \alpha \sin \phi + y_w \cos \phi + p_y) + \cdots \\
+ D_{uRW} (-x_w \cos \lambda \sin \alpha \sin \phi + p_y \cos \phi \sin \lambda + y_w \sin \lambda - p_z \sin \phi) \\
L_{uRW} (-x_w \cos \phi \cos \alpha + p_x \cos \lambda - p_z \sin \lambda + y_w \sin \phi) + \cdots \\
- D_{uRW} \cos \phi (p_z \cos \lambda + x_w \sin \alpha + p_x \sin \lambda) \\
- L_{uRW} \sin \lambda (x_w \cos \phi \cos \alpha + y_w \cos \phi \cos \lambda + p_x \sin \phi) + \cdots \\
- D_{uRW} (x_w \sin \alpha \sin \phi \cos \phi \cos \lambda + y_w \cos \lambda + p_x \sin \phi)
\end{bmatrix} \\
M_{dRW}^B &= \begin{bmatrix}
-L_{dRW} \cos \lambda (-x_w \cos \alpha \sin \phi + y_w \cos \phi + p_y) + \cdots \\
- D_{dRW} (-x_w \cos \lambda \sin \alpha \sin \phi + p_y \cos \phi \sin \lambda + y_w \sin \lambda - p_z \sin \phi) \\
L_{dRW} (x_w \cos \phi \cos \alpha + p_x \cos \lambda - p_z \sin \lambda + y_w \sin \phi) + \cdots \\
+ D_{dRW} \cos \phi (p_z \cos \lambda + x_w \sin \alpha + p_x \sin \lambda) \\
L_{dRW} \sin \lambda (-x_w \cos \phi \cos \alpha \sin \phi + y_w \cos \phi \cos \lambda + p_x \sin \phi) + \cdots \\
+ D_{dRW} (x_w \sin \alpha \sin \phi \cos \lambda \cos \phi \cos \lambda + y_w \cos \phi \cos \lambda + p_x \sin \phi)
\end{bmatrix}
\end{align*}
\]
3.3 Novel Wingbeat Forcing Function

The selection of a wingbeat pattern that can be manipulated by a suitable set of control parameters is perhaps the most fundamental step towards a successful design of a flight control system for FW-MAVs. Any candidate wingbeat function shall be amenable to generate a net lift and drag in an average sense, to be able to produce sufficient control authority. On the other hand, such wingbeat function shall depend in a continuously differentiable manner on the kinematic parameters used as control inputs to shape the wingbeat waveform independently during upstroke and downstroke. Previous studies [27] [67] [14] [19] [59] [47] have considered using a sinusoidal waveform forcing function. Symmetric sinusoidal wingbeat function, however, will only produce non-zero cycle-averaged lift, while drag forces cancel each other out, which is desirable for hovering. Nonetheless, a non-zero cycle averaged drag forces are crucial in achieving 6-DOF maneuvers requiring yawing and rolling moments [90] [19]. Several control strategies attempted to induce asymmetric drag forces by employing different wing kinematics, aimed at modifying the basic sinusoidal
waveform. A net averaged drag forces can be produced by varying the wing angle of attack \[13\] [33], the stroke-plane angle \[69\], and split-cycle parameters \[19\], which aimed at producing a distortion in the waveform between upstroke and downstroke. Hence, a candidate wingbeat forcing function must be capable of producing lift as well as drag forces in order to produce moments about all three body axes. In addition, a rigorous application of averaging theory necessitates that wingbeat waveform to be periodic and continuously differentiable with respect to the kinematic parameters. It should be noted that among all the aforementioned approaches, wing-bias is the only mechanism that has been utilized within the available micro-technology into an insect-scale FW-MAV. The Harvard Robofly \[29\] equipped with one actuator per wing is capable of manipulating flapping frequency as well as wing-bias. Candidate control design methodologies must take into consideration using smallest possible number of actuators due to the constraints posed by extreme miniaturization.

In this section, a novel approach for designing controlled periodic wingbeat function for flapping-wing MAVs is presented. The proposed waveform, inspired by the results in \[32\], is characterized by two continuously-variable control parameters, namely, a phase-shift (whose time-derivative can be assigned, acting as frequency control) and a bias. The method removes technical difficulties related to averaging discontinuous and non-periodic signals that affect most of the work found in the literature, for instance \[19\][54], and yields forces which are in appropriate form for averaging. The novel wingbeat forcing function employ two wing kinematic parameters which only requires one physical actuator per wing. The proposed wing kinematics are defined independently for right and left wings and treated as control variables. The novel wingbeat forcing function modulates wing-bias and flapping frequency. It is worth
noting that the resulting waveform has a normalized period of $4\pi$ in place of the normalized period of $2\pi$ commonly found in the literature.

The proposed wingbeat forcing function, $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}$, that drives the rotation of each wing is given by

$$
\mu(t) = \begin{cases} 
\cos(\tau(t)) + b(t) \tanh(\tau(t))^3 & : \tau(t) \in [0, 2\pi) \\
\cos(\tau(t)) + b(t) [1 - \tanh(\tau(t) - 2\pi)^3] & : \tau(t) \in [2\pi, 4\pi) 
\end{cases}
$$

(3.11)

where

$$
\tau(t) = \omega_0 t + \Delta(t) \mod 4\pi, \quad t \geq 0
$$

is the normalized time, $\omega_0 > 0$ is a given constant carrier frequency, $\Delta(\cdot)$ is the manipulated phase-shift and $b(\cdot)$ is the manipulated wing-bias. It is required that $|d\Delta(t)/dt| < \omega_0$ to maintain periodicity of the wing beat, whereas wing-bias is assumed to be $|b(t)| \leq 1$, for all $t \geq 0$, for reasons that will become clear in the sequel. Letting $\delta := \dot{\Delta}/\omega_0$, where $|\delta| < 1$, we obtain the parameters $\delta(t)$ and $b(t)$ that can be manipulated by the control system. As a result, the wing beat is frequency-controlled about the carrier frequency. This wingbeat forcing function involves a hyperbolic tangent in order to ensure continuity and smooth transition of wing kinematics between consecutive flapping cycles [32].

**Properties of $\mu(\cdot)$**

- For $b(t) \equiv \text{const}$, $\mu(\cdot)$ is a periodic function in the normalized time variable $\tau$ of period $T = 4\pi$.

- $\mu(\cdot)$ is a continuously differentiable function of time if $\Delta(\cdot), b(\cdot) \in \mathcal{C}^1$
Since by assumption $|\delta(t)| < 1$, the time re-scaling $\tau = \omega_0 t + \Delta(t)$ is well defined. It is also easy to verify that

\[
\frac{d}{dt} = \frac{d\tau}{d\tau} \frac{d\tau}{dt} = \omega_0 (1 + \delta) \frac{d}{d\tau} \quad \text{and} \quad \frac{d}{d\tau} = \frac{dt}{d\tau} \frac{d\tau}{d\tau} = \left(1 - \frac{d\delta}{d\tau}\right) \frac{d}{dt} = \frac{1}{\omega_0} \cdot \frac{1}{(1 + \delta)} \frac{d}{dt} \tag{3.12}
\]

Differentiating Equation (3.11), the wingbeat function $\mu(\tau, b)$, with respect to $t$ while using Equation (3.12), the wing beat derivatives are given by

\[
\dot{\mu}(\tau, b) = \begin{cases} 
- \sin(\tau(t)) + 3b(t) \tanh(\tau(t))^2 \left[1 - \tanh(\tau(t))^2\right] + \cdots \\
\frac{\dot{b}(t) \tanh(\tau(t))^3}{\omega_0 (1 + \delta)} : \tau(t) \in [0, 2\pi) \\
- \sin(\tau(t)) - 3b(t) \tanh(\tau(t) - 2\pi)^2 \left[1 - \tanh(\tau(t) - 2\pi)^2\right] + \cdots \\
\frac{\dot{b}(t) \tanh(\tau(t) - 2\pi)^3}{\omega_0 (1 + \delta)} : \tau(t) \in [2\pi, 4\pi) 
\end{cases}
\tag{3.13}
\]

For $b(t) \equiv \text{const}$, the wingbeat function $\mu(\cdot)$ and its time derivative $\dot{\mu}(\cdot)$ become functions of $t$ only, which read respectively as

\[
\mu(t) = \begin{cases} 
\cos(\tau) + b \tanh(\tau)^3 : \tau \in [0, 2\pi) \\
\cos(\tau) + b[1 - \tanh(\tau - 2\pi)^3] : \tau \in [2\pi, 4\pi)
\end{cases}
\quad \text{and} \quad 
\dot{\mu}(t) = \begin{cases} 
\omega_0 (1 + \delta) \left(- \sin(\tau) + 3b \tanh(\tau)^2 \left[1 - \tanh(\tau)^2\right]\right) : \tau \in [0, 2\pi) \\
\omega_0 (1 + \delta) \left(- \sin(\tau) - 3b \tanh(\tau - 2\pi)^2 \left[1 - \tanh(\tau - 2\pi)^2\right]\right) : \tau \in [2\pi, 4\pi)
\end{cases}
\tag{3.14}
\]

A plot of the functions $\mu(\cdot)$ and $\dot{\mu}(\cdot)$ for opposite (constant) values of $b$ is shown in Figure 3.3.

### 3.4 Averaged Dynamics

One of the most outstanding issues inherent in flapping wing flight is the requirement of periodically-varying actuation, which gives rise to time-varying models that
Figure 3.3: Plots of wingbeat, $\mu(\tau)$, and its derivative, $\dot{\mu}(\tau)$, for opposite values of $b$ are difficult to analyze. The quasi-periodic nature of aerodynamic forces and moments generated by the oscillatory motion of the flapping wings has prompted the use of averaging methods for analysis and design of controllers for flapping-wing MAVs. One should emphasize that averaging techniques can unveil some of the important features of the time-varying system by obtaining a suitable time-invariant model. Then, a control design law can be developed on the basis of the averaged system that would be applied to the higher fidelity time-accurate model to prove effectiveness. However, averaging theory can only guarantee that the original system trajectories will have a bounded error from the desired one with range of validity of $O(\varepsilon)$ and the averaging parameter, $\varepsilon$, depends on the flapping frequency, $\omega_0$, where $\varepsilon = 1/\omega_0$. Therefore, the error bound can be made sufficiently small by reducing the value of the averaging
parameter $\varepsilon$. In other words, this dictates increasing the flapping frequency since $1/\omega_0 = O(\varepsilon)$. Validity of the averaged approximation on the infinite interval $[0, \infty)$ requires additional exponential stability properties of a desired attractor that must be provided by the control system. These issues will be discussed in the next chapter.

To compute the averages of the aerodynamic forces and moments, we make the following preliminary assumptions:

1. $\tanh(\pi) \approx 1$

2. $b(t) = b = \text{const}$ for all $t \geq 0$

The first assumption is made to facilitate identifying upstrokes and downstrokes. The second assumption will be removed by showing that $\dot{b}$ is $\varepsilon$-small with respect to the rest of the dynamics. The wing beat forcing functions, driving the positions of the right and left wings of the flapping wing MAV, has the following intervals for upstroke and downstroke

$$
\mu(t) = \begin{cases} 
1^{\text{st}} \text{ upstroke} & : \tau \in [0, \pi) \\
1^{\text{st}} \text{ downstroke} & : \tau \in [\pi, 2\pi) \\
2^{\text{nd}} \text{ upstroke} & : \tau \in [2\pi, 3\pi) \\
2^{\text{nd}} \text{ downstroke} & : \tau \in [3\pi, 4\pi) 
\end{cases} \tag{3.15}
$$

The wing beat forcing functions, driving the positions of the right and left wings of the flapping wing MAV, has the following intervals for upstroke and downstroke

$$
\mu(t) = \begin{cases} 
1^{\text{st}} \text{ upstroke} & : \tau \in [0, \pi) \\
1^{\text{st}} \text{ downstroke} & : \tau \in [\pi, 2\pi) \\
2^{\text{nd}} \text{ upstroke} & : \tau \in [2\pi, 3\pi) \\
2^{\text{nd}} \text{ downstroke} & : \tau \in [3\pi, 4\pi) 
\end{cases} \tag{3.16}
$$

Using Equation (3.1) and Equation (3.14), the instantaneous lift and drag forces of the FW-MAV can be written as

$$ L = k_L \dot{\mu}(\tau) \tag{3.17} $$
\[ D = k_D \dot{\mu}(\tau) \]  

(3.18)

### 3.4.1 Cycle-Averaged Forces and Moments

This section addresses the aerodynamic forces and moments that arise from wing motion only. Thus, an actuated abdomen shall not be considered, since it has been showed in the literature [20] [32] [58] that it can only affect the vehicle pitching moment. The considered set of kinematic parameters includes all possible mechanisms that can be employed by a pair of wings. The cycle-averaged aerodynamic forces and moments in the body frame are computed to evaluate of multiple actuation mechanisms over FW-MAV controllability. The significance of various wing kinematic parameters are evaluated by computing the cycle-averaged control derivatives about hover. The control suite comprises the following actuation mechanisms:

- Independent wingbeat waveforms for each wing, \( \mu(\delta, b) \), with manipulated inputs given by the phase modulation \( \delta = (\delta_{RW}, \delta_{LW}) \) and wing-bias \( b = (b_{RW}, b_{LW}) \).

- Independent orientation of the stroke-plane for each wing, with manipulated inputs given by the stroke-plane angles, \( \lambda = (\lambda_{RW}, \lambda_{LW}) \).

- Independently actuated angle-of-attack of each wing, \( \alpha = (\alpha_{RW}, \alpha_{LW}) \).

The variation of wing angle of attack can generate asymmetric lift and drag force profiles in the upstroke and downstroke of a wingbeat, thus giving rise to non-zero
cycle-averaged aerodynamic forces and moments. For example, a fore/aft translational force can be induced by applying asymmetric wing angle of attack in downstroke and upstroke of each flapping cycle [86, 87, 33, 48, 55, 77]. Following the proposed approach in [55], the upstroke and downstroke angle of attack of each wing can be described by

\[ \alpha_u = \alpha_0 - \Delta\alpha \]
\[ \alpha_d = \alpha_0 + \Delta\alpha \]

(3.19)

where \( \alpha_0 \) is the nominal value of angle of attack and considered to be constant. By manipulating \( \Delta\alpha \), an asymmetric angle of attack can be generated in the upstroke and downstroke of each wing.

The general method of [19] used here to compute the cycle-averaged aerodynamic forces and moments is now explained. Let \( G(t) \) be a generalized force or moment either in the \( x \), \( y \), or \( z \) body-axis direction. The cycle-averaged generalized force or moment for each wing is computed, by solving a definite integral of the form

\[ \bar{G} = \frac{\omega}{4\pi} \int_0^{\frac{4\pi}{\omega}} G(\mu(\tau))d\tau \]

(3.20)

Since the expression for the generalized force or moment, \( G(\tau) \), is different for the upstroke and the downstroke, as seen from Figure 3.3 and Equation (3.16), we split the integral as follows

\[ G = \frac{\omega}{4\pi} \left( \int_0^{\frac{\pi}{\omega}} G_u(\mu(\tau))d\tau + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} G_d(\mu(\tau))d\tau + \int_{\frac{2\pi}{\omega}}^{\frac{3\pi}{\omega}} G_u(\mu(\tau))d\tau + \int_{\frac{3\pi}{\omega}}^{\frac{4\pi}{\omega}} G_d(\mu(\tau))d\tau \right) \]

(3.21)
For simplification of the above equation we substitute, \( s = \omega \tau \), in Equation (3.21). Thus, we have \( ds = \omega \, d\tau \), and

\[
\begin{align*}
&\tau \to 0, \quad s \to 0; \\
&\tau \to \frac{\pi}{\omega}, \quad s \to \pi; \\
&\tau \to \frac{2\pi}{\omega}, \quad s \to 2\pi; \\
&\tau \to \frac{3\pi}{\omega}, \quad s \to 3\pi; \\
&\tau \to \frac{4\pi}{\omega}, \quad s \to 4\pi;
\end{align*}
\]

Thus Equation (3.21) can be written as

\[
\bar{G} = \frac{1}{4\pi} \left( \int_0^\pi G_u(\mu(\tau)) \, ds + \int_0^{2\pi} G_d(\mu(\tau)) \, ds \right)
\]

(3.22)

In order to calculate the cycle-averaged forces and moments, several integrals were computed using polynomial approximation for all terms containing the wing-bias parameter, where it was restricted to be \(|b| < 1\).

**X-Body Axis Force**

Substituting the expression for the instantaneous \( x \)-body axis force for the right wing from Equation (3.4) into Equation (3.23) we get

\[
\bar{F}_{xRW} = \frac{1}{4\pi} \left( \int_0^\pi L_{RW} \sin \lambda_{RW} + D_{RW} \cos \lambda_{RW} \cos \mu(t) \, d\tau \\
+ \int_0^{2\pi} L_{RW} \sin \lambda_{RW} - D_{RW} \cos \lambda_{RW} \cos \mu(t) \, d\tau \\
+ \int_0^{3\pi} L_{RW} \sin \lambda_{RW} + D_{RW} \cos \lambda_{RW} \cos \mu(t) \, d\tau \\
+ \int_0^{4\pi} L_{RW} \sin \lambda_{RW} - D_{RW} \cos \lambda_{RW} \cos \mu(t) \, d\tau \right)
\]

(3.24)
Computing the above definite integral, the following expressions of the averaged $x$-body axis forces from the right and left wings are obtained

$$F_{xRW}^B = \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 \sin \lambda_{RW} \left[ C_L(\alpha_u)(\pi + 1.03b_{RW}^2) + C_L(\alpha_d)(\pi) \right] + \cdots$$

$$+ \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 \cos \lambda_{RW} \left[ C_D(\alpha_u)(2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3) + \cdots \right.$$

$$\left. + C_D(\alpha_d)(-2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3) \right]$$

$$F_{xLW}^B = F_{xRW}^B$$

(3.25)

where all the terms containing $b_{RW}$ and $b_{LW}$ where computed using the polynomial approximation for $|b| < 1$.

A non-zero cycle-averaged $x$-body forces can, therefore, be generated, which can be used to produce fore and aft translational motions of the vehicle by either varying wing bias parameter, $b_{RW}$, or wing angle of attack during upstroke $\alpha_u$, and downstroke, $\alpha_d$, or stroke plane angle, $\lambda_{RW}$. A nonzero cycle-averaged $x$-body forces can still be produced even when, $b_{RW} = b_{LW} = 0$, as long as $\alpha_u \neq \alpha_d$. That is, the magnitude of wing angle of attack during the upstroke is different than its magnitude during the downstroke. Furthermore, a larger (smaller) angle of attack during the downstroke relative to the upstroke generates a backward (forward) thrust. Thus, applying a larger angle of attack during the upstroke relative to the downstroke will result in generating larger drag force profile during the upstroke than the one generated by the downstroke, thereby resulting in a net forward motion.
Y-Body Axis Force

Substituting the expression for the instantaneous \( y \)-body force for the right wing from Equation (3.4) into Equation (3.23) yields

\[
\bar{F}_{yRW}^B = \frac{1}{4\pi} \left( \int_0^\pi -D_u \sin (\tau) + b \tanh (\tau \tau^3) \right) ds + \int_{2\pi}^{3\pi} -D_u \sin (\tau) + b[1 - \tanh (\tau - 2\pi \tau^3)] ds + \int_{3\pi}^{4\pi} D_d \sin (\tau) + b[1 - \tanh (\tau - 2\pi \tau^3)] ds \]  

(3.26)

The averaged \( y \)-body axis forces from the right and left wings are given as follows

\[
\bar{F}_{yRW}^B = \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 [C_D(\alpha_u)(-1.38b_{RW} - 0.306b_{RW}^2 - 0.178b_{RW}^3) + C_D(\alpha_d)(1.39b_{RW} + 0.00583b_{RW}^2 - 0.227b_{RW}^3)] 

+ \bar{F}_{yLW}^B = -\bar{F}_{yRW}^B 

(3.27)

Note that when, \( b_{RW} = b_{LW} = 0 \), it is not possible to produce non-zero cycle-averaged side forces. Moreover, when \( b_{RW} = b_{LW} \), the cycle-averaged side force for right and left wings are equal in magnitude and have opposite directions. Thus, cycle-averaged side forces for right and left wings are opposing and balance one another, which is a desirable feature for maintaining hover.
Z-Body Axis Force

Substituting the expression for the instantaneous z-body force for the right wing from Equation (3.4) into Equation (3.23), one obtains

$$F_{zRW} = \frac{1}{4\pi} \left( \int_0^\pi -L \cos \lambda + D \sin \lambda \cos(\cos(\tau)) + b \tanh(\tau^3) d\tau 
+ \int_\pi^{2\pi} -L \cos \lambda - D \sin \lambda \cos(\cos(\tau)) + b \tanh(\tau^3) d\tau 
+ \int_{2\pi}^{3\pi} -L \cos \lambda + D \sin \lambda \cos(\cos(\tau)) + b[1-\tanh(\tau-2\pi)^3] d\tau 
+ \int_{3\pi}^{4\pi} -L \cos \lambda - D \sin \lambda \cos(\cos(\tau)) + b[1-\tanh(\tau-2\pi)^3] d\tau \right)$$ (3.28)

The cycle-averaged z-body forces from the right and left wings are computed as follows

$$F_{zRW} = \frac{k}{4\pi \omega_0^2}(1 + \delta_{RW})^2 \cos \lambda[C_L(\alpha_u)(-\pi - 1.03b_{RW}^2) + C_L(\alpha_d)(-\pi)] + \cdots$$

$$+ \frac{k}{4\pi \omega_0^2}(1 + \delta_{RW})^2 \sin \lambda[C_D(\alpha_u)(2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3) + C_D(\alpha_d)(-2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3)]$$ (3.29)

$$F_{zLW} = F_{zRW}$$

At a hover condition, the z-forces produced by the wings act to counter the vehicle weight where the z-body axis is pointing down to earth surface.

Rolling Moment

Substituting the expression for the instantaneous x-body moment for the right wing from Equation (3.9) into Equation (3.23) yields

$$M_{xRW} = \frac{1}{4\pi} \left( \int_0^\pi M_{xu} d\tau + \int_\pi^{2\pi} M_{xd} d\tau + \int_{2\pi}^{3\pi} M_{xu} d\tau + \int_{3\pi}^{4\pi} M_{xd} d\tau \right)$$ (3.30)
The cycle-averaged rolling moments for the right wing is

\[ \bar{M}_{xRW}^B = - \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_L(\alpha_u) \cos \lambda p_y \left( \pi + 1.03b_{RW}^2 \right) + \cdots + x_{w} \cos \alpha_u \left( 1.38b_{RW} + 0.308b_{RW}^2 + 0.178b_{RW}^3 \right) + \cdots + y_{w} \left( 2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3 \right) \right] + \cdots - \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_L(\alpha_d) \cos \lambda p_y \left( \pi + 1.03b_{RW}^2 \right) + \cdots + x_{w} \cos \alpha_d \left( -1.39b_{RW} - 0.00583b_{RW}^2 + 0.227b_{RW}^3 \right) + \cdots + y_{w} \left( 2.76 - 0.685b_{RW}^2 - 0.00292b_{RW}^3 \right) \right] + \cdots + \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_D(\alpha_d) \left[ p_y \left( 1.39b_{RW} + 0.00583b_{RW}^2 - 0.227b_{RW}^3 \right) + \cdots + x_{w} \sin \alpha_d \cos \lambda \left( 1.39b_{RW} + 0.00583b_{RW}^2 - 0.227b_{RW}^3 \right) + \cdots + y_{w} \sin \lambda \left( -2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3 \right) + y_{w} \sin \lambda \left( -\pi \right) \right] \] (3.31)

The cycle-averaged rolling moments for the left wing is

\[ \bar{M}_{xLW}^B = -\bar{M}_{xRW}^B \] (3.32)

A non-zero cycle-averaged rolling moments can be generated by either varying, \( \delta \), or the wing-bias parameter \( b \), or the stroke plane angle, \( \lambda \). It is worth to mention that even when the wing bias is not applied, \( b_{RW} = b_{LW} = 0 \), there still exists a non-zero rolling moments. Having said that, when both wings have the same frequencies with \( \delta_{RW} = \delta_{LW} \), the generated rolling moments by the right and left wings are opposing and cancel each other which is, again, desirable feature for hovering condition.
Pitching Moment

Substituting the expression for the instantaneous $y$-body moment for the right wing from Equation (3.9) into Equation (3.23) yields

$$M_{yRW}^B = \frac{1}{4\pi} \left( \int_0^\pi M_{yu} d\tau + \int_\pi^{2\pi} M_{yd} d\tau + \int_{2\pi}^{3\pi} M_{yu} d\tau + \int_{3\pi}^{4\pi} M_{yd} d\tau \right)$$  \hspace{1cm} (3.33)

The cycle-averaged pitching moments for the right wing is

$$\bar{M}_{yRW}^B = \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_L(\alpha_u) [p_x \cos \lambda (\pi + 1.03b_{RW}^2) + \cdots $$

$$+ x_w \cos \alpha_u (-2.76 - 0.125b_{RW}^2 + 0.154b_{RW}^3) + \cdots $$

$$+ y_w (1.38b_{RW} + 0.308b_{RW}^2 + 0.178b_{RW}^3) + p_z \sin \lambda (-\pi - 1.03b_{RW}^2)] + \cdots $$

$$+ \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_L(\alpha_d) [p_x \cos \lambda (\pi) + \cdots $$

$$+ x_w \cos \alpha_d (2.76 - 0.685b_{RW}^2 - 0.00292b_{RW}^3) + \cdots $$

$$+ y_w (1.39b_{RW} + 0.00583b_{RW}^2 - 0.227b_{RW}^3) + p_z \sin \lambda (-\pi)] + \cdots $$

$$- \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_D(\alpha_u) [p_z \cos \lambda (2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3) + \cdots $$

$$+ x_w \sin \alpha_u (2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3) + \cdots $$

$$+ p_x \sin \lambda (-2.76 - 0.125b_{RW}^2 + 0.154b_{RW}^3)] + \cdots $$

$$- \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_D(\alpha_d) [p_z \cos \lambda (-2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3) + \cdots $$

$$+ x_w \sin \alpha_d (-2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3) + \cdots $$

$$+ p_x \sin \lambda (2.76 + 0.685b_{RW}^2 - 0.00292b_{RW}^3)] $$

\hspace{1cm} (3.34)

The cycle-averaged pitching moments for the left wing is

$$\bar{M}_{yLW}^B = \bar{M}_{yRW}^B$$

\hspace{1cm} (3.35)
A non-zero pitching moment can be generated by either varying $b$, or $\lambda$, or $\alpha$. Note that there still exists a non-zero pitching moment even though when the wing bias parameter is not applied, i.e., when $b_{RW} = b_{LW} = 0$. In order to avoid this occurrence of this non-zero pitching moment, the wing root hinge point must be designed such that its $x$ coordinate coincide with the vehicle center of gravity, that is $p_x = 0$. Such an arrangement will prevent any undesired pitching moment during hovering condition when the wing bias parameter is not applied. Once $p_x$ is set to be zero, the wing bias parameter can still generate pitching moments effectively.

**Yawing Moment**

Substituting the expression for the instantaneous $z$-body moment for the right wing from Equation (3.9) into Equation (3.23) yields

$$\bar{M}_{zRW} = \frac{1}{4\pi} \left( \int_0^\pi M_{zu}d\tau + \int_\pi^{2\pi} M_{zd}d\tau + \int_{2\pi}^{3\pi} M_{zu}d\tau + \int_{3\pi}^{4\pi} M_{zd}d\tau \right) \quad (3.36)$$
The cycle-averaged yawing moments for the right wing are computed as

\[
\bar{M}_{zRW} = -\frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_L(\alpha_u) \sin \lambda [p_y (\pi + 1.03b_{RW}^2) + \cdots \\
+ x_w \cos \alpha_u (1.38b_{RW} + 0.308b_{RW}^2 + 0.178b_{RW}^3) + \cdots \\
+ y_w (2.76 + 0.125b_{RW}^3 - 0.154b_{RW}^3)] + \cdots \\
- \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_L(\alpha_d) \sin \lambda [p_y (\pi) + \cdots \\
+ x_w \cos \alpha_d (-1.39b_{RW} - 0.00583b_{RW}^2 + 0.227b_{RW}^3) + \cdots \\
+ y_w (2.76 - 0.685b_{RW}^3 - 0.00292b_{RW}^3)] + \cdots \\
+ \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_D(\alpha_u) [p_x (-1.38b_{RW} - 0.308b_{RW}^2 - 0.178b_{RW}^3) + \cdots \\
+ p_y \cos \lambda (-2.76 - 0.125b_{RW}^3 + 0.154b_{RW}^3) + y_w \cos \lambda (-\pi - 1.03b_{RW}^2) + \cdots \\
+ x_w \sin \alpha_u \sin \lambda (1.38b_{RW} + 0.308b_{RW}^2 + 0.178b_{RW}^3)] + \cdots \\
+ \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 C_D(\alpha_d) [p_x (1.39b_{RW} + 0.00583b_{RW}^2 - 0.227b_{RW}^3) + \cdots \\
+ p_y \cos \lambda (2.76 - 0.685b_{RW}^3 - 0.00292b_{RW}^3) + y_w \cos \lambda (\pi) + \cdots \\
+ x_w \sin \alpha_d \sin \lambda (-1.39b_{RW} - 0.00583b_{RW}^2 + 0.227b_{RW}^3)]
\]  

(3.37)

The cycle-averaged yawing moments for the left wing are

\[\bar{M}_{zLW} = -\bar{M}_{zRW}\]  

(3.38)

Note that a non-zero cycle-averaged yawing moments can only be generated by either the wing bias parameter \(b_{RW}\) and \(b_{LW}\) or stroke plane angle, \(\lambda\), giving rise to drag force profile only. Moreover, when, \(b_{RW} = b_{LW} = \lambda_{RW} = \lambda_{LW}\), the generated yawing moments by the right and left wings are opposing and cancel each other which is desirable feature for hovering condition. Thus, a different wing bias parameter need to be applied on the right and left wings, i.e. \(b_{RW} \neq b_{LW}\) or \(\lambda_{RW} \neq \lambda_{LW}\), in order to generate a non-zero yawing moment.
Moreover, reducing the wing angle of attack magnitude on the upstroke while having opposite effect on the downstroke on one wing, while reverse action on the other wing, will give rise to a net clockwise yaw torque.

The following observations are in order:

1. Control of the longitudinal dynamics can be achieved by modulating the wing kinematic parameters symmetrically.

2. Control of the lateral dynamics can be achieved by modulating the wing kinematic parameters asymmetrically.

3. The control variables that affect longitudinal and lateral dynamics are decoupled.

4. A coupling effect exists between the translational force (X-Force) and the pitching moment.

5. A coupling effect exists between the side force (Y-Force) and the rolling moment.

3.4.2 Aerodynamic Control Derivatives about Hover

The cycle-averaged aerodynamic forces and moments are used to evaluate the cycle-averaged control derivatives with respect to the control variables. For controllability analysis and control synthesis, i.e., to determine if sufficient control authority can be achieved to regulate the vehicle’s six degrees of freedom (position and altitude), the sensitivity of each cycle-averaged force and moment to each control input parameter must be determined. Therefore expressions for the cycle-averaged control
derivatives with respect to variations in the input parameters are derived. The expressions for the cycle-averaged aerodynamic forces and moments are linearized about the hover condition. Letting \( \bar{G} \) to be a generalized cycle-averaged force or moment

\[
\bar{G} = \bar{G}_0 + \Delta \bar{G}
\]  

(3.39)

where \( \bar{G}_0 \) is evaluated at the hover condition, and the total increment \( \Delta \bar{G} \) is

\[
\Delta \bar{G} = \frac{\partial \bar{G}}{\partial \delta_{RW}} \Delta \delta_{RW} + \frac{\partial \bar{G}}{\partial \delta_{LW}} \Delta \delta_{LW} + \frac{\partial \bar{G}}{\partial b_{RW}} \Delta b_{RW} + \frac{\partial \bar{G}}{\partial b_{LW}} \Delta b_{LW} + \cdots
\]  

(3.40)

\[
+ \frac{\partial \bar{G}}{\partial \Delta \alpha_{RW}} \Delta \alpha_{RW} + \frac{\partial \bar{G}}{\partial \Delta \alpha_{LW}} \Delta \alpha_{LW} + \frac{\partial \bar{G}}{\partial \lambda_{RW}} \Delta \lambda_{RW} + \frac{\partial \bar{G}}{\partial \lambda_{LW}} \Delta \lambda_{LW}
\]

The control derivatives in Equation (3.40) are evaluated about the hover condition. Also the increments \( \Delta \delta_{RW}, \Delta \delta_{LW}, \Delta b_{RW}, \Delta b_{LW}, \Delta \lambda_{RW}, \) and \( \Delta \lambda_{LW} \), are replaced by \( \delta_{RW}, \delta_{LW}, b_{RW}, b_{LW}, \lambda_{RW}, \) and \( \lambda_{LW} \), respectively. Thus Equation (3.40) can be written as

\[
\Delta \bar{G} = \frac{\partial \bar{G}}{\partial \delta_{RW}} \bigg|_{\text{hover}} \delta_{RW} + \frac{\partial \bar{G}}{\partial \delta_{LW}} \bigg|_{\text{hover}} \delta_{LW} + \frac{\partial \bar{G}}{\partial b_{RW}} \bigg|_{\text{hover}} b_{RW} + \frac{\partial \bar{G}}{\partial b_{LW}} \bigg|_{\text{hover}} b_{LW} + \cdots
\]  

(3.41)

\[
+ \frac{\partial \bar{G}}{\partial \Delta \alpha_{RW}} \bigg|_{\text{hover}} \Delta \alpha_{RW} + \frac{\partial \bar{G}}{\partial \Delta \alpha_{LW}} \bigg|_{\text{hover}} \Delta \alpha_{LW} + \frac{\partial \bar{G}}{\partial \lambda_{RW}} \bigg|_{\text{hover}} \lambda_{RW} + \frac{\partial \bar{G}}{\partial \lambda_{LW}} \bigg|_{\text{hover}} \lambda_{LW}
\]

We consider the control of the MAV in the vicinity about hover condition. At the hover condition, \( \delta_{RW} = \delta_{LW} = 0, b_{RW} = b_{LW} = 0, \Delta \alpha_{RW} = \Delta \alpha_{LW} = 0, \) and \( \lambda_{RW} = \lambda_{LW} = 0, \) Note that all of the cycle-averaged forces and moments contain the lift and drag coefficients which are functions of \( \alpha_{RW, \alpha_{LW}}. \) Hence, lift and drag coefficients derivatives with respect to the change in wing angle of attack must be computed using Equation (3.3). Substituting \( \alpha_u \) and \( \alpha_d \) of each wing from Equation (3.19) into
Equation (3.3) and computing the derivatives with respect to $\Delta \alpha$ yields

\[
\begin{align*}
\frac{\partial C_L(\alpha_{uRW})}{\partial \Delta \alpha_{RW}} &= \frac{\partial C_L(\alpha_0 - \Delta \alpha_{RW})}{\partial \Delta \alpha_{RW}} = -2C_T \cos(2\alpha) \\
\frac{\partial C_L(\alpha_{dRW})}{\partial \Delta \alpha_{RW}} &= \frac{\partial C_L(\alpha_0 + \Delta \alpha_{RW})}{\partial \Delta \alpha_{RW}} = 2C_T \cos(2\alpha) \\
\frac{\partial C_D(\alpha_{uRW})}{\partial \Delta \alpha_{RW}} &= \frac{\partial C_D(\alpha_0 - \Delta \alpha_{RW})}{\partial \Delta \alpha_{RW}} = -2C_D \sin(\alpha) \cos(\alpha) \\
\frac{\partial C_D(\alpha_{dRW})}{\partial \Delta \alpha_{RW}} &= \frac{\partial C_D(\alpha_0 + \Delta \alpha_{RW})}{\partial \Delta \alpha_{RW}} = 2C_D \sin(\alpha) \cos(\alpha)
\end{align*}
\] (3.42)

$X$-Body Axis Force Control Derivatives about Hover

The $x$-body force control derivatives for the right wing, using all the considered control parameters are

\[
\begin{align*}
\frac{\partial \vec{F}_x^{RB}}{\partial \delta_{RW}} &= \frac{k_D}{2\pi} \omega_0^2 (1 + \delta_{RW}) (0.81b_{RW}^2 - 0.151b_{RW}^3) \\
\frac{\partial \vec{F}_x^{RB}}{\partial b_{RW}} &= \frac{k_D}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 (1.62b_{RW} - 0.453b_{RW}^3) \\
\frac{\partial \vec{F}_x^{RB}}{\partial \Delta \alpha_{RW}} &= \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 \left[ -2C_D \sin(\alpha_u) \cos(\alpha_u) (2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3 + 2C_D \sin(\alpha_d) \cos(\alpha_d) (-2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3) \right] \\
\frac{\partial \vec{F}_x^{RB}}{\partial \lambda_{RW}} &= \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 \cos \lambda[C_L(\alpha_u) (\pi + 1.03b_{RW}^2) + C_L(\alpha_d)(\pi)] + \cdots \\
&- \frac{k}{4\pi} \omega_0^2 (1 + \delta_{RW})^2 \sin \lambda[C_D(\alpha_u) (2.76 + 0.125b_{RW}^2 - 0.154b_{RW}^3) + \cdots \\
&+ C_D(\alpha_d) (-2.76 + 0.685b_{RW}^2 + 0.00292b_{RW}^3)]
\end{align*}
\] (3.43)
On evaluating the above control derivatives about hover condition, the $x$-body force control derivatives about the hover condition for right are computed as follows

\[
\begin{align*}
\frac{\partial \bar{F}^B_{xRW}}{\partial \delta_{RW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{xRW}}{\partial b_{RW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{xRW}}{\partial \Delta \alpha_{RW}}_{\text{hover}} &= -\frac{k \omega_0^2}{\pi} (2.76) C_D \sin(\alpha_0) \cos(\alpha_0) \\
\frac{\partial \bar{F}^B_{xRW}}{\partial \lambda_{RW}}_{\text{hover}} &= \frac{k L}{2} \omega_0^2
\end{align*}
\] (3.44)

Following a similar procedure as the right wing, the $x$-body axis force control derivatives for the left wing about the hover condition are computed as

\[
\begin{align*}
\frac{\partial \bar{F}^B_{xLW}}{\partial \delta_{LW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{xLW}}{\partial b_{LW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{xLW}}{\partial \Delta \alpha_{LW}}_{\text{hover}} &= -\frac{k \omega_0^2}{\pi} (2.76) C_D \sin(\alpha_0) \cos(\alpha_0) \\
\frac{\partial \bar{F}^B_{xLW}}{\partial \lambda_{LW}}_{\text{hover}} &= \frac{k L}{2} \omega_0^2
\end{align*}
\] (3.45)

**Y-Body Axis Force Control Derivatives about Hover**

The $y$-body axis force control derivatives for right about the hover condition are

\[
\begin{align*}
\frac{\partial \bar{F}^B_{yRW}}{\partial u_{RW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{yRW}}{\partial b_{RW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{yRW}}{\partial \Delta \alpha_{RW}}_{\text{hover}} &= 0 \\
\frac{\partial \bar{F}^B_{yRW}}{\partial \lambda_{RW}}_{\text{hover}} &= 0
\end{align*}
\] (3.46)
and the $y$-body axis force control derivatives for the left wing about the hover condition are computed as

\[
\begin{align*}
\left. \frac{\partial \vec{F}_{yLW}^B}{\partial u_{LW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{yLW}^B}{\partial b_{LW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{yLW}^B}{\partial \Delta \alpha_{LW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{yLW}^B}{\partial \lambda_{LW}} \right|_{\text{hover}} &= 0
\end{align*}
\] (3.47)

**Z-Body Axis Force Control Derivatives about Hover**

The $z$-body axis force control derivatives for the right about the hover condition are computed as

\[
\begin{align*}
\left. \frac{\partial \vec{F}_{zRW}^B}{\partial u_{RW}} \right|_{\text{hover}} &= -k_L \omega_0^2 \\
\left. \frac{\partial \vec{F}_{zRW}^B}{\partial b_{RW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{zRW}^B}{\partial \Delta \alpha_{RW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{zRW}^B}{\partial \lambda_{RW}} \right|_{\text{hover}} &= 0
\end{align*}
\] (3.48)

and the $z$-body axis force control derivatives for the left wing about the hover condition are computed as

\[
\begin{align*}
\left. \frac{\partial \vec{F}_{zLW}^B}{\partial u_{LW}} \right|_{\text{hover}} &= -k_L \omega_0^2 \\
\left. \frac{\partial \vec{F}_{zLW}^B}{\partial b_{LW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{zLW}^B}{\partial \Delta \alpha_{LW}} \right|_{\text{hover}} &= 0 \\
\left. \frac{\partial \vec{F}_{zLW}^B}{\partial \lambda_{LW}} \right|_{\text{hover}} &= 0
\end{align*}
\] (3.49)
Rolling Moment Control Derivatives about Hover

Rolling moment control derivatives for the right wing about hover are

\[
\frac{\partial \tilde{M}^B_{xRW}}{\partial u_{RW}} \bigg|_{\text{hover}} = -\frac{k_L}{2\pi} \omega_0^2 (2\pi p_y + 5.52 y_w)
\]

\[
\frac{\partial \tilde{M}^B_{xRW}}{\partial b_{RW}} \bigg|_{\text{hover}} = 0
\]

\[
\frac{\partial \tilde{M}^B_{xRW}}{\partial \Delta \alpha_{RW}} \bigg|_{\text{hover}} = 0
\]

\[
\frac{\partial \tilde{M}^B_{xRW}}{\partial \lambda_{RW}} \bigg|_{\text{hover}} = 0
\]

(3.50)

and rolling moment control derivatives for the left wing about hover are

\[
\frac{\partial \tilde{M}^B_{xLW}}{\partial u_{LW}} \bigg|_{\text{hover}} = \frac{k_L}{2\pi} \omega_0^2 (2\pi p_y + 5.52 y_w)
\]

\[
\frac{\partial \tilde{M}^B_{xLW}}{\partial b_{LW}} \bigg|_{\text{hover}} = 0
\]

\[
\frac{\partial \tilde{M}^B_{xLW}}{\partial \Delta \alpha_{LW}} \bigg|_{\text{hover}} = 0
\]

\[
\frac{\partial \tilde{M}^B_{xLW}}{\partial \lambda_{LW}} \bigg|_{\text{hover}} = 0
\]

(3.51)

Pitching Moment Control Derivatives about Hover

For the right wing, pitching moment control derivatives about hover are

\[
\frac{\partial \tilde{M}^B_{yRW}}{\partial u_{RW}} \bigg|_{\text{hover}} = 0
\]

\[
\frac{\partial \tilde{M}^B_{yRW}}{\partial b_{RW}} \bigg|_{\text{hover}} = \frac{k_L}{4\pi} \omega_0^2 (2.77 y_w)
\]

\[
\frac{\partial \tilde{M}^B_{yRW}}{\partial \Delta \alpha_{RW}} \bigg|_{\text{hover}} = \frac{k\omega_0^2}{\pi} (2.76) C_T \cos(2\alpha_0) x_w \cos \alpha_0 + \cdots
\]

\[
+ \frac{k\omega_0^2}{\pi} (2.76) C_D \sin(\alpha_0) \cos(\alpha_0) (p_z + x_w \sin \alpha_0)
\]

\[
\frac{\partial \tilde{M}^B_{yRW}}{\partial \lambda_{RW}} \bigg|_{\text{hover}} = -\frac{k_L}{2} \omega_0^2 p_z
\]

(3.52)
Similarly, pitching moment control derivatives for the left wing about hover are

\[ \frac{\partial \bar{M}_{yLW}}{\partial u_{LW}} \bigg|_{\text{hover}} = 0 \]
\[ \frac{\partial \bar{M}_{yLW}}{\partial b_{LW}} \bigg|_{\text{hover}} = \frac{k_L}{4\pi} \omega_0^2 (2.77) y_w \]
\[ \frac{\partial \bar{M}_{yLW}}{\partial \Delta \alpha_{LW}} \bigg|_{\text{hover}} = \frac{k_L}{2} \omega_0^2 \left( \sin(\alpha_0) \cos(\alpha_0) p_y + \cdots \right) \]
\[ \frac{\partial \bar{M}_{yLW}}{\partial \lambda_{LW}} \bigg|_{\text{hover}} = -\frac{k_L}{2} \omega_0^2 p_z \] (3.53)

**Yawing Moment Control Derivatives about Hover**

For the right wing, yawing moment control derivatives about hover are

\[ \frac{\partial \bar{M}_{zRW}}{\partial u_{RW}} \bigg|_{\text{hover}} = 0 \]
\[ \frac{\partial \bar{M}_{zRW}}{\partial b_{RW}} \bigg|_{\text{hover}} = 0 \]
\[ \frac{\partial \bar{M}_{zRW}}{\partial \Delta \alpha_{RW}} \bigg|_{\text{hover}} = \frac{k_L}{2} \omega_0^2 \left( \sin(\alpha_0) \cos(\alpha_0) p_y + \cdots \right) \]
\[ \frac{\partial \bar{M}_{zRW}}{\partial \lambda_{RW}} \bigg|_{\text{hover}} = -\frac{k_L}{4\pi} \omega_0^2 \left[ 2\pi p_y + (5.52) y_w \right] \] (3.54)

and yawing moment control derivatives for the left wing about hover are

\[ \frac{\partial \bar{M}_{zLW}}{\partial u_{LW}} \bigg|_{\text{hover}} = 0 \]
\[ \frac{\partial \bar{M}_{zLW}}{\partial b_{LW}} \bigg|_{\text{hover}} = 0 \]
\[ \frac{\partial \bar{M}_{zLW}}{\partial \Delta \alpha_{LW}} \bigg|_{\text{hover}} = -\frac{k_L}{2} \omega_0^2 \left( \sin(\alpha_0) \cos(\alpha_0) p_y + \cdots \right) \]
\[ \frac{\partial \bar{M}_{zLW}}{\partial \lambda_{LW}} \bigg|_{\text{hover}} = \frac{k_L}{4\pi} \omega_0^2 \left[ 2\pi p_y + (5.52) y_w \right] \] (3.55)
Control Effectiveness Matrix

Expressing all aerodynamic forces and moments as per the generalized force $\bar{G}$ given in Equation (3.41), and expressing them in matrix form yields

$$
\begin{bmatrix}
\Delta F_{x \text{des}}^B \\
\Delta F_{y \text{des}}^B \\
\Delta F_{z \text{des}}^B \\
\Delta M_{x \text{des}}^B \\
\Delta M_{y \text{des}}^B \\
\Delta M_{z \text{des}}^B
\end{bmatrix}
= \mathbf{B}
\begin{bmatrix}
u_{RW} \\
u_{LW} \\
b_{RW} \\
b_{LW} \\
\Delta \alpha_{RW} \\
\Delta \alpha_{LW} \\
\lambda_{RW} \\
\lambda_{LW}
\end{bmatrix}

(3.56)

In the above equation, $\mathbf{B}$ is the control effectiveness matrix that contains the aerodynamic control derivatives evaluated at hover, and the vector on the left side of the equation is a desired set of forces and moments, which are generated by a cycle-averaged control law. The control effectiveness matrix $\mathbf{B}$ is given by

$$
\mathbf{B} =
\begin{bmatrix}
0 & 0 & 0 & 0 & B_{15} & B_{15} & \frac{k_L \omega_0^2}{2} & \frac{k_L \omega_0^2}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{k_L \omega_0^2}{2} & -\frac{k_L \omega_0^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
B_{41} & -B_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_{53} & B_{53} & B_{55} & B_{55} & -\frac{k_L \omega_0^2}{2} \omega_0^2 p_z & -\frac{k_L \omega_0^2}{2} \omega_0^2 p_z \\
0 & 0 & 0 & 0 & B_{63} & -B_{63} & B_{67} & -B_{67}
\end{bmatrix}

(3.57)
where

\[ B_{15} = -\frac{k\omega_0^2}{\pi} (2.76) C_D \sin(\alpha_0) \cos(\alpha_0) \]

\[ B_{41} = -\frac{k_L}{2\pi} \omega_0^2 [2\pi p_y + (5.52) y_w] \]

\[ B_{53} = \frac{k_L}{4\pi} \omega_0^2 [2.77 y_w] \]

\[ B_{55} = \frac{k\omega_0^2}{\pi} (2.76) C_T \cos(2\alpha_0) x_w \cos \alpha_0 + \cdots \]

\[ + \frac{k\omega_0^2}{\pi} (2.76) C_D \sin(\alpha_0) \cos(\alpha_0) (p_z + x_w \sin \alpha_0) \]

\[ B_{65} = \frac{k\omega_0^2}{\pi} (2.76) C_D \sin(\alpha_0) \cos(\alpha_0) p_y + \cdots \]

\[ + k\omega_0^2 C_D \sin(\alpha_0) \cos(\alpha_0) y_w \]

\[ B_{67} = -\frac{k_L}{4\pi} \omega_0^2 [2\pi p_y + (5.52) y_w] \]

where \( k_L = k C_L(\alpha) \) and \( k_D = k C_D(\alpha) \).

By examining the structure of matrix \( B \), it is possible to determine that rank of \( B = 5 \). Note that the stroke plane angle and the wing angle of attack have exactly the same effects on the aerodynamic forces and moments as seen from the control effectiveness matrix. On the other hand, the stroke plane angle and the wing angle of attack have similar contribution as of the wing-bias, as they both provide control over the yaw-axis.

### 3.5 Averaged Longitudinal Model for a Minimally-Actuated FW-MAV

In this section, the averaged longitudinal model for the minimally-actuated configuration of the FW-MAV shall be computed, for the purpose of developing the control design in next chapter. When restricted to the longitudinal plane \([6, 67]\), the
equations of motion of the 6-DOF flapping wing MAV dynamics of [7] are given by

\[
\ddot{x} = \frac{2k}{m} \dot{\mu}^2(t) [C_L(\alpha_0) \sin(\lambda - \theta) - \text{sign}(\dot{\mu}(t))C_D(\alpha_0) \cos(\mu(t)) \cos(\lambda - \theta)]
\]

\[
\ddot{z} = g - \frac{2k}{m} \dot{\mu}^2(t) [C_L(\alpha_0) \cos(\lambda - \theta) + \text{sign}(\dot{\mu}(t))C_D(\alpha_0) \cos(\mu(t)) \sin(\lambda - \theta)]
\]

\[
\ddot{\theta} = \frac{2k}{I_{yy}} \dot{\mu}^2(t) C_L(\alpha_0) [p_x \cos(\lambda) - x_w \cos(\mu(t)) \sin(\psi(t)) - p_z \sin(\lambda) + y_w \sin(\mu(t))] + \frac{2k}{I_{yy}} \dot{\mu}^2(t) C_D(\alpha_0) \cos(\mu(t))\text{sign}(\dot{\mu}(t)) [x_w \cos(\psi(t)) + p_z \cos(\lambda) + p_x \sin(\lambda)]
\]

(3.59)

where \((x, z, \theta)\) are respectively the longitudinal and vertical position of the center of mass and the pitch angle of the vehicle with respect to an inertial coordinate system. For convenience, we denote by \(\mathbf{x} = [x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}]\) the state vector of (3.60). The output to be controlled is selected as the position of the center of mass, \(\mathbf{y} = [x, z]\).

The vector \([x_w, y_w]\) locates the center of pressure within the wing configuration, while the vector \([p_x, p_z]\) are constant vehicle design parameters that locates the wing root hinge point with respect to the vehicle center of mass, as discussed earlier in Chapter 2. The inertial parameters \(g, m\) and \(I_{yy}\) denote respectively acceleration of gravity, mass and moment of inertia about the body-\(y\) axis, whereas \(k = \rho I_{\dot{z}}/2\), where \(I_{\dot{z}}\) is the wing moment of inertia and \(\rho\) is the air density, assumed constant. It is also assumed that the angle-of-attack of the wings, \(\alpha_0\), is constant along the span of both the upstroke and downstroke, with an instantaneous change at the end of each half stroke. For this study, it is assumed that \(\alpha_0 \in (0, \pi/4]\). The exogenous signal \(\mu(t)\) is the time-varying wing beat, whereas \(\psi(t) = -\text{sign}(\dot{\mu}(t)) (\pi/2 - \alpha_0)\). The lift and drag coefficients of the wings are obtained from the slender body approximation as

\[
C_L(\alpha_0) = C_T \sin(2\alpha_0) \quad \text{and} \quad C_D(\alpha_0) = C_{D0} \sin^2(\alpha_0).
\]

Finally, we define the constant \(k_L = kC_L(\alpha_0)\) and \(k_D = kC_D(\alpha_0)\) to denote respectively the overall lift and drag force coefficients.

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In this investigation, as opposed to the original model in [7], it is assumed that the tail is not actuated, hence it does constitute neither a mechanical degree of freedom or a control input providing independent control of the pitch dynamics. The main purpose of the stroke plane angle $\lambda$ is to modulate the stroke plane orientation relative to the body and, hence, control the pitching dynamics of the vehicle, analogous to controlling the tilt of helicopter main rotor [7]. Since wing-bias parameter has similar feature by producing sufficient pitching moment[27, 30], it seems prudent to keep unnecessary complexity to a minimum, as the wing-bias can be applied using the same actuator that generates flapping motion. As per the definition given by Bolender in [7], the stroke plane angle is zero only if its plane of motion is perpendicular to the vehicle vertical axis. Thus, the longitudinal equations of motion of the flapping wing MAV, after eliminating the dependence on the stroke plane angle, $\lambda = 0$, are given by

\[
\ddot{x} = \frac{2k}{m} \dot{\mu}^2(t) \left[-C_L(\alpha_0) \sin(\theta) - \text{sign}(\dot{\mu}(t)) C_D(\alpha_0) \cos(\mu(t)) \cos(\theta)\right]
\]

\[
\ddot{z} = g - \frac{2k}{m} \dot{\mu}^2(t) \left[C_L(\alpha_0) \cos(\theta) - \text{sign}(\dot{\mu}(t)) C_D(\alpha_0) \cos(\mu(t)) \sin(\theta)\right]
\]

\[
\dot{\theta} = \frac{2k}{I_{yy}} \dot{\mu}^2(t) C_L(\alpha_0) \left[p_x - x_w \cos(\mu(t)) \sin(\psi(t)) + y_w \sin(\mu(t))\right]
\]

\[
+ \frac{2k}{I_{yy}} \dot{\mu}^2(t) C_D(\alpha_0) \cos(\mu(t)) \text{sign}(\dot{\mu}(t)) [x_w \cos(\psi(t)) + p_z]
\]

(3.60)

The novel wingbeat forcing function, $\mu(t)$, presented previously provides the longitudinal model with two control inputs, through the frequency modulation, $\delta$, and the wing-bias, $b$. System (3.60) can be represented as

\[
\dot{x} = f(t, x, v)
\]

(3.61)

\[
y = H x
\]

(3.62)
where \( \mathbf{v} = [\delta, b] \) is the vector of control inputs. Note that (3.61) is time-varying and non-affine in the control. The control input is to be provided by a (possibly dynamic) smooth time-invariant state-feedback controller of the form

\[
\dot{\xi} = f_c(\xi, x) \\
v = h_c(\xi, x)
\]

(3.63)

where \( \xi \in \mathbb{R}^\nu \) and \( H(\cdot) \) is a bounded function of its arguments. Note that the controller does not depend explicitly on \( t \).

Following the procedure outlined in the previous section, the dynamics (3.61)-(3.63) are time-rescaled and averaged to obtain a suitable time-invariant model. The novel wingbeat function, \( \mu(t) \), and its time-derivative, \( \dot{\mu}(t) \) were discussed thoroughly in the previous sections and are briefly recalled here for the sake of completeness:

\[
\mu(t) = \begin{cases} 
\cos(\tau) + b \tanh(\tau)^3 & : \tau \in [0, 2\pi) \\
\cos(\tau) + b[1 - \tanh(\tau - 2\pi)^3] & : \tau \in [2\pi, 4\pi)
\end{cases}
\]  

(3.64)

\[
\dot{\mu}(\tau, b) = \begin{cases} 
- \sin(\tau(t)) + 3b(t) \tanh(\tau(t))^2 \left[ 1 - \tanh(\tau(t))^2 \right] + \cdots \\
+ \frac{\dot{b}(t) \tanh(\tau(t))^3}{\omega_0(1 + \delta)} & : \tau(t) \in [0, 2\pi) \\
- \sin(\tau(t)) - 3b(t) \tanh(\tau(t) - 2\pi)^2 \left[ 1 - \tanh(\tau(t) - 2\pi)^2 \right] + \cdots \\
+ \frac{\dot{b}(t) \tanh(\tau(t) - 2\pi)^3}{\omega_0(1 + \delta)} & : \tau(t) \in [2\pi, 4\pi)
\end{cases}
\]  

(3.65)

Since typically \( \omega_0 >> 1 \), it makes sense in our case to consider \( 1/\omega_0 = \varepsilon = O(\varepsilon) \), where \( \varepsilon \) is the averaging parameter. Recalling that, in closed loop, \( \mathbf{v} \) is a bounded function of the augmented state, \( \mathbf{v} = h_c(\mathbf{x}, \xi) \), the closed-loop system (3.61)-(3.63)
expressed in the new time variable \( \tau = \omega_0 t + \Delta \) reads as

\[
\begin{align*}
\frac{d\mathbf{x}}{d\tau} &= \frac{\varepsilon}{1 + \delta(\mathbf{x}, \xi)} f(\mathbf{x}, h_c(\mathbf{x}, \xi)) \\
\frac{d\xi}{d\tau} &= \frac{\varepsilon}{1 + \delta(\mathbf{x}, \xi)} f_c(\mathbf{x}, \xi)
\end{align*}
\] (3.66)

Denoting by \( \overset{\circ}{(\cdot)} \) differentiation with respect to \( \tau \), one obtains the following expression of the closed-loop system in the new time variable

\[
\begin{align*}
\overset{\circ}{\mathbf{x}} &= \frac{\varepsilon}{1 + \delta(\mathbf{x}, \xi)} \left( -\frac{2k_L}{m} \mu^2(\tau, b) \sin \theta - \frac{2k_D}{m} \mu^2(\tau, b) \operatorname{sign}(\dot{\mu}(\tau, b)) \cos \mu(\tau, b) \cos \theta \right) \\
\overset{\circ}{\mathbf{z}} &= \frac{\varepsilon}{1 + \delta(\mathbf{x}, \xi)} \left( g - \frac{2k_L}{m} \mu^2(\tau, b) \cos \theta - \frac{2k_D}{m} \mu^2(\tau, b) \operatorname{sign}(\dot{\mu}(\tau, b)) \cos \mu(\tau, b) \sin \theta \right) \\
\overset{\circ}{\theta} &= \frac{\varepsilon}{1 + \delta(\mathbf{x}, \xi)} \left( \frac{2k_L}{I_{yy}} \mu^2(\tau, b) \left[ p_x - x_w \cos \mu(\tau, b) \sin \psi(t) + y_w \sin \mu \right] + \cdots \\
&\quad + \frac{2k_D}{I_{yy}} \dot{\mu}(t)^2(\tau, b) \operatorname{sign}(\dot{\mu}(\tau, b)) \cos \mu(\tau, b) \left[ x_w \cos \psi(t) + p_z \right] \right)
\end{align*}
\] (3.67)

The expression in (3.67) is in the appropriate form for averaging with respect to the new time variable \( \tau \). However, it worth mentioning that the model used for control design is obtained by neglecting the contribution of \( \dot{b} \). The reason behind this choice is that the contribution of \( \dot{b} \) term is \( \varepsilon \)-small as it shall be shown in the sequel. For the sake of clarity, the first part of the \( x \)-dynamics of (3.67) shall be evaluated as follows

\[
\overset{\circ}{\mathbf{x}} = \frac{\varepsilon}{1 + \delta(\mathbf{x}, \xi)} \frac{k_L}{m} \dot{\mu}^2(\tau, b) \] (3.68)

The time derivative of \( \mu(\tau, b) \) is given by

\[
\begin{align*}
\frac{d}{dt} \mu(\tau, b) &= \frac{\partial \mu}{\partial \tau} \cdot \frac{d\tau}{dt} + \frac{\partial \mu}{\partial b} \cdot \frac{db}{dt} \\
&= \left( -\sin(\tau) + 3b \tanh(\tau)^2 \left[ 1 - \tanh(\tau)^2 \right] \right) \cdot \omega_0(1 + \delta) + \frac{\tanh(\tau)^3}{\varphi_2(\tau)} \cdot \dot{b} \\
&= \varphi_1(\tau, b) \cdot \omega_0(1 + \delta) + \varphi_2(\tau) \cdot \dot{b}
\end{align*}
\] (3.69)
where the square of $\mu(\tau, b)$ reads as

$$
\left( \frac{d}{dt} \mu(\tau, b) \right)^2 = \varphi_1^2(\tau, b) \cdot \omega_0^2(1 + \delta)^2 + \varphi_2^2(\tau) \cdot \dot{b}^2 + 2 \omega_0(1 + \delta) \varphi_1(\tau, b) \varphi_2(\tau) \dot{b}
$$

and substituting Equation (3.70) into Equation (3.68), one obtains

$$
\begin{aligned}
\dot{x} &= \varepsilon \frac{1}{1 + \delta(x, \xi)} \frac{k_L}{m} \mu^2(\tau, b) \\
&= \varepsilon \frac{k_L}{1 + \delta(x, \xi)} \frac{1}{m} \left( \varphi_1^2(\tau, b) \cdot \omega_0^2(1 + \delta)^2 + \varphi_2^2(\tau) \cdot \dot{b}^2 + 2 \omega_0(1 + \delta) \varphi_1(\tau, b) \varphi_2(\tau) \dot{b} \right) \\
&= \varepsilon \frac{k_L}{1 + \delta(x, \xi)} \frac{1}{m} \left( \omega_0^2 \varphi_1^2(\tau, b) \cdot (1 + \delta)^2 + \frac{k_L}{m} \varphi_2^2(\tau) \cdot \dot{b}^2 + \cdots \right) \\
&\quad + 2 \omega_0 \frac{k_L}{m} (1 + \delta) \varphi_1(\tau, b) \varphi_2(\tau) \dot{b}
\end{aligned}
$$

An important parameter required for hovering is the trim flapping frequency, defined as

$$
\omega_* = \sqrt{\frac{mg}{k_L}}
$$

where $k_L = k C_L(\alpha)$ and recalling that $\omega_* \gg 1$ and $\omega_*^2$ scales with the ratio $m/k_L$. By setting $\omega_* = 1/\varepsilon$ and regarding $\sqrt{m/k_L} = O(\varepsilon)$, one can notice that the averaging parameter $\varepsilon$ becomes a control design parameter which can be used to enlarge the basin of attractor of the original closed-loop system. Keeping in mind

$$
\frac{k_L}{m} = \frac{1}{\omega_*^2} = O(\varepsilon^2), \quad \omega_* = O(1/\varepsilon)
$$

the following can be verified

$$
\frac{k_L}{m} \omega_0^2 \varphi_1^2(\tau, b) \cdot (1 + \delta)^2 = \frac{1}{\omega_*^2} \cdot \frac{\omega_0^2}{\omega_*^2} \varphi_1^2(\tau, b) \cdot (1 + \delta)^2
$$
Equation (3.71) can be expressed as follows

\[
\dot{x} = \frac{\varepsilon}{1 + \delta(x)} \left( \frac{k_L}{m} \omega_0^2 \varphi_1^2(\tau, b) \cdot (1 + \delta)^2 + \frac{k_L}{m} \varphi_2^2(\tau) \cdot b^2 + \cdots \right) \\
+ 2 \frac{k_L}{m} \omega_0 (1 + \delta) \varphi_1(\tau, b) \varphi_2(\tau) \cdot \dot{b} \right) + \mathcal{O}(\varepsilon) 
\]

(3.73)

\[
\dot{x} = \frac{\varepsilon}{1 + \delta(x)} \left( \frac{\varphi_1^2(\tau, b)}{f_0(\tau, x, \nu)} + \varepsilon \frac{2 \varphi_1(\tau, b) \varphi_2(\tau) \cdot \dot{b} + \varepsilon^2 \varphi_2^2(\tau) \cdot \dot{b}^2}{f_1(\tau, x, \nu) f_2(\tau, x, \nu)} \right) 
\]

The above averaged dynamics can be re-written as

\[
\dot{x} = \frac{\varepsilon}{1 + \delta(x)} \left( f_0(\tau, x, \nu) + \varepsilon f_1(\tau, x, \nu) \cdot \dot{b} + \varepsilon^2 f_2(\tau, x, \nu) \cdot \dot{b}^2 \right) 
\]

(3.74)

In order to find the averaged values of \( f_1(\tau, x, \nu) \) and \( f_2(\tau, x, \nu) \), we get

\[
f_{1}^\text{avg}(x, \nu) = \frac{1}{4\pi} \left( \int_0^\pi f_1(\tau, x, \nu)d\tau + \int_\pi^{2\pi} f_1(\tau, x, \nu)d\tau + \cdots \\
+ \int_1^{2\pi} f_1(\tau, x, \nu)d\tau + \int_2^{3\pi} f_1(\tau, x, \nu)d\tau + \cdots \right) 
\]

(3.75)

\[
f_{2}^\text{avg}(x, \nu) = \frac{1}{4\pi} \left( \int_0^\pi f_2(\tau, x, \nu)d\tau + \int_\pi^{2\pi} f_2(\tau, x, \nu)d\tau + \cdots \\
+ \int_1^{2\pi} f_2(\tau, x, \nu)d\tau + \int_2^{3\pi} f_2(\tau, x, \nu)d\tau + \cdots \right) 
\]

On computing the above definite integral we get the following expressions of the averaged \( f_{1}^\text{avg}(x, \nu) \) and \( f_{2}^\text{avg}(x, \nu) \)

\[
f_{1}^\text{avg}(x, \nu) = a_0 + a_1 b 
\]

(3.76)

\[
f_{2}^\text{avg}(x, \nu) = a_2
\]

where

\[
a_0 = 0.0615, \quad a_1 = 0.0521, \quad a_2 = 0.4456
\]

The only terms which contains time derivative of the wing-bias, \( \dot{b} \), are the functions \( f_1(\tau, x, \nu) \) and \( f_2(\tau, x, \nu) \). As a matter of fact, the effect of \( f_1(\tau, x, \nu) \) and \( f_2(\tau, x, \nu) \)
on the averaged model can be neglected since they correspond to bounded term that are scaled by $\varepsilon$ and $\varepsilon^2$, respectively. This can be achieved by making $\varepsilon$ sufficiently small, as it is a control design parameter that depends on the baseline angle of attack $\alpha_0$ and the flapping frequency.

The same arguments are applicable to the rest of the model dynamics since

$$\frac{k_L}{m} = \frac{1}{\omega^2} = \mathcal{O}(\varepsilon^2) \quad \frac{k_L}{I_{yy}} = \frac{1}{\omega^2} = \mathcal{O}(\varepsilon^2)$$

$$\frac{k_D}{m} = \frac{1}{\omega^2} = \mathcal{O}(\varepsilon^2) \quad \frac{k_D}{I_{yy}} = \frac{1}{\omega^2} = \mathcal{O}(\varepsilon^2)$$

Since it has been shown that all the terms that include the wing-bias time derivative, $\dot{b}$, are $\varepsilon$-small and negligible, the averaging of (3.67) shall be carried out with the assumption that $b(t) \equiv \text{const}$. However, the value of $|\dot{b}|$ is assumed to be bounded, where this feature shall be provided by the control system. Reverting back to the natural time scale, one obtains the *averaged closed-loop system*

$$\dot{x} = f_{\text{avg}}(x, h_c(x, \xi))$$

$$\dot{\xi} = f_c(x, \xi) \quad (3.77)$$

where the *averaged plant model* is given by

$$\ddot{x} = -\frac{k_L}{m} \omega_0^2 (1 + \delta)^2 \beta_1(b) \sin \theta - \frac{k_D}{m} \omega_0^2 (1 + \delta)^2 \beta_2(b) \cos \theta$$

$$\ddot{z} = g - \frac{k_L}{m} \omega_0^2 (1 + \delta)^2 \beta_1(b) \cos \theta + \frac{k_D}{m} \omega_0^2 (1 + \delta)^2 \beta_2(b) \sin \theta$$

$$\ddot{\theta} = \frac{k_L}{I_{yy}} \omega_0^2 (1 + \delta)^2 \left( \beta_1(b) p_x + \beta_2(b) x_w \cos \alpha_0 + \beta_3(b) y_w \right) + \cdots$$

$$+ \frac{k_D}{I_{yy}} \omega_0^2 (1 + \delta)^2 \left( \beta_2(b) [ p_z + x_w \sin \alpha_0] \right) \quad (3.78)$$

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\[
\beta_1(b) = 1 + 0.1639b^2 \\
\beta_2(b) = -0.1289b^2 + 0.024b^3 \\
\beta_3(b) = 0.4409b + 0.05b^2 - 0.00078b^3
\]

and recall that these terms, \( \beta(\cdot) \), were obtained using polynomial approximations in computing the cycle-averaged aerodynamic forces and moments and the associated integrals discussed earlier in this chapter.

The pitching dynamics of the averaged model (3.78) provide further insight into the vehicle design. Note that when no control is applied to the system, there still exists an undesired pitching moment introduced through the constant design parameter, \( p_x \), to the vehicle. Since \( p_x \) is a vehicle design parameter, this suggests that the wing attachment point should be placed such that its \( x \)-body location is coincident with the nominal vehicle center of mass, i.e. \( p_x = 0 \). With this choice, the pitching dynamics of the averaged model (3.78) can be re-written as

\[
\ddot{\theta} = \frac{k_L}{I_{yy}} \omega_0^2 (1 + \delta)^2 \left( \beta_2(b) x_w \cos \alpha_0 + \beta_3(b) y_w \right) + \cdots + \frac{k_D}{I_{yy}} \omega_0^2 (1 + \delta)^2 \left( \beta_2(b) [ p_z + x_w \sin \alpha_0 ] \right) \quad (3.79)
\]

The averaged plant model (3.78), in more concise formulation, can be re-written as

\[
\ddot{x} = -\mu_1(b)(1 + \delta)^2 \sin[\theta + \gamma(b)] \\
\ddot{z} = g - \mu_1(b)(1 + \delta)^2 \cos[\theta + \gamma(b)] \\
\ddot{\theta} = \mu_2(1 + \delta)^2 \left[ \beta_3(b) + d_\theta(b) \right] \quad (3.80)
\]

where
\[ \mu_1(b) = \frac{\omega_0^2}{m} \rho(b) \]

\[ \mu_2 = \frac{k_L \omega_0^2}{I_{yy}} y_w \]

\[ d_{\theta}(b) = \beta_2(b) \left[ \frac{x_w}{y_w} \cos \alpha_0 + \frac{k_D}{k_L} \left( \frac{p_z}{y_w} + \frac{x_w}{y_w} \sin \alpha_0 \right) \right] \]

\[ \rho(b) = \sqrt{k_L^2 \beta_1^2(b) + k_D^2 \beta_2^2(b)} \]

\[ \gamma(b) = \arctan \frac{k_D \beta_2(b)}{k_L \beta_1(b)} \]
The longitudinal instantaneous aerodynamic forces and moments of the vehicle will be used to compute the cycle-averaged aerodynamic forces and moments, and thus the development of the control-oriented model of the MAV. It is important to note that the control-oriented cycle-averaged model is solely used for controller design and stability analysis. The nonlinear controller thus designed for the control-oriented model is then used for validation on a nonlinear higher-fidelity model instantaneous model of the aerodynamic forces and moments produced by the MAV.

4.1 Vehicle Model and Problem Formulation

In this chapter, we present the control design for the longitudinal model developed in the previous chapter. It is worth mentioning that the control design developed based on the control-oriented model (3.80), while the performance of the controller evaluated on the simulation model (3.60). The control design methodology, adopted in this work, builds upon the results in [67], borrowing some of the underlying design philosophy, albeit with a significant difference, stemming from the novel actuation mechanism, as well as providing a thorough stability analysis of the overall interconnected system. As far as the novel actuation mechanism is concerned, a remarkable
novel feature of the proposed approach is the ability to eliminate the need for a stroke plane actuator, which results in significant potential reduction in terms of weight as well as the added complexity of such mechanism. The vehicle model that have been developed in the previous chapter is briefly recalled here for the sake of completeness.

The *averaged longitudinal model* is given by

\[
\begin{align*}
\ddot{x} &= -\mu_1(b)(1 + \delta)^2 \sin[\theta + \gamma(b)] \\
\ddot{z} &= g - \mu_1(b)(1 + \delta)^2 \cos[\theta + \gamma(b)] \quad (4.1) \\
\dot{\theta} &= \mu_2(1 + \delta)^2 \left[ \beta_3(b) + d_\theta(b) \right]
\end{align*}
\]

where

\[
\begin{align*}
\mu_1(b) &= \frac{\omega_0^2}{m} \rho(b) \\
\mu_2 &= \frac{k_L \omega_0^2 y_w}{I_{yy}} \\
d_\theta(b) &= \beta_2(b) \left[ \frac{x_w}{y_w} \cos \alpha_0 + \frac{k_D}{k_L} \left( \frac{p_z}{y_w} + \frac{x_w}{y_w} \sin \alpha_0 \right) \right] \\
\rho(b) &= \sqrt{k_L^2 \beta_1^2(b) + k_D^2 \beta_2^2(b)} \\
\gamma(b) &= \arctan \frac{k_D \beta_2(b)}{k_L \beta_1(b)}
\end{align*}
\]

and \(d_\theta(0) = 0\) when \(\beta_2(0) = 0\). The control inputs \(\delta\) and \(b\) are bounded, where \(\delta \in (-1, 1)\) and \(b \in (-1, 1)\). Note that \(\mu_1(b) > 0\) and \(\mu_2(b) > 0\), are affected by uncertainty, and are hereby assumed to range over appropriate known compact intervals. In particular, we set

\[
0 < \underline{\mu}_1 \leq \mu_1(b) \leq \overline{\mu}_1, \quad 0 < \underline{\mu}_2 \leq \mu_2(b) \leq \overline{\mu}_2, \quad 0 < \omega_* \leq \omega_s \leq \overline{\omega}_s \quad (4.2)
\]
4.1.1 Problem Formulation

The following proposition, proved in Teel et al.[85], establishes the connection between systems (3.66) and (3.77) in terms of the results attainable by a control policy designed on the basis of the averaged model. For ease of exposition, we let \( x^a := [x, \xi] \) denote the augmented state of both the original and averaged models (it will be clear from the context which system is being considered) and assume that the (desired) equilibrium for the averaged model is at the origin of the coordinate system, \( x^a = 0 \), which is always possible via a rigid translation.

**Proposition 4.1.1 [85, Theorem 2]** Assume that the origin of system (3.77) is globally asymptotically stable. Then, the origin of system 3.66 is semi-globally practically asymptotically stable in the parameter \( \varepsilon \). Specifically, for system (3.66), for any number \( \varsigma, \varrho \) with \( 0 < \varsigma < \varrho < \infty \) there exists \( \varepsilon^* \) such that for \( \varepsilon \in (0, \varepsilon^*) \) the following hold:

1. For each \( \psi > \varsigma \) there exists \( \sigma(\psi) > 0 \) such that, for all \( \tau_0 \in [0, 2\pi) \), \( \|x^a(\tau_0)\| \leq \sigma(\psi) \implies \|x^a(\tau)\| \leq \psi \quad \forall \tau \geq \tau_0 \);

2. For each \( \rho \in (0, \varrho) \) there exists a finite \( \nu(\rho) > 0 \) such that \( \|x^a(\tau_0)\| \leq \rho \implies \|x^a(\tau)\| \leq \nu(\rho) \quad \forall \tau \geq \tau_0 \);

3. For each \( \rho \in (0, \varrho) \) and each \( \psi > \varsigma \) there exists a finite \( T(\rho, \psi) \) such that, for all \( \tau_0 \in [0, 2\pi) \), \( \|x^a(\tau_0)\| \leq \rho \implies \|x^a(\tau)\| \leq \psi \quad \forall \tau \geq \tau_0 + T(\rho, \psi) \).

In addition, the following classic result can be invoked to determine the local behavior of solutions of (3.66) near the origin:

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Proposition 4.1.2 [41, Theorem 10.4] If the origin $x^a = 0$ is an exponentially stable equilibrium of system (3.77), then there exist positive constants $\varepsilon^*$ and $\kappa$ such that for all $\varepsilon \in (0, \varepsilon^*)$ system 3.66 has a unique, exponentially stable, periodic solution $\bar{x}^a(\tau, \varepsilon)$ with the property that $\|\bar{x}^a(\tau, \varepsilon)\| \leq \kappa \varepsilon$.

It must be kept in mind that for the origin of system (4.1) to be an equilibrium, it is necessary that $\delta$ satisfies $(1 + \delta(0,0))^2 = \omega^2_* / \omega^2_0$. Since $\omega_* / \omega_0 > 0$ and $|\delta| < 1$, necessarily, $\delta(0,0) = \omega_* / \omega_0 - 1$ and $\omega_* / \omega_0 \in (0, 2)$. This, in turn, shows that the origin is not an equilibrium of system (3.66), hence the periodic solution $\bar{x}^a(\tau, \varepsilon)$ can not be the origin itself (see the discussion in [41, Theorem. 10.4].) Also, recall that $1/\omega_* = \mathcal{O}(\varepsilon)$ and $\omega_* = \sqrt{mg/k_L(\alpha)}$. Accordingly, the averaging parameter, $\varepsilon$, is a control design parameter, since it is related to the mechanical design of the MAV, specifically to the wing angle of attack, $\alpha_0$, which is a control design parameter. As a result, the basin of the attractor of the original closed-loop system can in principle be enlarged by the controller designed on the basis of the averaged model by reducing the value of $\varepsilon$. Consequently, the control objective can only be stated as follows:

**Control Objective:** Given the averaged model (4.1) with the control input $v \in (-1, 1) \times (-1, 1)$, find a dynamic controller (3.63) such that the resulting closed-loop system has a globally asymptotically and locally exponentially stable equilibrium at the origin$^1$, robustly with respect to all values of the model parameters satisfying (4.2).

$^1$Clearly, the origin of the axes will be centered at a desired point in the longitudinal plane, $(x_d, z_d)$.
4.2 Controller Design

The overall multi-loop controller architecture proposed in this work, shown in Figure 4.1, is comprised of two control loops. In the outer-loop, the wingbeat frequency and the pitch dynamics act as servo-controllers for the longitudinal and vertical dynamics generating the output to be regulated. The inner-loop regulates the pitch dynamics servo-controller via the wingbeat frequency and the wing-bias. To begin, we determine the trim condition of the averaged model 4.1 corresponding to the desired set-point \((x, z) = (x_d, z_d)\). A simple computation shows that, at trim,

\[
\delta_* = \frac{\omega_* - \omega_0}{\omega_0}, \quad b_* = 0, \quad \theta_* = 0
\]

where \(\omega_*\) is the trim flapping frequency at hovering and \(\omega_0\) is the constant carrier flapping frequency. Previous studies [18] have emphasized on the importance of the trim frequency at hovering where the vertical dynamics exhibit an unstable limit cycle when the wingbeat frequency is selected to be exactly \(\omega_*\).
4.2.1 Outer-Loop Controller: Forward and Vertical Dynamics

Applying the change of variables $\theta \mapsto \eta := \theta + \gamma(b)$ and the “polar-to-Cartesian” transformation $(\delta, \eta) \mapsto (v_x, v_z)$ defined as

\[
\begin{align*}
v_x &= (1 + \delta)^2 \sin(\eta) \\
v_z &= 1 - (1 + \delta)^2 \cos(\eta)
\end{align*}
\] (4.3)

the $(x, z)$-dynamics assume the particularly simple expression

\[
\begin{align*}
\ddot{x} &= -\mu_1(b) \ v_x \\
\ddot{z} &= \mu_1(b) \ [v_z + r_\omega + \nu(b)]
\end{align*}
\] (4.4)

where

\[
r_\omega = \frac{\omega^2}{\omega_0^2} - 1, \quad \nu(b) = \frac{\omega^2}{\omega_0^2} \left( \frac{k_L - \rho(b)}{\rho(b)} \right)
\]

and $r_\omega$ is a constant uncertain term to be compensated, while $\nu(b)$ is a bounded disturbance due to the presence of control variable $b$ and $\nu(0) = 0$. In (4.4), $v_x$ and $v_z$ play the role of (bounded) virtual inputs, as neither of them can be directly assigned due to their implicit dependence on $\theta$ through $\eta$. The transformation (4.3) is defined globally on $(\delta, \eta) \in \mathbb{R} \times \mathbb{R}$ and is a diffeomorphism in the domain $\mathcal{D} = (-1, 1) \times (-\pi/2, \pi/2)$ onto its range, $\mathcal{R}(\mathcal{D})$, where $v_z$ must necessarily satisfy $|v_z| < 1$, since the inverse map entails $\eta = \text{arctan}(v_x/(1 - v_z))$. Next, we determine bounded controls

\[
|v_{x,cmd}| \leq \bar{v}_x, \quad |v_{z,cmd}| \leq \bar{v}_z
\] (4.5)

that robustly stabilize a given set-point $(x_d, z_d) \in \mathbb{R}^2$ when $[v_x, v_z] = [v_{x,cmd}, v_{z,cmd}]$. The bounds $\bar{v}_x > 0$ and $\bar{v}_z > 0$ is selected to satisfy

\[
|v_x| \leq \bar{v}_x \quad \text{and} \quad |v_z| \leq \bar{v}_z \quad \Rightarrow \quad |\delta| \leq \bar{\delta} \quad \text{and} \quad |\eta| \leq \bar{\eta}
\]
for some $\bar{\delta} \in (0, 1)$ and $\bar{\eta} \in (0, \pi/2)$. Using the identity $(1 + \delta)^4 = v_x^2 + (1 - v_z)^2$, the selection

$$\bar{v}_x = \frac{1}{\sqrt{2}}(1 + \bar{\delta})^2, \quad \bar{v}_z = \frac{1}{\sqrt{2}}(1 + \bar{\delta})^2 - 1$$

parameterized by the bound $\bar{\delta}$ is a valid choice as long as one takes $\bar{\delta} \in (2^{1/4} - 1, \sqrt{2}\sqrt{2} - 1) \approx (0.19, 0.68)$. Note that the second bound in (4.5) inevitably restricts to a fraction of $\bar{v}_z$ the maximum parametric uncertainty $r_\omega$ that can be tolerated.

Due to the uncertainty on $\mu_1(b)$, the design of the virtual controls $v_{x,cmd}, v_{z,cmd}$ can not be accomplished by the methodologies originally proposed in [83, 45] and successive refinements [36, 49]. Conversely, the approach proposed in [50] for uncertain chains of integrators applies. As in [50], a unit saturation function $\text{sat} : \mathbb{R} \to \mathbb{R}$ is an at least twice-differentiable function satisfying the following properties:

**Property 4.2.1 (Properties of the saturation function)**

$$|\text{sat}'(s)| := |d\text{sat}(s)/ds| \leq 2 \quad \text{for all } s \in \mathbb{R}$$

$$|\text{sat}''(s)| := |d^2\text{sat}(s)/ds^2| \leq K \quad \text{for all } s \in \mathbb{R}, \ K > 0$$

$s\text{sat}(s) > 0$ for all $s \neq 0$, sat(0) = 0

$s\text{sat}(s) = \text{sign}(s)$ for all $|s| \geq 1$

$|s| < |\text{sat}(s)| < 1$ for all $|s| < 1$

**Design for the $x$-dynamics**

For the $x$-dynamics, we first augment the first equation in (4.4) with an integrator in anticipation of the need to compensate for a steady-state error due to any external disturbances or model uncertainties. Let $x_1$ be the state of the integrator dynamics,
\[ \dot{x}_1 = x - x_d \]

and define the change of coordinates

\[ x_2 := x - x_d + \ell_1^x \text{ sat } \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) \]

\[ x_3 := \dot{x} + \ell_2^x \text{ sat } \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) \]

Selecting the virtual control

\[ v_{x,cmd} := \ell_3^x \text{ sat } \left( \frac{\kappa_3^x}{\ell_3^x} x_3 \right) \]

where \( \kappa_i^x > 0 \) and \( \ell_i^x > 0 \), \( i = 1, \ldots, 3 \), are respectively gain parameters and saturation levels to be chosen, the \( x \)-dynamics can be written as

\[ \dot{x}_1 = -\ell_1^x \text{ sat } \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) + x_2 \]

\[ \dot{x}_2 = -\ell_2^x \text{ sat } \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) + x_3 + \kappa_1^x \text{ sat } \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) \dot{x}_1 \]

(4.8)

\[ \dot{x}_3 = -\mu_1(b) \left[ \ell_3^x \text{ sat } \left( \frac{\kappa_3^x}{\ell_3^x} x_3 \right) + d_x \right] + \kappa_2^x \text{ sat } \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) \dot{x}_2 \]

where \( d_x := v_x - v_{x,cmd} \). Since \( v_x \) is given by the bound on \( v_{x,cmd} \), where

\[ |v_{x,cmd}| \leq \overline{v}_x \implies 0 < \ell_3^x \leq \overline{v}_x \]

Proposition 4.2.2 Let the gains \( \kappa_1^x, \kappa_2^x \) and the saturation levels \( \ell_1^x, \ell_2^x \) be parameterized as follows

\[ \kappa_1^x = \epsilon_1^x \kappa_2^x \]

\[ \ell_1^x = (1 - a_1^x) \frac{\ell_2^x}{\kappa_2^x} + a_1^x \frac{\ell_2^x}{8\epsilon_1^x \kappa_2^x} \]

\[ \kappa_2^x = \epsilon_2^x \kappa_3^x \]

\[ \ell_2^x = (1 - a_2^x) \frac{2\ell_3^x}{\kappa_3^x} + a_2^x \frac{\mu_1 \ell_3^x}{8\epsilon_2^x \kappa_3^x} \]

where

\[ \epsilon_1^x \in \left( 0, \frac{1}{8} \right), \quad \epsilon_2^x \in \left( 0, \frac{\mu_1}{16} \right), \quad a_1^x \in (0,1), \quad a_2^x \in (0,1) \]

(4.10)

Then, for any \( \kappa_3^x > 0 \) and any \( \ell_3^x \in (0, \overline{v}_x] \) the state of system (4.8) satisfies an \( \alpha-L_\infty \) bound from the set \( \mathcal{X}_x = \mathbb{R}^3 \) with restriction \( \Delta_x = \ell_3^x / 2 \) on the
disturbance $d_x$. Specifically, the following asymptotic bounds are obtained:

$$
\|x_1\|_a \leq \frac{4}{\kappa_1^x \kappa_2^x \kappa_3^x} \|d_x\|_a, \quad \|x_2\|_a \leq \frac{4}{\kappa_2^x \kappa_3^x} \|d_x\|_a, \quad \|x_3\|_a \leq \frac{2}{\kappa_3^x} \|d_x\|_a \tag{4.11}
$$

**Proof** Since the vector field of system (4.8) is globally Lipschitz, it is forward complete and do not exhibit a finite escape time. Following the standard results in the literature on robust control of perturbed chains of integrators by means of nested-saturation control, in particular the one outlined in [50], we begin the proof by defining the following nested sets $\Omega_1, \Omega_2$ and $\Omega_3$, as

$$
\Omega_1 := \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_1| \leq \frac{\ell_1^x}{\kappa_1^x}, \ |x_2| \leq \frac{\ell_2^x}{\kappa_2^x}, \ |x_3| \leq \frac{\ell_3^x}{\kappa_3^x} \right\} \tag{4.12}
$$

$$
\Omega_2 := \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_2| \leq \frac{\ell_2^x}{\kappa_2^x}, \ |x_3| \leq \frac{\ell_3^x}{\kappa_3^x} \right\} \tag{4.13}
$$

$$
\Omega_3 := \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_3| \leq \frac{\ell_3^x}{\kappa_3^x} \right\} \tag{4.14}
$$

where $\Omega_1 \subset \Omega_2 \subset \Omega_3$. First, we establish conditions for forward invariance of $\Omega_1, \Omega_2$ and $\Omega_3$. Assume that $\mathbf{x} \in \Omega_1$ and $|x_1| = \ell_1^x / \kappa_1^x$, and define the following Lyapunov function candidate

$$
V_1(x_1) := \frac{1}{2} x_1^2 \tag{4.15}
$$

The derivative of $V_1(x_1)$ along the trajectories of the system can be estimated as follows

$$
x_1 \dot{x}_1 = \left( -\ell_1^x \text{ sat} \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) + x_2 \right) x_1
$$

$$
= -\ell_1^x \ |x_1| + x_2 x_1
$$

$$
\leq -\ell_1^x \ |x_1| + |x_2| |x_1|
$$

$$
\leq -\ell_1^x \frac{\ell_1^x}{\kappa_1^x} + \frac{\ell_2^x \ell_1^x}{\kappa_2^x \kappa_1^x} \tag{4.16}
$$
In order to guarantee forward invariance of system (4.16), a sufficient condition for
\( x_1 \dot{x}_1 \) is readily seen to be
\[
- \ell_1^x \frac{\ell_1^x}{\kappa_1^x} + \frac{\ell_2^x \ell_1^x}{\kappa_2^x \kappa_1^x} < 0 \quad \implies \quad \ell_1^x > \frac{\ell_2^x}{\kappa_2^x} \tag{4.17}
\]
The condition (4.17) means that if \( \mathbf{x}(t_0) \in \Omega_1 \) and \( \ell_1^x > \ell_2^x / \kappa_2^x \), then \( \mathbf{x}(t) \), for \( t \geq t_0 \),
can not leave \( \Omega_1 \) through \( |x_1| = \ell_1^x / \kappa_1^x \). In addition, \( |\dot{x}_1| \) on the set \( \Omega_1 \) satisfy the
following condition
\[
|\dot{x}_1| \leq \ell_1^x + \frac{\ell_2^x}{\kappa_2^x} \leq 2 \ell_1^x \tag{4.18}
\]
Assume that \( \mathbf{x} \in \Omega_2 \) and \( |x_2| = \ell_2^x / \kappa_2^x \) and consider the following Lyapunov function candidate for \( x_2 \) and its time derivative as follows
\[
V_2(x_2) := \frac{1}{2} x_2^2
\]
\[
x_2 \dot{x}_2 = \left( -\ell_2^x \text{sat} \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) + x_3 + \kappa_1^x \text{sat'} \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) \dot{x}_1 \right) x_2
\]
\[
\leq -\ell_2^x |x_2| + |x_2||x_3| + 4 \kappa_1^x \ell_1^x |x_2|
\]
\[
\leq \frac{\ell_2^x}{\kappa_2^x} \left( -\ell_2^x + \frac{\ell_3^x}{\kappa_3^x} + 4 \kappa_1^x \ell_1^x \right)
\]
where for the last term of the right-hand side, we have used the fact that \( |\text{sat'} (\kappa_1^x x_1 / \ell_1^x)| \)
vanishes for \( |x_1| > \ell_1^x / \kappa_1^x \), hence the bound of (4.18) can be used since \( |x_2| = \ell_2^x / \kappa_2^x \)
and \( |x_1| \leq \ell_1^x / \kappa_1^x \) implies that \( \mathbf{x} \in \Omega_1 \).
In order to guarantee forward invariance of system (4.19), the following conditions must hold
\[
\frac{\ell_2^x}{\kappa_2^x} \left( -\ell_2^x + \frac{\ell_3^x}{\kappa_3^x} + 4 \kappa_1^x \ell_1^x \right) < 0 \quad \implies \quad \ell_1^x < \frac{\ell_2^x}{8 \kappa_1^x}, \quad \ell_2^x > \frac{2 \ell_3^x}{\kappa_3^x} \tag{4.20}
\]
Moreover, \( |\dot{x}_2| \) on the set \( \Omega_2 \) satisfy the following condition
\[
|\dot{x}_2| \leq \ell_2^x + \frac{\ell_3^x}{\kappa_3^x} + 4 \kappa_1^x \ell_1^x \leq 2 \ell_2^x \tag{4.21}
\]
Assume that \( \mathbf{x} \in \Omega_3 \) and \( |x_3| = \ell_3^x / \kappa_3^x \), and consider the following Lyapunov function candidate for \( x_3 \) and its time derivative as follows

\[
V_3(x_3) = \frac{1}{2} x_3^2
\]

\[
x_3 \dot{x}_3 = -\mu_1(b) \left( \ell_3^x \text{sat} \left( \frac{\kappa_3^x x_3}{\ell_3^x} \right) + d_x \right) x_3 + \kappa_2^x \text{sat}' \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) \dot{x}_2 x_3
\]

(4.22)

where we assume a restriction for \( d_x \) as \( |d_x| < \Delta x \).

In order to guarantee forward invariance of system (4.22), the following conditions must hold

\[
\mu_1(b) \ell_3^x > \mu_1(b) \Delta x + 4 \kappa_2^x \ell_2^x \implies \Delta x < \frac{\ell_3^x}{2}, \quad \ell_2^x < \frac{\mu_1 \ell_3^x}{8 \kappa_2^x}
\]

(4.23)

By combining the bounds of conditions (4.17), (4.20) and (4.23), we obtain the following bounds

\[
\frac{\ell_2^x}{\kappa_2^x} < \ell_1^x < \frac{\ell_2^x}{8 \kappa_1^x}
\]

\[
\frac{2\ell_3^x}{\kappa_3^x} < \ell_2^x < \frac{\mu_1 \ell_3^x}{8 \kappa_2^x}
\]

(4.24)

which is a necessary condition for the existence of such \( \ell_1^x \) and \( \ell_2^x \), independently on the value of \( \ell_3^x \). One may reparameterize \( \kappa_1^x \) and \( \kappa_2^x \) as \( \kappa_1^x = \epsilon_1^x \kappa_2^x \) and \( \kappa_2^x = \epsilon_2^x \kappa_3^x \), which introduces the following bound

\[
\frac{\ell_2^x}{\kappa_2^x} < \ell_1^x < \frac{\ell_2^x}{8 \kappa_1^x} \implies 0 < \epsilon_1^x < \frac{1}{8}
\]

\[
\frac{2\ell_3^x}{\kappa_3^x} < \ell_2^x < \frac{\mu_1 \ell_3^x}{8 \kappa_2^x} \implies 0 < \epsilon_2^x < \frac{\mu_1}{16}
\]

(4.25)

The bounds of the system (4.24), (4.25) make sure that all trajectories of the system converge to a neighborhood where the saturation is not active. Now, one needs to show global attractivity and local ISS within these sets. Attractivity of \( \Omega_3 \) will be
proved first. Recall that by design $x_3 \dot{x}_3 < 0$ at the boundary of the set $\Omega_3$. Then, the claim is that there exist $T_3 > 0$ such that $|x_3(T_3)| \leq \ell_3^x / \kappa_3^x$. The previous claim is proved by contradiction. Namely, assume that $x \not\in \Omega_3$, which implies that the initial condition lies outside the set $\Omega_3$, that is, $|x_3(0)| > \ell_3^x / \kappa_3^x$, and there does not exist such $T_3 > 0$. This implies that time derivative of the auxiliary function is

$$\frac{d}{dt} V_3(t) = x_3(t) \dot{x}_3(t) \leq -\mu_1(b) \ell_3^x |x_3(t)| + \mu_1(b) |x_3(t)| \Delta_x + 4 \kappa_2^x \ell_2^x |x_3(t)|$$

$$\leq - [\mu_1(b) \ell_3^x - \mu_1(b) |d_x| - 4 \kappa_2^x \ell_2^x] |x_3(t)|$$

(4.26)

where $\ell_0 = \mu_1(b) \ell_3^x - \mu_1(b) |d_x| - 4 \kappa_2^x \ell_2^x$ and $\ell_0 \geq \ell_0^{\text{min}} > 0$. Equation (4.26) implies that $x_3^2(t) \to -\infty$ as $t \to \infty$ which is impossible. According to the comparison lemma, Equation (4.26) implies that

$$\frac{d}{dt} V_3(t) \leq - \ell_0 \ell_3^x \frac{\kappa_3^x}{\kappa_2^x} \quad \forall t \geq 0$$

(4.27)

$$V_3(t) \leq V_3(0) - \ell_0 \ell_3^x t \quad \lim_{t \to \infty} V_3(t) = -\infty$$

which is a contradiction since $V_3(t) \geq 0$. Hence $T_3$ does exist. Recall that the system is globally Lipschitz which means that the trajectory $x(t)$ is globally defined on $[0, \infty)$. Since $T_3$ does exist, let $t_0 = T_3$ and repeat the analysis for $\Omega_2$, where

$$\dot{x}_2 := -\ell_2^x \text{sat} \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) + x_3 + \kappa_1^x \text{sat}' \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) \dot{x}_1 \quad \forall t \geq T_3$$

(4.28)

The claim now is that for all $x_2(0)$ there exists $T_2 > T_3$ such that $|x_2(t)| \leq \ell_2^x / \kappa_2^x$. Using the same arguments, assume that this is not the case, then $|x_2(t)| > \ell_2^x / \kappa_2^x$ for all $t \geq T_3$, hence

$$x_2(t) \dot{x}_2(t) \leq - \ell_2^x |x_2(t)| + \ell_3^x |x_2(t)| + 4 \kappa_1^x \ell_1^x |x_2(t)|$$

$$\frac{d}{dt} V_2(t) \leq - \left( \ell_2^x - \ell_3^x \frac{\kappa_3^x}{\kappa_2^x} - 4 \kappa_1^x \ell_1^x \right) \frac{\ell_2^x}{\kappa_2^x} \quad \lim_{t \to \infty} V_2(t) = -\infty$$

(4.29)
which is a contradiction since $V_2(t) \geq 0$.

Since $T_2$ does exist, let $t_0 = T_2$ and repeat the analysis for $\Omega_1$,

$$\dot{x}_1 := -\ell_1^x \text{ sat} \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) + x_2 \quad \forall t \geq T_2 \quad (4.30)$$

The claim now is that for all $x_1(0)$ there exists $T_1 > T_2$ such that $|x_1(t)| \leq \ell_1^x / \kappa_1^x$. By the same arguments, assume not, then $|x_1(t)| > \ell_1^x / \kappa_1^x \quad \forall t \geq T_2$.

$$x_1(t)x_1(t) \leq -\ell_1^x |x_1(t)| + \frac{\ell_2^x}{\kappa_2^x} |x_1(t)|$$

$$\frac{d}{dt} V_1(t) \leq - \left( \ell_1^x - \frac{\ell_2^x}{\kappa_2^x} \right) \frac{\ell_1^x}{\kappa_1^x} \implies \lim_{t \to \infty} V_1(t) = -\infty \quad (4.31)$$

which is a contradiction since $V_1(t) \geq 0$. Hence $\exists T_1 : \mathbf{x}(t) \in \Omega_1 \quad \forall t \geq T_1$.

Therefore, for all $t \geq T_1$, the trajectory $\mathbf{x}(t, x_0)$ evolves on the set $\Omega_1$, system (4.8) takes the form,

$$\dot{x}_1 := -\kappa_1^x x_1 + x_2$$

$$\dot{x}_2 := -\kappa_2^x x_2 + x_3 + \kappa_1^x \text{ sat} \left( \frac{\kappa_1^x}{\ell_1^x} x_1 \right) \dot{x}_1$$

$$\dot{x}_3 := -\mu_1(b) \kappa_3^x x_3 - \mu_1(b) d_x + \kappa_2^x \text{ sat} \left( \frac{\kappa_2^x}{\ell_2^x} x_2 \right) \dot{x}_2$$

Following Teel [84], the a-$L_\infty$ bound of $x_1$ is found as

$$\|x_1\|_a \leq \frac{1}{\kappa_1^x} \|x_2\|_a$$

Hence,

$$\|\dot{x}_1\|_a \leq \kappa_1^x \|x_1\|_a + \|x_2\|_a \leq 2\|x_2\|_a$$

The a-$L_\infty$ bound of $x_2$ is

$$\|x_2\|_a \leq \frac{2}{\kappa_2^x} \max \{ \|x_3\|_a , 4\kappa_1^x \|x_2\|_a \}$$

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The composition of the channel-two gains form a simple contraction if $\epsilon_1^x$ is chosen such that $\epsilon_1^x < 1/8$ which implies that

$$\frac{8\kappa_1^x}{\kappa_2^x} < 1$$

Thus,

$$\|x_2\|_a \leq \frac{2}{\kappa_2^x} \|x_3\|_a$$

As a result,

$$\|\dot{x}_2\|_a \leq \kappa_2^x \|x_2\|_a + \|x_3\|_a + 2\kappa_1^x \|\dot{x}_1\|_a$$

$$\leq 3\|x_3\|_a + 4\kappa_1^x \|x_2\|_a$$

$$\leq 3\|x_3\|_a + \frac{8\kappa_1^x}{\kappa_2^x} \|x_3\|_a$$

$$\leq 4\|x_3\|_a$$

The $a\mathcal{L}_\infty$ bound of $x_3$ is computed as follows

$$\|x_3\|_a \leq \frac{2}{\kappa_3^x} \max\{ \|d_x\|_a, \frac{2}{\kappa_3^x} \|\dot{x}_2\|_a \}$$

$$\leq \frac{2}{\kappa_3^x} \max\{ \|d_x\|_a, \frac{8\kappa_2^x}{\mu_1} \|x_3\|_a \}$$

Applying the small-gain condition, one obtains

$$\frac{16 \kappa_3^x}{\mu_1 \kappa_3^x} = \frac{16 \epsilon_2^x \kappa_3^x}{\mu_1 \kappa_3^x} < 1 \iff \epsilon_2^x < \frac{\mu_1}{16}$$

This yields the following asymptotic bounds

$$\|x_1\|_a \leq \frac{4}{\kappa_1^x \kappa_2^x \kappa_3^x} \|d_x\|_a, \quad \|x_2\|_a \leq \frac{4}{\kappa_2^x \kappa_3^x} \|d_x\|_a, \quad \|x_3\|_a \leq \frac{2}{\kappa_3^x} \|d_x\|_a$$

(4.33)
Design for the $z$-dynamics

In a similar fashion to the previous subsystem, we first augment the $z$-dynamics in (4.4) with an integrator in anticipation of the need to compensate for a steady-state error due to imprecise knowledge of the trim frequency at hovering $\omega_*$. An integrator is augmented to the set-point tracking error $\dot{z}_1 = z - z_d$ and then apply the change of coordinates

$$z_2 := z - z_d + \ell_1^z \text{sat} \left( \frac{\kappa_1^z}{\ell_1^z} z_1 \right)$$

$$z_3 := \dot{z} + \ell_2^z \text{sat} \left( \frac{\kappa_2^z}{\ell_2^z} z_2 \right)$$

and select the virtual control

$$v_{z,cmd} := -\ell_3^z \text{sat} \left( \frac{\kappa_3^z}{\ell_3^z} z_3 \right)$$

where $\kappa_i^z > 0$ and $\ell_i^z > 0$, $i = 1, \ldots, 3$, are respectively gain parameters and saturation levels to be chosen. This yields the following system

$$\begin{align*}
\dot{z}_1 &= -\ell_1^z \text{sat} \left( \frac{\kappa_1^z}{\ell_1^z} z_1 \right) + z_2 \\
\dot{z}_2 &= -\ell_2^z \text{sat} \left( \frac{\kappa_2^z}{\ell_2^z} z_2 \right) + z_3 + \kappa_1^z \text{sat}' \left( \frac{\kappa_1^z}{\ell_1^z} z_1 \right) \dot{z}_1 \\
\dot{z}_3 &= \mu_1(b) \left[ d_z - \ell_3^z \text{sat} \left( \frac{\kappa_3^z}{\ell_3^z} z_3 \right) + r_\omega \right] + \kappa_2^z \text{sat}' \left( \frac{\kappa_2^z}{\ell_2^z} z_2 \right) \dot{z}_2
\end{align*}$$

where $d_z := v_z - v_{z,cmd} + \nu(b)$. Recall that $\overline{v}_z$ is given by the bound on $v_{z,cmd}$, that is

$$|v_{z,cmd}| \leq \overline{v}_z \implies 0 < \ell_3^z \leq \overline{v}_z$$

The analogous to Proposition 4.2.2 for system (4.36) is stated as follows:
Proposition 4.2.3 Let the gains and saturation levels $\kappa_i^z, \ell_i^z, i = 1, 2,$ be parameterized as follows

\[
\kappa_1^z = \epsilon_1^z \kappa_2^z, \quad \ell_1^z = (1 - a_1^z) \frac{\ell_2^z}{\kappa_2^z} + a_1^z \frac{\ell_2^z}{8 \epsilon_1^z \kappa_2^z}, \\
\kappa_2^z = \epsilon_2^z \kappa_3^z, \quad \ell_2^z = (1 - a_2^z) \frac{2 \ell_3^z}{\kappa_3^z} + a_2^z \frac{\mu_1 \ell_3^z}{12 \epsilon_2^z \kappa_3^z}
\]

where

\[
\epsilon_1^z \in \left(0, \frac{1}{8}\right), \quad \epsilon_2^z \in \left(0, \frac{\mu_1}{24}\right), \quad a_1^z \in (0, 1), \quad a_2^z \in (0, 1) \quad (4.38)
\]

Then, for any $\kappa_3^z > 0$ and any $\ell_3^z \in (0, \tau_z)$ the state of system (4.36) satisfies an $\alpha$-$L_\infty$ bound from the set $X_z = \mathbb{R}^3$ with restriction $\Delta_z = \ell_3^z / 2, \Delta_\omega = \ell_3^z / 2$ on the disturbance inputs $d_z$ and $r_\omega$, respectively. Furthermore, for any constant $r_\omega$ satisfying $|r_\omega| < \Delta_\omega$, system (4.36) has a globally asymptotically and locally exponentially stable equilibrium at

\[
z_1^* = \frac{r_\omega}{\kappa_1^z \kappa_2^z \kappa_3^z}, \quad z_2^* = \frac{r_\omega}{\kappa_2^z \kappa_3^z}, \quad z_3^* = \frac{r_\omega}{\kappa_3^z}
\]

whenever $d_z = 0$. For arbitrary local essentially bounded signals $d_z$ satisfying $\|d_z\|_\alpha < \Delta_z$, the state of (4.36) in the error coordinates

\[
\tilde{z}_1 := z_1 - z_1^*, \quad \tilde{z}_2 := z_2 - z_2^*, \quad \tilde{z}_3 := z_3 - z_3^*
\]

satisfies the asymptotic bounds

\[
\|\tilde{z}_1\|_\alpha \leq \frac{4}{\kappa_1^z \kappa_2^z \kappa_3^z} \|d_z\|_\alpha, \quad \|\tilde{z}_2\|_\alpha \leq \frac{4}{\kappa_2^z \kappa_3^z} \|d_z\|_\alpha, \quad \|\tilde{z}_3\|_\alpha \leq \frac{2 \mu_1}{\kappa_3^z} \|d_z\|_\alpha \quad (4.39)
\]

Proof The proof is similar to that of Proposition 4.2.2, and for the reader’s convenience, the proof is given in Appendix A. \qed
4.2.2 Inner-Loop Controller: Pitch Dynamics

Starting from the virtual controls $v_{x,cmd}$ and $v_{z,cmd}$ defined in the previous section, define reference command for the frequency shift $\delta_{cmd}$ and the pitch angle $\theta_{cmd}$ to satisfy

\[
v_{x,cmd} := (1 + \delta_{cmd})^2 \sin(\theta_{cmd})
v_{z,cmd} := 1 - (1 + \delta_{cmd})^2 \cos(\theta_{cmd})
\] (4.40)

where
\[
|v_{x,cmd}| \leq \bar{v}_x := \ell_x^3 
|v_{z,cmd}| \leq \bar{v}_z := \ell_z^3 
\]

Since $\delta$ can be directly manipulated, $\delta$ is assigned $\delta_{cmd}$. By way of the transformation (4.40), $\delta_{cmd}$ is selected as

\[
\delta_{cmd} := \left( \frac{v_{x,cmd}^2 + (1 - v_{z,cmd})^2}{1 - v_{z,cmd}} \right)^{1/4} - 1 
\] (4.41)

Due to the implicit dependence of $v_{x,cmd}$ and $v_{z,cmd}$ on $\theta$, the control policy involves letting $\theta$ to track asymptotically its commanded trajectory\(^2\)

\[
\theta_{cmd} := \arctan \left( \frac{v_{x,cmd}}{1 - v_{z,cmd}} \right) 
\] (4.42)

However, $\theta_{cmd}$ can not be assigned directly, thus, the control input $b_{cmd}$ need to be selected such that $|\theta(t) - \theta_{cmd}(t)| \to 0$, while maintaining closed-loop stability. To this end, define the tracking error using the change of coordinates $(\theta, \dot{\theta}) \mapsto (\theta_1, \theta_2)$ as follows

\[
\theta_1 := \theta - \theta_{cmd} \\
\theta_2 := \dot{\theta} + \ell_1^\theta \text{sat} \left( \frac{k_1^\theta}{\ell_1^\theta} \theta_1 \right) 
\] (4.43)

\(^2\)Recall that (4.40) is an invertible mapping for the “commanded variables”.

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where $\ell_1^\theta > 0$ and $\kappa_1^\theta$ are design parameters. The resulting pitch dynamics in the new coordinates read as

\begin{align}
\dot{\theta}_1 &= -\ell_1^\theta \, \text{sat}\left(\frac{\kappa_1^\theta}{\ell_1^\theta} \theta_1\right) + \theta_2 - \dot{\theta}_{\text{cmd}} \\
\dot{\theta}_2 &= \mu_2 (1 + \delta)^2 \tau(b) + \kappa_1^\theta \, \text{sat}'\left(\frac{\kappa_1^\theta}{\ell_1^\theta} \theta_1\right) \dot{\theta}_1
\end{align}

(4.44)

where $\tau(b) := \beta_3(b) + d_\theta(b)$ and the wing bias, $b$, is the control input selected as

$$b := -\ell_2^\theta \, \text{sat}\left(\frac{\kappa_2^\theta}{\ell_2^\theta} \theta_2\right)$$

(4.45)

System (4.44) is written in the following form

\begin{align}
\dot{\theta}_1 &= -\ell_1^\theta \, \text{sat}\left(\frac{\kappa_1^\theta}{\ell_1^\theta} \theta_1\right) + \theta_2 - \dot{\theta}_{\text{cmd}} \\
\dot{\theta}_2 &= -\mu_2 (1 + \delta)^2 \tau\left(\ell_2^\theta \, \text{sat}\left(\frac{\kappa_2^\theta}{\ell_2^\theta} \theta_2\right)\right) + \kappa_1^\theta \, \text{sat}'\left(\frac{\kappa_1^\theta}{\ell_1^\theta} \theta_1\right) \dot{\theta}_1
\end{align}

(4.46)

Recall that the averaging theorem requires that the time derivative of the wing bias control input $\dot{b}$ to be bounded only and need not to be saturated at rate by a small arbitrary number. Moreover, the time derivative $\dot{b}$ is already scaled by the averaging parameter $\varepsilon$ which can be made sufficiently small as it is a control design parameter where it depends on the baseline angle of attack and the flapping frequency. The time derivative of the wing bias control input (4.45) reads as

$$\dot{b} = -\kappa_2^\theta \, \text{sat}'\left(\frac{\kappa_2^\theta}{\ell_2^\theta} \theta_2\right) \left[\mu_2 (1 + \delta)^2 \tau(b) + \kappa_1^\theta \, \text{sat}'\left(\frac{\kappa_1^\theta}{\ell_1^\theta} \theta_1\right) \dot{\theta}_1\right]$$

(4.47)

The first term $\left[\kappa_2^\theta \, \text{sat}'\left(\frac{\kappa_2^\theta}{\ell_2^\theta} \theta_2\right)\right]$ of (4.47) is always bounded and zero when $\theta_2 > \ell_2^\theta / \kappa_2^\theta$. Hence, we only need to consider the case when $\theta_2 \leq \ell_2^\theta / \kappa_2^\theta$.

Note that $\mu_2$ depends on uncertain parameters and assumed to range over appropriate known compact interval. That is,

$$0 < \underline{\mu}_2 \leq \mu_2 \leq \overline{\mu}_2$$

(4.48)
Since $|\delta| \leq \bar{\delta}$ and $|b| \leq \bar{b}$, the following implications can be inferred

\[
|\delta| \leq \bar{\delta} \quad \Rightarrow \quad 0 < (1 - \bar{\delta})^2 \leq (1 + \delta)^2 \leq (1 + \bar{\delta})^2
\]

\[
|b| \leq \bar{b} \quad \Rightarrow \quad \tau \leq \tau(b) \leq \bar{\tau}
\]

where

\[
\tau = \beta_3(\theta) + d_\theta(\theta)
\]

\[
\tau = \beta_3(1) + d_\theta(1)
\]

Using (4.48) and (4.49), it’s obvious that $[\mu_2(1 + \delta)^2 \tau(b)]$ of (4.47) is bounded. The last term of (4.47) is written as

\[
k_1^\theta \text{ sat}' \left( \frac{k_1^\theta}{\ell_1^\theta} \theta_1 \right) \left[ -\ell_1^\theta \text{ sat} \left( \frac{k_1^\theta}{\ell_1^\theta} \theta_1 \right) + \theta_2 - \dot{\theta}_{cmd} \right]
\]

(4.52)

The first two terms of (4.52) are already bounded while $\theta_2$ is bounded since $\theta_2 \leq \ell_2^\theta / k_2^\theta$. The last term of (4.52), $\dot{\theta}_{cmd}$, is shown to be bounded in Proposition 4.2.4.

The asymptotic behavior of system (4.46) is described in the following proposition, which gives the guidelines for the selection of the control parameters.

**Proposition 4.2.4** Let the gain $k_1^\theta$, and the saturation level $\ell_1^\theta$ be parameterized as follows

\[
k_1^\theta = \epsilon_1^\theta k_2^\theta,
\]

\[
\ell_1^\theta = (1 - a_1^\theta) \frac{2\ell_2^\theta}{k_2^\theta} + a_1^\theta \frac{\mu_2 (1 - \bar{\delta})^2 \tau \ell_2^\theta}{4\epsilon_1^\theta k_2^\theta}
\]

where

\[
\epsilon_1^\theta \in \left( 0, \frac{\mu_2 (1 - \bar{\delta})^2 \tau}{8} \right), \quad a_1^\theta \in (0, 1)
\]

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Then, for any $\kappa_2^\theta > 0$ and any $\ell_2^\theta \in (0, \bar{b}]$, system (4.46) is ISS from the set $X_\theta = \mathbb{R}^2$ with restriction $\Delta_{\dot{\theta}_{cmd}} = \ell_1^\theta/2$ on $\dot{\theta}_{cmd}$. Furthermore, the state satisfies the following asymptotic bounds

$$
\|\theta_1\|_a \leq \frac{2}{\kappa_1^\theta} \|\dot{\theta}_{cmd}\|_a
$$

$$
\|\theta_2\|_a \leq \frac{8}{\mu_2 (1-\delta)^2 \kappa_1^\theta \kappa_2^\theta} \|\dot{\theta}_{cmd}\|_a
$$

(4.53)

**Proof** Since the vector field of system (4.46) is globally Lipschitz, it is forward complete and does not exhibit a finite escape time. Following [50], define the sets

$$
\Omega_1 := \left\{ \theta \in \mathbb{R}^2 : |\theta_1| \leq \frac{\ell_1^\theta}{\kappa_1^\theta}, |\theta_2| \leq \frac{\ell_2^\theta}{\kappa_2^\theta} \right\}
$$

(4.54)

$$
\Omega_2 := \left\{ \theta \in \mathbb{R}^2 : |\theta_2| \leq \frac{\ell_2^\theta}{\kappa_2^\theta} \right\}
$$

(4.55)

where $\Omega_1 \subset \Omega_2$. First, we establish conditions for forward invariance of both sets $\Omega_1$ and $\Omega_2$. Assume that $\theta \in \Omega_1$ where $|\theta_1| = \ell_1^\theta/\kappa_1^\theta$ and $|\theta_2| = \ell_2^\theta/\kappa_2^\theta$, and define the following Lyapunov function candidate

$$
V_1(\theta_1) := \frac{1}{2} \theta_1^2
$$

(4.56)

The derivative of $V_1(\theta_1)$ along trajectories of the system can be found as follows

$$
\theta_1 \dot{\theta}_1 = \left( -\ell_1^\theta \text{sat} \left( \frac{\kappa_1^\theta}{\ell_1^\theta} \tilde{\theta}_1 \right) + \theta_2 - \dot{\theta}_{cmd} \right) \theta_1
$$

$$
\leq -\ell_1^\theta |\theta_1| + |\theta_2| |\theta_1| + |\dot{\theta}_{cmd}| |\theta_1|
$$

$$
\leq |\theta_1| \left[ -\ell_1^\theta + |\theta_2| + |\dot{\theta}_{cmd}| \right]
$$

$$
\leq \frac{\ell_1^\theta}{\kappa_1^\theta} \left[ -\ell_1^\theta + \frac{\ell_2^\theta}{\kappa_2^\theta} + \Delta_{\dot{\theta}_{cmd}} \right]
$$

(4.57)

where we assume a restriction for $\dot{\theta}_{cmd}: |\dot{\theta}_{cmd}| < \Delta_{\dot{\theta}_{cmd}}$. In order to guarantee forward invariance of system (4.57), a sufficient condition for $\theta_1 \dot{\theta}_1$ is readily seen to be

$$
\ell_1^\theta + \frac{\ell_2^\theta}{\kappa_2^\theta} + \Delta_{\dot{\theta}_{cmd}} < 0
$$

(4.58)
which implies that
\[ \frac{\ell_2^\theta}{\kappa_2^\theta} < \frac{\ell_1^\theta}{2}, \quad \text{and} \quad \Delta_{\dot{\theta}_{cmd}} < \frac{\ell_1^\theta}{2} \] (4.59)

The condition (4.59) implies that if the initial condition \( \theta(t_0) \in \Omega_1 \), then the trajectory \( \theta(t) \), for \( t \geq t_0 \), can not leave \( \Omega_1 \) through \( |\theta_1| = \ell_1^\theta/\kappa_1^\theta \). In addition, \( |\dot{\theta}_1| \) on the set \( \Omega_1 \) satisfy the following condition
\[ |\dot{\theta}_1| \leq \ell_1^\theta + \frac{\ell_2^\theta}{\kappa_2^\theta} + \Delta_{\dot{\theta}_{cmd}} < 2\ell_1^\theta \] (4.60)

Assume that \( \theta \in \Omega_2 \) and \( |\theta_2| = \ell_2^\theta/\kappa_2^\theta \) and consider the following Lyapunov function candidate for \( \theta_2 \) and its time derivative as follows
\[
V_2(\theta_2) := \frac{1}{2} \theta_2^2 \\
\theta_2 \dot{\theta}_2 = \left( -\mu_2 (1 + \delta)^2 \right) \tau \left( \ell_2^\theta \text{sat} \left( \frac{\kappa_2^\theta \dot{\theta}_2}{\ell_2^\theta} \right) + \kappa_1^\theta \text{sat}' \left[ \frac{\kappa_2^\theta}{\ell_2^\theta} \theta_1 \right] \dot{\theta}_1 \right) \theta_2 \\
\leq -\mu_2 (1 + \delta)^2 \tau \left( \ell_2^\theta \text{sat} \left[ \frac{\kappa_2^\theta \dot{\theta}_2}{\ell_2^\theta} \right] \right) |\theta_2| + \kappa_1^\theta \text{sat}' \left[ \frac{\kappa_2^\theta}{\ell_2^\theta} \theta_1 \right] \dot{\theta}_1 |\theta_2| \\
\text{(4.61)}
\]

In order to find a suitable bound for \( \tau \left( \ell_2^\theta \text{sat} \left[ \frac{\kappa_2^\theta \dot{\theta}_2}{\ell_2^\theta} \right] \right) \) term in Equation (4.61), the following property can be defined.

**Property 4.2.5** Since \( \forall b : |b| \leq \bar{b} \), the following condition
\[ |\tau(b)| = \tau(|b|) \geq \tau |b| \]
is verified since
\[ b \geq 0 \implies \tau(b) \geq \tau b \]
\[ b \leq 0 \implies \tau(b) \leq \tau b \]

and
\[ \tau(b \text{ sign}(b)) = \tau(|b|) \text{ sign}(b) \]

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which implies the following

\[ |\theta_2| \geq \frac{\ell_2^0}{\kappa_2^0} \implies \tau \left( \ell_2^0 \text{ sat} \left[ \frac{\kappa_2^0}{\ell_2^0} \theta_2 \right] \right) = \tau \left( \ell_2^0 \text{ sign} \theta_2 \right) = \tau \left( \ell_2^0 \text{ sign} \theta_2 \right) \]

Using Property 4.2.5 discussed above, Equation (4.61) satisfies the following bound

\[ \dot{\theta}_2 \leq -\mu_2 (1 + \delta)^2 \tau \left( \ell_2^0 \text{ sat} \left[ \frac{\kappa_2^0}{\ell_2^0} \theta_2 \right] \right) |\theta_2| + \kappa_1^0 \text{ sat}' \left[ \frac{\kappa_1^0}{\ell_1^0} \theta_1 \right] \dot{\theta}_1 |\theta_2| \]

\[ \leq -\mu_2 (1 - \delta)^2 \tau \ell_2^0 |\theta_2| + 2\kappa_1^0 \dot{\theta}_1 |\theta_2| \]

\[ \leq -\mu_2 (1 - \delta)^2 \tau \ell_2^0 + 4\kappa_1^0 \ell_1^0 |\theta_2| \]

(4.62)

where for the last term of the right-hand side, we have used the fact that \([\text{sat}' \left( \frac{\kappa_1^0 \theta_1}{\ell_1^0} \right)]\) vanishes for \(|\theta_1| > \ell_1^0/\kappa_1^0\), hence the bound of (4.60) can be used since \(|\theta_2| = \ell_2^0/\kappa_2^0\)

and \(|\theta_1| \leq \ell_1^0/\kappa_1^0\) implies that \(\theta \in \Omega_1\).

A sufficient condition that guarantees forward invariance of system (4.62) is readily seen to be

\[ -\mu_2 (1 - \delta)^2 \tau \ell_2^0 + 4\kappa_1^0 \ell_1^0 < 0 \implies \mu_2 (1 - \delta)^2 \tau \ell_2^0 > 4\kappa_1^0 \ell_1^0 \]

(4.63)

Combining the bounds of conditions (4.59) and (4.63), we have the following

\[ \frac{2\ell_2^0}{\kappa_2^0} < \ell_1^0 < \frac{\mu_2 (1 - \delta)^2 \tau \ell_2^0}{4 \kappa_1^0} \]

(4.64)

which is a necessary condition for the existence of such \(\ell_1^0\), which is independent on the value of \(\ell_2^0\). One may reparameterize \(\kappa_1^0 = \epsilon_1^0 \kappa_2^0\), which introduces the following bound

\[ \frac{2}{\kappa_2^0} < \frac{\mu_2 (1 - \delta)^2 \tau}{4 \epsilon_1^0 \kappa_2^0} \implies 0 < \epsilon_1^0 < \frac{\mu_2 (1 - \delta)^2 \tau}{8} \]

(4.65)

The restrictions (4.64) and (4.65) make sure that the trajectories of system (4.46) eventually converges to a neighborhood where saturation is not active.
Therefore, when the trajectory $\theta(t, \theta_0)$ evolves on the sets $\Omega$, system (4.46) takes the form

$$
\begin{align*}
\dot{\theta}_1 &= -\kappa_1^\theta \theta_1 + \theta_2 - \dot{\theta}_{cmd} \\
\dot{\theta}_2 &= -\mu_2(1 + \delta)^2 \tau \left( \kappa_2^\theta \theta_2 \right) + \kappa_1^\theta \text{sat}' \left( \frac{\kappa_1^\theta}{\ell_1^\theta} \theta_1 \right) \dot{\theta}_1
\end{align*}
$$

(4.66)

Following Teel [84], the $\mathcal{L}_\infty$ bound of $\theta_1$ is

$$
\|\theta_1\|_a \leq \frac{1}{\kappa_1^\theta} \left\{ \|\theta_2\|_a - \|\dot{\theta}_{cmd}\|_a \right\}
$$

$$
\leq \frac{1}{\kappa_1^\theta}\|\theta_2\|_a + \frac{1}{\kappa_1^\theta}\|\dot{\theta}_{cmd}\|_a
$$

while

$$
\|\dot{\theta}_1\|_a \leq \kappa_1^\theta \|\theta_1\|_a + \|\theta_2\|_a + \|\dot{\theta}_{cmd}\|_a
$$

$$
\leq 2\|\theta_2\|_a + 2\|\dot{\theta}_{cmd}\|_a
$$

$$
\leq 4 \max\{\|\theta_2\|_a, \|\dot{\theta}_{cmd}\|_a\}
$$

The $\mathcal{L}_\infty$ bound of $\theta_2$ is

$$
\|\theta_2\|_a \leq \frac{2 \kappa_1^\theta}{\mu_2(1 - \delta)^2 \tau \kappa_2^\theta} \|\dot{\theta}_1\|_a
$$

$$
\leq \frac{8 \kappa_1^\theta}{\mu_2(1 - \delta)^2 \tau \kappa_2^\theta} \max\{\|\theta_2\|_a, \|\dot{\theta}_{cmd}\|_a\}
$$

Using the small-gain condition, the composition of the channel-two gains form a simple contraction if $\epsilon_1^\theta$ is chosen such that

$$
\frac{8 \kappa_1^\theta}{\mu_2(1 - \delta)^2 \tau \kappa_2^\theta} = \frac{8 \epsilon_1^\theta \kappa_2^\theta}{\mu_2(1 - \delta)^2 \tau \kappa_2^\theta} < 1 \implies \epsilon_1^\theta < \frac{\mu_2(1 - \delta)^2 \tau}{8}
$$

Lastly, this yields the following asymptotic bounds of $\theta_1$ and $\theta_2$

$$
\|\theta_1\|_a \leq \frac{2}{\kappa_1^\theta} \|\dot{\theta}_{cmd}\|_a
$$

$$
\|\theta_2\|_a \leq \frac{8 \kappa_1^\theta}{\mu_2(1 - \delta)^2 \tau \kappa_2^\theta} \|\dot{\theta}_{cmd}\|_a
$$

□
4.3 Analysis of the Interconnection

The stability analysis of the overall interconnected system will be carried out using the small-gain theorem, and in particular the “asymptotic” formulation for systems with saturated feedback due to Teel [84]. The analysis of the feedback interconnection of the overall system involve enforcing a small gain condition between locally ISS systems. The overall structure of the closed-loop system is shown in Figure 4.2, where the relevant coupling inputs are highlighted. Since each individual subsystem has been shown to be ISS with or without restrictions, the task is to show that for each loop there exists a choice of the available degrees of freedom (that is, those gains that can be freely assigned) such that:

1. The restrictions for the corresponding disturbance inputs are attained in finite time;

Figure 4.2: Structure of the interconnected subsystems.
2. A small-gain involving the asymptotic norms of the coupling signals can be enforced.

If this is the case, then global asymptotic stability follows from Teel’s “nonlinear small-gain theorem” [84]. To begin, we obtain a suitable bound for $\dot{\theta}_{cmd}$. Since

$$\dot{\theta}_{cmd} = \frac{1 - v_{z,cmd}}{(1 + \delta)^4} \dot{v}_{x,cmd} + \frac{v_{x,cmd}}{(1 + \delta)^4} \dot{v}_{z,cmd}$$

one obtains

$$\left| \dot{\theta}_{cmd} \right| \leq \frac{1 + v_{z}}{(1 - \delta)^4} |\dot{v}_{x,cmd}| + \frac{v_{x}}{(1 - \delta)^4} |\dot{v}_{z,cmd}|$$

(4.67)

where $B_x$ and $B_z$ are fixed numbers once $v_x$, $v_z$ and $\delta$ have been selected. Keeping in mind (4.9) and the properties of the saturation function, it can be shown that

$$|\dot{v}_{x,cmd}| \leq \kappa_x (2\ell_x^3 \bar{\mu}_1 + 2\bar{\mu}_1 |d_x| + 4\ell_x^2 \kappa_x + 4\ell_x^2 \epsilon_x + 8\kappa_x \kappa_x \ell_x + 8\kappa_x \ell_x^2)$$

Since $|d_x| \leq 2(1 + \bar{\delta})^2$, the above inequality yields

$$|\dot{v}_{x,cmd}| \leq \kappa_x^3 N_x$$

(4.68)

where $N_x := 2\ell_x^3 \bar{\mu}_1 + 4\bar{\mu}_1 (1 + \bar{\delta})^2 + 4\ell_x^2 \kappa_x + 4\ell_x^2 \epsilon_x + 8\kappa_x \kappa_x \ell_x + 8\kappa_x \ell_x^2$. Similarly, one can find a bound on $\dot{v}_{z,cmd}$ using (4.37) and the properties of the saturation function as follows

$$|\dot{v}_{z,cmd}| \leq \kappa_z^3 \left(2\ell_z^3 \bar{\mu}_1 + 2\bar{\mu}_1 r_{\omega} + 2\bar{\mu}_1 |d_z| + 4\kappa_z \ell_z^2 + 4\ell_z^2 \epsilon_z + 8\kappa_z \kappa_z \ell_z + 8\kappa_z \ell_z^2 \right)$$

Since $|d_x| \leq 2(1 + \bar{\delta})^2 + c_r \bar{b}^2$, the above inequality yields

$$|\dot{v}_{z,cmd}| \leq \kappa_z^3 N_z$$

(4.69)

where $N_z := 2\ell_z^3 \bar{\mu}_1 + 2\bar{\mu}_1 r_{\omega} + 2\bar{\mu}_1 \left(2(1 + \bar{\delta})^2 + c_r \bar{b}^2 \right) + 4\kappa_z \ell_z^2 + 4\ell_z^2 \epsilon_z + 8\kappa_z \kappa_z \ell_z + 8\kappa_z \ell_z^2$. As a matter of fact, it can be shown that $N_x$ and $N_z$ are fixed numbers once all
design parameters are fixed and saturation levels are chosen, for instance, $\ell^x_3 = v_x$ and $\ell^z_3 = v_z$. Consequently, from (4.67), (4.68) and (4.69) it follows that

$$|\dot{\theta}_{cmd}| \leq \kappa^x_3 M_x + \kappa^z_3 M_z$$

(4.70)

where $M_x := B_x N_x$ and $M_z := B_z N_z$. Equation (4.70) shows that $\dot{\theta}_{cmd}$ can be bounded \textit{a priori} and scalable by the gains $\kappa^x_3$ and $\kappa^z_3$. In order to analyze the interconnection and satisfying Proposition 4.2.4 conditions, we need necessarily to enforce the restriction on $\dot{\theta}_{cmd}$ such that

$$|\dot{\theta}_{cmd}| \leq \kappa^x_3 M_x + \kappa^z_3 M_z < \Delta_{\dot{\theta}_{cmd}} = \frac{\ell^{\theta}_1}{2}$$

(4.71)

The restriction $\Delta_{\dot{\theta}_{cmd}} = \ell^{\theta}_1/2$ can be enforced for any given $\ell^{\theta}_1/2$ by selecting $\kappa^x_3$ and $\kappa^z_3$ sufficiently small (recall from Propostion 4.2.2 and Propostion 4.2.3 that these gains can be chosen arbitrarily). Recall from Propostion 4.2.4, the saturation $\ell^{\theta}_1$ can be expressed as follows

$$\ell^{\theta}_1 = \left[2 \left(1 - a^\theta_1\right) + a^\theta_1 \frac{\mu_2 (1 - \bar{\delta})^2 \tau}{4 \epsilon^\theta_1}\right] \frac{\ell^{\theta}_2}{\kappa^\theta_2}$$

Since $\ell^{\theta}_1$ determine the restriction bound on $|\dot{\theta}_{cmd}|$, it is crucial that it is independent on any other gain. One can notice that once the design parameters $\{a^\theta_1, \mu_2, \bar{\delta}, \epsilon^\theta_1\}$ have been selected, $\ell^{\theta}_1$ can be written in the following form

$$\ell^{\theta}_1 = \frac{\tau}{\kappa^\theta_2} \frac{\ell^{\theta}_2}{\kappa^\theta_2}$$

(4.72)

where

$$\ell^{\theta}_1 := \left[2 \left(1 - a^\theta_1\right) + a^\theta_1 \frac{\mu_2 (1 - \bar{\delta})^2 \tau}{4 \epsilon^\theta_1}\right]$$
and $\ell_1^\theta$ is a fixed number. However, the value of $\ell_1^\theta$ in (4.72) is not fixed, since it depends on the ratio $\ell_2^\theta/\kappa_2^\theta$. To that end, it would be advantageous to keep $\ell_2^\theta$ and $\kappa_2^\theta$ parameterized by a single parameter and define

$$\ell_2^\theta = \sigma_2^\theta \kappa_2^\theta$$  \hspace{1cm} (4.73)

where $\sigma_2^\theta$ is a fixed number. Thus, Equation (4.72) can be re-written as follows

$$\ell_1^\theta = \ell_1^\theta \sigma_2^\theta$$

which ensures that $\ell_1^\theta$ is a fixed number. Hence, the restriction on $\Delta_{\dot{\theta}_{cmd}}$ can be expressed as follows

$$|\dot{\theta}_{cmd}| < \Delta_{\dot{\theta}_{cmd}} = \frac{\ell_1^\theta \sigma_2^\theta}{2}$$ \hspace{1cm} (4.74)

By combining (4.71) and (4.74), the following condition can be formulated

$$(M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\} < \frac{\ell_2^\theta \sigma_2^\theta}{2} \implies \max \{\kappa_3^x, \kappa_3^z\} < M_1$$ \hspace{1cm} (4.75)

where

$$M_1 = \frac{\ell_1^\theta \sigma_2^\theta}{2} \cdot \frac{1}{(M_x + M_z)}$$

Hence $\kappa_3^x$ and $\kappa_3^z$, of the upper subsystem in Figure 4.2, must be taken small until condition (4.75) is met. Consequently, the saturations, of the $\theta$-subsystem, become eventually inactive and the asymptotic bounds are valid. Thus, by combining the asymptotic norm $\|\theta\|_a$ from Proposition 4.2.4 and condition (4.75), one obtains

$$\|\theta_1\|_a \leq \frac{2}{\kappa_1^\theta} \|\dot{\theta}_{cmd}\|_a$$

$$\leq \frac{2}{\kappa_1^\theta} (M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\}$$

$$\leq \frac{2}{\ell_1^\theta \kappa_2^\theta} (M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\}$$
However, the disturbances $d_x$ and $d_z$ depends on $\theta_1$, which can be bounded as follows

$$
|d_x| \leq (1 + \bar{\delta})^2 |\theta_1| + (1 + \bar{\delta})^2 |\gamma(b)| \\
\leq (1 + \bar{\delta})^2 |\theta_1| + (1 + \bar{\delta})^2 c_\gamma |b|^2 \\
\leq (1 + \bar{\delta})^2 |\theta_1| + (1 + \bar{\delta})^2 \bar{c}_\gamma |b|
$$

Equation (4.76) utilizes the bound shown in Figure 4.3 which illustrates that $\gamma(b)$ is globally Lipschitz in a compact set, as $|\gamma(b)|$ can be bounded by the linear function of the form $c_\gamma |b|$, as shown in Figure 4.3. Similarly, $|d_z|$ can be bounded as follows

$$
|d_z| \leq (1 + \bar{\delta})^2 |\theta_1| + (1 + \bar{\delta})^2 |\gamma(b)| + |\nu(b)| \\
\leq (1 + \bar{\delta})^2 |\theta_1| + (1 + \bar{\delta})^2 |\gamma(b)| + c_\nu |b|^2 \\
\leq (1 + \bar{\delta})^2 |\theta_1| + (1 + \bar{\delta})^2 |\gamma(b)| + \bar{c}_\nu |b|
$$

Figure 4.3: An illustration that $|\gamma(b)|$ in $d_x$ is globally Lipschitz in a compact set, where $c_\gamma = 0.07$, and $\bar{c}_\gamma = 0.06$. 

| $|\gamma(b)|$ | $c_\gamma |b|^2$ |
|-----------------|-----------------|
| (a) An illustration of $|\gamma(b)|$ and $c_\gamma |b|^2$ | (b) An illustration of $|\gamma(b)|$ and $\bar{c}_\gamma |b|$ |
Figure 4.4 shows that $|\nu(b)|$ can be bounded by the linear function $\bar{c}_\nu|b|$ for some $\bar{c}_\nu$. Recall that all trajectories are well-defined and do not have finite escape time by virtue of the fact that the system is globally Lipschitz. Equations (4.76) and (4.77) shows that $|d_x|$ and $|d_z|$ are globally bounded as well and that $d_x = d_z = 0$ when $\theta_1 = 0$ and $b = 0$. The next step is to show that $\|d_x\|_a < \Delta_x$ and $\|d_z\|_a < \Delta_z$. Recall

\begin{align*}
\|d_x\|_a &\leq \frac{2(1+\bar{\delta})^2}{\epsilon_1^2 \kappa_2^2} (M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\} + (1+\bar{\delta})^2 c_\gamma \left(\sigma_2^\theta \kappa_2^\theta\right)^2 \\
\|d_z\|_a &\leq \frac{2(1+\bar{\delta})^2}{\epsilon_1^2 \kappa_2^2} (M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\} + \left[(1+\bar{\delta})^2 c_\gamma + c_\nu\right] \left(\sigma_2^\theta \kappa_2^\theta\right)^2
\end{align*}

Figure 4.4: An illustration that $|\gamma(b)|$ in $d_z$ is globally Lipschitz in a compact set, where $c_\nu = 0.16$, and $\bar{c}_\nu = 0.12$. That all saturations become inactive, and we only need to fulfill the asymptotic norms of the relative restrictions. Thus, the asymptotic norms of the disturbances $d_x$ and $d_z$ can found as follows

\begin{align*}
\|d_x\|_a &\leq \frac{2(1+\bar{\delta})^2}{\epsilon_1^2 \kappa_2^2} (M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\} + (1+\bar{\delta})^2 c_\gamma \left(\sigma_2^\theta \kappa_2^\theta\right)^2 \\
\|d_z\|_a &\leq \frac{2(1+\bar{\delta})^2}{\epsilon_1^2 \kappa_2^2} (M_x + M_z) \max \{\kappa_3^x, \kappa_3^z\} + \left[(1+\bar{\delta})^2 c_\gamma + c_\nu\right] \left(\sigma_2^\theta \kappa_2^\theta\right)^2
\end{align*}
Recall that from Proposition 4.2.2 and Proposition 4.2.3 that $\|d_x\|_a < \ell_3^x/2$ and $\|d_z\|_a < \ell_3^z/3$, one obtains

$$\max\{\kappa_3^x, \kappa_3^z\} < \kappa_2^\theta M_2$$

$$\max\{\kappa_3^x, \kappa_3^z\} < \kappa_2^\theta M_3$$

where

$$M_2 = \frac{\ell_3^x}{4} \cdot \frac{\ell_1^\theta}{2(1+\delta)^2(M_x+M_z)}, \quad M_3 = \frac{\ell_3^z}{6} \cdot \frac{\ell_1^\theta}{2(1+\delta)^2(M_x+M_z)}$$

and

$$\frac{(1+\delta)^2 c_\gamma \left(\sigma_2^\theta \kappa_2^\theta\right)^2}{4} < \frac{\ell_3^x}{4} \Rightarrow \kappa_2^\theta < \frac{1}{\sigma_2^\theta} \sqrt{\frac{\ell_3^x}{4(1+\delta)^2 c_\gamma}}$$

$$\left[(1+\delta)^2 c_\gamma + c_\nu\right] \frac{(\sigma_2^\theta \kappa_2^\theta)^2}{6} < \frac{\ell_3^z}{6} \Rightarrow \kappa_2^\theta < \frac{1}{\sigma_2^\theta} \sqrt{\frac{\ell_3^z}{6 \left[(1+\delta)^2 c_\gamma + c_\nu\right]}}$$

Therefore, we need to satisfy both conditions (4.75) and (4.78) where $\kappa_3^x$ and $\kappa_3^z$ should be kept small until (4.75) and (4.78) are met after we select $\kappa_2^\theta$ using condition (4.79). Keep in mind that condition (4.79) and (4.80) are only due to the presence of the wing-bias control variable $b$. Accordingly, condition (4.75) makes sure that $\dot{\theta}_{cmd}$ should be able to satisfy its restriction, that is, $|\dot{\theta}_{cmd}| < \ell_1^\theta/2$ while conditions (4.78), (4.79) and (4.80) are to make sure that the asymptotic norms $\|d_x\|_a$ and $\|d_z\|_a$ are smaller than their restrictions $\|d_x\|_a < \ell_3^x/2$ and $\|d_z\|_a < \ell_3^z/2$, respectively.

It is crucial that all signals, eventually, have asymptotic norms smaller than their restrictions. To that end, select $\ell_3^x$ and $\ell_3^z$ to be fixed, for instance, $\ell_3^x := \overline{\nu}_x$ and $\ell_3^z := \overline{\nu}_z$. Moreover, using the definition in (4.73) and conditions (4.79) and (4.80), one obtains

$$\ell_2^\theta = \sigma_2^\theta \kappa_2^\theta < \sqrt{\frac{\overline{\nu}_x}{4(1+\delta)^2 c_\gamma}}$$

$$\ell_2^\theta = \sigma_2^\theta \kappa_2^\theta < \sqrt{\frac{\overline{\nu}_z}{6 \left[(1+\delta)^2 c_\gamma + c_\nu\right]}}$$
where both right-hand sides are fixed and independent on the gain parameters. Recall that $\ell^o_2$ is not fixed, but it is parameterized by $\sigma^o_2$ which is a fixed number. Moreover, we defined $\ell^o_2$ in (4.73) to avoid the dependence on both $\ell^o_2$ and $\kappa^o_2$ in (4.72) which is necessary in order to meet condition (4.71). By combining all conditions for $\ell^o_2$ that need to be met, we have the following

$$\ell^o_2 = \sigma^o_2 \kappa^o_2 < \min \left\{ \overline{b}, \sqrt{\frac{\nu_x}{4(1+\delta)^2c_\gamma}}, \sqrt{\frac{\nu_z}{6[(1+\delta)^2c_\gamma+c_\nu]}} \right\}$$

where $\overline{b}$ is due to the actuator constraint.

To recap, in order to enforce the restriction on $d_x$ and $d_z$ in finite time, the selection of the gains proceeds as follows:

- set $\ell^x_3 := \nu_x$ and $\ell^z_3 := \nu_z$
- fix $a^x_1, a^x_2, a^z_1, a^z_2 \in (0, 1)$ arbitrarily.
- set $\epsilon^x_1, \epsilon^x_2, \epsilon^z_1, \epsilon^z_2$ to satisfy the small-gain conditions as described in Equation (4.10) and Equation (4.38).
- Then, $\kappa^x_3$ and $\kappa^z_3$ can be selected freely where all the other parameters are functions of either $\kappa^x_3$ or $\kappa^z_3$, where

\[
\begin{align*}
\kappa^x_1 &= \epsilon^x_1 \kappa^x_2 \\
\ell^x_1 &= (1 - a^x_1) \frac{\ell^x_2}{\kappa^x_2} + a^x_1 \frac{\ell^x_2}{8\epsilon^x_1 \kappa^x_2} \\
\kappa^x_2 &= \epsilon^x_2 \kappa^x_3 \\
\ell^x_2 &= (1 - a^x_2) \frac{2\ell^x_3}{\kappa^x_3} + a^x_2 \frac{\mu_1 \ell^x_3}{8\epsilon^x_2 \kappa^x_3}
\end{align*}
\]

and

\[
\begin{align*}
\kappa^x_1 &= \epsilon^x_1 \kappa^x_2 \\
\ell^x_1 &= (1 - a^x_1) \frac{\ell^x_2}{\kappa^x_2} + a^x_1 \frac{\ell^x_2}{8\epsilon^x_1 \kappa^x_2} \\
\kappa^z_2 &= \epsilon^z_2 \kappa^z_3 \\
\ell^z_2 &= (1 - a^z_2) \frac{2\ell^z_3}{\kappa^z_3} + a^z_2 \frac{\mu_1 \ell^z_3}{12\epsilon^z_2 \kappa^z_3}
\end{align*}
\]
This implies that the gains $\kappa^x_3$ and $\kappa^z_3$ can be viewed as tuning knobs while all other parameters for the $x$-subsystem and the $z$-subsystem are set.

On the other hand, in order to enforce the restriction on $\dot{\theta}_{cmd}$ in finite time, the following procedure should be followed for the $\theta$-subsystem:

- Select $\kappa^\theta_2$ arbitrarily, then, select $\sigma^\theta_2$ such that the following condition is met.
  $$\sigma^\theta_2 < \frac{1}{\kappa^\theta_2} \min \left\{ \tilde{b}, \sqrt{\frac{v^x}{4(1 + \delta)^2c_c}}, \sqrt{\frac{v^z}{6[(1 + \delta)^2c_c + c_v]}} \right\}$$

- Select $\kappa^x_3$ and $\kappa^z_3$ to satisfy the following condition
  $$\max \{\kappa^x_3, \kappa^z_3\} < \min \left\{ \frac{\ell^\theta_1 \sigma^\theta_2}{2(M_x + M_z)}, M_2 \kappa^\theta_2, M_3 \kappa^\theta_2 \right\}$$

Following the above guidelines guarantee that all the restrictions are satisfied and saturations disappear eventually. We are looking now at the interconnection between $d_x, d_z$ and $\dot{\theta}_{cmd}$. Recall that $d_x$ and $d_z$ depend on $\theta_1$ and $b$. First, we need to find an appropriate bound on $\|\dot{\theta}_{cmd}\|_a$. In order to analyze the interconnection, we shall derive the relation between $\|\dot{\theta}_{cmd}\|_a$ and $\|d_x\|_a, \|d_z\|_a$ using (4.67).

$$\|\dot{\theta}_{cmd}\|_a \leq B_x \|\dot{v}_{x,cmd}\|_a + B_z \|\dot{v}_{z,cmd}\|_a$$

Using (4.8) and (4.11), one obtains

$$\|\dot{v}_{x,cmd}\|_a \leq \kappa^x_3 \left[ 3\bar{m}_1 + 12 \epsilon^x_2 + 32 \epsilon^x_1 \epsilon^x_2 \right] \|d_x\|_a$$

Similarly, using (4.36) and (4.39), one obtains $\|\dot{v}_{x,cmd}\|_a$ and $\|\dot{v}_{z,cmd}\|_a$ as follows

$$\|\dot{v}_{x,cmd}\|_a \leq \kappa^x_3 Q_x \|d_x\|_a$$

$$\|\dot{v}_{z,cmd}\|_a \leq \kappa^z_3 Q_z \|d_z\|_a$$

(4.81)
where \( Q_x := \kappa_x^3 [3\mu_1 + 12\epsilon_2^3 + 32\epsilon_1^3 \epsilon_2^3] \) and \( Q_z := \kappa_z^3 [3\mu_1 + 12\epsilon_2^3 + 32\epsilon_1^3 \epsilon_2^3] \). Using (4.76) and (4.77), one obtains the following asymptotic norms

\[
\|d_x\|_a \leq (1 + \bar{\delta})^2 \|\theta_1\|_a + (1 + \bar{\delta})^2 c\gamma \|b\|_a^2
\]

\[
\|d_z\|_a \leq (1 + \bar{\delta})^2 \|\theta_1\|_a + \left[ (1 + \bar{\delta})^2 c\gamma + c\nu \right] \|b\|_a^2
\]

Recall that to stabilize the closed-loop system, the design parameters are selected to obtain a small-gain interconnection between the upper and lower subsystems in Figure 4.2. From (4.67) and (4.81) it follows that

\[
\|\dot{\theta}_{cmd}\|_a \leq B_x \|\dot{v}_{x,cmd}\|_a + B_z \|\dot{v}_{z,cmd}\|_a
\]

\[
\leq \kappa_x^3 B_x Q_x \|d_x\|_a + \kappa_z^3 B_z Q_z \|d_z\|_a
\]

\[
\leq (\kappa_x^3 B_x Q_x + \kappa_z^3 B_z Q_z) (1 + \bar{\delta})^2 \|\theta_1\|_a + \cdots
\]

\[
+ (\kappa_x^3 B_x Q_x + \kappa_z^3 B_z Q_z) \left[ (1 + \bar{\delta})^2 c\gamma + \kappa_z^3 B_z Q_z c\nu \right] \|b\|_a^2
\]

The bound of \( \|\dot{\theta}_{cmd}\|_a \) in (4.82) can be given in a concise form as follows

\[
\|\dot{\theta}_{cmd}\|_a \leq \max \{\kappa_x^3, \kappa_z^3\} Q_\theta \|\theta_1\|_a + \max \{\kappa_x^3, \kappa_z^3\} Q_b \|b\|_a^2
\]

where

\[
Q_\theta := 2 (B_x Q_x + B_z Q_z) (1 + \bar{\delta})^2
\]

\[
Q_b := 2 \left[ B_x Q_x (1 + \bar{\delta})^2 c\gamma + B_z Q_z \left( (1 + \bar{\delta})^2 c\gamma + c\nu \right) \right]
\]

One should emphasize that to stabilize the closed-loop system, the design parameters are selected to obtain a small-gain interconnection between the upper and lower subsystems in Figure 4.2. The composition of the asymptotic gains forms a simple contraction if

\[
\max \{\kappa_x^3, \kappa_z^3\} Q_\theta \|\theta_1\|_a + \max \{\kappa_x^3, \kappa_z^3\} Q_b \|b\|_a^2 < 1
\]
Keeping in mind (4.83) and the following bound from Proposition 4.2.4

\[ \|\theta_1\|_a \leq \frac{2}{\kappa_1} \|\dot{\theta}_{cmd}\|_a \]

the small-gain condition due to the first part of (4.83) is

\[
\max \left\{ \kappa_3^x, \kappa_3^z \right\} Q_\theta \cdot \frac{2}{\kappa_1} < \frac{1}{2}
\]

where \( \kappa_1^\theta \) is fixed once \( \kappa_2^\theta \) has been selected.

The second part of (4.83) involves the asymptotic bound \( \|b\|_a^2 \). However, using (4.45), we obtain the following

\[
\|b\|_a \leq \kappa_2^\theta \|\theta_2\|_a \\
\leq \frac{8}{\mu_2} \|\dot{\theta}_{cmd}\|_a
\]

Thus, (4.84) can be re-written as follows

\[
\max \left\{ \kappa_3^x, \kappa_3^z \right\} Q_b \cdot \frac{8}{\mu_2} < \frac{1}{2}
\]

The small-gain condition of the overall system is

\[
\max \left\{ \kappa_3^x, \kappa_3^z \right\} < \frac{\kappa_1^\theta}{4Q_\theta} + \frac{\mu_2}{16Q_b\kappa_1^\theta}
\]

where the second term in (4.85) which contains \( Q_b \) is due to the dependence on the wing-bias control variable \( b \). Thus, the design parameters \( \kappa_3^x \) and \( \kappa_3^z \) need to satisfy the following final condition

\[
\max \left\{ \kappa_3^x, \kappa_3^z \right\} < \min \left\{ \frac{\bar{\kappa}_1}{2(M_x + M_z)}, \frac{\sigma_2^\theta}{M_2\kappa_2^\theta}, \frac{\sigma_2^\theta}{M_3\kappa_2^\theta}, \frac{\kappa_1^\theta}{4Q_\theta} + \frac{\mu_2}{16Q_b\kappa_1^\theta} \right\}
\]

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Table 4.1: Vehicle parameters

4.4 Simulations

In this section, simulation results concerning the performance and robustness of the closed-loop system are presented and discussed. To validate the performance of the controller derived in the previous section, simulations have been carried out on the higher fidelity model (3.60) which will be referred to as the “truth model”. It is important to note here that the averaged model (4.1) is solely used to develop the control design and for tuning of the control parameters. Thus, there exist inevitable modeling errors between the control design model and the truth model where the simulations have been performed. The values of the model parameters are given in Table 4.1. Note the significant mismatch between the value of the wing-beat frequency at hover, $\omega_*$, and the carrier frequency of the wing beat, $\omega_0$, which underestimates
the correct value by about 16%. For all other parameters, a 15% uncertainty about the nominal value is assumed. Various robustness tests of the proposed control design were performed in order to validate the assumptions adopted in modeling simplifications. For reasons of space limitation, only two representative case studies will be presented here

- Simulations on the nominal model.
- Simulations on the perturbed model.

The vehicle is commanded to perform a simultaneous step change in both the \( x \) and \( z \) coordinates from different initial conditions to the desired set point \( x_d = 10 \text{ [m]} \), \( z_d = -5 \text{ [m]} \), with \( \theta^* \). Since in [7] the reference frame is assumed positive downward, this maneuver corresponds to the vehicle raising its initial altitude while translating forward. The simulations were performed in the MATLAB Simulink environment.

**4.4.1 Simulations on the Nominal Model**

The controller is first tested on the nominal truth model. The vehicle is commanded to perform a simultaneous step change in both the \( x \) and \( z \) coordinates from an initial condition \((x_0, \dot{x}_0) = (0, 0), (z_0, \dot{z}_0) = (0, 0), (\theta_0, \dot{\theta}_0) = (0, 0)\) to the desired set point \( x_d = 10 \text{ [m]} \), \( z_d = -5 \text{ [m]} \). The simulation results for the longitudinal and vertical motion are shown in Fig. 4.5 and Fig. 4.6, respectively. The vehicle position converges asymptotically to the set point in about 30 \([s]\). Notice the limit attained by the vertical velocity within \( t \in [0.7, 12.5] \text{ [s]} \) due to the saturated control law. Figure 4.7 shows that during the maneuver the internal pitch dynamics remain well-behaved, and asymptotic convergence to the equilibrium value, \( \theta_* = 0 \), is observed.
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Table 4.2: Controller gains and bounds.

The behavior of the physical control inputs, $\delta(t)$ and $b(t)$, is shown in Figure 4.8 and Figure 4.9. From these plots it is observed that $\delta(t)$ fluctuates between 0.19 and 0.21, whereas $b(t)$ varies between $-0.012$ and $0.012$. The (unknown) constant steady-state correction required to maintain the vehicle in hover is seen in the long term behavior of $\delta(t)$ to compensate for $\omega_0 < \omega_*$. On the other hand, pitch stability is guaranteed by virtue of the wing-bias control variable. Figure 4.9 shows a highly dynamic initial behavior in order to track the new set-point, $x_d$ and then bounded oscillations to ensure tracking of the desired direction. The set of controller parameters used in the simulation is given in Table 4.2.

4.4.2 Simulations on the Perturbed Model

To further verify the robustness and performance of the proposed control design, the second case study considers the presence of modeling uncertainties, especially on the location of the wing root hinge point. Furthermore, a more stringent reference
trajectory is considered, which requires the vehicle to perform a figure-eight motion in the plane. The misalignment of the wing root hinge point results in displacement of the wing center of pressure location from its nominal position with respect to the vehicle center of mass. This misalignment produces an undesired offsets in the aerodynamic pitching moment. The vector \([p_x, p_y, p_z]\) defines the wing root hinge position with respect to the vehicle center of mass as illustrated in Figure 3.2. Only the \(p_x\) and \(p_z\) terms affect the pitching moment. A series of simulation tests were performed on the truth model to study the robustness of the proposed control design. In order to provide a qualitative assessment of the ability of the proposed scheme to deal with such misalignment, the value of \(p_x\) is increased as a percentage of the value of \(p_z\). The first simulation result was obtained by commanding the vehicle to

Figure 4.5: Simulation results: Longitudinal motion \((x(t), \dot{x}(t))\).
Figure 4.6: Simulation results: Vertical motion \((z(t), \dot{z}(t))\).

perform a simultaneous step change in both the \(x\) and \(z\) coordinates from an initial configuration \((x_0, \dot{x}_0) = (0, 0), (z_0, \dot{z}_0) = (0, 0), (\theta_0, \dot{\theta}_0) = (0, 0)\) to the desired set point \(x_d = 10 \text{ [m]}, z_d = -5 \text{ [m]}\). Figure 4.10 and Figure 4.11 show the simulation results for the longitudinal and vertical motion, respectively. The vehicle position converges asymptotically to the set point in about 31 [s]. It can be observed in Figure 4.12 that the pitch angle converges to a constant value in order to compensate the offset in the aerodynamic pitching moment due to displacement \(p_x = 0.15p_z\). The behavior of the physical control inputs, \(\delta(t)\) and \(b(t)\), is shown in Figure 4.13 and Figure 4.14. However, the oscillation in the wing-bias \(b\) is quite interesting, since it fluctuates between \(-0.687\) and \(-0.759\) in order to ensure stability of the pitching dynamics. The set of controller parameters used in the simulation is given in Table 4.2.
The second case study uses the same initial and final configurations of the previous perturbed simulations but has different initial condition \((\theta_0, \dot{\theta}_0) = (\pi/6, 0)\). Figure 4.15 and Figure 4.16 show the simulation results for the longitudinal and vertical motion, respectively. The vehicle position converges asymptotically to the set point in about 33 [s]. However, the initial error in \(\theta\) affected the transient response which have resulted in overshoot at the beginning of the longitudinal trajectory as seen in Figure 4.15. The behavior of the physical control inputs, \(\delta(t)\) and \(b(t)\), is shown in Figure 4.18 and Figure 4.19. The set of controller parameters used in the simulation is given in Table 4.2.

In order to emphasize on the effectiveness of the proposed controller on a complex trajectories, a simultaneous displacement along the longitudinal and vertical were
Figure 4.8: Simulation results: Physical control input $\delta(t)$

demonstrated. Figure 4.20 shows the simulation results of the vehicle’s performance when commanded to track a figure-eight trajectory. It is observed that the vehicle position successfully converges asymptotically to the desired trajectory even with the misalignment of $p_x = 0.15p_z$ exists. There is an initial offset in Figure 4.20 due to the initial vehicle position at $(x, z) = (0, 0)$. The behavior of the physical control inputs, $\delta(t)$ and $b(t)$, is shown in Figure 4.21 and Figure 4.22. The wing-bias $b$ varies between $-0.62$ and $-0.8$ in order to ensure stability of the pitching dynamics. In addition, Figure 4.23 illustrates the vehicle tracking a figure-eight trajectory with initial condition $(\theta_0, \dot{\theta}_0) = (\pi/8, 0)$ as well as displacement in $p_x = 0.15p_z$. It is observed that the vehicle position successfully converges asymptotically to the desired trajectory. There is a noticeable initial offset in Figure 4.23 due to the initial vehicle position was at $(x, z) = (0, 0)$ and initial error in $\theta$. The behavior of the physical
control inputs, $\delta(t)$ and $b(t)$, is shown in Figure 4.24 and Figure 4.25. The wing-bias $b$ varies between $-0.58$ and $-0.78$ in order to ensure stability of the pitching dynamics. The set of controller parameters used in the both simulations of figure-eight trajectory is given in Table 4.2.

In order to provide more insight into the wing root hinge design parameters, simulations were performed by increasing the value of $p_z$. Figure 4.26 shows the effect of the design parameter $p_z$ on the wing-bias control input. It can be observed that increasing the value of $p_z$ have a significant effect on the amount on the wing-bias to be applied on the vehicle. The control input $b$ observed to increase proportionally with the value of $p_z$. The maximum value of the design parameter $p_z$ that can be tolerated by the proposed controller is $p_z = 400 \times 10^{-3}$ [m], unless different wing dimensions were considered. Note that the wing-bias $b$ was saturated in Figure 4.26d.
Figure 4.10: Simulation results: Longitudinal motion \((x(t), \dot{x}(t))\) for \(p_x = 15\%\) of \(p_z\).

Figure 4.11: Simulation results: Vertical motion \((z(t), \dot{z}(t))\) for \(p_x = 15\%\) of \(p_z\).
Figure 4.12: Simulation results: Pitch dynamics $(\theta(t), \dot{\theta}(t))$ for $p_x = 15\%$ of $p_z$.

Figure 4.13: Simulation results: Physical control input $\delta(t)$ for $p_x = 15\%$ of $p_z$. 
Figure 4.14: Simulation results: Physical control input $b(t)$ for $p_x = 15\%$ of $p_z$.

Figure 4.15: Simulation results: Longitudinal motion $(x(t), \dot{x}(t))$ for $p_x = 15\%$ of $p_z$ and $\theta(0) = \pi/6$. 
Figure 4.16: Simulation results: Vertical motion \((z(t), \dot{z}(t))\) for \(p_x = 15\% \) of \(p_z\) and \(\theta(0) = \pi/6\).

Figure 4.17: Simulation results: Pitch dynamics \((\theta(t), \dot{\theta}(t))\) for \(p_x = 15\% \) of \(p_z\) and \(\theta(0) = \pi/6\).
Figure 4.18: Simulation results: Physical control input $\delta(t)$ for $p_x = 15\%$ of $p_z$ and $\theta(0) = \pi/6$

Figure 4.19: Simulation results: Physical control input $b(t)$ for $p_x = 15\%$ of $p_z$ and $\theta(0) = \pi/6$
Figure 4.20: Simulation results: Figure-8 Trajectory with $p_x = 15\%$ of $p_z$.

Figure 4.21: Simulation results: Physical control input $\delta(t)$ with $p_x = 15\%$ of $p_z$ for figure-8 Trajectory.
Figure 4.22: Simulation results: Physical control input $b(t)$ with $p_x = 15\%$ of $p_z$ for figure-8 Trajectory.

Figure 4.23: Simulation results: Figure-8 Trajectory with $p_x = 15\%$ of $p_z$ and $\theta(0) = \pi/8$. 

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Figure 4.24: Simulation results: Physical control input for $\delta(t)$ with $p_x = 15\%$ of $p_z$ and $\theta(0) = \pi/8$ for figure-8 Trajectory.

Figure 4.25: Simulation results: Physical control input $b(t)$ with $p_x = 15\%$ of $p_z$ and $\theta(0) = \pi/8$ for figure-8 Trajectory.
Figure 4.26: Simulation results: Wing-bias $b(t)$ for different values of the wing root hinge design parameter $p_z$
Chapter 5: Conclusions

In this dissertation, a theoretical framework towards the analysis and control design of flapping-wing micro air vehicles (FW-MAV) was presented. A key achievement of this work is the proposed novel wingbeat function which manifests the significant features of an effective wing actuation mechanism for dynamic stability in the FW-MAVs. The procedure begins with the development of a nonlinear dynamic model for a prototypical bio-inspired multi-body FW-MAV towards the derivation of a control-design model and a simulation model for control validation. Three main research objectives are established in this work: to develop a mathematically precise framework for evaluating various wing kinematics parameters, to develop a continuous wingbeat forcing function that is amenable to rigorous application of averaging techniques, and to employ the proposed novel wingbeat function in a nonlinear control design that is robust with respect to modeling uncertainties.

The primary motivation of the presented framework, in this dissertation, is to provide a formulation of a wingbeat forcing function that is mathematically precise. The majority of the proposed schemes in the literature do not satisfy the fundamental conditions for the applicability of averaging method. Compared to previous wingbeat functions [54, 89, 2, 8], the novelty in the approach suggested in this dissertation is that the proposed wingbeat is differentiable, does not require manipulations of the
waveform within each period, and is amenable to a rigorous application of averaging theory [63, 26]. The proposed novel wingbeat function produces aerodynamic forces and moments by employing phase-shift of flapping frequency and wing-bias, that only requires one physical actuator for each wing.

Part of this dissertation was conducted to analyze the most important features of various wing actuation mechanisms towards achieving highly maneuverable FW-MAVs. This analysis involved the evaluation of all possible actuation configurations resulting from all possible contributions of the available wing kinematics parameters used as control inputs. This is particularly important where the choice of an input from the set of wing kinematics must be carefully vetted against the constraints imposed by stringent size and weight constraints for such vehicles, alongside achieving robust stable maneuvers for FW-MAVs. The developed framework affords additional insight into the critical role of certain wing kinematics parameters.

A nonlinear controller was developed for the control of the longitudinal model of minimally-actuated FW-MAV model using the proposed wingbeat function. The flight controller composed of inner-outer loop architecture. The inner-loop is responsible for the stabilization of the vehicle attitude by controlling pitching moment and evaluates the appropriate wing-bias command in the wingbeat function. Moreover, inner loop is responsible for achieving the desired maneuver by tracking the reference commands generated by the outer-loop. The outer-loop determines the set-point for the regulation of translational motion. Accordingly, the commanded pitch angle will be a function of the desired translational dynamics. A paramount advantage of the developed control design over the traditional linear controllers is its capability in tolerating modeling uncertainties to a certain extent. The robustness of the
developed controller was evaluated through considering large initial conditions and modeling uncertainties of the vehicle design parameters. Simulations demonstrated that the proposed controller possesses a certain degree of robustness to parametric uncertainties and modeling errors. Trajectory tracking performance was successfully implemented in simulations for figure-eight trajectory.

Part of the simulation tests was performed to provide a qualitative assessment about the significance of the wing root hinge location relative to the vehicle center of gravity on the FW-MAV stability. To investigate the effect of such design parameter, the vertical location of the wing root hinge point, $p_z$, was assumed to be a free parameter. Simulation results have revealed that increasing the vertical distance between the wing root hinge point and the vehicle center of gravity have prompted an extra effort in the wing-bias control variable to compensate for the undesirable gravitational pitching moment. Although this analysis was applied to the longitudinal model of the prototypical FW-MAV, it can still serve as a good estimate of the control effort to be applied on the roll and yaw dynamics. By examining the instantaneous aerodynamic moments in the body frame, one can notice that the design parameter $p_z$ do not contribute in the yawing moment, while rolling moment can be controlled using the $p_y$ and $p_z$ parameters, keeping in mind the initial assumption that $p_x$ is set to be identically zero. Thus, the reduction of the $p_z$ design parameter do not affect the yawing moment, yet the rolling and yawing moments can be affected using the vehicle width parameter, $p_y$. This situation gives rise to the coupling that exists between the rolling and yawing dynamics which might requires that coordinated turn maneuvers to be employed similar to the remarkable ones observed in fruit flies [53] [52]. These results offer further insight into vehicle design in order to provide passive stability.
of the FW-MAVs. The developed theoretical framework in this work could serve as a guideline towards the goal of designing successful FW-MAV in performing various maneuvers analogous to their biological counterparts. In parallel, the developed model modularity can serve as a tool for investigating FW-MAV performance and achieve passive stability by examining the best location of the wing root hinge relative to vehicle center of gravity.

Important topics that are left for a future study are considering auxiliary control inputs in order to enhance maneuvering performance of such vehicles. One may look at augmenting the control suite with additional wing kinematics such as wing angle of attack and the stroke plane angle as well as the postural changes of abdomen movements. FW-MAV could be enhanced from being under-actuated system with the proposed wingbeat forcing function to partially actuated and fully actuated with the auxiliary control inputs. It would be of interest to incorporate an actuated appendage, abdomen, as an additional control input to enhance the maneuverability performance. Such control design that incorporates abdomen would be similar to that of acrobot and pendubot control designs found in the literature [73] [25]. On the other hand, it will also be of interest to assume a methodology that stabilizes lateral dynamics of the vehicle by including the wing angle of attack as an additional control variable. This introduces, however, redundancy in the input space when all control variables are employed. Analyzing the over-actuated FW-MAVs is also suggested for future research directions for allocating different control effectors [68] [31].
Appendix A: Proof of Proposition 4.2.3

The statement of Proposition 4.2.3 is repeated here for the sake of completeness.

Proposition 4.2.3: Let the gains and saturation levels $\kappa^z_i, \ell^z_i, i = 1, 2$, be parameterized as follows

$$
\kappa^z_1 = \epsilon^z_1 \kappa^z_2, \quad \ell^z_1 = (1 - a^z_1) \frac{\ell^z_3}{\kappa^z_2} + a^z_1 \frac{\ell^z_2}{8 \epsilon^z_1 \kappa^z_2} \\
\kappa^z_2 = \epsilon^z_2 \kappa^z_3, \quad \ell^z_2 = (1 - a^z_2) \frac{2 \ell^z_3}{\kappa^z_3} + a^z_2 \frac{\mu^z_1 \ell^z_3}{12 \epsilon^z_2 \kappa^z_3}
$$

where

$$
\epsilon^z_1 \in \left(0, \frac{1}{8}\right), \quad \epsilon^z_2 \in \left(0, \frac{\mu^z_1}{24}\right), \quad a^z_1 \in (0, 1), \quad a^z_2 \in (0, 1)
$$

Then, for any $\kappa^z_3 > 0$ and any $\ell^z_3 \in (0, \bar{v}_z]$ the state of system 4.36 satisfies an $a$-$L_\infty$ bound from the set $\mathcal{X}_2 = \mathbb{R}^3$ with restriction $\Delta_z = \ell^z_3/2, \Delta_\omega = \ell^z_3/2$ on the disturbance inputs $d_z$ and $r_\omega$, respectively. Furthermore, for any constant $r_\omega$ satisfying $|r_\omega| < \Delta_\omega$, system 4.36 has a globally asymptotically and locally exponentially stable equilibrium at

$$
z^*_1 = \frac{r_\omega}{\kappa^z_1 \kappa^z_2 \kappa^z_3}, \quad z^*_2 = \frac{r_\omega}{\kappa^z_2 \kappa^z_3}, \quad z^*_3 = \frac{r_\omega}{\kappa^z_3}
$$

whenever $d_z = 0$. For arbitrary local essentially bounded signals $d_z$ satisfying $\|d_z\|_a < \Delta_z$, the state of 4.36 in the error coordinates...
\[
\begin{align*}
\tilde{z}_1 &:= z_1 - z^*_1 \\
\tilde{z}_2 &:= z_2 - z^*_2 \\
\tilde{z}_3 &:= z_3 - z^*_3 
\end{align*}
\]
satisfies the asymptotic bounds

\[
\|\tilde{z}_1\|_a \leq \frac{4}{\kappa_1^2 \kappa_2^2 \kappa_3^3} \|d_z\|_a, \quad \|\tilde{z}_2\|_a \leq \frac{4}{\kappa_2^2 \kappa_3^3} \|d_z\|_a, \quad \|\tilde{z}_3\|_a \leq \frac{2}{\kappa_3^3} \|d_z\|_a \quad (A.1)
\]

**Proof** The vector field of system (4.36) is globally Lipschitz, where it is forward complete and does not exhibit a finite escape time. Following the proof of [50], we begin the proof by defining the following sets \(\Omega_1, \Omega_2\) and \(\Omega_3\),

\[
\begin{align*}
\Omega_1 &:= \{ z \in \mathbb{R}^3 : |z_1| \leq \frac{\ell_1^z}{\kappa_1^2}, \ |z_2| \leq \frac{\ell_2^z}{\kappa_2^3}, \ |z_3| \leq \frac{\ell_3^z}{\kappa_3^3} \} \quad (A.2) \\
\Omega_2 &:= \{ z \in \mathbb{R}^3 : |z_2| \leq \frac{\ell_2^z}{\kappa_2^3}, \ |z_3| \leq \frac{\ell_3^z}{\kappa_3^3} \} \quad (A.3) \\
\Omega_3 &:= \{ z \in \mathbb{R}^3 : |z_3| \leq \frac{\ell_3^z}{\kappa_3^3} \} \quad (A.4)
\end{align*}
\]

where \(\Omega_1 \subset \Omega_2 \subset \Omega_3\). First, we establish conditions for forward invariance of \(\Omega_1, \Omega_2\) and \(\Omega_3\). Assume that \(z \in \Omega_1\) and \(|z_1| = \ell_1^z / \kappa_1^2\) and define the following Lyapunov function candidate

\[
V_1(z_1) := \frac{1}{2} z_1^2
\]

\[
\dot{V}_1(z_1) = -\ell_1^z \operatorname{sat}\left(\frac{\kappa_1^2}{\ell_1^z} z_1\right) + z_2 z_1 \\
= -\ell_1^z |z_1| + z_2 z_1 \\
\leq -\ell_1^z |z_1| + |z_2||z_1| \\
\leq -\ell_1^z \frac{\ell_2^z}{\kappa_1^2} + \frac{\ell_2^z \ell_1^z}{\kappa_2^3 \kappa_1^4}
\]

(A.5)

In order to guarantee forward invariance of system (A.5), the following conditions must hold

\[
-\ell_1^z \frac{\ell_1^z}{\kappa_1^2} + \frac{\ell_2^z \ell_1^z}{\kappa_2^3 \kappa_1^4} < 0 \quad \Rightarrow \quad \ell_1^z > \frac{\ell_2^z}{\kappa_2^3} \quad (A.6)
\]

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The condition (A.6) means that if \( z(t_0) \in \Omega_1 \) and \( \ell_1 > \ell_2 / \kappa_2^z \), then \( z(t) \), for \( t \geq t_0 \),
can not leave \( \Omega_1 \) through \( |z_1| = \ell_1^z / \kappa_1^z \). Now, assume that \( z \in \Omega_2 \) and \( |z_2| = \ell_2^z / \kappa_2^z \).

Using the Lyapunov function candidate \( V_2(z_2) = \frac{1}{2} z_2^2 \), one obtains

\[
\dot{V}_2(z_2) = \left( -\ell_2^z \ \text{sat} \left( \frac{\kappa_2^z}{\ell_2^z} z_2 \right) + z_3 + \kappa_1^z \ \text{sat}' \left( \frac{\kappa_1^z}{\ell_1^z} z_1 \right) \right) z_2 \\
\leq -\ell_2^z |z_2| + |z_2| |z_3| + 4 \kappa_1^z \ell_1^z |z_2| \\
\leq \frac{\ell_2^z}{\kappa_2^z} \left( -\ell_2^z + \frac{\ell_2^z}{\kappa_3^z} + 4 \kappa_1^z \ell_1^z \right)
\]

(A.7)

In order to guarantee forward invariance of system (A.7), the following conditions must hold

\[
\frac{\ell_2^z}{\kappa_2^z} \left( -\ell_2^z + \frac{\ell_2^z}{\kappa_3^z} + 4 \kappa_1^z \ell_1^z \right) < 0 \quad \Rightarrow \quad \ell_1^z < \frac{\ell_2^z}{8 \kappa_1^z} \\
\ell_2^z > \frac{2 \ell_3^z}{\kappa_3^z}
\]

(A.8)

Now, assume that \( z \in \Omega_3 \) and \( |z_3| = \ell_3^z / \kappa_3^z \), and consider the following Lyapunov function candidate for \( z_3 \) and its time derivative as follows

\[
\dot{V}_3(z_3) = \frac{1}{3} z_3^3 \\
\dot{V}_3(z_3) = \mu_1(b) \left[ -\ell_3^z \ \text{sat} \left( \frac{\kappa_3^z}{\ell_3^z} z_3 \right) + d_z + r_\omega \right] z_3 + \kappa_2^z \ \text{sat}' \left( \frac{\kappa_2^z}{\ell_2^z} z_2 \right) \dot{z}_2 z_3 \\
\leq -\mu_1(b) \ell_3^z |z_3| + \mu_1(b) |z_3| |d_z| + \mu_1(b) |z_3| |r_\omega| + 4 \kappa_2^z \ell_2^z |z_3| \\
\leq -\mu_1(b) \ell_3^z |z_3| + \mu_1(b) |z_3| \Delta_z + \mu_1(b) |z_3| \Delta_\omega + 4 \kappa_2^z \ell_2^z |z_3|
\]

(A.9)

where we assume a restriction for \( d_z, r_\omega \) as \( |d_z| < \Delta_z, \ |r_\omega| < \Delta_\omega \).

In order to guarantee forward invariance of system (A.9), the following conditions must hold

\[
\mu_1(b) \ell_3^z > \mu_1(b) \Delta_z + \mu_1(b) \Delta_\omega + 4 \kappa_2^z \ell_2^z \quad \Rightarrow \quad \Delta_z < \frac{\ell_3^z}{3 \mu_1(b)} \\
\Delta_\omega < \frac{\ell_3^z}{3} \\
\ell_2^z < \frac{\mu_1(b)}{12 \kappa_2^z}
\]

(A.10)
Combining the bounds of conditions (A.6), (A.8) and (A.10), we obtain the following

\[
\frac{\ell_z}{\kappa_z} < \frac{\ell_z}{8 \kappa_z} \quad \text{(A.11)}
\]

One may reparameterize \( \kappa_z \) and \( \kappa_z \) as \( \kappa_z = \epsilon_1 \kappa_z \) and \( \kappa_z = \epsilon_2 \kappa_z \), which introduce the following bound

\[
\frac{\ell_z}{\kappa_z} < \frac{\ell_z}{8 \epsilon_1 \kappa_z} \quad \Rightarrow \quad 0 < \epsilon_1 < \frac{1}{8}
\]

\[
\frac{2\ell_z}{\kappa_z} < \frac{\mu_1 \ell_z}{12 \epsilon_1 \kappa_z} \quad \Rightarrow \quad 0 < \epsilon_2 < \frac{\mu_1}{24}
\]

(A.12)

The bounds of the system (A.11), (A.12) make sure that the trajectories of the system converge to a neighborhood where the saturation is not active. Now, one needs to show that for any constant \( r_\omega \) satisfying \( |r_\omega| < \Delta_\omega \), system (4.36) has a globally asymptotically and locally exponentially stable equilibrium. Due to the existence of the uncertainty \( r_\omega \), system (4.36) doesn’t have equilibrium at zero. By defining the tracking error \( e_z := z - z_d \), we can find an equilibrium \( z_i^\ast \) in terms of uncertainty \( r_\omega \).

One may find \( e_z \) from (4.34) as follows

\[
e_z = z_2 - \ell_z \text{ sat} \left( \frac{\kappa_z}{\ell_1} z_1 \right)
\]

where at equilibrium, we need \( e_z^\ast = 0 \). When the saturation is inactive, the equilibrium of (4.34) can be found as

\[
r_\omega = \kappa_z z_3^\ast \quad \Rightarrow \quad z_3^\ast = \frac{r_\omega}{\kappa_z}
\]

\[
z_3^\ast = \kappa_z z_2^\ast \quad \Rightarrow \quad z_2^\ast = \frac{r_\omega}{\kappa_z}
\]

\[
z_2^\ast = \kappa_z z_1^\ast \quad \Rightarrow \quad z_1^\ast = \frac{r_\omega}{\kappa_z}
\]

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whenever \( d_z = 0 \). For arbitrary local essentially bounded signals \( d_z \) satisfying \( \|d_z\|_a < \Delta_z \), the state of (4.34) in the error coordinates

\[
\tilde{z}_1 := z_1 - z_1^* \\
\tilde{z}_2 := z_2 - z_2^* \\
\tilde{z}_3 := z_3 - z_3^*
\]

System (4.36) can be expressed in the error coordinates as follows

\[
\dot{\tilde{z}}_1 = -\kappa_1 \tilde{z}_1 + \tilde{z}_2 \\
\dot{\tilde{z}}_2 = -\kappa_2 \tilde{z}_2 + \tilde{z}_3 + \kappa_1 \text{ sat}' \left( \frac{\kappa_1}{\ell_1} \tilde{z}_1 \right) \tilde{z}_1 \\
\dot{\tilde{z}}_3 = \mu_1(b) [d_z - \kappa_3 \tilde{z}_3] + \kappa_2 \text{ sat}' \left( \frac{\kappa_2}{\ell_2} \tilde{z}_2 \right) \tilde{z}_2 
\]

(A.13)

Following Teel [84], the asymptotic bound of \( \tilde{z}_1 \) is found as

\[
\|\tilde{z}_1\|_a \leq \frac{4}{\kappa_1^z} \|\tilde{z}_2\|_a
\]

and

\[
\|\dot{\tilde{z}}_1\|_a \leq \kappa_1^z \|\tilde{z}_1\|_a + \|\tilde{z}_2\|_a \leq 2\|\tilde{z}_2\|_a
\]

The asymptotic bound of \( \tilde{z}_2 \) is

\[
\|\tilde{z}_2\|_a \leq \frac{2}{\kappa_2^z} \max \{ \|\tilde{z}_3\|_a, 4 \kappa_1^z \|\tilde{z}_2\|_a \}
\]

The composition of the channel-two gains form a simple contraction if \( \epsilon_1^z \) is chosen such that \( \epsilon_1^z < 1/8 \) which implies that

\[
\frac{8\kappa_1^z}{\kappa_2^z} < 1
\]

Thus,

\[
\|\tilde{z}_2\|_a \leq \frac{2}{\kappa_2^z} \|\tilde{z}_3\|_a
\]
As a result,

\[
\|\dot{\hat{z}}_2\|_a \leq \kappa_2^z \|\hat{z}_2\|_a + \|\hat{z}_3\|_a + 2\kappa_1^\varepsilon \|\dot{\hat{z}}_1\|_a \\
\leq 2\kappa_2^z \|\hat{z}_3\|_a + \|\hat{z}_3\|_a + 4\kappa_1^\varepsilon \|\dot{\hat{z}}_2\|_a \\
\leq 3\|\hat{z}_3\|_a + \frac{8\kappa_1^\varepsilon}{\kappa_2^z} \|\dot{\hat{z}}_3\|_a \\
\leq 4\|\hat{z}_3\|_a
\]

The asymptotic bound of \(\hat{z}_3\) is computed as follows

\[
\|\hat{z}_3\|_a \leq \frac{2}{\kappa_3^z} \max \{\|d_z\|_a, \frac{2}{\mu_1} \kappa_2^z \|\dot{\hat{z}}_2\|_a\} \\
\leq \frac{2}{\kappa_3^z} \max \{\|d_z\|_a, \frac{8}{\mu_1} \kappa_2^z \|\dot{\hat{z}}_3\|_a\}
\]

Applying the small-gain condition, one obtains

\[
\frac{16 \kappa_2^z}{\mu_1 \kappa_3^z} = \frac{16 \varepsilon_2^z \kappa_3^z}{\mu_1 \kappa_3^z} < 1 \iff \varepsilon_2^z < \frac{\mu_1}{16}
\]

This yields the following asymptotic bounds

\[
\|\hat{z}_1\|_a \leq \frac{4}{\kappa_1^\varepsilon \kappa_2^z \kappa_3^z} \|d_z\|_a, \quad \|\hat{z}_2\|_a \leq \frac{4}{\kappa_2^z \kappa_3^z} \|d_z\|_a, \quad \|\hat{z}_3\|_a \leq \frac{2}{\kappa_3^z} \|d_z\|_a
\]

\[\square\]
Bibliography


