BOUNDS FOR HECKE EIGENFORMS AND THEIR ALLIED L-FUNCTIONS

DISSERTATION

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ABSTRACT

We consider Hecke eigenforms and their allied L-functions from three aspects in this thesis.

First we generalize the Iwaniec’s spectral large sieve estimates of Maass cusp form to the local version for all congruence groups $\Gamma_0(q)$. Our approach is based on an inequality [33, Proposition] for a general bilinear form involving Kloosterman sums and Bessel functions. The exceptional eigenvalues emerge in the course of the proof.

In the second part, we extend Luo’s result [29] to prove a general optimal bound for $L^4$–norms of the dihedral Maass forms associated to Hecke’s grossencharacters of a fixed real quadratic field. Given a fixed quadratic field $K = Q(\sqrt{D})$ with discriminant $D$, we remove the condition that the narrow class number of $K$ is 1. The key ingredients are Watson [45] and Ichino [15]’s formula and the local spectral large sieve inequality established in the first part.

Finally we obtain a long equation intended to establish an upper bound for the second moment of symmetric square $L$-functions. Petersson trace formula plays an important role and we study thoroughly an analogue of Estermann series using Hurwitz zeta function and establish its meromorphic extension and functional equation. This work provides a useful approach to the further study of $L(\frac{1}{2}, \text{sym}^2(f))$. 

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To my parents
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CHAPTER 1
INTRODUCTION

In this chapter, we review the basic theory of automorphic forms and closely related background. Then we formulate the problems we are focusing on in this dissertation and state the main results.

1.1 Upper Half Plane $\mathbb{H}$ and the Congruence Group $\Gamma_0(q)$

Let $\mathbb{H}$ denote the upper half plane $\mathbb{H} = \{x + iy | y > 0; x, y \in \mathbb{R}\}$ and $q$ be a positive integer. Consider the Hecke congruence group $\Gamma_0(q)$ of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries, $c \equiv 0 \pmod{q}$, and determinant 1. This group acts on $\mathbb{H}$ through $z \mapsto \frac{az + b}{cz + d}$.

Let $\sigma$ be an element in $\Gamma_0(q)$. It is called parabolic if and only if it has exactly one fixed point in $\mathbb{C} \cup \infty$, and the fixed point of a parabolic element in $\Gamma_0(q)$ is called a cusp. If $a$ is a cusp of $\Gamma_0(q)$, its stabilizer $\Gamma_a$ is defined by $\Gamma_a = \{\sigma \in \Gamma_0(q) : \sigma a = a\}$.

In particular, $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{Z} \right\}$. Two cusps $a$ and $b$ are equivalent if and only if there exists $\tau \in \Gamma_0(q)$ such that $b = \tau a$.

Every cusp of $\Gamma_0(q)$ is equivalent to some $\frac{u}{w}$ with positive coprime $u$ and $w$ such that $w|q$, two such cusps $\frac{u}{w}$ and $\frac{u'}{w'}$ being equivalent if and only if $w = w'$ and $u \equiv u'$.
(mod \((w, \frac{q}{w})\)). For later use, when \(a\) is equivalent to some \(\frac{u}{w}\) with positive coprime \(u\) and \(v\) such that \(w|q\), we define

\[
\mu(a) = (w, \frac{q}{w})q^{-1}.
\]

The groups \(\Gamma_0(q)\) have finite index in \(SL(2, \mathbb{Z})\), precisely

\[
[\Gamma_0(1) : \Gamma_0(q)] = q \prod_{p|q}(1 + p^{-1}).
\]

A fundamental domain for the action of \(\Gamma_0(q)\) on \(\mathbb{H}\) is a subset \(F \subset \mathbb{H}\) satisfying the following conditions:

1. \(F\) is an open subset in \(\mathbb{H}\);

2. The closure \(\overline{F}\) of \(F\) in \(\mathbb{C}\) intersects every orbit of \(\Gamma_0(q)\), i.e. for all \(z \in \mathbb{H}\), there exists \(\gamma \in \Gamma_0(q)\) with \(\gamma z \in \overline{F}\);

3. No two points in \(F\) are \(\Gamma_0(q)\)-equivalent.

Particularly, the fundamental domain for \(SL(2, \mathbb{Z})\) can be given as

\[
F_1 = \{z \in \mathbb{H} \mid |\Re(z)| < \frac{1}{2}, |z| > 1\},
\]

and

\[
\text{Vol}(F_1) = \frac{\pi}{3}.
\]

Thus

\[
\text{Vol}(\Gamma_0(q) \backslash \mathbb{H}) = \frac{\pi}{3} [\Gamma_0(1) : \Gamma_0(q)] = \frac{\pi}{3} q \prod_{p|q}(1 + p^{-1}).
\]

### 1.2 Double Cosets Decomposition and Kloosterman Sum

Let \(a\) and \(b\) be two cusps of \(\Gamma_0(q)\) and we choose a scaling matrix \(\sigma_a \in SL(2, \mathbb{R})\) such that

\[
\sigma_a \infty = a \quad \text{and} \quad \sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_{\infty}
\]
and similarly $\sigma_b$. We have the following double cosets decomposition:

$$\sigma_a^{-1}\Gamma_0(q)\sigma_b = \delta_{ab}\Omega_\infty \cup \bigcup_{c > 0 \mod c} \Omega_{d/c},$$

where

$$\delta_{ab} = \begin{cases} 
1 & \text{if } a, b \text{ are equivalent} \\
0 & \text{otherwise} 
\end{cases} \tag{1.2.1}$$

and

$$\Omega_\infty = \Gamma_\infty w_\infty \Gamma_\infty \text{ for some } w_\infty = \begin{pmatrix} 1 & * \\
& 1 \end{pmatrix} \in \sigma_a^{-1}\Gamma_0(q)\sigma_b$$

and

$$\Omega_{d/c} = \Gamma_\infty w_{d/c} \Gamma_\infty \text{ for some } w_{d/c} = \begin{pmatrix} * & * \\
& c & d \end{pmatrix} \in \sigma_a^{-1}\Gamma_0(q)\sigma_b.$$

Given any $m, n \in \mathbb{Z}$, the Kloosterman sum is defined as follows:

$$S_{ab}(m, n; c) = \sum_{a \Gamma_\infty \setminus \sigma_a^{-1}\Gamma_0(q)\sigma_b / \Gamma_\infty} e\left(\frac{ma + nd}{c}\right).$$

In particular, when $q = 1$, $a = b = \infty$, we obtain the classical Kloosterman sum

$$S(m, n; c) = \sum_{ad \equiv 1 \mod c} e\left(\frac{ma + nd}{c}\right).$$

We list some properties of $S(m, n; c)$ below:

$$S(m, n; c) = S(n, m; c),$$

$$S(am, n; c) = S(m, an; c) \text{ if } (a, c) = 1,$$

$$S(m, n; c) = \sum_{d \mid (c, m, n)} dS(mnd^{-2}, 1; cd^{-1}).$$
In particular the last equation gives
\[ S(m, n; c) = S(mn, 1; c) \text{ if } (c, m, n) = 1. \]

Moreover, we have
\[ S(m, n; c) = S(\bar{q}m, \bar{q}n; r)S(\bar{r}m, \bar{r}n; q) \]
if \( c = qr \) with \((q, r) = 1\), where \( \bar{q}, \bar{r} \) are multiplicative inverses of \( q, r \) to moduli \( r, q \) respectively.

We also have the Weil's bound:

**Theorem 1.2.1** ([18], Theorem 4.5). For any integer \( c \geq 1 \), \( m, n \) we have
\[
|S(m, n; c)| \leq (m, n, c)^{1/2}e^{1/2}\tau(c)
\]
where \( \tau(c) \) denotes the divisor function.

### 1.3 Holomorphic Modular Forms

Let \( k \geq 1 \) be an integer. Then we have an action (the "weight k action") of \( SL(2, \mathbb{Z}) \) on functions \( f : \mathbb{H} \to \mathbb{C} \) by
\[
(f|_k g)(z) = j(g, z)^{-k}f(gz), \text{ where } j(g, z) = cz + d \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Let \( q \geq 1 \) be an integer, and \( \chi \) a Dirichlet character modulo \( q \) which induces a character of \( \Gamma_0(q) \) by \( \chi(g) = \chi(d) \) for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \)

A modular form of weight \( k \), level \( q \), and nebentypus (or character) \( \chi \) is a holomorphic function \( f \) on \( \mathbb{H} \) which satisfies
\[
f|_k \gamma = \chi(\gamma)f, \text{ for all } \gamma \in \Gamma_0(q)
\]
and is holomorphic at all cusps of \( \Gamma_0(q) \).
The holomorphy at cusp $a$ is explained as follows. For a cusp $a$ of $\Gamma_0(q)$, the function $f_a = f|_{k\sigma_a}$ is seen by modularity to be periodic of period one, $f_a(z + 1) = f_a(z)$. This implies that $f$ is a function of the parameter $q = e(z)$, namely $f_a(z) = g_a(z)$, where $g_a$ is holomorphic in a punctured disc $\{z \in \mathbb{C} \mid 0 < |z| < r\}$. We then say that $f$ is meromorphic at $a$ if $g_a$ is meromorphic at 0. Therefore $f_a$ has a Laurent series expansion

$$f_a(z) = \sum_{n \geq n_a} \hat{f}_a(n)e(nz)$$

for some integer $n_a$, and $\hat{f}_a(n_a) \neq 0$. We say that $f$ is holomorphic at $a$ if it is meromorphic and $n_a \geq 0$. If, moreover, we have $n_a > 0$, we say that $f$ vanishes at $a$. If $f$ vanishes at all cups, it is called a cusp form.

Modular forms of level $q$, weight $k$ and character $\chi$ form a vector space, which is denoted $M_k(q, \chi)$, and the set of the cusp forms is a subspace denoted $S_k(q, \chi)$. And we assume the character $\chi$ modulo $q$ satisfies the consistency property $\chi(-1) = (-1)^k$.

### 1.4 Hecke Operators and Hecke Eigenforms

We now sketch the theory of Hecke operators and Hecke forms. And we only describe the theory for holomorphic automorphic forms.

Fix integers $k \geq 1$, $q \geq 1$ and a character $\chi$ modulo $q$ satisfying the consistency property $\chi(-1) = (-1)^k$. The operator $T(n)$ is defined by the formula

$$T(n)f(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \leq b < d} f\left(\frac{az + b}{d}\right)$$

for $n \geq 1$.

The following propositions are fundamental:
Proposition 1.4.1 ([16], Proposition 14.8). For any \( n \geq 1 \), \( T(n) \) acts on modular forms and on cusp forms: in other words, it induces linear maps

\[
T(n) : M_k(q, \chi) \rightarrow M_k(q, \chi)
\]
\[
T(n) : S_k(q, \chi) \rightarrow S_k(q, \chi)
\]

Proposition 1.4.2 ([16], Proposition 14.9). The Hecke operators commute. More precisely, for any \( n \geq 1 \) and \( m \geq 1 \) we have

\[
T(m)T(n) = \sum_{d | (n,m)} \chi(d)d^{k-1}T\left(\frac{mn}{d^2}\right)
\]
or equivalently

\[
T(mn) = \sum_{d | (n,m)} \mu(d)\chi(d)d^{k-1}T\left(\frac{m}{d}\right)T\left(\frac{n}{d}\right)
\]

In particular, the \( T(n) \) are multiplicative: \( T(mn) = T(m)T(n) \) if \( (m,n) = 1 \).

Proposition 1.4.3 ([16], Proposition 14.10). Let \( n \) be coprime with \( q \). Then the operator \( T(n) \) acting on the space of cusp forms \( S_k(q, \chi) \) is normal with respect to the Petersson inner product; more precisely, its adjoint is

\[
T(n)^* = \overline{\chi(n)}T(n)
\]

And we have

\[
<T(n)f, g> = \chi(n) <f, T(n)g>,
\]
for \( (n, q) = 1 \).

Proposition 1.4.4 ([16], Proposition 14.11). There is an orthonormal basis of the space \( S_k(q, \chi) \) of cusp forms which consists of eigenfunctions of all the Hecke operators \( T(n) \) for \( (n, q) = 1 \).
Any non-zero modular form \( f \in M_k(q, \chi) \) for which there exist complex numbers \( \lambda(n) \) such that
\[
T(n) f = \lambda(n) f
\]
for \((n, q) = 1\) is called a Hecke eigenform.

Now we introduce Hecke newforms (or primitive forms). Let \( S^b(q, \chi) \) be the subspace of \( S_k(q, \chi) \) spanned by all cusp forms of the type \( f(dz) \) where \( f \in S_k(q', \chi') \) with \( q' < q \) and \( dq' \mid q \). Then let \( S^*(q, \chi) \) be the orthogonal complement of \( S^b(q, \chi) \) with respect to the Petersson inner product, so
\[
S_k(q, \chi) = S^b(q, \chi) \bigoplus S^*(q, \chi).
\]

Note that if \( \chi \) is primitive, we have \( S^*(q, \chi) = S_k(q, \chi) \).

We call \( f \) a Hecke new form if \( f \in S^*(q, \chi) \).

**Proposition 1.4.5.** (Multiplicity one principle)[[16], Proposition 14.12] Given a sequence \((\lambda(n))\) of complex numbers, the subspace of \( S^*(q, \chi) \) spanned by Hecke forms with eigenvalues \( \lambda(n) \) for all \((n, q) = 1\) is at most one-dimensional. In other words, such a Hecke eigenform \( f \), if it exists, is unique up to multiplication by a scalar.

**Proposition 1.4.6** ([16], Proposition 14.13). Let \( f \in S^*(q, \chi) \) be a Hecke newform. Then \( f \) is an eigenfunction of all Hecke operators,
\[
T(n) f = \lambda_f(n) f, \text{ for all } n \geq 1,
\]
and for all \( n \geq 1 \), \( a_f(n) = \lambda(n)a_f(1) \). Hence the first Fourier coefficient \( a_f(1) \) is non-zero.

It is natural to normalize a Hecke new form. We say that \( f \) is Hecke-normalized if its first Fourier coefficient is \( a_f(1) = 1 \), so that the Fourier coefficients are the same as the Hecke eigenvalues. We say \( f \) is Petersson-normalized if
\[
\|f\| = \left( \int_{\Gamma_0(q) \backslash \mathbb{H}} |f(z)|^2 y^k \frac{dx \, dy}{y^2} \right)^{1/2} = 1.
\]
1.5 Eisenstein series

We define Eisenstein series $E_\alpha(z, s)$ associated with a cusp $\alpha$ of $\Gamma_0(q)$

$$E_\alpha(z, s) = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} (\Im \sigma_\alpha^{-1} \gamma z)^s,$$

where $\Re(s) > 1$ and $z \in \mathbb{H}$.

We denote that the Eisenstein series have meromorphic continuation to $\mathbb{C}$ and satisfy certain matrix functional equations. Although they are not square-integrable, they are eigenfunctions of the laplacian operator $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ . Thus the Eisenstein series share many properties with the Maass cusp forms discussed in the next section.

It has the Fourier expansion \cite{[16], P. 388}

$$E_\alpha(z, s) = \delta_{\alpha \infty} y^s + \varphi_{\alpha \infty}(s)y^{1-s} + \sum_{n \neq 0} 2|n|^{\frac{1}{2}} y^{\frac{1}{2}} \varphi_{\alpha \infty}(n, s) K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),$$

where

$$\varphi_{\alpha \infty}(s) = \pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{c > 0} c^{-2s} S_{\alpha \infty}(0, 0; c),$$

$$\varphi_{\alpha \infty}(n, s) = \pi^s \frac{\Gamma(s)}{\Gamma(s)} |n|^{s-1} \sum_{c > 0} c^{-2s} S_{\alpha \infty}(0, n; c)$$

for $\Re(s) > 1$.

1.6 Maass Forms and the Spectral Decomposition

Let $\mathbb{H}$ be the upper half plane of complex numbers, and $\Gamma = \Gamma_0(q)$ be the Hecke congruence group.

We first define the space of automorphic functions

$$\mathcal{A}(\Gamma \backslash \mathbb{H}) = \{ f : \mathbb{H} \rightarrow \mathbb{C} \mid f(\gamma z) = f(z) \text{ for all } \gamma \in \Gamma \text{ and } f \text{ is real analytic} \}. $$
We also define the space of square-integrable automorphic functions

\[ L^2(\Gamma \backslash \mathbb{H}) = \{ f \in \mathcal{A}(\Gamma \backslash \mathbb{H}) \mid \int_F |f(z)|^2 \frac{dx\,dy}{y^2} < +\infty \} \]

and the space of Maass forms with eigenvalue \( \lambda = s(1 - s) \)

\[ \mathcal{A}_s(\Gamma \backslash \mathbb{H}) = \{ f \in \mathcal{A}(\Gamma \backslash \mathbb{H}) \mid \Delta f = s(1 - s)f \}. \]

Let \( a \) be any cusp of \( \Gamma \) and \( \sigma_a \) a scaling matrix. We further define the space of cuspidal automorphic functions

\[ L^2_0(\Gamma \backslash \mathbb{H}) = \{ f \in L^2(\Gamma \backslash \mathbb{H}) \mid \hat{f}_{a,0} = 0 \text{ for any cusp } a \} \]

where

\[ f(\sigma_a z) = \sum_{n \in \mathbb{Z}} \hat{f}_{a,n}(y)e(nx), \]

and the space of Maass cusp forms with eigenvalue \( s(1 - s) \)

\[ \mathcal{A}^0_0(\Gamma \backslash \mathbb{H}) = L^2_0(\Gamma \backslash \mathbb{H}) \cap \mathcal{A}_s(\Gamma \backslash \mathbb{H}). \]

We have the following full spectral decomposition theorem for the automorphic laplacian on \( \Gamma_0(q) \backslash \mathbb{H} \):

**Theorem 1.6.1 ([16], Theorem 15.5).** Let \( u_0, u_1, u_2, \ldots \) be an orthonormal basis of the residual and cuspidal spaces, i.e. \( u_0 = \text{constant} = \text{Vol}(\Gamma_0(q) \backslash \mathbb{H})^{-1/2} \) with eigenvalue \( \lambda_0 = 0 \) and \( u_j \in \mathcal{A}^0_0(\Gamma \backslash \mathbb{H}) \) with eigenvalue \( \lambda_j = s_j(1 - s_j) \) for \( j = 1, 2, \ldots \). Then any \( f \in L^2(\Gamma_0(q) \backslash \mathbb{H}) \) has the spectral decomposition

\[ f(z) = \sum_{j \geq 0} < f, u_j > u_j(z) + \sum_{\alpha} \frac{1}{4\pi} \int_{\mathbb{R}} < f, E_\alpha(\cdot, 1/2 + it) > E_\alpha(z, 1/2 + it) dt \]

where \( \{\alpha\} \) runs over all cusps of \( \Gamma_0(q) \) and the above decomposition is valid in \( L^2 \)-sense, and also converging absolutely and uniformly on compact sets.

Moreover, the Parseval formula holds

\[ ||f||^2 = \sum_{j \geq 0} | < f, u_j > |^2 + \sum_{\alpha} \frac{1}{4\pi} \int_{\mathbb{R}} | < f, E_\alpha(\cdot, 1/2 + it) > |^2 dt. \]
1.7 Petersson Trace Formula and Kuznetsov’s Trace Formula

We state the Petersson trace formula and Kuznetsov’s formula in this section.

**Theorem 1.7.1. (Petersson’s Trace Formula)** [[16], Proposition 14.5] Let $B_k(q)$ be a Hecke-normalized basis of $S_k(\Gamma_0(q))$, the space of holomorphic cusp form of weight $k > 2$ and level $q$. Assume $f(z) = \sum_{n \geq 1} \lambda_f(n)n^{(k-1)/2}e(nz)$. Then for any positive integers $m, n$ we have

$$\sum_{f \in B_k} w_f \lambda_f(n)\lambda_f(m) = \delta_{m,n} + 2\pi i^k \sum_{c \equiv 0 (mod q)} c^{-1} S(m, n, c) J_{k-1}(4\pi \sqrt{mn}/c),$$

where $w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\||f||^2}$ and

$$\||f||^2 = \int_{\Gamma_0(q) \backslash \mathbb{H}} \|f(z)\|^2 y^k dx dy/y^2.$$

Let $\{u_j(z)\}_{j \geq 1}$ be an orthonormal basis of Maass cusp form for $\Gamma_0(q)$ and $\{E_\alpha(z, \frac{1}{2} + ir); r \in \mathbb{R}\}$ is the eigenpacket of Eisenstein series associated with a system of inequivalent cusps. Suppose $u_j(z)$ has the Fourier expansion

$$u_j(z) = y^{1/2} \sum_{n \neq 0} \rho_j(n) K_{\frac{1}{2} + it_j}(2\pi |n|y)e(nx),$$

where $\lambda_j = s_j(1 - s_j)$, $s_j = \frac{1}{2} + it_j$ for each $j$.

We remark that either $t_j$ is real and positive, or $0 \leq it_j \leq \frac{1}{4}$.

We recall the Fourier expansion for the Eisenstein series in [Chapter 1, Section 1.5]:

$$E_\alpha(z, s) = \delta_{\alpha \infty} y^s + \varphi_{\alpha \infty}(s) y^{1-s} + \sum_{n \neq 0} 2|n|^{\frac{1}{2}} y^{\frac{1}{2}} \varphi_{\alpha \infty}(n, s) K_{\frac{s-1}{2}}(2\pi |n|y)e(nx),$$

where

$$\varphi_{\alpha \infty}(s) = \pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{c > 0} c^{-2s} S_{\alpha \infty}(0, 0; c),$$
\[ \varphi_{\alpha\infty}(n, s) = \frac{\pi^s}{\Gamma(s)} |n|^{s-1} \sum_{c > 0} c^{-2s} S_{\alpha\infty}(0, n; c) \]

where \( \Re(s) > 1 \).

We also define

\[ \tau_\alpha(n, t) = (2|n|^\frac{1}{2}) \varphi_{\alpha\infty}(n, \frac{1}{2} + it), \]

**Theorem 1.7.2. (Kuznetsov’s Trace Formula)[[16], Theorem 16.3]**

Suppose \( h(t) \) satisfies the following conditions:

\[
\begin{aligned}
&\begin{cases}
  h(t) = h(-t), \\
  h \text{ is holomorphic in } |\Im(t)| \leq \frac{1}{2} + \delta, \\
  h(t) \ll (|t| + 1)^{-2-\delta}
  \end{cases}
\end{aligned}
\]

for some \( \delta > 0 \).

Then for any \( m, n > 0 \) we have

\[
\begin{align*}
\sum_j \overline{\rho}_j(m) \rho_j(n) \frac{h(t_j)}{\cosh \pi t_j} &+ \sum_\alpha \frac{1}{4\pi} \int_{-\infty}^\infty \overline{\tau}_\alpha(m, r) \tau_\alpha(n, r) \frac{h(r)dr}{\cosh \pi r} \\
&= \delta(m, n) g_0 + \sum_{c \equiv 0 (\text{mod } q)} \overline{c}^{-1} S(m, n; c) g\left(\frac{4\pi \sqrt{mn}}{c}\right)
\end{align*}
\]

where

\[ g_0 = \pi^{-2} \int_{-\infty}^\infty rh(r) \tanh(\pi r) dr \]

and

\[ g(x) = \frac{2i}{\pi} \int_{-\infty}^\infty J_{2ir}(x) \frac{rh(r)}{\cosh(\pi r)} dr. \]

### 1.8 Classical automorphic L-functions

An L-function \( L(f, s) \) has the following data and conditions in general:
1. A Dirichlet series with Euler product of degree \( d \geq 1 \),

\[
L(f, s) = \sum_{n \geq 1} \lambda_f(n) n^{-s} = \prod_p (1 - \alpha_1(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1}
\]

with \( \lambda_f(1) = 1 \), \( \lambda_f(n) \in \mathbb{C} \), \( \alpha_i(p) \in \mathbb{C} \). The series and Euler products must be absolutely convergent for \( \Re(s) > 1 \). The \( \alpha_i(p) \), \( 1 \leq i \leq d \), are called the local parameters of \( L(f, s) \) at \( p \), and they must satisfy

\[ |\alpha_i(p)| < p \text{ for all } p. \]

2. A gamma factor

\[
\gamma(f, s) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right)
\]

where the numbers \( \kappa_j \in \mathbb{C} \) are called the local parameters of \( L(f, s) \) at infinity. These numbers must be either real or come in conjugate pairs and \( \Re(\kappa_j) > -1 \).

3. An integer \( q(f) \geq 1 \), called the conductor of \( L(f, s) \), such that \( \alpha_i(p) \neq 0 \) for \( p \nmid q(f) \) and \( 1 \leq i \leq d \). A prime \( p \nmid q(f) \) is said to be unramified.

We also define

\[
q_{\infty}(s) = \prod_{j=1}^{d} (|s + \kappa_j| + 3),
\]

and the analytic conductor

\[
q(f, s) = q(f)q_{\infty}(s) = q(f) \prod_{j=1}^{d} (|s + \kappa_j| + 3).
\]

We denote \( q(f) = q(f, 0) = q(f) \prod_{j=1}^{d} (|\kappa_j| + 3) \).

From these, we define the complete \( L \)-function

\[
\Lambda(f, s) = q(f)^{\frac{1}{2}} \gamma(f, s)L(f, s)
\]
It is holomorphic in the half-plane $\Re(s) > 1$, and it has an analytic continuation to a meromorphic function for $s \in \mathbb{C}$, with at most poles at $s = 0$ and $s = 1$. Moreover, it satisfies the functional equation

$$\Lambda(f, s) = \epsilon(f)\Lambda(\bar{f}, 1 - s),$$

where $\bar{f}$ is the dual of $f$, and $\epsilon(f)$ is a complex number of absolute value 1, called the root number.

Let $f$ be a primitive holomorphic cusp form of weight $k \geq 1$, level $q$, with nebentypus $\chi$ for $SL_2(\mathbb{Z})$. Let

$$f(z) = \sum_{n \geq 1} \lambda_f(n)n^{(k-1)/2}e(nz)$$

be its Hecke-normalized Fourier expansion at the cusp $\infty$. Then

$$L(f, s) = \sum_n \lambda_f(n)n^{-s} = \prod_p (1 - \lambda_f(p)p^{-s} + \chi(p)p^{-2s})^{-1}$$

is an L-function of degree 2 with conductor $q$ and gamma factor given by

$$\gamma(f, s) = \pi^{-s}\Gamma\left(\frac{s + (k-1)/2}{2}\right)\Gamma\left(\frac{s + (k+1)/2}{2}\right).$$

Similarly, let $\varphi$ be a primitive Maass form of level $q$ with nebentypus $\chi$ which is an eigenfunction of the Laplace operator with eigenvalue $\lambda = \frac{1}{4} + r^2$, where $r \in \mathbb{R}$ or $ir \in [0, \frac{1}{2}]$. Writing its Fourier expansion at infinity in the form

$$\varphi(z) = \sqrt{y}\sum_{n \neq 0} \rho(n)K_{ir}(2\pi|n|y)e(nx),$$

we associate with $\varphi$ the L-function

$$L(\varphi, s) = \sum_{n \geq 1} \rho(n)n^{-s} = \prod_p (1 - \rho(p)p^{-s} + \chi(p)p^{-2s})^{-1}$$

of conductor $q$, with gamma factor

$$\gamma(\varphi, s) = \pi^{-s}\Gamma\left(\frac{s + \delta + ir}{2}\right)\Gamma\left(\frac{s + \delta - ir}{2}\right)$$

where $\delta = 0$ if $\varphi$ is even and $\delta = 1$ if $\varphi$ is odd.
1.9 Approximate functional equation

The following Theorem gives analytically convenient expressions for \( L(f,s) \) in the critical strip where the series does not converge absolutely.

**Theorem 1.9.1** ([16], Theorem 5.3). Let \( L(f,s) \) be an L-function. Let \( G(u) \) be any function which is holomorphic and bounded in the strip \(-4 < \text{Re}(u) < 4\), even, and normalized by \( G(0) = 1 \). Let \( X > 0 \). Then for \( s \) in the strip \( 0 \leq \sigma \leq 1 \) we have

\[
L(f,s) = \sum_n \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{X\sqrt{q}}\right) + \epsilon(f,s) \sum_n \frac{\overline{\lambda}_f(n)}{n^{1-s}} V_{1-s}\left(\frac{Xn}{\sqrt{q}}\right) + R
\]

where \( V_s(y) \) is a smooth function defined by

\[
V_s(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(f,s+u)}{\gamma(f,s)} \frac{\gamma(f,1-s)}{\gamma(f,s)} \frac{\Lambda(f,s+u)}{q^{s/2} \gamma(f,s)} \frac{1}{u} du
\]

and

\[
\epsilon(f,s) = \epsilon(f) q(f)^{1/2-s} \frac{\gamma(1-s)}{\gamma(s)}. \]

The last term \( R = 0 \) if \( \Lambda(f,s) \) is entire, otherwise

\[
R = (\text{res}_{u=1-s} + \text{res}_{u=-s}) \frac{\Lambda(f,s+u)}{q^{s/2} \gamma(f,s)} \frac{G(u)}{u} X^u. \]

We choose suitable test functions \( G(u) \), such as \( G(u) = (\cos \frac{\pi u}{4A})^{-4dA} \), where \( A \) is a positive integer, so that the sums can be effectively limited to the terms with \( n \ll q(f,s) \) because of the following proposition.

**Proposition 1.9.2** ([16], Proposition 5.4). Suppose \( \text{Re}(s + \kappa_j) \geq 3\alpha > 0 \) for \( 1 \leq j \leq d \). Then the derivatives of \( V_s(y) \) satisfy

\[
y^a V_s^{(a)}(y) \ll (1 + \frac{y}{\sqrt{q_\infty}})^{-A},
\]

\[
y^a V_s^{(a)}(y) = \delta_\alpha + O\left(\frac{y}{\sqrt{q_\infty}}\right)^\alpha,
\]

where \( \delta_0 = 1, \delta_\alpha = 0 \) if \( \alpha > 0 \) and the implied constants depend only on \( \alpha, a, A, d \).
1.10 L-functions on number fields

Let $K/\mathbb{Q}$ be a finite extension of rationals of degree $d = [K : \mathbb{Q}]$. There are exactly $d$ isomorphisms of $K$ into the field of complex numbers. Among these there are $r_1$ real embeddings and $2r_2$ complex embeddings, pairwise complex conjugate, where $r_1 + 2r_2 = d$. We say that $K$ is totally real if $r_1 = d$ or totally imaginary if $r_1 = 0$.

Let $\mathcal{O} \subset K$ denote the ring of integers, so $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank $n$:

$$\mathcal{O} = w_1 \mathbb{Z} + \cdots + w_d \mathbb{Z}$$

where $\{w_1, \ldots, w_d\}$ is an integral basis. This gives us the discriminant

$$D = \det(\text{Tr } w_i w_j) \in \mathbb{Z}$$

where $\text{Tr} : K \to \mathbb{Q}$ denotes the trace. Furthermore we introduce the different $\delta \subset \mathcal{O}$; its inverse is given by the fractional ideal

$$\delta^{-1} = \{x \in K : \text{Tr } x\mathcal{O} \subset \mathbb{Z}\}.$$

The prime ideals $\mathcal{P} \subset \mathcal{O}$ which divide the different also divide the discriminant. Prime divisors of $D$, and only they, are ramified in the extension $K/\mathbb{Q}$. We have

$$\text{sign}D = (-1)^{r_2}.$$

The group of units $U \subset \mathcal{O}$ is the direct product of a finite cyclic group $E \subset K$ of roots of unity and a free abelian group of rank $r = r_1 + r_2 - 1$, i.e. every unit can be written uniquely as the product

$$\eta = \zeta^{n_1} \cdots \epsilon_r^{n_r}$$

with $\zeta \in E$ and $n_1, \ldots, n_r \in \mathbb{Z}$, where $\{\epsilon_1, \ldots, \epsilon_r\}$ is a fixed system of free generators, called the fundamental units of $K$. We then define the regulator of $K$ by

$$R = \left| \det(\log |\epsilon_i^{n_j}|) \right|$$
where \( \sigma_j \) ranges over \( r = r_1 + r_2 - 1 \) isomorphisms, no two of which are complex conjugate.

Let \( I \neq 0 \) be the group of fractional ideals,

\[
I = \left\{ \frac{a_1}{a_2} : a_1, a_2 \subset \mathcal{O}, a_1a_2 \neq 0 \right\}
\]

and \( P \subset I \) the subgroup of principal ideals

\[
P = \{(a) = a\mathcal{O} : a \in K^*\}.
\]

Then \( \mathcal{H} = I/P \) is the class group, and its order \( h = [I : P] < \infty \) is called the class number of \( K \).

For any integral ideal \( m \subset \mathcal{O} \) we consider the groups

\[
I_m = \{a \in I : (a, m) = 1\}
\]

\[
P_m = \{(a) \in P : a \equiv 1 (\text{mod } m)\}.
\]

Here \((a, m) = 1\) means \( a = a_1/a_2, a_1, a_2 \subset \mathcal{O}, (a_1a_2, m) = 1\); and \( a \equiv 1 (\text{mod } m)\) means \( a = a_1/a_2, a_1, a_2 \in \mathcal{O}, (a_1a_2, m) = 1, a_1 \equiv a_2 (\text{mod } m)\), i.e. \( a_1 - a_2 \in m \).

Clearly \( P_m \subset I_m \), and \( h_m = [I_m : P_m] \) is finite. The factor group \( I_m/P_m \) is called the ray class group.

Let \( \{\sigma\} \) range over a fixed system of isomorphisms having exactly one isomorphism from each pair of complex conjugates. For any \( \sigma \) from the system we take numbers \( u_\sigma, v_\sigma \) subject to the following restrictions:

\[
\begin{cases}
  u_\sigma = 0, 1 & \text{if } \sigma \text{ is real,} \\
  u_\sigma \in \mathbb{Z} & \text{if } \sigma \text{ is complex,} \\
  v_\sigma \in \mathbb{R} & \text{such that } \sum_\sigma v_\sigma = 0.
\end{cases}
\]

Let \( S_1 = \{z \in \mathbb{C} : |z| = 1\} \). Define the homomorphism \( \xi_\infty : K^* \to S_1 \) by the product

\[
\xi_\infty(a) = \prod_\sigma (a^\sigma / |a^\sigma|)^{u_\sigma} |a^\sigma|^{iv_\sigma}.
\]
Let \( m \) be an integral ideal, small enough, such that the group

\[
U_m = \{ \eta \in U : \eta \equiv 1(\text{mod } m) \}
\]

is in the kernel of \( \xi_\infty \). Then \( \xi_\infty \) can be regarded as a function on \( P_m \), and we call \( m \) a modulus for \( \xi_\infty \).

A group homomorphism \( \xi : I_m \to S^1 \) is said to be a character to modulus \( m \) if it coincides with \( \xi_\infty \) on \( P_m \); i.e.

\[
\xi(a) = \xi_\infty(a) \quad \text{if } a = (a) \text{ with } a \equiv 1(\text{mod } m)\}.
\]

Of course if \( \xi \) is a character to modulus \( m \), then it is also a character to any modulus \( n \subset m \). Moreover, given \( \xi \) (mod \( m \)), there may exist a character \( \xi^* \) (mod \( m^* \)) with \( m \subset m^* \) such that \( \xi^*(a) = \xi(a) \) on \( I_m \). The largest ideal \( m^* \) with this property is called the conductor of \( \xi \). If \( m^* = m \), then \( \xi \) is called primitive. For any character \( \xi \) (mod \( m \)) there exists a unique primitive character \( \xi^* \) (mod \( m^* \)) such that \( \xi^*(a) = \xi(a) \) if \( (a, m) = 1 \).

Any Hecke character \( \xi : I_m \to S^1 \) extends to \( \xi : I \to \mathbb{C} \) by setting

\[
\xi(a) = 0 \quad \text{if } (a, m) \neq 1.
\]

These so-called Grossencharacters were introduced by E. Hecke[12].

The \( L \)-function associated with a Hecke character \( \xi \) (mod \( m \)) is defined by

\[
L(s, \xi) = \sum_{0 \neq a \subset O} \xi(a)(Na)^{-s}
\]

where \( a \) ranges over non-zero integral ideals and \( N : I \to \mathbb{Q}^* \) is the norm defined by

\[
Na = \#(O/a),
\]

and it extends to fractional ideals by multiplicativity. The series converges absolutely in \( \Re s > 1 \), and it has the Euler product over prime ideals

\[
L(s, \xi) = \prod_p (1 - \xi(p)(Np)^{-s})^{-1}.
\]
For the trivial character we get the Dedekind zeta function

\[ \zeta_K(s) = \sum_{a \neq 0} (Na)^{-s} = \prod_p (1 - (Np)^{-s})^{-1}. \]

It is a self-dual L-function of degree \( d = [K : Q] \) with conductor \( D = |d_K| \), the absolute value of the discriminant of \( K \), and root number 1. The gamma factor is

\[ \gamma(s) = \pi^{-ds/2} \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2} \]

where \( r_1 \) is the number of real embeddings of \( K \) and \( r_2 \) the number of pairs of complex embeddings so that \( d = r_1 + 2r_2 \). The zeta function has simple pole at \( s = 1 \) with residue

\[ \text{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{D}}, \]

where \( h \) is the class number of \( K \), \( R \) the regulator and \( w \) the number of roots of unity.

For \( \xi \pmod{m} \) a non-trivial primitive Hecke Grossencharacter. The Hecke L-function is defined by

\[ L(\xi, s) = \sum_{a \neq 0} \xi(p)(Na)^{-s} = \prod_p (1 - \xi(p)(Np)^{-s})^{-1} \]

for \( \Re(s) > 1 \). Hecke showed that \( L(\xi, s) \) is an L-function of degree \( d = [K : Q] \) which is entire if \( \xi \neq 1 \) and coincides with \( \zeta_K(s) \) for \( \xi = 1 \). The conductor is \( \Delta = |d_K|Nm \) and the gamma factor is given by

\[ \gamma(\xi, s) = \pi^{-ds/2}2^{-r_2s} \prod_\sigma \Gamma\left(\frac{1}{2}(|u_\sigma| + n_\sigma(s + iv_\sigma))\right) \]

where \( n_\sigma = 1 \) if \( \sigma \) is real and \( n_\sigma = 2 \) if \( \sigma \) is complex.
1.11 Quadratic field $\mathbb{Q}(\sqrt{D})$

In this section, we state some basic facts about the quadratic field $K = \mathbb{Q}(\sqrt{D})$. We assume that $D$ is the discriminant of $K$. The field character is given by the Kronecker symbol

$$\chi_D(n) = \left(\frac{D}{n}\right).$$

Note that $\chi_D(-1) = \text{sign}(D) = 1, -1$ if $K$ is real or imaginary, respectively. Moreover, the values of $\chi_D$ at primes characterize the factorization into prime ideals in $K$[[18], P. 212], namely

$$\begin{cases} p = p^2 & \text{if } \chi_D(p) = 0 \quad \text{(ramified prime)}, \\ p = pp' & \text{if } \chi_D(p) = 1 \quad \text{(split prime)}, \\ p = p & \text{if } \chi_D(p) = -1 \quad \text{(inert prime)}. \end{cases}$$

Next we will construct Hecke automorphic forms on the quadratic field $\mathbb{Q}(\sqrt{D})$. For clarity we state the imaginary field and the real field cases separately[[18], Theorem 12.5] and [29].

1. Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with discriminant $D < 0$ and $\xi \pmod{m}$ a Hecke character such that

$$\xi((a)) = \left(\frac{a}{|a|}\right)^u \quad \text{if } a \equiv 1 \pmod{m}$$

where $u$ is a nonnegative integer. Then

$$f(z) = \sum_a \xi(a)(Na)^\frac{u}{2}e(zNa) \in \mathcal{M}_k(\Gamma_0(N), \chi)$$

where $k = u + 1, N = |D|Nm$, and $\chi \pmod{N}$ is the Dirichlet character given by

$$\chi(n) = \chi_D(n)\xi((n)) \quad \text{if } n \in \mathbb{Z}.$$ 

Moreover, $f$ is cusp form if $u > 0$. 

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2. Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D > 0$, fundamental unit $\epsilon_D > 0$, and the ring of integers $\mathcal{O}_K$. For integer $k \neq 0$, we consider the Hecke’s grossencharacter $\Psi_k$ of $K$ defined by

$$\Psi_k((\alpha)) = \left| \frac{\alpha}{\alpha'} \right|^{\frac{2ik}{\log \epsilon_D}}$$

for principle ideal $(\alpha) \subset \mathcal{O}_K$, with generator $\alpha$, where $\alpha'$ is the conjugate of $\alpha$ under the nontrivial automorphism of $K$.

Consider

$$\Phi_k(z) = y^{1/2} \sum_{0 \neq a \in \mathcal{O}_K} \Psi_k(a)K \frac{x^{ik}}{\log \epsilon_D} (2\pi |N(a)|y) e(N(a)x)$$

where $\{a\}$ runs over all non-zero integral ideals in $\mathcal{O}_K$.

Then $\Phi_k(z)$ is a Hecke-Maass cusp form on $\Gamma_0(D)$, with eigenvalue $\frac{1}{4} + \left( \frac{k\pi}{\log \epsilon_D} \right)^2$, and nebentypus character the Kronecker symbol $\chi_D$.

1.12 Hurwitz Zeta Function $\zeta(s,a)$

In this section we state some basic facts about Hurwitz zeta function $\zeta(s,a)$ which we need to use in Chapter 4.

The Hurwitz zeta function $\zeta(s,a)$ is defined as

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

for $\Re s > 1$ and $\Re a > 0$, and is meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1.

Moreover $\zeta(s,a)$ has the following functional equation:

$$\zeta(s,\alpha) = i^{-1} \Gamma(1-s)(2\pi)^{s-1} \left( e\left(\frac{s}{4}\right)\varphi(1-s,\alpha) - e\left(-\frac{s}{4}\right)\varphi(1-s,-\alpha) \right)$$
where
\[ \varphi(s, \alpha) = \sum_{m=1}^{\infty} \frac{e(m\alpha)}{m^s} \]
for \( \Re s > 1 \).

We write \( \zeta(s, a) = \frac{1}{s-1} + \gamma_0(a) + \cdots \), and the constant term is given by
\[ \gamma_0(a) = \lim_{s \to 1} \left[ \zeta(s, a) - \frac{1}{s-1} \right] = -\varphi(a) \]
where \( \varphi \) is the digamma function defined as
\[ \varphi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \]

In particular, when \( x = 1 \), \( \varphi(1) = \gamma \) where \( \gamma \) is the Euler constant.

When \( x \) is large, we have the following asymptotic formula:
\[ \varphi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \]

When \( \Re x > 0 \), we have
\[ \varphi(x) = -\gamma + \sum_{n=0}^{\infty} \frac{z - 1}{(n+1)(n+z)} = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right). \]

In particular, for positive \( r < m \), we have the Gauss’s digamma theorem
\[ \varphi\left(\frac{r}{m}\right) = -\gamma - \ln(2m) - \frac{\pi}{2} \cot \frac{r\pi}{m} + 2 \sum_{n=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \cos \frac{2\pi nr}{m} \ln \sin \left( \frac{\pi n}{m} \right). \]
Moreover, we have
\[ \sum_{r=1}^{m} \varphi\left(\frac{r}{m}\right) = -m(\gamma + \ln m). \]

### 1.13 Statement of results

The goal of this section is to state the main results in this dissertation.

Theorem 1.13.1 is to establish a general large sieve inequality to any interval for any congruence group \( \Gamma_0(q) \), together with a continuous spectral counterpart which comes from the Kuznetsov’s formula.
For \(f,g \in S(\Gamma_0(q))\), we define the Petersson inner product

\[
<f, g> = \frac{1}{\text{vol}(X_0(q))} \int_{X_0(q)} f(z)\overline{g(z)} \frac{dx dy}{y^2},
\]

where

\[X_0(q) = \Gamma_0(q) \setminus \mathbb{H},\]

Let \(S(\Gamma_0(q))\) be the infinite dimensional space of Maass cusp forms of level \(q\), and \(\{u_j(z)\}_{j \geq 1}\) be an orthonormal Hecke basis of \(S(\Gamma_0(q))\) consisting of common eigenfunctions for all Hecke operators \(T_n\) with \((n,q) = 1\). Thus \(u_j(z)\) is an eigenfunction of the hyperbolic Laplace-Beltrami operator \(\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\) with eigenvalue \(\lambda_j = s_j(1 - s_j), s_j = \frac{1}{2} + it_j\) for each \(j\), and it has the Fourier expansion

\[u_j(z) = y^{1/2} \sum_{n \neq 0} \rho_j(n) K_{\frac{1}{2} + it_j} (2\pi |n| y) e(nx).\]

**Theorem 1.13.1.** Assume \(1 \leq A \leq T\), then we have

\[
\sum_{T < t_j \leq T + A} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \\
+ \text{vol}(X_0(q)) \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{t_j}{\alpha}\right)^2} + e^{-\left(\frac{t_j}{\alpha}\right)^2}}{\cosh(\pi t)} \left| \sum_{N < n \leq 2N} a_n \tau_{\alpha}(n, t) \right|^2 dt \\
\ll (N + qAT)(qNT)^c \sum_{N < n \leq 2N} |a_n|^2,
\]

where \(\{\alpha\}\) run over all inequivalent cusps of the congruence group \(\Gamma_0(q)\), and \(\{a_n\}\) is a square-summable sequence. \(\{\tau_{\alpha}(n, t)\}\) are defined in Page 11. The inequality is uniform in \(q\).

Theorem 1.13.2 is to prove the optimal bound for \(L^4\)-norms of the dihedral Maass forms associated to Hecke’s grossencharacters of a fixed real quadratic field, as their Laplacian eigenvalues tend to infinity.
Let $K = Q(\sqrt{D})$ be a fixed real quadratic field with discriminant $D > 0$, fundamental unit $\epsilon_D > 0$, and the ring of integers $\mathcal{O}_K$. We assume $D > 8$ is a prime. For integer $k \neq 0$, we consider the Hecke’s grossencharacter $\Psi_k$ of $K$ defined by

$$\Psi_k((\alpha)) = \left| \frac{\alpha}{\alpha'} \right|^k \pi i k \log \epsilon_D$$

for principle ideal $(\alpha) \subset \mathcal{O}_K$, with generator $\alpha$, where $\alpha'$ is the conjugate of $\alpha$ under the nontrivial automorphism of $K$.

Consider

$$\Phi_k(z) = y^{1/2} \sum_{0 \neq \beta \subset \mathcal{O}_K} \Psi_k(\beta) K_{\frac{\pi i k}{\log \epsilon_D}} (2\pi N(\beta)y) e(N(\beta)x),$$

where $N(\beta)$ is the norm of $\beta$, $K_\nu(z)$ is the modified Bessel function, and $e(\xi) = e^{2\pi i \xi}$.

Then $\Phi_k(z)$ is a Hecke-Maass cusp form on $\Gamma_0(D)$, eigenvalue $\frac{1}{4} + \left( \frac{k\pi}{\log \epsilon_D} \right)^2$, and with nebentypus character the Kronecker symbol $\chi_D$.

**Theorem 1.13.2.** With the notion as given above, we have

$$\frac{\|\Phi_k\|_4}{\|\Phi_k\|_2} \ll_\epsilon T_k^\epsilon,$$

for any $\epsilon > 0$.

Theorem 1.13.3 is to establish a long equation intended to give an upper bound for the second moment of symmetric square L-functions $L(s, \text{sym}^2(f))$, which is predicted to be $\log^3 k$.

Let $S_k(\Gamma(1))$ be the space of all holomorphic cusp forms of weight $k$ with respect to the modular group $\Gamma(1)$, and $\mathcal{B}_k$ is a basis of $S_k(\Gamma(1))$ consisting of Hecke-normalized cusp forms which are simultaneously eigenforms for all Hecke operators $T_n$.

The Fourier series expansion of $f$ is

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e(nz),$$

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and we have
\[ T_n f = \lambda_f(n)n^{(k-1)/2} f. \]

Then the symmetric square L-function \( L(s, \mathrm{sym}^2(f)) \) is defined as
\[
L(s, \mathrm{sym}^2(f)) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{m^2=n} \lambda_f(m^2)}{n^s}, \text{ for } \Re(s) > 1.
\]

**Theorem 1.13.3.** Let \( f \in S_k(\Gamma(1)) \), \( k \) even, then we have the following expression for the second moment of the symmetric square L-functions:
\[
\sum_{f \in B_k} w_f L^2(1/2, \mathrm{sym}^2(f)) = V(1) + M_1 + M_2
\]
\[
= V(1) + M_1 + M'_1 + M''_1
\]

where
\[
w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2};
\]
\[
V(x) = \frac{1}{\pi i} \int (2) \frac{L^2_{\infty}(1/2 + w)}{L^2_{\infty}(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{x^w} dw,
\]
\[
M_1 = x^k \sum_{d_1} \frac{1}{d_1^2} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{L^2_{\infty}(1/2 + w)}{L^2_{\infty}(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^w}
\]
\[
\times \text{Res}_{s=-1/2-w} \left( \frac{c}{2\pi d_1} \right)^s \frac{\Gamma(k-s)}{\Gamma(k-s/2)} E\left( \frac{3}{2} + w + s, d_1^2, c \right) dw,
\]
\[
M_2 = x^k \sum_{d_1} \frac{1}{d_1^2} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{L^2_{\infty}(1/2 + w)}{L^2_{\infty}(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^w}
\]
\[
\times \int (-7) \left( \frac{c}{2\pi d_1} \right)^s \frac{\Gamma(k-s)}{\Gamma(k-s/2)} E\left( 3/2 + w + s, d_1^2, c \right) ds dw,
\]
\[
M'_2 = 2\sqrt{2\pi} x^k \sum_{d_1} \frac{1}{d_1^{3/2}} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{L^2_{\infty}(1/2 + w)}{L^2_{\infty}(1/2)} K(w) \zeta^3(1 + 2w) \left( \frac{2\pi}{cd_1} \right)^w
\]
\[
\times \sum_{n=1}^{\infty} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c) \sum_{m|n} e(m\alpha/c + \frac{n\beta}{m})
\]
\[
\times \int (9/2) \left( \frac{2\pi n}{cd_1} \right)^{-z} \frac{\Gamma\left( \frac{k-w-z-1/2}{2} \right)}{\Gamma\left( \frac{k+w+z+1/2}{2} \right)} \Gamma^2(z) dz dw,
\]
\[ 24 \]
\[ M_2'' = -4\pi^{3/2}i^k \sum_{d_1} \frac{1}{d_1^{3/2}} \sum_{c=1}^\infty \frac{1}{c^{5/2}} \left( \frac{1}{2\pi i} \right)^2 \int_2^{(2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) \left( \frac{4\pi}{cd_1} \right)^w \]

\[ \times \sum_{n=1}^\infty \sum_{\alpha=1}^c \sum_{\beta=1}^c S(\alpha^2 \beta^2 d_1^2, 1; c) \sum_{m|n} e(m\alpha/c + \frac{n}{m} \beta/c) \]

\[ \times \int_0^{\infty} J_{k-1}(x)x^{-w-1/2}Y_0(\sqrt{2xy})dx \, dw. \]

Furthermore, we have the following estimates

\[ V(1) \ll \log^3 k, M_1 \ll 1, M_2' \ll 1. \]
CHAPTER 2
A LOCAL LARGE SIEVE INEQUALITY FOR THE MASS CUSP FORM

In this chapter, we generalize the Iwaniec’s spectral large sieve estimates of Maass cusp form to the local version for all congruence groups $\Gamma_0(q)$. The exceptional eigenvalues emerge in the course of the proof.

2.1 Introduction

Let $S(\Gamma_0(q))$ be the infinite dimensional space of Maass cusp forms of level $q$, and $\{u_j(z)\}_{j \geq 1}$ be an orthonormal Hecke basis of $S(\Gamma_0(q))$ consisting of common eigenfunctions for all Hecke operators $T_n$ with $(n,q) = 1$. Thus $u_j(z)$ is an eigenfunction of the hyperbolic Laplace-Beltrami operator $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ with eigenvalue $\lambda_j = s_j(1-s_j)$, $s_j = \frac{1}{2} + it_j$ for each $j$, and it has the Fourier expansion

$$u_j(z) = y^{1/2} \sum_{n \neq 0} \rho_j(n) K_{\frac{1}{2} + it_j} (2\pi|n|y) e(nx).$$

Remark 2.1.1. Either $t_j$ is real and positive, or $0 \leq it_j \leq \frac{1}{4}$ since $\frac{1}{2} \leq s_j < 1$ and we have the well-known result $\lambda_j \geq \frac{3}{16}$ due to Selberg[41]. There can be only finitely many $\lambda_j$ for the latter case, which are called exceptional eigenvalues. Selberg conjectured that they do not exist for congruence groups $\Gamma_0(q)$. And the conjecture is
known to be true for groups of small level \( q \). Thus in either cases, we have \( \cosh(\pi t_j) > 0 \).

For \( f, g \in S(\Gamma_0(q)) \), we define the Petersson inner product

\[
<f, g> = \frac{1}{\text{vol}(X_0(q))} \int_{X_0(q)} f(z) \overline{g(z)} \frac{dx dy}{y^2},
\]

where

\[
X_0(q) = \Gamma_0(q) \backslash \mathbb{H},
\]

and

\[
\text{vol}(X_0(q)) = \frac{\pi}{3} q \prod_{p/q} (1 + p^{-1}).
\]

We define

\[
\nu_j(n) = \rho_j(n) \Gamma(s_j)
\]

Modern spectral methods in analytic number theory play an important role in leading to spectral sums of linear forms involving Fourier coefficients of Mass forms. H. Iwaniec and J. Deshouillers were the first to obtain spectral large sieve estimates using Kuznetsov's trace formula. Deshouillers and Iwaniec [9], Theorem 2] showed the following large sieve inequality:

**Theorem 2.1.2.** Let \( T \geq 1, N \geq \frac{1}{2} \) and \( \epsilon > 0 \) be real numbers, \( \{a_n\}_{n \geq 1} \) a sequence of complex numbers and \( \alpha \) a cusp of \( \Gamma_0(q) \). Then we have

\[
\sum_{|t_j| \leq T} \frac{1}{\cosh \pi t_j} \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \ll (N + qT^2)(qNT)^\epsilon \sum_{N < n \leq 2N} |a_n|^2.
\]

\[
\sum_{\alpha} \int_{-T}^T \left| \sum_{N < n \leq 2N} a_n n^{ir} \varphi_{\alpha \infty}(\frac{1}{2} + ir) \right|^2 dr \ll (N + qT^2)(qNT)^\epsilon \sum_{N < n \leq 2N} |a_n|^2
\]

where

\[
\varphi_{\alpha \infty}(s) = \sum_{c>0} c^{-2s} S_{\alpha \infty}(0,n;c).
\]
A "local" variant for short interval proved by W. Luo [[33], Theorem 2] is given below when $q = 1$:

**Theorem 2.1.3.**

$$\sum_{T < t_j \leq T + 1} \left| \sum_{N < n \leq 2N} a_n \nu_j(n) \right|^2 \ll (N + T)(NT)^\epsilon \sum_{N < n \leq 2N} |a_n|^2.$$  

Later, Jutia [23] and Motohashi [37] formulated and proved the following large sieve inequality for any interval when $q = 1$:

**Theorem 2.1.4.**

$$\sum_{T < t_j \leq T + A} \left| \sum_{N < n \leq 2N} a_n \nu_j(n) \right|^2 \ll (N + AT)(NT)^\epsilon \sum_{N < n \leq 2N} |a_n|^2.$$  

In the applications of Kuznietsov's formula for bounding sums of Kloosterman sums, it will turn out that the contribution of the exceptional eigenvalues $\lambda_j$'s is dominating in some cases, and more over, the smaller $\lambda_j$ is, the more important its contribution is; the following weighted large sieve inequality is used to limit this effect.

**Theorem 2.1.5** ([9], Theorem 4). For any $X \geq 1$ and $\epsilon > 0$, we have

$$\sum_{\lambda_j - \text{except}} X^{2i\lambda_j} \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \ll q(1 + \sqrt{NX/q})(1 + \sqrt{N^{1+\epsilon}/q}) \sum_{N < n \leq 2N} |a_n|^2.$$  

The following lemma is crucial in the proof of our main theorem in this chapter:

**Lemma 2.1.6** ([33], Proposition). For any $r \geq 1$ and $c \leq N^{1-\epsilon}$, we have

$$\sum_{N < m, n \leq 2N} a_n b_m S(n, m; c) J_{i\pi}(\frac{4\pi \sqrt{nm}}{c}) \ll c \cosh(\pi r/2) \|a\| \|b\|,$$

where $a = \{a_n\}_{N}^{2N}, b = \{b_n\}_{N}^{2N}$ are two sequences of complex numbers, $\| \cdot \|$ stands for the $l^2$-norm, and the constant implied in $\ll$ depends upon $\epsilon$ alone.
2.2 Statement of Theorem

Theorem 2.2.1. Assume \( 1 \leq A \leq T \), then we have

\[
\sum_{T < t_j \leq T + A} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 + \text{vol}(X_0(q)) \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{1+T}{A}\right)^2} + e^{-\left(\frac{1+T}{A}\right)^2}}{\cosh(\pi t)} \left| \sum_{N < n \leq 2N} a_n \tau_\alpha(n,t) \right|^2 \, dt \\
\ll (N + qAT)(qNT)^\epsilon \sum_{N < n \leq 2N} |a_n|^2,
\]

where \( \{\alpha\} \) run over all inequivalent cusps of the congruence group \( \Gamma_0(q) \). The inequality is uniform in \( q \).

2.3 Proof of Theorem

Proof. We assume \( N^\epsilon \ll T \), and \( A \ll T^{1-\epsilon} \). Otherwise the theorem is trivial according to Theorem [2.1.1]. We define

\[
h(t) = e^{-\left(\frac{1+T}{A}\right)^2} + e^{-\left(\frac{1+T}{A}\right)^2}.
\]

Then we have
\[
\sum_{T < t_j \leq T + A} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \\
\ll \sum_{t_j \in \mathbb{R}} \left( e^{-\frac{(t_j - T)^2}{A}} + e^{-\frac{(t_j + T)^2}{A}} \right) \cosh(\pi t_j) \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \\
= \sum_{\text{all } t_j} \left( e^{-\frac{(t_j - T)^2}{A}} + e^{-\frac{(t_j + T)^2}{A}} \right) \cosh(\pi t_j) \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \\
- \sum_{t_j - \text{exceptional}} \left( e^{-\frac{(t_j - T)^2}{A}} + e^{-\frac{(t_j + T)^2}{A}} \right) \cosh(\pi t_j) \left| \sum_{N < n \leq 2N} a_n \rho_j(n) \right|^2 \\
= \sum_{N < n_1 \leq 2N} \sum_{N < n_2 \leq 2N} a_{n_1} a_{n_2} \sum_{\text{all } t_j} h(t_j) \frac{\rho_j(n_1) \rho_j(n_2)}{\cosh(\pi t_j)} \\
+ O \left( (N + q) N^\epsilon \sum_{N < n \leq 2N} |a_n|^2 \right)
\]

**Remark 2.3.1.** The error term comes from Theorem [2.1.5], when \( X = 1 \).

Next we apply Kuznetsov’s formula

\[
\frac{1}{\text{vol}(X_0(q))} \sum_{t_j} \frac{h(t_j) \rho_j(n_1) \rho_j(n_2)}{\cosh(\pi t_j)} + \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(t)}{\cosh(\pi t)} \tau_\alpha(n_1, t) \tau_\alpha(n_2, t) dt \\
= \delta_{n_1, n_2} \frac{1}{\pi^2} \int_{-\infty}^{\infty} t \tanh(\pi t) h(t) dt \\
+ \sum_{c \equiv 0 \pmod{q}} c \cdot S(n_1, n_2; c) \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2it} \left( \frac{4\pi \sqrt{n_1 n_2}}{c} \right) \frac{\tanh(t)}{\cosh(\pi t)} dt.
\]
and since
\[
\sum_{N<n_1<2N} \sum_{N<n_2<2N} a_{n_1} \overline{a_{n_2}} \int_{-\infty}^{\infty} \frac{h(t)}{\cosh(\pi t)} \tau_{\alpha}(n_1, t) \overline{\tau_{\alpha}(n_2, t)} dt
\]
\[
= \int_{0}^{\infty} \frac{h(t)}{\cosh(\pi t)} \left| \sum_{N<n\leq 2N} a_n \tau_{\alpha}(n, t) \right|^2 dt
\]
which is always non-negative, by positivity we have
\[
\sum_{T<t_j\leq T+A} \left| \sum_{N<n\leq 2N} a_n \nu_j(n) \right|^2
\]
\[
+ \text{vol}(X_0(q)) \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(t)}{\cosh(\pi t)} \left| \sum_{N<n\leq 2N} a_n \tau_{\alpha}(n, t) \right|^2 dt
\]
\[
\ll \sum_{n_1} \sum_{n_2} a_{n_1} \overline{a_{n_2}} \left( \sum_{t_j} h(t_j) \rho_j(n_1) \rho_j(n_2) \frac{1}{\cosh(\pi t_j)} \right)
\]
\[
+ \text{vol}(X_0(q)) \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(t)}{\cosh(\pi t)} \tau_{\alpha}(n_1, t) \overline{\tau_{\alpha}(n_2, t)} dt
\]
\[
= \text{vol}(X_0(q)) \sum_{n_1} \sum_{n_2} a_{n_1} \overline{a_{n_2}} (Q_1 + Q_2),
\]
where
\[
Q_1 = \delta_{n_1, n_2} \frac{1}{\pi^2} \int_{-\infty}^{\infty} t \tanh(\pi t) h(t) dt,
\]
\[
Q_2 = \sum_{c \equiv 0 (\mod q)} c^{-1} S(n_1, n_2; c) \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2i} \left( \frac{4\pi \sqrt{n_1 n_2}}{c} \right) \frac{th(t)}{\cosh(\pi t)} dt.
\]
It is easy to see that
\[
\int_{-\infty}^{\infty} t \tanh(\pi t) h(t) dt \ll \int_{-T-A \log T}^{T+A \log T} t dt + \int_{-T-A \log T}^{-T-A \log T} t dt \ll AT^{1+\epsilon}.
\]
Therefore the diagonal term \(Q_1\) contributes at most \(O(qAT(q)\epsilon \sum_{N<n\leq 2N} |a_n|^2)\).
It remains to estimate the sum of Kloosterman sums in the off-diagonal term $Q_2$ which we shall do for each modulus $c$ separately. And we write $Q_2$ in the following way:

$$Q_2 = \sum_{c=1}^{\infty} (qc)^{-1} S(n_1, n_2; qc) \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2it} \left( \frac{4\pi \sqrt{n_1 n_2}}{qc} \right) \frac{th(t)}{\cosh(\pi t)} dt.$$

1. If $c \geq \frac{NT^{\epsilon-1}}{q}$, we consider $t$ to be a complex variable $x + yi$, and evaluate the integral transform of Bessel’s function

$$\int_{-\infty}^{\infty} J_{2it} \left( \frac{4\pi \sqrt{n_1 n_2}}{qc} \right) \frac{th(t)}{\cosh(\pi t)} dt$$

by moving the line of integration to $\Im(t) = -K$, where $K$ is a sufficiently large integer, depending only upon $\epsilon$.

We have the Poisson’s Bessel function formula \[22\]

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1/2) \Gamma(1/2)} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta$$

for $\Re \nu > -1/2$,

thus when $t = x - Ki$, $2it = 2K + 2xi$, by Stirling formula for $\Gamma(s)$, we have

$$\left| J_{2it} \left( \frac{4\pi \sqrt{n_1 n_2}}{qc} \right) \right| \ll \left( \frac{\pi N}{xqc} \right)^{2K} e^{\pi |x|}$$

for $|x| > 1$.

This leads to

$$\begin{align*}
\int_{\Im(t)=-K} J_{2it} \left( \frac{4\pi \sqrt{n_1 n_2}}{qc} \right) \frac{th(t)}{\cosh(\pi t)} dt & \ll \int_{T-A \log T}^{T+A \log T} \left( \frac{\pi N}{xqc} \right)^{2K} \sqrt{x^2 + K^2 e^{\frac{K^2}{\pi x}}} dx \\
& + \int_{-T-A \log T}^{-T+A \log T} \left( \frac{\pi N}{xqc} \right)^{2K} \sqrt{x^2 + K^2 e^{\frac{K^2}{\pi x}}} dx \\
& \ll \left( \frac{\pi N}{qTc} \right)^{2K} \sqrt{T^2 + K^2 e^{\frac{K^2}{\pi T}}} AT^\epsilon.
\end{align*}$$
When K is large enough, the sum of $c$ is convergent. Since $\frac{N}{qTC} \leq T^{-\epsilon}$, we see the resulting integral can be made as small as wish. On the other hand, the residues of the integrand are also very small at the simple poles which are of the form $(k + 1/2)i, k \in \mathbb{Z}$.

2. If $c < \frac{NT^{\epsilon-1}}{qA}$, which implies $N \geq T^{1-\epsilon}$. We still truncate the integral as before,

$$\int_{-\infty}^{\infty} J_{2it}(\frac{4\pi \sqrt{n_1n_2}}{qc}) \frac{t \cosh(\pi t)}{\cosh(\pi t)} \, dt$$

$$= \int_{T-A\log T}^{T+A\log T} J_{2it}(\frac{4\pi \sqrt{n_1n_2}}{qc}) \frac{t}{\cosh(\pi t)} \, dt$$

$$+ \int_{-T-A\log T}^{-T+A\log T} J_{2it}(\frac{4\pi \sqrt{n_1n_2}}{qc}) \frac{t}{\cosh(\pi t)} \, dt + O(e^{-\pi T}).$$

We then apply Lemma[2.1.6] to get

$$\sum_{c < \frac{NT^{\epsilon-1}}{qA}} \frac{1}{qc} \sum_{n_1} \sum_{n_2} a_{n_1} \overline{a_{n_2}} S(n_1, n_2; qc) J_{2it}(\frac{4\pi \sqrt{n_1n_2}}{qc})$$

$$\ll \frac{NT^{\epsilon-1}}{qA} \cosh(\pi t) \sum_{N < n \leq 2N} |a_n|^2.$$

Then integrating in $t$ in the truncated interval, we have

$$\frac{NT^{\epsilon-1}}{qA} \int_{T-A\log T}^{T+A\log T} t \cosh(\pi t) \cosh(\pi t) \, dt \ll \frac{NT^{\epsilon-1}}{qA} TAT^\epsilon \ll \frac{NT^\epsilon}{q}.$$

This will contribute at most $O(N(qT)^\epsilon \sum_{N < n \leq 2N} |a_n|^2)$.

**Remark 2.3.2.** If $A = 1$, then we have already completed the proof. Therefore we shall assume $A \geq T^\epsilon$ for the last part of the proof.
3. If \( \frac{NT^{\epsilon-1}}{qA} \leq c < \frac{NT^{\epsilon-1}}{q} \), we shall make use of the following asymptotic expansion for \( J_{ir}(x) \) when \( x, r > 0 \):

\[
\pi \sqrt{2} (r^2 + x^2)^{1/4} J_{ir}(x) = e^{\pi r/2 - \pi i/4 + i \omega_r(x)} \left( \sum_{m=0}^{M-1} t_m (r^2 + x^2)^{-m/2} + O \left( (r^2 + x^2)^{-M/2} \right) \right),
\]

where

\[
\omega_r(x) = \sqrt{r^2 + x^2} + r \log \frac{\sqrt{r^2 + x^2} - r}{x}
\]

and \( t_m \) are constants. The implied constant in \( O \)-symbol depends on \( M \) only.

When \( r < 0 \), the asymptotic expansion is completely analogous since we have

\( J_{ir}(x) = J_{-ir}(x) \).

It is easy to calculate that

\[
\frac{d(\omega_r)}{dr} = \log \left( \frac{x}{\sqrt{r^2 + x^2} + r} \right).
\]

In the following discussion, we shall consider the function

\[
f_r(x) = (r^2 + x^2)^{-1/4} e^{\pi r/2 + i \omega_r(x)}
\]

instead of \( J_{ir}(x) \), and

\[
\int_0^\infty f_{2i} \left( \frac{4\pi \sqrt{n_1 n_2}}{qc} \right) \frac{th(t)}{\cosh(\pi t)} dt
\]

instead of

\[
\int_{-\infty}^\infty J_{2it} \left( \frac{4\pi \sqrt{n_1 n_2}}{qc} \right) \frac{th(t)}{\cosh(\pi t)} dt.
\]

**Remark 2.3.3.** \( f_r(x) \) is essentially the \( m = 0 \) term in the asymptotic expansion of \( J_{ir}(x) \) above and we can similarly discuss the other terms.

**Remark 2.3.4.** The error term is also negligible since

\[
O \left( \int_0^\infty e^{\pi t} \left( t^2 + \left( \frac{N}{qc} \right)^2 \right)^{-1/4-M/2} \frac{th(t)}{\cosh(\pi t)} dt \right) \ll T^{-M+2}
\]
can be made as small as wish, where $M$ depends on $\epsilon$ alone. We shall only use the lower bound $\frac{NT^{\epsilon-1}}{qA} \leq c$ in the following proof, but we still need the upper bound $c < \frac{NT^{\epsilon-1}}{q}$ to estimate this error term.

We start with

\[
\int_0^\infty f_2(t) \left( \frac{4\pi \sqrt{n_1n_2}}{qc} \right) \frac{\text{th}(t)}{\cosh(\pi t)} dt
\]

\[
= \int_0^\infty \left( 4t^2 + \left( \frac{4\pi}{qc} \right)^2 n_1n_2 \right)^{-1/4} e^{i\omega_2(\frac{4\pi \sqrt{n_1n_2}}{qc})} \text{th}(t) e^{\pi t} dt
\]

\[
= \int_0^\infty \left( 4t^2 + \left( \frac{4\pi}{qc} \right)^2 n_1n_2 \right)^{-1/4} e^{i\omega_2(\frac{4\pi \sqrt{n_1n_2}}{qc})} \text{th}(t) dt + O(e^{-\pi T}),
\]

and then we have

\[
\int_0^\infty \left( 4t^2 + \left( \frac{4\pi}{qc} \right)^2 n_1n_2 \right)^{-1/4} e^{i\omega_2(\frac{4\pi \sqrt{n_1n_2}}{qc})} \text{th}(t) dt
\]

\[
= \int_0^\infty \left( 4t^2 + \left( \frac{4\pi}{qc} \right)^2 n_1n_2 \right)^{-1/4} \frac{(e^{i\omega_2(\frac{4\pi \sqrt{n_1n_2}}{qc})})'}{2i \log \left( \frac{\frac{4\pi \sqrt{n_1n_2}}{qc}}{\sqrt{\left( \frac{4\pi \sqrt{n_1n_2}}{qc} \right)^2 + 4t^2 + 2t}} \right)} \text{th}(t) dt.
\]

We shall use integration by parts to estimate the above integral.

It is easy to get

\[
\left( \log\left( \frac{x}{\sqrt{x^2 + 1} + 1} \right) \right)' = \frac{1}{x \sqrt{x^2 + 1}}.
\]

And the function

\[
\left| \log\left( \frac{x}{\sqrt{x^2 + 1} + 1} \right) \right|^{-1}
\]

is strictly increasing on $[0, \infty)$.

We also have

\[
T^{-\epsilon} < \frac{N}{qcT} \leq AT^{-\epsilon},
\]

since $\frac{NT^{\epsilon-1}}{qA} \leq c < \frac{NT^{\epsilon-1}}{q}$.

Let

\[
f_1(t) = t(4t^2 + \left( \frac{4\pi}{qc} \right)^2 n_1n_2)^{-1/4},
\]

\[
f_2(t) = h(t) = e^{-(\frac{t-A}{A})^2} + e^{-(\frac{t+A}{A})^2},
\]

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\[ f_3(t) = \log^{-1}\left( \frac{\frac{4\pi\sqrt{n_1n_2}}{qc}}{\sqrt{\left(\frac{4\pi\sqrt{n_1n_2}}{qc}\right)^2 + 4t^2 + 2t}} \right), \]

then by partial integration once, we have

\[
\int_0^\infty f_1(t)f_2(t)f_3(t)(e^{i\omega_2t(\frac{4\pi\sqrt{n_1n_2}}{qc})})' \, dt = \int_0^\infty (f_1(t)f_2(t)f_3(t))'(e^{i\omega_2t(\frac{4\pi\sqrt{n_1n_2}}{qc})}) \, dt.
\]

We write \( A \triangle B \) if \( A = \sum_{n=1}^N a_nC_n, \ B = \sum_{n=1}^N b_nC_n \) are two finite sums with only possibly different positive coefficients \( \{a_n\} \) and \( \{b_n\} \).

Using the above notation, we have

\[ (f_1f_2f_3)'f_3' \triangle (f_1f_2f_3')'' \]

In general, if we apply integration by parts \( p \) times, we just need to consider the integral which has the form

\[
\int_{-\infty}^{\infty} f_1(t)^{(p_1)}f_2(t)^{(p_2)}(f_3(t)^{(p_3)}e^{i\omega_2t(\frac{4\pi\sqrt{n_1n_2}}{qc})}) \, dt,
\]

where \( p_1, p_2, p_3 \) are nonnegative integers and \( p_1 + p_2 + p_3 = p \).

In the truncated interval \((T - A \log T, T + A \log T)\), it is easy to get

\[ f_1(t)^{(p_1)} = O(T^{1/2-p_1}), \]

\[ f_2(t)^{(p_2)} = O(A^{-p_2}), \]

To calculate \( f_3 \), we need to consider two cases:

1. when \( \frac{N}{qcT} \gg 1 \), say \( \frac{N}{qcT} \geq 10^5 \), we have the following estimate from Tayler expansion

\[
\log \left( \frac{\frac{N}{qcT}}{\sqrt{\left(\frac{N}{qcT}\right)^2 + 1 + 1}} \right) \sim \frac{qcT}{N}.
\]
We have
\[
(f_3(t)^p)^{(p_3)} \triangleq \sum_{r_1 + \cdots + r_p = p_3} f_3^{(r_1)} \cdots f_3^{(r_p)}
\]
and
\[
f_3(t)^{(r)} \ll ((\frac{N}{qcT})T^{-r}).
\]
Therefore
\[
(f_3(t)^p)^{(p_3)} \ll (\frac{N}{qcT})^{pT^{-p_3}}.
\]
Thus
\[
\begin{align*}
 f_1(t)^{(p_1)} f_2(t)^{(p_2)} (f_3(t)^p)^{(p_3)} & \ll T^{1/2-p_1} A^{-p_2} (\frac{N}{qcT})^{pT^{-p_3}} \\
 & \ll T^{1/2-p_1} A^{-p_2} (AT^\epsilon)^{pT^{-p_3}} \\
 & \ll T^{1/2-\epsilon} (\frac{A}{T})^{p_1+p_3} \\
 & \ll T^{1/2-\epsilon}
\end{align*}
\]
When $p$ is large enough, it is easy to see the integral
\[
\int_{-\infty}^{\infty} f_1(t)^{(p_1)} f_2(t)^{(p_2)} (f_3(t)^p)^{(p_3)} e^{i\omega_2 (\frac{4\pi}{\sqrt{\frac{N}{qcT}^2 + 1 + 1}})} dt,
\]
can be made as small as we wish.

2. If $\frac{N}{qcT} \leq 10^5$, then there exists a positive number $M_1$, such that
\[
|f_3| = \left| \log \left( \frac{\frac{N}{qcT}}{\sqrt{(\frac{N}{qcT})^2 + 1 + 1}} \right) \right|^{-1} \leq M_1.
\]

It is easy to get
\[
f_3(t)^{(r)} \ll \frac{M_1^{r+1}}{T^r}
\]
and
\[
(f_3(t)^p)^{(p_3)} \ll \frac{M_1^{p+p_3}}{T^{p_3}}.
\]
Then we have
\[ f_1(t)^{(p_1)} f_2(t)^{(p_2)} (f_3(t)^p)^{(p_3)} \]
\[ \ll T^{1/2-p_1} A^{-p_2} \left( \frac{M_1^{p+p_3}}{T^{p_3}} \right) \]
\[ \ll T^{1/2-p_1} T^{-p_2} T^{-p_3} M_1^{2p} \]
\[ \ll T^{1/2-p_1-p_2-p_3} M_1^{2p}. \]

When \( p \) is large enough, Again the resulting integral
\[
\int_{-\infty}^{\infty} f_1(t)^{(p_1)} f_2(t)^{(p_2)} (f_3(t)^p)^{(p_3)} e^{i\omega_2 t \left( \frac{4\pi n_1}{c} \right)} dt,
\]
can be made as small as we wish, since we assume \( T \gg N^\epsilon \). Now we have completed the proof.
CHAPTER 3

$L^4$-NORMS OF THE DIHEDRAL MAASS FORMS

The $L^4$-norms of automorphic forms have a remarkable connection with central values of the triple product L-functions by Watson’s formula\cite{45}. In this chapter, we prove the optimal bound for $L^4$-norms of the dihedral Maass forms associated to Hecke’s grossencharacters of a fixed real quadratic field in the spectral aspect.

3.1 Introduction

Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D > 0$, fundamental unit $\epsilon_D > 0$, and the ring of integers $\mathcal{O}_K$. We assume $D > 8$ is a fixed large prime. For integer $k \neq 0$, we consider the Hecke’s grossencharacter $\Psi_k$ of $K$ defined by

$$
\Psi_k((\alpha)) = \left| \frac{\alpha}{\alpha'} \right|^{\frac{\pi ik}{\log(\epsilon_D)}}
$$

for principle ideal $(\alpha) \subset \mathcal{O}_K$, with generator $\alpha$, where $\alpha'$ is the conjugate of $\alpha$ under the nontrivial automorphism of $K$.

Consider

$$
\Phi_k(z) = y^{1/2} \sum_{\emptyset \neq \beta \subset \mathcal{O}_K} \Psi_k(\beta) K_{\frac{\pi ik}{\log(\epsilon_D)}}(2\pi N(\beta)y)e(N(\beta)x),
$$

where $N(\beta)$ is the norm of the ideal $\beta$, $K_\nu(z)$ is the modified Bessel function, and $e(\xi) = e^{2\pi i \xi}$. 

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Then \( \Phi_k(z) \) is a Hecke-Maass cusp form on \( \Gamma_0(D) \), eigenvalue \( \frac{1}{4} + (\frac{k\pi}{\log \epsilon_D})^2 \), and with nebentypus character the Kronecker symbol \( \chi_D \).

We rewrite \( \Phi_k(z) \) as below:

\[
\Phi_k(z) = y^{1/2} \sum_{n \neq 0} a_{n,k} K_{iT_k}(2\pi ny)e(nx),
\]

where

\[
a_{n,k} = \sum_{N_K(\beta) = n} \Psi_k(\beta),
\]

and \( T_k = \frac{k\pi}{\log \epsilon_D} \).

**Remark 3.1.1.** \( \Phi_k(z) \) has real coefficients and is either odd or even depending on whether \( a_{n,k} = \pm a_{-n,k} \).

We also define the Petersson inner product on \( \Gamma_0(D) \) as below:

\[
\langle f, g \rangle_D = \int_{\Gamma_0(D) \backslash \mathbb{H}} f(z) \overline{g(z)} \frac{dx \, dy}{y^2}
\]

Our main goal is to prove the following theorem:

**Theorem 3.1.2.** With the notion as given above, we have

\[
\frac{\|\Phi_k\|_4}{\|\Phi_k\|_2} \ll \epsilon T_k^\epsilon,
\]

for any \( \epsilon > 0 \).

**Remark 3.1.3.** Sarnak and Watson [40] established the following optimal bound for the \( L^4 \)-norms:

If \( f \) is an Hecke-Maass cusp form on \( SL(2, \mathbb{Z}) \) with Laplacian eigenvalue \( \lambda_f = 1/4 + t_f^2 \), then

\[
\frac{\|f\|_4}{\|f\|_2} = \left( \frac{\int_{\Gamma(1) \backslash \mathbb{H}} |f(z)|^4 \, dx \, dy}{\int_{\Gamma(1) \backslash \mathbb{H}} |f(z)|^2 \, dx \, dy} \right)^{1/4} \ll \epsilon (|t_f| + 1)^\epsilon,
\]

for any \( \epsilon > 0 \).
for any \( \epsilon > 0 \).

They are the first to observe the remarkable connection of the \( L^4 \)-norm of \( f \) with central \( L \)-values of the triple product \( L \)-functions, via the Plancherel theorem and Watson formula \([45]\). This observation becomes the starting point to study the \( L^4 \)-norm of Maass forms by means of analytic theory of automorphic \( L \)-functions.

3.2 Spectral decomposition

The spectrum of \( L^2(\Gamma_0(D)\backslash \mathbb{H}) \) consists of the constant function, Maass forms and Eisenstein series \( E_0(\cdot, \frac{1}{2} + it), E_\infty(\cdot, \frac{1}{2} + it) \) for \( t \in \mathbb{R} \), corresponding to the two cups \( \alpha = 0, \infty \), since in our case \( D \) is a prime. For any Maass form \( g \) we denote by

\[
t_g = \sqrt{\lambda_g - \frac{1}{4}} \in \mathcal{T} := \mathbb{R} \cup \left(-\frac{1}{2}, \frac{1}{2}\right)i
\]

its spectral parameter.

Let \( \{u_j\}_{j \geq 0} \) be the orthonormal Hecke-Maass basis with trivial nebentypus for the discrete spectrum in \( L^2(\Gamma_0(D)\backslash \mathbb{H}) \) such that

\[
<u_j, u_k> = \int_{\Gamma_0(D)\backslash \mathbb{H}} u_j(z) \overline{u_k(z)} \frac{dx \, dy}{y^2} = \delta_{jk},
\]

and

\[
u_0(z) = \frac{1}{\sqrt{\text{vol}(\Gamma_0(D)\backslash \mathbb{H})}}.
\]

By the spectral decomposition of \( \Phi_k^2(z) \) in \( L^2(\Gamma_0(d)\backslash \mathbb{H}) \) and the Parseval identity,

\[
\|\Phi_k\|_4^4 = \int_{\Gamma_0(D)\backslash \mathbb{H}} |\Phi_k(z)|^4 \frac{dx \, dy}{y^2} = <\Phi_k^2, \Phi_k^2>_D
\]

\[
= |<\Phi_k^2, u_0>_D|^2 + \sum_{j \geq 1} |<\Phi_k^2, u_j>_D|^2 + \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} |<\Phi_k^2, E_\alpha(\cdot, 1/2 + it)>_D|^2 dt,
\]

where

\[
E_\alpha(z, s) = \frac{1}{2} \sum_{\gamma \in \Gamma_0 \setminus \Gamma_0(D)} \mathbb{I}^s(\sigma_\alpha^{-1}\gamma z),
\]
is the Eisenstein series associated to the cusp $\alpha$ as $\alpha$ ranges over a complete set of inequivalent cusps for $\Gamma_0(D)$, which are $0, \infty$ in our case; $\Gamma_\alpha \subset \Gamma_0(D)$ is the stability group of the cusp $\alpha$, and $\sigma_\alpha \in \text{SL}(2, \mathbb{R})$ is the scaling matrix such that $\sigma_\alpha(\infty) = \alpha$ and $\sigma_\alpha^{-1}\Gamma_\alpha\sigma_\alpha = \Gamma_\infty$, the translation subgroup in $\text{SL}(2, \mathbb{Z})$.

Then we have

$$\frac{\|\Phi_k\|^4}{\|\Phi_k\|^2} \ll 1 + \sum_{j \geq 1} \frac{<\Phi_k^2, u_j >_D}{<\Phi_k, \Phi_k >_D^2} + \sum_{\alpha} \int_{-\infty}^{\infty} \frac{|<\Phi_k^2, E_\alpha(\cdot, 1/2 + it) >_D|^2}{<\Phi_k, \Phi_k >_D^2} dt.$$ 

We define the Fricke involution:

$$Wf = f(\sigma_0 z), \quad \text{where } \sigma_0 = \begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}.$$ 

Then we have $W\Phi_k = \eta_{\Phi_k} \Phi_k$ with $|\eta_{\Phi_k}| = 1$. Since $\sigma_0 \in \text{GL}_2^+(\mathbb{Z})$, the Fricke involution is an isometry. It follows

$$|<\Phi_k^2(z), E_0(z, s) >_D| = |<\Phi_k^2(\sigma_0 z), E_0(\sigma_0 z, s) >_D| = |<\Phi_k^2(z), E_\infty(z, s) >_D|$$

Therefore, we deduce that

$$\frac{\|\Phi_k\|^4}{\|\Phi_k\|^2} \ll 1 + \sum_{j \geq 1} \frac{|<\Phi_k^2, u_j >_D|^2}{<\Phi_k, \Phi_k >_D^2} + \int_{-\infty}^{\infty} \frac{|<\Phi_k^2, E_\infty(\cdot, 1/2 + it) >_D|^2}{<\Phi_k, \Phi_k >_D^2} dt.$$
3.3 Treatment of the continuous spectrum part

By unfolding method, we obtain

\[
< \Phi_k^2, E_\infty(\cdot, s) >_D = \int_{\Gamma_0(D) \backslash \mathbb{H}} \Phi_k^2(z) E_\infty(z, s) \frac{dx \, dy}{y^2}
\]

\[
= \int_{\Gamma_\infty \backslash \mathbb{H}} \Phi_k^2(z) y^s \frac{dx \, dy}{y^2}
\]

\[
= \int_0^\infty \int_0^1 \Phi_k^2(z) y^s \frac{dx \, dy}{y^2}
\]

\[
= \sum_{n \neq 0} a_{n,k}^2 (2\pi |n|)^s \int_0^\infty K_{iT_k}(y) y^{s-1} \, dy
\]

\[
= \frac{\Gamma^2(s) \Gamma\left(\frac{s+2iT_k}{2}\right) \Gamma\left(\frac{s-2iT_k}{2}\right)}{8\pi^s \Gamma(s)} \sum_{n \neq 0} \frac{a_{n,k}^2}{|n|^s}
\]

where

\[
\Phi_k(z) = y^{1/2} \sum_{n \neq 0} a_{n,k} K_{iT_k}(2\pi ny) e(nx),
\]

and we use a well-known formula

\[
\int_0^\infty K_{iT_k}(y) y^{s-1} \, dy = \frac{2^s \Gamma^2\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2iT_k}{2}\right) \Gamma\left(\frac{s-2iT_k}{2}\right)}{8 \Gamma(s)}.
\]

By Comparing the local Euler factors, we have

\[
L(s, \text{sym}^2(\Phi_k)) = \zeta_D(s) L(s, \Phi_{2k}),
\]

\[
L(s, \text{ad}^2(\Phi_k)) = L(s, \chi_D) L(s, \Phi_{2k}),
\]

where \(\zeta_D(s) = (1 - D^{-s}) \xi(s)\), and \(\Phi_{2k}\) is constructed similarly to \(\Phi_k\) with Hecke’s grossencharacter \(\psi_{2k}\).

Therefore, we have

\[
< \Phi_k^2, E_\infty(\cdot, s) >_D = \frac{\Gamma^2\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2iT_k}{2}\right) \Gamma\left(\frac{s-2iT_k}{2}\right)}{4\pi^s \Gamma(s)} \frac{L(s, \chi_D) \zeta_D(s) L(s, \Phi_{2k})}{L(2s, \chi_D^2)}.
\]

Taking the residue at 1, we have
\[
\frac{1}{\text{Vol}(\Gamma_0(D)\setminus \mathbb{H})} \| \Phi_k \|_2^2 = \frac{\Gamma\left(\frac{1+2iT_k}{2}\right)\Gamma\left(\frac{1-2iT_k}{2}\right)}{4} (1 - D^{-1}) \frac{L(1, \chi_D)L(1, \Phi_{2k})}{L(2, \chi_D^2)}
\]

Thus

\[
\int_{-\infty}^{\infty} \left| < \Phi_k^2, E_a(\cdot, 1/2 + it) >_D \right|^2 dt
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\pi(D+1)^2} \left| L^2(2, \chi_D^2) \right| \left| \Gamma\left(\frac{1/2+it}{2}\right) \right|^4 \left| \Gamma\left(\frac{1/2+it+2iT_k}{2}\right) \right|^2 \left| \Gamma\left(\frac{1/2+it-2iT_k}{2}\right) \right|^2
\]

\[
\cdot \left| L(1/2 + it, \chi_D) \right|^2 |\zeta(1/2 + it)|^2 |L(1/2 + it, \Phi_{2k})|^2 dt
\]

By Stirling’s formula, we have the following estimates for the gamma factor

\[
\frac{\left| \Gamma\left(\frac{1/2+it}{2}\right) \right|^4 \left| \Gamma\left(\frac{1/2+it+2iT_k}{2}\right) \right|^2 \left| \Gamma\left(\frac{1/2+it-2iT_k}{2}\right) \right|^2}{\left| \Gamma\left(\frac{1+2iT_k}{2}\right) \right|^4 |\Gamma(1/2 + it)|^2}
\]

\[
\ll \begin{cases} 
\frac{1}{(t+1)T_k^{1/2}(|t-2T_k|+1)^{1/2}} & \text{if } 0 \leq t \leq 2T_k \text{ or } t_u \in i\mathbb{R} \\
\frac{e^{-\pi/(t-2T_k)}}{tT_k^{1/2}(|t-2T_k|+1)^{1/2}} & \text{if } t > 2T_k 
\end{cases}
\]

Hence

\[
\int_{-\infty}^{\infty} \left| < \Phi_k^2, E_a(\cdot, 1/2 + it) >_D \right|^2 dt
\]

\[
\ll T_k^\epsilon \int_0^{2T_k+10 \log T_k} \frac{|L(1/2 + it, \chi_D)|^2 |\zeta(1/2 + it)|^2 |L(1/2 + it, \Phi_{2k})|^2}{(t+1)T_k^{1/2}(|t-2T_k|+1)^{1/2}} dt
\]

where we have the fact that \( L(1, \Phi_k) \gg T_K^{-\epsilon} \) since \( L(s, \Phi_k) \) has no Siegel zeros[14].

Applying Cauthy’s inequality, we get

\[
\int_0^{2T_k+10 \log T_k} \frac{|L(1/2 + it, \chi_D)|^2 |\zeta(1/2 + it)|^2 |L(1/2 + it, \Phi_{2k})|^2}{(t+1)T_k^{1/2}(|t-2T_k|+1)^{1/2}} dt
\]

\[
\ll \left( \int_0^{3T_k} \frac{|L(1/2 + it, \chi_D)|^4}{t+1} dt \right)^{1/2} \left( \int_0^{3T_k} \frac{|\zeta(1/2 + it)|^4}{t+1} dt \right)^{1/2}
\]

\[
\times \max_{0 \leq t \leq 3T_k} \frac{|L(1/2 + it, \Phi_{2k})|^2}{T_k^{1/2}(|t-2T_k|+1)^{1/2}}
\]

\[
\ll T_k^\epsilon
\]

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by the convexity bound and the fourth moment bounds for $L(s, \chi)$ and $\zeta(s)$.

We deduce that

$$\int_{-\infty}^{\infty} | < \Phi_k^2, E_\infty(\cdot, 1/2 + it) > D |^2 dt \ll T_k^\epsilon$$

### 3.4 Treatment of the discrete spectrum part

The orthonormal Hecke-Maass basis for the cuspidal spectrum in $L^2(\Gamma_0(D) \backslash \mathbb{H})$ is as following:

1. $B_1 = \{ g_j(z) \}_{j \geq 1}$, the orthonormal Hecke-Maass basis for the new form spectrum in $L^2(\Gamma_0(D) \backslash \mathbb{H})$;

2. $B_2 = \frac{f_j(z)}{[\Gamma_0(1) : \Gamma_0(D)]^{1/2}}$, where $B_2' = \{ f_j(z) \}_{j \geq 1}$ is the orthonormal Hecke-Maass basis for the cuspidal spectrum in $L^2(\Gamma_0(1) \backslash \mathbb{H})$;

3. 

$$B_3 = \left\{ \frac{\beta f_j(z) + f_j(Dz)}{\| \beta f_j(z) + f_j(Dz) \|_2} \right\}_{j \geq 1},$$

for $f_j \in B_2'$, where

$$\beta f_j = -\frac{< f_j(Dz), f_j(z) >_D}{< f_j(z), f_j(z) >_D} = \lambda f_j(D) D^{1/2}(D + 1)^{-1}.$$

Here we use the fact \cite{20}, Lemma 2.4:

$$< f_j(Dz), f_j(z) >_D = \lambda f_j(D) D^{1/2} < f_j(Dz), f_j(z) >_1,$$

and

$$< f_j(Dz), f_j(Dz) >_D = (D + 1) < f_j(Dz), f_j(z) >_1.$$
Note that

\[
\| \beta f_j(z) + f_j(Dz) \|_2 = \langle \beta f_j(z) + f_j(Dz), \beta f_j(z) + f_j(Dz) \rangle_D \\
= \left( (D + 1) - \chi_{f_j}^2(D) \frac{D}{D + 1} \right) \langle f_j(Dz), f_j(z) \rangle_1 \\
\gg \langle f_j(Dz), f_j(z) \rangle_1
\]

for \( D \geq 8 \) in view of the bound [25]: \( |\chi_{f_j}^2(D)| \leq 2D \frac{z}{D} \).

Note that \( \Phi_k(-1/(Dz)) = \eta_{\Phi_k} \Phi_k(z) \) with \( |\eta_{\Phi_k}| = 1 \) and \( f(-1/z) = f(z) \) for \( f_j(z) \in B'_2 \). Moreover, if \( F_D \) is a fundamental domain for \( \Gamma_0(D) \), then \( \sigma_0(F_D) \) is a fundamental domain for \( \sigma_0 \Gamma_0(D) \sigma_0^{-1} = \Gamma_0(D) \). Hence,

\[
| \langle \Phi_k^2(z), f_j(Dz) \rangle_D | = | \int_{\Gamma_0(D) \setminus H} \Phi_k^2(z) f_j(Dz) \frac{dx \, dy}{y^2} | \\
= | \int_{\Gamma_0(D) \setminus H} \Phi_k^2(-1/(Dz)) f_j(-1/z) \frac{dx \, dy}{y^2} | \\
= | \int_{\Gamma_0(D) \setminus H} \Phi_k^2(z) f_j(z) \frac{dx \, dy}{y^2} | \\
= | \langle \Phi_k^2(z), f_j(z) \rangle_D |.
\]

Thus

\[
\sum_{j \geq 1} | \langle \Phi_k^2, u_j \rangle_D |^2 = \sum_{u \in B_1 \cup B_2 \cup B_3} | \langle \Phi_k^2, u \rangle_D |^2 \\
\ll \sum_{u \in B_1 \cup B_2} | \langle \Phi_k^2, u \rangle_D |^2
\]

Applying Watson [45] and Ichino [15]'s formula, we have, for \( u \in B_1 \bigcup B_2 \),

\[
\frac{| \langle \Phi_k^2, u \rangle_D |^2}{| \langle \Phi_k^2, \Phi_k^2 \rangle_D |^2} \leq \frac{\Lambda(1/2, \Phi_k \otimes \Phi_k \otimes u)}{\Lambda^2(1, \text{ad}^2(\Phi_k)) \Lambda(1, \text{ad}^2(u))} \\
= \frac{\Lambda(1/2, u)\Lambda(1/2, u \otimes \chi_D)\Lambda(1/2, \Phi_{2k} \otimes u)}{\Lambda^2(1, \Phi_{2k})\Lambda^2(1, \chi_D)\Lambda(1, \text{sym}^2(u))}
\]

where we use the factorization

\[
\Lambda(s, \Phi_k \otimes \Phi_k \otimes u) = \Lambda(s, u)\Lambda(s, u \otimes \chi_D)\Lambda(s, \Phi_{2k} \otimes u).
\]
Thus we have

\[ | < \Phi_k^2, u >_D |^2 \ll \frac{\left| \Gamma \left( \frac{1}{2} + i \frac{t_u + \delta_1}{2} \right) \right|^4 \left| \Gamma \left( \frac{1}{2} + i \frac{(2T_k + t_u) + \delta_1}{2} \right) \right|^2 \left| \Gamma \left( \frac{1}{2} + i \frac{(2T_k - t_u) + \delta_1}{2} \right) \right|^2}{\left| \Gamma \left( \frac{1}{2} + i \frac{t_u - \delta_1}{2} \right) \right|^4 \left| \Gamma \left( \frac{1}{2} + i \frac{(2T_k - t_u) + \delta_1}{2} \right) \right|^2} \]

\times \frac{L(1/2, u) L(1/2, u \otimes \chi_D) L(1/2, \Phi_{2k} \otimes u)}{L^2(1, \Phi_{2k}) L^2(1, \chi_D) L(1, \text{sym}^2(u))}.

where \( \delta_1, \delta_2, \delta_3 \) depends on the parity of \( u \) and \( \Phi_{2k} \).

By Stirling’s formula, we have the following estimates for the gamma factor

\[
\left| \frac{\left| \Gamma \left( \frac{1}{2} + i \frac{t_u + \delta_1}{2} \right) \right|^4 \left| \Gamma \left( \frac{1}{2} + i \frac{(2T_k + t_u) + \delta_1}{2} \right) \right|^2 \left| \Gamma \left( \frac{1}{2} + i \frac{(2T_k - t_u) + \delta_1}{2} \right) \right|^2}{\left| \Gamma \left( \frac{1}{2} + i \frac{t_u - \delta_1}{2} \right) \right|^4 \left| \Gamma \left( \frac{1}{2} + i \frac{(2T_k - t_u) + \delta_1}{2} \right) \right|^2} \right|
\]

\[
\ll \begin{cases} 
1 & \text{if } 0 \leq t \leq 2T_k \text{ or } t_u \in i\mathbb{R} \\
\frac{(|t_u| + 1) T_k^{1/2} (|t_u - 2T_k| + 1)^{1/2}}{e^{-\pi \left( |t_u - 2T_k| + \frac{1}{4} \right)}} & \text{if } t > 2T_k.
\end{cases}
\]

By the standard method of approximate functional equations, for \( T \leq t_j \leq 2T, T \geq 2 \), and for any \( \epsilon > 0 \), we have

\[ L\left( \frac{1}{2}, u_j \right) \ll T^\epsilon \left( \left| \sum_{1 \leq n \leq T^{1+\epsilon}} \lambda_j(n) a(n) n^{-1/2} \right| + 1 \right), \]

\[ L\left( \frac{1}{2}, u_j \otimes \chi_D \right) \ll T^\epsilon \left( \left| \sum_{1 \leq n \leq T^{1+\epsilon}} \lambda_j(n) \chi_D(n) a(n) n^{-1/2} \right| + 1 \right), \]

for certain \( a(n) \) with \( a(n) \leq T^\epsilon \) for \( 1 \leq n \leq T^\epsilon \).

Similarly, we have, for \( 1 \leq A \leq 2T_k + 10 \log T_k, A \leq |t_j - 2T_k| \leq 2A \),

\[ L\left( \frac{1}{2}, \Phi_{2k} \otimes u_j \right) \ll T^\epsilon_k \left( \left| \sum_{1 \leq n \leq T_k^{1+\epsilon} A^{1+\epsilon}} \lambda_j(n) a_{n, 2k} b(n) n^{-1/2} \right| + 1 \right), \]

for certain \( b(n) \ll T_k^\epsilon \) for \( 1 \leq n \leq T_k^{1+\epsilon} A^{1+\epsilon} \).
Hence, by Hölder’s inequality, we deduce that

\[
\sum_{u \in B_1 \cup B_2, 0 \leq t_u \leq 2T_k + 10 \log T_k} \frac{\left| L(1/2, u) L(1/2, u \otimes \chi_D) L(1/2, \Phi_{2k} \otimes u) \right|}{(t_u + 1) T_k^{1/2} (|2T_k - t_u| + 1)^{1/2}} \leq T_k^2
\]

by the above bounds for the central L-values and the lemma, after splitting the spectral range in \( t_u \) to \( A \leq t_u \leq 2A, 1 \leq A \leq 2T_k + 10 \log T_k \) for the first two sums, and to : (1) \( t_u \leq T_k \); (2) \( 2T_k - 1 \leq t_u \leq 2T_k + 10 \log T_k \); and (3) \( A \leq 2T_k - t_u \leq 2A, 1 \leq A \leq T_k/2 \), respectively, for the third sum above.
CHAPTER 4
SECOND MOMENT OF SYMMETRIC SQUARE
\(L\)-FUNCTIONS

In this chapter, we study the family of degree three \(L\)-functions: the symmetric square \(L\)-functions \(L(s, \text{sym}^2(f))\) and obtain a long equation intended to give an upper bound to the second moment of symmetric square \(L\)-functions.

4.1 Introduction

Let \(S_k(\Gamma(1))\) be the space of all holomorphic cusp forms of weight \(k\) with respect to the modular group \(\Gamma(1)\), and \(B_k\) is a basis of \(S_k(\Gamma(1))\) consisting of Hecke-normalized cusp forms which are simultaneously eigenforms for all Hecke operators \(T_n\).

The Fourier series expansion of \(f\) is

\[ f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e(nz), \]

and we have

\[ T_n f = \lambda_f(n)n^{(k-1)/2}f. \]

Note that \(\lambda_f(1) = 1\), and each \(\lambda_f(n)\) is real. The associated \(L\)-function

\[ L(s, f) = \sum_{n \geq 1} \lambda_f(n)n^{-s}, \Re(s) > 1, \]
has Euler product $L(s, f) = \prod_p L_p(s, f)$ with the local factors

$$L_p(s, f) = (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1} = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}.$$ 

It admits analytic continuation to the whole complex plane $\mathbb{C}$ and satisfies the functional equation

$$(2\pi)^{-s}\Gamma\left(\frac{k}{2} - s\right)L(s, f) = i^{k}(2\pi)^{s-1}\Gamma\left(\frac{k}{2} + 1 - s\right)L(1 - s, f).$$

The symmetric square L-function associated to $f$ is defined by

$$L(s, \text{sym}^2(f)) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - \alpha_p\beta_p p^{-s})^{-1}(1 - \beta_p^2 p^{-s})^{-1}, \Re(s) > 1.$$ 

Equivalently, the symmetric square L-function $L(s, \text{sym}^2(f))$ is defined as

$$L(s, \text{sym}^2(f)) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}$$

$$= \sum_{n=1}^{\infty} \sum_{m^2 = n} \lambda_f(m^2), \text{for } \Re(s) > 1.$$ 

The complete symmetric square L-function $\Lambda(s, \text{sym}^2(f))$ is defined as

$$\Lambda(s, \text{sym}^2(f)) = L_\infty(s)L(s, \text{sym}^2(f))$$

where

$$L_\infty(s) = \pi^{-\frac{s}{2}+1/2}2^{2s-k}\Gamma\left(\frac{s+1}{2}\right)\Gamma(s+k-1).$$

Note that both $L(s, \text{sym}^2(f))$ and $\Lambda(s, \text{sym}^2(f))$ are entire in $\mathbb{C}$, and the complete L-function $\Lambda(s, \text{sym}^2(f))$ satisfies the functional equation:

$$\Lambda(s, \text{sym}^2(f)) = \Lambda(1 - s, \text{sym}^2(f)).$$

**Remark 4.1.1.** The functional equation has three gamma factors, which is consistent with the fact that $L(s, \text{sym}^2(f))$ is an automorphic form on $GL(3)$, but only two of them involve $k$. Therefore $L(s, \text{sym}^2(f))$ looks rather like a form on $GL(2)$ in the weight aspect.
The main theorem in this chapter is as follows:

**Theorem 4.1.2.** Let \( w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2} \), we have

\[
\sum_{f \in B_k} w_f L^2(1/2, \text{sym}^2(f)) = V(1) + M_1 + M_2
\]

where

\[
V(x) = \frac{1}{\pi i} \int (2) \frac{L^2_\infty(1/2 + w)}{L^2_\infty(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{x^w} dw.
\]

\[
M_1 = i^k \sum_{d_1=1}^{\infty} \frac{1}{d_1} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{L^2_\infty(1/2 + w)}{L^2_\infty(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^2 w} \]

\[
\times \text{Res}_{s=-1/2-w} \left( \frac{c^2}{2\pi d_1} \right)^s \frac{\Gamma\left(\frac{k+z}{2}\right)}{\Gamma\left(\frac{k-z}{2}\right)} E\left(\frac{3}{2} + w + s, d_1^2, c\right) \right) dw,
\]

\[
M_2 = i^k \sum_{d_1=1}^{\infty} \frac{1}{d_1} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{L^2_\infty(1/2 + w)}{L^2_\infty(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^2 w} \]

\[
\times \int (\gamma) \left( \frac{c^2}{2\pi d_1} \right)^s \frac{\Gamma\left(\frac{k+z}{2}\right)}{\Gamma\left(\frac{k-z}{2}\right)} E\left(\frac{3}{2} + w + s, d_1^2, c\right) ds dw,
\]

\[
M'_2 = 2\sqrt{2\pi i}^k \sum_{d_1=1}^{\infty} \frac{1}{d_1^{3/2}} \sum_{c=1}^{\infty} \frac{1}{c^{5/2}} \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{L^2_\infty(1/2 + w)}{L^2_\infty(1/2)} K(w) \zeta^3(1 + 2w) \frac{2\pi}{cd_1}^w
\]

\[
\times \sum_{n=1}^{\infty} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} S(\alpha^2 \beta^2 d_1^2, 1; c) \sum_{m|n} c \left( m \alpha / c + \frac{n}{m} \beta / c \right)
\]

\[
\times \int (\gamma) \left( \frac{2\pi n}{cd_1} \right)^{-z} \frac{\Gamma\left(\frac{k+w-z-1/2}{2}\right)}{\Gamma\left(\frac{k+w+z+1/2}{2}\right)} \Gamma(z) dz dw,
\]
\[ M_2'' = -4\pi^{3/2} k \sum_{d_1 \text{ square free}}^{\infty} \frac{1}{d_1^{3/2}} \sum_{c=1}^{\infty} \frac{1}{e^{5/2} (2\pi i)^2} \int_{(2)}^{\infty} \frac{L_c^2(1/2 + w)}{L_c^2(1/2)} K(w) \zeta^3(1 + 2w) \left( \frac{4\pi}{cd_1} \right)^w \times \sum_{n=1}^{\infty} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} S(\alpha^2 \beta^2 d_1^2, 1; c) \sum_{m|n} e(m\alpha/c + \frac{n}{m}\beta/c) \times \int_0^{\infty} J_{k-1}(x) x^{-w-1/2} Y_0(\sqrt{2xy}) dx dw. \]

Furthermore, we have the following estimates

\[ V(1) \ll \log^3 k, M_1 \ll 1, M_2' \ll 1. \]

We remark that in comparison, Iwaniec and Sarnark [21] have the following theorem in the study of the non-vanishing of automorphic \( L \)-functions at the central value:

**Theorem 4.1.3.** Let \( \Gamma = \text{SL}_2(\mathbb{Z}) \) and \( k > 2, k \equiv 0 \pmod{2} \). For any \( m \geq 1 \), we have

\[ \frac{12}{k-1} \sum_{f \in \mathcal{S}_k} \lambda_f(m) L^2(1/2, f) = 2(1 + i^k) \frac{\tau(m)}{\sqrt{m}} \left( \sum_{0 < l < k/2} l^{-1} - \log 2\pi \sqrt{m} \right) \]

\[ - \frac{2\pi i^k}{\sqrt{m}} \sum_{h \neq m} \tau(h) \tau(h - m) p_k \left( \frac{h}{m} \right) + \frac{2\pi i^k}{\sqrt{m}} \sum_h \tau(h) \tau(h + m) q_k \left( \frac{h}{m} \right) \]

where \( p_k(x) \) and \( q_k(x) \) are given by Hankel transforms of Bessel functions,

\[ p_k(x) = \int_0^{\infty} Y_0(y\sqrt{x}) J_{k-1}(y) dy, \quad q_k(x) = \int_0^{\infty} K_0(y\sqrt{x}) J_{k-1}(y) dy. \]

**Remark 4.1.4.** V. Bykovskii [6] also has a similar formula intended to estimate the derivative of twisted Hecke \( L \)-functions on the critical line.
4.2 Some preparations

We start with

\[ L^2(s, \text{sym}^2(f)) = L(s, \text{sym}^2(f))L(s, \text{sym}^2(f)) \]
\[ = \zeta^2(2s) \sum_{m=1}^{\infty} \frac{\lambda_f(m^2)}{m^s} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} \]
\[ = \zeta^2(2s) \sum_{m,n=1}^{\infty} \frac{1}{(mn)^s} \sum_{d|(m^2,n^2)} \lambda_f\left(\frac{m^2n^2}{d^2}\right). \]

We write \( d = d_1 d_2^2 \), where \( d_1 \) is square free. Then we have

\[ L^2(s, \text{sym}^2(f)) = \zeta^2(2s) \sum_{r=1}^{\infty} \frac{1}{r^s} \sum_{d_1d_2|(m,n)} \frac{\lambda_f\left(\frac{r^2}{d_1^2d_2^2}\right)}{d_1d_2} \]
\[ = \zeta^2(2s) \sum_{d_1=1}^{\infty} \frac{1}{d_1^2} \sum_{d_2} \frac{1}{d_2^2} \sum_{d_1d_2^2|\nu} \frac{\lambda_f\left(\frac{r^2}{d_1^2d_2^2}\right)}{t^s} \left( \sum_{d_1d_2|(m,n)} 1 \right) \]
\[ = \zeta^3(2s) \sum_{d_1=1}^{\infty} \frac{1}{d_1^2} \sum_{n=1}^{\infty} \frac{\lambda_f(d_1^2n^2)d(n)}{n^s}. \]

Define

\[ K(w) = \frac{\Gamma(2(A-w))\Gamma(2(A+w))}{\Gamma^2(2A)} \frac{1}{w} \]
where \( 2 < A < 3 \). Note that \( K(w) \) is odd, and has only a simple pole at \( w = 0 \), residue 1 inside the strip \(-A < \Re w < A\).

We have

\[ \Lambda^2(s, \text{sym}^2(f)) = \frac{1}{\pi i} \int(2) \Lambda^2(1/2 + w, \text{sym}^2(f))K(w)dw \]
and

\[ L^2(1/2, \text{sym}^2(f)) = \frac{1}{\pi i} \int(2) \frac{L^2(1/2 + w)}{L^2(1/2)} L^2(1/2 + w, \text{sym}^2(f))K(w)dw \]
\[ = \sum_{d_1=1}^{\infty} \frac{1}{d_1} \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} \lambda_f(d_1^2n^2) V(d_1^2n) \]

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where
\[
V(x) = \frac{1}{\pi i} \int_{(2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{x^w} dw.
\]

By Petersson’s trace formula
\[
\sum_{f \in \mathcal{B}_k} w_f \lambda_f(n) \lambda_f(m) = \delta_{m,n} + 2\pi i^k \sum_{c=1}^\infty c^{-1} S(m, n, c) J_{k-1}(\frac{4\pi \sqrt{mn}}{c}),
\]
where \(w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1} \|f\|^2} \ll \frac{\log k}{k}\), we have
\[
\sum_{f \in \mathcal{B}_k} w_f L^2(1/2, \text{sym}^2(f)) = \sum_{d_1=1}^\infty \sum_{n=1}^\infty \frac{d(n)}{\sqrt{n}} V(d_1^2 n) \sum_{f \in \mathcal{B}_k} w_f \lambda_f(d_1^2 n^2)
\]
\[
= V(1) + M
\]
where
\[
M = 2\pi i^k \sum_{d_1=1}^\infty \sum_{n=1}^\infty \frac{d(n)}{\sqrt{n}} \frac{S(d_1^2 n^2, 1, c)}{c} J_{k-1}(\frac{4\pi nd_1}{c}) V(d_1^2 n).
\]

Using the Mellin transform formula
\[
J_{k-1}(x) = \frac{1}{2\pi i} \int_{(-1)}^{(\frac{1}{2})} \frac{\Gamma((k+s)/2)}{\Gamma((k-s)/2)} 2^s x^{-s-1} ds,
\]
we have
\[
M = i^k \sum_{d_1=1}^\infty \sum_{n=1}^\infty \frac{1}{d_1^2} \int_{(-1)}^{(\frac{1}{2})} \frac{L^2(1/2 + w)}{L^2(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^w} \left[ (\frac{c}{2\pi d_1})^s \frac{\Gamma(k+s)}{\Gamma(k-s)} \right] \sum_{n=1}^\infty \frac{d(n) S(d_1^2 n^2, 1, c)}{n^{3/2 + w + s}} dw.
\]

Define
\[
E(s, d_1^2, c) = \sum_{n=1}^\infty \frac{d(n) S(n^2 d_1^2, 1, c)}{n^s}.
\]

Then we have
\[
M = i^k \sum_{d_1=1}^\infty \sum_{n=1}^\infty \frac{1}{d_1^2} \int_{(-1)}^{(\frac{1}{2})} \frac{L^2(1/2 + w)}{L^2(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^w} \left[ (\frac{c}{2\pi d_1})^s \frac{\Gamma(k+s)}{\Gamma(k-s)} \right] E(3/2 + w + s, d_1^2, c) dw.
\]
4.3 Treatment of $V(1)$

$$V(1) = \frac{1}{\pi i} \int_{\gamma_2} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) dw$$

$$= 2 \text{Res}_{w=0} \left( \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) \right)$$

$$+ \frac{1}{\pi i} \int_{(-A/2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) dw.$$  

We have

$$\text{Res}_{w=0} \left( \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) \right) \ll \log^3 k,$$

and

$$\frac{1}{\pi i} \int_{(-A/2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) dw = O_A(k^{-A}).$$

4.4 Treatment of $E(s, d_1^2, c)$

We have

$$E(s, d_1^2, c) = \sum_{n=1}^{\infty} \frac{d(n) S(n^2 d_1^2, 1; c)}{n^s}$$

$$= \sum_{a \pmod{c}}^{*} e\left(\frac{a}{c}\right) \sum_{n=1}^{\infty} \frac{d(n) e\left(\frac{a n^2 d_1^2}{c}\right)}{n^s}$$

$$= \sum_{a \pmod{c}}^{*} e\left(\frac{a}{c}\right) \sum_{n=1}^{\infty} \frac{e\left(\frac{a r t^2 d_1^2}{c}\right)}{r^s t^s}.$$  

Let

$$r = r_1 c + \alpha,$$

$$t = t_1 c + \beta$$

where $r_1, t_1 \geq 0$, and $1 \leq \alpha, \beta \leq c.$
Then we have

\[
E(s, d_1^2, c) = \sum_{a \pmod{c}} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} e\left(\frac{\alpha^2 \beta^2 d_1^2}{c}\right) \frac{1}{\Gamma(1-s)} \sum_{r_1=0}^{\infty} \frac{1}{(r_1 c + \alpha)^s} \sum_{t_1=0}^{\infty} \frac{1}{(t_1 c + \beta)^s}
\]

\[
= \sum_{a \pmod{c}} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} e\left(\frac{\alpha^2 \beta^2 d_1^2}{c}\right) \zeta(s, \frac{\alpha}{c}) \zeta(s, \frac{\beta}{c})
\]

\[
= -2s \ln(c) - 2s \ln(\Gamma(2s-1)) - (2\pi)^{2s-2} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c) \zeta(s, \frac{\alpha}{c}) \zeta(s, \frac{\beta}{c})
\]

where the Hurwitz zeta function \(\zeta(s, a)\) is defined as

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}
\]

for \(\Re s > 1\) and \(\Re a > 0\).

From the functional equation of Hurwitz zeta function \(\zeta(s, a)\), we can also get the functional equation of \(E(s, d_1^2, c)\):

\[
E(s, d_1^2, c) = -2s \ln(c) - 2s \ln(\Gamma(2s-1)) - (2\pi)^{2s-2} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c) \zeta(s, \frac{\alpha}{c}) \zeta(s, \frac{\beta}{c})
\]

\[
= -2s \ln(c) - 2s \ln(\Gamma(2s-1)) \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c) \zeta(s, \frac{\alpha}{c}) \zeta(s, \frac{\beta}{c})
\]

It is easy to see that \(E(s, d_1^2, c)\) only has a double pole at \(s = 1\). Therefore the Laurent expansion of \(E(s, d_1^2, c)\) at \(s = 1\) is of the form

\[
E(s, d_1^2, c) = \frac{A_{-2}(d_1^2, c)}{(s-1)^2} + \frac{A_{-1}(d_1^2, c)}{s-1} + \ldots
\]

where

\[
A_{-2}(d_1^2, c) = \frac{1}{c^2} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c)
\]

and

\[
A_{-1}(d_1^2, c) = \frac{1}{c^2} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c)(\gamma_0(\alpha/c) + \gamma_0(\beta/c))
\]

Define the quadratic Gauss sum

\[
G(a, c) = \sum_{n \pmod{c}} e\left(\frac{an^2}{c}\right).
\]
When \((a, c) = 1\), the values of Gauss sums are explicitly given by the Gauss formula

\[
G(a, c) = \begin{cases} 
0 & c \equiv 2 \pmod{4} \\
\epsilon_c \sqrt{c} \left( \frac{a}{c} \right) & c \text{ odd} \\
(1 + i) \epsilon_a^{-1} \sqrt{c} \left( \frac{c}{a} \right) & a \text{ odd}, 4 \mid c,
\end{cases}
\]

(4.4.1)

where

\[
\epsilon_a = \begin{cases} 
1 & a \equiv 1 \pmod{4} \\
i & a \equiv 3 \pmod{4}
\end{cases}
\]

(4.4.2)

for odd integer \(a\).

**Lemma 4.4.1.**

\(A_{-2}(d_1^2, c) \ll \sigma_{\frac{1}{2}}(c)\) for all \(c \geq 1\),

where \(\sigma_x(c) = \sum_{c_1 \mid c} c^x\).

**Proof.** 1. If \(c\) is 1, then \(A_{-2}(d_1^2, 1) = 1\).

2. If \(c\) is odd, then

\[
A_{-2}(d_1^2, c) = \frac{1}{c^2} \sum_{\alpha \pmod{c}} \sum_{\alpha \pmod{c}} \star e\left( \frac{\bar{a}}{c} \right) \sum_{\beta \pmod{c}} e\left( \frac{a \alpha^2 d_1^2}{c} \beta^2 \right)
\]

\[
= \frac{1}{c^2} \sum_{c_1 \mid c} \sum_{\alpha \pmod{c}} \star e\left( \frac{\bar{a}}{c} \right) \sum_{\alpha \pmod{c}} e\left( \frac{a \alpha^2 d_1^2}{c} / c_1 \beta^2 \right)
\]

\[
= \frac{1}{c^2} \sum_{c_1 \mid c} \sum_{\alpha \pmod{c}} \star e\left( \frac{\bar{a}}{c} \right) c_1 \epsilon_{\frac{c}{c_1}} \sqrt{c_1} \left( \frac{a}{c/c_1} \right) \left( \frac{a^2 d_1^2 / c_1}{c/c_1} \right)
\]

\[
= \frac{\sqrt{c}}{c^2} \sum_{c_1 \mid c} \sum_{\alpha \pmod{c}} \sqrt{c_1} \epsilon_{\frac{c}{c_1}} \left( \frac{a^2 d_1^2 / c_1}{c/c_1} \right) \sum_{a \pmod{c}} \star e\left( \frac{a}{c} \right) \left( \frac{a}{c/c_1} \right).
\]
Define a Dirichlet character \( \chi_{c,c_1} \pmod{c} \)

\[
\chi_{c,c_1}(a) = \begin{cases} 
\left( \frac{a}{c/c_1} \right), & (a, c) = 1 \\
0, & (a, c) > 1.
\end{cases}
\]  

(4.4.3)

Then the sum

\[
\sum_a \ast e\left( \frac{a}{c} \right) \left( \frac{a}{c/c_1} \right) = \sum_{a \pmod{c}} \chi_{c,c_1}(a)e\left( \frac{a}{c} \right)
\]

is a Gauss sum, thus

\[
|\sum_a \ast e\left( \frac{a}{c} \right) \left( \frac{a}{c/c_1} \right)| \leq \sqrt{c}.
\]

And we have

\[
|A_{-2}(d_1^2, c)| \leq \frac{\sqrt{c}}{c^2} \sum_{c_1 \mid c} \sum_{c_1 = (\alpha^2 d_1^2, c)} \sqrt{c_1} \sqrt{c} \\
= \frac{1}{c} \sum_{c_1 \mid c} \sum_{c_1 = (\alpha^2 d_1^2, c)} \sqrt{c_1} \\
\leq \sigma_{\frac{1}{2}}(c).
\]

3. If \(2 \mid c\), then

\[
A_{-2}(d_1^2, c) = \frac{1}{c^2} \sum_{a \pmod{c}} \sum_{a \pmod{c}} \ast e\left( \frac{\bar{a}}{c} \right) \sum_{\beta \pmod{c}} e\left( \frac{a\alpha^2 d_1^2}{c} \beta^2 \right) \\
= \frac{1}{c^2} \sum_{c_1 \mid c} \sum_{2c_1 \mid c} \ast e\left( \frac{\bar{a}}{c} \right) \sum_{c_1 = (\alpha^2 d_1^2, c)} \sum_{\beta \pmod{c}} e\left( \frac{a\alpha^2 d_1^2}{c/c_1} \beta^2 \right) \\
+ \frac{1}{c^2} \sum_{c_1 \mid c} \sum_{2c_1 \mid c} \ast e\left( \frac{\bar{a}}{c} \right) \sum_{c_1 = (\alpha^2 d_1^2, c)} \sum_{\beta \pmod{c}} e\left( \frac{a\alpha^2 d_1^2}{c/c_1} \beta^2 \right).
\]

The first sum can be treated similarly as in case 2, and the second sum is zero.
4. If $4|c$, then
\[
A_{-2}(d_1^2, c) = \frac{1}{c^2} \sum_{\alpha \pmod{c}} \sum_{a \pmod{c}}^* \sum_{\beta \pmod{c}} e\left(\frac{\bar{\alpha}}{c}\right) e\left(\frac{a\alpha^2 d_1^2}{c}\right) + \frac{1}{c^2} \sum_{\alpha \mid c} \sum_{\beta \pmod{c}} e\left(\frac{\bar{\alpha}}{c}\right) e\left(\frac{a\alpha^2 d_1^2}{c}\right) + \frac{1}{c^2} \sum_{4|c} \sum_{\alpha \pmod{c}}^* \sum_{\beta \pmod{c}} e\left(\frac{\bar{\alpha}}{c}\right) e\left(\frac{a\alpha^2 d_1^2}{c}\right).
\]

The first sum can be treated similarly as in case 2, and the second sum is zero. As for the third sum, fix $\alpha, d_1, c_1$, there exist $t_1(\alpha, d_1, c_1), t_2(\alpha, d_1, c_1) \in \mathbb{C}$, such that we have
\[
\epsilon^{-1}_{\alpha d_1^2} = t_1 \chi_1(a) + t_2 \chi_2(a)
\]
where
\[
\chi_1(a) = \begin{cases}
1, & a \equiv 1 \pmod{4} \\
1, & a \equiv 3 \pmod{4} \\
0, & \text{otherwise}
\end{cases}
\]
(4.4.4)

and
\[
\chi_2(a) = \begin{cases}
1, & a \equiv 1 \pmod{4} \\
-1, & a \equiv 3 \pmod{4} \\
0, & \text{otherwise}
\end{cases}
\]
(4.4.5)
Since $4 | c$, $\chi_1, \chi_2$ induces $\chi_1', \chi_2'$ mod $c$.

$$
\frac{1}{c^2} \sum_{c_1 | c} \sum_{\substack{a \equiv \bar{b} \pmod{c} \atop 4 | \bar{c}_1}} ^* e\left(\frac{\bar{a}}{c}\right) \sum_{\substack{\beta \equiv a^2 \alpha^2 \pmod{c} \atop \alpha \equiv \bar{b} \pmod{c}}} e\left(\frac{c \beta}{c_1} \bar{a} \alpha^2 \beta^2 / a \alpha^2 \beta^2 / c_1 \right)
$$

$$
= \frac{1}{c^2} \sum_{c_1 | c} \sum_{\substack{a \equiv \bar{b} \pmod{c} \atop 4 | \bar{c}_1}} ^* e\left(\frac{\bar{a}}{c}\right) \sum_{\substack{\beta \equiv a^2 \alpha^2 \pmod{c} \atop \alpha \equiv \bar{b} \pmod{c}}} c_1 (1 + i) e_{1/a}^{-1} \sqrt{\frac{c}{c_1}} \left(\frac{c / c_1}{a / c_1 \alpha^2 \beta^2 / c_1}\right)
$$

$$
= \frac{\sqrt{c}}{c^2} \sum_{c_1 | c} \sum_{\substack{a \equiv \bar{b} \pmod{c} \atop 4 | \bar{c}_1}} \sqrt{c_1} (1 + i) \left(\frac{c / c_1}{a / c_1 \alpha^2 \beta^2 / c_1}\right)
$$

$$
\leq \frac{1}{c} \sum_{c_1 | c} \sum_{\substack{\alpha \equiv \bar{b} \pmod{c} \atop 4 | \bar{c}_1}} \sqrt{c_1}
$$

$$
\leq \sigma_1(c).
$$

\[\square\]

Lemma 4.4.2.

$$
A_{-1}(d^2_1, c) \ll \sqrt{c} \sigma(c) (\gamma + \ln c) \text{ for all } c \geq 1,
$$

where $\sigma(c) = \sum_{c_1 | c} 1$.

Proof. Apply Weil’s bound, we have

$$
S(\alpha^2 \beta^2 d_1^2, 1; c) \leq \sqrt{c} \sigma(c).
$$

Therefore

$$
|A_{-1}(d^2_1, c)| = \left| \frac{1}{c^2} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} S(\alpha^2 \beta^2 d_1^2, 1; c)(\gamma_0(\alpha/c) + \gamma_0(\beta/c)) \right|
$$

$$
\leq \frac{\sqrt{c} \sigma(c)}{c^2} \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} |\gamma_0(\alpha/c) + \gamma_0(\beta/c)|
$$
Since $\gamma_0(\alpha/c) > 0$ and
\[ \sum_{\alpha=1}^{c} \gamma_0(\frac{\alpha}{c}) = c(\gamma + \ln c), \]
we have
\[ |A_{-1}(d^2_1, c)| \ll \sqrt{c} \sigma(c) (\gamma + \ln c). \]

4.5 Treatment of $M$

By moving the line of integration of the inner integral from $\Re s = -1$ to $\Re s = -7$, we have
\[
M = i^k \sum_{d_1} \frac{1}{d^2_1} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_2^2(1/2 + w)}{L_2^2(1/2)} K(w) \zeta^3 (1 + 2w) \frac{1}{d_1^{2w}} \times \int_{(-1)} \left( \frac{c}{2\pi d_1} \right)^s \frac{\Gamma(\frac{k+s}{2})}{\Gamma(\frac{k-s}{2})} E(3/2 + w + s, d_1^2, c) \, dw 
\]
where
\[
M_1 = i^k \sum_{d_1} \frac{1}{d^2_1} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_2^2(1/2 + w)}{L_2^2(1/2)} K(w) \zeta^3 (1 + 2w) \frac{1}{d_1^{2w}} \times \text{Res}_{s=-\frac{1}{2}} \left( \frac{c}{2\pi d_1} \right)^s \frac{\Gamma(\frac{k+s}{2})}{\Gamma(\frac{k-s}{2})} E\left( \frac{3}{2} + w + s, d_1^2, c \right) \, dw
\]
and
\[
M_2 = i^k \sum_{d_1} \frac{1}{d^2_1} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_2^2(1/2 + w)}{L_2^2(1/2)} K(w) \zeta^3 (1 + 2w) \frac{1}{d_1^{2w}} \times \int_{(-7)} \left( \frac{c}{2\pi d_1} \right)^s \frac{\Gamma(\frac{k+s}{2})}{\Gamma(\frac{k-s}{2})} E(3/2 + w + s, d_1^2, c) \, dw
\]
To bound $M_2$, by changing viable $s = -w - z - 1/2$, we have
\[
M_2 = i^k \sum_{d_1} \frac{1}{d^2_1} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_2^2(1/2 + w)}{L_2^2(1/2)} K(w) \zeta^3 (1 + 2w) \frac{1}{d_1^{2w}} \times \int_{(9/2)} \left( \frac{c}{2\pi d_1} \right)^{-w-z-1/2} \frac{\Gamma(\frac{k-w-z+1/2}{2})}{\Gamma(\frac{k+w+z+1/2}{2})} E(1 - z, d_1^2, c) \, dw.
\]
Then apply the functional equation of $E(s, d^2_1, c)$, we get

$$M_2 = i^k \sum_{d_1} \frac{1}{d_1^2} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int \frac{L^2_\infty(1/2 + w)}{L^2_\infty(1/2)} K(w) \zeta(1 + 2w) \frac{1}{d_1^w}$$

$$\times \int_{(9/2)} \left( \frac{c}{2\pi d_1} \right)^{-w-z-1/2} \frac{\Gamma(k-w-z-1/2)}{\Gamma(k+w+z+1/2)} E(1-z, d^2_1, c) ds dw$$

$$= 2\sqrt{2\pi i} \sum_{d_1} \frac{1}{d_1^{3/2}} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int \frac{L^2_\infty(1/2 + w)}{L^2_\infty(1/2)} K(w) \zeta(1 + 2w) \frac{1}{d_1^w}$$

$$\times \sum_{n=1}^{\infty} \sum_{\alpha, \beta} S(\alpha^2 \beta^2 d^2_1, 1; c) \sum_{m|n} e(m\alpha/c + n\beta/c)$$

$$\times \int_{(9/2)} \frac{2\pi n}{cd_1} \Gamma(k-w-z-1/2) \Gamma^{2}(z) (1 + \cos \pi z) dz dw.$$
And we notice that the functions \( J_{k-1}(x) \) and \( 2^s \frac{\Gamma\left(\frac{k+s}{2}\right)}{\Gamma\left(\frac{k-s}{2}\right)} \) are Mellin transform pairs, that is,

\[
J_{k-1}(x) = \frac{1}{2\pi i} \int_{-1}^{(-1)} \frac{\Gamma\left(\frac{k+s}{2}\right)}{\Gamma\left(\frac{k-s}{2}\right)} 2^s x^{-s-1} ds,
\]

\[
\frac{\Gamma\left(\frac{k+s}{2}\right)}{\Gamma\left(\frac{k-s}{2}\right)} = \int_0^{\infty} J_{k-1}(x) \left(\frac{x}{2}\right)^s dx \quad (-k < \Re(s) < -\frac{1}{2}).
\]

Now consider the integral

\[
\frac{1}{2\pi i} \int_{(9/2)} \int_0^{\infty} \frac{\Gamma\left(\frac{k-w-z-1/2}{2}\right)}{\Gamma\left(\frac{k+w+z+1/2}{2}\right)} \Gamma^2(z) e^{\pi z} y^{-z} dz.
\]

Moving the line of integration to \( \Re(z) = 1/2 \), we have

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \int_0^{\infty} J_{k-1}(x) \left(\frac{x}{2}\right)^{-w-1/2} dx \Gamma^2(z) e^{\pi z} y^{-z} dz
\]

\[
= \int_0^{\infty} J_{k-1}(x) x^{-w-1/2} \frac{1}{2\pi i} \int_{(1/2)} 2^z \Gamma^2(z) e^{\pi z} (xy)^{-z} dz dx
\]

\[
= \pi^{2w+1/2} \int_0^{\infty} J_{k-1}(x)x^{-w-1/2}(iJ_0(\sqrt{2xy}) - Y_0(\sqrt{2xy}))dx.
\]

This completes the proof by comparing the real part. 

\[\square\]

Summarize all the discussion above, we have

\[
\sum_{f \in B_k} w_f L^2(1/2, \text{sym}^2(f)) = V(1) + M_1 + M_2
\]

\[
= V(1) + M_1 + M'_1 + M''_1
\]

where

\[
V(x) = \frac{1}{\pi i} \int_{(2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{x^w} dw,
\]

\[
M_1 = i^k \sum_{d_1=1}^{\infty} \sum_{d_1 \text{square free}} \frac{1}{d_1^2} \int_{c=1}^{\infty} \frac{(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta^3(1 + 2w) \frac{1}{d_1^w} \times \Res_{s=-1/2-w} \left( \frac{c}{2\pi d_1} \right)^s \frac{\Gamma\left(\frac{k+s}{2}\right)}{\Gamma\left(\frac{k-s}{2}\right)} E\left(\frac{3}{2} + w + s, d_1^2, c) \right) dw,
\]

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\[ M_2 = \quad i^k \sum_{d_1 = 1}^{\infty} \sum_{d_1 \text{ square free}} \frac{1}{d_1^2} \sum_{c=1}^{\infty} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta(1 + 2w) \frac{1}{d_1^{2w}} \]
\[
\times \int (\frac{-c}{2\pi d_1})^s \frac{\Gamma(k+s)}{\Gamma(k-s)} \left( 1 + w \right) \left( 1 + 2w \right) \frac{1}{c} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} S(\alpha^2 \beta^2 d_1^2, 1; c) e(ma/c + \frac{n}{m} \beta/c) \sum_{m|n} \Gamma(\frac{k-w-1/2}{2}) \Gamma(\frac{k+k+w+z+1/2}{2}) \Gamma^2(z) dz dw, \]
\[
M'_2 = \quad 2\sqrt{2\pi i}^k \sum_{d_1 = 1}^{\infty} \sum_{d_1 \text{ square free}} \frac{1}{d_1^{3/2}} \sum_{c=1}^{\infty} \frac{1}{c^{5/2}} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta(1 + 2w) \left( \frac{2\pi}{cd_1} \right)^w \]
\[
\times \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{\beta=1}^{\infty} S(\alpha^2 \beta^2 d_1^2, 1; c) e(ma/c + \frac{n}{m} \beta/c) \sum_{m|n} \frac{2\pi n}{cd_1} \int_{(9/2)} \Gamma(\frac{k-w-z-1/2}{2}) \Gamma^2(z) dz dw, \]
\[
M''_2 = \quad -4\pi^{3/2} i^k \sum_{d_1 = 1}^{\infty} \sum_{d_1 \text{ square free}} \frac{1}{d_1^{3/2}} \sum_{c=1}^{\infty} \frac{1}{c^{5/2}} \left( \frac{1}{2\pi i} \right)^2 \int_{(2)} \frac{L_\infty^2(1/2 + w)}{L_\infty^2(1/2)} K(w) \zeta(1 + 2w) \left( \frac{4\pi}{cd_1} \right)^w \]
\[
\times \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{\beta=1}^{\infty} S(\alpha^2 \beta^2 d_1^2, 1; c) e(ma/c + \frac{n}{m} \beta/c) \sum_{m|n} \frac{2\pi n}{cd_1} \int_0^\infty J_{k-1}(x)x^{-w-1/2}Y_0(\sqrt{2xy})dx dw. \]

Applying Sterling formula, we can easily get

\[ M_1 \ll 1, M'_2 \ll 1. \]

**Remark 4.5.2.** Y. Lau and K. Tsang [27] discussed a similar integral

\[ \int_0^\infty J_{k-1}(x)x^{-s}Y_0(ax)dx \]

for large \( k \).

However in our case, the integral is much more difficult to deal with since the transition range of the J-bessel function is about \( k^2 \) instead of \( k \). However, this work provides a useful approach to the further study of \( L(1/2, \text{sym}^2(f)) \).


[27] Y. Lau and K. Tsang, A mean square formula for central values of twisted automorphic $L$-functions, ACTA ARITHMETICA 118.3(2005)


