Essays on Business Cycles with Credit Shocks

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy in the Graduate School of The Ohio State
University

By

In Hwan Jo, B.A., M.A.

Graduate Program in Economics

The Ohio State University

2015

Dissertation Committee:
Professor Aubhik Khan, Advisor
Professor Julia Thomas
Professor Lu Zhang
© Copyright by

In Hwan Jo

2015
Abstract

A recent but growing literature in macroeconomics works to reconcile microeconomic data with the micro-level predictions of dynamic stochastic equilibrium models in an effort to improve the aggregate performance of macroeconomic models. My dissertation follows in this new tradition. It also contributes to a recently revitalized literature attempting to understand the links between financial markets and real economic activity. The essays discussed below examine how real and financial shocks affect the distribution of production in an economy, and how that distribution in turn influences aggregate quantity variables. The models I explore involve rich, time-varying distributions of firms differing in their capital stocks, debt, and productivities.

In the first chapter, “Firm Size and Business Cycles with Credit Shocks,” I highlight the importance of empirically consistent firm-level heterogeneity in shaping aggregate dynamics. A large empirical literature documents that the firm size distribution is highly-skewed, a finding that business cycle studies typically ignore. I quantitatively investigate the macroeconomic implications of a realistic size distribution in an economy with real and financial shocks. Specifically, I build a tractable model including heterogeneous firms in which each firm faces persistent shocks to both individual and aggregate productivity. Firms choose employment, investment and debt, and these decisions are made in a setting with forward-looking collateral constraints that limit loan sizes as a function of firms chosen capital stocks. In
addition to reproducing the empirical firm size distribution, I jointly estimate the parameters governing aggregate productivity and financial shocks using a simulated method of moments approach. The estimated shock processes drive plausible business cycle dynamics in my model and allow me to isolate the importance of financial shocks in driving business cycles. A central finding of my study is that the aggregate elasticity of response to a financial shock is underestimated when the size distribution of firms fails to reproduce the degree of skewness observed in firm-level data. In other words, existing studies under-predict the severity of financially-generated recessions and so must rely on very large financial shocks to explain recessions like the 2007 U.S. recession. Furthermore, I find that including a realistic firm size distribution in my model helps to explain the slow employment recovery following a financial recession, a second puzzle of particular interest since the 2007 recession. By contrast, my model predicts that the aggregate response following a common productivity shock is largely unaffected by the underlying size distribution of firms. Thus, existing models are equally successful in explaining business cycles arising from real shocks. A financial shock in my model produces dissimilar responses across firms of different sizes and ages, whereas a productivity shock does not. In particular, financial shocks have disproportionate negative impacts on small firms in my model, which is consistent with the relative employment declines among such firms over the last U.S. recession.

In the second chapter, “Production Heterogeneity with Borrowing Constraints and Working Capital”, I consider the extent to which the weak labor market recovery seen following the 2007 recession may be explained by a credit shock. A notable aspect of this recession has been the persistent widening of the aggregate labor wedge, the gap between the marginal product of labor and the marginal rate of substitution of
leisure for consumption. I explore this using an equilibrium model with frictional hiring decisions by heterogeneous firms. In my model, external financing for a firm is constrained by both a working capital requirement and a collateralized borrowing limit. As a result, each firm’s credit is affected by both its employment and collateral, and some choose to hire less than the statically optimal labor input in order to obtain more external funds for investment. Following a sudden tightening of credit in my model economy, this firm-level labor demand distortion is exacerbated, driving an increased aggregate labor wedge. The greater misallocation of labor that this reflects itself amplifies the decline in aggregate employment and production and, because reallocation following the credit shock is not immediate, it delays and protracts the economic recovery.
To my wife, Yun-mi Lee.
Acknowledgments

I would like to thank my advisor, Aubhik Khan, for his guidance. Aubhik has continuously provided me with his great insights and encouragements during my doctoral study at The Ohio State University. This work has enormously benefited from his time, patience, and support. I also thank Julia Thomas for her comments and suggestions that have significantly improved the quality of this work. She has always been my intellectual mentor, actively involved in every breakthrough of my research. I am also grateful to my committee member, Lu Zhang, for his helpful advice.

I have largely benefited from constant discussions with my colleagues at the Macroeconomics Workshop. In addition to expanding my knowledge of other research topics, I learned from my colleagues how to effectively communicate with others. Lastly, I thank my family for their sacrifice, love, and belief in me. My wife, Yun-mi Lee, and young son, Ian Yunje, have always been there, giving me the reason to work hard as well as the meaning of being a family.
Vita

October 12, 1981 .......................... Born - Ulsan, South Korea

2007 .................................. B.A. Economics
                               Korea University
2009 .................................. M.A. Economics
                               Korea University
2010 .................................. M.A. Economics
                               The Ohio State University
2010-2014 .............................. Graduate Associate
                               The Ohio State University

Fields of Study

Major Field: Economics
Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>v</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>vi</td>
</tr>
<tr>
<td>Vita</td>
<td>vii</td>
</tr>
<tr>
<td>List of Tables</td>
<td>x</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xi</td>
</tr>
<tr>
<td>1. Firm Size and Business Cycles with Credit Shocks</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Related Literature</td>
<td>7</td>
</tr>
<tr>
<td>1.3 Model</td>
<td>10</td>
</tr>
<tr>
<td>1.3.1 Firms</td>
<td>10</td>
</tr>
<tr>
<td>1.3.2 Households and Equilibrium</td>
<td>14</td>
</tr>
<tr>
<td>1.3.3 Firm Types and Decision Rules</td>
<td>17</td>
</tr>
<tr>
<td>1.3.4 Reducing State Space</td>
<td>19</td>
</tr>
<tr>
<td>1.4 Model Parameters</td>
<td>21</td>
</tr>
<tr>
<td>1.4.1 Calibration</td>
<td>22</td>
</tr>
<tr>
<td>1.4.2 Estimation of Aggregate Shocks</td>
<td>23</td>
</tr>
<tr>
<td>1.5 Results</td>
<td>27</td>
</tr>
<tr>
<td>1.5.1 Steady State</td>
<td>27</td>
</tr>
<tr>
<td>1.5.2 Business Cycles</td>
<td>31</td>
</tr>
<tr>
<td>1.5.3 Implications of Firm Size Distribution</td>
<td>33</td>
</tr>
<tr>
<td>1.5.4 Firm Dynamics</td>
<td>37</td>
</tr>
<tr>
<td>1.6 Conclusion</td>
<td>41</td>
</tr>
</tbody>
</table>
## 2. Production Heterogeneity with Borrowing Constraint and Working Capital 56

2.1 Introduction ........................................ 56
2.2 Related Literature ........................................ 59
2.3 Model .................................................. 62
  2.3.1 Firms ............................................ 62
  2.3.2 Households and Equilibrium ............................. 67
2.4 Analysis .............................................. 70
  2.4.1 Decisions of Unconstrained Firms ......................... 71
  2.4.2 Decisions of Constrained Firms ......................... 73
2.5 Results .............................................. 75
  2.5.1 Parameters and Steady State ........................... 75
  2.5.2 Aggregate Dynamics .................................. 77
2.6 Concluding Remarks .................................... 82

Bibliography ............................................. 92

## Appendices 97

A. Appendix to Chapter 2 .................................... 97

  A.1 Computational Methods .................................. 97
    A.1.1 Solving Steady State ................................ 98
    A.1.2 Solving Perfect Foresight Transition ................. 102
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Calibration</td>
</tr>
<tr>
<td>1.2</td>
<td>SMM Estimation</td>
</tr>
<tr>
<td>1.3</td>
<td>Firm Size Distribution</td>
</tr>
<tr>
<td>1.4</td>
<td>Business Cycles in the Full Economy</td>
</tr>
<tr>
<td>1.5</td>
<td>Business Cycles Only with Productivity Shocks</td>
</tr>
<tr>
<td>2.1</td>
<td>Calibration</td>
</tr>
<tr>
<td>2.2</td>
<td>Aggregate Variables by Firm Type</td>
</tr>
</tbody>
</table>
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Firm Size Distribution in the U.S.</td>
<td>45</td>
</tr>
<tr>
<td>1.2</td>
<td>Entire Firm Distribution at the Steady State</td>
<td>46</td>
</tr>
<tr>
<td>1.3</td>
<td>Decision Rules at the Steady State</td>
<td>47</td>
</tr>
<tr>
<td>1.4</td>
<td>Comparison of Model Firm Size Distribution</td>
<td>48</td>
</tr>
<tr>
<td>1.5</td>
<td>Aggregate Productivity Shock</td>
<td>49</td>
</tr>
<tr>
<td>1.6</td>
<td>Credit Shock</td>
<td>50</td>
</tr>
<tr>
<td>1.7</td>
<td>Firm Size Distribution with Credit Shock</td>
<td>51</td>
</tr>
<tr>
<td>1.8</td>
<td>Cash-on-hand and Decisions (Comparative Statics)</td>
<td>52</td>
</tr>
<tr>
<td>1.9</td>
<td>Firm Dynamics with a Productivity Shock</td>
<td>53</td>
</tr>
<tr>
<td>1.10</td>
<td>Firm Dynamics with a Credit Shock, part 1</td>
<td>54</td>
</tr>
<tr>
<td>1.11</td>
<td>Firm Dynamics with a Credit Shock, part 2</td>
<td>55</td>
</tr>
<tr>
<td>2.1</td>
<td>The 2007 Recession in the U.S.</td>
<td>84</td>
</tr>
<tr>
<td>2.2</td>
<td>Entire Firm Distribution at the Steady State</td>
<td>85</td>
</tr>
<tr>
<td>2.3</td>
<td>Unconstrained Choices over $\varepsilon$ Values</td>
<td>86</td>
</tr>
<tr>
<td>2.4</td>
<td>A TFP Shock</td>
<td>87</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>2.5 A TFP Shock with Unconstrained Firms Only</td>
<td>88</td>
<td></td>
</tr>
<tr>
<td>2.6 A Credit Shock</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td>2.7 A Credit Shock without Working Capital</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>2.8 Business Cycle Accounting, Labor Wedge</td>
<td>91</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 1: Firm Size and Business Cycles with Credit Shocks

1.1 Introduction

The evidence on the size distribution of firms is rarely addressed in macroeconomic studies.\textsuperscript{1} Moreover, small firms experience relatively more difficulties in external financing.\textsuperscript{2} I build a tractable business cycle model with financial frictions and production heterogeneity which captures the substantial differences in firm size seen in the data. I then quantitatively investigate the macroeconomic effects of credit shocks in this model economy with a realistic firm size distribution. In my model, the misallocation of resources across firms is amplified when most firms are small and financially constrained, and matching the size distribution magnifies the impact of financial shocks relative to productivity shocks. To examine this distinct feature of financial shocks, I measure the model employment dynamics at the firm level both in size and age. This confirms the disproportionate effect of financial shocks particularly on small firms in different age groups.

\textsuperscript{1}The recent studies on production heterogeneity focus on micro-level moments or partly match the empirical size distribution. For example, Khan and Thomas (2013) reproduce the establishment-level investment dynamics with irreversibilities of investments, whereas Buera, Fattal-Jaef, and Shin (2015) target the largest firms’ earnings and employment shares.

\textsuperscript{2}Beck, Demirguc-Kunt, and Maksimovic (2005) use extensive survey data from multiple countries to examine how firm growth is affected by legal and financial constraints. They highlight that it is small firms that are more constrained by financial obstacles and collateral requirements.
I begin with a motivating empirical fact on the size distribution of firms in employment. The well-known shape of the size distribution is characterized by the population share and the employment share in each firm size group. Figure 1.1 summarizes data from the Business Dynamics Statistics (BDS), and shows the distribution of firms across size groups. Most firms tend to be small whereas it is the largest firms that dominate aggregate employment. That is, more than 85 percent of firms are relatively small hiring less than 20 employees, whereas about 42 percent of U.S. employment is accounted for by the largest 0.2 percent of firms. Nonetheless, those small firms still account for a substantial employment share which is about 20 percent of aggregate employment.

Standard business cycle models of production heterogeneity are not typically designed to generate the empirical firm size distribution described above. In this class of models, firms are not sufficiently different from each other, and the resulting size distribution exhibits insufficient dispersion relative to its empirical counterpart. The assumption of decreasing returns to scale at the firm level fundamentally limits the size of the largest firms. In addition, shocks to firm-level productivity are roughly symmetric. The logarithm of a firm’s individual productivity is typically assumed to follow a first order autoregressive process, AR (1), with Gaussian innovations. In this paper, I introduce an exogenous stochastic process for firm productivity that is better able to reproduce the empirical size distribution of firms.

I study a dynamic stochastic general equilibrium model with both aggregate productivity and financial shocks. Firms are heterogeneous in their levels of productivity, capital, and financial position. Their decisions on investment and employment are
endogenously affected by a forward-looking collateral constraint in the spirit of Kiyotaki and Moore (1997). A credit shock is represented by an unexpected change in firms’ borrowing capacity. One distinctive feature of the model is the introduction of a new variable, firms’ cash-on-hand, which summarizes the endogenous states. By reducing the state space of the model, I derive firm-level decision rules in a convenient and tractable way. This allows me to jointly estimate the aggregate shocks in the model using a simulated method of moments (SMM) approach.

When the firm size distribution is highly-skewed as observed in the data, the recession triggered by a credit shock is more severe. I compare the aggregate dynamics of the model from two different specifications of the idiosyncratic productivity process. The benchmark specification is targeted to match the empirical size distribution, while the other uses a common lognormal process. Following a tightened credit condition, the benchmark economy exhibits a deeper and more persistent recession. This is because the resource misallocation arising from the financial friction is larger in an economy with a realistic size distribution of firms. My result implies that the existing studies under-predict the severity of financial recessions. For this reason, they must rely on a large financial shock to explain the observed loss in aggregate total factor productivity.

3In my model, a firm’s cash-on-hand is a sufficient statistic for its individual states in a setting of forward-looking collateral constraints. I derive firm-level decision rules in a general business cycle framework by introducing this new state variable, unlike other applications in Buera and Moll (2014) or Midrigan and Xu (2014).

4The forward-looking collateral constraint with a reduced state space differentiates my model from the similar framework in Khan and Thomas (2013).

5To my best knowledge of the literature, this is the first attempt to estimate financial shocks in a model with production heterogeneity. Jermann and Quadrini (2012) measure financial shocks using a representative firm model. Takamura (2013) estimates both aggregate productivity and financial shocks in the banking sector. Sunakawa (2012) applies the explicit aggregation (Xpa) algorithm to estimate only the TFP shocks in an economy with heterogeneous firms.

6Studies featuring this specification include Khan and Thomas (2008) and Bloom (2009).
productivity (TFP) during the 2007 U.S. recession. My benchmark economy also more closely resembles the recent U.S. experience in predicting a slower recovery following the recession. This happens because the tighter fit to the empirical firm size distribution going into the recession implies greater persistence in the increased misallocation of resources following a financial shock.

The key mechanism that distinguishes my model’s response to a financial shock, as opposed to a real shock, is its uneven effects on firms. In the model, each firm responds to a credit shock by adjusting its investment and employment. These decisions may be influenced by a firm’s borrowing limit, which is determined by the amount of collateral held as capital. Following a sudden credit tightening, firms with sufficient collateral are able to externally finance their ideal levels of investment. Other firms with less financial assets or capital have their investments limited by their ability to borrow. Such firms suffer more from the shock, because limited credit further stifles their use of capital and labor. In the aggregate, the result is an unusually large misallocation of production. When an economy has relatively more financially constrained firms, the impact of a credit shock becomes larger. Even though each of these firms is small, they collectively contribute to aggregate output and employment, as suggested by the considerable employment share of small firms in the data. Therefore, from a quantitative perspective, the skewness of the firm size distribution determines the severity of recessions caused by financial shocks.

On the other hand, aggregate dynamics following a productivity shock are largely unaffected by the underlying distribution of firms. This is because the shock itself does not distort firms’ borrowing conditions. In other words, real shocks evenly affect firm-level decisions as a function of size. It follows that existing models are
largely successful in explaining business cycles when only real aggregate shocks are considered.

The shock processes I estimate in this paper lead to a model-generated business cycle that is mainly dominated by productivity shocks in frequency, while financial shocks substantially compound aggregate volatility. In contrast to the arbitrary approach based on Reinhart and Rogoff (2009) banking crises dates taken by Khan and Thomas (2013), my estimation suggests that credit shocks are far more common and thus relevant to an understanding of the typical business cycle comovement of GDP, employment, and TFP.

With the size distribution in hand, I use my model to study how different firms respond to aggregate shocks. In particular, I quantitatively examine the disproportionate impact of financial shocks at the firm level to address the empirical evidence on differential firm dynamics over business cycles. Specifically, I measure the employment dynamics of each firm size group following an aggregate shock in the model. In addition, I further disaggregate the dynamics of firms by controlling for age. This is motivated by Fort, Haltiwanger, Jarmin, and Miranda (2013) who emphasize the age dimension of firm heterogeneity during the recent recession in the U.S.

Simulations of a large panel of firms show that there are substantial differences in employment dynamics across firms depending on the source of aggregate fluctuations. While firms respond homogeneously to a productivity shock, the net employment growth following a credit shock varies with firm size and age. In particular, small firms exhibit very different responses according to their ages. While small-young firms are less responsive during the economic downturn, small-mature firms have volatile employment dynamics. This is because most young firms start small and gradually
build up their assets over time. Through the financial friction, a credit shock hits these firms more severely than others in the economy. Again, it is the difference in the nature of aggregate shocks that determines firm dynamics as a function of size and age.

The difference between small and large firms over business cycles, therefore, depends on the relative importance of financial shocks compared to productivity shocks. I relate this observation with the empirical findings on firm dynamics over business cycles. In the literature, several empirical works focus on how small firms respond differently from large firms to aggregate fluctuations. The consensus appears to be that small firms are more negatively affected during episodes of monetary contraction, but they exhibit similar patterns to large firms in other periods.\footnote{I briefly review the empirical evidence in the following subsection.} These findings are not easily addressed in a standard business cycle analysis without heterogeneous firms and financial shocks. Because my model includes these crucial elements while being sufficiently tractable so that I can estimate both its aggregate shocks, I am able to contribute to this discussion. My estimation, when combined with model-generated data from a panel of firms, suggests that the dissimilar response of small firms is less pronounced when both real and financial shocks are present. As such, my results show that the disagreement between empirical findings can be reconciled in a unified theoretical framework.

The rest of this paper is organized as follows. Below, I briefly provide a review of the literature most closely related to my paper. The model environment is described in Section 2 together with a description of firm-level decision rules. I parameterize...
the model using both calibration and estimation methods in Section 3. Section 4 summarizes the quantitative results of this paper. I conclude in Section 5.

1.2 Related Literature

This paper belongs to a literature that studies the macroeconomic consequences of financial shocks. Following the 2007 recession in the U.S., there have been growing research on how disturbances in the financial sector affect real activity. Jermann and Quadrini (2012) highlight the role of financial shocks as a source of business cycle fluctuations. In their work, the financial friction arises from the limited enforcement of debt contracts. Khan and Thomas (2013) investigate the resource misallocation channel of financial shocks in an economy with heterogeneous firms subject to individual credit limits. A credit shock generates substantial efficiency loss due to the misallocation of production factors across firms, which leads to a financially driven recession consistent with several empirical observations. Buera and Moll (2014) study the misallocation channel in relation to time-varying aggregate wedges, and they emphasize the importance of the source of a model's underlying heterogeneity. The model presented in this paper is located at the intersection of these two papers. It maintains all the necessary elements for a quantitative business cycle analysis as in Khan and Thomas (2013), but also achieves the tractability in Buera and Moll (2014), and thus can be estimated. At the same time, firms in my model are not risk-averse entrepreneurs in Buera and Moll (2014), and so can be more directly mapped to the firm size distribution observed in the data. Moreover, I need not make the assumption of linear production precluding an optimal firm size, nor permitting advance observation of shocks at the time investments are undertaken.
Regarding the size distribution of firms, this paper is largely motivated by the empirical findings in Cabral and Mata (2003). They also suggest a role for financial constraints in shaping the skewed firm size distribution in relation with the evolution of the dispersion across young firms over time. Beck, Demirguc-Kunt, and Maksimovic (2005) provide the evidence that small firms experience significantly more financial obstacles including collateral requirements than large firms. The recent evidence by Schmaltz, Sraer, and Thesmar (2014) implies that business startups and firm growth are closely related to the access to external finance and to the value of housing collateral. A theoretical investigation of the effects of financial shocks, therefore, is required to carefully address the rich heterogeneity in firm size in a model with financial frictions.

My paper takes a similar approach in matching the empirical size distribution to that in Guer, Ventura, and Xu (2008). They consider a log-truncated distribution of managerial abilities in production units augmented by extreme values for the largest establishments. Castaneda, Diaz-Gimenez, and Rios-Rull (2003) also adopt a similar strategy to account for household wealth inequality. Gabaix (2011) provides a motivation for departing from the on-average specification commonly used for firm-level productivity. He studies the role of the largest firms in amplifying aggregate fluctuations when a fat-tailed size distribution is considered, while I focus on the misallocation from financial shocks.

My work also relates to a series of papers examining the severity of financial recessions and the subsequent slow recovery. There is a growing body of research focused on production heterogeneity and, in particular, changes in size and age distributions as a possible explanation for the slow recovery observed following the 2007 recession.
Siemer (2013) studies the employment dynamics of small and young firms in connection with the effects of financial frictions on the firm entry margin. Sedlacek (2013) also considers the implication of endogenous firm entry in driving a time-varying firm age distribution and unemployment fluctuations. Buera, Fattal-Jaef, and Shin (2015) emphasize the interaction between financial and labor market frictions toward explaining the persistent rise in the unemployment rate during the recent recession. My paper stands apart from these in that it considers both aggregate productivity and financial shocks estimated in a standard framework for quantitative business cycle analysis.

Finally, as noted above, my paper is related to the empirical literature on the responsiveness of small firms to aggregate shocks. Gertler and Gilchrist (1994) document that small firms suffer more in their sales than do large firms during periods of monetary contractions in the U.S. However, Chari, Christiano, and Kehoe (2013) find that the cyclical differences in firms’ behavior as a function of size is less pronounced during NBER recessions. They emphasize the potential of introducing an additional shock that affects firms asymmetrically, a financial shock, in order to reconcile their evidence with that in Gertler and Gilchrist (1994). Elsewhere, Moscarini and Postel-Vinay (2012) find that the negative correlation of large firms’ job creation with aggregate unemployment is stronger than that of small firms. My work helps to bring these disparate findings together. By examining firm-level employment dynamics following aggregate shocks in my model, with these shocks parsimoniously estimated, I provide a theoretical foundation in which to reconcile the seemingly contradictory evidence on firms’ relative responsiveness over business cycles.
1.3 Model

The model economy has a large number of heterogeneous firms. An individual firm’s external borrowing is subject to a collateral constraint. Choices of investment and employment, across firms, generate substantial heterogeneity, alongside persistent differences in productivities. I complete the model by describing the household side of the economy, and then define a recursive competitive equilibrium. Next, I transform the model into a tractable form by introducing a new state variable that summarizes firms’ financial positions. I explore the quantitative implications of the model in the following sections.

1.3.1 Firms

Production Environment

The production side of the economy is populated by a continuum of firms. Each firm owns its predetermined capital stock, \( k \), and hires labor, \( n \), in a competitive labor market to produce a homogeneous good. The production technology is given by

\[
y = z \epsilon \cdot F(k, n),
\]

where \( F(\cdot) \) exhibits decreasing returns to scale (DRS). The exogenous component of total factor productivity is given by \( z \), which is common across firms. \( z \) follows a Markov chain with

\[
\pi_{fg}^z \equiv Pr(z' = z_g | z = z_f) \geq 0 \quad \text{and} \quad \sum_{g=1}^{N_z} \pi_{fg}^z = 1
\]

for each \( f = 1, \ldots, N_z \). \( \epsilon \) represents firm-specific idiosyncratic productivity, and I assume that \( \epsilon \) also follows a Markov chain such that \( \epsilon \in E \equiv \{\epsilon_1, \ldots, \epsilon_{N_\epsilon}\} \) with

\[
\pi_{ij}^\epsilon \equiv Pr(\epsilon' = \epsilon_j | \epsilon = \epsilon_i) \geq 0 \quad \text{with} \quad \sum_{j=1}^{N_\epsilon} \pi_{ij}^\epsilon = 1 \quad \text{for each} \quad i = 1, \ldots, N_\epsilon.
\]

Financial Friction

To introduce financial frictions into the model, I assume that each firm faces an individual borrowing constraint for one-period discount debt at the price of \( q \).
Specifically, the amount of newly issued debt $b'$ is limited by a firm’s collateral in the future period. This is due to the limited enforceability of financial contracts, and the amount of collateral is given by the firm’s future capital stock $k'$. For each unit of newly issued debt $b'$ that needs to be repaid in the next period, firms only receive $q$ units of output from a competitive financial market. Then the collateral constraint takes the form of,

$$b' \leq \theta \cdot k', \tag{11}$$

where $\theta \in [0, q^{-1})$ captures the economy-wide level of financial frictions. $\theta = q^{-1}$ corresponds to a perfect credit market in which firms are always allowed to finance their desirable investment. I assume that $\theta$ also follows a Markov chain with $\theta \in \Theta \equiv \{\theta_1, \ldots, \theta_N\}$. A low value of $\theta$ corresponds to a credit shock. Notice that the above collateral constraint is forward-looking in the spirit of Kiyotaki and Moore (1997).\(^8\)

Lastly, there is no sign restriction on $b$; firms can accumulate financial assets held as negative debt. This also implies that the above borrowing constraint is occasionally binding depending on a firm’s individual state.

**Entry and Exit**

If firms exist indefinitely, they eventually accumulate sufficient wealth ($b < 0$) so that the collateral constraint becomes irrelevant for their decisions. In this case, the model’s aggregate implications are analogous to those from the standard one sector growth model; the only difference is in firms’ productivities and capital stocks. To avoid this case of a Modigliani and Miller (1958) environment, I impose exogenous

\(^8\)In Khan and Thomas (2013), the borrowing constraint depends on the current capital stock, while Buera and Moll (2014) assume that firms are informed about their future productivities in advance. In contrast, the borrowing constraint in my model is consistent with the theory of financial friction, without relying on an exotic timing assumption.
entry and exit of firms in each period. Firms are subject to a fixed probability of exit in each period given by \( \pi_d \in (0, 1) \). The arrival of exit information is known at the beginning of a period, and exiting firms liquidate all their remaining earnings and assets after their production. Without loss of generality, only surviving firms choose investment and borrowing. To maintain a stationary distribution of firms in the model’s equilibrium, exiting firms are replaced by an equal measure of new firms. These entrants are endowed with an initial capital stock which is a \( \chi \) fraction of the average capital stock in the economy.

Timing and Firm Distribution

I illustrate the timing of exogenous shocks and decisions made by firms within a given period of the model. At the beginning of a period, a firm is identified by its individual state \((k, b, \epsilon)\); its predetermined capital stock \(k \in K \subset \mathbb{R}_+\), the amount of existing debt to be repaid \(b \in B \subset \mathbb{R}\), and the current idiosyncratic productivity level \(\epsilon \in E\). I summarize the distribution of firms by a probability measure \(\mu(k, b, \epsilon)\) which is defined on a Borel algebra \(\mathcal{S} \equiv K \times B \times E\). For simplicity, I use \(s \equiv (z, \theta)\) to summarize the exogenous aggregate state of the model with its transition probability given by \(\pi^{*}_{lm} \geq 0\). Then the aggregate state of the model economy becomes \((s, \mu)\), and the evolution of firm distribution follows a mapping \(\Gamma\) such that \(\mu' = \Gamma(s, \mu)\).

Just after the exogenous shocks are realized on \(s\) and \(\epsilon\), a firm learns whether it will survive to the next period or exit at the end of current period. Given its individual and aggregate states \((k, b, \epsilon; s, \mu)\), the firm maximizes the expected discounted value of all future dividend payments. After production is completed, the firm pays its wage bill and its existing debt \(b\) is cleared. Conditional on its survival into the next period, the firm chooses its investment level \(i\) for the future capital stock \(k'\) alongside
newly issued debt $b'$. At the same time, the firm needs to determine current period dividends $D$ paid to shareholders. Capital accumulation is standard, $i = k' - (1 - \delta)k$.

Firms take the wage rate $w(s, \mu)$ and the discounted debt price $q(s, \mu)$ as given.

**Firm’s Problem**

Given the possibility of exit, I organize the description of a firm’s problem by whether or not its exit status is known. Let $v_0(k, b, \epsilon; s, \mu)$ be the expected discounted value of the firm before the exit shock is known, at the beginning of a period. If the firm is allowed to move on to the next period, its within-the-period value is given by $v(k, b, \epsilon; s, \mu)$. I present the firm’s optimization problem by defining the value functions $v_0$ and $v$ as follows.

\[
v_0(k, b, \epsilon; s, \mu) = \pi_d \cdot \max_n \left[ z_l \epsilon_i F(k, n) - w(s_l, \mu) n + (1 - \delta) k - b \right] + (1 - \pi_d) \cdot v(k, b, \epsilon; s, \mu) \tag{1}\]

With the current realizations of $(s_l, \epsilon_i)$ at the beginning of the period, the firm takes a binary expectation over the values before its exit or survival is known. This is given by Equation (1) in which the exit probability $\pi_d$ serves as the weight on the value of exiting. In case of exit, the firm maximizes its liquidation value after production.

\[
v(k, b, \epsilon; s_l, \mu) = \max_{n, k', b', D} \left[ D + \sum_{m=1}^{N_s} \pi_{lm} d_m(s_l, \mu) \sum_{j=1}^{N_s} \pi_{ij} v_0(k', b', \epsilon_j; s_m, \mu') \right] \tag{2}\]

subject to

\[
0 \leq D = z_l \epsilon_i F(k, n) - w(s_l, \mu) n + (1 - \delta) k - b - k' + q(s_l, \mu) b'
\]

\[
b' \leq \theta k'
\]

\[
\mu' = \Gamma(s_l, \mu)
\]
The continuation value of the firm in Equation (2) is more involved. The firm optimally chooses its employment \( n \), future capital \( k' \), and new debt level \( b' \) to maximize the sum of the firm’s current dividends \( D \) and the \textit{beginning-of-the-period value} at the future period \( v_0(k', b', \epsilon; s_m, \mu') \). The dividend payment is the residual from the firm’s budget constraint, and it is restricted to be non-negative. The firm discounts its future value by the stochastic discount factor \( d_m(s_l, \mu) \). The above firm decisions are subject to the collateral constraint given a realized value of the financial parameter \( \theta \in \Theta \).

### 1.3.2 Households and Equilibrium

#### Representative Household

In the other side of the economy, I assume that there is a unit measure of identical households. Each household participates in the labor market by supplying a fraction of its time endowment in return for wage income. The households hold their wealth as a comprehensive portfolio of one period firm shares with measure \( \lambda \) and non-contingent bonds \( \phi \). The period utility function is given by \( U(C, 1 - N) \), with the subjective discount factor \( \beta \in (0, 1) \). The representative household maximizes the lifetime expected discounted utility by choosing the aggregate consumption demand \( C^h \), labor supply \( N^h \), while adjusting the asset portfolio in each period.

\[
V^h(\lambda, \phi; s_l, \mu) = \max_{C^h, N^h, \lambda', \phi'} \left[ U(C^h, 1 - N^h) + \beta \sum_{m=1}^{N_s} \pi_{lm}^s V^h(\lambda', \phi'; s_m, \mu') \right] \tag{3}
\]

subject to

\[
C^h + q(s_l, \mu)\phi' + \int_S p_1(k', b', \epsilon; s_l, \mu)\lambda'(d[k' \times b' \times \epsilon']) \\
\leq w(s_l, \mu)N^h + \phi + \int_S \rho_0(k, b, \epsilon; s_l, \mu)\lambda(d[k \times b \times \epsilon']) \\
\mu' = \Gamma(s_l, \mu)
\]
Due to the timing issue related to dividends, I use \( \rho_1(k', b', \epsilon'; s_t, \mu) \) to denote the ex-dividend prices of firm shares, whereas \( \rho_0(k, b, \epsilon; s_t, \mu) \) is the dividend-inclusive values of current share holding. Let \( \Phi^h(\lambda, \phi; s, \mu) \) be the household’s decision for bond holding, and denote \( \Lambda^h(k', b', \epsilon', \lambda, \phi; s, \mu) \) as the new choice of firms share holding with future state \((k', b', \epsilon')\).

**Equilibrium and Prices**

Having described the model environment, I define recursive competitive equilibrium (RCE) as follows.

A RCE is a set of functions: prices \( (w, q, (d_m)_{m=1}^{N_s}, \rho_0, \rho_1) \), quantities \( (N, K, B, D, C^h, N^h, \Phi^h, \Lambda^h) \), and values \( (v_0, v, V^h) \) that solve the optimization problems, clear each market, and the associated policy functions are consistent with the aggregate law of motion as in the following conditions.

1. \( v_0 \) and \( v \) solve Equation (1) and (2), and \((N, K, B, D)\) are the associated policy functions for firms.

2. \( V^h \) solves Equation (3), and \((C^h, N^h, \Phi^h, \Lambda^h)\) are the associated policy functions for households.

3. The labor market clears, \( N^h = \int_S^N N(k; \epsilon; s, \mu) \cdot \mu(d[k \times b \times \epsilon]) \).

4. The goods market clears,

\[
C^h = \int_S \left[ z \varepsilon F(k, N) - (1 - \pi_d)(K(k, b, \epsilon; s, \mu) - (1 - \delta)k) + \pi_d(1 - \delta)k \right] \cdot \mu(d[k \times b \times \epsilon]).
\]

5. The law of motion for the firm distribution is consistent with the policy functions, where \( \Gamma \) defines the mapping from \( \mu \) to \( \mu' \) with \( \pi_d, K(k, b, \epsilon; s, \mu), \) and \( B(k, b, \epsilon, s, \mu) \).
As noted by Khan and Thomas (2013), it is convenient to modify the firm’s value functions by using the equilibrium prices at the market clearing quantities. Let $C$ and $N$ be the equilibrium quantities for aggregate consumption and labor supply of households. Then the equilibrium value of output can be expressed using marginal utility of consumption, $D_1U(C, 1 - N)$. Further, the real wage $w(s, \mu)$ is equal to the marginal rate of substitution between leisure and consumption. The inverse of the discounted bond price $q^{-1}$ equals the expected gross real interest rate. Lastly, the stochastic discount factor $d_m(s, \mu)$ is the households’ intertemporal marginal rate of substitution across states. The following equations summarize the equilibrium prices in terms of $D_1U(C, 1 - N)$ and $D_2U(C, 1 - N)$.

\[
\begin{align*}
  w(s, \mu) &= \frac{D_2U(C, 1 - N)}{D_1U(C, 1 - N)} \\
  q(s, \mu) &= \frac{\beta \sum_{s=1}^{N_s} \pi^s \pi_m D_1U(C'_m, 1 - N'_m)}{D_1U(C, 1 - N)} \\
  d_m(s, \mu) &= \frac{\beta D_1U(C'_m, 1 - N'_m)}{D_1U(C, 1 - N)}
\end{align*}
\]

By using the above price definitions, I solve the equilibrium allocations from the firm’s problem in a consistent manner with the households’ optimal decisions. Let $p(s, \mu)$ be the marginal utility associated with the market clearing level of consumption by households, i.e., $p(s, \mu) \equiv D_1U(C, 1 - N)$. Firms value their current period output and dividends by $p(s, \mu)$. The value functions $v_0$ and $v$ can be re-written in terms of this price without carrying the stochastic discount factor. Specifically, define the new value functions $V_0$ and $V$ by multiplying $p(s, \mu)$ to the original values, $v_0$ and $v$. 

Then the firm’s problem is given by the following equations.

\[ V_0(k, b, \epsilon_i; s_l, \mu) = \pi_d \cdot \max_n p(s_l, \mu) \left[ z_i \epsilon_i F(k, n) - w(s_l, \mu)n + (1 - \delta)k - b \right] \quad (4) \]

\[ + (1 - \pi_d) \cdot V(k, b, \epsilon_i; s_l, \mu) \]

\[ V(k, b, \epsilon_i; s_l, \mu) = \max_{n, k', b', D} \left[ p(s_l, \mu) D + \beta \sum_{m=1}^{N_s} \sum_{j=1}^{N_s} \pi_{im}^{s} \pi_{ij}^{c} V_0(k', b', \epsilon_j; s_m, \mu') \right] \quad (5) \]

subject to

\[ 0 \leq D = z_i \epsilon_i F(k, n) - w(s_l, \mu)n + (1 - \delta)k - b - k' + q(s_l, \mu)b' \]

\[ b' \leq \theta k' \]

\[ \mu' = \Gamma(s_l, \mu) \]

For the remainder of this paper, I suppress the aggregate state \((s, \mu)\) in the price functions and the decision rules whenever needed for simplicity.

1.3.3 Firm Types and Decision Rules

The collateral constraint in Equation (5) is a challenging object because it is not always binding. I adopt the definition of firm types in Khan and Thomas (2013) to analyze the firm-level decision rules in the model. Define a firm as un\(\text{constrained}\) when it has already accumulated enough wealth such that it never worries about experiencing a binding borrowing constraint in any possible future state. With all the Lagrangian multipliers on the borrowing constraints being zero, unconstrained firms become indifferent between paying dividends and saving because the shadow value of internal saving equals to \(p(s, \mu)\). On the other hand, the remaining firms in the distribution are defined as \(\text{constrained}\). A constrained firm may have a currently binding borrowing constraint or not, and it puts non-zero probability of having a
binding constraint in the future. The constrained firms choose $D = 0$ because the shadow value of retained earning is greater than the value of dividends.

I begin with the optimal decision for labor demand. Firms are allowed to adjust their employment flexibly. Thus, all firms with the same $(k, \epsilon)$ choose $n = N^w(k, \epsilon; s, \mu)$ which solves the static condition $z\epsilon D_2 F(k, n) = w(s, \mu)$. Next, I consider the choice of future capital stock $k'$ by the unconstrained firms. Since the collateral constraint is not relevant for these firms, I can easily derive their optimal level of $k' = K^w(\epsilon; s, \mu)$ as follows. Let $\Pi^w(k, \epsilon; s, \mu) \equiv z\epsilon F(k, N^w) - wN^w$ be current earnings net of the wage bill. Given Markov property for stochastic processes and the lack of any capital adjustment costs, $K^w$ is the solution to the equation below.

$$
\max_{k'} \left[ -p(s_l, \mu) k' + \beta \sum_{m=1}^{N_s} \pi^s_{lm} p(s_m, \mu') \sum_{j=1}^{N_s} \pi^t_{ij} \left( \Pi^w(k', \epsilon_j; s_m, \mu') + (1 - \delta) k' \right) \right]
$$

With the policy functions $N^w$ and $K^w$ on hand, I exploit the definition of an unconstrained firm to derive an optimal policy for debt, $b'$, that I assign to all financially indifferent firms using the following recursive equations.

$$
B^w(\epsilon_i; s_l, \mu) = \min_{(s_m, \epsilon_j)} \left( \tilde{B} \left( K^w(\epsilon_i; s_l, \mu), \epsilon_j; s_m, \mu' \right) \right)
$$

$$
\tilde{B}(k, \epsilon_i; s_l, \mu) = z\epsilon_i F(k, N^w) - wN^w + (1 - \delta) k - K^w(\epsilon_i)
$$

Let $\tilde{B}(K^w, \epsilon_j; s_m, \mu')$ be the maximum level of debt that an unconstrained firm can hold at the beginning of next period and still remain unconstrained, having chosen the unconstrained choice of capital $K^w$ at the current period and then realized the exogenous state $(\epsilon_j, s_m)$. The minimum savings policy $B^w$ ensures that the firm’s debt never exceeds this level, given all possible realizations of $(s, \epsilon)$. Furthermore, $\tilde{B}$ itself as a function of the firm’s state vector may be retrieved using $B^w$ as shown in
Thus, in (6), the minimum savings policy $B^w(\epsilon; s, \mu)$ is defined to ensure that an unconstrained firm will be unaffected by the borrowing constraint over any future path of $(s, \epsilon)$.

1.3.4 Reducing State Space

In practice, I solve the equilibrium of the model using the firm distribution over the two continuous endogenous variables $(k, b)$ as well as idiosyncratic productivity $\epsilon$, given the value functions described above. A theoretical contribution of my work is to reduce the state space of the firm’s problem in a novel way without altering the equilibrium allocation implied by solving the firm’s problem in Equation (4) and (5). Towards this, I introduce a new individual state variable that summarizes a firm’s financial position in terms of $k$ and $b$. Then I transform the value functions to characterize decision rules in a simpler way.

By using the optimal labor demand function $N^w$, define the cash-on-hand of a firm with $(k, b, \epsilon)$ as $m(k, b, \epsilon; s, \mu) \equiv z\epsilon F(k, N^w) - wN^w + (1 - \delta)k - b$.

Notice that all information relevant for a firm’s decisions on current investment and borrowing is contained in $m(k, b, \epsilon)$, and the future cash-on-hand for the next period is affected by the firm’s intertemporal choices. In particular, the firm’s chosen $k'$ and $b'$, alongside the realizations of $\epsilon'$ and $s'$, determine the level of future $m$. Given the level of cash-on-hand of a firm, I transform Equation (4) and (5) by defining $W_0$ and $W$ as

9The definition of cash-on-hand is exactly the same as the firm’s net worth after production.
follows.

\[ W_0(m, \epsilon; s_t, \mu) = \pi_d \cdot p \cdot m + (1 - \pi_d) \cdot W(m, \epsilon; s_t, \mu) \]  

(8)

\[ W(m, \epsilon; s_t, \mu) = \max_{m', \epsilon', s', D} \left[ pD + \beta \sum_{m=1}^{N_s} \sum_{j=1}^{N_e} \pi_{t,m}^{l} \pi_{ij}^{r} W_0(m_{jm}', \epsilon_j; s_m', \mu') \right] \]  

(9)

subject to

\[ 0 \leq D = m - k' + qb' \]

\[ b' \leq \theta k' \]

\[ \mu' = \Gamma(s_t, \mu) \]

\[ m_{jm}' = z_{m} \epsilon_{j} F(k', N^{w}(k', \epsilon_j)) - w' N^{w}(k', \epsilon_j) + (1 - \delta)k' - b' \]

Given the unconstrained decision rules, \( K^{w} \) and \( B^{w} \) at each \( \epsilon \), I define the threshold level of cash-on-hand that distinguishes unconstrained firms in the model. From the budget constraint in (9), an unconstrained firm pays non-negative dividends \( D^{w} = m - K^{w} + qB^{w} \). This implies that the unconstrained firm’s cash-on-hand is greater than or equal to a certain threshold level \( \tilde{m}(\epsilon; s, \mu) \equiv K^{w}(\epsilon; s, \mu) - qB^{w}(\epsilon; s, \mu) \). This threshold is a function of the aggregate state and idiosyncratic productivity \( \epsilon \).

Accordingly, I recognize a constrained firm as any with \( m < \tilde{m} \) regardless of the firm’s current state of \( k \) and \( b \). Recall that, in a given period, some constrained firms experience binding borrowing constraints while the others do not. Constrained firms without binding borrowing constraints (Type-1 firms) are those that can adopt the unconstrained capital choice \( K^{w} \), but not \( B^{w} \). Their debt policies are determined by the zero-dividend policy discussed earlier. On the other hand, firms experiencing currently binding borrowing constraint (Type-2 firms) invest to the extent their borrowing limits allow. This constrained choice of future capital can be derived from the cash-on-hand \( m \) held by these firms. Specifically, \( D = 0 \) implies that Type-2 firms’
decisions on $k'$ and $b'$ should be feasible given their available cash-on-hand. From the collateral constraint in (9) together with $m = k' - q b'$, I can define the upper bound for capital chosen by Type-2 firms as $\bar{K}(m) \equiv \frac{m}{1 - q \theta}$. Notice that a Type-1 firm has enough cash-on-hand such that $K^w \leq \bar{K}(m)$. Also, the upper bound approaches infinity as the degree of financial friction $\theta$ gets closer to the gross real interest rate $q^{-1}$, which corresponds to the perfect credit market case. In sum, the decision rules by each firm type as a function of cash-on-hand and productivity are as follows.

- Firms with $m \geq \bar{m}(\epsilon; s, \mu)$ are unconstrained, and adopt $K^w(\epsilon)$ and $B^w(\epsilon)$.
- For constrained firms with $m < \bar{m}(\epsilon; s, \mu)$, the upper bound for future capital choice is $\bar{K}(m) \equiv \frac{m}{1 - q \theta}$.
  - Firms with $K^w(\epsilon) \leq \bar{K}(m)$ are Type-1, and adopt $k' = K^w(\epsilon)$ and $b' = \frac{1}{q}(K^w(\epsilon) - m)$.
  - Firms with $K^w(\epsilon) > \bar{K}(m)$ are Type-2, and adopt $k' = \bar{K}(m)$ and $b' = \frac{1}{q}(\bar{K}(m) - m)$.

1.4 Model Parameters

I parameterize the model to quantitatively investigate the implications of firm heterogeneity with credit shocks. First, I set the model parameters other than those for the aggregate shocks to match the key aggregate moments in the U.S. data. Firm specific productivity shocks follow an exogenous stochastic process which reproduces the empirical firm size distribution. In the second subsection, I estimate both the aggregate productivity shocks and credit shocks in the model using the simulated method of moments (SMM).
1.4.1 Calibration

I begin with the assumed functional forms of the model. The period utility function is standard as in the literature (Rogerson (1988)), \( U(C, 1-N) = \log C + \psi(1-N) \). The DRS production function takes the form of Cobb-Douglas, \( F(k, n) = k^\alpha n^\nu \) with \( \alpha, \nu > 0 \) and \( \alpha + \nu < 1 \).

The model is annual, and the parameters are set to be consistent with the existing literature and the U.S. data. Table 1.1 summarizes the calibrated parameter values in the model. The subjective discount factor, \( \beta \), is set to imply a real interest rate of 4 percent. Following Cooley and Prescott (1995), I set the production parameter, \( \nu \), to be the average share of labor income at 0.6. The depreciation rate, \( \delta \), is set to match the average investment to capital ratio during the postwar U.S. period. Given the value of \( \delta \), the capital share of output, \( \alpha \), is determined to yield the observed average capital to output ratio of 2.3. I use the time series data of investment, output, and private capital stock from the Fixed Asset Tables and National Income and Product Accounts (NIPA) in the Bureau of Economic Analysis (BEA). The preference parameter for the disutility from labor, \( \psi \), is set to imply total hours of worked to be one third, on average.

I set the period exogenous exit probability, \( \pi_d \), to be consistent with the average firm exit rates in BDS database. The parameter for the initial capital stock of new firms, \( \chi \), is set to 0.1, which implies that the entrants in the model hold about 10 percent of the average capital stock in the economy. The steady state level of the financial parameter \( \theta_{ss} \) in the model is set to imply the aggregate debt to asset ratio of 0.33. This is consistent with the average non-farm, non-financial debt to asset ratio in the Flow of Funds from 1954 to 2007.
The idiosyncratic productivity $\epsilon$ is drawn from a time invariant distribution $G(\epsilon; \epsilon_m, \epsilon_M, \xi)$ which is a bounded Pareto distribution with $0 < \epsilon_m < \epsilon_M$ and $\xi \geq 1$.\textsuperscript{10} The bounds of $\epsilon$ support, $(\epsilon_m, \epsilon_M)$, and the shape parameter, $\xi$, are chosen to match the empirical firm size distribution at the steady state of the model. I assume that a firm retains its current individual productivity level with a fixed probability $\pi_\epsilon$ in each period. I set $\pi_\epsilon = 0.7$ so that the implied persistence of $\epsilon$ is consistent with the empirical evidence on firm-level productivity.\textsuperscript{11} I discuss the implication of $G(\cdot)$ in the next section.

1.4.2 Estimation of Aggregate Shocks

To evaluate the model performance over business cycles, I jointly estimate the parameters of the model’s aggregate shock processes using SMM. I iteratively solve the model’s stochastic equilibrium until the simulated model moments are consistent with the empirical target moments. SMM is a simulated version of generalized method of moments (GMM) in Hansen (1982), without relying on closed form solutions from the model.\textsuperscript{12} Because SMM is computationally intensive, I keep the calibrated parameters fixed in the estimation of aggregate shocks.

\textsuperscript{10}Unlike its standard form, the bounded Pareto distribution has the upper bound of its support. The density functions can be derived by normalizing a standard Pareto distribution by its density on the interval. The cumulative density is given by $G(\epsilon) = \frac{1 - (\frac{\epsilon_m}{\epsilon_M})^\xi}{1 - (\frac{\epsilon_m}{\epsilon_M})^\xi}$.

\textsuperscript{11}Foster, Haltiwanger, and Syverson (2008) estimate the persistence of establishment-level TFP using the Census of Manufacturing. It ranges from 0.7 to 0.8 using various measures of TFP. The average persistence of $\epsilon$ in my calibration is 0.77.

\textsuperscript{12}Rigorous theoretical foundations of SMM can be found in Lee and Ingram (1991) and Duffie and Singleton (1993) among others. Ruge-Murcia (2012) estimates nonlinear DSGE models using SMM.
Functional Forms and Data

Following the literature, I assume that the logarithm of the exogenous TFP component \( z \) follows an AR(1) process.

\[
\log z' = \rho_z \log z + \eta_z', \quad \text{where } \eta_z \sim N(0, \sigma_z^2)
\]

Regarding the stochastic process for financial shocks, there is no consensus in the literature. I assume that the credit shocks directly hit the financial parameter \( \theta \) in the model. Specifically, \( \theta \) follows a two-state Markov switching process with \( \theta \in \{\theta_{ss}, \theta_l\} \), and the transition matrix is given by,\(^{13}\)

\[
\Pi^\theta = \begin{bmatrix} p_o & 1 - p_o \\ 1 - p_l & p_l \end{bmatrix}
\]

Note that \( \theta_l \) reflects the tightened borrowing condition when a credit shock hits the economy. I set the value of \( \theta_l \) to yield a 5 percent drop in aggregate output when \( \theta \) stays at this value permanently. This is roughly the observed drop in detrended GDP over the recent recession. \( p_o \) is the probability of remaining at the steady state value of \( \theta \), whereas the economy escapes from a credit crunch with \( 1 - p_l \). In sum, I estimate a vector of model parameters \( \gamma \equiv (\rho_z, \sigma_z, p_o, p_l) \) that generates plausible model moments.

The estimation uses the real and financial data from NIPA and the Flow of Funds between 1954 to 2007. In particular, I target the moments from HP filtered real per capita GDP \( (Y) \) and total assets \( (A) \) in non-financial corporations.\(^{14}\) The empirical

\(^{13}\)I also estimate a standard AR(1) process of \( (\theta - \theta_{ss}) \). This specification, however, also involves persistent periods of credit booms as well as credit crunches in the model. The resulting estimates are available upon request. Moreover, I am currently extending the parameter set \( \gamma \) to be estimated. In particular, I focus on estimating the credit shock parameter \( \theta_l \) and the cross correlations between credit and productivity shocks, while additionally selecting appropriate target moments.

\(^{14}\)The resulting estimates are sensitive to the choice of target financial moments. For example, the estimated persistence of normal periods, \( \bar{p}_o \), converges to an unlikely value (0.15), when the model is instead targeted to match the empirical moments of debt-asset ratio or total debt.
moments in the estimation include the autocorrelations and standard deviations of the target variables, \((\rho_Y, \sigma_Y, \rho_A, \sigma_A)\). Total assets in the model are computed as the sum of aggregate capital stock and the total saving (i.e., negative debt) held by firms. The optimal weighting matrix in the objective function of SMM is set to be the identity matrix. That is, I minimize the distance between the target moments and simulated moments by equally weighting each parameter, as is appropriate in an exactly identified case.

**Estimated Parameters**

Table 1.2 summarizes the estimation results. Overall, the model generates the simulated data quite close to the target moments. I present more detailed business cycle statistics in the next section, and then discuss the model’s performance.

The persistence of aggregate productivity shocks is lower than the conventional value from the Solow residual approach. This is because the model is annual, and the AR(1) coefficient of detrended real GDP falls when compared to its quarterly counterpart. Given the steady state level of financial frictions \(\theta_{ss}\), the model economy proceeds to the next period without experiencing a credit shock with a probability of \(p_o = 0.7\). The recovery from a credit shock occurs, on the other hand, with a probability of \(1 - p_l = 0.39\). These probability values are different from those \((p_o = 0.98, p_l = 0.69)\) in Khan and Thomas (2013). In that, they calibrated these parameters to be consistent with the observed banking crises in advanced economies. Here, my estimates reflect the fluctuations in U.S. financial markets at annual frequency, jointly with the exogenous productivity shocks.

From the estimated transition matrix \(\Pi^\theta\), I simulate the stochastic credit shock process to get an average duration and relative frequency for credit crises. On average,
a credit crunch lasts for about 2.5 years and the economy spends about 40 percent of
time in such tough borrowing conditions.\footnote{15} Therefore, the estimated credit shocks are
relatively common and they substantially alter the aggregate dynamics of the model
economy. In other words, credit shocks are relevant to a proper understanding of the
business cycle.

**Numerical Overview**

I briefly describe the estimation procedure of SMM and the numerical methods
implemented to solve the stochastic equilibrium of the model. Explicitly, let \( \gamma \) be
a vector of model parameters to be estimated and denote \( M_T(x) \) as a vector of em-
pirical data moments from the time series \( x \) with \( T \) periods. The corresponding
moments \( M_N(\gamma) \) can be generated from \( N \) repetitions of a \( T \) period simulation of the
model. Given a symmetric weighting matrix \( W \), SMM estimates the parameters by
minimizing the following objective function.

\[
\hat{\gamma} = \arg \min_{\gamma} (M_T(x) - M_N(\gamma))^\prime W (M_T(x) - M_N(\gamma))
\]

That is, \( \hat{\gamma} \) minimizes the weighted distance between target data moments and sim-
ulated moments. The estimator is consistent and asymptotically normal, and be-
comes efficient when the weighting matrix is optimal by \( W = S^{-1} \), where \( S \) is the
variance-covariance matrix. The standard errors can be computed either by using a
Monte-Carlo method or by the asymptotic variance.

Given initial parameter values, I solve the above minimization problem using the
Nelder-Mead simplex method which does not assume differentiability of the objective.

\footnote{The relative frequency of a credit shock is quite large in the model simulation. Reinhart and
Rogoff (2009) find that the share of time spent in banking crises in the U.S. is about 13 percent.
Here, I focus on the credit shocks as the time varying efficiencies of broader financial markets.}
Notice that the stochastic equilibrium of the model needs to be solved at each iteration of the optimization algorithm. This is because the parameters of the aggregate shock processes are critical in determining simulated moments.

When solving the equilibrium of the model, the firm distribution in the model is a high-dimensional object, so the curse of dimensionality applies. I apply the Krusell-Smith algorithm which approximates \( \mu(k, b, \epsilon) \) by the first moment of the distribution of capital across firms, \( K \). In that, the aggregate law of motion \( \Gamma(s, \mu) \) is replaced by a simple forecasting rule \( \hat{\Gamma}(s, K) \). I follow the approach in Khan and Thomas (2008) which develops forecasting rules specific to a model of heterogeneous firms.

To estimate \( (\rho_z, \sigma_z) \), I discretize the AR(1) process of \( z \) using the Rouwenhorst method with \( N_z = 3 \). The length of each simulation, \( T \), is set to 109 consistent with the empirical data series. I discard the initial 50 observations of each simulation to eliminate the model dependency to the steady state. Lastly, the weighting matrix of the SMM estimator is the identity matrix.

1.5 Results

1.5.1 Steady State

Heterogeneity and Decisions

In the model, the financial friction generates substantial heterogeneity across firms alongside idiosyncratic productivity. This is readily observed in the stationary distribution of firms at the steady state of the model. Figure 1.2 shows the entire firm distribution over current capital and debt level \( (k, b) \). In the distribution, all three types of firms analyzed in the previous section can be identified; unconstrained, Type-1, and Type-2 firms. The distribution involves a large spike near \( (0, 0) \), of entrants,
which begins a diagonal line from which there are several perpendicular lines due to \( N_x = 13 \). The unconstrained firms are located at the point masses where the perpendicular lines from the main body (the primary diagonal beginning around \((0,0)\)) of the distribution end. At each mass point distinguished by the current level of firm productivity \( \epsilon \), the debt level reflects the minimum savings policy \( B^w \). The firms on the perpendicular lines ending with unconstrained firms are Type-1. They accumulate assets by first reducing their debt then saving to become unconstrained. Notice that these firms hold the same amount of current capital stock as the unconstrained firms do. This is because their investment decisions in the previous period were not affected by binding borrowing constraints. Type-2 firms are on the diagonal line which is the main body of the distribution. These firms chose capital that was constrained by their cash-on-hand in the previous period, and the straight line indicates the inverse relationship between these firms’ capital and financial savings.

Firm types are easily distinguishable from the decision rules as a function of the new state variable, cash-on-hand. Figure 1.3 displays the decision rules for capital \( k' \), debt \( b' \), and dividends \( D \) as functions of the cash-on-hand \( m(k, b, \epsilon) \) for the maximum value of idiosyncratic productivity.\(^{16}\) In the figure, the two vertical lines separate the firm types by the level of \( m \). The line on the left distinguishes Type-1 and Type-2 firms, the latter being those with the lowest \( m \) and to the left of the threshold line. The other vertical line near \( m = 70 \) represents the threshold \( \tilde{m} \) for being an unconstrained firm for this \( \epsilon \) value.

Figure 1.3 can be interpreted as illustrating the growth dynamics of a firm when the exogenous exit risk is ignored. Suppose a firm was born with a small amount

\(^{16}\)The overall pattern of decision rules is similar across \( \epsilon \) values. I chose the maximum value of \( \epsilon \) for a clear illustration.
of capital, so its cash-on-hand is negligible. This firm is Type-2 and is only able to invest to the extent allowed by its borrowing limit as given by $\bar{K}(m)$. As the firm grows by accumulating cash-on-hand, it eventually makes past the first vertical line in the figure and becomes Type-1. Its investment does not suffer from a binding borrowing constraint and it chooses the optimal capital stock $K^\ast$. Nonetheless, to be unconstrained, the firm must accumulate more financial savings. Thus it pays no dividend and accumulates all profits by reducing debt. Overtime, its cash-on-hand, $m$, rises further. This is represented by the negatively sloped debt policy between the two vertical lines. When the level of $m$ becomes greater than the threshold $\bar{m}$, the firm is now unconstrained. While following both $K^\ast$ and $B^\ast$ policies, this firm starts to pay positive dividends.

**Firm Size Distribution**

I compare the model-generated firm size distribution with its empirical counterpart. The empirical size distribution is constructed using the Business Dynamics Statistics (BDS).\(^\text{17}\) As already shown in Figure 1.1, I compute the average employment and population shares of each firm size group (in terms of employees) between 1977 and 2007. Note that the employment size of a firm in the model is not discrete as in the data. Thus, I use the empirical employment shares as a proxy to distinguish firm size groups at the stationary equilibrium of the model. Specifically, given the cumulative density of employment shares in the data,\(^\text{18}\) I use bisection to find the

\(^{17}\)BDS is based on the Census data from Longitudinal Business Database (LBD). It covers about 90 percent of the U.S. private employment starting from 1977. Firms and establishments are categorized by size and age, and their annual job creations and destructions are reported. Each firm size bin is defined as the number of employees and held fixed over time. For further details, see Haltiwanger, Jarmin, and Miranda (2009).

\(^{18}\)The employment shares in BDS are stable over time even with the entry and exit dynamics of firms during the sample period.
threshold levels of employment, in the model, that correspond to size groups in the data. The resulting model population shares of firms are reported in the last column of Table 1.3.

The model generates a substantial degree of firm-level heterogeneity, certainly enough to match the size distribution observed in the BDS data. It is important to note that only 3 parameters, \((\epsilon_m, \epsilon_M, \xi)\), are used to match the 6 size bins.\(^{19}\) In particular, the lower tail of the distribution of population shares is almost perfectly matched using the bounded Pareto distribution of idiosyncratic productivity \(\epsilon\).\(^{20}\) The share of the largest firms in the model is somewhat larger than in the data, but it is mainly driven by the assumption of decreasing returns to scale (DRS) production.\(^{21}\) Overall, the model is able to capture the huge degree of inequality across firms in the U.S. economy. As will be discussed in the next section, I further investigate the aggregate implications of such a realistic size distribution over business cycles.

Together with the financial friction, the assumption of Pareto distributed \(\epsilon\) is critical for generating an empirically consistent firm size distribution. Existing models of heterogeneous firms, however, typically assume that the logarithm of a firm’s individual productivity follows an AR(1) process with Gaussian innovations; \(\log \epsilon' = \rho \log \epsilon + \eta'_t\) with \(\eta_t \sim N(0, \sigma^2_{\eta_t})\). The parameters of the AR(1) process are set to match micro data moments, other than the size distribution of firms. Here, I argue that this

---

\(^{19}\) The lower and upper bounds of idiosyncratic productivity, \(\epsilon_m\) and \(\epsilon_M\), jointly imply the mean value and the standard deviation of \(\epsilon\) at a given \(\xi\) value. The skewness parameter, \(\xi\), largely determines the number of firms in each size bins.

\(^{20}\) I consider 6 different firm size groups. The size groups were chosen to be consistent with the empirical definitions of small and large firms. To the best of my knowledge, however, this is one of the first attempts to match detailed size bins without relying on a simulation method.

\(^{21}\) Sedlacek and Sterk (2013) calibrates the returns to scale parameters by each firm size group.
conventional identification of $\epsilon$ process is not enough to generate the observed size distribution in the data.

Figure 1.4 compares the population shares of firms when using different identifications of the $\epsilon$ process in the model.\textsuperscript{22} Clearly from the figure, the case of lognormal $\epsilon$ process is relatively poor in matching the empirical size distribution. There are less small firms in the distribution whereas more than 5 percent of firms are among the largest. While not reported here, the lower tail of the empirical size distribution in the lognormal case can be matched by enormously increasing the volatility parameter, but the population shares in the other size categories move away from their targets. On the other hand, the size distribution from the case of Pareto $\epsilon$ preserves its well-known shape with changes in the parameter values.\textsuperscript{23}

1.5.2 Business Cycles

The model includes two different sources of aggregate fluctuations; productivity and credit shocks. By using the estimated parameters of the aggregate shocks from the previous section, I simulate the model for 5000 periods. Table 1.4 presents the business cycle moments from the simulation. The simulated moments are broadly similar to the results from a standard real business cycle model which is only driven by exogenous TFP shocks. Output volatility is about 2.2 percent, while the relative volatility of consumption is about 40 percent of the output series. Due to the absence

\textsuperscript{22}I control the persistence and volatility parameters of the lognormal identification case to be same as in the assumed Pareto $\epsilon$ process. Also, I recalibrate the other model parameters to get the same aggregate moments in each specification. If these values were chosen to match the micro-level investment moments as in Khan and Thomas (2013), on the other hand, the resulting model size distribution also fails to match its empirical counterpart.

\textsuperscript{23}I do not mean that the bounded Pareto distribution is the only way to match the size distribution. For example, Guner, Ventura, and Xu (2009) use a truncated lognormal productivity process augmented with extreme values for the largest firms.
of adjustment costs in production factors, the relative volatilities of investment and labor are somewhat larger than the typical values.

A notable difference with a conventional heterogeneous firm model driven by productivity shocks is the contemporaneous correlation of consumption with output. Aggregate consumption is only weakly procyclical suggesting a strong effect of credit shocks at least where consumption is concerned. This is mainly because the estimated credit shocks are relatively frequent and persistent. As discussed earlier, the estimated probability of experiencing a credit crunch is relatively high, which implies households actively making use of investment to smooth their consumption over time. This less cyclical consumption is, however, a result of my identification of credit shocks using aggregate firm assets and GDP. When the credit shock parameters are set to reproduce the evidence of banking crises as in Reinhart and Rogoff (2009), the simulated consumption series become more procyclical. This becomes even clearer when I simulate the model using only the estimated process for real shocks. Table 1.5 shows the business cycle statistics are very similar to the results from a standard model. Consumption there is very procyclical, and the economy is less volatile overall while the mean aggregates are slightly larger than in the full benchmark economy. These results suggest that a precise identification of financial shocks is essential to the model’s quantitative performance.

By jointly estimating the aggregate real and financial shocks, the model achieves much of the typical business cycle patterns as in the literature. Although the estimated productivity shock parameters are not close to the typical values measured using Solow residuals, Table 1.4 shows that the model exhibits plausible business cycles in the presence of both real and credit shocks. This is because the estimated
productivity shocks are relatively persistent affecting all firms in the model symmetrically, unlike credit shocks. Therefore aggregate dynamics of the model are mainly driven by the shocks to $z$, which is the exogenous component of total factor productivity (TFP). Figure 1.5 confirms that the model in this paper is comparable, at least following a productivity shock, to a standard business cycle model. Following a persistent negative shock to $z$, the model generates almost isomorphic aggregate responses (the red lines) to the results from a frictionless model in which firms are not constrained by financial frictions. At impact, all key macroeconomic variables fall immediately and gradually recover to their steady state levels. In addition, measured TFP is almost identical to the exogenous series of $z$, indicating that the misallocation of resources implied by collateral constraints does not interact in a significant way with the productivity shock.

I conclude that the underlying idiosyncratic productivity process is not relevant for aggregate dynamics when only aggregate TFP shocks are considered. Figure 1.5 also compares the transitional dynamics of the model from a different specification of $\epsilon$ process. There is no pronounced difference in each aggregate series following a negative shock to $z$. Again, this is because the aggregate productivity shocks in the model affect all firms with the same magnitude through the production function.

1.5.3 Implications of Firm Size Distribution

I investigate the implication of a realistic firm size distribution for financial recessions. In the model, a financial recession is triggered by a sudden tightening of the economy-wide borrowing condition. This is captured by a drop in the financial parameter $\theta$, which is referred as a credit shock in this paper. I conduct an exercise
aimed at the recent recession in the U.S. by hitting the model economy with a credit shock. Note that my benchmark economy is also calibrated to match the empirical firm size distribution at the steady state. Figure 1.6 presents the impulse responses of the model when $\theta$ falls about 17 percent from its steady state value. This magnitude is intended to generate about 26 percent drop in the aggregate lending at the impact which is consistent with the empirical observation in the 2007 recession. After date 3, $\theta$ gradually recovers to its pre-crisis level with persistence of 0.386.

First, measured TFP following the credit shock is consistent with the observed pattern in the recent recession. It gradually falls from the impact date and reaches its trough which is about 2 percent below the steady state level. This reflects the slow reallocation of production factors across firms following the shock. Due to tightened credit, Type-2 firms lose capital to Type 1 and unconstrained firms, and the share of Type-2 firms grows. The persistent response of output also characterizes the financial recessions distinct from those following the real shocks as illustrated in Figure 1.5. It initially drops by 1 percent while the greatest decline is at date 4. Consumption, on the other hand, slightly rises at impact and gradually falls during the recession. In sum, the model is able to generate a convincing financial recession as in the existing models of production heterogeneity with financial shocks.

\textsuperscript{24}Various measures of business lending suggest a reduction in lending to businesses during the recent recession. For example, Ivashina and Scharfstein (2010) report the newly issued volume of syndicated loan fell more than 50 percent in 2008. Khan and Thomas (2013) compute the reduction in real lending at commercial banks between 2008 and 2011. In that, the fall in lending is 26 percent, and I follow this conservative value in the above.

\textsuperscript{25}This persistence value is from my estimation of credit shocks identified as AR(1). Note that the probability of escaping from a credit crisis in the two-state Markov process, $1 - p_t$, is close to this value.

\textsuperscript{26}Khan and Thomas (2013) document the gradual fall in measured TFP from 2007 to 2011. They also discuss the notable features of aggregate dynamics in the Great Recession.
More importantly, I focus on the quantitative differences when the firm size distribution is considered. In Figure 1.7, I compare the aggregate dynamics upon a credit shock by switching the specification of the idiosyncratic productivity process from the benchmark. Specifically, I use the conventional lognormal $\epsilon$ process instead of the assumed Pareto distribution. As shown earlier, this specification of firm-level productivity does not generate a realistic size distribution of firms.

Figure 1.7 shows that the financial recession in an economy with the lognormal $\epsilon$ is not as much severe as in my benchmark.\textsuperscript{27} In particular, the greatest decline in measured TFP is about half of the benchmark model with a Pareto distribution for firm-level total factor productivity. This observation implies that the misallocation from the tighter borrowing condition is less pronounced in the economy with a less dispersed size distribution. Because of this small TFP loss, it is natural to have relatively short periods of recovery in this economy.

From the above comparison, I emphasize that the underlying firm size distribution is critical when quantitatively evaluating the macroeconomic effects of a financial shock. This is because the size distribution is closely related to the degree of efficiency loss due to misallocation. In the benchmark economy with a Pareto distribution, most firms are small and the probability of their acquiring a high level of individual productivity is very low. These small firms tend to be financially constrained because of their insufficient cash-on-hand.\textsuperscript{28} A credit shock further restricts the upper bound of their choice sets from the fall in $\theta$. The number of firms with a currently binding

\textsuperscript{27}I control the size of the credit shock as well as the persistence and volatility of both processes. The shock is calibrated to yield the same amount of decrease in the aggregate lending, while matching the aggregate moments at the steady state. The resulting fall in $\theta$ with lognormal $\epsilon$ is about 20 percent.

\textsuperscript{28}Cash-on-hand $m$ is a concave function of current capital $k$. 

35
borrowing limit rises sharply during this episode. The fraction of all firms that are Type-2 rises from its steady state value of 14.2 percent to 29.9 percent by date 3. On the other hand, only a small number of firms are productive enough to be large, and they are able to exploit the low interest rate during the recession. The aggregate recovery from the recession is mainly driven by these large firms. However, the fall in $\theta$ also reduces the number of unconstrained firms in the model by increasing the threshold cash-on-hand $\bar{m}$. Firm heterogeneity in size, therefore, naturally entails huge differences in returns to capital across firms, and the credit shock intensifies this resource misallocation through the collateral constraint. When a model size distribution is not consistent with its empirical counterpart, the recessions from financial shocks can be underestimated.\textsuperscript{29}

To understand the propagation of a credit shock across heterogeneous firms, I illustrate the model mechanics using firm-level decisions. Figure 1.8 shows the comparative statics of the model with a change in $\theta$, at a given level of idiosyncratic productivity. Here, I use the level of cash-on-hand, $m$, as a proxy for firm size so that the density over the horizontal axis (red-dotted line) reflects the size distribution of firms. This is because firm size is positively correlated with cash-on-hand in the model. The left panel in Figure 1.8 shows the decision of future capital $k'$ at the steady state. Note that the upper limit of $k$ choice set, $\bar{K}$, is linear in $m$ which is represented by the straight line from the origin. Given the level of unconstrained choice of future capital $K^w$ and firm-level productivity $\epsilon$, a firm with sufficient cash-on-hand greater than $m(\bar{K})$ is able to invest efficiently. However, a large number of small firms with $m < m(\bar{K})$ suffer from the binding borrowing constraint even at the

\textsuperscript{29}Recall that the model in this paper abstracts from any micro-level adjustment frictions which can further magnify the aggregate responses from a financial shock.
steady state. The right panel of Figure 1.8 presents the change in the upper bound when \( \theta \) falls, holding the size distribution and the interest rate fixed. Following a credit shock, the slope of \( \tilde{K} \) becomes less steep and more firms are constrained by the upper bound imposed on their choice of capital through the collateral constraint. In this regard, the size distribution of firms determines the strength of the misallocation channel from the credit shocks in the model.

1.5.4 Firm Dynamics

Financial shocks affect firms unevenly amplifying the misallocation in an economy. This is because a firm responds to the changes in its borrowing condition by adjusting investment and employment given its financial asset holding. I examine firm-level employment dynamics to confirm the differential nature of credit shocks in the model.

In the estimated model economy, however, the aggregate productivity shocks are dominant in shaping business cycles as discussed earlier. I also present the dynamics of firms at disaggregated level following an aggregate productivity shock. By investigating the difference in firm behavior by size conditional on the source of aggregate shocks, I provide a theoretical basis to reconcile the differing views on firm dynamics over the U.S. business cycles.

As noted by Gertler and Gilchrist (1994), small firms are more responsive to a monetary contraction which is a form of negative financial shock. During a typical U.S. recession, however, the dynamics of small firms are not largely different from those of large firms according to Chari, Christiano, and Kehoe (2013). These two previous findings seem to contradict each other, but it is the difference in focus that distinguishes the latter finding; Gertler and Gilchrist (1994) study the responsiveness
of firms as a function of size conditional on monetary shocks. And Moscarini and Postel-Vinay (2012) focus on the unconditional behavior of small and large firms. They find that employment growth by large employers is more negatively correlated with unemployment rate over business cycles. On balance, I show that it is crucial to consider both aggregate real and financial shocks for understanding the empirical evidence on cyclical firm dynamics.

I simulate a panel of 75,000 firms. In addition to the detailed firm size categories, I further disaggregate employment dynamics by firm age.\textsuperscript{30} At each firm size or age group, I measure the job flows as defined in Davis, Haltiwanger, and Schuh (1996).\textsuperscript{31} Given a group index $s$ for a firm $i$, job creation rate ($JC$) and job destruction rate ($JD$) are defined as,

$$JC_{s,t} = \sum_{i \in s} \frac{n_{i,t} - n_{i,t-1}}{0.5(N_{s,t} + N_{s,t-1})} \text{ if } n_{i,t} \geq n_{i,t-1}$$

$$JD_{s,t} = \sum_{i \in s} \frac{|n_{i,t} - n_{i,t-1}|}{0.5(N_{s,t} + N_{s,t-1})} \text{ if } n_{i,t} < n_{i,t-1}$$

$n_{i,t}$ is the employment of firm $i$ at date $t$, and $N_{s,t}$ is the aggregated employment in group $s$. Net employment growth ($NEG$) is the difference between $JC$ and $JD$ rates in firm group $s$, i.e., $NEG_{s,t} = JC_{s,t} - JD_{s,t}$.

Figure 1.9 shows $NEG$ of each firm group following a productivity shock. In the upper panel of Figure 1.9, I disaggregate firms into three size groups following

\textsuperscript{30}Exogenous entry and exit with $\pi_d$ in my model generate a firm age distribution consistent with the BDS data. For example, the population share of young firms aged less than 5 years in the model is about 0.39, while its counterpart in BDS is about 0.42.

\textsuperscript{31}Buera, Fattal-Jaef, and Shin (2015) have a similar analysis of employment changes by firm size and age. They focus on the credit shocks in a model with frictions in labor markets. However, I follow the empirical definition of small and large firms here and also consider aggregate productivity shocks.
the definition in Fort, Haltiwanger, Jarmin, and Miranda (2013). Clearly, firms respond similarly, across size groups, to a change in exogenous TFP. At impact, firms of different size decrease their employment by destroying jobs at almost the same rate. Thereafter, they gradually increase employment as the economy reverts back to steady state. A similar pattern appears in the lower panel of the figure, in which I distinguish firms by age. These results imply that resource allocation is not affected by a productivity shock as mentioned above. From this exercise, I confirm the neutral effects of aggregate productivity shocks. If an economy is only driven by such productivity shocks, Figure 1.9 is consistent with the finding in Chari, Christiano, and Kehoe (2013). However, they also identify a higher sensitivity of small firms during certain periods in the U.S., indicating the possibility of financial shocks.

Following a credit shock, on the other hand, firms adjust employment differently by each size and age group. In Figure 1.10, large firms decrease their hiring during the credit crunch, and gradually adjust back to their steady state level. On the other hand, net employment growth by small firms is relatively volatile. Small firms increase their employment while the wage rate is below its steady state level. This is because they expect credit conditions to gradually recover in the future. The upper panels of Figure 1.9 and 1.10 together imply that employment growth of large firms is more strongly correlated with business cycles. That is, the negative relationship between employment growth and economic contraction is robust among these firms, regardless of the source of aggregate fluctuations. This finding is in favor of the evidence in Moscarini and Postel-Vinay (2012) because the cyclical behavior of small firms is more affected by financial shocks in my model which weakens the unconditional correlation.

39
The differential response is more pronounced when I examine the age dimension of firm heterogeneity. The lower panel of Figure 1.10 indicates that it is young firms that suffer from the tightened borrowing condition. Since firms are born with a small amount of capital in the model, young firms are more likely to be financially constrained and more vulnerable to credit shocks. This becomes clear when the size and age dimensions are jointly considered in Figure 1.11. Large firms, both young and mature, experience very similar employment dynamics during the credit crunch. Net employment growth by small-mature firms have the most volatile responses, while small-young firms are relatively stable in their labor adjustment. It follows that the increased employment growth of small firms in the upper panel of Figure 1.10 is mainly driven by the relatively large share of small-mature firms in the economy. These results are at odds with those from the similar exercise in Buera, Fattal-Jaef, and Shin (2015). They consider frictions in the hiring market, and young firms in their model, both large and small, behave very similarly by reducing employment following a credit shock.\footnote{Buera, Fattal-Jaef, and Shin (2015) focus on the slow recovery in labor markets in the Great Recession, which addresses the empirical finding in Fort, Haltiwanger, Jarmin, and Miranda (2013). It is small-young firms that lost mostly from the collapse of housing markets in the recent U.S. recession.}

Taking the observations from Figure 1.9 - 1.11, I confirm that distinguishing an aggregate source of economic fluctuations is important towards reconciling existing empirical evidence on firm dynamics. That is, a heterogeneous agent business cycle model calls for multiple aggregate shocks that affect economic agents disproportionately at times. Financial frictions and the associated shocks allow us to understand the differential dynamics at firm level during certain recessions in the U.S., even though the estimated productivity shocks still dominate the overall business cycles.
Therefore, my model provides a theoretical foundation for understanding the different evidence on cyclical sensitivity of firms by size and age.

1.6 Conclusion

The evidence on the size distribution of firms is often ignored in the existing business cycle literature. In this paper, I highlighted the importance of the skewness in the firm size distribution for assessing the aggregate effect of financial shocks. This is because the very nature of such shocks implies differing impact on firms as a function of their size and age.

I have quantitatively investigated the macroeconomic implications of a realistic firm size distribution in an economy with both real and financial shocks. I have built an equilibrium business cycle model with heterogeneous firms in which financial frictions are nested in forward-looking collateral constraints. The model is highly tractable and consistent with existing theories of financial friction, while maintaining all the necessary elements for quantitative business cycle analysis. I specified an exogenous idiosyncratic process of firm productivity which is able to reproduce the empirical firm size distribution. Furthermore, the real and financial shock processes were estimated by matching moments from U.S. data.

This paper suggests that the importance of financial shocks may be under-predicted in existing models. This is because small firms in the lower tail of the size distribution are most affected by financial frictions that impede their borrowing. To the extent the existing literature fails to capture the large number of such firms, it will understate the real effects of financial shocks. Compared to earlier work, the parameterized model presented in this paper shows that the misallocation from financial shocks is
larger and, importantly, the subsequent recovery is slower, when the model firm size
distribution carefully matches its empirical counterpart. On the other hand, shocks
to aggregate TFP do not imply different aggregate dynamics under different firm size
distributions. This allows the model to remain capable of explaining typical U.S.
business cycles.

To examine the impact of financial shocks at micro-level, I measured employment
dynamics of each firm size and age group in the model. Firms exhibit differential
dynamics following a credit shock depending on their financial situation. In particular,
small firms show dramatic differences in employment adjustment by age, a finding
that has been emphasized in recent empirical work. This result suggests how the
recent recession in the U.S. was different from previous episodes, and also provides a
theoretical explanation to the empirical debate on firm dynamics over business cycles.
In this regard, this paper also has implications for the quantitative evaluation of firm
size- or age-targeted government policies. This is left for future research.
### Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target Value and Description</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.96</td>
<td>Annual real interest rate</td>
<td>1.04</td>
</tr>
<tr>
<td>$\psi$</td>
<td>2.14</td>
<td>Average hours worked</td>
<td>0.33</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.60</td>
<td>Labor share of income</td>
<td>0.60</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.069</td>
<td>Investment to capital ratio (BEA)</td>
<td>0.069</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.276</td>
<td>Capital to output ratio (BEA)</td>
<td>2.30</td>
</tr>
<tr>
<td>$\theta_{ss}$</td>
<td>0.81</td>
<td>Debt to asset ratio (Flow of Funds)</td>
<td>0.34</td>
</tr>
<tr>
<td>$\pi_d$</td>
<td>0.10</td>
<td>Average firm exit rate (BDS)</td>
<td>0.10</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.10</td>
<td>Relative size of entrants</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 1.1: Calibration

### Estimated Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Model Moment</th>
<th>Data Moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_z$</td>
<td>0.5702</td>
<td>0.5576</td>
<td>0.5379</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.0083</td>
<td>0.0282</td>
<td>0.0203</td>
</tr>
<tr>
<td>$p_o$</td>
<td>0.7005</td>
<td>0.7214</td>
<td>0.7245</td>
</tr>
<tr>
<td>$p_l$</td>
<td>0.6019</td>
<td>0.0252</td>
<td>0.0377</td>
</tr>
</tbody>
</table>

Table 1.2: SMM Estimation
### Firm Size Distribution

<table>
<thead>
<tr>
<th>Employees</th>
<th>Employment Share</th>
<th>Population Share</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Model</td>
<td>Data Model</td>
</tr>
<tr>
<td>1 to 4</td>
<td>0.0584 0.0584</td>
<td>0.5506 0.5459</td>
</tr>
<tr>
<td>5 to 19</td>
<td>0.1455 0.1455</td>
<td>0.3342 0.3232</td>
</tr>
<tr>
<td>20 to 99</td>
<td>0.1814 0.1814</td>
<td>0.0964 0.0884</td>
</tr>
<tr>
<td>100 to 499</td>
<td>0.1395 0.1395</td>
<td>0.0153 0.0230</td>
</tr>
<tr>
<td>500 to 2499</td>
<td>0.1179 0.1179</td>
<td>0.0026 0.0097</td>
</tr>
<tr>
<td>2500+</td>
<td>0.3573 0.3573</td>
<td>0.0009 0.0098</td>
</tr>
</tbody>
</table>

Table 1.3: Firm Size Distribution

### Business Cycle Moments

<table>
<thead>
<tr>
<th></th>
<th>$Y$</th>
<th>$C$</th>
<th>$I$</th>
<th>$N$</th>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$mean(x)$</td>
<td>0.309</td>
<td>0.265</td>
<td>0.044</td>
<td>0.331</td>
<td>0.644</td>
<td>0.042</td>
</tr>
<tr>
<td>$\sigma_x/\sigma_Y$</td>
<td>(2.298)</td>
<td>0.402</td>
<td>7.098</td>
<td>1.028</td>
<td>0.696</td>
<td>0.353</td>
</tr>
<tr>
<td>$corr(x,Y)$</td>
<td>1.000</td>
<td>0.132</td>
<td>0.941</td>
<td>0.921</td>
<td>0.013</td>
<td>0.733</td>
</tr>
</tbody>
</table>

Table 1.4: Business Cycles in the Full Economy

### Business Cycle Moments

<table>
<thead>
<tr>
<th></th>
<th>$Y$</th>
<th>$C$</th>
<th>$I$</th>
<th>$N$</th>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$mean(x)$</td>
<td>0.319</td>
<td>0.271</td>
<td>0.048</td>
<td>0.333</td>
<td>0.696</td>
<td>0.042</td>
</tr>
<tr>
<td>$\sigma_x/\sigma_Y$</td>
<td>(1.382)</td>
<td>0.306</td>
<td>5.428</td>
<td>0.788</td>
<td>0.539</td>
<td>0.289</td>
</tr>
<tr>
<td>$corr(x,Y)$</td>
<td>1.000</td>
<td>0.773</td>
<td>0.979</td>
<td>0.969</td>
<td>-0.075</td>
<td>0.822</td>
</tr>
</tbody>
</table>

Table 1.5: Business Cycles Only with Productivity Shocks
Figure 1.1: Firm Size Distribution in the U.S.
Figure 1.2: Entire Firm Distribution at the Steady State
Figure 1.3: Decision Rules at the Steady State
Figure 1.4: Comparison of Model Firm Size Distribution
Figure 1.5: Aggregate Productivity Shock
Figure 1.6: Credit Shock
Figure 1.7: Firm Size Distribution with Credit Shock
Note: The above is an illustration of the initial impact of a credit shock on firm decisions, holding the equilibrium prices fixed. In equilibrium, the unconstrained choice of capital $K^w$ falls and the discount bond price $q$ rises (but less than the fall in $\theta$).

Figure 1.8: Cash-on-hand and Decisions (Comparative Statics)
Note: There are 3 groups of firm size, small (less than 20), medium (20 to 499), and large (more than 499) by the number of employees. Firms less than 5 years old are defined as young. These size and age groups are consistent with Fort, Haltiwanger, Jarmin, and Miranda (2013).

Figure 1.9: Firm Dynamics with a Productivity Shock
Note: There are 3 groups of firm size, small (less than 20), medium (20 to 499), and large (more than 499) by the number of employees. Firms less than 5 years old are defined as young. These size and age groups are consistent with Fort, Haltiwanger, Jarmin, and Miranda (2013).

Figure 1.10: Firm Dynamics with a Credit Shock, part 1
Note: There are 3 groups of firm size, small (less than 20), medium (20 to 499), and large (more than 499) by the number of employees. Firms less than 5 years old are defined as young. These size and age groups are consistent with Fort, Haltiwanger, Jarmin, and Miranda (2013).

Figure 1.11: Firm Dynamics with a Credit Shock, part 2
Chapter 2: Production Heterogeneity with Borrowing Constraint and Working Capital

2.1 Introduction

The Great Recession in the U.S. is characterized by its unusual dynamics of the observed macroeconomic aggregates. Large declines in the aggregate employment and investment following a financial distress are noticeable when compared to the previous recessions. In addition to the magnitude, persistent deterioration and slow recovery in labor market conditions are in contrast with the predictions from the standard business cycle analysis (see Figure 2.1). Accordingly, a growing literature now incorporates credit market imperfection in a model economy, and analyzes macroeconomic implications of a financial shock. Regarding the aggregate employment dynamics, however, there are only few models that relate financial frictions with labor market conditions. The goal of this paper is, therefore, to build a model with both frictional borrowing and hiring at firm-level in order to investigate the aggregate dynamics upon a credit shock.

The model economy in this paper consists of heterogeneous firms that face with persistent differences in individual productivity, and their decisions on investment and borrowing are endogenously subject to a collateralized borrowing constraint. The endogenous borrowing limit is also affected by a firms scale of hiring. I assume that
firms need to finance wage bill payments before production as working capital, and this assumption serves as a type of friction in employment through the borrowing constraint. Unlike the existing approaches, the pay-in-advance requirement embodies a tradeoff between efficient level of borrowing for investment and optimal labor demand for production. When a firm is under a tighter borrowing condition, for instance, the firm has an incentive to reduce its employment level than the static optimum because of the limited credit availability. In the recovery stage of an economic downturn, the firm also finds it better to accumulate its capital stock quickly from external financing, further restricting its additional hiring despite relatively low wage. The assumption of working capital embedded in the borrowing constraint, thus generates distortions at firm-level employment and emerges as labor wedge in aggregate. Even though this assumption may not exactly reflect the existing frictions in labor market or the payroll structure of firms, I focus on the aggregate implications of the misallocation of production factors generated from the frictions in the model.

I quantitatively investigate the impulse responses of the model upon exogenous shocks. Following a persistent shock to aggregate Total Factor Productivity (TFP), the model exhibits almost identical responses to the results from the standard business cycle model. One exception is slightly more persistent dynamics of the aggregate employment due to the working capital requirement. In this paper, a credit shock is modeled as a sudden tightening of firms borrowing capacity. Upon a credit shock, the aggregate employment initially responds with a substantial decline, and shows persistent dynamics at the recovery stage of the recession. It also shows that reallocation is further exacerbated during a credit crunch because of the frictional labor demand when compared to the dynamics in the absence of the working capital assumption.
In addition to the well-known interest rate channel, the pay-in-advance requirement for wage bills reduces the amount of available credit for intertemporal borrowing in this model. Hence, financially constrained firms are no longer able to choose their desired levels of employment, because their decisions on investment and borrowing are inter-related with their labor demand.

I then conduct the Business Cycle Accounting (BCA) exercise to compare the aggregate wedges generated from the model with the empirical wedges in the U.S. In particular, the dynamics of model-generated labor wedge is qualitatively consistent with its empirical counterpart in the recent recession. The unique combination of borrowing constraint and working capital made in this paper is represented by micro-level distortions in labor and capital inputs. In turn, the increased misallocation among heterogeneous firms upon a credit shock appears as the fluctuations in aggregate wedges of the model. Therefore, I provide an endogenous propagation of distortions at firm-level decisions into the aggregate wedges over business cycles.

Along with the findings mentioned above, the numerical method used to solve the model is worth to note. This is due to the multi-dimensional state-vector of the model, the non-convexity arising from occasionally binding borrowing constraint, and the consideration of working capital requirement in firms external financing. I establish a consistent approach to solve each firms decisions on labor, investment, and borrowing by categorizing firms into groups depending on the bindingness of the borrowing constraint. The latter consistency is obtained from mapping the feasibility of unconstrained decisions into each firms resource constraint and collateral constraint. In particular, I numerically solve the constrained choice of labor demand that affects
the borrowing limit as well. The details of computational methods are illustrated in
the appendix of this paper.

The rest of this paper is organized as follows. In the next section, I briefly discuss
the literature that is closely related to this paper. The model environment is presented
in Section 3, and I characterize the decision rules of heterogeneous firms in Section
4. Section 5 summarizes the results from the model, and I conclude in Section 6.

2.2 Related Literature

This paper is related to several different strands of the literature. Regarding
the recent recession in the U.S., a typical TFP shock in the standard business cycle
model is not able to generate the observed dynamics of the aggregate variables, as
noted by Ohanian (2010). There has been a growing research on the macroeconomic
consequences of financial shocks by incorporating financial frictions thereafter. Jer-
emann and Quadrini (2012) emphasize the role of financial shocks as another source
of business cycle fluctuations in an economy of representative agents. Broadly speak-
ing, the model in this paper can be understood as the heterogeneous version of their
model, because of my introduction of working capital requirement into the borrowing
constraint. In their model, the intra-period loan for wage bill payments affect the rep-
resentative firms available credit. However, I rather focus on the misallocation arising
from the additional friction in heterogeneous firms hiring decisions along the similar
structure of the borrowing constraint from their paper. My modeling approach on
production heterogeneity is closely related to Khan and Thomas (2013), where risk-
neutral firms are subject to persistent differences in individual productivity as well as
to a collateral constraint for intertemporal borrowing. By considering both financial
and real frictions in firm-level investment decisions, they provide a propagation mechanism of financial shocks into real aggregates via misallocations of resources. Buera and Moll (2014) take a slightly different approach on the investigation of a credit shock by considering heterogeneous entrepreneurs maximizing their lifetime utilities. They consider different variants of a model by its underlying heterogeneity, and analyze the dynamics of aggregate wedges from a credit shock. Even though they confirm that a BCA exercise using aggregate data within a representative framework may be invalidated due to the existence of heterogeneity, I focus on direct implications on the aggregate employment from the borrowing constraint augmented with frictional labor demand. This is because firms decisions are related each other from the working capital requirement, so that the mapping from the firm-level heterogeneity to aggregate wedges is more sophisticated in my model. To my best knowledge, the only model that considers both financial and labor frictions is Buera, Fattal-Jaef, and Shin (2015). They consider a frictional hiring market for entrepreneurs from a simple match function. While their focus is on the persistent increase in the unemployment rate upon a credit shock, I introduce a different type of firm-level distortion in hiring that is consistent with Jermann and Quadrini (2012).

The introduction of working capital requirement is relatively common in the literature, which is intended to generate substantial aggregate responses that resemble data. Neumeyer and Perri (2005) focus on business cycle fluctuations in emerging economies where the interest rate channel for financing working capital is critical. In their model, a representative firm needs to borrow its wage bill in advance by paying net interest rate, and a shock to international interest rate creates large and volatile responses in small open economies. The timing assumption made in this paper is
slightly different from theirs, but firms in my model are assumed to pay the interest rate for intra-period loans for working capital. Chari, Kehoe, and McGrattan (2005) consider the pay-in-advance restrictions of wage bills to explain the sudden stops in emerging economies that are associated with their simultaneous output drops. Even though the working capital assumption alone cannot generate substantial wedges from their model, they also recognize that it needs to be combined with other sources of friction. In this paper, the way I incorporate the assumption of working capital is an application for a closed economy case with financial shocks. As a general approach for an advanced economy, Christiano, Eichenbaum, and Evans (2005) also include the working capital assumption in order to generate persistent employment dynamics upon a monetary shock. Here, I focus on the business cycle dynamics upon a credit shock.

Lastly, I briefly discuss the recent research focused on the weak recovery in the U.S. labor market after 2007, particularly in relation with firm-level evidence. The empirical finding by Fort, Haltiwanger, Jarmin, and Miranda (2013) especially highlights the large decline of job creation by small and young firms in the recent recession. Siemer (2014) builds a model with financial frictions on firm entry and labor adjustment costs, to analyze the employment dynamics of small firms during the periods of tighter borrowing condition. Sedlacek (2014) rather focus on the role of firm age in shaping the firm size distribution and aggregate dynamics upon an exogenous shock. The theoretical framework in my paper, however, is more consistent with the existing literature, and considers more general environment of firm heterogeneity.
2.3 Model

I model an economy with heterogeneous firms in their productivity, capital stock, and financial structure. I assume that a firm needs to finance its investment as well as wage bill externally, using a non-contingent one-period debt. Each firm is subject to an individual borrowing constraint, where the available credit is limited by the firms collateral value. In addition, wage bills must be paid in advance of production, and I further assume that firms separately use within-period loans to finance the working capital at a common interest rate. This formulation is necessary to introduce the additional margin on firms intertemporal borrowing decisions. Other than the borrowing constraint and working capital requirement, I abstract from micro-level adjustment frictions in this paper. The following subsections describe the model environment in detail and then I define a recursive competitive equilibrium of the model.

2.3.1 Firms

Production Environment

There is a large number of firms in the model, and they produce homogenous output goods using a Decreasing Returns to Scale (DRS) technology. Each firm owns its predetermined capital stock $k$, and hires labor $n$ from a competitive labor market. The production technology is given by $y = z\varepsilon F(k; n)$, where $F(\cdot)$ is increasing in both arguments and concave. Productivity level of a firm is determined as the multiplicative term $z\varepsilon$, where $z$ represents the level of aggregate TFP common across firms and $\varepsilon$ is an idiosyncratic component of the firm-level productivity. I assume that shocks to the idiosyncratic productivity $\varepsilon$ follows a Markov chain such that
$\varepsilon \in \mathbf{E} \equiv \{\varepsilon_1, \ldots, \varepsilon_{N_\varepsilon}\}$ with transition probabilities $\pi_{ij}^\varepsilon \equiv Pr(\varepsilon' = \varepsilon_j|\varepsilon = \varepsilon_i) \geq 0$.

Independently from $\varepsilon$, the aggregate TFP $z$ also follows a Markov chain with $\pi_{lm}^z \geq 0$.

I begin with describing the working capital requirement of the model. Given a market wage rate $\omega$, a firm must pay a fraction $\theta_w \in [0, 1]$ of its current period wage bill $\omega n$ as working capital, before production takes place. The case of $\theta_w = 1$ indicates that firms pay the full amount of wage bills in advance. I assume that the working capital is financed using intra-period loans. Then firms repay $\theta_w \omega n/q$ at the end of the period, where $q$ is the discount price of one-period debt. A type of financial friction is introduced to firms’ intertemporal borrowing decisions. Specifically, a firm’s newly issued debt level $b'$ is limited by its collateral in the current period. The amount of collateral depends on each firm’s accumulated capital stock $k$, net of the existing within-period debt for working capital to be repaid. I assume that the repayment of working capital is completed after the decision on $b'$, in order to analyze its effects on the borrowing constraint. That is, the collateralized borrowing constraint endogenously affects investment decisions made by firms, while the choice of labor demand is also related to firms’ borrowing limits. For each unit of new debt $b'$ that needs to be repaid in the next period, a firm only receives $q$ units of output from its borrowing at the current period. Then the borrowing constraint is given as follows.

$$b' \leq \theta \cdot \left( k - \theta_w \frac{\omega n}{q} \right)$$

The parameter $\theta$ captures the degree of collateralization of the economy, and it is assumed to be common across firms. A credit shock is then modeled as an unexpected drop in $\theta$, indicating tighter borrowing conditions in economy-wide. As discussed earlier, the available credit for a firm is limited by its capital stock less the down
payment of wage bill, and the firm’s hiring decision in turn affects its borrowing limit
together with the parameters $\theta$ and $\theta_w$. This formulation of borrowing constraint is
consistent with a variant of the benchmark model in Jermann and Quadrini (2012),
and is now applied to an environment with heterogeneous firms in this paper. There is
no sign restriction on $b'$, so a negative value of $b'$ implies savings. The above borrowing
constraint is a challenging object to solve because it is occasionally binding, and the
second term in the parenthesis creates an additional non-convexity in the model.

Since firms are allowed to accumulate enough wealth, either in terms of $k$ or $-b$,
the borrowing constraint may become irrelevant to firms’ decisions. When all firms
have such sufficient wealth, the model’s aggregate implications is analogous to the
standard business cycle model with differences in firm-level productivity and capital.
To avoid the latter case of Modigliani and Miller (1958), I assume that firms face
with exogenous possibility of exit in each period, and the risk of exit is given by a
constant probability $\pi_d \in (0, 1)$. As in Khan and Thomas (2013), the arrival of exit
shock is known to each firm at the beginning of a given period. Without loss of
generality, only surviving firms are allowed to invest for future capital stock as well as
to borrow across periods. Exiting firms, on the other hand, can fully uninstall their
undepreciated capital after production and must clear their existing debt. I assume
that the exiting firms are replaced with the equal measure of new firms in each period,
in order to characterize a stationary equilibrium of the model. New firms initially
begin with small capital, which is a $\chi$ fraction of the average level of capital stock in
the economy.

The timing within a given period is summarized in the following figure.
At the beginning of each period, both the aggregate and idiosyncratic productivities are realized, \((z, \varepsilon)\). A firm’s predetermined individual state is given by its current capital stock \(k \in K \subset \mathbb{R}_+\) and existing debt \(b \in B \subset \mathbb{R}\) carried from the previous period. Hence each firm is identified by its individual state vector \((k, b, \varepsilon)\). Just after the realization of \(z\) and \(\varepsilon\), the exit status at the end of the period is known. Before starting production, the firm borrows the working capital \(\theta \omega n\) at \(q\). After production, the existing debt is cleared and the firm issues new debt \(b'\) for its investment \(i\), conditional on its survival into the next period. Finally, the firm repays the intra-period loan for working capital. The capital accumulation is standard with a depreciation rate \(\delta\). The distribution of firms can be characterized as a probability measure \(\mu\) on a Borel algebra generated by \(S \equiv K \times B \times E\). Then the aggregate states are given by \((z, \mu)\), and the evolution of firm distribution follows a mapping \(\Gamma\) from the economy’s current period aggregate states, such that \(\mu' = \Gamma(z, \mu)\). Firms take the wage rate \(\omega(z, \mu)\) and the debt price \(q(z, \mu)\) as given from \((z, \mu)\).

**A Firm’s Problem**

In this subsection, I formulate the firm’s dynamic optimization problem. Due to the exit shock in each period, I separate the problem by whether a firm’s exit status is known within a given period. Let \(v_0(k, b, \varepsilon; z, \mu)\) be the expected discounted value of a firm with its individual states \((k, b, \varepsilon)\) at the beginning of a period, just before
the arrival of its exit shock. In case that the firm is determined to survive in the next period, the firm’s within-period continuation value is defined as \( v(k, b, \varepsilon; z, \mu) \).

\[
v_0(k, b, \varepsilon; z, \mu) = (1 - \pi_d) \cdot v(k, b, \varepsilon; z, \mu) + \pi_d \cdot \max_n \left[ z\varepsilon F(k, n) - \omega(z, \mu)n \left( 1 + \frac{\theta_w}{q(z, \mu)} - \theta_w \right) + (1 - \delta)k - b \right]
\]

In Equation (1), \( v_0(k, b, \varepsilon; z, \mu) \) is the value from a binary expectation over the cases of exit and survival with the probability of exit \( \pi_d \). The second line of the equation corresponds to the case of exit, where the firm maximizes its liquidation in the bracket value without considering \( k' \) and \( b' \). The liquidation value is the sum of output produced and undepreciated capital stock, net of its wage bill and debt clearing. The total amount of wage bill payment is augmented with the degree of working capital requirement \( \theta_w \). The following problem then defines \( v(k, b, \varepsilon; z, \mu) \), where a continuing firm determines its hiring level \( n \), future capital \( k' \), and new debt level \( b' \). The firm maximizes the sum of its current dividends \( D \) and future expected discounted value \( v_0(k', b', \varepsilon, j; z_m, \mu) \).

\[
v(k, b, \varepsilon; z_l, \mu) = \max_{n, k', b', D} \left[ D + \sum_{m=1}^{N_z} \pi^z m d_m(z_l, \mu) \sum_{j=1}^{N_z} \pi^z j v_0(k', b', \varepsilon, j; z_m, \mu') \right]
\]

subject to

\[
b' \leq \theta(k - \theta_w \frac{\omega(z_l, \mu)n}{q(z_l, \mu)})
\]

\[
D \leq z\varepsilon F(k, n) - \omega(z_l, \mu)n \left( 1 + \frac{\theta_w}{q(z_l, \mu)} - \theta_w \right) + (1 - \delta)k - b - k' + q(z_l, \mu)b'
\]

\[
D \geq 0, \quad \mu' = \Gamma(z_l, \mu)
\]
The current dividend payment is given by the firm’s resource constraint, and assumed to be non-negative. The future expected value is represented by the summation operators and the transitional probabilities \( \pi^z_{lm}, \pi^\varepsilon_{ij} \), due to the assumption of Markov chains on \( z \) and \( \varepsilon \). The firm discounts its future value by using the stochastic discount factor \( d_m(z_t, \mu) \). The collateral constraint mentioned earlier is embedded in this problem.

2.3.2 Households and Equilibrium

I assume that there is a unit measure of identical households, or a representative household in the model economy. The representative households period utility is given by \( U(C^h, 1 - N^h) \), where \( C^h \) is the choice of aggregate consumption and \( N^h \) represents the labor supply. Each household holds its wealth as a comprehensive portfolio of one-period firm shares \( \lambda \) and non-contingent bonds \( \phi \). Households have access to a complete set of state-contingent claims for consumption smoothing across periods with a subjective discount factor \( \beta \in (0, 1) \). The equilibrium quantity of these assets becomes zero, so that I illustrate a simpler version of the representative households problem that maximizes the lifetime expected discounted utility.

\[
V^h(\lambda, \phi; z_t, \mu) = \max_{C^h, N^h, \lambda', \phi'} \left[ U(C^h, 1 - N^h) + \beta \sum_{m=1}^{N_z} \pi^z_{lm} V^h(\lambda', \phi'; z_m; \mu') \right]
\]

subject to

\[
C^h + q(z_t, \mu)\phi' + \int_S \rho_1(k', b', \varepsilon'; z_t, \mu)\lambda'(d[k' \times b' \times \varepsilon']) \\
\leq \omega(z_t, \mu)N^h + \phi + \int_S \rho_0(k, b, \varepsilon; z_t, \mu)\lambda(d[k \times b \times \varepsilon]) \\
\mu' = \Gamma(z_t, \mu)
\]
Due to the timing issue on dividends pricing, $\rho_1(k', b'; \varepsilon'; z_t, \mu)$ above denotes the ex-dividend prices of firm shares and $\rho_0(k, b, \varepsilon; z_t, \mu)$ is the dividend-inclusive values of current shares. Let $\Phi^h(\lambda, \phi; z, \mu)$ and $\Lambda^h(k', b'; \varepsilon', \lambda, \phi; z, \mu)$ be the households policies for bond holding and firm shares, respectively. Then a Recursive Competitive Equilibrium (RCE) of the model is defined as below.

A recursive competitive equilibrium is a set of functions: prices $(\omega, q, d_m, \rho_0, \rho_1)$, quantities $(N, K, D, C^h, N^h, \Phi^h, \Lambda^h)$, and values $(v_0, V^h)$ that solve the optimization problems, clear each market, and the associated policy functions are consistent with the aggregate law of motion:

1. $v_0$ solves Equation (1) and (2), and $(N, K, D)$ are the associated policy functions for firms.

2. $V^h$ solves Equation (3), and $(C^h, N^h, \Phi^h, \Lambda^h)$ are the associated policy functions for households.

3. The labor market clears with $N^h = \int_S N(k, \varepsilon; z, \mu) \cdot \mu(d[k \times b \times \varepsilon])$

4. The output market clears with

$$C^h = \int_S \left[ z\varepsilon F(k, N(k, \varepsilon; z, \mu)) - (1 - \pi_d)(K(k, b, \varepsilon; z, \mu)
- (1 - \delta)k + \pi_d(1 - \delta)(1 - \chi)k \right] \cdot \mu(d[k \times b \times \varepsilon])$$

5. The law of motion for the firm distribution is consistent with the policy functions, where $\Gamma$ defines the mapping from $\mu$ to $\mu'$, with $\pi_d$, $K(k, b, \varepsilon; z, \mu)$, and $B(k, b, \varepsilon; z, \mu)$.

As noted in Khan and Thomas (2013), it is convenient to modify the firms problem by using the equilibrium prices resulting from the market-clearing quantities. Let $C$
and $N$ be the equilibrium quantities of consumption and labor hours, satisfying the above definition of RCE. The equilibrium prices then can be expressed in terms of the marginal utility of consumption at $(C, N)$, under the assumption of differentiability. The real wage $\omega(z, \mu)$ is equal to the marginal rate of substitution between leisure and consumption, $D_2 U(C, 1 - N) / D_1 U(C, 1 - N)$. The inverse of the discount debt price $q^{-1}$ is the expected real interest rate, $(\beta \sum_{m=1}^{N_z} \pi_{tm} D_1 U(C_{m}', 1 - N_{m}') / D_1 U(C, 1 - N)$.

Lastly, the stochastic discount factor equals to the intertemporal rate of substitution across states, $d_m(z, \mu) = (\beta D_1 U(C_{m}', 1 - N_{m}') / D_1 U(C, 1 - N)$. Using these prices, the equilibrium allocation from the firms problem can be solved consistently with the representative households optimal decisions. Define $p(z, \mu)$ as the marginal utility from market-clearing consumption, at which firms value their current period output and dividends. Then the equilibrium price functions are given by,

\[
p(z, \mu) \equiv D_1 U(C, 1 - N)
\]

\[
\omega(z, \mu) = \frac{D_2 U(C, 1 - N)}{p(z, \mu)}
\]

\[
q(z, \mu) = \beta \sum_{m=1}^{N_z} \pi_{tm} p(z_{m}', \mu') / p(z, \mu)
\]

The stochastic discount factor collapses into $\beta$ in the firm’s problem, by augmenting the value functions with $p(z, \mu)$ as follows.

\[
V_0(k, b, \varepsilon; z, \mu) \equiv p(z, \mu) \cdot v_0(k, b, \varepsilon; z, \mu)
\]

\[
V(k, b, \varepsilon; z, \mu) \equiv p(z, \mu) \cdot v(k, b, \varepsilon; z, \mu)
\]

For the remainder of this paper, I suppress the notation for the aggregate states in the price functions and decision rules whenever needed for brevity. Then, the following
equations summarize the modified problem solved by a firm with \((k, b, \varepsilon)\).

\[
V_0(k, b, \varepsilon; z, \mu) = (1 - \pi_d) \cdot V(k, b, \varepsilon; z, \mu) \\
+ \pi_d \cdot \max_n p(z, \mu) \left[ z \varepsilon F(k, n) - \omega n \left( 1 + \frac{\theta_w}{q} - \theta_w \right) + (1 - \delta) k - b \right]
\]

\[
V(k, b, \varepsilon_i; z_l, \mu) = \max_{n, k', b', D} \left[ p(z, \mu) \cdot D + \beta \sum_{m=1}^{N_s} \sum_{j=1}^{N_x} \pi_{imn} \pi_{ij}^z V_0(k', b', \varepsilon_j; z_m, \mu') \right]
\]

subject to

\[
b' \leq \theta (k - \theta_w \frac{\omega n}{q}) \\
D \leq z_l \varepsilon_i F(k, n) - \omega n \left( 1 + \frac{\theta_w}{q} - \theta_w \right) + (1 - \delta) k - b - k' + q b' \\
D \geq 0, \quad \mu' = \Gamma(z_l, \mu)
\]

### 2.4 Analysis

In this section, I characterize the optimal decision rules adopted by firms. Due to the occasionally binding borrowing constraint, I distinguish firm types by whether the borrowing constraint is affecting the decision rules as in Khan and Thomas (2013). First, define a firm as unconstrained when it already has enough wealth accumulated, so that the borrowing constraint becomes irrelevant as in Modigliani and Miller (1958). In particular, an unconstrained firm is assumed not to experience a binding constraint at any possible future state. Hence all the Lagrangian multipliers on the borrowing constraints become zero for this type of firms. In this case, firms are indifferent between paying dividends and saving internally because the shadow value of retained earnings is equal to \(p(z, \mu)\). On the other hand, constrained firms are the complement set of all unconstrained firms. The definition of a constrained firm
is somewhat broad, in the sense that the firm does not necessarily face with a currently binding borrowing constraint. In other words, constrained firms put non-zero probabilities of having a binding constraint in the future. Then the shadow value of internal saving becomes greater than that of dividends, and constrained firms find it better to have zero dividend payment. The following subsections discuss the decision rules by unconstrained firms and constrained firms, respectively.

2.4.1 Decisions of Unconstrained Firms

By definition, an unconstrained firm is able to optimally adjust its future capital stock $k$ without affected by its borrowing constraint. In this case, the working capital requirement only affects the firms resource constraint marginally via the interest paid on the intra-period loan. Thus, the firms labor demand is simply derived from the static condition below, where $\theta_w = 0$ leads to a case without the working capital.

$$z \varepsilon D_2 F(k; n) = \omega (1 + \frac{\theta_w}{q} - \theta_w)$$

All firms with the same $(k, \varepsilon)$ choose the solution of the above condition, and let $n = N^w(k, \varepsilon; z, \mu)$ be the unconstrained choice of labor demand. I substitute $N^w(k, \varepsilon)$ into Equation (4) and (5), and illustrate the unconstrained value functions $W_0$ and $W$ as below.
\[ W_0(k, b, \varepsilon; z, \mu) = (1 - \pi_d) \cdot W(k, b, \varepsilon; z, \mu) \]
\[ + \pi_d \cdot p \left[ z \varepsilon F(k, N^w(k, \varepsilon)) - \omega N^w(k, \varepsilon) \left( 1 + \frac{\theta_w}{q} - \theta_w \right) + (1 - \delta) k - b \right] \]  

\[ W(k, b, \varepsilon_i; z_l, \mu) = \max_{k', b', D} \left[ p \cdot D + \beta \sum_{m=1}^{N_x} \sum_{j=1}^{N_e} \pi_{im}^x \pi_{ij}^z W_0(k', b', \varepsilon_j; z_m, \mu') \right] \]  
subject to
\[ D \leq z_l \varepsilon_i F(k, N^w(k, \varepsilon)) - \omega N^w(k, \varepsilon) \left( 1 + \frac{\theta_w}{q} - \theta_w \right) + (1 - \delta) k - b - k' + qb' \]
\[ D \geq 0, \quad \mu' = \Gamma(z_l, \mu) \]

I first solve for the unconstrained choice of \( k = K^w(\varepsilon; z, \mu) \) because it is not affected by the borrowing constraint anymore in this case. The analytical solution for \( K^w(\varepsilon) \) is provided in the appendix. Using \( N^w(k, \varepsilon) \) and \( K^w(\varepsilon) \), the optimal policy of \( b \) by unconstrained firms is established as the minimum savings policy. This is defined by exploiting the definition of unconstrained firms. Specifically, an unconstrained firm already holds enough wealth to remain unconstrained at all future states. And this firm is indifferent between paying dividends and saving. The unconstrained choice of \( b \) is then characterized by defining the maximum level of new debt, or the minimum level of savings that ensures a firm to remain unconstrained in any future path of \((z, \varepsilon)\). Let \( \tilde{B}(k, \varepsilon; z, \mu) \) be the threshold of debt level that a firm can still adopt the unconstrained policy of \( b \). Then the minimum savings policy \( b = B^w(\varepsilon; z, \mu) \) is given
recursively by the following equations.

\[ B^w(\epsilon_i; z_l; \mu) = \min_{(\epsilon_j, z_m)} \left( \tilde{B}(K^w(\epsilon_i), \epsilon_j; z_m, \Gamma(z_l, \mu)) \right) \quad (8) \]

\[ \tilde{B}(k, \epsilon_i; z_l, \mu) = z_l \epsilon_i F(k, N^w) - \omega N^w(1 + \frac{\theta_w}{q} - \theta_w) + (1 - \delta)k - K^w(\epsilon_i) + q \min \{ B^w(\epsilon_i; z_l, \mu), \theta(k - \theta_w \frac{\omega N^w}{q}) \} \quad (9) \]

On the RHS of Equation (9), the minimum operator imposes the borrowing constraint.

In Equation (8), the minimum savings policy is defined as the minimum value of firms existing debt level at the next period when it adopts the unconstrained decisions at the current period. Equation (9) is a functional equation that features a contraction mapping due to \( q < 1 \). The dividend policy from the firm is then the residual from the resource constraint when choosing \( N^w(k, \epsilon) \), \( K^w(\epsilon) \), and \( B^w(\epsilon) \).

### 2.4.2 Decisions of Constrained Firms

Since constrained firms recognize the risk of having binding borrowing constraint in the future, they prefer to retain current earnings instead of paying dividends, so that \( D = 0 \). In this subsection, I further distinguish the remaining constrained firms.

In particular, I sort the firms into two types, by whether their borrowing constraints are currently binding (Type-2) or not (Type-1). There is the other type of constrained firms that just become unconstrained by accumulating wealth, but their decision rules are analogous to the unconstrained firms discussed in the previous subsection. In this case, the firms value is also the same as \( W_0 \) and \( W \).

Since the unconstrained decision rules are already established, I can use them as the criteria of distinguishing the constrained firms. In case of having a currently binding borrowing constraint, a Type-2 firms labor demand is affected as well as investment. On the other hand, decisions of hiring and investment by Type-1 firms
are not affected by the bindingness. Therefore, they still can adopt the unconstrained policies $N^w(k, \varepsilon)$ and $K^w(\varepsilon)$. The optimal debt policy $b'$ for this type of firms is then given by forcing $D = 0$ from the resource constraint in Equation (5). The following equation denotes the continuation value function in terms of the feasibility of the unconstrained policies, given $(k, b, \varepsilon)$.

$$V(k, b, \varepsilon; z, \mu) = \begin{cases} W(k, b, \varepsilon; z, \mu) & \text{if all } (N^w, K^w, B^w) \text{ are feasible} \\ V_1^c(k, b, \varepsilon; z, \mu) & \text{if } (N^w, K^w) \text{ are feasible } \text{(Type-1)} \\ V_2^c(k, b, \varepsilon; z, \mu) & \text{otherwise } \text{(Type-2)} \end{cases}$$

The difference between $V_1^c$ and $V_2^c$ is at whether the borrowing constraint in Equation (5) is binding or not, as mentioned above.

The hiring decision by Type-2 firms is more involved. Since the binding borrowing constraint affects the labor demand as well, I begin with illustrating the optimal condition with Lagrange multipliers $\lambda_1$ and $\lambda_2$ on the constraints in (5).

$$\lambda_1 \theta q \frac{w \omega}{q} = \lambda_2 (z \varepsilon D_2 F(k, n) - \omega (1 + \frac{\theta w}{q} - \theta w))$$

Since the multipliers are positive due to the bindingness, the constrained labor demand becomes less than $N^w(k, \varepsilon)$, and its value depends on the tightness of the borrowing constraint. In this paper, this micro-level distortion in employment generates further misallocation of factors and eventually emerges as the aggregate wedge. Consider a firm with small $k$, for instance. The firm needs to finance its investment externally in order to grow over time. Given the binding credit limit with its small collateral value, the firm may have an incentive to reduce its hiring in the current period to accumulate capital more quickly. This distorted labor demand affects the allocation of labor as well as capital across firms, mainly driven by the combination of the borrowing constraint with the working capital requirement. I use a grid method.
to numerically solve for the constrained labor decision. Once such \( n \) is determined for Type-2 firms, their debt policy is given by the borrowing constraint and \( k \) is now the residual from \( D = 0 \).

2.5 Results

2.5.1 Parameters and Steady State

I parameterize the model economy to match the key macroeconomic moments in the U.S. A period in the model is set to one year. The functional forms of the production function and the period utility function are consistent with the literature. The production function is assumed to be \( F(k, n) = k^{\alpha}n^{\nu} \), with \( \alpha + \nu < 1 \) and \( \alpha, \nu > 0 \). The preference is from the indivisible labor; \( U(C, 1 - N) = \log C + \psi(1 - N) \) as in Rogerson (1988). The logarithm of idiosyncratic productivity \( \epsilon \) follows an AR(1) process; \( \log \epsilon' = \rho_{\epsilon} \log \epsilon + \eta', \) where \( \eta' \sim N(0, \sigma_{\eta}) \). The standard deviation \( \sigma_{\eta} \) and the persistency \( \rho_{\epsilon} \) of \( \epsilon \) are from Khan and Thoams (2013). I discretize the process into \( N_{\epsilon} \) values, using Tauchen (1986).

First, I set the subjective discount factor \( \beta \) to imply the annual real interest rate of 4 percent. The labor share of income \( \nu \) is set to 0.6, following Cooley and Prescott (1995). From the average investment to capital ratio during the postwar U.S. periods, the depreciation rate \( \delta \) is chosen while the private capital stock is measured in BEA Fixed Asset Tables. The capital share of output \( \alpha \) is determined to get the average capital-output ratio of 2.3, jointly with \( \delta \). The disutility of labor parameter \( \psi \) is set to yield the total hours worked to be one third.

I assume that the total measure of firms is constant over time. Regarding the exogenous exit rate, I set \( \pi_{d} = 0.1 \) to match the average firm exit rate in Business
Dynamics Statistics (BDS). In each period, new firms enter the economy with small enough capital stock. The parameter $\chi$ is set to 0.1, implying that a new firm begins with 10 percent of the average capital stock held by incumbent firms. The parameter for collateralization $\theta$ is determined to match the aggregate debt to asset ratio of 0.37. The debt to asset ratio is from nonfarm nonfinancial businesses in the Flow of Funds. In the model, the aggregate asset is calculated by summing the aggregate capital and negative debt holdings. I assume that the degree of working capital requirement $\theta_w$ is equal to 1, and I also provide the results with smaller $\theta_w$ values in the next subsection. Table 2.1 below summarizes the parameter values of the model.

The stationary distribution of firms is shown in Figure 2.2. At first glance, the distribution itself illustrates the rich firm-level heterogeneity generated from the model. Further, it displays all three types of firms, which is discussed in the previous section. The unconstrained firms are located at the discrete masses along the curved area, starting from the first mass at the level of capital slightly less than 1. A bulk of firms is at the minimum level of capital and they are spread over $k$, on the straight line at the maximum level of debt to capital ratio. The latter firms are Type-2, where the straight line implies that their borrowing constraints are currently binding. Type-1 firms connects the unconstrained firms and Type-2 firms in the distribution. Even though their decisions on investment and employment are not affected by the borrowing constraint, Type-1 firms accumulate their wealth toward the masses of unconstrained firms. Table 2.2 below summarizes the aggregates by each firm type.

Percentage shares of the aggregate variables are reported in the second row of each firm type, where $B^+ \equiv \int_{S_{b \geq 0}} b \cdot \mu(d[k \times b \times \varepsilon])$ and $B^- \equiv \int_{S_{b < 0}} b \cdot \mu(d[k \times b \times \varepsilon])$. The next row in the table shows the average value of each firm type at the steady
Among the entire firms, only 13 percent are unconstrained whereas more than 37 percent are under the binding constraint. Intuitively, the unconstrained firms are relatively larger and their savings account for more than 70 percent of the entire economy. On the other hand, the ratio of average capital to labor among Type-2 firms is relatively smaller, implying that firm-level investment and hiring decisions are more distorted from the working capital and the borrowing constraint.

The unconstrained decisions of $K^w(\varepsilon)$ and $B^w(\varepsilon)$ are shown in Figure 2.3. Obviously, both of the decision rules are increasing functions of the idiosyncratic productivity $\varepsilon$. In particular, when an unconstrained firm becomes productive enough, the minimum savings policy in the figure implies that the firm doesn’t even need to save.

### 2.5.2 Aggregate Dynamics

In this subsection, I discuss the transitional dynamics of the model upon exogenous shocks. The impulse responses presented in this paper are computed using the perfect foresight transition method. Before an unexpected shock hits the economy at date 1, the model is assumed to be at the steady state. During a transition, the future path of the shock is perfectly known to the economic agents. In particular, I apply the method of Guerrieri and Lorenzoni (2015) and Khan (2011), which iteratively solves transitional dynamics in a model with heterogeneous agents. Detailed approaches for the numerical methods used in this paper are provided at the computational appendix. In the following exercises, I also conduct the Business Cycle Accounting (BCA), by simplifying the benchmark model in Chari, Kehoe, and McGrattan (2007). The simplified benchmark model with representative agents is also reported in the appendix.
A TFP Shock

Before diving into the models aggregate dynamics, I illustrate the recent recession in the U.S. in Figure 2.1. In that, each aggregate variable is reported as the deviations from its 2007Q4 level. All macroeconomic series are log-detrended using Hodrick-Prescott filter with the smoothing parameter of 1600, and the last 8 observations are discarded. GDP, consumption, and private investment data are retrieved from BEA NIPA tables in real terms, where private investment is the sum of business fixed investment, residential investment, and consumer durables. Total hours worked are from my updates based on Cociuba, Prescott, and Ueberfeldt (2012). As mentioned in Section 1, the dynamics of employment and investment in the recent recession are noticeable. First, the initial drop in GDP is relatively small and it gradually reaches its trough around 2009Q2. On the other hand, private investment and total hours worked exhibit rather persistent dynamics with relatively larger magnitudes. When the residential investment is excluded, the private investment shows more persistency reaching its trough almost at 2009Q4. While the recovery of GDP, measured TFP, and consumption roughly starts at 2009Q1, investment and labor hours do not recover quickly enough. Even in 2010Q4, they are still 4 percent and 10 percent below their 2007Q4 levels, respectively.

The above unusual patterns in data are in contrast to the prediction from a standard business cycle model with a TFP shock. Figure 2.4 reports the impulse responses of the aggregate variables in my model upon a persistent TFP shock. The initial decline in TFP is 2.18 percent of its steady state level, which makes the results comparable with their empirical counterparts. The persistency of the TFP shock is set to $\rho_z = 0.91$ from the Solow residual. Obviously, the model in this paper yields
almost similar responses as in an otherwise standard model upon a TFP shock. In
the figure, the aggregate consumption initially drops about 1.5 percent at the im-
pact, and further deceases until date 5. The decline in employment is rather mild
with about 1.5 percent less than its steady state level, and gradually recovers along
the transition. These results are not consistent with data, because it is the nature
of an exogenous shock that differentiates the most recent recession in the U.S. as
illustrated by Jermann and Quadrini (2012). Since all firms in the model economy
are affected by the TFP shock in the same way, the measured TFP (mTFP) in Figure
2.4 is identical to the exogenous TFP process (TFP), implying that the degree of re-
source misallocation remains the same. Therefore, the investment and labor wedges
from BCA (not reported) do not significantly vary with the TFP shock, only affected
by the working capital requirement of the model. I also report the same exercise in
a model only with unconstrained firms in Figure 2.5, by eliminating the borrowing
constraint. This frictionless model corresponds to the standard one-sector growth
model of heterogeneous firms, augmented by the working capital requirement. I pro-
vide the latter model in the appendix. In sum, I confirm that a TFP shock in this
type of model can not generate the observed dynamics in the recent recession even
with financial frictions, as discussed in Khan and Thomas (2013).

A Credit Shock

I turn to analyzing the aggregate implications of a credit shock, when the working
capital requirement affects firms decisions. Specifically, the collateral parameter \( \theta \)
unexpectedly falls to the level of 0.75 at the impact, and then gradually recovers to
its steady state level starting from date 2. I calibrate the magnitude of the shock to
match a decline of about 50 percent in aggregate lending \( B^+ \) of the model economy.
This is a relatively large shock, but less persistent when compared with Khan and Thomas (2013), although it is still within the range of the recent empirical evidence on the reduction in corporate loans in 2007. In this paper, however, I abstract from any micro-level adjustment frictions in labor or capital that may amplify the effects of a credit shock.

The resulting aggregate responses upon a credit shock are in Figure 2.6. When compared to the results from a TFP shock (Figure 2.4), the measured TFP in the top-left panel indicates that the misallocation across firms in the economy becomes more severe than its steady state level. The measured TFP (mTFP) falls slightly less than 1 percent and gradually recovers until date 10. On the other hand, the exogenous component of the aggregate productivity $z$ remains at its steady state level in this exercise. Unlike the case of a TFP shock, the dynamics of aggregate employment becomes more persistent and the magnitude on its fall is around 2 percent which is greater than that of consumption. In addition, both employment and investment become more volatile showing ups and downs along the transition and recovery, when the fall in $\theta$ remains the same until date 3 (not reported).

The above results imply that the credit concerns of the financially constrained firms are quite huge, and the tradeoff between hiring and investment under a tighter borrowing condition is indicated during a credit crunch. Under a perfect foresight on $\theta$, firms in the model are aware that the borrowing condition gradually recovers in the future. Hence, they tend to limit their labor demand during a credit crunch from the impact, in order to increase their borrowing limit as much as possible. This is because of the initial fall in the real interest rate and the time-to-build nature of investment. Upon a credit shock, the real interest rate $q^{-1}$ immediately falls and
gradually recovers. Then firms are willing to invest more in advance while enduring the current tightening of borrowing at the relatively low interest rate. Thus, they have incentives to reduce hiring especially at the recovery stage of the recession, even thought the wage rate is still low. This becomes clear when I eliminate the working capital assumption from the model. In Figure 2.7, I implement the same size of a credit shock, after recalibrating the model with $\theta_w = 0$. This version of the model corresponds to Khan and Thomas (2013) without the investment irreversibility. In the figure, the aggregate measure of the misallocation (mTFP) looks similar to that in Figure 2.8. The aggregate responses, however, are relatively larger with $\theta_w = 0$. In this case, the aggregate employment falls more than 2 percent at the impact and the response of investment is also more volatile. This latter comparison is the first attempt to identify the differences in aggregate dynamics with and without the consideration of frictional hiring in this class of models. The results suggest that the aggregate volatility as well as persistency from a financial shock can be amplified when another source of friction for firms labor demand is introduced into a financial friction.

As in the previous subsection, I also conduct the BCA from the model-generated data. In particular, I compare the labor wedge with its empirical counterpart in Figure 2.8 with other aggregates in annual term. In the bottom-right panel, I set the level of the empirical wedge to zero at 2007 level whereas the models labor wedge is denoted as the differences from its steady state level. Even though the labor wedge from data is much larger and volatile in the recent recession, the model is able to account for the dynamics upon a credit shock, at least qualitatively. In relative to the recent data, the impulse responses from the model imply a shallow but more persistent recession. This is partly due to the lack of micro-level adjustment frictions, and also due to the
difference of the model structure. Regarding the total hours worked, for instance, the model in this paper does not take into account the unemployment and the discouraged workers leaving the labor force. In overall, however, this BCA exercise confirms that the results from the model are consistent with the recent recession experienced in the U.S.

2.6 Concluding Remarks

The unusual dynamics in the recent labor market has been addressed in this paper, in relation with the existing literature on the macroeconomic effects of a financial shock. I have introduced the working capital requirement into a model of credit shocks with production heterogeneity. The calibrated model yields more distorted equilibrium allocations of labor and capital, due to the combination of frictional hiring and investment via a collateral constraint. Upon a financial shock in the model, the dynamics of aggregate employment and investment become distinguishable from the results without working capital. I related the micro-level distortions from the borrowing constraint with the aggregate wedges, and also compared with the empirical wedges in the recent recession in the U.S. A substantial rise in labor wedge from the model is consistent with the evidence on the large decline and slow recovery in the labor market.
## Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.27</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.96</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.6</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1.97</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.065</td>
</tr>
<tr>
<td>$N_\varepsilon$</td>
<td>7</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.7</td>
</tr>
<tr>
<td>$\rho_\varepsilon$</td>
<td>0.659</td>
</tr>
<tr>
<td>$\pi_d$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>0.057</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\theta_w$</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 2.1: Calibration

## Aggregates by Firm Type

|          | Mass | $K$  | $N$  | $Y$  | $B^+$ | $|B^-|$ |
|----------|------|------|------|------|-------|--------|
| Unc.     | 0.131| 0.261| 0.063| 0.103| 0.012 | 0.102  |
| (\%)     | 19.3 | 18.8 | 18.5 | 2.13 | 72.0  |
| (Avg.)   | 1.986| 0.477| 0.785| 0.090| 0.775 |

<table>
<thead>
<tr>
<th></th>
<th>Type-1</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Unc.</td>
<td>0.498</td>
<td>0.904</td>
<td>0.216</td>
<td>0.355</td>
<td>0.434</td>
<td>0.039</td>
</tr>
<tr>
<td>(%)</td>
<td>67.1</td>
<td>64.8</td>
<td>64.0</td>
<td>78.0</td>
<td>27.9</td>
<td></td>
</tr>
<tr>
<td>(Avg.)</td>
<td>1.815</td>
<td>0.434</td>
<td>0.713</td>
<td>0.872</td>
<td>0.079</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Type-2</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Unc.</td>
<td>0.371</td>
<td>0.183</td>
<td>0.055</td>
<td>0.097</td>
<td>0.111</td>
<td>0.000</td>
</tr>
<tr>
<td>(%)</td>
<td>13.6</td>
<td>16.4</td>
<td>17.5</td>
<td>19.9</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>(Avg.)</td>
<td>0.493</td>
<td>0.147</td>
<td>0.262</td>
<td>0.298</td>
<td>0.000</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Aggregate Variables by Firm Type
Recent Recession (HP)

Figure 2.1: The 2007 Recession in the U.S.
Figure 2.2: Entire Firm Distribution at the Steady State
Figure 2.3: Unconstrained Choices over $\varepsilon$ Values
Figure 2.4: A TFP Shock
Figure 2.5: A TFP Shock with Unconstrained Firms Only
Figure 2.6: A Credit Shock
Figure 2.7: A Credit Shock without Working Capital
Figure 2.8: Business Cycle Accounting, Labor Wedge
Bibliography


Appendix A: Appendix to Chapter 2

A.1 Computational Methods

In this appendix, I describe the computational method to solve the steady state equilibrium and the perfect foresight transition of the model. Since the model features firm-level decisions on capital accumulation and debt financing, as well as the individual productivity shocks, there are three-dimensional heterogeneities across firms. Further, the non-convexity, which is arising from an individual firms collateralized borrowing constraint that involves another choice variable $n$, requires a nonlinear solution method in order to solve the model. Solving the steady state of the model in the absence of aggregate uncertainty allows me to characterize the equilibrium results, including the stationary distributions and the aggregate variables. In addition, the steady state solution becomes the basis for the perfect foresight dynamics of the model upon an exogenous TFP shock or a credit shock. The numerical algorithm of transitional dynamics under perfect foresight is largely based on Guerrieri and Lorenzoni (2012), and Khan (2011). The first subsection below explains the solution method of the steady state, and the next comes with the transitional dynamics.
A.1.1 Solving Steady State

I focus on a stationary equilibrium of the model that has a time-invariant distribution of firms, which is denoted by $\mu$. Accordingly, equilibrium prices $(p, w, q)$, are consistent with the aggregate resource constraint of the model. Since there is no distortionary cost or tax, the aggregate resource constraint is,

$$C + K - (1 - \delta)K = Y$$

, where $C = 1/p$, $K = \int_S k \cdot \mu(d[k \times b \times \varepsilon])$, and so on. The aggregate productivity $z$ and the price of one-period debt $q$ are assumed to be at their steady state values in the remaining of this subsection. For the idiosyncratic productivities, I discretize them using Tauchen method.

The solution algorithm utilizes the bisection method to find the equilibrium $p$ which is defined as the marginal utility of household consumption in section 3. Given the initial price range that contains the equilibrium price, I iterate the three steps described below until the aggregate excess demand equals zero. First, I solve the unconstrained policies $(N^w, K^w, B^w)$ and the corresponding values, $W$ and $W_0$. Next, the constrained firms values $V$ and $V_0$, as well as its policies $(k, b, n)$ are solved. In the latter step, distinguishing the constrained firms by the feasibility of the unconstrained policies is critical, as described in section 4. Finally, I update the distribution of firms by using the optimal policies from the previous steps, while managing the exogenous exit and entry of firms. Note that once a firm becomes unconstrained, it will permanently remain as unconstrained by definition, until it exits exogenously. So it is more convenient for me to maintain two separate distributions each for unconstrained and constrained firms, $(\mu, \mu_c)$. Only Case 1 firms are transferred to $\mu$, when
the constrained firm distribution is updated. After iterating until the convergence of both distributions, it is possible to get the aggregate variables \((K, N, I, Y)\), and the corresponding excess demand. The remainder of this subsection explains each step mentioned above more in detail. Before that, one thing to note is that I solve the constrained firms value, \(V(k, b/k, \varepsilon)\) instead of \(V(k, b, \varepsilon)\) following Khan and Thomas (2011). This is due to a technical reason to avoid the irrelevant regions of the state vector, where firms directly dive into insolvency due to their high debt level. I use univariate splines for \(k\) and evenly-spaced grids for \(b/k\) to solve the value functions.

In step 1, the unconstrained labor and next periods capital choices are relatively easy to get, since they have the following analytical solutions from the structure of the unconstrained problem itself.

\[
N^w(k, \varepsilon) = \left(\frac{\varepsilon \nu q}{\omega} k^\alpha\right)^{\frac{1}{1-\nu}}
\]

\[
K^w(\varepsilon) = \left(\frac{1}{1 - \beta(1 - \delta)} \right) \{ \alpha \beta \left(\frac{\nu q}{\omega}\right)^{\frac{\nu}{1-\nu}} \left(\sum_j \pi_{ij} \varepsilon_j^{\frac{1}{1-\nu}}\right) \}^{\frac{1-\nu}{1-\alpha-\nu}}
\]

For the minimum savings policy, \(B^w\), I apply \(K^w\) and \(N^w\) in the equation below and then recursively solve \(\tilde{B}\) by contraction mapping. To do so, I need to find the value of \(B^w\) given \(K^w\) and \(\varepsilon_j\), and then substitute it to the above equation to get the value of \(T\tilde{B}\).

\[
T\tilde{B}(k, \varepsilon) = \varepsilon F(k, N^w) - \frac{\omega N^w}{q} + (1 - \delta)k - K^w(\varepsilon)
\]

\[
+ q \min_{\varepsilon_j} \{ \min \tilde{B}(K^w(\varepsilon), \varepsilon_j), \theta(k - \frac{\omega N^w}{q}) \}
\]

Next in step 2, I need to solve the constrained value function \(V(k, b/k, \varepsilon)\) by sorting the firms into Case 1, Case 2-(i), and Case 2-(ii). I first check if a constrained firm falls into Case 1 by whether it is capable of unconstrained policies while keeping
the constraints, \( D \geq 0 \) and \( b' < \theta(k - \omega n/q) \). If not, then the firm belongs to Case 2. Note that Case 2 firms pay zero dividends, and that they are only distinguished by the bindingness of current period borrowing constraint. Since the borrowing constraint of a Case 2-(i) firm is nonbinding, I substitute the unconstrained policies, \( K^w \) and \( N^w \) into the firms resource constraint \( D = 0 \) to get the constrained savings policy \( b \).

For Case 2-(ii) firms, I find it useful to focus on the labor decision because both of their constraints are binding. Specifically, once I have the optimal choice of labor for these firms, the new debt level \( b \) is determined by the binding borrowing constraint and in turn I can get the new capital stock \( k \) through \( D = 0 \).

For the labor choice of Case 2-(ii) firms \( N^c(k, b, \varepsilon) \), it is crucial to understand that there exists a huge amount of non-convexity in the borrowing constraint to determine \( b' \). This applies when one considers a small constrained firm with substantial amount of existing debt to repay. Since firms are not allowed to pay negative dividends, the small firm must externally finance its debt repayment as well as its capital accumulation to grow, if any. In this sense, the current labor choices of this type of firms are intertemporally related with other policies through both constraints. Furthermore, this non-convexity tends to drive these relatively small firms to choose the lowest allowed level of employment, since lower current labor hiring implies that lower debt level or higher capital stock are carried to the next period for higher expected future value. Note that firms are not allowed to exit endogenously in this model. Therefore, these small firms maintain the lowest level of production and employment, until they happen to get high levels of persistent individual productivity or until they are faced with the exogenous exit. In practice, I use a pure grid method to solve the constrained labor policies \( N^c \) instead of the conventional golden section search method,
since there can be corner solutions due to the non-convexity mentioned above. When
the minimum level of labor is the optimal choice of a firm which sometimes implies
a negative value of $k$ depending on the firms state $(k, b, \varepsilon)$, I assign the firm to have
the lowest level of $k'$ to prevent it from exiting by choosing zero-capital stock. This
is a slight violation of the individual resource constraint for the constrained firms,
$D = 0$. So I admit that there is a chance of overstating the aggregate response of the
model in this paper, since the aggregate resource constraint requiring that the ex-
cess demand must be zero in equilibrium should also include a transfer term to make
these exit-desiring firms survive. Therefore, this issue motivates further research that
allow firms to endogenously exit or default in the model, while maintaining the labor
friction and the borrowing constraint as in this paper.

Finally, the policy functions from the previous steps can be used to update the
two separate distributions $(\mu, \mu_c)$. I use a weighted grid method with finer grids of
$k$ and $b/k$. The exogenous exit of firms applies to both distributions just before
updating, since the exiting firms are liquidated right after their current production.
Updating the constrained firm distribution also follows the consistent procedure that
distinguishes the firms by their feasibility of the unconstrained policies. Then, the new
born firms enter the constrained distribution at the end of current period according
to the ergodic distribution of $\varepsilon$, and they are assumed to begin with small capital
stock and zero debt level. Since I focus on the steady state distributions, I iterate the
distributions until convergence while applying the same policy functions, given the
current price level.
A.1.2 Solving Perfect Foresight Transition

The numerical algorithm for the transitional dynamics of a model with multi-dimensional heterogeneity often comes with high costs. The perfect foresight transition method in this subsection enables me to make use of equilibrium price adjustments along the exogenous process of interest, without relying on simulation method which usually requires more computational capacity. In particular, I feed the sequences of \( \{z_t\}_{t=1}^T \) and \( \{\theta_t\}_{t=1}^T \) as the exogenous processes into the model, and assume that the economy reaches its steady state in \( T \) periods. I iterate on an equilibrium price vector \( \{p_t\}_{t=1}^T \) over the transition to ensure that the excess demand equals zero in every period. The price vector is updated over the iterations depending on the size of excess demand each period, and this is for having a smooth sequence of prices over the transition. One thing to be careful is that the steady state solution method and the algorithm over the given periods should be consistent with this method, since the aggregate excess demand responds sensitively to a small change of equilibrium prices.

I initialize the price vector \( \{p_t\}_{t=1}^T \) from the steady state equilibrium price \( p \), and solve the model with the exogenous processes of \( \{z_t\}_{t=1}^T \) and \( \{\theta_t\}_{t=1}^T \) over the given \( T \) periods. The algorithm follows the three steps which will be described below. The first step is to get the policies at each date \( t \) by recursively solving the value functions starting from \( t = T - 1 \) to \( t = 1 \). This backward pass iteratively updates the RHS of the functional equation for \( V \) and \( V_0 \), while applying the similar method to solve the optimal policies mentioned in the previous subsection. The next step uses the optimal decision rules at each \( t \) to update the aggregate state and the distributions, starting from \( t = 0 \) and going forward to \( t = T - 1 \). In this forward pass, I use the steady state distributions as the initial at \( t = 0 \), and update them over the periods.
while computing the aggregate variables and the excess demands. Finally, I apply
the method by Guerrieri and Lorenzoni (2012) to update the price vector according
to the excess demand at each period. The price updating constant should be small
enough to smooth the price vector over the iterations.