Illuminated Scores and the Architectural Design of Musical Form

D.M.A Document

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Abstract

The project *Illuminated Scores and the Architectural Design of Musical Form* is designed as an approach to making scholarly editions for performing artists and educators that will portray the form and structure of a musical composition. It incorporates semiotic tools into the study of musical language intended to make the architectural design visible in a coherent format. This study develops a method of representing music graphically that differs from the established score layout, reorganizing content in a manner that allows one to overcome the constraints within which musical skills are developed under the current model of publishing music.

The motivation for such a study grew out of the need to enable students to experience an immediate representation of the overall formal design of a musical composition. Initially thought of as overlaying a Schenkerian analytical sketch over the published work horizontally aligned, the project has developed into a new publishing format containing the researcher’s analysis of works by Bach, Mozart and Schoenberg. It is the author’s hypothesis that this new perspective will impact the ways we learn and teach musical form and structure. The goal is to present music notation in a format similar to that of poetic verse rather than that of continuous prose, where one measure
follows another in no relevant way. Phrase lengths, melodic relationships, harmonic structure, and the number of measures in a system play a significant role in the visual layout of the work. In addition, diagrams and color-coding illustrate how mathematics and music are combined to explain the concept of balance in musical form, thereby revealing the inherent beauty of a composer’s cohesive thought process. The author points out mathematical concepts of numerical sequences, like Mersenne Primes and other patterns in Bach, and the Fibonacci sequence and golden proportion in sonata form movements of Mozart.

The current stage of this research is that of a scholarly work aiming to develop the representational method, using new engraving software programs capable of digitizing scores in a new format. Once this has been done we can implement the appropriate curriculum materials on a larger scale and measure their effect on students’ achievement through individual lessons and piano labs. Through this project, the author hopes to show the intrinsic beauty of form with the objective of impacting performance and music pedagogy. The goal of the Illuminated Scores is to present a practical, holistic method that leaves students excited about the possibility of exploring the fascinating tools employed by the most brilliant minds in the world of music.
This document is dedicated to my family, friends and all those who lent an ear in the countless hours of reflection on music and its meaning.

My Parents, Orlando B. Alonso and Maria Isabel Alonso

My Grandmother, Amelia L. Alonso Tarancón

My Brother, Orlando Alonso

My Wife, Emily B. Alonso-Taub

My Children, Sofia Lilian Alonso-Taub and Oliver Peter Alonso-Taub

My Friends, Evan Drummond, Evan Wels and Andrea Lam
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Chapter 1: Introduction

During a lesson with pianist and Professor Claude Frank, as I was working on Beethoven Piano Sonata Op. 110, he turned and asked “can you get through a piece of music in your mind without playing it?” After considering the implication of this question, my response was a resounding “no.” In that moment, I was startled to realize that after many months of vigorous practicing to commit every note to memory, I certainly could not run through the piece in my mind without touching the instrument.

It was this realization that led me to question how one learns a piece of music, and in turn how the learning process might impact the level of performance. This revelation expanded from answering a single question “can you mentally get through any given piece of music?” to addressing a series of questions: “Do you know how the work was conceived? Are you aware of the compositional methods used? Do you understand the overall structure? Are you able to identify every single theme, key area, harmonic progression, rhythmic pattern, and marking on the score?”

Following Claude Frank’s belief that every performer should learn to compose in order to truly understand and convey the meaning of music, I set out to determine if there could be a way to approach the study of a musical composition that would spark the
student’s awareness of the intricate compositional techniques employed by the great composers. Could a different approach to the study of a musical composition that provided additional relevant musical information on the score, assist and enhance the learning process as well as challenge traditional teaching techniques?

I urge the learner to consider the hidden clues embedded in the score, unnoticeable to the untrained eye, that might have been forgotten with the passage of time or are just the patrimony of the few scholars and performers who have either had the privilege of discovering them or have been trained in the lineage of the great musicians and/or pedagogues.

Upon further inquiry, I discovered Howard Gardner’s\textsuperscript{1} theory of intelligences, in which he divides learning styles into eight categories, the first 5 of which are critical in understanding the way we learn music:

1. Verbal–Linguistic (Word Smart)
2. Logical–Mathematical (Logic Smart)
3. Visual–Spatial (Picture Smart)
4. Bodily–Kinesthetic (Body Smart)
5. Auditory–Musical (Music Smart)

While contemplating the pedagogical application of Howard Gardner’s theory in relation to the study of music, I ascertained that different individuals could have a

preferred learning style, sometimes without knowing, that would depend on favorable environmental learning conditions. In general, however, an individual’s learning style directly depends on that person’s training and education. In my own case, I realized that I had not consciously considered the Logical–Mathematical style as a means of understanding music as mathematical expression, and began to think about how to enhance one’s capacity for learning music in this style.

Concurrently, as I contemplated how to make sense of a musical work as a coherent whole, it became apparent that awareness of musical form is relevant to understanding the entire composition. Musical form and structure exist precisely under Gardner’s Logical-Mathematical category. By not purposely perceiving the true form of a work, I was missing an essential ingredient that could potentially raise my level of performance to that of a true artist.

From an abstract philosophical perspective, a musical composition is typically analyzed through its form and its content. In his *Critique of Judgment,* Immanuel Kant states that “it is music's form which raises it to the level of art: while music involves a ‘beautiful play of sensations’3, that is, sense impressions, it is not the fluctuations of mood bound to this play, but ‘the form of the arrangement of these sensations’4,

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3 Ibid., p. 188
4 Ibid., p. 194
dependent on the ‘proportion of the different degrees of tension’\textsuperscript{5} which is given by the relative speeds of vibration and accordingly it is ‘mathematical form to which alone belongs the delight’.”\textsuperscript{6}

The study of form entails the analysis of “musical structure, that is, the way the elements of a musical composition are put together.”\textsuperscript{7} It also involves pattern recognition (repetition or variation), perception of phrase structure (complete musical thoughts), and a detailed look at the organization of melodic, harmonic and rhythmic content. In considering methods for studying and analyzing musical form, and taking into consideration Gardner’s learning styles, I asked myself the following questions:

- Could a method be created that would immediately reveal musical form?
- Could a system be created whereby a physical “blueprint” for the music could be constructed to assist the performer, without introducing new abstract concepts or requiring a profound knowledge of musical analysis?
- Could a new visual representation of the score facilitate the visualization and organization of the content of a score, using traditional notation?
- Would such a method enable deeper understanding of the score and thereby enhance memorization and interpretation?
- Would answers to these questions enhance one’s capacity for getting through a score in one’s mind without actually playing it?

To address these questions, I embarked on developing a new method to visually depict the architectural design of a composition in order to reveal its musical form, one

\textsuperscript{5} Ibid., p. 188-9
that would lead to better identification of themes and interpretation of meaning. All of these would lend to greater comprehension of the “road map” the composer has created for the performer.

Upon analyzing the architectural design of a piece, I was not only able to retain larger amounts of information (musical content) but I also discovered hidden mathematical proportions that gave not only meaning, but a sense of awe and magic, to the findings. It was through the analysis of the elements of form that mathematical structure took shape. By applying color-coding to every single pattern discovered through an in-depth analysis of the score, I realized that a large number of abstract concepts could be effortlessly conveyed through visual representation. Such realizations generated the need to develop a new method for reconstructing the score, undefined by traditional printing parameters and page measurements. This is how the Illuminated Scores method was born.

The development of the Illuminated Scores provides a practical and holistic method for analyzing a score, including the identification of architectural design, musical themes, mathematical concepts and proportions, phrase structure, spatial relations and temporal proportions. Such concepts have proven invaluable in yielding a deeper understanding of the process of composition.
Chapter 2: The Development of the Illuminated Scores Method

The Illuminated Scores satisfies the need to provide an immediate representation of the overall formal design of a musical composition from the first moment one lays eyes on the score. The purpose of this new method is to attain a deeper understanding of the meaning of music and its form.

Drawing from Howard Gardner's theories I considered five of the eight learning styles to apply to understanding musical perspective:

1. Verbal–Linguistic (Word Smart): “People who possess this learning style learn best through reading, writing, listening, and speaking. Verbal students absorb information by engaging with reading materials and by discussing and debating ideas.”8 This category is represented by my poetic analysis, which treats instrumental music as a song without words.

2. Logical–Mathematical (Logic Smart): Those “who exhibit this type of intelligence learn by classifying, categorizing, and thinking abstractly about patterns, relationships, and numbers.”9 This category is represented in the architectural design of formal structure and mathematical proportions.

3. Visual–Spatial (Picture Smart): Those who “learn best by drawing or visualizing things using the mind’s eye. Visual people learn the most from pictures, diagrams, and

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9 loc. cit.
other visual aids.”¹⁰ This is incorporated in the visual representation of the Illuminated Scores method.

4. Bodily–Kinesthetic (Body Smart): Those who “learn best through touch and movement. These people are best at processing information through the body.”¹¹ Many performers heavily rely on this learning skill, which is represented in music as repetition and finger memory, which I would recommend as a tool for technically challenging, awkward or tricky spots, not as the main tool for performance and memorization.

5. Auditory–Musical (Music Smart): Those “who are music smart learn using rhythm or melody, especially by singing or listening to music.”¹² This obviously applies to all things herein. Nevertheless, the method provides a tool for critically thinking about the aesthetic and technical issues of a musical interpretation.

The Illuminated Scores method provides visual and analytical solutions that appeal to all of these learning styles by exercising each learning skill individually and in combination, independently of the performer’s preferred learning style. The method emphasizes the representation of abstract concepts in a comprehensible way that intentionally utilizes a more universal language, a visual language. In this sense, the first three of these categories became the focus of development of my new method.

¹⁰ loc. cit.
¹¹ loc. cit.
¹² loc. cit.
The Illuminated Scores method portrays the form and structure of a given musical composition in a format similar to that of poetic verse. For example, below is a transcription of a poem by Emily Dickinson in the layout of prose rather than verse.

“I HAVE no life but this, to lead it here; nor any death, but lest dispelled from there; nor tie to earths to come, nor action new, except through this extent, the realm of you.”¹³

The poetic form is not immediately recognizable in this format. By comparison, below (Fig. 1) is the same poem in its poetic layout, as originally penned by the author:

Figure 1: “I have no life but this” Poem by Emily Dickinson

Looking at the poem in its rightful shape, or originally intended design, one can immediately deduce that this work is composed of one poem made up of two stanzas with four verses per stanza, where green represents rhyme as in a Ballad, alternating six and four syllables per verse.

The application of the Illuminated Scores to a musical composition reveals the same structural details as in the poetic form, which immediately changes the way one identifies the form of the composition. In Chapter 3, page 21, I give a detailed example of this application.

This poetic analysis appeals to the Verbal-Linguistic (Gardner) style, identifying the poetic form of a musical composition in order to show a parallel comparison of what the author hopes to accomplish musically.

After analyzing the poem in both prose and poetic formats, it becomes clear that printed music has been published in “prose” for over five hundred years, written with the economy of space in mind and from a printing perspective. By musical prose I mean that the traditional organization of a score is one long, continuous succession of bar lines that has been laid out to best fit the page, without regard to the musical content, where one measure follows another in no relevant way.

In other words, the traditional method of publishing does not allow for an intuitive musical perception of poetic verse. If we consider, as did Johann Mattheson, that “an instrumentalist…must observe that which is required for a good melody and harmony even more assiduously than a singer…, since one has the aid of the clearest words in singing, while instruments are always lacking these,”\textsuperscript{15} then we begin to realize how important it is to account for the relationship between verbal-linguistic and musical concepts.

If we think of instrumental music as a song without words, a poetic analysis can be accomplished through the structural mathematical concepts that rule poetic form. I give detailed examples of musical representations of poetic verse in figures 9, 25 and 56.

When reading a score for the first time, one generally comes into contact with musical content (motives and phrases), which can typically be easily discerned. I felt the need to be able to visually organize such content instantly in a poetic format in order to be able to see what its formal structure would look like. Some performers can do such analysis in their minds; since I could not, however, I began to literally cut up the score, using a pair of scissors, into phrases and organizing them in order to solve the puzzle of the architectural design. This led me to the refinement of the visual-spatial parameters.

Mathematical Proportions: Logical-Mathematical

“Music is the arithmetic of the soul, which counts without being aware of it.”

– Leibnitz

After following the procedures of the visual-spatial analysis and the poetic approach, one discovers that mathematical proportions reveal themselves. Once one can see the intended structure of a piece, one realizes that the key structural features that represent the architectural design of a given work existed in the mathematical realm first, and manifested themselves through the poetic and spatial representation.

Music and mathematics have enjoyed a very long history together. They have been related since the time of Pythagoras, who proposed a sublime model of perfection and unity in mathematics, music, and cosmology through a universal music of the spheres.

With the institution of the Quadrivium\(^{16}\) as a learning curriculum, the marriage of music and mathematics was forged. The Quadrivium consisted of “the four classical liberal arts of Number, Geometry, Music & Cosmology”\(^{17}\) that together with the Trivium\(^{15}\) constituted the basis for classical education up to the Renaissance.

\(^{16}\) The term Quadrivium is believed to have been coined by Boethius, see Férdia J. Stone-Davis, 2011. Musical beauty: negotiating the boundary between subject and object. Eugene, Or: Cascade Books. p. 3

The history of the relationship between music and mathematics, boasts such renowned scholars as: Plato, Euclid, Nicomachus of Gerasa, Ptolemy, Theon of Smyrna, Boethius, Vincenzo Galilei, Mersenne, Sauveur, Leibnitz, Euler, Hugo Riemann, Gauss, Schoenberg, Hofstdter, Rothstein, Lerdahl, Jackendoff, and Tanay, among countless others.

The theoretical concepts of music discussed in this document are a useful tool not only for the performer but also the listener. When we unify music and mathematics for the purpose of organizing content (dealing with the abstract) we realize “that the formation of a piece of music is an ordering process in which specified musical elements are selected and arranged from an infinite variety of possibilities, i.e., from chaos.”¹⁸

(Hiller-Isaacson)

The history of music and mathematics traces the evolution of tuning, instrument building, form, structure, music notation, theories of harmony, rhythm, melody, analysis, and the definition of concepts such as consonances and dissonances; as a representation of the level of knowledge at a given historical point in time.

If we consider the creative process in music and mathematics, music could be considered an expression of the mathematical discoveries of its time. In general I have

always assumed that a mathematical discovery leads to a new philosophical idea (or
discourse), that in turn manifests itself through literature, art and the performing arts (in
this case music). The deep connection between music and mathematics lies in the study
of patterns. The mathematical proportions that appear through the analysis of form can
corroborate the architectural design of the composition.

As I ponder the words of Möbius: “Musical mathematicians are frequent ... but
there are wholly unmusical mathematicians and many more musicians without any
mathematical capability,”¹⁹ I am further motivated to develop the Illuminated Scores
method in order to bridge this gap.

“A mathematician, like a painter or a poet, is a maker of patterns.” – G.H. Hardy

**Analysis of Architectural Design: Visual-Spatial Style**

In further analyzing a score to identify its structure, and keeping in mind the
possibility of poetic verse, I established a set of “rules” for the Illuminated Scores
method of analysis. The list of rules below is in ascending order, from the very small (measure
number) to the very large (overall form). As illustrated in figures 7 and 8, procedures of
the method include:

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Mathematical Association of America. p. 42.
• All notes should be notated with an even beat spacing (must correspond with the time it takes to play)
• All measures are equally divided through space (keeping in mind that music is an architectural structure that takes place through time and is audible only through time)
• All measures are displayed with equal length from bar line to bar line in the same time signature (notice that the lines are perfectly aligned and equidistant in figure 8)
• All bar lines should create a straight line down the page as to create a perfect uniform visual experience
• The number of measures in a system is determined by the length of the phrase
• Measure numbers are displayed on every measure and the last measure in the system is highlighted with color in order to see mathematical proportions
• Phrase lengths\(^{20}\) are notated as they are, regardless of where they fall within the bar lines, and should be written as though the bar lines do not exist
• Phrases are self-contained (left justified)
• Phrases follow the rules of harmonic and formal structure intended by the composer (sentence structure, fugal, binary, ternary, poetic, etc.)
• The entire composition is color-coded at the melodic, harmonic, rhythmic, formal and fragment level
• Color-coding highlights the desired analytical aspect of the work
• Colors connect similar or equal melodic, harmonic or rhythmic material
• All concepts of form and structure are labeled on the score, i.e. exposition, development, subject, counter-subject, row, set, retrograde, etc.

These rules exist as a visual representation in space of what we normally associate with temporal proportions. Since “music is based on temporal succession and requires

\(^{20}\) It should be noted that musicians may disagree about phrase lengths.
alertness of memory,”21 (Stravinsky) as well as “instantaneous perception”22 (Hiller) in order to comprehend musical structure, I wanted to aid the reader (the one looking at the score) in realizing the implications of temporal proportion in a two-dimensional plane, i.e. the score.

Color-coding plays a significant role in the visual-spatial recognition of patterns. It allows for an immediate correlation of content into groups or families of groups. By graphically displaying an analysis of the score, it allows the reader to realize (on his/her own terms) how important the role of formal structure is.

In addition, diagrams and color-coding illustrate how mathematics and music are combined to explain the concept of balance in musical form, thereby revealing the inherent beauty of a composer’s cohesive thought process. Through the Illuminated Scores, color-coding has proven to be very helpful in aiding the reader to arrive at an understanding of form and structure, without requiring previous knowledge of the inner-workings of “musical form;” as well as in the demonstration of how compositional space is proportionally divided and utilized by the composer.

By visually portraying the score while keeping a strict measurement of the exact length of phrases, one can clearly see exactly how equal (or different) each phrase is. This will hopefully motivate the student’s need to question why the phrases are different in length and what it means, which would inevitably lead them to the logical-mathematical aspects of form.

Influential Works - Background

The works of Marius Van Crevel, Arthur Schnabel, Carl Schachter and Edwin F. Kalmus proved instrumental in the development of the Illuminated Scores.

Van Crevel, the Dutch musicologist who “in the controversial prefaces to his two Obrecht mass volumes… advanced a method of transcription, based on a new theory of tactus”…“Van Crevel adopts a type of scoring (NTM) in which the value of the tactus (semibreve) is given a fixed amount of space on the staff,”23 in addition he does not use bar lines. His equal spacing in notation stressed the relevance of keeping “the intervals of musical time, i.e. duration”24 proportional in order to more accurately execute complex rhythms and meter, which in his case “led to the discovery of secret mathematical structures.”25

23 Jacob Obrecht and M. van Crevel. 1964, Opera omnia Vol. I, VII. Opera Omnia. Amsterdam: [s.n.]. p. XII ff
24 Ibid., p. LV
25 loc. cit.
Schnabel, in his edition of Beethoven’s 32 Sonatas for the Pianoforte, addresses the issue of regular and irregular phrases. He introduces the use of Roman numerals to “point out those periods of bars which do not correspond to the traditional symmetrical form of 8 bars or twice 4 bars.”26 His point of view on phrase groupings is always worthy of detailed examination.

The analytical work in the Kalmus Open Score edition of The Well-Tempered Clavier Book I, is a monumental work of art. As described by the publisher: “by separating each line of the piano edition of the Fugues, and printing them in open score, the study of the Fugues is greatly facilitated. An extremely important contribution to the study of the fugue in general is thereby offered.”27 While the edition is entirely in German, the content is clear and explicit in purely musical terms. The polyphonic analysis of every single entrance of every single voice, subject, counter-subject and episode is clearly marked on the score.

The Illuminated Scores method draws from the rich tradition of critical and performance editions while introducing a new format that organizes musical content on the page in complete thoughts or phrases. The page layout works as a visual representation of the parameters prescribed by the method. Once music is organized in a

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coherent phrase structure, the form reveals itself. I believe that this new perspective will impact the ways we learn and teach musical form and its structure.

Through the Illuminated Scores, I hope to provide performers and educators with a new set of tools to raise the baseline of the learning process in music. Many performers typically learn a score by diving immediately into deciphering the notes on the score and memorizing the piece, doing little or no research. In turn, after years of practicing, performing and further analyzing the piece, more aspects of the work reveal themselves. My intent is to facilitate and expedite this process, by including more visual and analytical information in the score itself. Such Semiotic tools that are descriptive from a visual perspective will constitute the symbolic representation needed to aid those who do not have the means, ability, resources or time to research a musical work to a high standard.

The visual layout of a score plays a significant role in a person’s assimilation of the musical content, thus the proposed scores could potentially aid in visual memory, pattern recognition and performance in general. From a pedagogical perspective it could provide educators with a new set of tools to impart knowledge on crucial aspects of music.

The Illuminated Scores method reveals a format similar to poetic verse with the objective of aiding the performer in shaping musical phrases more intelligently. In the
words of Arnold Schoenberg, “the most important changes are necessary for the
distribution of the phrases of which a segment is composed. Over-accentuation of strong
beats shows poor musicianship, but to bring out the ‘center of gravity’ of a phrase is
indispensable to an intelligent and intelligible presentation of its contents. This may or
may not involve emotional expression, though it is done mainly by dynamic changes:
crescendi, decrescendi, sforzati, etc.”\(^{28}\) The idea of intelligent phrasing requires taking
into account the different factors that affect a musical phrase, such as melodic contour,
harmonic hierarchy, dissonance treatment, points of rest and tension for a complete
phrase, among other key factors. If we rely on melodic contour only, our overall phrasing
will be incomplete a large percentage of the time.

It appears that most students overlook the harmonic hierarchy of chords and the
relationship between the tonic and dominant polarity in tonal music. Knowledge of the
function of each chord is a crucial step in how we shape each phrase, i.e. the tonic I chord
(home key) is rest and has the least amount of tension while the dominant V has the most
tension. This seems completely rudimentary and simple, yet it continues to go unnoticed
with regularity. It can take hours of work to decipher a score harmonically. While
performers can analyze a piece while they prepare, they often avoid the painstaking task
of fully decoding any given composition in advance. The Illuminated Scores method
represents an attempt to enable the performer to gain greater understanding of the score
and its meaning by unveiling an organized representation of patterns that exist in music,

rather than a long continuous line that has been laid out to best fit the page from a printing point of view.

Refinement in musical playing is often associated with how well a performer can shape phrases, and change color and character accordingly, “without ever losing balance” or “violating good taste.”29 These concepts cannot manifest themselves in concert if the performer is not aware of their relative placement in the score. The new layout restructures the material into blocks of phrases regardless of length of measures, and intends to provide a practical organizational system for learning and approaching the score. In addition, it uses color-coding extensively to illuminate patterns and make them easily recognizable.

The Illuminated Scores is an ambitious project that focuses on the way we learn, teach and perceive the concept of “form” in music. One must keep in mind that it is only through an in-depth analysis of music and its complex series of events that we can hope to attain a deep understanding of form and advance our interpretative skills. By displaying a score according to this new method, I hope to enable performers and educators to obtain a different visual perspective. In the pages that follow, I shall demonstrate the application of the Illuminated Scores to three compositions, and reveal the layout for a proposed new analytical edition. Through this new method and the possibility of a more informed performance edition, I hope to assist musicians, both students and professionals, in maximizing their performance.

29 loc. cit.
Chapter 3: Application of the Illuminated Scores Method

J. S. Bach Fugue No. 2 in C minor, BWV 847

For the study of The Well-Tempered Clavier, I researched numerous editions which include the Neue Ausgabe sämtlicher Werke,30 Henle,31 Dover,32 and annotated performance editions such as the Czerny,33 Schirmer-Bussoni,34 Kalmus-Bischoff,35 Tovey-Jones36 and Palmer.37

While we may have the skills to read the music it is another thing to understand the composer’s intent and in turn be able to communicate the meaning of the score. These important issues are addressed in the proposed Illuminated Scores method from the first time we come into contact with a new score regardless of the performer’s level of expertise (Fig. 7).

Let me begin by applying poetic analysis to Bach’s c-minor Fugue, BWV 847, following the parameters and rules proposed in the Illuminated Scores method. For the sake of poetry, let us consider the entire fugue as a Sonnet (poem) (Fig. 9). At first glance we can see that Bach’s “Sonnet”\(^{38}\) is made up of two stanzas, with four verses per stanza. This horizontal analysis of the poetic form yields a (poem) work with eight poetic lines (octave). Now if we consider the vertical analysis of the work, by separating the episodes from their original context, these episodes take on a new meaning. It is through the vertical analysis that we can contextualize the episodes in the poetic form, and see that they become a new poetic stanza, as in ritornello and episode, or refrain and verse, alternating between ritornello (R) and episode (E) throughout this work, i.e.

\[
R-R-E- R-E- R-E- R-E- R-E- R-E- R. 
\]

When we analyze the episodes individually (Fig. 9), as they now represent a poetic stanza, we realize that the newly formed stanza is composed of six verses. The reason I have used the term sonnet in association with this fugue is because of the similarities to the structural design of Petrarchan Sonnets.\(^{39}\)

Through the comparative analysis of this fugue and poetry, the eight entrances of the subject (ritornello) represent the octave of a sonnet, and the six episodes represent the

\(^{38}\) Definition: An Italian Sonnet (Petrarch Sonnet) consists of an octave (8 lines, verses) (rhyming abba abba) and a sestet (6 lines) rhyming in any of various patterns (as cde cde or cde cde). See Merriam-Webster, Inc. 2003. Merriam-Webster's collegiate dictionary. Springfield, MA: Merriam-Webster, Inc.

\(^{39}\) Please note that the poetic model of a sonnet theoretically works as an exercise when practicing, but not in performance as it rearranges the temporal sequence of the work.
sestet. With regard to rhyme, such analysis is not applicable to the subject of a fugue since poetically it would constitute an exact repetition of content; however, such analysis could be performed in the episodes. By analyzing episodes in more detail (Fig. 9), if we consider the colors as symbolic representation of equality, phrases with the same colors “rhyme” within a musical context. In other words, Episode 1 (E1) rhymes with Episode 4 (E4), E2 rhymes with E5, and E3 rhymes with E6. They all rhyme poetically using the cde cde pattern of the sestet prescribed in a Petrarchan sonnet as well as dividing the sestet into two sub-groups of 3 lines (tercet).

Let us analyze the musical content of the fugue in detail. By the texture of the work we can clearly see that it is a three-voice fugue in the key of c minor. Looking at the exposition by itself (Fig. 10), one can see that the first three entrances of the subject represent the thematic exposition of the subject and its two complementary counter-subjects (green and blue). The fourth statement represents the development in the key of E-flat Major. In the example, each system is labeled with numbers 1 through 8 to enumerate each entrance of the subject, and letters M, U, and L (Middle = alto entrance of the subject, Upper = soprano, Lower = tenor) pertain to the voice in which it appears. When we compare the exposition (Fig. 10) with the re-exposition (Fig. 11) we can see how the same pattern is taken up identically in the re-exposition (MULU). This orderly presentation of the entrances of the subject shows a very highly evolved method of organization. The fact the pattern (MULU) is repeated in the re-exposition highlights Bach’s level of detail to the architectural design. Please note that in the re-exposition, the
last statement U8 represents the coda, in which Bach employs a pedal point and a Picardy third. In figures 10 and 11, one can also examine the tonality of each subject entrance. The three main key areas are c-minor - the tonic (i), g-minor - the minor dominant (v) which employs a tonal answer, and E-flat Major - the diatonic major mediant (III) and relative major.

The first counter-subject (Fig. 12) is mainly constructed of stepwise motion, which counteracts the angular nature of the subject. The order of the entrances of counter-subject 1 (green) also uses the same exact pattern (MULU) as the subject entrances in the exposition and re-exposition. The second counter-subject (blue) plays a supporting role in completing the harmony.

Figure 13 illustrates an analysis of the episodes by themselves. Note that on the left (in black) is shown the mathematical proportion and on the right (in red) measure numbers starting from one. The episodes make use of the rich materials from the subject and counter-subjects while they build up tension and drama and explore new harmonies, dissonances and sequences. I have enumerated each episode 1-6 in order to facilitate their analysis.

When we compare episodes two and three (E2 and E3) in figure 14, we can see with ease that the scales, represented in black, are a perfect inversion - a mirror image of each other. Furthermore, the mastery of Bach’s command of music can be seen in figure
15. Here we can see the second episode (E2) now upside down where a mirror retrograde almost worked. Though a perfect retrograde did not work in this instance, since it is one note short (seven vs. eight notes), the contour of high notes to low notes is perfect.

As previously discussed in the poetic analysis of this work (Fig. 9), Episode 1 (E1) is closely related to Episode 4 (E4), E2 to E5, and E3 to E6. Poetically these three pairs rhyme and musically their polyphonic content complements each pair ingeniously. Color-coding has proved invaluable in the process of comparing polyphonic phrases (Fig. 17).

By portraying a spatially accurate measurement of phrase lengths, in accordance with this new method, we notice that perhaps the episode’s most climactic effect lies in the delay of the return of the subject as in measure 14 (Fig. 13). The episodes that take place during the exposition of the work are of equal length, which is easily seen as a result of the spatial proportions of the Illuminated Scores method. However, we can also clearly see that the episodes in the re-exposition differ drastically in length; thus playing with the listener’s expectations. Episode 4 adds one whole measure to the original proportion (phrase length) established in the exposition, and episode 5 adds two and a half extra measures. It is through this episode (E5) that Bach completely switches the tactus\(^{40}\) (metric alignment) of the entire fugue. It seems as though E5 is an amalgamation

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\(^{40}\) Definition: (Lat.). A 15th- and 16th-century term for ‘beat.’ One tactus actually comprised two hand motions, a downbeat and an upbeat (positio and elevatio, or thesis and arsis).
of the procedures previously presented in E2. In other words it builds upon the material of the episodes in the exposition and develops it in the re-exposition. E5 is a perfect example of how bar lines are not always helpful, in fact they blur the meaning of the content drastically.

At first glance it appears that the tactus shift takes place somewhere between measure 24 and the middle of measure 26. However, by comparing E5 to E2 (Fig. 16) we realize that the tactus shift occurs during the first two measures of E5, between measure 22 and the middle of measure 23. The fact that Bach sneaks in the tactus shift so seamlessly made it very difficult to detect. In fact if one looks at the treble only (soprano & alto) the shift is unnoticeable, but if one examines the bass, one can notice that the scale pattern has lost one full tactus - two beats (2 quarters).

Now let us look at E5 from a completely different perspective (Fig. 17). Let us think of the episodes in terms of a prolation\(^41\) switch (or shift). We can see how Bach has gone through three different prolations in the episodes: Episodes 1-3 in the exposition are of imperfect mode, imperfect time, minor prolation; E4 has perfect mode, perfect time, minor prolation; and E5 has perfect mode, perfect time, major prolation.

\(^{41}\) Definition: Division in medieval mensural notation of the whole-note (semibreve) into 3 smaller time-units (major prolation) or 2 (minor).


It seems like Bach is shifting between prolations – going from duple meter 4/2 to triple meter 3/2 in E4 and 9/2 division in E5. Notice how the bracketed patterns (motivic fragments – with the value of a half note) have switched from repeating four times in the exposition to three times in the re-exposition (Fig. 17). In a sense, Bach is matching what he has done before and raising the stakes, with each new repetition. He has managed to overlap the concepts of E2 and E5 (treble material) and the number of repetitions of patterns as explored in the comparison of E4 and E5.

Now, through the understanding of a switch in prolation, E5 makes more sense as a triple subdivision of the long (dotted whole) made up of three half notes. In other words E5 in mensural notation has perfect mode, perfect time, major prolation.

Figure 2: Prolation Chart from: *History of Music* by Stanford and Forsyth

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This realization sheds new light on the meaning and function of measure 25 through the middle of measure 26. I believe it serves as an addition of material when compared to E4 and a Cadenza-like flourish that exists within the parameters of the overall structural design of the piece. The downbeat of measure 25, dominant (V\(^7\)) chord, followed by a rest, resembles the point where an elaborate cadenza would go.

When we turn the work on its side, as illustrated in Figure 18, we can clearly see the dramatic arch, as a skyline view where the higher one goes, the more tension in the piece- poetically, harmonically and dramatically. All of these details clearly come to light in this new layout as a reflection of the architectural design intended by the composer.

After following the procedures of the Illuminated Scores approach, one begins to notice that mathematical proportions come to light (Fig. 19). On the left side of the score in black are the mathematical proportions and on the right side in red the measure numbers that bookend each phrase. The work is 31 measures long with phrase (lengths) proportions of 7, 8 and 16 and (measure number) proportions of 3, 7, 15 and 31 in red.

Could Bach have based the formal architectural structure of this fugue on the Mersenne numbers? (Fig. 19). In order to answer this question, let us look at the mathematical aspects of form in this work in more detail (Figs. 20-21). In figure 20 Mersenne primes are shown in red. Note that even though 15 is not a prime number it is nonetheless an integral part of the Mersenne numbers’ sequence. As used herein:
- Mersenne Numbers are numbers in the form: \( M_n = 2^n - 1 \), \( n \geq 0 \).
- A geometric sequence is a sequence where the quotient of two consecutive terms is a constant, which is called the ratio of the geometric sequence.
- A Mersenne number is a term of the sequence obtained when subtracting 1 to the terms of the geometric series of ratio 2.
- A prime number (building blocks of numbers) is a number greater than 1 that is divisible only by 1 and by itself.
- Prime numbers up to 31 are: 2 3 7 11 13 17 19 23 29 31.
- A Mersenne prime is a Mersenne number which is a prime number.
- The first four Mersenne primes are 3, 7, 31 and 127.

While I have not found any documented evidence that Bach was familiar with the work of Mersenne \( (\text{Harmonie universelle})^{43} \) from 1636-37), I would consider it not only plausible but appropriate to correlate this work of Bach with Mersenne. Based on this fugue’s proportions, a coincidental manifestation of the Mersenne number sequence seems highly improbable. After analyzing a few other fugues it seems that reciprocal (sequential) proportions are commonly present, such as the golden proportion. Perhaps Bach is using proportions like the Mersenne numbers’ sequence for the same reason he uses the golden proportion: his love for mathematics and “a sense of Divinity” and “hidden meanings” from “an intellectual perception that can shape interpretation and affect audience perception of a score.”\(^{44}\) In addition, the observation that not only was


Mersenne a mathematician, instrument builder and theorist, but his contributions to equal temperament\textsuperscript{45} and tuning in general could tie him indirectly to Bach’s \textit{Well-Tempered Clavier}.

Contrary to our widely held belief that Bach was a great promoter, advocate, and supporter of equal temperament, there is no evidence that he either used or did not use equal temperament. Some records indicate that Bach “was not won over by it (equal temperament),”\textsuperscript{46} which to me represents that Bach was aware of the existence and capabilities of equal temperament, yet chose not to use it.

In his \textit{Génération harmonique},\textsuperscript{47} Jean-Philippe Rameau proposes a new system for equal tuning. In reaction to such a method, Rousseau reminds us that “equal temperament had long been described by Mersenne”\textsuperscript{48} (Dominic Eckersley) and that “this method Mr. Rameau proposes to us today, has already been proposed and was rejected by the famous Couperin.”\textsuperscript{49} (Rousseau)

\begin{footnotesize}
\textsuperscript{45} Frederick B. Hyde. 1954, \textit{The position of Marin Mersenne in the history of music}. New Haven, CT. p. 325. \\
Hyde states that Mersenne “certainly, recommended it (equal temperament) from the point of view of its usability.”
\textsuperscript{46} James Murray Barbour. 1972, \textit{Tuning and temperament; a historical survey}. New York: Da Capo Press. p. 87
\end{footnotesize}
Many scholars agree that “Bach probably favored some intermediate form of temperament which did indeed permit the extensive modulation the compositions in his Well-Tempered Clavier demand, but which did not rob thirds and minor sixths of their trueness as does complete equality of temperament.”\(^{50}\) (Hyde) Bach most likely “used a mean-tone or irregular temperament of some sort, possibly one of Werckmeister’s or one of his own devising. Which exact tuning he used is unknown, but it is certain that Bach used this composition (\textit{The Well-Tempered Clavier}) as a vehicle to systematically explore the tonal palettes of the keys of the temperament he was using.”\(^{51}\) (Gareth Loy)

It is difficult for me not to idolize a figure such as Mersenne for his accomplishments, but as Frederick Hyde reminds us “It would be agreeable to imagine Mersenne as a forward looking intellect, who saw the development toward tonality, and the need for modulatory facility, and who visualized equal temperament as the substance for these developments long before Kuhnau and Bach. It would be wiser, however, to see in Mersenne that ...(he)…at least saw in equal temperament, with the mechanist's eye, a direct means to a worthy end - an end not incompatible with the demands of true music-making in matters of pitch accuracy.”\(^{52}\)

\(^{50}\) Frederick Bill Hyde. 1954, \textit{The position of Marin Mersenne in the history of music}. New Haven, CT. p. 330


\(^{52}\) Frederick Bill Hyde. 1954, \textit{The position of Marin Mersenne in the history of music}. New Haven, CT. p. 464
After having reviewed the mathematical terminology (Figs. 20-21) one can apply such knowledge to the score. When one “formally” divides the fugue in sections horizontally, into exposition and re-exposition (see. 22), one notices that the exposition is 15 measures long and the re-exposition is 16 measures long. Mathematically (numerically) speaking one would be more satisfied with an equal division of 16 and 16, adding up to the enjoyable 32 which in music is so favored and easily remembered due to the subdivision of rhythm into whole, half, quarter, eighth, sixteenth, thirty-second, etc. Now, the mastery of Bach’s mathematical knowledge and capabilities are represented in the connection between the examples shown in figures 22 and 23.

The beauty of the layout lies in the reciprocity of form; it works horizontally as well as vertically. Thus we find Bach utilizing form as he does harmony and melody in counterpoint at a completely new level. When we divide vertically between subject entrances and episodes, one notices that the entrances of the subject, which now act as a ritornello or refrain, are 16 measures long and the episodic verses are 15 measures long (Figs. 22-23). Precisely because of this superbly proportional reciprocity at a variety of levels, I cannot help but believe that the use of the Mersenne numbers sequence is intentional.

Another way of representing the architectural structure of this fugue can be found in figure 3.
**Geometric Shape - Bach:** A visual representation of the mathematical proportions of the composition in measure numbers (Sequence of Mersenne numbers 3, 7, 15 and 31 that structurally bind the Fugue)

Figure 3: Geometric representation of phrases for Bach Fugue No. 2 in C minor, BWV 847.

The pedagogical benefits of this new analytical approach lie in the ability to phrase related melodies equally to each other, independently of all others, thus giving the performer a depth and variety of musical colors, dynamics and articulation by establishing a clear thematic hierarchy. In other words, each color in this fugue should have its own personal identity, phrasing, articulation, shape and direction, although not necessarily in the same dynamic range. Rarely do we hear performances that depict the rich variety of techniques utilized by Bach.

Each color may be orchestrated as the same instrument (or group of instruments) and different colors as different instruments (or stops if using an organ); or on the piano by changing character between subject entrances and episodes, polarizing the sections in a refrain and verse format. When deciding what phrasing to choose, one should make sure it is possible to maintain the same identical phrasing and articulation throughout the piece for each color, especially in intricate passages. Most musicians commonly settle for knowing where the entrances of the subject are, and that by no means is enough to do justice to these miniature masterpieces.
A. Schoenberg Piano Suite Op. 25, Prelude

The Piano Suite Op. 25 (Fig. 24) was composed in July 1921; in his own writings, Schoenberg considers it his “first large work in this (serial) style.”53 The entire work utilizes a single twelve-tone row for all six movements of the suite. Schoenberg’s discovery of a “Method of Composing with Twelve Tones which are Related Only with One Another,” became the “most important and influential compositional procedure of the century.” The method that emerges is a “technique for organizing total chromaticism.”54 (David Burge)

The prelude (Präludium) sits on the verge of “one of the most complex stylistic and technical changes —the passage from freely atonal to twelve-tone serial composition.”55 (Deborah How) It also “demonstrates powerful symmetries of rhythm, meter, register, and phrase formation and cohesion.”56 (Richard Kurth) Richard Kurth also attributes the cohesive balance of this movement to “mosaic polyphony, a characteristic association of order-number mosaics, pitch-class mosaics, and 12-tone

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rows with other musical features.”57 (Richard Kurth) The Piano Suite also serves as an example of Schoenberg’s experimentation in serial technique. One can trace the development of the method (serialism) and its rules throughout the suite, for example the prelude allows for certain freedom such as note repetitions while the gigue is strictly serial.

Although this prelude is very dense (highly concentrated content), its overall structure is simple: a six-line poem, made up of 3 stanzas composed of 2 verses each. (Fig. 25) In other words, the prelude is 24 measures long, evenly divided into three 8-measure long sections (Fig. 26), which resemble an exposition, development and re-exposition in a fugue. Furthermore, each individual section can be divided into 2 phrases, yielding even (1/2) and golden (3, 5) proportions.

When we apply the Illuminated Scores analytical method to the Prelude of Schoenberg’s Suite Op. 25 we realize that although it is titled prelude, in reality it is a four voice fugal structure based on serial technique (12-tone). The beauty of the underlying symmetric mathematical structure lies not only in the melodic and harmonic reciprocity expressed on Schoenberg’s matrix but in the thematic mastery of palindromes, juxtapositions, inversions, retrograde, overlapping, interlocking and many more techniques that comprise 12 notes only.

57 Ibid., p. 191.
Figure 27 shows the matrix for the entire suite, which Schoenberg used as the basic mathematical structure; however, we shall focus only on the prelude, which uses the outside edge of the matrix only (Fig. 28). In the score as well as the matrix the original set (P0) is red, the most important 12-note row on which the entire suite is based (Fig. 29). Inversion zero (I0) is blue (Fig. 30); transposition six (T6) is brown (Fig. 31), and inversion six (I6) is green (Fig. 32); all of which Schoenberg exploits in retrograde, particularly at the tetrachord level. In figure 33, one will notice the tritone polarity of the matrix – E to Bb - in which interval class six is used as the “Dominant” since it is the halfway point in serial technique.

Regarding the form and architectural design of this prelude, in figure 26 we have three groups or sections made up of eight bar phrases where the first section is divided equally in half (4 + 4) and the second and third sections in golden proportion (5 + 3). Thematically speaking, from a fugal perspective, we have a clear exposition and re-exposition treated in a very modernist way. A relevant question remains: why this particular layout; why divide the work like this since clearly there is not a melodic restatement of the main subject P0 (red) as in the example by Bach? Using the method of the present invention, referring to figure 34, we find the answer: rhythm takes precedence over melody. The rhythmic restatement of the subject in the second system is a transformation, presented in a retrograde inversion at the tetrachord level (T6), as shown in brown.

58 I believe that the tritone polarity of the matrix purposefully imitates the polarity of the tonic and the dominant in tonal music, similar to fugal writing.
In the second section (Fig. 35), Schoenberg utilizes two new techniques: the use of palindromes in measure 13, and voice overlapping in measure 14. Looking at measure 13 in detail (Fig. 36), we find three overlapping palindromes; the easiest to see is represented in green (tenor line) as 1 2 3 4 3 2 1, followed by (Fig. 37) 5 6 7 8 7 6 5 in red and (Fig. 38) 9 10 11 12 12 11 10 9 in purple. The overlapping technique in measure 14 (Fig. 39) at the tetrachord level is an ingenious use of accents and tetrachord distribution by Schoenberg.

Section three acts as the re-exposition (Fig. 40), which deals with palindromes, and complete presentations of each voice in this fugal structure. The second system represents a three-measure long coda developing and exploring palindromes, dyads, trichords and tetrachords. Figures 41-49 display a new mathematical mode of representation in which the four trichords of the set (P0) are proportionally built using the Clock Wheel. The first trichord is 1 9 5 (Fig. 41), the second trichord is 2 6 10 (Fig. 42), the third is 7 11 3 (Fig. 43), and the fourth is 12 8 4 (Fig. 44). Schoenberg continues by presenting the four trichords in retrograde order, thus creating a palindrome at the trichord level (Fig. 49).

Figure 50 shows additional techniques used in the last measure of this work. Here Schoenberg exploits the clock-wheel at the tetrachord level in a “mosaic-like” fashion. In blue we have the first tetrachord 3,12,6,9; in green 28,5,11,2 and in orange 4,7,10,1. This
section seems to function as a coda that is summarizing (juxtaposing) as many techniques as possible while bringing this piece to an explosive fortissimo ending. The composition as a whole grew in dynamics, texture and intensity. We experienced a three voice fugal structure that evolved into four voices in the re-exposition while introducing new and complex techniques that increased exponentially in difficulty.

My original impression was that this style of composition fell under the category of “music for the eye” and brain rather than for the ear. The first time I heard the prelude, I thought of it as an unrecognizable foreign language. I erroneously believed it to follow in the tradition of Op. 11 as a work that had plunged deeply into the abyss of atonality. While I enjoyed its rhythmic vitality and dance-like quality I must admit that my ear could not process the sounds that were coming to me. However, the more I analyze the work the better it sounds to me, as though knowledge of the process has enhanced my capacity for the sublime and the beautiful. The beauty of craftsmanship is undeniable; I was unprepared to realize that I was very wrong about something. In essence this composition is the most compact with the highest level of ‘economy of material’ I have ever encountered. The idea that out of a single 12-note row (set) one could devise an entire composition (Suite) was a revelation to me.

Through the Illuminated Scores method, I hope to provide performers (especially those unaware of the intricate techniques being applied) with a roadmap that would lead to an appreciation of dodecaphonic music, and aid those who would discard the work for
lack of understanding to dive deeper into the meaning of such compositions. Color-coding has made it much easier for me to change character and touch for each section. Schoenberg consistently highlights the changes of texture, dynamic and articulation whenever a new iteration of the row occurs, as though he is guiding the listener and performer through each motivic fragment.

This type of composition requires a high level of focus and concentration by both the performer and the listener. I would highly encourage practicing it four notes at a time (hexachord level) in the same way one would practice a Bach fugue. I would play each voice (color) individually, then I would practice every section of the same color in succession.

**Geometric Shape – Schoenberg:** A visual representation of the mathematical proportions of the composition (8 + 8 + 8 = 24 measures). Eight bar phrases divided into (4 + 4) and (5 + 3) measures.

4 + 4
5 + 3
5 + 3

Figure 4: Geometric representation of phrases for Schoenberg Piano Suite Op. 25, Prelude.
W. A. Mozart Piano Sonata No. 4 in E-flat major, K. 282, III

Many essays, articles and books have been written on the subject of Sonata-Allegro form. A number of sources have been crucial to my own understanding of and position on issues relating to sonata-allegro form and the architectural design of musical form. Of particular importance are the findings of mathematician John Putz and music engineer Timothy Warner.

A key issue that I shall attempt to address is that of temporal proportions, in particular the golden proportion. In general, proportional relations in music are very problematic. That is, how does a listener comprehend or a performer project proportion, since one can only compare relative lengths in memory, after they have been realized?

John Putz in his article on “The Golden Section and the Piano Sonatas of Mozart,”59 aims to show the possible connection between the architectural design of sonata-allegro form and the proportions of the golden ratio in all the piano sonatas of Mozart, while Timothy Warner in his article on “more appropriate ways to approach the structural analysis of classical music,”60 dives into the issues of temporal proportions.

Putz analyzed the first movement (sonata-allegro) of all 19 piano sonatas by Mozart, plus additional movements where he identified the golden proportion, totaling 28 movements in all. In figure 52 I have included Putz’s analyses and findings. Table 1 contains the number of measures in each section, where “a” represents the exposition; “b” represents the development, re-exposition and coda; and “a + b” represents the total number of measures in the entire composition.

In a scattered plot representation of his findings (Fig. 53), we can clearly see the overall level of approximation to the Phi (Φ) line (a line whose gradient is Φ) between structural sections of the sonatas and the golden proportion. One immediately notices that (Fig. 53, sub-figures 5 and 8) the larger numbers - the points (dots) in the scattered plot (b is to a + b) - provide the closest points to the Φ line, and in turn a more accurate approximation to the golden section than the smaller numbers (a is to b).

This issue is not new in any way but rather a characteristic of the golden proportion. If one looks at the Fibonacci Numbers (sequence) one would naturally run into exactly the same problem; in other words, the bigger the number value of the pair, the closer one gets to the approximation of Φ. In music analysis we are typically in the realm of small numbers, phrase lengths range from four to eight measures on average.

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61 Phi (Φ): Greek letter used to represent the golden proportion (1.6180339887...).
62 Please keep in mind that Phi is an irrational number that cannot be written as a whole number, whose infinite decimal places have no pattern.
and particularly in Mozart piano sonatas the longest sonata-allegro movement is 273 measures long.

What Putz found was “that the double bar (i.e. end of the exposition and beginning of the development sections) regularly appears around 38% of the way through the entire movement. In other words, the double bar – probably the most structurally significant moment in any ‘classic’ sonata form movement – is close to, and often coincides with, the point of the golden section.”63 (Timothy Warner)

\[ A=38\% \quad B=62\% \]

Figure 5: Golden Proportion expressed in percentages

To simplify the golden ratio even further, let us use the analogy that a complete movement is 100% and 38% of the way is the nearest approximation to the point where the golden section division occurs.

Timothy Warner reminds us that Putz, whose “interest and focus was primarily mathematical, responds rather cautiously to these seemingly remarkable findings: ‘Mozart may have known of the golden section and used it [...] and perhaps Mozart, through his consummate sense of form, gravitated to it as the perfect balance.’ Such caution is understandable: in the absence of contemporary written testimony, there is no

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evidence to suggest that Mozart consciously employed the golden mean as a structural device in many of the movements of his piano sonatas.”64 (Timothy Warner)

What Putz fails to account for is the temporal relevance of the piece. He counts measure numbers but does not account for the time it may take to play them. On the other hand, Timothy Warner analyzed the work of 5 recording artists: Mitsuko Uchida, Jean-Bernard Pommier, Klara Würtz, Georges Pludermacher and Glen Gould, with all the tools of modern audio technology (Fig. 54). What Warner discovered from all the recordings he analyzed of the complete Piano Sonatas of Mozart was that only the work of Glenn Gould “consistently retains the temporal proportions of the score.”65 Figure 55 provides a detailed graph of the data from Glenn Gould’s recordings of Mozart Piano sonatas. It shows how Timothy Warner converts Putz’s calculations from measure numbers into seconds, in order to show how “in every movement that Putz identifies, Gould manages to consistently recreate the temporal proportions as specified by Mozart in his scores.”66 (Timothy Warner) One should keep in mind that anything that is 60/40 or larger is more closely related to a ratio of 3:2 than to Phi.

These findings further sparked my imagination and guided additional research on the Illuminated Scores method through a path that would show a new representation of sonata form. If Glenn Gould could play the sonatas of Mozart, keeping the temporal

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64 loc. cit.
65 Ibid., p.21
66 loc. cit.
proportions, certainly the Illuminated Scores could target the same goal of visually portraying the spatial relation associated with temporal proportions of a score. Once again if we think of music as an architectural structure that takes place through time, Mozart is doing nothing more than erecting musical edifices to the pre-established parameters of architectural structures from classical antiquity. Thus we can dare to define sonata-allegro form in its most basic and simplest term as a proportional architectural design that utilizes the golden proportion as its basic structure or point of departure.

The poetic analysis of the third movement of Sonata No. 4 in E-flat major, K. 282, yields a poem made up of two stanzas, with fourteen poetic lines (Fig. 56). I can see the similarity to a Petrarchan sonnet again (as in Bach’s fugue Fig. 9), since in this case we also have an octave (8 poetic lines) that is composed of a five-line exposition and a three-line development (golden proportion), and a sestet that constitutes the recapitulation and coda.

Let us examine figure 57, which displays the entire third movement (102 measures long) of Mozart’s Piano Sonata No. 4 in E-flat major, K. 282. Notice that everything is visually organized (aligned) and that every phrase (verse/system) is eight measures long except for the three instances of the seven-measure-long phrase and the two-measure coda (codetta) (Fig. 58). No indication of the meaning of those seven-measure-long phrases has been found in historic or academic texts to provide a satisfactory explanation for them. They commonly occur in works by Mozart, Haydn,
Beethoven, and Schubert, and they are characteristically present in sonata form
movements (along with other “irregular” phrase lengths).

In music, as in poetry, irregular phrases (poetic meter) add drama and vitality to
the work while avoiding monotonous repetitions of symmetrical phrases. An irregular
phrase is one that does not contain dimensions of 4, 8 or 16 measures. The typical phrase
is 4 or 8 measures long, thus we find a larger number of choices for irregular phrases and
sub-phrases, 3, 5, 6, 7, 9, 10, etc., out of which the most common inherently irregular
phrase divisions are 3, 5, 6 and 7. A variety of techniques exist in the construction of
irregular phrases such as prolongation, extension, modification, repetition, and
anticipation.

In music, as Geoffrey Chew points out, “phrasing theory developed out of 17th-
century rhythmic theory which was conceived in terms of poetic metrical theory. The
18th century ... introduced ... the rhetorical analogy of punctuation... and in Mattheson
(1737) the idea of phrasing explicitly appeared. These and later writers show the modern
reader that various different degrees of articulation were required to make phrases,
sentences and so on perceptible.”

According to Grove Music Online, a phrase is “a term adopted from linguistic

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state.edu/subscriber/article/grove/music/40952.
syntax and used for short musical units of various lengths; a phrase is generally regarded as longer than a motif but shorter than a period. It carries a melodic connotation, in so far as the term ‘phrasing’ is usually applied to the subdivision of a melodic line. As a formal unit, however, it must be considered in its polyphonic entirety, like ‘period’, ‘sentence’ and even ‘theme’.  

According to the Merriam-Webster Dictionary, the definition of irregular is:

1. a: not being or acting in accord with laws, rules, or established custom
   b: not conforming to the usual pattern of inflection
   c: not following a usual or prescribed procedure
2. not belonging to or a part of a regular organized group
3. lacking perfect symmetry or evenness
4. lacking continuity or regularity especially of occurrence or activity

If we take our common understanding of an “irregular phrase,” I believe we are communicating (and reinforcing) a misconception. In other words, I do not consider any of the irregular phrases in my examples of Bach, Mozart and Schoenberg to fall under the category of an “irregular” phrase as defined by the Merriam-Webster Dictionary. Every single phrase I have examined within this document is proportional, symmetrical, organized, acting in accord and following the prescribed procedures of the overall architectural design of the work. I would consider these phrases to have an irregular meter as in poetic analysis but I would not consider them to be irregular. One should note that if a work does not have a cohesive architectural structure, it would not affect the analysis presented here.

In my analysis of the third movement of Mozart K. 282, the answer to the riddle of the seven bar phrase lies not only in the overall design of the piece, but also in the small details of each section: “One thing arises from all things, and all things arise from one thing” – Heraclitus. In the exposition (Fig. 59), the 7-measure long phrase (the transition) occurs in order to allow the large-scale architectural design of the piece to exist. This “phrase of irregular meter” (irregular phrase or 7-bar phrase) is not arbitrarily created, but is part of the greater poetic and formal scheme of the work. Please note that the dotted line represents the medial caesura (Hepokoski/Darcy) that separates the first theme area from the second. Further investigation into a few other sonatas has yielded that the medial caesura happens at the division of the golden proportion not only in this movement but in the first movement of K 281-284. In a comparative correlation of concepts, one could arrive at the conclusion that the medial caesura coincides with the golden cut in many of Mozart’s sonata-allegro movements. In other words the medial caesura is a musical (poetic) manifestation of the golden cut.

The golden proportion has been commonly used to describe the large-scale structure of sonata form. However, in the Illuminated Scores method, the golden proportion is used to describe themes and phrases; not to account for temporal

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70 Definition: The Medial Caesura is the dividing point (break, rest) in the thematic exposition.

proportions, but rather to calculate the mathematical length of themes (phrases) and their expected appearance.

In figure 60, we see a clear mathematical proportion emerge to clarify the division of themes and phrases. The first theme area is 15 measures long and the second theme area is 24 measures long; adding $15 + 24 = 39$ and $39$ (expo) + 63 (development/recap/coda) = 102. At this point we can no longer ignore the presence of the golden proportion, which is perhaps the most optimal way of combining binary and ternary form. Furthermore, if we used a different approach, in which we excluded the coda from our total calculations (Fig. 61) of the piece (as many analysts have done), we would still see the golden proportion manifested as 62 (exposition/development) and 38 (recapitulation) now delineating the arrival at the recapitulation and not the double bar from the exposition. It is almost as though we cannot escape the symmetrical proportion of the golden ratio no matter how we look at the piece. The numbers that provide the best approximation to the golden ratio can be examined in figure 62. The golden ratio in musical terms states that the exposition is to the development/recap/coda as the development/recap/coda is to the entire movement (the total sum of the exposition/development/recap/coda).

**Geometric Shape – Mozart:** A visual representation of the mathematical proportions of the composition in measure numbers (Golden Proportion Pyramid 15, 24, 39, 63 and 102 that structurally binds the Sonata)

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Figure 6: Geometric representation of phrases for Mozart Piano Sonata No. 4 in E-flat major, K. 282, III
Chapter 4: The Illuminated Editions: Emergence of a Publishing Method

Following the successful application of the Illuminated Scores method to these three pieces, it became clear that there is an opportunity to develop new analytical editions, utilizing this information. The Illuminated Editions serve to accommodate the boundless parameters of the newly reorganized score, without restriction by current publishing constraints.

In the digital age of the 21st Century the physical size of a score is no longer an issue. Digital sheet music is becoming more widely available and artists are becoming increasingly comfortable utilizing digital scores both in practice and performance. As performers and scholars we have become dependent on facsimile, urtext, critical and performance editions. However, a large number of performers and students commonly rely solely on performance editions and their editorial suggestions for phrasing, fingerings, dynamics etc., without verifying their authenticity by comparison to an urtext or facsimile. In this digital era there is also a predicament in the publishing industry with the rise in the use of public domain scores, easily accessible and downloadable at the click of a button. Young performers are largely unaware of the unreliability of the freely accessible scores, which lend no academic insight into the analysis of music to the untrained eye, nor do they highlight the relevance of the musical material.
Current critical editions (scholarly editions) while invaluable, present a number of challenges. First, they are expensive and hard to find; and second, they typically contain historical, comparative and/or analytical findings, and often utilize technical language that is unfamiliar to many students. Examples of these issues include understanding Roman numeral analysis, figured bass, articulations, ornamentation symbols, formal analysis, Schenkerian analysis etc., all of which take experience and understanding of the methods utilized.

The new Illuminated Editions would be designed as scholarly editions (analytical performance editions) for performing artists and educators, and would incorporate semiotic tools into the study of musical language intended to make the architectural design (formal structure) of a composition visible in a coherent format.

On the most basic level, the Illuminated Scores could provide a platform for the proposed visual score layout, with clearly organized content that would reveal the structure of the work. These editions are a graphically logical representation of the mathematical proportions that occur intentionally or inadvertently when a musical work is composed. This format overlays graphical representations of harmonic structure, rhythmic patterns, melodic relationships, etc., and the number of measures in a system.
A poetic “verse–like” analysis is applied to a piece of music to visually represent the architectural design of the composition. Musical content and form are visually represented to delineate musical phrases and fragments through the use of color-coding, which enables the performer to analyze the content of the piece in order to decipher the composer’s intent.

The view provided by the Illuminated Editions reorganizes the material into blocks of phrases, phrase lengths, melodic relationships, harmonic structure, rhythmic patterns, and the number of measures in a system. This new visual layout of the score differs significantly from the traditional printed score, representing music graphically and reorganizing content without the constraints of the current publishing model.

These editions also allow for the integration of traditional theoretical practices of harmony and voice leading, analysis of tonal and atonal music, and elements of form and meter in a holistic approach. This new method could provide a more practical system for learning the score. Creating scores in this format could contribute to greater understanding of the written music and its interpretation, therefore enhancing the caliber of one’s performance.
Conclusion

In my efforts to help fellow musicians develop a pathway for truly understanding meaning in music, I was delighted to discover that the application of poetic and mathematical analysis actually reveals the architectural design of musical form within a score.

In analyzing the works of Bach, Schoenberg and Mozart I examined three of the most revered composition techniques in music: fugal writing, sonata-allegro form and serial technique. Key concepts revealed in this analysis include form, phrase structure and mathematics. Music as an artistic expression is able to communicate the relationship between poetry, architecture and musical design with ease. Through the detailed analysis of these works, I was able to synthesize multiple theories and thus conceive the Illuminated Scores method.

The revelation of phrase structure in the visual representation of form demonstrates how phrases complement each other and generate harmony and unity of ideas. One can also interpret how themes and rhythms meld in such a way that they cannot stand alone without their counterparts.
Once music is organized in a coherent phrase structure, the form reveals itself, highlighting the beauty of the design and balance of proportions, while confirming the existence of order and thoughtfully premeditated calculations. Through this visual design, one can conceive the relevance of mathematical proportions in music.

The revelation of the extent to which mathematical proportions exist in the structural design of music is one of the most important outcomes of the Illuminated Scores. The identification of mathematical concepts helps us prove their relationships, while clarifying seemingly unexplainable musical ideas, such as the meaning of irregular phrases, intricate overlapping techniques, and extra-musical concepts. In the study of mathematical concepts, I found a number of patterns and sequences, which can enhance one’s appreciation for the mathematical formulations.

The works of Bach, Mozart and Schoenberg proved to be fertile ground for the study of music and mathematics. The fascinating relationship between music and mathematics demonstrates how “the language of mathematics is a convenient tool for comprehending and communicating (the) underlying structure. Pure mathematics provides a convenient framework for the music theorist to communicate good ways of hearing a work of music.”71 (Fiore)

The purpose of the Illuminated Scores is to provide a roadmap to understanding the Architectural Design of Musical Form. It is a method that provides the possibility of exploring the fascinating tools employed by some of the most brilliant composers in the world of music. I believe this new perspective has cognitive advantages, especially in regards to visual memory and a deeper understanding of the meaning of music. This method addresses the relationships among formal structure, thematic representation, temporal dimension, harmonic and melodic hierarchy, and how they function in a musical context. By developing an effective learning platform, the Illuminated Scores aspires to spark the performer’s desire to connect with extra-musical concepts.

The application of this new method could become a valuable tool for performers and educators. Used as a pedagogical tool, the Illuminated Scores could serve to enhance a student’s understanding of temporal proportions and their role in the overall structure of a piece. This method could also aid performers to think in larger units, moving away from attention to the individual beats, and rather visualize the larger picture. Certainly one could keep the temporal proportions perfectly by playing metronomically on a perfectly proportioned composition. But so can a machine, and that by no means is what my intentions are. What impassions me is the thought that we could be flexible and musical, following the direction of the piece with ups and downs, forward when needed, relaxed and calm when required, and still simultaneously do so within a structurally coherent framework.

72 Please note that I do not intend for students to calculate their playing time, but rather to realize that a larger work is truly a sum of its parts.
I would especially stress the pedagogical importance of approaching the overall structure of a work before we begin to learn a new piece (in general or in preparation for a performance), which in turn will facilitate the learning process.

Through the Illuminated Scores, I envisioned the creation of a platform in which I could translate my own analytical findings and simultaneously make it possible to incorporate the findings of other significant sources into the Illuminated Scores edition of musical masterworks.

These findings have inspired me to continue employing the Illuminated Scores method in the hope that even just one performance would convey such meaning that transcends human limitations and reveals our innermost need to connect to the world around us in a meaningful and thoughtful way.
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60
Appendix A: Figures
Fugue in C minor, BWV 847 (WTC I)

J. S. Bach
Illuminated by
Orlay Alonso

SUBJECT
Red = S, Subject, Theme
Green = CS1, 1st Counter-Subject
Dark Blue = CS2, 2nd Counter-Subject
Maroon = Half Step Motif (mordent)

EPISODES
Purple = Variation of the Subject
Orange = Retrograde of CS1 scale
Black = Scale Motif
Light Blue = Parallel Thirds Motif
Blue Aqua = Material not closely related
Maroon = Half Step Motif (mordent)

Fig. 7: J. S. Bach, Fugue in C minor (BWV 847), illuminated layout
Fig. 8: J. S. Bach, Fugue in C minor (BWV 847), aligned spatially

SUBJECT
Red = S, Subject, Theme
Green = CS1, 1st Counter-Subject
Dark Blue = CS2, 2nd Counter-Subject
Maroon = Half Step Motif (mordent)

EPISODES
Purple = Variation of the Subject
Orange = Retrograde of CS1 scale
Black = Scale Motif
Light Blue = Parallel Thirds Motif
Blue Aqua = Material not closely related
Maroon = Half Step Motif (mordent)
Fig. 9: J. S. Bach, Fugue in C minor (BWV 847), poetic "verse-like" analysis as a Petrarchan Sonnet
Fig. 10: J. S. Bach, Fugue in C minor (BWV 847), exposition and development

M = middle voice entrance
U = upper voice entrance
L = lower voice entrance
1-4 = entrance of the subject
Fig. 11: J. S. Bach, Fugue in C minor (BWV 847), re-exposition and coda

M = middle voice entrance
U = upper voice entrance
L = lower voice entrance
5-8 = entrance of the subject
Fig. 12: J. S. Bach, Fugue in C minor (BWV 847), counter-subject imitation of subject entrances order
Fig. 13: J. S. Bach, Fugue in C minor (BWV 847), episodic material analysis
Fig. 14: J. S. Bach, Fugue in C minor (BWV 847), inversion technique in episodes two (E2) and three (E3)
Fig. 15: J. S. Bach, Fugue in C minor (BWV 847), mirror retrograde in episode two (E2)
Fig. 17.1: J. S. Bach, Fugue in C minor (BWV 847), motivic pattern repetition in episodes (prolation)
Fig. 18: J. S. Bach, Fugue in C minor (BWV 847), dramatic arch, skyline view
Prime numbers up to 31: 2 3 7 11 13 17 19 23 29 31

Mersenne Prime Sequence

\[ M_p = 2^p - 1; \text{ whenever } M_p \text{ prime, } p \text{ is prime} \]

\[
\begin{align*}
1 + 2 &= 3 \\
1 + 2 + 4 &= 7 \\
1 + 2 + 4 + 8 &= 15 \\
1 + 2 + 4 + 8 + 16 &= 31 \\
2^2 - 1 &= 3 \\
2^3 - 1 &= 7 \\
2^4 - 1 &= 15 \\
2^5 - 1 &= 31 \\
3 + 4 &= 7 \\
7 + 8 &= 15 \\
15 + 16 &= 31
\end{align*}
\]

Fig. 20: Mathematical analysis, Mersenne numbers and prime numbers up to 31
- A geometric sequence is a sequence where the ratio of two consecutive terms is a constant, which is called the radius of the geometric sequence.

- A geometric series is a series (sequence of partial sums) whose terms (addends) form a geometric sequence.

- A Mersenne number is a partial sum in the geometric series of ratio 2.

- A prime number (building blocks of numbers) is a number greater than 1 that is divisible only by 1 and by itself.

- Prime numbers up to 31:

  2  3  7  11  13  17  19  23  29  31

- A Mersenne prime is Mersenne number which is a prime number. The first four Mersenne primes are 3, 7, 31 and 127

<table>
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<th>Number of Iterations (n)</th>
<th>Geometric sequence of radius ( r = 2 )</th>
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<th>Sum (Mersenne Numbers)</th>
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<td>5</td>
<td>( 2^4 = 16 )</td>
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<td>( 1 + 2^1 + 2^2 + \ldots + 2^{n-1} )</td>
<td>( 2^n - 1 )</td>
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</table>

Mersenne Numbers are numbers of the form:

\[
M_n = 2^n - 1 = \sum_{i=1}^{n} 2^{i-1}, \quad n \geq 1
\]

Fig. 21: Mathematical analysis, geometric sequence and formula for Mersenne Numbers
Fig. 22: J. S. Bach, Fugue in C minor (BWV 847), horizontal reciprocity of form between the exposition and re-exposition
Fig. 23: J. S. Bach, Fugue in C minor (BWV 847), vertical reciprocity of form between the subject and episodes
Matrix
Red = (P0) Set - Row
Brown = (P6) Transposition Sum Six
Green = (16) Inversion Sum Six
Dark Blue = (10) Inversion Zero

Fig. 24: A. Schoenberg, Piano Suite Op. 25, Prelude, illuminated layout
Fig. 25: A. Schoenberg, Piano Suite Op. 25, Prelude, poetic analysis
Fig. 26: A. Schoenberg, Piano Suite Op. 25, Prelude, eight bar phrase division
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Fig. 27: A. Schoenberg, Piano Suite Op. 25, Prelude, complete matrix
Fig. 28: A. Schoenberg, Piano Suite Op. 25, Prelude, matrix outside edge as the basis
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Fig. 29: A. Schoenberg, Piano Suite Op. 25, Prelude, twelve-tone Row (P0)
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Fig. 30: A. Schoenberg, Piano Suite Op. 25, Prelude, inversion zero (l0)
Fig. 31: A. Schoenberg, Piano Suite Op. 25, Prelude, transposition sum six (T6)
Fig. 32: A. Schoenberg, Piano Suite Op. 25, Prelude, inversion sum six (I6)
Fig. 33: A. Schoenberg, Piano Suite Op. 25, Prelude, tritone polarity of the matrix
Fig. 34: A. Schoenberg, Piano Suite Op. 25, Prelude, exposition, rhythmic vs. melodic precedence
Fig. 35: A. Schoenberg, Piano Suite Op. 25. Prelude, development, palindromes and overlapping technique.
Fig. 36: A. Schoenberg, Piano Suite Op. 25, Prelude, overlapping palindromes (tenor line)
Fig. 37: A. Schoenberg, Piano Suite Op. 25, Prelude, overlapping palindromes (alto line)
Fig. 38: A. Schoenberg, Piano Suite Op. 25, Prelude, overlapping palindromes (soprano and bass line)
Fig. 39: A. Schoenberg, Piano Suite Op. 25, Prelude, overlapping technique (bar 14)
Fig. 40: A. Schoenberg, Piano Suite Op. 25, Prelude, re-exposition, palindromes, dyads, trichords an tetrachords
Fig. 41: A. Schoenberg, Piano Suite Op. 25, Prelude, clock wheel trichord 1,9,5 (bar 24)
Fig. 42: A. Schoenberg, Piano Suite Op. 25, Prelude, clock wheel trichord 2,6,10
Fig. 43: A. Schoenberg, Piano Suite Op. 25, Prelude, clock wheel trichord 7,3,11
Fig. 49: A. Schoenberg, Piano Suite Op. 25, Prelude, clock wheel trichords palindrome
John Putz’s Table for Mozart Piano Sonatas:
Number of measures in each section

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Fig. 52: John Putz’s Table for Mozart Piano Sonatas, number of measures in each section
Fig. 53: John Putz’s Scattered Plot for Mozart Piano Sonatas, Golden proportion
Table 1. Temporal proportions of each of the movements in selected recordings of Mozart’s Piano Sonatas

The temporal proportions of each of the sonata form movements are represented as percentages of the whole where “a” is the first section (i.e. exposition) and “b” is the second section (i.e. development and recapitulation). The golden mean, represented as percentages of the whole, should correspond to a ratio of 38:62.

Ratio numbers **bold italic** denote that the “a” section is not repeated.
Ratio numbers in **bold** denote that the “b” section is repeated.

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<tr>
<th>Köchel No.</th>
<th>Mitsuko Uchida</th>
<th>Jean-Bernard Pommier</th>
<th>Klara Wurtz</th>
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Fig. 54: Timothy Warner’s analysis of 5 recording artists, temporal proportions
Table 2. Data for Glenn Gould’s recordings of the sonata form movements of Mozart’s Piano Sonatas

The golden mean, represented as percentages of the whole, should correspond to a ratio of 38:62.

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Fig. 55: Graph of Timothy Warner’s data from Glenn Gould’s recordings of Mozart Piano sonatas
Fig. 56: Mozart, Piano Sonata in E-flat major, K. 282. III, poetic analysis
Fig. 57: Mozart, Piano Sonata in E-flat major, K. 282. III, illuminated layout
Fig. 58: Mozart, Piano Sonata in E-flat major, K. 282. III, alignment, seven and two measure phrases
Fig. 59: Mozart, Piano Sonata in E-flat major, K. 282. III, exposition and meaning of the seven measure phrase
Fig. 60: Mozart, Piano Sonata in E-flat major, K. 282. III, mathematical proportions and division of themes and phrases
Fig. 61: Mozart, Piano Sonata in E-flat major, K. 282. III, mathematical proportions and division of themes and phrases
Golden Ratio \( \varphi = 1.618 \ldots \)

A is to B as B is to A + B

\[
\frac{A}{B} = \frac{1}{\varphi} = \frac{B}{A+B} = \varphi - 1 \approx 0.618
\]

Sequence: 39 63 102

\[
\begin{align*}
&\frac{102}{63} \approx 1.619 \\
&\frac{63}{39} \approx 1.615
\end{align*}
\]

Fig. 62: Mathematical explanation of the Golden Ratio