A Multidimensional Discontinuous Galerkin Modeling Framework for Overland Flow and Channel Routing

A Thesis

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Abstract

This thesis presents the development and application of a multidimensional (2-D & 1-D) kinematic wave model in a discontinuous Galerkin framework for simulating overland flow and runoff due to torrential rainfall. The objective of this work is to improve on (1) the modeling approach involving many small-scale rivers and channels and (2) the accuracy of flooding caused by storm surge coupled with torrential rainfall. The overland flow is modeled using the 2-D kinematic wave equations derived from the 2-D depth averaged shallow water equations. In areas of the domain where flow converges into channels, the edges of 2-D elements are used as 1-D channels for flow routing. To best represent complex topography, domains are discretized using an application co-developed by the author called Admesh+, an automatic unstructured mesh generator for shallow water models. Admesh+ produces high-quality meshes with the appropriate amount of refinement where it is needed to resolve all of the geometry and flow characteristics of the domain. The mesh generation technique utilizes high-resolution digital elevation maps (DEMs) to automatically produce unstructured meshes with elements arranged to best represent shorelines, channel networks and watershed delineations. The development of the multidimensional overland flow model and mesh generation techniques are presented along with comparisons of model results with various analytic solutions and experimental data.
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Chapter 1: Introduction

Recent storm events, for example Hurricane Sandy, which led to extensive flooding along much of the U.S. East Coast, have demonstrated the severe vulnerability of coastal lowlands and watersheds to storm surge combined with torrential rainfall. Since the 1980's the intensity, frequency, and duration of hurricanes in the North Atlantic, and their impacts on the U.S., have all increased and research suggests continued increases in storm intensity and significant potential for heavy rainfall in many regions [24]. These types of disasters highlight the rising importance of effective emergency management and hazard mitigation, and the need for advanced physics-based models to better understand their impacts.

Accurate and efficient computational modeling of these types of flooding events, which can provide crucial data to assess flood risk and aid in emergency management planning, presents significant challenges. These challenges are primarily due to the complex topology of coastal watersheds and floodplains, and in particular urban environments, which include numerous features relevant to flooding such as small-scale drainage channels, piping networks, etc, that receive stormwater from both the landfall of storm surge and runoff/overland flow due to rainfall.
This thesis is focused on the continued development of a computational hydrodynamic model to improve the predictive capability for flooding in complex coastal watersheds and floodplains using a multi-physics-multidimensional modeling approach. The focus of this research consists of two main tasks:

**Task 1: Mesh representation of watersheds**

One of the first steps required for the development of a hydrodynamic computer model is the discretization, or creation of a computational mesh, of the physical domain of interest, e.g., the Gulf of Mexico or Lake Erie, see Figures 1.1 and 1.2, respectively.

![Figure 1.1: An unstructured finite element mesh of the Gulf of Mexico (Countours show bathymetry/topography).](image-url)
The development of efficient and robust mesh generation techniques are needed to represent the complex geometric and topographic features of watersheds and floodplains.

**Task 2: Develop solution strategies of overland and channel equations**

The next step is to define a set of partial differential equations (PDEs) describing the overland flow and runoff process(es) due to torrential rainfall. The shallow water equations (SWE), a set of hyperbolic PDEs, are used to model free surface flow in the deep ocean, coastal ocean, rivers, open channels, and coastal floodplains. The SWE
can be used to develop a robust and efficient approach to modeling overland flow and runoff in a multidimensional Discontinuous Galerkin (DG) finite element framework.

1.1 Motivation and Relevant Background

The ADvanced CIRCulation continuous Galerkin (CG) finite element model, ADCIRC [22, 26], is a computational tool that has been adopted operationally by numerous local, state and federal entities, including, for example, the U.S. Army Corps of Engineers, the Federal Emergency Management Agency (FEMA), the National Oceanic and Atmospheric Administration (NOAA), and the Louisiana State University Hurricane Center. ADCIRC is widely used for simulating circulation and storm surge propagation from deep water to shelf to adjacent inland waterways. It is the state-of-the-art model in use today for studying hurricane storm surges and their impacts on coastal inundation, ecology and infrastructure [33, 8, 16, 10].

The computational tools developed in this thesis are for the development of the next generation coastal hydrodynamic model, the Discontinuous Galerkin Shallow Water Equation Model (DG-SWEM) in the ADCIRC framework. DG-SWEM solves the two-dimensional, depth-integrated shallow water equations. DG methods are favorable for several reasons such as their local (elemental) and global conservation, ability to handle advection-dominated flows, high parallelizability, and the ease with which mesh ($h$) and polynomial order ($p$) refinement can be implemented. Applications of DG-SWEM range from tidal simulations, inlet studies, sediment transport, hurricane storm surge and more [27, 18, 3, 7, 37].
In 2010-2011, Dawson, Kubatko, Westerink, Trahan, Mirabito, Michoski, and Panda [7] performed an extensive hindcast study on Hurricane Ike (2008) with DG-SWEM. Hurricane Ike was an intense category 4 hurricane that made landfall near Galveston, Texas in September of 2008. The hurricane caused an estimated $29.6 billion in damage and displaced many people in the coastal region. Figure 1.3 shows the path of Hurricane Ike.

![Figure 1.3: Hurricane Ike track (2008). Source: noaa.gov.](image)

Figure 1.4 shows the computational domain used by DG-SWEM for simulation. The mesh provided detailed coverage of the Texas coast and the mesh contained just over 6 million elements with mesh sizes down to about 30 meters in some areas. In
Figure 1.4: An Unstructured Finite Element Mesh of the Western North Atlantic Tidal Domain (Countours show Bathymetry). Insets shows close-ups of the Louisiana/Texas Gulf coast and downtown New Orleans.

general, the hindcast study showed quite good results, however, there is still room for improvement, mainly in many small-scale channels and hydraulic structures (natural and man-made) that have a strong influence on determining storm surge propagation and inland flooding. Figure 1.5 illustrates that the current modeling approach fails to capture storm surge propagating upstream when channels are under-resolved. Thus, two areas of improvement need to be explored. First, the poor model results seen in Figure 1.5 are due to the mesh resolution in this part of the domain; however, with a mesh already consisting of 6 million elements other mesh generation techniques must be considered in order to represent the domain adequately and reduce the mesh resolution. This could ultimately improve model results and computational costs in the model. Second, in other similar situations torrential rainfall plays a significant role in flooding caused by the overland flow and runoff. An example of such a case is
Hurricane Irene. In 2011, Hurricane Irene made landfall along the East Coast of the U.S. and much of the flooding was caused, not by the storm surge itself, but by the overland flow and runoff caused by the torrential rainfall. In order to be successful in capturing inland flooding, inclusion of these phenomena should be taken into account in the model development.

Figure 1.5: Satellite image of a narrow channel along the Texas coast (top left). Mesh resolution of the channel (bottom left). The current modeling approach fails to capture storm surge propagating upstream (right).

1.2 The Modeling Framework

In this section, a description of a modeling framework that addresses these areas of improvements is provided. To begin, a schematic of the modeling framework showing a hypothetical computational domain (Ω) can be seen in Figure 1.6. Reiterating the tasks that were laid out in the previous section, the first task would be to discretize the domain into an unstructured mesh consisting of triangular elements and possibly
triangular prisms (in future work when the model coupling is extended to 3D) that cover all of the relevant features within the domain boundary ($\partial\Omega$), e.g., topographic features such as ridges and channels. Generating a high-quality mesh that does not become too expensive from a computational perspective and that does not sacrifice mesh quality presents several challenges. The mesh generation techniques developed in this thesis are discussed in Chapter 3.

![Figure 1.6: Schematic of the multi-physics, multi-dimensional modeling framework showing part of a computational domain.](image)

**The Multidimensional Approach**

After the domain has been discretized, elements will be classified as one of three main types:
• Shallow Water Equation (SWE) elements (either 2D and/or 3D)
• Overland flow elements (either 2D or 1D)
• Channel elements (1D)

The SWE elements are triangular elements with more than one edge initially located in a water body. Elements over initially dry regions, referred to here as overland flow elements, will be triangular, however, overland flow calculations will be both 2D and 1D and governed by the kinematic overland flow equations. These overland flow elements are further classified based on the type of edge segment they possess – either ridge edge, flow edge, or channel edge based on the local topographic features, as shown in Figure 1.6. The channel edge segments will be used to represent channels or rivers that cannot be appropriately resolved in two-dimensions, due to the high resolution required to resolve channels in two-dimensions. These channel elements would contain the geometric properties needed to compute the cross-sectional area of flow using the 1D section-averaged St. Venant equations, e.g., channel depth, width, side slopes.

The Multi-Physics Approach

In an extreme storm event where flooding is caused by storm surge and/or runoff due to torrential rainfall, the elements described would adapt based on the local flow conditions. For example, water may propagate into a 1D channel via overland flow and/or storm surge back water effects. Once the channel reaches its carrying capacity it then ”floods” and the 1D channel edge and corresponding overland flow elements would adapt to SWE elements for flooding scenarios. Then once the water recedes the 1D and 2D elements would revert back to their original model equations.
In this thesis, the multidimensional aspect of this project is developed and tested using the kinematic wave equations with the DG formulation. The model developed is called DG-SAKE (Discontinuous Galerkin - Section Averaged Kinematic Equations).

1.3 Organization of this Thesis

The rest of this thesis is organized as follows. Chapter 2 will provide an overview of the governing equations used for the physical processes that are being modeled. Following the model description, Chapter 3 will describe the DG formulation in addition to the multidimensional coupling and implementation. Then, in Chapter 4, a brief overview will be given on an automatic mesh generator co-developed by the author called ADMESH+ and the latest developments in the mesh generator being used for discretization of overland domains will be presented. Chapter 5 will provide numerical examples and applications of the current state of the overland flow model. Finally, Chapter 6 concludes with discussion on future work for this project.
Chapter 2: Model Description

In this chapter, a basic description of the governing hydrodynamic equations are provided for the purpose of understanding the implementation of the computational model described in later chapters.

2.1 Governing Equations

The governing equations for shallow water hydrodynamics are derived from the Navier-Stokes equations. For shallow water flows, a general assumption used is the vertical length is much smaller than the horizontal length. For example, if $H$ is the depth of the ocean and $L$ is the typical wave length of a wave, the shallow water assumption says $H/L$ is much smaller than 1 (i.e., $(H/L) << 1$). With this assumption, a balance in the momentum in the vertical $z$-direction reduces to the hydrostatic approximation for the pressure, i.e.,

$$p = \rho g (\zeta - z).$$
The horizontal momentum equations, together with the continuity equation, form the three dimensional shallow water equations in Cartesian coordinates \((x, y, z)\),

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv = -g \frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} \right) \tag{2.2}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu = -g \frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial z} \left( \mu \frac{\partial v}{\partial z} \right) \tag{2.3}
\]

Figure 2.1 shows a schematic of the three-dimensional shallow water model where \(\zeta = \zeta(t, x, y)\) is the elevation \((L)\) of the free surface relative to the datum; \(b = b(x, y)\) is the bathymetry \((L)\), measured positive downward from the datum; \((u, v, w)\) are the fluid velocities \((L/T)\) in the \(x, y,\) and \(z\)-directions respectfully; \(g\) is the acceleration due to gravity \((L/T^2)\); \(s\) represents a source term \((L/T)\); and \(H = H(t, x, y) = \zeta + b\) is the total depth \((L)\) of the water column.

![Schematic of 3D Shallow Water Model](image)

**Figure 2.1**: Schematic of 3D Shallow Water Model.

The two dimensional form of the shallow water equations can be obtained by integrating over the depth and applying the appropriate boundary conditions. To account for overland flow from rainfall, the appropriate source terms are included.
in the continuity equation. This yields the depth-averaged shallow water equations, commonly referred to as either the two dimensional shallow water or dynamic wave equations,

\[
\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = r - i \tag{2.4}
\]

\[
\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{1}{2}gH^2\right) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) = gH(S_{0x} - S_{fx}) \tag{2.5}
\]

\[
\frac{\partial}{\partial t}(H\bar{v}) + \frac{\partial}{\partial x}(H\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(H\bar{v}^2 + \frac{1}{2}gH^2\right) = gH(S_{0y} - S_{fy}) \tag{2.6}
\]

Figure 2.2 shows a schematic of the two-dimensional shallow water model, with \(\bar{u}\) and \(\bar{v}\) representing the depth-averaged horizontal velocities \((L/T)\); \(r\) is the rainfall rate \((L/T)\); \(i\) is the infiltration rate \((L/T)\); \(S_{0x}, S_{0y}\) are the bed slopes in the \(x\) and \(y\)-directions \((L/L)\); \(g\) is the acceleration due to gravity \((L/T^2)\); and \(S_{fx}, S_{fy}\) are the bed friction terms in the \(x\) and \(y\) directions \((L/L)\), respectively.

![Schematic of 2D Shallow Water Model.](image)
For open-channel flow applications, equations 2.4, 2.5, and 2.6 can be integrated laterally to derive the one dimensional, section averaged, shallow water equations (also known as the St. Venant equations after the French mathematician),

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_L \tag{2.7}
\]

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( Q\bar{u} + gI_1 \right) = gA(S_0 - S_f) + gI_2. \tag{2.8}
\]

Figure 2.3 shows a schematic of the one-dimensional shallow water model, where \(x\) is the horizontal distance along a channel; \(A(x,t)\) = the wetted cross-sectional area of flow \((L^2)\); \(Q\) is the volumetric flow rate \((L^3/T)\); \(q_L\) is the lateral inflow into the channel \(((L^3/T)/L)\); \(S_0\) is the channel slope \((L/L)\); \(S_f\) is the friction slope \((L/L)\); and \(I_1\) and \(I_2\) represent the hydrostatic pressure force and the pressure force caused by the channel width variations, respectively.

![Figure 2.3: Schematic of 1D Shallow Water Model.](image)

Given the one-dimensional and two-dimensional governing equations presented above, simplification of these equations for overland flow applications will now be presented.
2.2 Kinematic Wave Approximation

In overland flow applications where the depth of flow can be very small, employing the full shallow water equations can be difficult due to the low flow depths and excessive since some terms in the momentum equation do not have a large influence on flow. The kinematic wave theory [21] is a widely accepted approach for modeling overland flow, which neglects the local acceleration, convective acceleration, and pressure terms in the momentum equations. Thus, the so-called kinematic flow on a plane or in wide channels occurs when there exists a balance between the force causing the flow (i.e., gravitational force, \( S_0 \)) and the force resisting the flow (i.e., drag force, \( S_f \)) in the momentum equations.

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( Q \bar{u} + gI_1 \right) - gI_2 - gA(S_0 - S_f) = 0
\]

The following sections provide the necessary details of the 1D and 2D kinematic wave approximation used for the overland flow model discussed in later Chapters.

2.3 1D Kinematic Overland Flow and Channel Routing

Applying the kinematic wave approximation to equation (2.8), the momentum equation is reduced to,

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( Q \bar{u} + gI_1 \right) - gI_2 - gA(S_0 - S_f) = 0
\]

\[
-gA(S_0 - S_f) = 0
\]

\[
S_0 = S_f
\]
The volumetric flow rate in the continuity equation (2.7) is approximated using Manning’s resistance formula for uniform flow,

\[ Q = \frac{k}{n} R_h^{2/3} \sqrt{S_0 A} \]  

(2.9)

where \( n \) is Manning’s roughness coefficient \((T/L^{1/3})\); \( R_h \) is the hydraulic radius \((L)\); and \( k = 1.486 \) for English units and \( k = 1 \) for SI units. The continuity equation (2.7) will be used to route converging overland flow (i.e., lateral inflow) to either channel networks or basin outlets in the domain. The flow rate, \( Q \), for channel flow and channel routing can be approximated using simple cross-sectional geometry for channels. Equation (2.9) can be re-written in the following general form,

\[ Q = \alpha \sqrt{S_0 A^m}. \]  

(2.10)

Where the coefficient \( \alpha \) and the constant \( m \) are defined based on the type of cross-sectional geometry, e.g., rectangular, triangular, or trapezoidal cross-sections. In this work, the two types of cross-sectional geometries used are rectangular and triangular.

**Triangular Channel Geometry**

Figure 2.4 shows a schematic of a triangular channel cross-section, where \( H \) is the water depth; \( z_L \) and \( z_R \) are the left and right side slopes of the channel, respectfully. Given the following geometric relationships for the wetted area \((A)\), wetted perimeter

![Figure 2.4: Schematic of Triangular Channel Geometry.](image-url)
(P), and hydraulic radius \((Rh)\),

\[
A = \frac{1}{2} H^2 (z_L + z_R)
\]

\[
P = H (\sqrt{1 + z_L} + \sqrt{1 + z_R})
\]

\[
Rh = \frac{A}{P} = \frac{\sqrt{(A/2)(z_L + z_R)}}{(\sqrt{1 + z_L} + \sqrt{1 + z_R})}
\]

and making the appropriate substitution into equation (2.9), the \(\alpha\) coefficient and constant \(m\) are,

\[
\alpha = \frac{k}{n} \left( \frac{(1/2)(z_L + z_R)^{1/3}}{(\sqrt{1 + z_L} + \sqrt{1 + z_R})} \right)
\]

\[
m = \frac{4}{3}
\]

**Rectangular Channel Geometry**

Figure 2.5 shows a schematic of a rectangular channel cross-section, where \(H\) is the water depth and \(B\) is the width of the channel. Given the following geometric relationships for the wetted area \((A)\), wetted perimeter \((P)\), and hydraulic radius \((Rh)\),

\[
A = BH
\]

\[
P = B + 2H
\]

\[
Rh = \frac{A}{P} = \frac{A}{B + 2(A/B)}
\]

Figure 2.5: Schematic of Rectangular Channel Geometry.
and making the appropriate substitution into equation (2.9), the $\alpha$ coefficient and constant $m$ are,

$$\alpha = \frac{k}{n} \left( \frac{1}{B + 2(A/B)} \right)^{2/3}$$
$$m = \frac{5}{3}$$

The 1D kinematic wave equations will govern the fluid flow provided by the lateral inflow from either the runoff of overland flow into a channel, or the flow converging on the surface due to the topographic features. To describe overland flow, the 2D kinematic wave equations will now be described.

## 2.4 2D Kinematic Overland Flow

In two-dimensional space, applying the kinematic wave approximation to equations (2.5) and (2.6), the momentum equations in the horizontal directions simplify to,

$$S_{0x} = S_{fx}$$
$$S_{0y} = S_{fy}$$

The bed friction, $S_{fx}$ and $S_{fy}$, can be determined by,

$$S_{fx} = \left( \frac{n}{k} \right)^2 \frac{u \sqrt{u^2 + v^2}}{H^{4/3}}$$
$$S_{fy} = \left( \frac{n}{k} \right)^2 \frac{v \sqrt{u^2 + v^2}}{H^{4/3}}$$

For shallow overland flows on wide planes, $R_h \approx H$, and Manning’s formula can be written as,

$$V = \frac{k}{n} \sqrt{S_0} H^{2/3}$$
where \( V \) is the cross-sectional average velocity \((L/T)\). Substituting equation (2.13) into equations (2.11) and (2.12) produces the following velocity equations,

\[
\bar{u} = \frac{k}{n} \frac{S_0 x}{\sqrt{S_0}} H^{2/3} \tag{2.14}
\]

\[
\bar{v} = \frac{k}{n} \frac{S_0 y}{\sqrt{S_0}} H^{2/3} \tag{2.15}
\]

Finally, substituting equations (2.14) and (2.15) into the 2D continuity equation gives,

\[
\frac{\partial H}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = r - i \tag{2.16}
\]

\[
q_x = H \bar{u} = \frac{k}{n} \frac{S_0 x}{\sqrt{S_0}} H^{5/3} \tag{2.17}
\]

\[
q_y = H \bar{v} = \frac{k}{n} \frac{S_0 y}{\sqrt{S_0}} H^{5/3} \tag{2.18}
\]

The 1D and 2D kinematic wave equations governing overland flow have been developed and the coupling of these models will be explained in further detail in Chapter 3. Due to the simplification made with the kinematic wave approximation, the limitation of these equations is now discussed.

### 2.5 Kinematic Wave Number

The kinematic wave number, denoted \( K \), is a common criteria used to judge whether the kinematic wave approximation is appropriate based on the flow conditions over a sloping plane subject to rainfall or lateral inflow [39]. The kinematic wave number is expressed as,

\[
K = \frac{S_0 L}{H F_0^2} \tag{2.19}
\]

where \( S_0 \) is the slope of the plane, \( L \) is the length of the plane, \( H \) is the normal flow depth, and \( F_0 \) is the Froude number, expressed as,

\[
F_0^2 = \frac{V^2}{gH} \tag{2.20}
\]
where $V$ is the velocity of normal flow. For large values of $K$, the kinematic wave approximation is valid. It’s been found that for $K = 10$, the maximum error as a result of neglecting the dynamic terms of the St. Venant equations was approximately 10% [39, 28]. This error decreased as $K$ increased. Small values of $K$ indicate the dynamic terms have a greater influence on the flow, thus indicating the kinematic wave approximations may not be as applicable for small values of $K$. The kinematic wave number would be a criteria used in the developed model for the multi-physics approach in adapting between the kinematic and dynamic wave approximations of overland and channel flow.

### 2.6 Green-Ampt Infiltration Model

At the beginning of a rainfall event, under the assumption that the soil is dry, the rainfall is infiltrated into the soil. The model used to describe soil infiltration is the so-called Green-Ampt model (1911). By neglecting ponding on the surface, the infiltration rate, $i$, can be expressed as,

\[
    i = K_s \left( 1 + \frac{H_f M_d}{F} \right) \\
    F = K_s t + H_f M_d \ln \left( 1 + \frac{F}{H_f M_d} \right)
\]

where $i$ is the infiltration rate ($L/T$), $K_s$ is the saturated hydraulic conductivity ($L/T$), $H_f$ is the capillary pressure head at the wetting front ($L$), $M_d$ is the moisture deficit ($L^3/L^3$) (i.e., the effective porosity subtracted by the soil initial moisture content, $\theta_e - \theta_i$), and $F$ is the total infiltrated depth ($L$).
Due to the fact that the infiltration model equation is implicit, an explicit solution is used as used in the CASC2D-SED model [30]. The explicit solution is as follows,

\[ i^{t+\Delta t} = \frac{1}{2\Delta t} \left( I_1 + \left( I_2^2 + 8I_2\Delta t\right)^{1/2} \right) \] (2.23)

\[ I_1 = K_s \Delta t - 2F_t \] (2.24)

\[ I_2 = K_2 F_t + K_s H_f M_d \] (2.25)

In the next chapter, a DG formulation of the equations presented in this section for 1D channel routing and 2D overland flow will be presented.
Chapter 3: Finite Element Discretization

The discontinuous Galerkin (DG) method is a finite element method. As such, it uses a weak, or variational form of the governing equations. DG methods possess several advantageous features ranging from easily varying polynomial order from one element to another, being highly parallelizable, and being locally conservative (i.e., satisfying conservation principles element by element) just to name a few. In this chapter, the DG method is applied to a general form of the governing equations presented in Chapter 2, followed by an explanation of the multidimensional coupling methodology.

3.1 Discontinuous Galerkin Formulation

The DG formulation will be presented in an all-encompassing formulation for the one-dimensional and two-dimensional models. To begin, a general time-dependent, hyperbolic equation is introduced,

\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u) = s, \quad x \in \Omega, \quad t > 0 \]  

(3.1)

with the initial condition

\[ u(x, 0) = u_o \text{ on } \Omega. \]

We begin by partitioning \( \Omega \) into \( N \) subdomains (i.e., elements), denoted \( \Omega_e \). The term element in this context refers to the element shape or geometry (1D line, 2D
triangle, quadrilateral, etc.). Next, we multiply (3.1) by a sufficiently smooth test function, \( v(x) \), and integrate over each element,

\[
\int_{\Omega_e} \left( \frac{\partial u}{\partial t} \right) v \, d\Omega_e + \int_{\Omega_e} (\nabla f(u)) \, v \, d\Omega_e = \int_{\Omega_e} s \, v \, d\Omega_e. \tag{3.2}
\]

The spatial derivative is then integrated by parts,

\[
\int_{\Omega_e} \left( \frac{\partial u}{\partial t} \right) v \, d\Omega_e - \int_{\Omega_e} (f(u) \nabla v) \, d\Omega_e + \int_{\partial \Omega_e} f(u) \cdot n v \, ds = \int_{\Omega_e} s \, v \, d\Omega_e \tag{3.3}
\]

where \( n \) is the outward unit normal to the element boundary \( \partial \Omega_e \) (representing the one dimensional end-points of an element or the boundary edges of two dimensional element). This further weakens the PDE by taking the derivative off of the flux function, \( f(u) \), and placing it on the test function, \( v \).

Next, an approximate form of equation (3.3) is made by taking \( u \approx u_h \) and choosing a test function \( v = v_h \), both of which belong to polynomial spaces of degree \( p \) denoted by \( \mathbb{P}^p \). This gives the discrete weak form of the general equation, where we want to find \( u_h \mid_{\Omega_e} \in \mathbb{P}^p \) such that,

\[
\int_{\Omega_e} \left( \frac{\partial u_h}{\partial t} \right) v_h \, d\Omega_e - \int_{\Omega_e} (f(u_h) \nabla v_h) \, d\Omega_e + \int_{\partial \Omega_e} f(u_h) \cdot n v_h \, ds = \int_{\Omega_e} s \, v_h \, d\Omega_e, \tag{3.4}
\]

\[\forall v_h \in \mathbb{P}^p \text{ and } \forall \Omega_e \in \Omega.\]

Moving the flux terms to the right-hand-side of equation (3.4) we have,

\[
\int_{\Omega_e} \left( \frac{\partial u_h}{\partial t} \right) v_h \, d\Omega_e = \int_{\Omega_e} s \, v_h \, d\Omega_e + \int_{\Omega_e} (f(u_h) \nabla v_h) \, d\Omega_e - \int_{\partial \Omega_e} f(u_h) \cdot n v_h \, ds. \tag{3.5}
\]

We will now define two new terms,

\[
M_e(u_h, v_h) \equiv \int_{\Omega_e} \left( \frac{\partial u_h}{\partial t} \right) v_h \, d\Omega_e \tag{3.6}
\]

\[
F_e(u_h, v_h) \equiv \int_{\Omega_e} s \, v_h \, d\Omega_e + \int_{\Omega_e} (f(u_h) \nabla v_h) \, d\Omega_e - \int_{\partial \Omega_e} f(u_h) \cdot n v_h \, ds. \tag{3.7}
\]
so equation (3.5) can be written compactly as

\[ M_e(u_h, v_h) = F_e(u_h, v_h), \forall v_h \in \mathbb{P}_p \text{ and } \forall \Omega_e \in \Omega. \tag{3.8} \]

where \( M_e(u_h, v_h) \) is a bilinear operator in \( u_h \) and \( v_h \), that is, linear in both \( u_h \) and \( v_h \) separately, and \( F_e(u_h, v_h) \) is a linear operator in \( v_h \) only.

Given a set of polynomial basis functions \( \Phi = [\phi_0, \phi_1, \ldots, \phi_n]^T \), any test function \( v_h \) can be written in the form, \( v_h = v^T \Phi = [\alpha_0 \phi_0 + \alpha_1 \phi_1 + \ldots + \alpha_n \phi_n] \), where \( \alpha_i \) are arbitrary scalars and \( v \) is a column vector containing the time-dependent degrees of freedom of \( v_h \). Substituting this expanded form of \( v_h \) into equation (3.8) we have,

\[
\alpha_0 [M_e(u_h, \phi_0)] + \alpha_1 [M_e(u_h, \phi_1)] + \cdots + \alpha_p [M_e(u_h, \phi_p)] \\
= \alpha_0 [F_e(u_h, \phi_0)] + \alpha_1 [F_e(u_h, \phi_1)] + \cdots + \alpha_p [F_e(u_h, \phi_p)],
\]

where we have made use of the linearity properties of both \( M_e \) and \( F_e \). In order for the equality above to hold, each of the bracketed terms multiplied by a given \( \alpha_i \) must be equal, that is,

\[
(\alpha_0) : \quad M_e(u_h, \phi_0) = F_e(u_h, \phi_0) \\
(\alpha_1) : \quad M_e(u_h, \phi_1) = F_e(u_h, \phi_1) \\
\vdots \\
(\alpha_p) : \quad M_e(u_h, \phi_p) = F_e(u_h, \phi_p).
\]

Note that the set of equations above are for each element \( \Omega_e \). Given this set of equations for each element, a substitution can also be made for the expanded form of \( u_h \), similar to the process for expanding \( v_h \). Over each element \( \Omega_e \) we have,

\[ u_h \big|_{\Omega_e} = u_0(t) \phi_0 + u_1(t) \phi_1 + \cdots + u_p(t) \phi_p. \]
where the \( u_i(t) \) are the set of time-dependent degrees of freedom we want to solve for.

For the left-hand-side of the equation, we write \( u_h \) with the time derivative, which appears in the set of equations, as,

\[
\frac{\partial u_h}{\partial t} = \frac{\partial}{\partial t} \left( u_0(t)\phi_0 + u_1(t)\phi_1 + \cdots + u_p(t)\phi_p \right),
\]

\[
= \dot{u}_0(t)\phi_0 + \dot{u}_1(t)\phi_1 + \cdots + \dot{u}_p(t)\phi_p,
\]

where \( \dot{u}_i = \frac{du_i}{dt} \).

Substituting the above expression into equation (3.8), as we did with \( v_h \), we have,

\[
\begin{align*}
(\alpha_0) : & \quad \dot{u}_0 M_e(\phi_0, \phi_0) + \dot{u}_1 M_e(\phi_0, \phi_1) + \cdots + \dot{u}_p M_e(\phi_0, \phi_p) = F_e(u_0\phi_0, \phi_0) \\
(\alpha_1) : & \quad \dot{u}_1 M_e(\phi_1, \phi_0) + \dot{u}_1 M_e(\phi_1, \phi_1) + \cdots + \dot{u}_p M_e(\phi_1, \phi_p) = F_e(u_1\phi_1, \phi_1) \\
& \vdots \\
(\alpha_p) : & \quad \dot{u}_0 M_e(\phi_p, \phi_0) + \dot{u}_1 M_e(\phi_p, \phi_1) + \cdots + \dot{u}_p M_e(\phi_p, \phi_p) = F_e(u_p\phi_p, \phi_p).
\end{align*}
\]

where

\[
M_e(\phi_i, \phi_j) \equiv \int_{\Omega_e} \phi_i \phi_j \, d\Omega_e \tag{3.10}
\]

\[
F_e(\phi_i u_0, \phi_j) \equiv \int_{\Omega_e} s \phi_j \, d\Omega_e + \int_{\Omega_e} (f(u_i\phi_i)\nabla \phi_j) \, d\Omega_e - \int_{\partial\Omega_e} f(u_i\phi_i) \cdot \mathbf{n} \phi_j \, ds \tag{3.11}
\]

The system of equations in (3.9) can be written in matrix-vector form as,

\[
\begin{bmatrix}
M_e(\phi_0, \phi_0), & M_e(\phi_0, \phi_1), & \cdots, & M_e(\phi_0, \phi_p) \\
M_e(\phi_1, \phi_0), & M_e(\phi_1, \phi_1), & \cdots, & M_e(\phi_1, \phi_p) \\
\vdots & \vdots & \ddots & \vdots \\
M_e(\phi_p, \phi_0), & M_e(\phi_p, \phi_1), & \cdots, & M_e(\phi_p, \phi_p)
\end{bmatrix}
\begin{bmatrix}
u_0 \\
u_1 \\
\vdots \\
u_p
\end{bmatrix}
= 
\begin{bmatrix}
F_e(u_0\phi_0, \phi_0) \\
F_e(u_1\phi_1, \phi_1) \\
\vdots \\
F_e(u_p\phi_p, \phi_p)
\end{bmatrix}
\]

\[
\equiv \mathbf{M}_e \quad \equiv \mathbf{u}_e \quad \equiv \mathbf{F}_e 
\]

where \( \mathbf{M}_e \) is referred to as the element mass matrix, \( \mathbf{u}_e \) is the element DG solution vector, and \( \mathbf{F}_e \) represents the element right-hand-side of the equation. The full set of
discrete equations for all $N$ elements can be written in the compact form,

$$ M\dot{u} = F(u) \quad (3.13) $$

Finally, inverting the mass matrix to solve for the DG solution vector gives,

$$ \dot{u} = M^{-1}F(u) = L(u) \quad (3.14) $$

where

$$ u = [u_1^T, u_2^T, \ldots, u_N^T]^T $$

and

$$ L(u) = [(M_1^{-1}F_1(u_1))^T, (M_2^{-1}F_2(u_2))^T, \ldots, (M_N^{-1}F_N(u_N))^T]^T, $$

where $L(u)$ is now a system of ordinary differential equations (ODE’s) and is referred to as the DG spatial operator.

### 3.1.1 The Numerical Flux

In the previous section, the DG formulation presented does not enforce continuity along element boundaries, thus allowing the numerical solution to be discontinuous across element edges. Consequently, the flux may not be uniquely defined which gives rise to the Riemann problem, i.e., seeking the solution of a hyperbolic equation at a discontinuity. The solution to this problem is the introduction of the so-called numerical flux, denoted $\hat{f}$, which computes a single-valued flux across element boundaries based on the direction information is propagating.

In the kinematic wave model, one of the implications of the first order continuity equation is that the flow travels in the downstream direction only and the formulation cannot accommodate flow that travels in the upstream direction as in the case of backwater flow. Thus, information from element to element will propagate in the
direction the element is sloped (i.e., flow direction). See, for example, Figure 3.1. Given this characteristic of the kinematic wave model, each element edge with a flow

\[ \Omega_{	ext{e}} u_{i} + h u_{i} - h u_{i} + h \]

Figure 3.1: Left: The 2D solution at the interior (+) and the exterior (-) of edge \( i \). Arrows denote flow direction. Right: The 1D solution at the interior (+) and the exterior (-) of node \( i \).

edge classification (see Figure 1.6) will have a designated exterior element (-) and designated interior element (+). Therefore, an upwind numerical flux can be used and the flux function in the boundary integral can be evaluated as,

\[ f(u_{h}) \cdot n \approx \hat{f}(u_{h}^{-}, u_{h}^{+}) = f(u_{h}^{-}) \cdot n \]  

(3.15)

3.1.2 Element Calculations

For each element \( \Omega_{e} \) in the domain \( \Omega \) there are a number of function evaluations and integrals to perform. It is useful here to define a reference, or master element, denoted by \( \hat{\Omega} \), over which we can define the polynomial basis and perform integrations, thereby removing the complexity of evaluating functions and integrals for each unique element. Computing the solution in the local element \( \Omega_{e} \) is then simply a matter of performing a coordinate transformation from the master element coordinates \( (\xi, \eta) \) to the local element coordinates \( x \). Figure 3.2 below shows the three master element
transformations for the 3 types of elements used in this research. For further details on the master element calculations, see [40, 23].

Figure 3.2: Master element transformation for a 1D element, a 2D triangular element and a 2D quadrilateral element.
3.1.3 Multidimensional Coupling

In the multidimensional DG framework, the model makes use of the Gauss–Lobatto (GLb) integration rule for numerical integration. In [40] the benefits of using Gauss–Lobatto rules over other commonly used integration rules are discussed.

In order to couple the runoff from the 2D overland flow with the lateral inflow of the 1D channels, the integration points along the 1D element must align with the 2D boundary integration points. The model makes use of a connectivity data structure for assigning the 2D radiating flux to the corresponding 1D integration points. Figure 3.3 illustrates how the 1D Gauss–Lobatto integration points align with the 2D Gauss–Lobatto boundary integration points of the element.

![Diagram showing 2D to 1D Gauss–Lobatto Rule for \( p = 1 \).]

Figure 3.3: 2D to 1D Gauss–Lobatto Rule for \( p = 1 \).
3.1.4 Runge-Kutta Time Discretization

The semi-discrete DG equations given by 3.14 are solved in time using explicit Runge–Kutta time steppers of order $p + 1$; details can be found in [19].
Chapter 4: Mesh Generation

This chapter will provide a brief overview of the existing methods employed in the mesh generation software called ADMESH+ and the further development required in order to automate the mesh generation process for overland applications.

4.1 ADMESH+ Overview

Prior and concurrent development in the area of mesh generation by the author and colleagues has resulted in a mesh generation application called ADMESH+ [5]. ADMESH+ is a two-dimensional, automatic unstructured mesh generator for shallow water models. Starting with only target minimum and maximum element sizes and points defining the boundary and bathymetry/topography of the domain, the goal of the mesh generator is to automatically produce a high-quality mesh from this minimal set of input. From the geometry provided, properties such as local features, curvature of the boundary, bathymetric/topographic gradients, approximate flow characteristics can be extracted, which are then used to determine local element sizes. The result is a high-quality mesh, with the correct amount of refinement where it is needed to resolve all the geometry and flow characteristics of the domain. For more details on the numerical methods employed in ADMESH+, see [5].
The ADMESH+ application is implemented in MATLAB and uses an interactive graphical user interface (GUI) that allows the user to view and edit input data and produce relevant output. The GUI also contains an option to extract coastline data via the National Oceanic and Atmospheric Administrations (NOAA) desktop coastline extractor database.

Prior to this work, ADMESH+ generated a mesh using the following (optional) criteria:

- **Shoreline Curvature:** In order to preserve the complexity of natural shorelines adequately, the user has the option to control the density of elements in areas of high curvature along the shoreline. This is done by specifying the number of elements per radian.
- **Local Feature Size:** The user can specify the number of elements that would span a channel or tributary, or come between two shorelines that are relatively close to one another, using the so-called medial axis of the domain.
- **Elevation Gradients:** A user prescribed value that controls the density of the elements in areas where there are sharp elevation gradients.
- **Dominate Tides:** The user can specify the number of elements per tidal wavelength.
- **Mesh Grading:** This feature controls the rate at which elements grow from the minimum element size to the maximum element size.
- **Max & Min Element Sizes:** Users can specify in ADMESH+ what the minimum and maximum element sizes should be in the domain.

Once a mesh is generated, ADMESH+ displays the mesh along with mesh stats such as the number of elements and nodes along with the mean and minimum elements.
quality. Element quality in ADMESH+ is based on a scale from 0 to 1. A value of 0 represent a completely degenerate triangle and a value 1 represents an equilateral triangle.

With the foundation of the mesh generation approach in ADMESH+, this work will be extended to aid in the domain discretization for the overland flow model discussed in previous chapters.

4.2 Watershed Delineation Approach

A watershed or drainage basin is defined as a geographic extent that shares a common single point, often referred to as a sink or basin outlet, to which all surface water from rain or melting snow converges. The surface water is transported to this point via a combination of overland runoff flow and free surface flow occurring in natural rivers and streams as well as man-made channels and piping networks. Watersheds are extremely complicated systems with a variety of complex spatial characteristics, for example, infiltration rates, friction coefficients, interception depths, channel networks, etc. Given these complexities, the construction of an accurate and computationally feasible representation of the most relevant features of the watersheds or drainage basins of a given geographical area presents several challenges. These challenges consist of:

- Extracting a smooth approximation of the watershed boundary of interest.
- Extracting a smooth approximation of channel networks with the flexibility of adding and/or removing streams and sub-networks not needed.
- Performing a constrained triangulation that would allow the element edges to vary in size based on the minimum element size specified by the user.
An effective way to obtain the criteria outlined above is to utilize high-resolution digital elevation maps (DEM) for extraction of watershed boundaries, channel networks, and the elevation for the domain. Several geographic information system (GIS) tools exist for DEM processing (e.g., ArcGIS, Manifold System). From experimentation with a variety of programs, an open source toolbox called TopoToolbox is used in combination with ADMESH+. TopoToolbox is a well-developed open source toolbox containing an extensive MATLAB library for the analysis of DEMs [32]. In later sections, the DEM processing procedure will be outlined through the process of (1) extracting a watershed boundary, (2) obtaining a smooth approximation of the boundary via hobby spline approximation [12], (3) extracting a channel network and (4) obtaining a smooth approximation of the channel network.

4.2.1 Constrained Mesh Generation

A constrained delaunay triangulation constrains triangular edges along user specified edges. Thus, the delaunay condition for triangulating at these constraints is ignored and the constraint specified by the user is honored. During the mesh generation process, the AMDESH+ application automatically distributes element edges along the channel segments based on the minimum element size input by the user. This ensures that no elements fall below the minimum element size specified by the user.

4.2.2 Overland Flow and Channel Routing

Although the channel network will have been defined by the user, the triangulated mesh topography could still produce channels where two neighboring elements share a common edge with converging flow. Due to the fact that the kinematic wave equations
cannot handle back water effects it is important that the fluid flow be routed from 
the converging edge to a channel or an outlet based on where the elevation gradients 
lead the flow.

These converging edges are identified by first, computing the slope vector, $\mathbf{s}$, of 
each element, and second, determining if the edge has flow coming in from both 
neighboring elements [2]. Figure 4.1 shows a schematic of two converging elements. 
The element slope vector is calculated by taking the cross product of the horizontal 
vector ($\mathbf{h}$) and the vector normal to the element surface ($\mathbf{n}$),

$$
\mathbf{s} = \mathbf{h} \times \mathbf{n} = [(y_1z_2 - y_2z_1)(x_1y_2 - x_2y_1)] \hat{i} 
+ [(z_1x_2 - z_2x_1)(x_1y_2 - x_2y_1)] \hat{j} 
- [(z_1x_2 - z_2x_1)^2 + (y_1z_2 - y_2z_1)^2] \hat{k}.
$$

The element slope vectors can then be used to classify the element edges as either 
being an outflow edge or an inflow edge. This criteria can then be used to classify 
the element edge types as either no-flow, flow, or channel.
4.3 Mesh Examples

4.3.1 Big Tujunga Creek Watershed Delineation

The first example is of a 30 meter spaced DEM of the Big Tujunga Creek in Los Angeles County, California. This is an example DEM provided in the TopoToolbox package. Figure 4.2 shows the DEM.

![DEM of Big Tujunga Creek](image)

Figure 4.2: DEM of Big Tujunga Creek in Los Angeles, California.

The first step would be to compute the drainage basins of the DEM. As in any case, multiple drainage basins will appear in the DEM, in this example the largest basin is chosen. The main watershed boundary of the DEM can be seen in Figure 4.3. Typically, with boundaries derived from a structured DEM grid, the boundary can have a stair-case like approximation and create sharp curvatures in the boundary. These sharp curvatures can potentially be problematic, causing ADMESH+ to force
Figure 4.3: Watershed boundary from DEM of Big Tujunga Creek in Los Angeles, California.

high resolution in areas of the domain where it may not necessarily be needed. To alleviate this problem, a smooth approximation of the boundary is made by making use of hobby splines [12]. Figure 4.4.

Now that the boundary has been obtained, the next step is to extract a channel network. Various options are available for selecting a channel network in the Topo-Toolbox, for example, selection can be made based on the Strahler stream order, the calculated flow accumulation, etc. Figure 4.5 shows the selected stream network. The final step is to obtain a smooth approximation of the stream network similar to how the refined watershed boundary was obtained. The algorithm preserves the stream junctions along with the starting and ending points on streams. Figure 4.6 shows a comparison of the original channel network and the new channel network.
The watershed boundary together with the channel network and corresponding elevation data will be saved in an edge structure file specific to ADMESH+. The Big Tujunga Creek domain is then discretized into a finite element mesh in ADMESH+ and can be seen in Figure 4.7.

The final mesh is obtained by defining all element edges with converging flow and defining a route for one-dimensional flow. Figure 4.8 shows the routing edges.
Figure 4.5: Big Tujunga Creek Channel Network (In Blue).

Figure 4.6: Spline approximation of the channel network. Original channels in blue. New channels in red.
highlighted in red, and the original channel network in blue. It should be noted that this process is completely automated, including the one-dimensional channel network connectivity and boundary condition specifications. Table 4.1 provides the settings used in ADMESH+ along with the mesh details.
Figure 4.8: Tujunga Creek finite element mesh with channel routing and flow direction.

Table 4.1: Summary of the Big Tujunga Creek watershed finite element mesh created with ADMESH+.

<table>
<thead>
<tr>
<th>Domain</th>
<th>No. of Elements</th>
<th>Run Time (min)</th>
<th>Max Element Size (m)</th>
<th>Minimum Element Size (m)</th>
<th>Mean Element Quality</th>
<th>Minimum Element Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big Tujunga Creek</td>
<td>34,411</td>
<td>2.46</td>
<td>405</td>
<td>50</td>
<td>0.98</td>
<td>0.70</td>
</tr>
</tbody>
</table>
4.3.2 Goodwin Creek Watershed Delineation

The Goodwin Creek watershed is an experimental watershed located in Panola County, Mississippi and has served as a platform for research in areas such as watershed hydrology, sediment transport, upstream erosion and more. This watershed has been used as a test case for several hydrology studies and model validations; see, for example, [4, 29, 42, 36]. In Chapter 5, this domain will be used as a numerical test case for the multidimensional overland flow modeling approach. The results of which will be compared with another overland flow model called CASC2D using the same DEM data used in a CASC2D simulation [1].

The DEM in this example is a 90-meter-spaced structured DEM including only elevation data within the watershed boundary. Figure 4.10 shows the DEM.

Figure 4.9: DEM of Goodwin Creek Watershed.
The boundary and channel network of the domain can be delineated, as discussed in the previous section, and a smooth approximation of the watershed features can be made. The delineated watershed boundary and channel network can be seen in Figure 4.10.

![Figure 4.10: DEM of Goodwin Creek Watershed with Boundary (black) and Channel Network (blue) Approximation.](image)

The Goodwin Creek domain is then discretized into a finite element mesh in ADMESH+ and can be seen in Figure 4.11. As illustrated in the previous example, converging flow between neighboring elements is likely and should be checked. The final finite element mesh, including channel routing and flow direction, can be seen in Figure 4.12. The routing edges in the mesh are highlighted in red, and the original channel network in blue. Table 4.2 provides a summary of the mesh generation results with ADMESH+.
Figure 4.11: Goodwin Creek Watershed finite element mesh. Red lines indicate channel network.
Figure 4.12: Goodwin Creek Watershed finite element mesh with channel routing and flow direction.

Table 4.2: Summary of the Goodwin Creek watershed finite element mesh created with ADMESH+.

<table>
<thead>
<tr>
<th>Domain</th>
<th>No. of Elements</th>
<th>Run Time (sec)</th>
<th>Max Element Size (m)</th>
<th>Minimum Element Size (m)</th>
<th>Mean Element Quality</th>
<th>Minimum Element Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodwin Creek</td>
<td>3,944</td>
<td>19</td>
<td>187</td>
<td>58</td>
<td>0.97</td>
<td>0.80</td>
</tr>
</tbody>
</table>
Chapter 5: Numerical Examples and Applications

In this section, numerical examples and results of the DG approach outlined above have been applied to five problems with both analytic solutions and experimental results. The first two problems demonstrate the accuracy and stability of the 1D DG approach using two synthetic rainfall-runoff experiments. The results are compared with analytic solutions of the kinematic wave equation. The third problem validates the 2D DG approach by numerically reproducing a laboratory experiment. The fourth problem validates the coupling approach of 1D and 2D methods using a synthetic V-shaped catchment test case. Finally, the model was used to simulate a rainfall-runoff event in the Goodwin Creek experimental watershed. The stability and accuracy of the coupled 1D and 2D DG approach are examined in this case.

5.1 One Dimensional: Rainfall-Runoff on a Plane

In this test case, consider a plane of length $L$, a unit width of 1 and slope $S_0$. A constant rainfall occurs for a fixed duration of time over the plane. For simplicity, infiltration is not considered thus it is assumed the rainfall is the excess rainfall or the plane is impervious. This is a common test case used in validating models for this type of practical application. See, for example, [15, 17, 42].
5.1.1 Rainfall-Runoff Analytic Solution

Singh [34] derived analytic solutions for an impervious plane subjected to constant uniform rainfall using the method of characteristics. There are two scenarios that can occur with this test case: (A) a rainfall duration that is long enough for the maximum discharge to be reached, also referred to as an equilibrium hydrograph, or (B) a rainfall duration that is short and causes what is referred to as a partial-equilibrium hydrograph. Since this model development is aimed at modeling intense rainfall events, we examine scenario (A).

Figure 5.1 illustrates the method of characteristics solution derived by Singh. The solution is divided into three domains $D_1$, $D_2$ and $D_3$ representing the rising limb, peak discharge and the recession limb of the hydrograph respectively. The $D_1$ region represents the discharge that occurs at the start of the rainfall event. The flow discharge falls within this domain and increases with time until the maximum discharge occurs defined by the time of equilibrium curve $t_e$. The discharge then falls into domain $D_2$ and the discharge is at its maximum until the rainfall stops. Once the rainfall stops the discharge is in the $D_3$ region and the discharge diminishes. The initial condition, boundary condition and domains $D_1$, $D_2$ and $D_3$ are expressed mathematically as follows:

\[
\text{IC: } h(0, t) = 0, \quad 0 \leq t \leq t_d, \quad (5.1)
\]
\[
\text{BC: } h(x, 0) = 0, \quad 0 \leq x \leq L, \quad (5.2)
\]
\[
\text{Domain } D_1: h(x, t) = rt, \quad t < t_e, \quad (5.3)
\]
\[
\text{Domain } D_2: h(x, t) = \left(\frac{rx}{\alpha}\right)^{(1/m)}, \quad t_e < t < t_d, \quad (5.4)
\]
\[
\text{Domain } D_3: \quad t = \left(\frac{x}{\alpha m}\right) h^{(1-m)} - \left(\frac{1}{rm}\right) h + t_d, \quad t > t_d \quad (5.5)
\]
Figure 5.1: Rainfall-runoff solution domain for the equilibrium case by Singh [34].

where \( h \) is the flow depth (\( L \)), \( x \) is the distance along the direction of flow (\( L \)), \( r \)
is the rainfall intensity (\( L/\text{T} \)), \( \alpha \) and \( m \) correspond to the coefficients used in Man-
ning’s formula, \( t \) is the time (\( \text{T} \)), \( t_d \) is the rainfall duration (\( \text{T} \)) and \( t_e \) is the time to
equilibrium constant (\( \text{T} \)). The time to equilibrium is defined as:

\[
t_e = r^{(1-m)/m} \left( \frac{x}{\alpha} \right)^{1/m}
\]

(5.6)

5.1.2 Domain Description and Numerical Solution

The parameters used in this test case were as follows: the length of the domain
\( L = 185.00 \text{ m} \); the bed slope \( S_o = 0.0016 \); the Manning’s roughness coefficient \( n =
0.025 \text{ s/m}^{1/3} \); the rainfall excess rate: \( r_o = 50.8 \text{ mm/h} \); and the rainfall duration: \( t_d = 30 \text{ min} \). A schematic of this test case can be seen in Figure 5.2.
\[ L = 185.00 \text{ m} \]
\[ S_o = 0.0016 \]
\[ n = 0.025 \text{ s/m}^{1/3} \]
\[ r_o = 50.8 \text{ mm/h} \]

Figure 5.2: A schematic of constant rainfall on an impervious surface test case.

In the DG modeling framework variations of mesh size \((h)\) and polynomial degree \((p)\) can be used. To demonstrate the advantage of using \(p\)-refinement this test case was simulated for \(p = 0, 1\) and \(2\) using a mesh size of \(h = 9.25\) meters (20 elements). A time step of \(\Delta t = 1.5\) seconds and total simulation time of \(t_{tot} = 1\) hour was used. The resultant discharge hydrographs are presented in Figure 5.3. Figure 5.3 shows the numerical solution converges to the analytical solution as \(p\) increases. The large differences at the peak discharge of the hydrograph between the \(p = 0\) numerical solution and analytical solution can be attributed to the the coarseness of the mesh. Taking an \(h\)-refinement approach will demonstrate that for \(p = 0\) the DG solution can still capture the peak discharge of the rising limb, however, the \(h\)-refinement is substantially larger in comparison to the \(p\)-refinement needed. Figure 5.4 shows a hydrograph comparison between \(p = 2\) with 20 elements and \(p = 0\) with 1280 elements. The time step for using 1280 elements had to be reduced to \(\Delta t = .2\) seconds to meet the Courant–Friedrichs–Lewy (CFL) condition. It can be shown in Figure 5.4 that \(p\)
$= 0$ performs significantly better with $h$-refinement, however, $p = 2$ still outperforms by capturing the peak discharge at the time of equilibrium. The refinement of $p$ with a larger element size $h$ in comparison to $h$-refinement is advantageous from the standpoint of computer memory and the number of calculations required for a simulation.
Figure 5.3: (a): Discharge hydrograph of 1D Constant Rainfall Test Case at $L = 185$ m. (b): Zoom view of discharge hydrograph at peak discharge.
Figure 5.4: (a): Discharge hydrograph of 1D Constant Rainfall Test Case at $L = 185$ m. (b): Zoom view of discharge hydrograph at peak discharge.
5.2 One Dimensional: Moving Storm on a Plane

Storm movement has shown to be a contributing factor in the shape of flow hydrographs and discharge peaks in watersheds. The effect storm movement has on surface runoff has been demonstrated in many laboratory, numerical and analytic case studies; see, for example, [41, 9, 20, 25, 13, 35]. In this test case, consider a plane of length $L$, a unit width of 1 and slope $S_o$. A moving storm with a velocity of $V_s$ propagates in the down-slope direction of the plane. The rainfall intensity $r$ is assumed to be constant in both space and time. For simplicity, infiltration is not considered thus it is assumed the rainfall is the excess rainfall or the plane is impervious.

5.2.1 Moving Storm Analytic Solution

Singh [35] derived analytic solutions using the method of characteristics for moving storms over a non-infiltrating surface subject to storm movement in either a down-stream direction (moving in the down-slope direction of the plane) or an upstream direction (moving in the up-slope direction of the plane). This approach is similar to what was described in section 5.1.1. In each scenario he provides solutions for equilibrium and partial equilibrium hydrographs. For this test case, the down-slope direction is examined.

Figure 5.5 illustrates the method of characteristics solution derived by Singh for a down-slope moving storm. Similar to the test case presented in section 5.1.1 the solution is divided into three domains $D_1$, $D_2$ and $D_3$ representing the rising limb, peak discharge and the recession limb of the hydrograph respectively. The $D_1$ region represents the discharge that occurs as the storm begins to cover the domain. It should be noted that no runoff occurs until the time reaches the lower bound, $t_L$, of
domain $D_1$. The flow discharge within $D_1$ increases with time until the maximum discharge occurs, defined by the time to equilibrium curve $t_\text{e}$. The discharge then falls into domain $D_2$ and the discharge is at its maximum until the storm passes over the domain defined by and upper bound, $t_U$. Once the storm completely passes over the plane the discharge is in the $D_3$ region and diminishes.

The initial condition, boundary condition and domains $D_1$, $D_2$ and $D_3$ are expressed mathematically as follows:

IC: $h(0,t) = 0, \quad 0 \leq t \leq t_d, \quad (5.7)$

BC: $h(x,0) = 0, \quad 0 \leq x \leq L, \quad (5.8)$

$D_1$: $h(x,t) = \left( \frac{r}{V_s - \alpha h^{(m-1)}} \right) (tV_s - x), \quad t_L < t < t_e, \quad (5.9)$

$D_2$: $h(x,t) = \left( \frac{r}{\alpha} \right)^{(1/m)}, \quad t_e < t < t_U, \quad (5.10)$

$D_3$: $t = t_d + \left( \frac{\alpha h^m}{rV_s} \right) + \left( \frac{h^{(1-m)}}{m\alpha} \right) \left( x - \frac{\alpha h^m}{r} \right), \quad t > t_U, \quad (5.11)$

where $h$ is the flow depth ($L$), $x$ is the distance along the direction of flow ($L$), $r$ is the rainfall intensity ($L/T$), $\alpha$ and $m$ correspond to the coefficients used in Manning’s formula, $t$ is the time ($T$), $t_d$ is the rainfall duration ($T$) and $t_e$ is the time to equilibrium ($T$). The lower time constraint $t_L$, time to equilibrium $t_e$, and upper time constraint $t_U$ are defined as:

$t_L = \frac{x}{V_s} \quad (5.12)$

$t_e = r^{(1-m)/m} \left( \frac{x}{\alpha} \right)^{1/m} \quad (5.13)$

$t_U = t_d + \frac{x}{V_s} \quad (5.14)$
Figure 5.5: Rainfall-runoff solution domain for the equilibrium case by [34].

5.2.2 Domain Description and Numerical Solution

The parameters used in this test case were as follows: the length of the domain \( L = 185 \) m; the bed slope \( S_o = 0.0016 \); the Manning’s roughness coefficient \( n = 0.025 \) \( s/m^{1/3} \); the rainfall excess rate: \( r_o = 50.8 \) mm/h; the moving storm velocity: \( V_s = .90 \) m/s; the storm length: \( L_s = 1850 \) m; and the rainfall duration: \( t_d(x = L) = 37.69 \) min. A schematic of this test case can be seen in Figure 5.6.

The numerical simulation presented used the following parameters: mesh size of \( h = 9.25 \) meters (20 elements); polynomial degree of \( p = 2 \); a total simulation time of 1.5 hours and a time step of \( \Delta t = 1.5 \) seconds. The discharge hydrograph for the analytical and numerical solutions at \( x = 185 \) m are presented in Figure 5.7.
Figure 5.6: A schematic of a moving storm on an impervious surface test case.

The DG-SAKE solution compared with the analytical solution presented in Figure 5.7 show good agreement in all three stages of the hydrograph.
Figure 5.7: (c): Discharge hydrograph of 1D Constant Rainfall Test Case at $L = 185$ m. (d): Zoom view of discharge hydrograph at peak discharge.
5.3 Two Dimensional: Iwagaki’s Experiment

In the literature, there are very few analytical test cases aimed at describing the rainfall-runoff process in 2D. In order to validate the DG-SAKE framework in 2D, a set of laboratory experiments conducted by Yuichi Iwagaki [14] were chosen for a 2D numerical test case. Iwagaki introduced an approximate method for calculating unsteady flow in open channels with lateral inflow. This work has been used for model comparison and validation in several publications; see, for example, [6, 31, 43].

5.3.1 Experimental Setup

One of Iwagaki’s experiments consisted of a 24 m long rectangular flume made of aluminum that was divided into a three plane cascade. The first plane, 0–8 m, had a slope of $S_y = 0.020$, the second, 8–16 m, had a slope of $S_y = 0.015$ and the final plane, 16–24 m, had a slope of $S_y = 0.010$. The flume was not sloped in the transverse direction (i.e., $S_x = 0$) and the cross-section of the flume was 19.6 cm wide and 9 cm tall. A schematic of the domain can be seen in Figure 5.8.

For this experiment, Iwagaki set up the experimental flume to simulate three different rainfall intensities, $r = 0.1080$, 0.0638 and 0.080 cm/s, simultaneously for each respective plane. The experiment was repeated for three different rainfall durations, $t_d = 10$, 20 and 30 seconds.

5.3.2 Domain Discretization and Numerical Solution

The domain as seen in Figure 5.8 was discretized using 2D rectangular elements. Each of the 3 cascades consist of eight elements with edge lengths of $dx = 0.196$ m.
and \( dy = 1 \) m for a total of 24 elements in the domain. The finite element mesh can be seen in Figure 5.9 below.

The modeling parameters used in this test case were as follows: a Manning’s roughness coefficient of \( n = 0.009 \) as reported by Iwagaki; polynomial degree of \( p = 1 \); a total simulation time of 70 seconds with a time step of \( \Delta t = 0.10 \) seconds. Simulations were performed for each of the three rainfall duration settings, \( t_d = 10, 20 \) and 30 seconds. The discharge hydrograph for the Iwagaki’s experimental data and the numerical solutions at the end of the domain, \( x = 24 \) m, are presented in Figure 5.10.

Figure 5.10a shows the 2D numerical approach performs well matching the rising limb, peak discharge and recession limb of the hydrograph. The slight dip in the numerical solution just before the rise to peak discharge (\( t \approx 23 \) sec) is an oscillation in the DG solution.
In Figures 5.10b and 5.10c where the rainfall duration is increased the numerical solution predicts the peak discharge early in both cases. In case (b), the numerical peak discharge is also slightly over estimated. The potential cause of this error could be due to the missing dynamic terms in the kinematic wave approximation. The kinematic wave number ($K$) was calculated at the outlet of the flume ($x = 24$ m) at each time step and is shown in Figure 5.11. For large values of $K$, the kinematic wave approximation is valid. It’s been found that for $K = 10$, the maximum error as a result of neglecting the dynamic terms of the St. Venant equations was approximately 10% [28]. This error decreased as $K$ increased. Small values of $K$ indicate the dynamic terms have a greater influence on the flow, thus indicating the kinematic wave approximations may not be as applicable for small values of $K$. 
From Figure 5.11, we see that for the first simulation, \( t_d = 10 \) seconds, the kinematic wave number decreases rapidly to \( \sim 5 \) at \( t = 10 \) seconds and eventually rises again as the discharge exits the flume. The DG-SAKE approximation matches well in this case because the rainfall duration is short enough that the dynamic terms do not dominate. In the other two terms the rainfall duration is long enough such that the discharge is increased significantly, as can be seen in Figure 5.10, and the kinematic wave approximation is not as applicable in these scenarios. In these situations, the \( K \) value can be used to monitor the applicability of the kinematic wave approximation and act at a threshold value for switching from the kinematic wave approximation to the dynamic wave approximation in future work.
Figure 5.10: (a): Discharge hydrograph for a rainfall duration of $t_d = 10$ sec. (b): Discharge hydrograph for a rainfall duration of $t_d = 20$ sec. (c): Discharge hydrograph for a rainfall duration of $t_d = 30$ sec.
Figure 5.11: The kinematic wave number \((K)\) evaluated at \(x = 24\) m for each numerical simulation.

### 5.4 Multidimensional: The Catchment-Stream Problem

The catchment-stream problem was investigated by Wooding in a series of publications in 1965 in an effort to develop an analytic and numerical approach for calculating a stream hydrograph by approximating a watershed domain as a V-shaped catchment. In the first publication of the three-part series, Wooding derived analytical solutions for (1) the discharge over a plane V-shaped catchment subjected to a constant rainfall for a finite duration and (2) the channel discharge emerging from the catchment discharge [38].

The V-shaped catchment domain can be described as two identical overland flow planes that share a common edge and lies perpendicular to a rectangular channel.
Wooding used the method of characteristics to derive analytical solutions for catchment flow and the stream flow. The derivation and analytical solution for stream flow is complex due to the time varying discharge into the stream from the catchment runoff. Another study, commonly used for model validation for a V-shaped catchment domain, was later published by Overton and Brakensiek [28].

5.4.1 The V-Shaped Catchment Solution

Overton and Brakensiek proposed a hydrograph solution for a V-shaped watershed in terms of the time \( t \), the time to equilibrium for the watershed \( T_{eq} \) and the lag modulus ratio \( \mu^* \). They define the time to equilibrium for the watershed as:

\[
T_{eq} = t_{ec} + \left( \frac{m}{m+1} \right) t_{eo}
\]

(5.15)

where \( t_{ec} \) is the time to equilibrium constant for the channel and \( t_{eo} \) is the time to equilibrium constant for overland flow. The constant \( t_{eo} \) and the overland flow hydrograph are calculated using the same approach presented in section 5.1.1. The constant \( t_{ec} \) is calculated similarly as:

\[
t_{ec} = \frac{q_{L_{max}} (1-m)/m}{L_c} \left( \frac{L_c}{\alpha_c} \right)^{1/m}
\]

(5.16)

where \( q_{L_{max}} = 2rL_o \) is the maximum lateral inflow that occurs from overland flow, \( L_o \) is the length of the overland flow plane, \( L_c \) is the length of the channel, and \( \alpha_c \) is the coefficient for Manning’s formula for a rectangular channel defined in previous chapters as:

\[
\alpha_c = \frac{k}{n} \left( \frac{b}{b^2 + A} \right)^{2/3}
\]

(5.17)

64
If it is assumed the channel width $b$ is wide and the cross-sectional area of flow $A$ is small then equation 5.17 can be re-written as:

$$\alpha_c = \frac{k}{n} \left( \frac{1}{b} \right)^{2/3}$$

Finally, the lag modulus time ($\mu_*$) is defined as:

$$\mu_* = \frac{t_{eo}}{T_{eq}}$$

The final form of the dimensionless watershed (stream) hydrograph is expressed as:

$$0 \leq \frac{t}{T_{eq}} \leq \mu_* \quad Q_* = \left( \frac{(t/T_{eq})^{(m+1)}}{(m + 1 - m\mu_*)(\mu_*^m)} \right)^m$$

$$\mu_* < \frac{t}{T_{eq}} \leq 1 \quad Q_* = \left( \frac{(t/T_{eq}) - (m/(m + 1))\mu_*}{1 - (m/(m + 1))(\mu_*)} \right)^m$$

$$1 < \frac{t}{T_{eq}} \leq t_d \quad Q_* = 1$$

Note that there is no expression for the recession limb of the hydrograph for stream flow in this case. Once the dimensionless form is solved, the dimensional solution is obtained by $Q = Q_*2rL_oL_c$. For further details on these studies, one can refer to [28] and [38].

5.4.2 Domain Description and Numerical Solution

A schematic of the V-shaped catchment can be seen in Figure 5.12. In this example, the overland flow planes slope in toward a rectangular channel and are assumed to have identical width and length. The analytical solutions for overland flow and stream flow have been presented in 1D. For validation purposes of the 1D–2D coupling, the overland flow planes will be represented in a 2D framework and the stream flow will be represented in a 1D framework. The domain characteristics are similar to previous studies using the V-Shaped catchment; see, for example [6, 11].
The overland flow planes are 1000 m x 800 m with a constant downward slope of $S_{ox} = 0.05$ and no slope in the transverse direction (i.e., $S_{oy} = 0.00$). The rectangular channel is 20 m wide with a constant slope of $S_o = 0.02$. The channel height is assumed to always be greater than the flow depth. It should be noted that since channels are treated as 1D it is assumed that the channel bed is flat and only sloped in one direction. The Manning’s roughness coefficients for overland and channel are $n_o = 0.015$ and $n_c = 0.15$, respectively. In this test case the rainfall parameters used were a constant uniform rainfall of 10.8 mm/h for a time of $t_d = 1.5$ hours. Infiltration is not considered in this test case thus it is assumed the rainfall is the excess rainfall or the plane is impervious.

![Figure 5.12: The V-Shaped Catchment.](image)

The V-shaped catchment is discretized into a finite element mesh consisting of triangular elements. The 2D element edges are constrained along the section of the
domain where the channel would be (i.e., the centerline of the channel). The mesh can be seen in Figure 5.13. The mesh contains 120 2D elements and 6 1D elements.

Figure 5.13: A Finite Element Mesh of a V-Shaped Catchment.

The numerical simulation presented used the following parameters: polynomial degree of \( p = 2 \); a total simulation time of 3.3 hours and a time step of \( \Delta t = 35.8 \) seconds. The discharge hydrograph for the analytical and numerical solutions at \( x = 800 \) m and \( y = 0 \) are presented in Figure 5.14.

As we see from Figure 5.7, the numerical solution for the overland flow hydrograph shows good agreement with the analytical solution. The numerical solution also shows good agreement with the stream discharge hydrograph.

### 5.5 Multidimensional: Goodwin Creek Watershed

To validate the DG-SAKE framework, DG-SAKE was tested on a real rainfall event that occurred October 17, 1981 in the Goodwin Creek watershed. In October
Figure 5.14: (a): Discharge hydrograph for 2D overland flow at $x = 800$ m, $y = 0$ m. Note: Discharge is only representative of the runoff for one plane. (b): Discharge hydrograph for 1D channel flow at $x = 800$ m, $y = 0$ m.
of 1981, the U.S. Department of Agriculture (USDA) National Sedimentation Laboratory (NSL) established the Goodwin Creek Experimental Watershed. The watershed is located in Panola County, Mississippi and has served as a platform for research in areas such as watershed hydrology, sediment transport, upstream erosion and more. A detailed report of the watershed digital elevation models (DEM), channel geometries, land use data, soil distribution, precipitation and stream-flow gauges have been presented in a study by Blackmarr [4].

The rainfall event began at 9:19 PM and continued for 4.8 hours. The DG-SAKE approach will be compared with observation data recorded by stream-flow gauges and the CASC2D model [29]. The CASC2D model is a finite difference model that solves for overland flow using the 2D kinematic wave equations and solves for channel flow using the 1D diffusive wave equations. For further details of the model description, see [30].

The following subsections will discuss the Goodwin Creek watershed and the data used for simulation, followed by a comparison of numerical solutions with measured data at six different stream gauges for three different simulations, and finally a discussion of the errors between the models and observational data.

5.5.1 Watershed Overview

The Goodwin Creek Watershed is approximately 21.3 km$^2$ with elevation ranging from 71 m – 128 m above sea level. The DEM, soil type and land use data used for this study was the same data used by the CASC2D model [1, 29]. The elevation, soil type, and land cover data are in 90 m x 90 m spaced grids and can be seen in Figure 5.15. The Goodwin Creek watershed land cover is classified into 4 types; forestland,
water, cultivated and pasture. The Manning’s \( n \) values and interception depths used in the CASC2D model are presented in Table 5.1. The 1D channels (excluding the 1D overland routing channels) in the domain are approximated as rectangular cross-sectioned channels and the overland routing channels are approximated as having a triangular cross section. All channels are given a uniform roughness coefficient of \( n_c = 0.035 \).

<table>
<thead>
<tr>
<th>Land Cover</th>
<th>Roughness, ( n )</th>
<th>Interception (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forest Land</td>
<td>0.25</td>
<td>3.00</td>
</tr>
<tr>
<td>Water</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>Cultivated</td>
<td>0.15</td>
<td>1.00</td>
</tr>
<tr>
<td>Pasture</td>
<td>0.20</td>
<td>1.50</td>
</tr>
</tbody>
</table>

Seven different soil types are identified in the watershed; Calloway, Falaya, Grenada, Loring, Colins, Memphis and Gullied Land. The infiltration parameters used for the Green-Ampt Model were those used by the CASC2D model in [1, 29] and are presented in Table 5.2. The details of these soil characteristics can be found the study by Blackmarr [4].

In this study, precipitation data is used from a total of 16 rain gauges distributed throughout the watershed. Stream discharge data is recorded at 6 stream gauges also distributed throughout the watershed, one of the stations being located at the outlet of the watershed. Figure 5.16 shows the location of the rain and stream gauges in the watershed.
Table 5.2: Soil Infiltration Parameters for the Green-Ampt Infiltration Model.

<table>
<thead>
<tr>
<th>Soil Series</th>
<th>Hydraulic Conductivity (cm/h)</th>
<th>Suction Head (cm)</th>
<th>Moisture Deficit (cm$^3$/cm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calloway</td>
<td>0.336</td>
<td>22</td>
<td>0.29</td>
</tr>
<tr>
<td>Falaya</td>
<td>0.307</td>
<td>14</td>
<td>0.29</td>
</tr>
<tr>
<td>Grenada</td>
<td>0.355</td>
<td>17</td>
<td>0.29</td>
</tr>
<tr>
<td>Loring</td>
<td>0.365</td>
<td>22</td>
<td>0.29</td>
</tr>
<tr>
<td>Collins</td>
<td>0.346</td>
<td>18</td>
<td>0.29</td>
</tr>
<tr>
<td>Memphis</td>
<td>0.432</td>
<td>22</td>
<td>0.29</td>
</tr>
<tr>
<td>Gullied Land</td>
<td>0.384</td>
<td>15</td>
<td>0.29</td>
</tr>
</tbody>
</table>

The rainfall intensity in this test case is computed at all of the points using the inverse squared distance interpolation scheme over the watershed,

\[ r(k) = \frac{\sum_{i=1}^{nrg} \frac{R_i}{d_i^2}}{\sum_{i=1}^{nrg} \frac{1}{d_i}} \]

where \( r \) is the rainfall intensity (m/s) at node \( k \), \( R_i \) is the rainfall intensity recorded at station \( i \), and \( d_i \) is the distance from node \( k \) to rain gauge \( i \).
Figure 5.15: **Top**: 90 m spaced Digital Elevation Model. **Middle**: 90 m spaced Soil Distribution Map. **Bottom**: 90 m spaced Land Use Map [1].
Figure 5.16: Location of Rain Gauges (*Top*) and Stream Gauges (*Bottom*) in the Goodwin Creek Watershed [1].
5.5.2 Numerical Simulation

For the DG-SAKE modeling framework, the computational domain used was that described in Chapter 4 and can be seen in Figure 5.17. The hydrograph results in this study are presented in units of stream velocity, $V$ (mm/h), which is computed by dividing the discharge at each station by the sub-basin area contributing flow to the station. For further details on the mesh generation techniques and mesh watershed analysis used, see Chapter 4.

![Finite Element Mesh of the Goodwin Creek Watershed](image)

Figure 5.17: A Finite Element Mesh of the Goodwin Creek Watershed.

Two numerical simulations were performed on this test case. Each simulation was performed on a 64-bit DELL Optiplex 990 machine with 16-GB RAM and a 2.93-GHz Intel i7 processor. The first simulation computes run-off using low order, $p = 0$. The second simulation will use a higher order of $p = 1$. Both simulations are
compared with observational data and CASC2D model results and the performance of the model is discussed.

Simulation # 1: \( p = 0 \)

The modeling parameters used for the first simulation presented were as follows:

- Polynomial Degree: \( p = 0 \).
- Total Simulation Time: \( T = 13 \) Hours.
- Time Step: \( \Delta t = 20.00 \) Seconds.

The simulation CPU time was 40.22 seconds for DG-SAKE. The discharge hydrographs for each of the six stream gauges are presented in Figure 5.19. The numerical results for watershed outlet, station 1, shows some additional lag time in the rising limb of the hydrograph. Although the peak discharge seems to have been well represented in the solution, the rising and recession limb of the hydrograph do not represent the measured data very well. Multiple factors could be influencing this. First, Figure 5.18 shows the kinematic wave number for each corresponding station. From Figure 5.18 we see that the \( K \) value drops rapidly and falls below 10, which indicates that the kinematic wave approximation may not be applicable here and the 1D SWE could be used to capture the additional dynamic effects the model is missing. The remaining stations show an underprediction in the hydrographs; however, the rising limbs, recession limbs, and time to peak appear to match with the measured data. Table 5.3 and Table 5.4 summarize the simulation results.

It should be noted that errors in these results could be due to a number of empirical variables associated with modeling overland flow such as the (1) the channel cross-section geometry used, (2) the coarse grid spacing of the input data which could
produce channel lengths much shorter than they really are, (3) the rainfall interpo-
lation method used, and/or (4) further Manning’s \( n \) calibration needed. These are areas
that can be further studied and resolved with data collection from a more recent time
frame. In this scope of work, these variables will be kept the same for comparison
with the CASC2D model.

As shown in previous test cases, \( p \)-refinement can be considered for improving
model results. The main stations that will be focused on are stations 4, 6, 7, 8, and
14.

Table 5.3: Peak Discharge Summary of Simulation #1 Model Results.

<table>
<thead>
<tr>
<th>Station</th>
<th>Peak Flow Rate, ( V ) (mm/h)</th>
<th>( p = 0 )</th>
<th>( \epsilon_{\text{CASC2D}} ) (%)</th>
<th>( \epsilon_{\text{DG-SAKE}} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Measured</td>
<td>CASC2D</td>
<td>DG-SAKE</td>
<td>DG-SAKE</td>
</tr>
<tr>
<td>1</td>
<td>7.059</td>
<td>7.014</td>
<td>0.63</td>
<td>6.902</td>
</tr>
<tr>
<td>4</td>
<td>9.939</td>
<td>10.445</td>
<td>5.09</td>
<td>7.463</td>
</tr>
<tr>
<td>6</td>
<td>11.714</td>
<td>8.622</td>
<td>26.40</td>
<td>7.431</td>
</tr>
<tr>
<td>7</td>
<td>11.613</td>
<td>10.870</td>
<td>6.40</td>
<td>7.784</td>
</tr>
<tr>
<td>8</td>
<td>17.066</td>
<td>8.501</td>
<td>50.19</td>
<td>8.477</td>
</tr>
<tr>
<td>14</td>
<td>15.240</td>
<td>8.308</td>
<td>45.49</td>
<td>8.192</td>
</tr>
</tbody>
</table>
Table 5.4: Time to Peak Summary of Simulation #1 Model Results.

<table>
<thead>
<tr>
<th>Station</th>
<th>Time, $t$ (min)</th>
<th>Measured</th>
<th>CASC2D</th>
<th>$\Delta t_{\text{CASC2D}}$</th>
<th>DG-SAKE</th>
<th>$\Delta t_{\text{DG-SAKE}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>278</td>
<td>277</td>
<td>1</td>
<td>290.79</td>
<td>12.79</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>207</td>
<td>202</td>
<td>5</td>
<td>210.42</td>
<td>3.42</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>182</td>
<td>210</td>
<td>28</td>
<td>210.42</td>
<td>28.42</td>
<td></td>
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<tr>
<td>7</td>
<td>214</td>
<td>183</td>
<td>31</td>
<td>202.75</td>
<td>11.25</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>193</td>
<td>221</td>
<td>28</td>
<td>202.75</td>
<td>9.75</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>189</td>
<td>189</td>
<td>0</td>
<td>200.09</td>
<td>11.09</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.18: The Kinematic Wave Number, $K$, corresponding to each Station for $p = 0$. 

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Figure 5.19: Hydrograph Results for DG-SAKE Simulation using a Polynomial Degree of $p = 0$. 
Simulation # 2: \( p = 1 \)

The modeling parameters used for this simulation are as follows:

- Polynomial Degree: \( p = 1 \).
- Total Simulation Time: \( T = 13 \) Hours.
- Time Step: \( \Delta t = 7 \) Seconds.

The simulation CPU time was 5.7 minutes for DG-SAKE and the discharge hydrographs for each of the six stream gauges are presented in Figure 5.21. Refining the polynomial degree from \( p = 0 \) to \( p = 1 \) improves the peak discharge significantly in all of the stations in comparison to the model results for \( p = 0 \), with the exception of station 1. Due to the very low values of \( K \) in the previous simulation, the error will only increase with \( p \)-refinement at station 1.

The remaining stations improve significantly. The percent error, tabulated in Table 5.5, is reduced at all stations with the exception of station 1. In comparison with CASC2D, the DG-SAKE framework outperforms at stations 6, 7, 8 and 14.

The kinematic wave number, \( K \), for each station can be seen in Figure 5.20. Figure 5.20 shows that for all stations the \( K \) values decrease to lower values in comparison to the first simulation. Although the \( K \) value is getting gradually smaller, the model performs much better for \( p = 1 \). Table 5.5 and table 5.6 summarize the simulation results.
Table 5.5: Summary of Simulation #2 Model Results.

Simulation # 2: $p = 1$

<table>
<thead>
<tr>
<th>Station</th>
<th>Peak Flow Rate, $V$ (mm/h)</th>
<th>Measured</th>
<th>CASC2D</th>
<th>$\epsilon_{CASC2D}$ (%)</th>
<th>DG-SAKE</th>
<th>$\epsilon_{DG-SAKE}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.059</td>
<td>7.014</td>
<td>0.63</td>
<td>12.249</td>
<td>73.54</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>11.714</td>
<td>8.622</td>
<td>26.40</td>
<td>10.577</td>
<td>9.71</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>11.613</td>
<td>10.870</td>
<td>6.40</td>
<td>12.111</td>
<td>4.29</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>17.066</td>
<td>8.501</td>
<td>50.19</td>
<td>12.90</td>
<td>24.44</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>15.240</td>
<td>8.308</td>
<td>45.49</td>
<td>12.39</td>
<td>18.68</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.20: The Kinematic Wave Number, $K$, corresponding to each Station for $p = 1$. 
Table 5.6: Time to Peak Summary of Simulation #2 Model Results.

<table>
<thead>
<tr>
<th>Station</th>
<th>Measured</th>
<th>CASC2D</th>
<th>$\Delta t_{CASC2D}$</th>
<th>DG-SAKE</th>
<th>$\Delta t_{DG-SAKE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>278</td>
<td>277</td>
<td>1</td>
<td>250.43</td>
<td>27.57</td>
</tr>
<tr>
<td>4</td>
<td>207</td>
<td>202</td>
<td>5</td>
<td>200.25</td>
<td>6.75</td>
</tr>
<tr>
<td>6</td>
<td>182</td>
<td>210</td>
<td>28</td>
<td>179.36</td>
<td>2.64</td>
</tr>
<tr>
<td>7</td>
<td>214</td>
<td>183</td>
<td>31</td>
<td>190.8</td>
<td>23.20</td>
</tr>
<tr>
<td>8</td>
<td>193</td>
<td>221</td>
<td>28</td>
<td>189.05</td>
<td>3.95</td>
</tr>
<tr>
<td>14</td>
<td>189</td>
<td>189</td>
<td>0</td>
<td>193.95</td>
<td>4.95</td>
</tr>
</tbody>
</table>
Figure 5.21: Hydrograph Results for DG-SAKE Simulation using a Polynomial Degree of $p = 1$. 
Chapter 6: Conclusions and Future Work

A multidimensional discontinuous Galerkin finite element model has been developed and tested on a variety of test cases. The mesh generation techniques developed make the pre-processing work less user intensive by automating boundary extraction and channel network extraction from digital elevation models. Then, using these watershed features, a user can quickly generate a finite element mesh using the ADMESH+ program.

The DG-SAKE framework performed well, but improvements are needed for adequately representing channel flow where the kinematic wave equations are no longer valid, such as in the case of back water effects from storm surge. The next stage of this project will be to couple the overland flow modeling framework with a one dimensional shallow water equation model. By making use of the kinematic wave number ($K$) a multi-physics approach can be implemented by monitoring the value $K$ and switching between the models when needed. This will improve the overall efficiency and accuracy of the model.

Other future work in the area of mesh generation will be developing techniques for automated quadrilateral mesh generation and curved element mesh generation for both triangular and quadrilateral elements; see, for example, Figure 6.1.
Figure 6.1: A Finite Element Mesh of the Western North Atlantic Tidal Domain. Straight Sided Quadrilateral Elements can be used in open areas of the Domain, while channels can be approximated with curved quadrilateral elements.
Bibliography


[27] The University of Texas at Austin. Computational hydraulics group. [https://wiki.ices.utexas.edu/chg/](https://wiki.ices.utexas.edu/chg/).


