CONTROL OF OVER-ACTUATED SYSTEMS WITH APPLICATION TO ADVANCED TURBOCHARGED DIESEL ENGINES

Dissertation

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ABSTRACT

The automotive industry is currently striving to improve vehicle fuel economy, while complying to stringent emission standards and keeping affordable costs. Engine downsizing with advanced turbochargers is one of the main technical solutions nowadays implemented by the industry in response to the above challenges. In particular, the adoption of advanced actuation techniques brings more degrees of freedom to improve the efficiency of breathing and combustion in internal combustion engines. However, it is understood that improvements in engine design can be effective only if matched by the ability to closely control engine breathing and combustion performance.

This dissertation aims at exploiting systematic methodologies for control and optimization of over-actuated systems. Following the path of the state of the art in control theory and practice, a novel control design methodology is proposed that hinges on the characterization of redundancy for a class of over-actuated systems in geometric terms to determine the reconfigurable structure of the controller. The method relies upon the concept of inverse model allocation, where the integration of allocation module is effectively separated from stabilization of the regulated outputs, allowing one to shape the transient response by optimizing on-line a given cost function without affecting the output tracking performance. The proposed approach overcomes fundamental limitations found in current techniques and specifically addresses the need for simultaneous control and optimization of the air-path systems for advanced downsized engines.
The proposed control methodology is applied to the case study of a Diesel engine air-path system equipped with Variable Geometry Turbine (VGT), Variable Geometry Compressor (VGC) and Exhaust Gas Recirculation (EGR) systems. A comparative study against different conventional control methods shows the effectiveness of the proposed methodology to improve the engine performance variables, as well as the overall reduction in design and calibration costs.
To my family.
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CHAPTER 1
INTRODUCTION

1.1 Background and Motivation

One of the major challenges in the automotive industry is meeting the conflicting goals of improving fuel economy and complying with stringent emission standards. The U.S. Environmental Protection Agency (EPA) and the Department of Transportation (DOT) have recently finalized new fuel economy standards that will increase fleet average fuel economy to the equivalent of 54.5 mpg for cars and light trucks by 2025 [1], and reduce emission for heavy-duty Diesel trucks [2] by 50%. As electric and plug-in hybrid electric vehicles are slowly penetrating the market, internal combustion engines will remain the mainstay of the automobile powertrain systems in the foreseeable future. As a result, automakers are required to massively advance engine technology developments to meet the new targets for fuel economy and emissions, without compromising on driveability and customer appeal.

Downsized boosted engines are currently the key solution adopted by the major automakers to improve the fuel economy. By reducing the engine displacement, lower fuel consumption is attained while engine peak performance and driveability are retained through the use of boosting techniques and advanced powertrain control systems. Several studies have shown that 18% fuel economy improvement can be
achieved with a 40% downsizing and turbocharging [3–5], and that downsized engines are a promising way to reduce CO$_2$ emissions, up to 25% [4–6].

In spite of the high market penetration of gasoline engines in the U.S. market, Diesel engines are becoming more popular nowadays, for both light-duty trucks and passenger cars applications, due to their high thermal efficiency and specific power output, resulting from their high compression ratio and fuel lean operation [7]. Diesel cars deliver 30% to 35% better fuel economy than comparable vehicles with gasoline engines [8]. However, noise, smoke and emissions are the most detrimental effects of a Diesel engine [9]. In the past years, Diesel engine development has been focusing on reduction of emissions [10–12], initially through the utilization of exhaust gas recirculation (EGR) [13]. As higher rates of EGR are introduced, higher boost levels are required to maintain the oxygen content. As a result, exhaust turbocharging, in conjunction with engine downsizing, is typically used on Diesel engines to manage emission legislations, while improving fuel economy [14].

To retain peak performance, modern turbochargers should be designed to achieve both fast transient response and high boost pressure over a wide engine operating range. Several solutions have been proposed, including Variable Geometry Turbochargers (VGT) [15, 16], multi-stage turbochargers [17, 18], electrically assisted turbochargers [19]. More recently, Variable Geometry Compressors (VGC) have been considered as a possibility to improve the compressor performance and stability [20–22].

With these advancements in engine design and system integration, the air-path systems of boosted engines have become more sophisticated. Although the added degrees of freedom offer opportunities to further improve fuel economy and emissions, they make the control problem considerably more complex. Due to limitations in the conventional control design methods, the extra-degrees of freedom available for
optimizing engine performance are nowadays not fully exploited. In fact, the common practice today in use for evaluation and optimization of a powertrain control strategy in the automotive industry highly relies upon engine experimental calibration. As the number of degrees of freedom in modern engine designs increases, the controller calibration and validation process become significantly complicated and time consuming. Moreover, the control design optimization is often limited to steady-state conditions, which could lead to poor transient performance.

In light of the opportunities and challenges in the control of modern Diesel engine air-path systems, a novel and comprehensive methodology, that achieves simultaneous control and optimization of the system level performance and significantly shortens the development and integration of new technologies, is of critical importance for the automotive industry to fully exploit the potential benefits of the advanced engine hardware components.

In the next section, a brief overview of the modern air-path system technologies and the existing literature of control methodologies for the air-path systems will be presented.

1.2 Literature Survey

1.2.1 Engine Air-path Systems Technology

An increasingly stringent legislation related to fuel economy and emissions requires constant innovation of engine and powertrain systems technology. Complying with the increasingly stringent fuel economy mandates in a cost effective manner and without impact on the emissions, has resulted in the improvement of engine air, fuel and combustion technology, and in the introduction of new components. For instance,
high pressure common-rail fuel injection systems reduce soot emissions through enhanced fuel atomization, and injection rate-shaping [23]. Moreover, Homogenous Charge Compression Ignition (HCCI) was introduced as a way to considerably reduce the NOx and soot emissions, namely by optimizing the combustion process based on a lean, homogeneous and well-mixed charge that reacts and burns simultaneously throughout the bulk of the mixture during the compression stroke [24,25].

Engine modifications alone clearly are not sufficient to meet the stringent emission legislations. For this reason, the NOx and Particulate Matter (PM) emissions have been substantially reduced through the introduction of advanced aftertreatment technologies. For instance, Diesel Oxidation Catalysts (DOC) and Diesel Particle Filters (DPF) are commonly adopted on light-duty and heavy-duty Diesel engines, where DOC promotes oxidation of several exhaust gas components (i.e., carbon monoxide and hydrocarbons) by oxygen, and DPF captures particles on a substrate, often ceramic monoliths or woven fibers. In addition, Selective Catalytic Reduction (SCR) catalysts employ nitrogen-containing compounds, such as ammonia or urea, to trigger NOx reduction reactions in an oxidizing atmosphere, and Lean NOx Traps (LNT) capture and store NOx during lean burn operation, then reduce NOx to N₂ during periodic regeneration [10–12].

In addition to the advances in fuel injection and aftertreatment systems, the air-path systems also play a critical role in improving the combustion process and engine efficiency, and thus have attracted particular interest. Modern automotive Diesel engines are universally equipped with Variable Geometry Turbine (VGT) and Exhaust Gas Recirculation (EGR) systems [15,16]. The turbocharger, driven by the exhaust gas energy to compress the air into the intake manifold via a compressor on a connected shaft, contributes to increasing the boost pressure and thus increasing the air mass drawn into the combustion chamber. The VGT actuator consists of
adjustable inlet guide vanes that control the amount of the exhaust flow through the
turbine by changing the inflow angle and inflow speed at the runner inlet. Comparing
to conventional turbochargers with fixed geometry and waste gate, the VGT typically
has a greater control authority to achieve high boost pressure compatible with an
extended operating range of the engine [26]. Moreover, the VGT also provides an
improved transient response, higher torque and better fuel economy [27].

On the other hand, the EGR recirculates the exhaust gas into the intake manifold,
diluting the incoming fresh air with inert gases. The mixture of air and exhaust
gas reduces the availability of oxygen during combustion, hence lowering the flame
temperature [28–31]. As a result, the NOx emissions decrease significantly. At the
same time, the intercooler and EGR cooler reduce the temperatures of the fresh
air and the recirculated exhaust gas, and consequently decrease the temperature of
the inlet charge and increase the charge density. This typically allows to further
reduce the NOx emissions, as well as to slightly increase the thermal efficiency of the
combustion due to lower heat losses.

Such VGT-EGR systems, however, are limited in the control authority due to the
slow system response (turbo lag) and the high sensitivity of the boost pressure to
the exhaust manifold pressure and temperature. Moreover, downsized engines are
further penalized by the risk of operating the compressor close to the surge limit,
due to the need of maintaining high boost pressure conditions. While the compressor
is typically sized to prevent the operating point from getting too close to the surge
line in steady state conditions, surge may still occur during rapid load transients (for
instance, during gear shifts).

One of the solutions currently in production or advanced development is the ap-
application of two-stage turbochargers [17,18], whereby a relatively small turbocharger
is usually used as the high-pressure stage to achieve good launch performance and
low speed end torque while a larger unit is typically used as the low-pressure stage to allow higher exhaust energy extraction in rated power conditions. Two-stage turbochargers are known to improve the engine low-end torque and power densities simultaneously. On the other hand, they inevitably cause the air-path system to become more complex, and consequently the control of such system particularly during transient becomes complicated. Another recent concept to improve the performance of the air-path system is the use of electrically assisted turbochargers [32, 33]. For the VGT system, while the boost pressure can be raised by controlling the variable nozzle, the amount by which the pressure can be increased at low engine speeds is limited due to the low exhaust energy. However, the electrically assisted turbocharger is capable of attaining torque characteristics at low engine speeds by adding motor assistance. Moreover, an electrically assisted turbocharger can also recover thermal exhaust energy by acting as a generator at high engine speeds [19]. As a result, electric turbocharger assistance improves engine transient response and torque curve in the low-engine speed zone, thus further enhancing engine downsizing with an improvement of fuel economy while maintaining the original vehicle performance [33].

More recently, Variable Geometry Compressors (VGC) have been considered as an opportunity to improve the compressor performance and stability by introducing a direct actuation on the compressor side [20, 21]. Centrifugal compressors can be equipped with either a variable inlet guide vanes device or with a variable geometry diffuser. Both devices allow one to control the passage area (either at the inlet or at the outlet of the impeller) by means of rotating vanes. Since the characteristic curves can be “scaled” in relation with the vanes position, the surge line can be properly repositioned, allowing one to maximize the boost pressure and compressor efficiency at low flow rate conditions without inducing stability issues [20, 22]. One of the pioneering studies on the VGC technology is presented in [34], where different
vaned diffuser designs are investigated for a centrifugal compressor, focusing on the operating range and surge margin. The paper shows that the progressive reduction of the diffuser flow area has a stabilizing effect on the compressor operating map, moving the surge line to lower flow rates at the cost of a decrease in efficiency due to higher friction losses. Another approach to control the compressor flow is the variable inlet guide vanes (VIGV) [20–22, 35–37]. The VIGV is a device located upstream the impeller that imposes a swirl motion to the inlet flow. The pre-rotation allows for extending the flow range of the compressor at the expense of pressure ratio capability and compressor efficiency, as shown in the experimental study conducted in [20,38]. Results from CFD analysis and experimental studies show that a negative pre-whirl improves the stability region of the compressor by about 25% [21]. For turbocharging applications, the proposed device leads to a maximum 6% improvement of fuel consumption and an increment of over 20% in boost pressure for a direct injection gasoline engine at small flow range [35].

In spite of the potential opportunities provided by the above technologies advancement, key issue is the development of control and optimization algorithms to coordinate the various air-path system actuators in achieving the improvements of the engine performance while meeting the stringent emission standards.

1.2.2 Overview of Control Approaches for Diesel Engine Air-path Systems

During the last decades, a considerable research effort has been dedicated to the control of modern Diesel engines. Coordinated decentralized and multivariable control strategies were first proposed to manipulate VGT and EGR openings for intake manifold pressure and air mass flow rate regulation [39–41]. Nonlinear control approaches
were then exploited based on Constructive Lyapunov Function Approach [42], dynamic feedback linearization [43,44], and sliding mode control [44,45]. Furthermore, robust gain-scheduled controllers have been proposed based on a linear parameter-varying (LPV) model formulated or identified for turbocharged Diesel engines [46–48]. Other control designs, for instance, hybrid gain scheduled control based on polytopic error model [49], optimal control based on dynamic optimization and neural networks [50], were also investigated in literature.

In recent times, Model Predictive Control (MPC) has gained interest in the engine control community [51–54], due to its effectiveness to deal with multivariable constrained control problem. In [52,55,56], an explicit MPC algorithm was developed and showed impressive results in tracking of MAF and MAP (and also in reduction of emissions), compared with the standard ECU algorithm. In [51], model predictive control was implemented to a real-world Diesel engine prototype controller, by employing an extension of the online active set strategy, which leads to computationally feasible solution to online optimization problem. In [57], a MPC is developed to optimize the system transient performance, where the control objective is formulated to track the set-point of EGR ratio and minimize the pumping losses, while satisfying a minimum limit of oxygen/fuel ratio.

On the other hand, a Nonlinear Model Predictive Control (NMPC) was proposed in [58] for the air-path control to resolve the system nonlinearities and constraints on both the inputs and process variables. The simulation results demonstrated the improved performances compared to a linear state feedback controller and input-output linearization based control method. However, as pointed out in [58], the main drawback of the NMPC techniques lies in the fact that they typically require far more computational power than what is typically available on production automotive ECUs.
The air-path control design problem for advanced Diesel powertrain systems, such as multi-stage turbocharger [59, 60], variable-valve actuation [43] and multi-combustion modes [61] has also attracted substantial research efforts over the past years. Through the addition of extra degrees of actuation into the system, it is in principle possible to optimize overall performances, such as emission, turbo lag and fuel efficiency. However, nearly all the control methodologies proposed in the literature, rely on the definition of extra outputs, e.g. exhaust pressure [61] or EGR fraction in the intake [43], to make the system square. This is not appealing for the automotive industry, as it is extremely costly and time consuming to install such sensors and obtain reference values for the extra outputs via experimental optimization. Moreover, this approach completely foregoes opportunity to achieve a better transient performance through the coordination of the available actuators. Therefore, an innovative control scheme needs to be developed to fully exploit the advancements in engine designs.

1.2.3 Control Allocation for Over-actuated System

Compared to the conventional EGR-VGT control problem [40, 62], which aims at regulation of air mass flow rate and intake manifold pressure, the VGC adds one extra degree of freedom to the system, making it over-actuated. An over-actuated system is defined as a dynamical system in which there are more independent control inputs than outputs to be regulated [63]. To illustrate the notation, without loss of generality, consider a nonlinear system modeled by equations of the form:

\[
\dot{x} = f(x, u) \quad y = h(x)
\] (1.1)

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) is the control input and \(y \in \mathbb{R}^p\) is the regulated output. System (1.1) is said to be over-actuated when \(m > p\).
In the literature, the presence of a redundant set of actuators has been traditionally addressed by means of Control Allocation (CA) [64]. Numerous methodologies, including direct allocation [65, 66], optimal control [67], dynamic [68, 69], model-predictive [70, 71] and adaptive allocation [72], linear [73], quadratic [74, 75] and multi-parametric [76] programming, have been proposed to solve the CA problem in the past years. However, it is noteworthy to observe that most of the proposed CA approaches rely upon the following standing assumptions [67]:

1. The control vector field is affine in the control, i.e., \( f(x,u) = F(x) + G(x)u \) in system (1.1).

2. The control vector field can be further factorized as \( G(x) = G_1(x)G_2(x) \), where \( G_1(x) \in \mathbb{R}^{n \times p} \) has full column rank and \( G_2(x) \in \mathbb{R}^{p \times m} \) has full row rank.

As a consequence, a virtual control input \( u_v = G_2(x)u \in \mathbb{R}^p \) can be defined such that the resulting system is square, where a controller \( u^*_v \) can then be found using conventional design techniques. Since \( p < m \), \( G_2(x) \) has a nullspace of dimension
$m - p$, in which $u$ can range while mapping into the same value $u^*_u$. The redundancy is then solved as an online optimization problem using the CA algorithm

$$u^* = \arg\min_u \left\{ \|G(x)u - u^*_u\|_p + \Lambda \|u - u_{des}\|_p \right\}$$

such that $u_{min} \leq u \leq u_{max}$ (1.2)

where $u_{des}$ is a desired control vector, $\Lambda^{m \times m}$ is a positive definite weighting matrix and $p \in [1, 2]$ denotes $l_1$ or $l_2$ norm.

The CA method has been studied in numerous areas, for instance aerospace applications [64, 77], ships and underwater vehicles [78], robotics [79] and automotive industry [80, 81]. More pertinent to automotive applications, the input allocation method of [82] is reported in [83] to the speed control of an internal combustion engine. In spite of its successful applications, the applicability of available CA methodology is limited, as most of the nonlinear over-actuated systems do not satisfy the above assumptions. The turbocharged Diesel engine with VGC system considered in this study is one such systems.

Recently, Zaccarian [82] recognized the distinction between the null space of the control input matrix (yielding strong redundancy) versus the null space of the plant transfer function (yielding weak redundancy) for LTI system. A geometric approach was exploited in [84] to solve an output regulation problem by allocating the weak input redundancy within the controllable weakly unobservable subspace, where the redundancy was characterized by a parameterization of all solutions to the regulator equations. However, the proposed allocation policy only shapes a family of steady-state trajectories that are compatible with a given reference output. Possibilities are still left open to use the proposed methodology to optimize the transient behavior by dynamic allocation methodologies.
1.2.4 Summary

As advanced engine technologies are being progressively deployed to production, the increased number of actuators offers the opportunity to improve engine performance, such as fuel economy, emission and drivability. However, there is still a very significant gap between the technology improvements and the advancement of the control algorithms developed to manage the air-path system, which still rely on basic feed-forward and feedback schemes and costly calibration procedures.

Due to limitations in the conventional control design methods, the extra degrees of freedom available for optimizing engine performance have nowadays not been fully exploited. Most of the control methodologies rely on the definition of additional outputs to make the system square, which aims essentially at mitigating, rather than exploiting the presence of redundancy. Furthermore, the control design optimization of those methods is often limited to steady-state conditions. On the other hand, the necessary conditions required by the established control allocation techniques are rather restrictive, and typically not satisfied in control allocation problems associated to the engine air-path system. As a consequence, there is considerable interest today in developing innovative control approaches that specifically address the potential benefits of systems characterized by redundant actuation.

The dissertation is dedicated to the development of a novel and comprehensive approach to achieve simultaneous control and optimization for a general class of over-actuated systems. This research is inspired by the fundamental conflicts between technology advancements and limitations of the current control techniques for advanced automotive applications. A case study will focus on the application of the proposed methodology to a turbocharged Diesel engine equipped with VGT-EGR-VGC system, seeking design of a feedback controller that coordinates the actuators
to meet desired system-level performance in both steady-state and transient conditions. It is evident that, in light of the limitations of conventional control approaches, it is necessary to extend significantly the state of the art in the theory and practice of control of complex over-actuated systems.

1.3 Dissertation Contributions

Based on the review of the state of the art, this dissertation aims to tackle an open challenge in control and optimization of over-actuated systems. In particular, the main contributions include: the investigation of advanced system theories to resolve the key technical issues for over-actuated systems; and the application of the theoretical findings to a novel and practical engineering problem, namely the control of a Diesel engine air-path system with a variable geometry compressor. The outcome of the dissertation is briefly outlined below.

1.3.1 Contributions on Over-actuated Systems and Model Predictive Control Allocation

The dissertation contributions with respect to the theoretical aspects on the control of over-actuated systems are summarized as follows:

1. A complete characterization of solutions to over-actuated linear output regulation problems using geometric terms, where the input redundancy exists in a prescribed subspace that has null influence on the output, entailing the existence of multiple trajectories in the state space that yield the same given reference output;

2. A modular control design scheme, as shown in Figure 1.2, based on the concept...
of inverse model allocation, which assigns the reference and input trajectories and enables the stabilizer to be designed independently;

3. A model predictive control strategy to resolve a dynamic allocation policy for systems subject to state/input constraints, where sufficient conditions for stability and feasibility of the closed-loop system are presented.

1.3.2 Contributions on Diesel Engine Air-path System Control

In addition to the theoretical results on control of over-actuated systems, the contributions of this dissertation also encompass the integration and implementation of a new actuator to the conventional VGT-EGR control problem in Diesel engine air-path systems. Moreover, a comprehensive systematic study, including a combination of engine modeling, model order reduction, linearization, system analysis and model based optimization, is performed to investigate the challenges and opportunities associated with this additional degree of freedom. Then, various methodologies are explored in the design of an optimal control strategy that coordinates the three actuators and
allows one to evaluate the benefits on engine performance, computation complexity and calibration efforts. In particular, the outcome of this dissertation shows the benefits of the proposed modular control design methodology, resulting from exploiting the geometry of over-actuated systems. The proposed design approach has the advantage of being an “add-on” solution, hence retaining the prior engineering calibration efforts while simultaneously coordinating the VGC system optimization with the existing framework of VGT-EGR control.

1.4 Dissertation Outline

The dissertation is structured as follows.

In Chapter 2, a description of the Diesel engine air-path system with the variable geometry compressor is presented as a case study for illustrating the principles and theoretical findings of this work. A detailed introduction of the overall control objectives and the system modeling approach with experimental validation results are then presented.

In Chapter 3, a systematic analysis of the air-path system dynamics is presented, based on a reduced order model that has been validated in both frequency and time domain. In particular, an analysis of the steady-state and transient behavior is conducted at several engine operating conditions based on linearized models, followed by a steady-state system optimization to identify the full potential of a coordinated EGR-VGT-VGC control on the air-path system variables, resulting into an open-loop control strategy. Moreover, a nonlinear state and parameter estimation based on Extended Kalman Filter (EKF) is conducted, for the purpose of developing a full-information control design.

In Chapter 4, a model predictive control design for the air-path system is proposed, considering different control problem formulations. The first is formulated in a
conventional way to track the desired set-points of air mass flow rate and intake manifold pressure, which are obtained from steady-state optimization. The other focuses on the coordination of the VGC system optimization with the existing VGT-EGR control algorithm for the air-path system.

In Chapter 5, a systematic characterization of system redundancy in the state space for over-actuated systems is presented. An introduction of the principles of inverse model allocation for output regulation problems, based on geometric characterizations, is then outlined. Moreover, a dynamic allocation strategy for constrained over-actuated linear systems based on receding horizon optimization is presented. In particular, the study focuses on the stability and feasibility properties of the proposed scheme and on the characterization of suitable sufficient conditions for stability in geometric terms.

Chapter 6 presents the simulation results and verification of the adopted methodology on the Diesel engine air-path system, which demonstrates the benefits compared to the previously developed MPC approach.

Finally, the conclusions and contributions of this dissertation are summarized in Chapter 7. Based on the work presented in the dissertation, relevant research topics and directions are proposed as potential future work.
CHAPTER 2
TURBOCHARGED DIESEL ENGINES-A CASE STUDY
FOR CONTROL DEVELOPMENT

In this chapter, a Diesel engine air-path system with a variable geometry compressor is introduced and a mean value engine model is developed and validated against experimental data. The model, which captures the low frequency dynamic behavior of the engine system, shows that the new actuator allows one to directly reshape the compressor characteristic map and actively shift the surge margin, consequently allowing one to increase the system efficiency and enlarge the stable operating region.

The system is depicted schematically in Figure 2.1, where comparing to the conventional structure of Diesel air-path system, the presented system possesses three actuators (VGT, EGR, VGC). However, only two sensors are present in the system, which measure the relevant variables in the intake manifold, namely air mass flow rate and intake manifold pressure. Such system, termed as over-actuated system, serves as an important and practical example in automotive applications, where more and more actuation technologies are introduced in modern engines to improve the overall system performance, but only limited number of sensors are available to limit the production costs. Due to its relevance, this system is treated as a test bench for studying the control methodologies that are proposed in this dissertation.

The chapter is organized as follows. Section 2.1 presents a systematic description of the Diesel engine air-path system, equipped with VGT, EGR and VGC actuators.
In Section 2.2, the control objectives for air-path system and requirements of the control algorithms are discussed with specific details. A mean value engine model is then designed in Section 2.3, and semi-physical models of variable geometry turbine and variable geometry compressor are developed and calibrated based on steady state maps provided by the manufacturer. Section 2.4 presents the results of model validation against experimental data from steady state and transient tests.

2.1 System Description

Figure 2.1 illustrates the typical scheme of an air-path system for Diesel engines. Conventional Diesel engines are normally equipped with Variable Geometry Turbocharger (VGT) and Exhaust Gas Recirculation (EGR) system. In addition to the VGT and EGR, this study considers an additional actuator in the engine air-path system, namely a Variable Geometry Compressor (VGC), which consists of a set of

Figure 2.1: Scheme of a Turbocharged Diesel Engine with VGT, EGR and VGC Actuators.
Variable Inlet Guide Vanes (VIGV) located at the inlet of the impeller, as shown in Figure 2.2. The device allows one to control the effective flow area and the direction of the flow velocity [20, 22]. The VIGV imposes a swirl motion to the flow at the impeller eye, which reduces the cross-sectional area and the static pressure at the impeller eye, with a consequent reduction of the pressure ratio of the machine, and a marginal reduction of the isentropic efficiency [34]. Another important effect is that VIGV produces a shift of the surge limit towards the lower flow rates, and consequently allowing for an extension of the flow range of the compressor.

Figure 2.3 shows a typical schematic diagram of fuel-path and air-path systems control for Diesel engines, where the controlled inputs of the system include the position of the VGT (\(u_{VGT}\)) and the position of the EGR valve (\(u_{EGR}\)). An additional control input in this study includes the position of the VIGV, \(u_{VGC}\). The measured outputs, namely the air mass flow rate \(m_c\) and the boost pressure \(p_{IM}\) are used, often in conjunction with estimation methods, to monitor the desired steady-state and transient response of air/fuel Ratio (AFR) and EGR ratio within the system.
Operating conditions are defined by the exogenous inputs to the system, namely the fuel mass flow rate $\dot{m}_f$ and the engine speed $N_{eng}$.

### 2.2 Objectives of Diesel Engine Air-Path Control

While the fuel-path of a Direct Injection (DI) Diesel engine is typically controlled by the common-rail system [23], the main objective of control design for air-path system is to achieve the following tasks:

- **Emission Regulation**: Since the combustion products (Soot or NOx emissions) are highly affected by the mixture composition (fresh air and EGR) in the engine cylinders, one objective of air-path control is to provide a charge of the desired composition and with sufficient oxygen into the cylinders with respect to a given operating condition [16, 42, 85].

- **Engine (Brake) Efficiency**: Despite the efficiency for Diesel engines is largely affected by the fuel injection control and incylinder combustion process, the air-path system also plays an important role in the improvement of fuel economy, namely by influencing the air charge composition, volumetric efficiency, level of boost and pumping losses at part load conditions.

- **Drivability/Turbo lag**: Turbo lag issues typically occur when the driver
demands a fast change from low to high torque conditions. Due to the shaft inertia, the turbocharger can not provide a desired boost and air flow instantly, to respond to the driver torque demand. However, this phenomenon can be mitigated by controlling the available actuators (primarily by VGT), which directly adjust the amount of flow through the turbine/compressor, to accelerate the process of converting exhaust energy into the boost pressure. Therefore, it is critical for the air-path control system to achieve the desired level of drivability, namely complying with the driver’s torque demand, while not mitigating the emission regulation.

- **Constraints**: Constraints are typically introduced to account for noise, safety limits, stability and durability. Constraints typically include (but are not restricted to) maximum pressure, temperature, speed, maximum/minimum air/fuel and EGR ratio, compressor surge/choke and position and rate limits for actuators.

In spite of the above goals for the Diesel engine air-path systems, certain performance requirements for the control algorithm itself have to be fulfilled when it is integrated to production engines [86]. These aspects for control development, including the synergy with engine hardware, environmental conditions, technology reproduction and extension, should always be kept in mind:

- **Computational Complexity**: The Engine Control Unit (ECU) is limited in computational power and available memory. The control algorithm should therefore neither consume too much computational load nor demand too large storage space, to be compatible with the current production ECUs.

- **Robustness**: The real-world engine operation may be subject to unknown disturbances and uncertainties, such as environmental variability (temperature,
humidity), fuel composition, measurement noises, product tolerance and ageing, and so on. This requires the control system to be adequately robust to meet the above goals even under these harsh conditions.

- **Modularity:** The air-path control system constitutes only a small part of the entire engine control system, for which the development and calibration take years and require the collaboration from different departments. Therefore, any novel control development for air-path system should be easy to be integrated with other subsystems, without the need to recalibrate the entire control system.

- **Development Period:** An increase of the development time and efforts ultimately obstructs the technology advancements into the market and leads to an increase of the overall costs for automakers.

- **Extensibility:** In order to balance the technology advancements and production costs to satisfy various customers’ appeal, modern Diesel engines could have very different topologies, sensors and actuators. Therefore, the control strategy should be flexible and easily reconfigurable.

### 2.3 Mean Value Engine Modeling

Since the evaluation and optimization of a powertrain control strategy based on an engine testing dynamometer is costly and time consuming, model-based design approaches certainly provide advantageous features for establishing a common framework to support the development cycle, including control design, calibration and early verification.

For model-based control, the modeling of the air-path system of internal combustion engines (both spark-ignition and compression-ignition engines) is a well-established practice in automotive research. In particular, mean value engine models
(MVEM) represent a reasonable compromise between accuracy and complexity [87]. The modeling approach is based on the assumption that the air-path system (comprising the intake and exhaust manifold, restrictions and engine cylinders) can be decomposed into few interconnected receivers and restrictions [88–90]. Therefore, only the low-frequency dynamics of the engine air-path are relevant, namely the intake and exhaust manifold filling dynamics as the engine speed and torque demand vary in time [91]. For control applications, mean value engine models have been largely applied to the control of Gasoline engines [92–95], and Diesel engines [16,18,95–98].

The model considered in this study was developed based on a GM Duramax medium-duty Diesel engine, whose specifications are summarized in Table 2.1. The engine model, originally consisting of VGT/EGR, was modified to include a variable geometry compressor. The mean value model for this engine was previously developed in [22], and validated on experimental data collected on the production system with fixed-geometry compressor. The model predicts the low-frequency dynamics of the engine air-path through a 5-state nonlinear system, which results from applying the mass and energy conservation laws to the intake and exhaust manifolds, and a power balance to the turbocharger shaft [89–91,95,99,100]. The subsequent sections briefly

Table 2.1: Summary of Engine Specifications

<table>
<thead>
<tr>
<th>Specification</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>6.599l</td>
</tr>
<tr>
<td>Valvetrain</td>
<td>OHV 4-V</td>
</tr>
<tr>
<td>Injection</td>
<td>Direct (Common Rail)</td>
</tr>
<tr>
<td>Max Torque</td>
<td>820Nm @ 1800 rpm</td>
</tr>
<tr>
<td>Max Power</td>
<td>231kW @ 3200 rpm</td>
</tr>
</tbody>
</table>
describe the basic modeling procedures for the main engine subsystems, which are joined together to form the overall model and validated with the experimental data.

**Intake and Exhaust Manifolds**

The intake and exhaust manifolds are modeled as receivers using a filling-and-emptying approach [101]. The methodology consists of applying the mass and energy conservations to the control volumes.

\[
\frac{dm}{dt} = \sum \dot{m}_{in} - \sum \dot{m}_{out} \\
\frac{dU}{dt} = \sum \dot{m}_{in}h_{in} - \dot{m}_{out}h_{out} - \dot{Q} \tag{2.1}
\]

where \( m \) and \( U \) represent the accumulated masses and internal energy in the control volumes, respectively, while \( \dot{m}_{in}(\dot{m}_{out}) \) denotes the inlet (outlet) mass flow rates, \( h_{in}(h_{out}) \) denotes the inlet (outlet) enthalpy and \( \dot{Q} \) denotes the rate of heat losses through the manifold walls. The specific thermodynamic properties, such as internal energy and enthalpy, are evaluated as functions of temperature, with reference to a homogeneous mixture of perfect gases with constant specific heats:

\[
u = c_v T \quad h = c_p T \tag{2.2}
\]

where \( c_p \) and \( c_v \) represent the specific heat at constant pressure and constant volume respectively. Substituting the idea gas law into the equation (2.1), one can obtain:

\[
\frac{dT}{dt} = \frac{\gamma \gamma c_p}{pV} \left[ \sum \dot{m}_{in}T_{in} - \sum \dot{m}_{out}T_{out} - \frac{T}{\gamma} \left( \sum \dot{m}_{in} - \sum \dot{m}_{out} \right) - \frac{\dot{Q}}{c_p} \right] \tag{2.3}
\]

where \( \gamma = c_p/c_v \). The pressure dynamics can also be derived from the ideal gas law:

\[
\frac{dp}{dt} = \frac{\gamma R}{V} \left( \sum \dot{m}_{in}T_{in} - \sum \dot{m}_{out}T_{out} - \frac{\dot{Q}}{c_p} \right) \tag{2.4}
\]

As regards the intake manifold, the heat transfer can be neglected from the energy balance, due to the typically low temperature gradients. With reference to
Figure 2.4: Block Diagram of a Turbocharged Engine with VGC System.
Figure 2.4, Equation (2.3) and (2.4), the pressure and temperature dynamics for the
intake manifold are described as follows:

\[
\frac{dp_{IM}}{dt} = \frac{\gamma R}{V_{IM}} (\dot{m}_c T_{ic} + \dot{m}_{EGR} T_{EGR} - \dot{m}_{eng} T_{IM})
\]

\[
\frac{dT_{IM}}{dt} = \frac{\gamma R T_{IM}}{p_{IM} V_{IM}} \left[ \dot{m}_c T_{ic} - \dot{m}_{eng} T_{IM} - (\dot{m}_c + \dot{m}_{EGR} - \dot{m}_{eng}) \frac{T_{IM}}{\gamma} \right]
\]

(2.5)

Regarding the exhaust manifold, a heat transfer term should be considered in the
energy balance equation to account for the losses since exhaust gas typically has high
temperature. From Figure 2.4, Equation (2.3) and (2.4), the pressure and tempera-
ture dynamics read as:

\[
\frac{dp_{EM}}{dt} = \frac{\gamma R}{V_{EM}} \left[ (\dot{m}_{eng} + \dot{m}_{fuel}) T_{exh} - (\dot{m}_{EGR} + \dot{m}_t) T_{EM} \right]
\]

\[
\frac{dT_{EM}}{dt} = \frac{\gamma R T_{EM}}{p_{EM} V_{EM}} \left[ (\dot{m}_{eng} + \dot{m}_{fuel}) T_{exh} - (\dot{m}_{EGR} + \dot{m}_t) T_{EM} \right.
\]

\[
- (\dot{m}_{eng} + \dot{m}_{fuel} - \dot{m}_{EGR} - \dot{m}_t) \frac{T_{EM}}{\gamma} - \dot{Q} \frac{c_p}{\gamma}
\]

(2.6)

where \(V_{IM}\) and \(V_{EM}\) are the volumes of the intake and exhaust manifolds, respectively.

**Engine Charge Mass Flow Rate**

The mass flow rate trapped by the engine cylinders is modeled with an approach
based on the speed-density equation [91]:

\[
\dot{m}_{eng} = \lambda_v (p_{IM}, N_{eng}) \frac{p_{IM} V_d \ N_{eng}}{R T_{IM} 120}
\]

(2.7)

where the volumetric efficiency \(\lambda_v\) is estimated with an approach developed for aspi-
rated gasoline engines and extended to automotive turbocharged Diesel engine. The
model results from the integration of the mass and energy equations during the charge
exchange process and, with proper simplification, states that the product of the vol-
umetric efficiency by the intake manifold pressure is a linear function of the pressure
itself [91]:

\[
\lambda_v p_{IM} = S_i(N) p_{IM} - Y_i(N)
\]

(2.8)
where $S_i$ and $Y_i$, both defined positive, are weakly dependent on the engine speed.

**EGR Mass Flow Rate**

The EGR flow rate is predicted through the steady-state isentropic nozzle flow equation corrected by a discharging coefficient $C_d$ [101]. The actuator is characterized as a static map, which correlates the opening command to the actual valve opening. Under the assumption that no reverse flow exists, the EGR mass flow is given by the expression

$$\dot{m}_{EGR} = \frac{C_d A p_{EM}}{\sqrt{R T_{EM}}} \sqrt{\gamma} \cdot f(P_r),$$

\[ (2.9) \]

$$f = \begin{cases} \sqrt{\frac{2}{\gamma - 1} \left[ (P_r)^{\frac{\gamma}{2}} - (P_r)^{\frac{\gamma+1}{\gamma}} \right]} & \text{if } P_r \geq \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma-1}{\gamma}} \\ \sqrt{\left( \frac{2}{\gamma + 1} \right)^{(\gamma+1)/(\gamma-1)}} & \text{otherwise} \end{cases}$$

where $P_r = p_{EM}/p_{IM}$. The equivalent area $C_dA$ depends on the EGR valve position, and is calibrated as a static function of the EGR opening area [18,101].

**Turbocharger**

Since the manufacturer’s characteristic maps for turbine/compressor are typically available only for turbocharger shaft speeds within limited regions, modeling the turbocharger performance is of critical importance, especially when predicting the behavior outside the data region. As a consequence, a dedicated modeling procedure must be established to adequately predict the turbine/compressor maps of mass flow rate and efficiency, extending their validity to entire operating range.

The performance variables for turbine/compressor, provided by the manufacturer, are typically presented using the normalized form as shown in Table (2.2), to avoid the dependance of the characteristic maps on the temperature and pressure upstream
of the inlet wheels. The reference temperature $T_{ref}$ and pressure $p_{ref}$ are normally chosen as 298$K$ and 101.3$kPa$, respectively. With the values of $p_{ref}$ and $T_{ref}$ at hand, the corrected and reduced variables can be easily converted. To this extent, in order not to complicate the notations, the subsequent study uses the corrected variables uniformly, since they preserve better the physical properties of the associated parameters.

Starting from the steady-state characteristic maps provided by the manufacturer, the turbocharger has been modeled assuming quasi-static behavior. In particular, the corrected mass flow rate through the turbine is estimated by considering the machine as equivalent to a flow restriction [18,97]. This leads to a simple and sufficiently accurate characterization of the corrected mass flow rate through a polytropic expansion

Table 2.2: Summary of Normalized Turbocharger Performance Variables

<table>
<thead>
<tr>
<th>Performance Variables</th>
<th>Corrected Variables</th>
<th>Reduced Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass Flow Rate</td>
<td>$\dot{m}<em>{corr} = \dot{m} \frac{\sqrt{T</em>{in}/T_{ref}}}{p_{in}/p_{ref}}$</td>
<td>$\dot{m}<em>{red} = \dot{m} \frac{\sqrt{T</em>{in}}}{p_{in}}$</td>
</tr>
<tr>
<td>Turbocharger Shaft Speed</td>
<td>$N_{corr} = \frac{N}{\sqrt{T_{in}/T_{ref}}}$</td>
<td>$N_{red} = \frac{N}{\sqrt{T_{in}}}$</td>
</tr>
<tr>
<td>Power</td>
<td>$P_{corr} = \frac{P}{p_{in}/p_{ref} \sqrt{T_{in}/T_{ref}}}$</td>
<td>$P_{red} = \frac{P}{p_{in} \sqrt{T_{in}}}$</td>
</tr>
</tbody>
</table>
as a function of the pressure ratio:

\[ M_{\text{corr}} = \frac{C_d A_t p_{\text{ref}}}{\sqrt{R T_{\text{ref}}} \sqrt{\gamma} \cdot f(\epsilon)}, \quad (2.10) \]

\[
f(\epsilon) = \begin{cases} 
\sqrt{\frac{2}{\gamma - 1} \left[ \epsilon - \frac{2}{m} - \epsilon \frac{m+1}{m} \right]} & \text{if } \epsilon \leq \left( \frac{2}{m + 1} \right)^{-\frac{m-1}{m}} \\
\left( \frac{2}{m + 1} \right)^{\frac{1}{m-1}} \sqrt{\frac{2}{\gamma - 1} \frac{m - 1}{m + 1}} & \text{otherwise}
\end{cases}
\]

where \( \epsilon = p_{EM}/p_b \), \( p_b \) represents the turbine outlet pressure, \( C_d A_t \) is the turbine equivalent area and \( m \) is the coefficient of the polytropic transformation approximating the turbine expansion process. Since the turbocharger has a variable geometry nozzle, the flow characteristic curves are strongly affected by the stator blade position. Therefore, the model parameters \( C_d A_t, m \) have to be evaluated as a function of the VGT positions [18]. The results of the identified model are presented in Figure 2.5, where the characteristic curves predicted by the proposed model are compared to the experimental flow maps.

The turbine outlet temperature and power can be calculated from the isentropic efficiency. For a given pressure ratio across the turbine, the relationship between the temperature and pressure at the inlet \( (T_{EM}, p_{EM}) \) and at the outlet \( (T_{b,s}, p_b) \) of the turbine can be derived:

\[
\frac{T_{b,s}}{T_{EM}} = \left( \frac{p_b}{p_{EM}} \right)^{\frac{\gamma - 1}{\gamma}} \quad (2.11)
\]

The isentropic efficiency \( \eta_t \) is introduced to account for heat losses:

\[
\eta_t = \frac{T_{EM} - T_b}{T_{EM} - T_{b,s}} \quad (2.12)
\]

Therefore, the temperature downstream of the turbine can be derived:

\[
T_b = T_{EM} - \eta_t T_{EM} \left[ 1 - \left( \frac{p_b}{p_{EM}} \right)^{\frac{\gamma - 1}{\gamma}} \right] \quad (2.13)
\]
Figure 2.5: Turbine Corrected Mass Flow Rate: Experimental Data and Model Prediction.

The turbine power $P_t$ is derived based on the first law of thermodynamics by neglecting the heat losses, and is expressed as a function of the mass flow rate through the turbine $\dot{m}_t$ and the total change of enthalpy:

$$P_t = \dot{m}_t (h_{EM} - h_b) = \dot{m}_t c_p (T_{EM} - T_b) = \dot{m}_t c_p \eta_t T_{EM} \left[ 1 - \frac{\epsilon}{\gamma} \right]$$  \hspace{1cm} (2.14)

The traditional approach adopted to model the turbine isentropic efficiency is based on the definition of blade speed ratio, which has a quadratic type relationship with the turbine efficiency [26]. However, this type of approach does not give satisfactory agreement with turbine data in most practical applications, where the limited data availability is not sufficient for a reliable characterization. On the other hand, since the efficiency data tends to cluster around a small region, a direct fit of the turbine efficiency with respect to pressure ratio is impractical due to the lack of a
suitable mathematical representation. For these reasons, a different approach has been adopted, starting from the definition of corrected turbine power:

\[ P_{corr} = \eta_t \dot{m}_{corr} c_p T_{ref} \left( 1 - \varepsilon \right) \left( 1 - \frac{1 - \gamma}{\gamma} \right) \]  

(2.15)

Recall that both the turbine efficiency and the corrected mass flow rate are functions of the pressure ratio and the VGT opening. Therefore, by calibrating the corrected power as a function of the turbine pressure ratio and the VGT openings, it is possible to represent the efficiency with a more convenient form for data interpolation. Furthermore, the representation of the corrected power allows for extending the available data to a larger range of pressure ratios. According to the similarity theory, the turbine power can be well approximated by applying the following relation:

\[ P_{corr}(\varepsilon) = a_1 (\varepsilon - \varepsilon_0)^{a_2} \]  

(2.16)

where \( \varepsilon_0 \) represents the pressure ratio at zero power condition, and \( a_1, a_2, \varepsilon \) can be

Figure 2.6: Turbine Corrected Power (Left) and Efficiency (Right): Experimental Data and Model Prediction.
correlated as functions of the VGT positions. Figure 2.6 presents the calibration results of the corrected turbine power. Moreover, Figure 2.6 shows the turbine efficiency converted from the calibrated power and flow models.

In order to characterize the flow and efficiency maps of the VGC, a predictive, physics-based model of a centrifugal compressor stage was used, based on the work described in [102]. The predictive compressor model was initially calibrated on the characteristic map of a production (fixed geometry) compressor, then used to simulate the effects of the variable inlet guide vanes on the flow and efficiency outputs [22]. Then, the predictive model was used to calibrate a “grey-box” model developed with the Jensen & Kristensen (J&K) method [96]. The model is essentially based on scaling the compressor flow and efficiency data to the following dimensionless variables:

\[
\Psi = \frac{c_p T_{ref} \left( \frac{p_c}{p_a} \right)^{\frac{2-1}{2}}} {0.5 U_{tip}^2},
\]

\[
U_{tip} = \frac{\pi}{60} D_{tip} N_{tc},
\]

\[
\Phi = \frac{\dot{m}_c R T_{ref}} {p_{ref} \frac{D_{tip}^2 U_{tip}}{4}},
\]

\[
Ma = \frac{U_{tip}} {\sqrt{\gamma R T_{ref}}},
\]

where \(D_{tip}\) denotes the blade tip diameter, \(p_c\) and \(p_a\) represent the compressor outlet and inlet pressure, respectively. The explanation of the defined notation above can be found in [96]. Using the above scaling, the flow and efficiency maps of a fixed geometry compressor can be easily interpolated with the following correlations [96]:

\[
\Phi = \frac{k_3 \Psi - k_1}{\Psi + k_2}, \quad k_i = k_{i1} + k_{i2} Ma
\]

\[
\eta_c = h_1 \Phi^2 + h_2 \Phi + h_3, \quad h_i = h_{i1} + h_{i2} Ma
\]

where \(i = 1, 2, 3\). The model is proven to show good fitting ability for most automotive
compressors. Figure 2.7 shows the correlation results for the characteristic maps of the fixed geometry compressor (VGC=0%).

Then, the procedures are applied considering the VGC actuation, based on the data generated from the first-principle compressor model [22]. The effects of the VGC actuation can be accounted for by modifying (2.18) as

$$k_i = k_{i1}(u_{VGC}) + k_{i2}(u_{VGC})Ma$$

$$h_i = h_{i1}(u_{VGC}) + h_{i2}(u_{VGC})Ma$$  \(2.19\)

Figure 2.8 and Figure 2.9 show the compressor characteristic maps at different VGC positions, which present an example of how the VGC actuation affects the compressor characteristic map. As it can be seen, an increase of the VGC actuator position leads to a closing of the VIGV, which in turn affects the output of the compressor, specifically, causing the operating range of the compressor to shift towards lower
Figure 2.8: Characteristic Maps of Centrifugal Compressor with Effects of VGC Actuation: 20% VGC (Left) and 50% VGC (Right).

Figure 2.9: Characteristic Maps of Centrifugal Compressor with Effects of VGC Actuation: 80% VGC (Left) and 100% VGC (Right).
values of boost pressure and air flow rate. This provides the opportunity, in concert with the other actuators, to actively control the stability of the compressor operating point, by shifting the surge line depending on the desired flow rate and boost pressure.

The derivation of the compressor power is similar with the turbine. Assuming that the compression is isentropic, the following relation between the temperature and pressure at the inlet \((T_a, p_a)\) and at the outlet \((T_{c,is}, p_c)\) of the compressor can be derived:

\[
\frac{T_{c,is}}{T_a} = \left(\frac{p_c}{p_a}\right)^{\frac{\gamma - 1}{\gamma}}
\]

However, due to the irreversibility across the compressor (e.g. incidence and friction losses), the compression process is not isentropic in reality. Therefore, the compressor isentropic efficiency is introduced, which relates the theoretical temperature rise (leading to \(T_{c,is}\)) to the actual (resulting in \(T_c\)):

\[
\eta_c = \frac{T_{c,is} - T_a}{T_c - T_a}
\]

Substituting equation (2.21) to equation (2.20) yields the expression for the temperature downstream of the compressor:

\[
T_c = T_a + \frac{1}{\eta_c}T_a \left[\left(\frac{p_c}{p_a}\right)^{\frac{\gamma - 1}{\gamma}} - 1\right]
\]

The compressor power \(P_c\) is derived based on the first law of thermodynamics by neglecting the heat losses, and is expressed as a function of the mass flow rate through the compressor \(\dot{m}_c\) and the total change of enthalpy:

\[
P_c = \dot{m}_c(h_c - h_a) = \dot{m}_c c_p(T_c - T_a) = \frac{1}{\eta_c} \dot{m}_c T_a \left[\left(\frac{p_c}{p_a}\right)^{\frac{\gamma - 1}{\gamma}} - 1\right]
\]

Furthermore, in order to evaluate the stability of compressor, a Surge Index (SI) is defined as the position of the operating condition relative to the limits of the compressor operating range, namely the surge and choke lines, as shown in Figure 2.8 and
Figure 2.9. The $SI$ can therefore be used to estimate the stability of the compressor, and is defined as follows:

$$SI = 1 - \frac{\dot{m}_c - \dot{m}_{surge}}{\dot{m}_{choke} - \dot{m}_{surge}}$$  \hspace{1cm} (2.24)$$

where $\dot{m}_{choke}$ and $\dot{m}_{surge}$ are functions of the turbocharger shaft speed.

Finally, the turbocharger rotational dynamics is modeled by applying Newton second law:

$$\frac{dN_{tc}}{dt} = \left(\frac{30}{\pi}\right)^2 \frac{1}{J_{tc} N_{tc}} (P_t - P_c)$$  \hspace{1cm} (2.25)$$

where $J_{tc}$ represents the inertia of the turbocharger. Note that the turbocharger mechanical efficiency $\eta_m$ does not explicitly appear in the equation (2.25) because, as shown in [100], it is typically lumped into the turbine efficiency $\eta_t$.

**EGR and Intercooler**

The downstream temperatures of both the EGR and the intercooler are estimated using static models. By neglecting the pressure drops across the coolers, the temperatures can be calculated using the heat exchanger effectiveness, the upstream gas temperatures and the appropriate coolant temperatures:

$$T_{ic} = \eta_{ic} T_{i,cool} + (1 - \eta_{ic}) T_c$$

$$T_{EGR} = \eta_{ec} T_{e,cool} + (1 - \eta_{ec}) T_{EM}$$  \hspace{1cm} (2.26)$$

**Engine Torque Output**

The engine combustion and torque production model assumes that the processes occurring in the engine cylinders are cycle-averaged. Therefore, the calculations of the incylinder processes is typically reduced to quasi-steady, algebraic relationships, mostly data-driven.
The engine output torque is based on phenomenological model that predicts the engine indicated efficiency [98]. The engine indicated efficiency is defined as:

\[
\eta_{\text{ind}} = \frac{\text{IMEP} \cdot V_d \cdot N_{\text{eng}}}{LHV_f \cdot 120 \cdot \dot{m}_f}
\]

(2.27)

where \(LHV_f\) is the fuel lower heating value. The indicated efficiency (\(\eta_{\text{ind}}\)) is predicted using a model that was derived by analyzing experimental data for different Diesel engines. Starting from basic principles of Diesel engine combustion, a heuristic model for the indicated efficiency is formulated to achieve physically consistent results:

- at constant engine speed, a decrease of the fuel/air ratio causes an increment of the indicated efficiency that is consequent to the increase of air quantity in the charge, thereby improving the ignition delay and combustion duration;

- at constant engine speed, an increase of the differential pressure between exhaust and intake manifolds leads to an increase of the pumping losses, reducing the indicated efficiency.

Such considerations lead to the definition of \(\eta_{\text{ind}}\) as a function of the engine speed, fuel/air ratio (FAR) and the differential pressure as:

\[
\eta_{\text{ind}} = \eta_0 \left( k_1 + k_2 N_{\text{eng}} - k_3 N_{\text{eng}}^2 \right) \cdot \left( 1 - k_4 FAR^{k_5} \right) \cdot \left[ 1 - k_6 (p_{EM} - p_{IM}) \right]
\]

(2.28)

where the last term accounts for the back pressure that occurs at high engine load conditions.

In [98] it was shown that, although the model does not explicitly predict the effect of several physical variables (for instance, the fuel injection system parameters), it provides a reasonable representation of how the indicated efficiency relates to the air-path system variables in different turbocharged Diesel engines, once it is calibrated on steady-state data. The parameters of the indicated efficiency model for the engine considered in this study are summarized in Table 2.3.
Table 2.3: Summary of the Indicated Efficiency Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\eta_0$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$k_5$</th>
<th>$k_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>0.58</td>
<td>0.71</td>
<td>0.45</td>
<td>0.26</td>
<td>1.26</td>
<td>0.78</td>
<td>0.0055</td>
</tr>
</tbody>
</table>

The exhaust gas temperature is estimated through the definition of the incremental temperature $\Delta T$, which is given by an empirical model based on a nonlinear regression of the charge composition (fresh air, fuel, EGR), the engine speed and the fuel injection parameters [22]:

$$T_{exh} = T_{IM} + \Delta T(\dot{m}_{eng}, \dot{m}_{EGR}, \dot{m}_{fuel}, N_{eng})$$ (2.29)

While the model does not explicitly predict emissions, it is well known that nitrogen oxides (NO$_x$) and particulate matter (PM) can be controlled by monitoring the in-cylinder air/fuel Ratio (AFR) and the EGR Ratio ($z_{EGR}$). These variables can be approximated considering quasi-static conditions, leading to [40]:

$$z_{EGR} = \frac{\dot{m}_{EGR}}{\dot{m}_c + \dot{m}_{EGR}}$$

$$AFR = (1 - z_{EGR}) \frac{\dot{m}_{eng}}{\dot{m}_{fuel}}.$$ (2.30)

2.4 Experimental Validation

The complete engine air-path system model was validated against experimental data in steady-state and transient conditions, without considering the effect of VGC actuation [22].

The operating conditions of the data set used for steady state validation, as shown in Figure 2.10, have been chosen to cover most of the engine operating range, in terms of engine speed and torque. Figure 2.10 (right) also shows the actuator commands for the VGT and EGR valve at the corresponding operating conditions, while the
Figure 2.10: Model Validation for Steady State Operating Conditions: Engine Speed and Torque (Left); VGT Position (Top Right); EGR Position (Bottom Right).

Figure 2.11: Model Validation Results for Steady State Operating Conditions: Intake Manifold Pressure (Left); Air Mass Flow Rate (Right).
Figure 2.12: Model Validation Results for Steady State Operating Conditions: Exhaust Manifold Pressure (Left); Torque (Right).

VGC command is set as zero for validation. The results of the steady state model validation are presented in Figure 2.11 and Figure 2.12, where the model correctly captures the engine behavior at different operating conditions. In particular, the mean errors for the intake and exhaust manifold pressures are less than 2%, while the level of inaccuracy for the air mass flow rate is close to 3%. Although the torque error predicted by the model is close to 6%, it is noticed that most of such errors come from the low torque conditions, due to the empirical and simplified assumptions made in the study.

Furthermore, a validation of the model is also performed considering the transient conditions, based on the data from the initial portion of the US-FTP driving cycle. Figure 2.13 shows the engine operating conditions and actuator commands in the FTP driving cycle. Figure 2.14 shows a comparison of the performance variables
Figure 2.13: Model Validation During a Portion of the FTP Cycle: Engine Speed (Top Left); Fuel Mass Flow Rate (Bottom Left); VGT Position (Top right); EGR Position (Bottom Right).

Figure 2.14: Model Validation Results During a Portion of the FTP Cycle: Intake Manifold Pressure (Top Left); Exhaust Manifold Pressure (Bottom Left); Air Mass Flow Rate (Top Right); Torque (Bottom Right).
predicted by the model against the experimental data. As it can be observed, the model is capable to capture the engine transient behavior with sufficient accuracy. In particular, the mean error calculated on the entire driving cycle shows that the boost pressure and air mass flow rate are predicted with an error smaller than 3% and 2%, respectively. As a result, this proves that the model can be used with confidence as a control design and verification tool.
CHAPTER 3
SYSTEM ANALYSIS, OPTIMIZATION AND
PARAMETER ESTIMATION

Compared to the conventional VGT-EGR control problem [16, 40], in which VGT and EGR are used to regulate Mass Air Flow (MAF) and intake manifold pressure, the inclusion of VGC in the system adds one degree of freedom to the air-path that can be exploited to optimize the system performance, for instance, to improve the compressor stability and efficiency.

The study in this chapter is a system-level analysis of the EGR/VGT/VGC system, to provide insights into the system dynamics and ultimately derive a coordinated control for the three actuators that aims at optimizing the system performance. Variables of interest include, in addition to the conventional intake manifold pressure $p_{IM}$ and air mass flow $\dot{m}_c$, the compressor performance variables (Surge Index $SI$, turbocharger shaft speed $N_{tc}$ and compressor efficiency $\eta_c$). Since some of the most powerful tools for system analysis can only be applied to linear systems, the study starts from a model order reduction, followed by linearization around different operating conditions. Step response, frequency response and DC gains are used to determine the control authority of the actuators with respect to the different performance variables, which contributes to optimizing the actuators interaction. Then, a model-based optimization is conducted, leading to an open-loop control law that improves fuel economy and compressor stability, based on simulation results. Finally,
this chapter illustrates the development of parameter estimation strategies, including state and performance variables of interest, to provide a prerequisite for the design of full-information control.

### 3.1 Model Order Reduction

Even though the 5-state model developed in previous chapter has been proven to be a good representation of the engine system, it can hardly be used for control design due to its complexity. As a result, model order reduction was applied to obtain a simplified model, which will be referred to as “reduced-order model”. In contrary, the 5-state model is termed as “full-order model” in the following study. Similarly to [103], singular perturbation theory was applied to decompose the model into slow and fast dynamics. The singular perturbation model of a dynamical system can be expressed as [104]:

\[
\begin{align*}
\dot{x} &= f(t, x, z, \varepsilon) \\
\varepsilon \dot{z} &= g(t, x, z, \varepsilon)
\end{align*}
\]

where \( f \) and \( g \) are continuously differentiable functions in their arguments for \((t, x, z, \varepsilon) \in [0, T] \times \mathcal{X} \times \mathcal{Z} \times [0, \varepsilon_0] \) and \( \varepsilon_0 \) is a small positive parameter. The peculiarity of a singularly perturbed model is that, since \( \varepsilon \) is small, the derivative \( \dot{z} = g/\varepsilon \) is large, thus, \( z \) may converge to an equilibrium point faster than \( x \). The state variables are hence characterized by different dynamic responses, which include fast \((z)\) and slow \((x)\) transients. As a result, one could either preserve only the slower dynamics \( x \) (by replacing the faster dynamics \( z \) with an algebraic function of \( x \)) or only the faster dynamics \( z \) (by considering quasi-steady values for the slower dynamics \( x \)).
Based on the philosophy of singular perturbation, the Diesel engine model developed in previous section can be rewritten as:

\[
\frac{dp_{IM}}{dt} = \left(\frac{\gamma R}{V_{IM}}\right) (\dot{m}_c T_c + \dot{m}_{EGR} T_{EGR} - \dot{m}_{eng} T_{IM})
\]

\[
\frac{dp_{EM}}{dt} = \left(\frac{\gamma R}{V_{EM}}\right) (\dot{m}_{eng} + \dot{m}_{fuel}) T_{exh} - (\dot{m}_{EGR} + \dot{m}_t) T_{EM}
\]

\[
\frac{dT_{IM}}{dt} = \varepsilon_{IM} \cdot \left(\frac{\gamma R}{V_{IM}}\right) (\dot{m}_c T_c + \dot{m}_{EGR} T_{EGR} - \dot{m}_{eng} T_{IM} - (\dot{m}_c + \dot{m}_{EGR} - \dot{m}_{eng}) \frac{T_{IM}}{\gamma})
\]

\[
\frac{dT_{EM}}{dt} = \varepsilon_{EM} \cdot \left(\frac{\gamma R}{V_{EM}}\right) ((\dot{m}_{eng} + \dot{m}_{fuel}) T_{exh} - (\dot{m}_{EGR} + \dot{m}_t) T_{EM}
\]

\[
- (\dot{m}_{eng} + \dot{m}_{fuel} - \dot{m}_{EGR} - \dot{m}_t) \frac{T_{EM}}{\gamma})
\]

\[
\frac{dN_{tc}}{dt} = \left(\frac{30}{\pi}\right)^2 \frac{1}{J N_{tc}} (P_t - P_c)
\]

(3.2)

where \(x(t) = [p_{IM}, p_{EM}, T_{IM}, T_{EM}, N_{tc}]^T\) represents the states, \(u(t) = [u_{VGT}, u_{EGR}, u_{VGC}]^T\) the control inputs, \(w(t) = [N_{eng}, \dot{m}_{fuel}]^T\) the exogenous signals and \(\varepsilon_{IM} = T_{IM}/p_{IM}, \varepsilon_{EM} = T_{EM}/p_{EM}\). Since the values of the pressure \(p_{IM}\) and \(p_{EM}\) are typically much larger than the temperature values \(T_{IM}\) and \(T_{EM}\), the parameters \(\varepsilon_{IM}\) and \(\varepsilon_{EM}\) are very small [103]. This implies that the temperature dynamics in the control volumes are hence significantly slower than the pressure and shaft speed dynamics. As a result, the two differential Equations in (3.2) that define the temperature dynamics can be removed and the prediction of the temperatures \(T_{IM}\) and \(T_{EM}\) can be done using quasi-steady maps scheduled as functions of engine speed and fuel flow rate. As a result, the reduced-order model reads as:

\[
\frac{dp_{IM}}{dt} = \frac{\gamma R}{V_{IM}} (\dot{m}_c T_c + \dot{m}_{EGR} T_{EGR} - \dot{m}_{eng} T_{IM})
\]

\[
\frac{dp_{EM}}{dt} = \frac{\gamma R}{V_{EM}} (\dot{m}_{eng} + \dot{m}_{fuel}) T_{exh} - (\dot{m}_{EGR} + \dot{m}_t) T_{EM})
\]

\[
\frac{dN_{tc}}{dt} = \left(\frac{30}{\pi}\right)^2 \frac{1}{J N_{tc}} (P_t - P_c)
\]

(3.3)

The accuracy of the so obtained reduced-order model with respect to the full-order
Figure 3.1: Reduced Order Model Verification of Intake Manifold Pressure via Frequency Response: From Engine Speed (Top Left), Fuel Mass Flow Rate (Top Right), VGT position (Bottom Left) and VGC position (Bottom Right), Respectively.

Figure 3.2: Reduced Order Model Verification of Exhaust Manifold Pressure via Frequency Response: From Engine Speed (Top Left), Fuel Mass Flow Rate (Top Right), VGT position (Bottom Left) and VGC position (Bottom Right), Respectively.
model has been tested first around specific nominal conditions and then considering transient profiles. In particular, the analysis around different operating conditions has been performed in both frequency and time domain. For instance, Figure 3.1 to Figure 3.4 present the results for the operating point at engine speed 1500\(r/min\) and fuel flow rate 2.6\(g/s\) with 80\% VGC opening (shown in Table 3.1). Similar results were obtained for the other operating conditions. The analysis in the frequency domain has been done by estimating the frequency response of the nonlinear systems around an operating condition.

Figure 3.1 shows how the frequency response of the intake manifold pressure with respect to both exogenous signals and control inputs of the reduced-order model ("Reduced") matches well the full-order model ("Full"). On the other hand, the exhaust manifold pressure frequency response predicted by the reduced model with respect to the exogenous signals does not match well the full order model at high frequency. This behavior is expected and can be explained considering that high frequency changes in engine speed and fueling rate cause rapid variations of the exhaust temperature. As a result, the reduced-order model exhibits a lower exogenous- signals-to-\(p_{EM}\) gain because it considers a quasi-steady value of the exhaust temperature. Finally, the last plots in Figure 3.1 show how the frequency response of the \(p_{EM}\) with respect to the control inputs of the reduced-order model matches well that of the full-order model.

The analysis around an operating condition in the time domain was performed by applying steps in the exogenous signals and control inputs to the system. In particular, for this case study, the following steps were considered:

- Engine speed (\(N_{eng}\)): from 1500 to 1600 [rpm]
- Fuel Flow Rate (\(m_{fuel}\)): from 2.6 to 3.1 [g/s]
Figure 3.3: Reduced Order Model Verification of Intake Manifold Pressure via Step Response: From Engine Speed (Top Left), Fuel Mass Flow Rate (Top Right), VGT position (Bottom Left) and VGC position (Bottom Right), Respectively.

Figure 3.4: Reduced Order Model Verification of Exhaust Manifold Pressure via Step Response: From Engine Speed (Top Left), Fuel Mass Flow Rate (Top Right), VGT position (Bottom Left) and VGC position (Bottom Right), Respectively.
• VGT Opening ($u_{VGT}$): from 33 to 27 [%]

• VGC Opening ($u_{VGC}$): from 80 to 60 [%]

Figure 3.3 shows response of the $p_{IM}$ and $p_{EM}$ to those steps for both reduced-order and full-order models. It is possible to see how the responses produced by the two models match well. The largest steady-state error is observed in the $p_{EM}$ dynamics and it can be explained with the same arguments introduced before. Even though neglecting the temperature dynamics introduces inaccuracies in the model, it is worth noticing that the transient behavior is preserved and the steady error observed is smaller than 0.5%.

Figure 3.5: Reduced Order Model Verification of Intake and Exhaust Pressure Considering an Acceleration Profile: Exogenous Inputs of Fuel Flow Rate and Engine Speed (Left); Intake and Exhaust Manifold Pressure (Right).
Finally, the reduced-order model is validated considering different transient profiles. The previous analysis highlighted how the accuracy of the reduced-order model could be affected by high frequency changes in engine speed and fueling rate. As a result, an acceleration profile that includes several gear shifts and enforces fast transients of the two exogenous inputs (Figure 3.5) is selected as a representative case study. Figure 3.5 also compares the $p_{IM}$ and $p_{EM}$ predicted by the two models during the acceleration profile. As it is possible to see, the pressure traces produced by the reduced-order model follow the full-order model ones very closely even during the sharp variations of $N_{eng}$ and $\dot{m}_{fuel}$ caused by the gear shifts. As a result, the reduced-order model proved itself to be a good representation of the system and can be adopted with confidence for system analysis and control design.

3.2 System Linearization and Dynamic Analysis

3.2.1 System Linearization

As a first step of the linearization process, the nonlinear algebraic functions (from Equation (2.7) to Equation (2.24)) that define the variables $\dot{m}_{eng}, \dot{m}_{EGR}, \dot{m}_{t}, \dot{m}_{c}, P_{t}, P_{c}$ have been linearized and expressed as functions of states, control inputs and exogenous signals:

\[
\begin{align*}
\delta \dot{m}_t &= \alpha_{15} \delta p_{EM} + \alpha_{17} \delta \dot{m}_{fuel} + \alpha_{11} \delta u_{VGT}, \\
\delta \dot{m}_c &= \alpha_{34} \delta p_{IM} + \alpha_{36} \delta N_{tc} + \alpha_{33} \delta u_{VGC}, \\
\delta \dot{m}_{EGR} &= \alpha_{54} \delta p_{IM} + \alpha_{55} \delta p_{EM} + \alpha_{52} \delta u_{EGR}, \\
\delta \dot{m}_{eng} &= \alpha_{64} \delta p_{IM} + \alpha_{65} \delta p_{EM} + \alpha_{68} \delta N_{eng}
\end{align*}
\]  

(3.4)

The linear algebraic equations have been substituted into the differential equations (3.3) which have then been linearized analytically. As a result, the “linearized
The reduced-order model reads as:

\[
\dot{x}(t) = Ax(t) + Bu(t) + Pw(t)
\]

\[
y(t) = Cx(t) + Du(t)
\]

\[
z(t) = Cz x(t) + Dz u(t)
\]

where

\[
x(t) = [p_{IM}, p_{EM}, N_{tc}]^T, \quad u(t) = [u_{VGT}, u_{EGR}, u_{VGC}]^T
\]

\[
w(t) = [N_{eng}, \dot{m}_{fuel}]^T, \quad y(t) = [p_{IM}, \dot{m}_c]^T
\]

and the matrices \(A, B, C, D, P\) are shown in Equation (3.7). Note that, even though the measured and controlled outputs \(y\) are represented by \(p_{IM}\) and \(\dot{m}_c\) as in the traditional EGR-VGT problem, other performance variables \(z = [SI, N_{tc}, \eta_c]^T\) that characterize the system performance are considered in this study.

The system linearization and analysis has been performed around different equilibrium points, \(eq = [x^0, u^0, \omega^0]\), that have been chosen according to the following criterion. Since the VGC is expected to provide fuel economy benefits at low to medium torque and engine speed conditions, the operating points bounded by the grey box in Figure 3.6 are of higher interest in this study. It is worthwhile noticing...
that the selected region is of key importance as most of the driving cycles operating points lie in it. Figure 3.6, for instance, shows how the operating points of the FTP cycle, mostly fall inside the range of interest. As representative points, the six operating conditions highlighted in Figure 3.6 have been considered for the system linearization and analysis.

In Table 3.1, the engine speed and fuel flow rate associated to each of these operating points are summarized. The corresponding VGT and EGR openings are taken from the feed-forward maps of the “original architecture” (corresponding to production compressor with the fixed geometry). Since the VGC was not included in the “original architecture”, for each of these 6 points, three nominal values of VGC opening were considered, namely, 20%, 50% and 80%. Results for representative
Table 3.1: Summary of Model Linearization Set Points

<table>
<thead>
<tr>
<th>Point</th>
<th>Speed</th>
<th>( \dot{m}_{\text{fuel}} )</th>
<th>( u_{\text{VGT},0} )</th>
<th>( u_{\text{EGR},0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1200rpm</td>
<td>3.4g/s</td>
<td>30%</td>
<td>2.1%</td>
</tr>
<tr>
<td>2</td>
<td>1200rpm</td>
<td>0.9g/s</td>
<td>27%</td>
<td>5.0%</td>
</tr>
<tr>
<td>3</td>
<td>1500rpm</td>
<td>2.6g/s</td>
<td>33%</td>
<td>6.4%</td>
</tr>
<tr>
<td>4</td>
<td>1800rpm</td>
<td>5.5g/s</td>
<td>44%</td>
<td>17.5%</td>
</tr>
<tr>
<td>5</td>
<td>1800rpm</td>
<td>1.5g/s</td>
<td>39%</td>
<td>10.1%</td>
</tr>
<tr>
<td>6</td>
<td>2100rpm</td>
<td>6.7g/s</td>
<td>58%</td>
<td>14.0%</td>
</tr>
</tbody>
</table>

operating conditions 2, 3 and 4 will be presented in the following, and similar results were obtained for the other operating conditions.

3.2.2 Step Response

A step response analysis around the equilibrium points of interest was performed for mainly two reasons. First, to evaluate the accuracy of the reduced-order linearized model with respect to the nonlinear one; second, to evaluate the control authority of the different actuators on the performance variables. Results obtained around the equilibrium point defined by

\[
x^0 = [145kPa, 181.6kPa, 58kr/min]
\]

\[
u^0 = [33\%, 6.4\%, 80\%], \quad \omega^0 = [1500r/min, 2.6g/s]
\]

which corresponds to operating condition 3 and a VGC opening of 80%, are presented in the following. For the case study presented here, starting from the equilibrium point, percentage openings of the actuators were increased by 5% from their nominal value one at a time and then decreased to their initial value after 2 seconds. In
particular, the VGT position was increased at $t = 2s$, the EGR position at $t = 6s$ and finally the VGC at $t = 10s$.

As shown in Figure 3.7, the transient dynamics predicted by the linearized model matches the nonlinear model very closely. The steady-state behavior is preserved as well and in particular negligible errors can be observed in the $p_{IM}$ and MAF profiles. On the other hand, a larger error can be observed in the compressor performance variables ($SI$, $N_{tc}$, $\eta_c$) in correspondence to steps in VGT openings.

Figure 3.7: Step Response to VGT, EGR and VGC for the Reduced Order Model and the Linearized Model.
However, it is important to notice that, for the case study considered here, the 5% increase in the VGT opening (which corresponds to a 13% relative variation of the VGT opening) induces a steady state error prediction of 1.65% for the $SI$, of 2.8% for the $N_{tc}$, and of 0.9% for the $\eta_c$. As a result, the linearized model can be considered a good approximation of the nonlinear dynamics and hence it can be used with confidence for system analysis.

If we look at the SISO properties of the system, from Figure 3.7 it is possible to recognize the well-known nonminimum phase (NMP) behavior from the EGR valve to the $p_{IM}$ and from the VGT to the MAF [16]. Furthermore, a NMP behavior can also be observed from the VGC to the boost pressure and air mass flow rate. This can be explained considering that variations in the VGC position have an immediate effect on the air mass flow rate, as illustrated in Equation (2.19). As the VGC actuator opens, the compressor operating map immediately shifts towards lower flow rate conditions for the same boost pressure and shaft speed, as shown in Figure 2.8 and Figure 2.9. On the other hand, the reduced compressor air flow causes an initial decrease of the boost pressure (due to the manifold filling dynamics) and a shift of the compressor operating point towards slightly higher efficiency regions, away from the surge line. This leads to a decrease of the power absorbed by the compressor, which results in a gradual increase of the rotational speed that is modeled by the turbocharger shaft dynamics equation. The increase in the turbocharger speed eventually results into a slight increase in the air mass flow rate processed by the compressor.

Finally, other NMP behaviors can be observed between the single inputs and the performance variables, namely, the turbocharger speed, the compressor efficiency and the surge index. All the NMP properties described above are confirmed by the existence of right-half plane zeros in the respective SISO transfer functions. Differently from the VGT-EGR control problem, where the MIMO system with inputs
(\(u_{\text{VGT}}, u_{\text{EGR}}\)) and outputs \((p_{\text{IM}}, \dot{m}_c)\) can also be shown to be NMP, the addition of the VGC actuator renders the MIMO system minimum phase when considering the same outputs. This can be seen observing that the Smith-McMillan form of the MIMO transfer function matrix, computed at different operating conditions, has no transmission zeros. For the operating condition (3.8), the Smith-McMillan form of the MIMO transfer function matrix from the inputs \((u_{\text{VGT}}, u_{\text{EGR}}, u_{\text{VGC}})\) to the outputs \((p_{\text{IM}}, \dot{m}_c)\) is given by

\[
M(s) = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{s^3 + 539s^2 + 43700s + 3.49e5} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & s - 47.64
\end{bmatrix}
\]

and it is straightforward to see that \(M(s)\) has no transmission zeros. On the other hand, if we make the system square to design the control by adding a third output (e.g. turbocharger shaft speed, compressor efficiency or \(SI\)), the system becomes non-minimum phase. For instance, for the operating condition (3.8), the Smith-McMillan form of the MIMO transfer function matrix from the inputs \((u_{\text{VGT}}, u_{\text{EGR}}, u_{\text{VGC}})\) to the outputs \((p_{\text{IM}}, \dot{m}_c, \eta_c)\) is given by

\[
M(s) = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{s^3 + 539s^2 + 43700s + 3.49e5} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & s - 47.64
\end{bmatrix}
\]

and the presence of the positive transmission zero makes the system non-minimum phase.

The step response has also been used to visualize the control authority of the different actuators on the variables of interest. As shown in Figure 3.7, the VGT has a higher control authority than EGR and VGC, on almost all the performance variables. On the other hand, the EGR has good control authority on the MAF and the VGC on the \(SI\) and \(\eta_c\). In the next section, a more detailed analysis that focuses
\[ H_{2,0} = \begin{bmatrix} -0.70 & -0.17 & -0.0072 \\ -0.33 & -0.57 & -0.0039 \\ 0.22 & 0.28 & -0.30 \\ -1.29 & -0.33 & 0.035 \\ 0.6 & 0.078 & -0.011 \end{bmatrix}, \quad H_{2,80} = \begin{bmatrix} -1.13 & -0.30 & 0.038 \\ -0.56 & -0.64 & 0.021 \\ -0.40 & 0.22 & -0.79 \\ -0.50 & -0.14 & 0.05 \\ 0.19 & -0.05 & 0.08 \end{bmatrix} \]

\[ H_{3,0} = \begin{bmatrix} -1.72 & -0.59 & 0.075 \\ -1.21 & -1.06 & 0.063 \\ -0.20 & 0.16 & -0.11 \\ -0.53 & -0.19 & 0.034 \\ 0.5 & 0.043 & 0.05 \end{bmatrix}, \quad H_{3,80} = \begin{bmatrix} -1.88 & -0.63 & 0.13 \\ -1.33 & -1.07 & 0.10 \\ -0.74 & 0.07 & -0.88 \\ -1.36 & -0.47 & 0.18 \\ 0.17 & -0.09 & 0.09 \end{bmatrix}, \quad H_{4,80} = \begin{bmatrix} -1.85 & -1.47 & -0.74 \\ -1.80 & -2.22 & -0.71 \\ 0.34 & 0.63 & -1.20 \\ -0.68 & -0.59 & 0.038 \\ 0.27 & 0.11 & -0.10 \end{bmatrix} \tag{3.9} \]

on the DC gains and frequency response will be presented to complete the evaluation of the actuators control authority.

### 3.2.3 DC Gains and Frequency Response

The DC gains of the system from the three inputs to the performance variables of interest were derived at different operating conditions. The advantage of considering the DC gains first, over just analyzing Bode plots, is that, by comparing the entries of the matrices, it is straightforward to numerically compare the control authority of the different actuators. On the other hand, Bode plots help visualizing the control authority over the entire frequency range of the inputs, so they will be considered later. The results obtained for representative operating conditions 2 and 3, with 0% and 80% VGC opening (labeled as conditions 2, 0 – 2, 80 – 3, 0 – 3, 80), and operating condition 4 with 80% VGC opening (4, 80) are reported here. The DC gains of the system with inputs \((u_{VGT}, u_{EGR}, u_{VGC})\) and outputs \((p_{IM}, \dot{m}_c, SI, N_{lc}, \eta_c)\) for the five operating conditions mentioned before are shown in Equation (3.9).

A quick inspection of the matrices allows to confirm what highlighted in the step
response analysis. The VGT has a significant impact on all the performance variables and in particular it has the strongest control authority on the intake manifold pressure, and turbocharger speed. Both VGT and EGR play an important role for what the air mass flow is concerned, while the VGC has higher influence on the Surge Index and $\eta_c$.

From the matrices it is also possible to see how the VGT gains relative to the SI change sign. The gains from the EGR to the SI are sign indefinite as well, for instance, in operating conditions 2 and 3 with 20% VGC, they are negative. On the other hand, since increasing the VGC position causes the compressor maps to shift towards the left, the VGC gains relative to the SI are always negative. This extra
information provided by the DC-gains analysis suggests the use of VGC to ensure the compressor stability.

From the DC-gains it is possible to see how the control authority of the VGC with respect to the intake manifold pressure and air mass flow rate increases at high power operating condition. For instance, for a given VGC opening, the gains get larger going from operating condition 2 to 3 and from 3 to 4. This can be explained by considering the position of the operating conditions on the compressor map. The high power operating conditions are characterized by medium-high values of $p_{IM}$ and $\dot{m}_c$. As a consequence, the compressor map shift caused by a variation in the VGC opening, leads to a significant shift of the operating condition on the map, hence a substantial change of $p_{IM}$ and $\dot{m}_c$. On the other hand, for low power conditions, a shift of the compressor maps does not significantly change the position of the operating condition on the map and hence $p_{IM}$ and $\dot{m}_c$ are less sensitive to VGT variations.

The DC-gains also highlight how the control authority of the VGT and EGR with respect to the intake manifold pressure and air mass flow rate increase at either high power conditions or high VGC opening. While the first effect is known in the literature [16], the second can be explained considering that at high VGC positions the characteristic curves in the compressor maps become steeper, hence, the two variables become more sensitive to the VGC.

The Bode plots were generated using the linearized models around the operating conditions of interest (Table 3.1). Figure 3.8 shows the magnitude of the Bode plots from the inputs ($u_{VGT}, u_{EGR}, u_{VGC}$) to the outputs ($p_{IM}, \dot{m}_c, SI, N_{tc}, \eta_c$) for the operating conditions mentioned before. The plots confirm what was found in the DC-gains analysis and show how those results hold for a frequency range for the inputs that goes up to 1Hz. On the other hand, it is important to notice how the gains relative to the VGC actuator have a more flat behavior around the whole frequency
range and become comparable to those of the VGT and EGR at high frequencies. This suggests us the possibility of using the VGC to improve the system performance during transients.

The analysis conducted provides an insight on the static and dynamic behavior of the system, particularly highlighting the different control authority of the actuators and their different behavior in frequency. Based on these results, model-based optimization problem was set up to derive coordinated control for the three actuators, with results presented in the following section.

3.3 Nonlinear Steady-State Optimization

In the typical framework of engine control unit, the air-path control is commonly dealt with by first conducting a static optimization on the air-path system to determine desired targets for the measured outputs in steady-state conditions, namely \( y^* = [p^*_{IM}, \dot{m}^*_c] \), and then developing a feedback tracking control to regulate the performance variables to the desired targets in transient condition.

In this proposed engine scheme, the additional degree of freedom provided by the VGC leads to the over-actuation of the system, and affects directly the compressor flow rate and output power in the plant dynamics model (3.3). While the complexity of the system is inevitably increased, the over-actuation poses an added opportunity to optimize system performances. In particular, the analysis conducted above has highlighted the potential of the VGC to improve the stability and efficiency of the compressor at the expense of minor changes in the boost pressure, air mass flow rate and, indirectly, EGR ratio.

To this extent, the optimization is designed to derive a feed-forward control law, aiming at improving the fuel economy and the compressor stability while preserving the desired EGR ratio, by coordinating the VGC actuator with the EGR and VGT.
However, because of the non-convex nature of this specific optimization problem, an arbitrary initial condition will not guarantee to find the global optimal solution. As a result, an analysis based on a full factorial Design of Experiment (DoE) is initially performed at different operating conditions, to identify the trends and trade-offs in the system outputs for various positions of the three actuators. Furthermore, the analysis provides the initial condition to conduct a gradient-based optimization and identify the optimal solutions, which allows to derive the feed-forward maps for the VGT, EGR and VGC as a function of $N_{eng}$ and $\dot{m}_{fuel}$.

The analysis of the EGR-VGT-VGC system plays a crucial role reducing the complexity of running a full factorial DoE analysis in this study. In particular, because of the different control authority of the actuators, the optimization is run considering two steps: first the VGT and VGC positions are chosen to minimize the system BSFC and ensure system stability, secondly the EGR valve opening is chosen to keep the EGR ratio inside a desired range as shown in Figure 3.9.

The analysis is conducted with respect to the operating conditions considered inside the boxed region shown in Figure 3.6. Since the VGC is expected to provide fuel economy benefits at low torque and medium engine speed conditions, the operating points bounded by the grey box in Figure 3.6 are of higher interest. It is worth noting that the selected region is of key importance for urban drive cycle conditions.

The feed-forward control maps of the original engine air-path system were obtained after a long process of calibration and validation in a simulation environment,
Figure 3.10: Levels Considered for the VGT and VGC Positions in the Full Factorial DoE.

To leverage this effort and experience, the production maps are chosen as a starting point for the DoE analysis. The maps were then refined to optimize the system performance while making sure that all constraints are still satisfied. The DoE is chosen to be a $11 \times 11$ full factorial design with VGT and VGC as factors. In particular, the VGT levels are chosen to be within a $\pm 15\%$ variation from the corresponding values in the map of the original architecture in order not to waste too much exhaust energy and not over-speed the turbocharger. On the other hand, the VGC is allowed to vary in its entire feasible range [0\%, 100\%].

As a representative case, the full factorial design for the operating point at $N_{eng} = 1800 r/min$, $m_{fuel} = 5.5 g/s$ is shown in Figure 3.10, and the results of the analysis are shown in Figure 3.11a and Figure 3.11b. The convention used in these plots for the markers is the same adopted in Figure 3.10. From the plots it is possible to see that, for a fixed VGC opening and increasing VGT position, the energy transported from exhaust gas drops, resulting into a monotonic decrease of both the AFR and the intake manifold pressure. For a Diesel engine, a lower AFR bound is typically imposed to reduce the soot production. In this case, the minimum AFR requirement
Figure 3.11: Performance Variables Considered for the DoE Analysis.

actually limits the maximum VGT opening, while the boost pressure upper bound results into a limit on the minimum VGT opening allowed for a given VGC position.

On the other hand, for fixed VGT position, the AFR and intake manifold pressure are not monotonic with respect to the VGC. Both variables rise when increasing VGC position up to 80%, then drop for higher VGC positions. The BSFC is a non-monotonic function of both VGT and VGC, with the likely presence of a stationary (local minimum) point as evidenced in Figure 3.11a.

The analysis shows that, for fixed VGC position, there is a VGT position that minimizes the BSFC. At that condition, an increase in the VGC position from 0% to 80% results into a reduction of the BSFC by only 0.9%. This condition also satisfies all the constraints of the problem, and can be chosen as the initial condition for the optimization of the system.

The above analysis was repeated at each operating point belonging to the region
shown in Figure 3.6, and the combinations of VGT, EGR and VGC positions obtained from the analysis were used as an initial guess to solve the following nonlinear constrained optimization problem

\[
(u_{VGT}^*, u_{EGR}^*, u_{VGC}^*)^* = \arg\min_{(u_{VGT}, u_{VGC}, u_{EGR})} (BSFC) \\
\text{s.t.} \quad f(x, u, w) = 0 \\
\quad z_{EGR} = z_{EGR}^*, \quad AFR \geq AFR_p \\
\quad p_{IM} \leq \bar{p}_{IM}, \quad SI \leq SI \leq \overline{SI}
\]

where the constraints of the optimization problem are as follows:

1. system model at steady-state conditions. In particular, \(f(x, u, w)\) represents the right hand side of the nonlinear state equations shown in Equation (3.3), where \(x, u, w\) represent the system state, control input and exogenous signals.

2. EGR ratio at desired value for every operating condition. This constraint, corresponding to the inner loop of the block scheme shown in Figure 3.9, ensures that the resulting open-loop strategy does not lead to an increase in NO\(_x\) production.

3. steady-state in-cylinder AFR being greater than the minimum threshold \(AFR\), to limit soot formation. The value \(AFR\) is determined from experimental data and depends on the engine operating point.

4. intake manifold pressure \(p_{IM}\) limited below a threshold \(\bar{p}_{IM}\) to avoid over-boost [40]. In this study \(\bar{p}_{IM}\) is set to 220kPa.

5. Surge Index within a safe operating range \([SI, \overline{SI}]\), where \(SI > SI\) to ensure that the compressor is away from the choke limit and \(SI < \overline{SI}\) ensures that the compressor is not surging. For this case study \(SI = 5\%\) and \(\overline{SI} = 95\%\) for every operating condition.
The steady-state values of the AFR, BSFC, Surge Index (SI), Turbocharger Speed ($N_{tc}$) and Compressor Efficiency ($\eta_c$) for each operating point and actuator positions are derived using the nonlinear model shown in Equation (2.30), Equation (2.18) and Equation (2.24). The optimization problem (3.10), highly nonlinear with substantial sets of both equality and inequality constraints, is solved using the MATLAB solver “fmincon” with a sequential quadratic programming algorithm. The algorithm has relatively fast convergence speed and is capable to enforce the constraints at every iteration. Note that due to the complexity of the nonlinear constrained optimization (3.10), the choice of initial condition is critical for the solution to converge and to guarantee the optimality. Therefore, the results obtained from the full factorial DoE analysis, being sufficiently close to the optimal points, are used as the initial conditions to setup the optimization problem. Then, the optimal solution ($u_{VGT}^\star$, $u_{EGR}^\star$, $u_{VGC}^\star$) at each operating condition forms the feed-forward maps for the coordinated control of the air-path system actuators.

In Figure 3.12, the performance improvement and maps for VGT and VGC obtained from the optimization procedure are shown. It is worth mentioning that, while not shown here, the optimization results in a small variation of EGR valve opening at all points within the considered region. This confirms the initial assumption of the decoupled optimization strategy. In Figure 3.12 (b) and Figure 3.12 (c) it is possible to see the percentage improvement in the engine BSFC and compressor efficiency obtained with the new maps. In particular, engine BSFC shows improvement between 0.6% and 1%, while the compressor efficiency improvement is within 6%. Figure 3.12(d), on the other hand, shows the values of the SI before and after the system optimization. As shown in the figure, original compressor (fixed-geometry) operates very close to the surge line in the engine operating range considered for this study.
Figure 3.12: Comparison Between Baseline and Optimized Values for the VGT and VGC System, and Related Performance Outputs.
In this sense, the VGC actuator provides the opportunity to operate the compressor away from the surge line and close to the maximum efficiency region.

Finally the performance of the system with the new feed-forward maps is tested in transient conditions, considering an acceleration profile for a light-duty truck in trailer towing conditions. Note that the only experimental data available in this study were used to validate the developed model for steady state conditions and the FTP driving cycle. The data presented for the acceleration test have been generated using a vehicle drivetrain simulator based on MATLAB/Simulink. Figures 3.13a and 3.13b show the results of the simulation, with the main conclusions summarized as follows:

- an improvement in the compressor efficiency can be observed throughout the whole profile, leading to a marginal improvement in the engine fuel economy;
- a slight overall increase of the turbocharger speed and a reduction of the turbo

(a) Compressor Efficiency Vs Surge Index.  
(b) Turbo Speed Vs BSFC.

Figure 3.13: Performance of the Steady-State Optimization on the Acceleration Test.
lag during the gear shift transients can be observed. This is explained by the improved efficiency resulting from the optimized VGC actuation, which leads to a reduction of the compressor power demand and a more rapid changing rate of the shaft speed;

- a substantial reduction of the Surge Index is achieved while the compressor is operating close to the surge limit under the harsh condition of the vehicle test profile. This result is a consequence of the compressor efficiency optimization. Since the efficiency usually peaks away from the surge line, the optimization always results in the Surge Index far lower than the stability limit;

- a marginal improvement of the engine fuel consumption can be observed throughout the test, and particularly during the transients that immediately follow each gear shift.

The simulation results confirm the potential of a coordinated VGT-EGR-VCG control to improve the turbocharger performance and the compressor stability.

3.4 State and Parameter Estimation

After static feed-forward maps were derived based on the steady state optimization, a feedback controller should be designed to track the desired set-points during transient, while optimizing transient performance like efficiency and turbo lag. As a result, a candidate cost function for transient optimization can be selected as a combination of the compressor efficiency, the turbocharger speed and the surge index. Since some of the performance outputs like the stability constraints ($SI$) and optimization parameters ($\eta_c, N_{tc}$) are not measurable, an estimation algorithm is required. To this extent, two different estimator designs [105] will be presented here.
Open-loop Estimator Design

The first estimator design relies on an open-loop scheme (see Figure 3.14) that is based on the analytical inversion of J&K model shown in Equation (2.17a) for the prediction of the compressor flow and efficiency maps. In Figure 3.14, $p_{IM}$ and $\dot{m}_c$ represent the two measured outputs (intake manifold pressure and air mass flow), and $\hat{N}_{tc}$, $\hat{\eta}_c$ and $\hat{SI}$ are estimated turbocharger speed, compressor efficiency and surge index, respectively.

Since the model variables $\Phi$, $\Psi$, $Ma$ in Equation (2.17a) are explicit functions of the corrected mass flow rate, pressure ratio and compressor speed, the model (2.18) could be rewritten in such an implicit form as:

$$
\left[ k_{12}\alpha_1\alpha_3 N_{tc}^4 + (k_{11}\alpha_1 + k_{22}\alpha_2\dot{m}_c) N_{tc}^3 + \left[ k_{21}\dot{m}_c - k_{32}\alpha_1\alpha_2\alpha_3 \left( \frac{p_{IM}}{p_{amb}} \frac{\gamma-1}{\gamma-1} - 1 \right) \right] N_{tc}^2 + \ldots 
\right]
- k_{31}\alpha_1\alpha_2 \left( \frac{p_{IM}}{p_{amb}} \frac{\gamma-1}{\gamma-1} - 1 \right) N_{tc} + \alpha_2\dot{m}_c \left( \frac{p_{IM}}{p_{amb}} \frac{\gamma-1}{\gamma-1} - 1 \right) = 0
$$

(3.11)

where $\alpha_1$, $\alpha_2$, $\alpha_3$ are constant parameters, which are functions of the compressor geometric parameters

$$
\alpha_1 = \frac{p_{ref}\pi^2D_{tip}^3}{240RT_{ref}}, \quad \alpha_2 = \frac{7200c_pT_{ref}}{\pi^2D_{tip}^2}, \quad \alpha_3 = \frac{\pi D_{tip}}{60\sqrt{\gamma RT_{ref}}}
$$

(3.12)

Since the compressor flow map is invertible, by assigning a feasible region for the solution of $N_{tc}$ it is possible to determine the root of this polynomial equation, hence

Figure 3.14: Open Loop Estimator Block Diagram.
Figure 3.15: Estimation of Turbocharger Shaft Speed and Surge Index in the Operating Range of the Compressor Based on Non-causal Model Inversion (Case \( u_{VGC} = 100\% \)).

generating a map of the shaft speed with respect to the corrected mass flow and pressure ratio. The estimated shaft speed can then be implemented into Equation (2.24) to obtain an open-loop estimator of the surge index. Figure 3.15a shows the inverted compressor speed map as a function of pressure ratio and compressor corrected mass flow rate. The predicted surge index value is presented in Figure 3.15b, which shows that 100% surge line matches with the surge limit in the compressor operating range.

**Extended Kalman Filter Observer Design**

To add robustness with respect to uncertainties and sensor noise a closed-loop estimator based on an Extended Kalman Filter (EKF) observer was derived according to the scheme shown in Figure 3.16. In particular, given the discretized reduced-order
model (3.3) of the system

\[
x_k = f(x_{k-1}, u_{k-1}) + \omega_k \\
y_k = h(x_k) + \nu_k
\]

(3.13)

where \(x_k, u_k, y_k\) represent the system states, inputs and outputs respectively, and \(\omega_k \in \mathbb{R}^n\) denotes the process noise which captures uncertainties in the model, and \(\nu_k \in \mathbb{R}^m\) is measurement noise. Both are uncorrelated (white noise), zero-mean random sequences, and hold covariances of \(Q_k\) and \(R_k\), respectively. The implementation of Extended Kalman Filter estimation is commonly conceptualized as a predictor-corrector scheme [106, 107], based on the linearization of the functions appeared in (3.13) around the current estimate. In the predictor step, states from the previous time step are used to produce an estimate of the state at current time step

Predict: \[
\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_{k-1}) \\
P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^T + Q_{k-1}
\]

(3.14)

where \(\hat{x}_{k|k-1}\) and \(P_{k|k-1}\) are the predicted state and covariance estimates at time step \(k\) from the previous time step, respectively. The corrector step refines the predicted
state and covariance estimates with the current observation information.

Update: \[ \tilde{y}_k = y_k - h(\hat{x}_{k|k-1}) \]
\[ K_k = P_{k|k-1}H_k^T(H_kP_{k|k-1}H_k^T + R_k)^{-1} \]
\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\tilde{y}_k \]
\[ P_{k|k} = (I - K_kH_k)P_{k|k-1} \] (3.15)

where \( P_{k|k-1}, P_{k|k} \) represent the predicted and updated covariance estimate, respectively. Moreover, \( \tilde{y}_k \) is the measurement residual and \( K_k \) the an optimal Kalman gain. In particular, the Jacobian linearization of process and observation vector field

\[ F_{k-1} = \frac{\partial f}{\partial x} \bigg|_{\hat{x}_{k-1|k-1}, u_{k-1}} \quad H_k = \frac{\partial h}{\partial x} \bigg|_{\hat{x}_{k|k-1}} \] (3.16)

is applied to obtain the covariance and optimal Kalman gain at each predictor and corrector phase.

The two estimator designs were tested in simulation studies considering the mild acceleration profile in Figure 3.5. Figure 3.17 compares the turbocharger shaft speed, the surge index and compressor efficiency profiles generated by the complete Diesel engine model with the ones predicted by the estimators. As it can be observed, in absence of sensor noise, the open loop estimator performs well, and the Extended Kalman Filter is capable of predicting the turbocharger speed, the surge index and compressor efficiency profiles, despite the initial offset in the initial condition of the estimator.

Finally, the open-loop and closed-loop estimators were compared considering a case study in which the measurements were corrupted by noise (Figure 3.18(a)). In order to account for the possible sensor noise at the measured outputs, a digital low pass filter is introduced for the open loop estimator design. Savitzky and Golay [108] proposed a simplified least-squares-fit convolution for data smoothing. The idea of
Savitzky-Golay filtering can be understood as a weighted moving average filter with weighting given as a polynomial of a certain degree (2nd degree in this study). This polynomial is designed to preserve higher moments and to reduce the bias introduced by the filter. The weight coefficients (referred to Equation (3.17) as coefficients), when applied to a signal, perform a polynomial least-squares fit within the filter window. \( n_L = 5, n_R = 0 \) is set to realize a causal filter. At each time step, we least-squares fit a polynomial (quadratic) to all \( n_L + 1 \) points in the moving window, and then set \( g_i \) to be the value of that polynomial at position \( i \).

\[
g_i = \sum_{n=-n_L}^{n_R} c_n f_{i+n}
\]  \hspace{1cm} (3.17)

Figure 3.18 presents the estimation results for the proposed estimators, where the

**Figure 3.17:** (a) Open-loop Estimator Performance in Simulation. (b) Closed-Loop Estimator Performance in Simulation.
measurements are corrupted by noise. As shown in Figure 3.18, in the open-loop estimator, a small noise in the measured variables reflects into a good estimate of the turbocharger speed, but also noisy estimate of the Surge Index (up to ±20% error) and compressor efficiency, which might lead to false detection of compressor stability status. In particular, The RMS error for the Surge Index estimation is 3.8% for the open loop estimator and 1.2% for the Kalman filter observer. The implementation of digital filter enables the open loop estimation to reduce the noise, however, it also diminished the peak value during the fast transient, and moreover, it’s performance

Figure 3.18: (a) Example of System Outputs With Applied Noise. (b) Performance Comparison of Open-loop and Extended Kalman Filter Estimators In Case of Measurements Corrupted by Noise.
is easily affected by the noise. On the other hand, the closed-loop estimator shows high robustness with respect to the additive measurement noise.

### 3.5 Summary

Starting from the validated model of the engine air-path system in previous chapter, a reduced-order model was then developed and further linearized to facilitate system analysis and control design. The control authorities of the different actuators, as well as the dynamic property of the system were then analyzed. The DC gain and frequency analysis shows that VGT and EGR have significant control authorities on BSFC and EGR ratio respectively. On the other hand, the VGC was shown to have higher sensitivity on Surge Index. Furthermore, an interesting property of the EGR-VGT-VGC system was observed, in that the addition of the VGC actuation renders the system minimum phase with respect to the boost pressure and air mass flow rate outputs. This departure from the well-known EGR-VGT control problem was explained through a physical interpretation of the system dynamics.

Then, based on the results of the system analysis, a multivariable optimization was formulated and performed to derive a simple open-loop control strategy for the air-path system, with focus on improving the compressor efficiency and stability, the turbocharger response and ultimately the fuel economy. Simulation results for an acceleration test show the potential benefits to the turbocharger and engine performance, verifying that the adoption of VGC actuator allows for improving the fuel economy, compressor stability and turbo lag at part load conditions, while retaining peak performance at full load and enlarging the compressor operating range.

In the following, different parameter estimation methods were developed and evaluated in the study, aimed at estimating the trajectory of two important performance variables, namely the Surge Index and the compressor efficiency. The first estimator
design relied upon an open-loop scheme that was based on the analytical inversion of a grey-box model for the prediction of the compressor flow and efficiency maps. The second estimator was based on an Extended Kalman Filter observer developed using the reduced order model. The simulation results show that the two estimator designs perform equally well in absence of sensor noise. However, when the measurement is corrupted by noise, the Extended Kalman Filter shows a more robust performance.

The results of system analysis, optimization and estimation in this chapter will be largely adopted in the following chapters, which focus on the development of feedback control strategies to achieve better transient performance.
CHAPTER 4
MODEL PREDICTIVE CONTROL OF OVER-ACTUATED TURBOCHARGED DIESEL ENGINE

Despite the steady-state optimization provides reasonable results, the system can still react poorly in transient conditions. Moreover, the existence of disturbances may cause the actual operating conditions to considerably deviate from the desired set points. As a result, a feedback controller needs to be designed to achieve regulation of the performance variables to the desired targets.

In the standard EGR-VGT control problem for Diesel engines, the EGR valve and VGT actuator are used to control the boost pressure $p_{IM}$ and the air mass flow rate $\dot{m}_c$. A static optimization is firstly conducted on the air-path system to determine desired targets $y_r = [p_{IM}^*, \dot{m}_c^*]^T$ for the measured outputs in steady-state conditions. The desired targets are then scheduled, as shown in the block diagram representation in Figure 4.3, based on the engine speed and torque demand from the driver. A feedback controller is then designed to track the desired targets in transient conditions.

Since the VGC adds an extra degree of freedom to the control design problem, causing the system to be over-actuated, additional performance metrics can be considered for feedback control design, in addition to tracking the measured outputs. In particular, the analysis conducted above implies that the additional degree of freedom
provided by the VGC actuator can be primarily exploited to optimize the compressor performance variables during transient conditions.

While the use of the VGC brings improvements to the engine performance, it inevitably increases the complexity of the control design, due to the nonlinear dynamics of the VGT-EGR-VGC system, the presence of conflicting control objectives as well as the inclusion of additional constraints in the design [109]. To this extent, a model-based approach can be adopted to facilitate system level development, calibration and optimization in engine applications, due to the flexibility, limited tuning effort and reduced costing offered by this method [110].

Among various model-based control design techniques, Model Predictive Control (MPC) appears particularly indicated to handle constrained optimization problems in a systematic way for systems characterized by nonlinear dynamics. Moreover, advances in optimization theory and computational algorithms made MPC applicable to engine control applications in modern automotive ECUs [52–54], by improving the implementation of the constrained finite horizon optimization [111]. Recently, several contributions have been made in the application of MPC to the engine air-path control problem, for instance see [52,53,57,112].

To this extent, the study in this chapter proposes two different frameworks of model predictive control, to deal with the over-actuation for the Diesel air-path with VGC system. In particular, section 4.2 presents a conventional MPC tracking control, by defining turbo speed $N_{tc}$ as a third output to make the control system square. On the other hand, section 4.3 proposes an online optimization framework that coordinates the optimization of compressor transient performances with the MPC tracking of output variables. Various simulation results are presented in the study to verify the performances of the proposed control strategies.
4.1 Model Description for Control Development

In this study, the model predictive control is designed based on the linearized model (3.5) discussed in chapter 3. While the nonlinear MPC generally requires far more computational power than what is typically available on modern automotive ECUs [58], a possible solution to deal with the nonlinear dynamics in the engine system is to apply linear MPC techniques. In this case, the engine operating range must be divided into separate regions. In each region, a linear model is obtained to locally approximate the nonlinear dynamics, and a set of controllers is developed based on the linearized models. Then, the individual linear controllers are integrated together to the entire operating range using gain scheduling techniques [52,53].

To deal with the nonlinear nature of the engine dynamics, the engine operating range is divided into 16 regions, as shown in Figure 4.1, and multiple linearized models are obtained at different operating points centered at each region. Figure 4.2 shows

![Figure 4.1: The Partition of Engine Operating Range for Multiple Linearized Models.](image-url)
Figure 4.2: Piecewise-linear Model (PWL-Model) Validation Compared with the Reduced Order Nonlinear Model (NLIN-Model) During the Acceleration Simulation.

the comparison of performance variables between the piecewise linear model and the reduced order nonlinear model, considering the acceleration profile simulation. The set of linearized models captures well the nonlinear behavior of the system, hence it can be used with confidence as a tool for control design.

A set of local linear controllers is designed and switched online based on the engine speed and torque conditions. In practice, modeling error and unmeasured disturbances may directly affect closed-loop performance. Therefore, the linearized plant model (3.5) is augmented with a disturbance model [113, 114] at the output equations in order to capture the mismatch between (3.3) and (3.5). The overall
model considered for the MPC control design is discretized as,

\[
x(k + 1) = A_\varrho x(k) + B_\varrho u(k) + E_\varrho w(k)
\]
\[
d(k + 1) = d(k)
\]
\[
y(k) = C_\varrho x(k) + D_\varrho u(k) + d(k)
\]

where \( A_\varrho, B_\varrho, C_\varrho, D_\varrho, E_\varrho \) are the discretized system matrices, \( \varrho \in \{1, \cdots, 16\} \) represents the linear model in the associated partition region and \( d(k) \) is the added vector to account for the disturbance at the output for offset-free tracking.

### 4.2 Model Predictive Tracking Control Development

#### 4.2.1 Problem Statement

In the first case study, the turbocharger shaft speed \( N_{tc}^* \) is introduced as a third output, in addition to the typical controlled outputs \( \dot{m}_c^* \) and \( p_{IM}^* \), due to the over-actuation, thus making the system square. This additional objective aims at providing a faster turbocharger response, hence mitigating the turbo lag issues. The overall structure of the control is shown in Figure 4.3. Note that \( N_{tc} \) is typically not measured by production sensors. However, the nonlinear observer proposed in previous section provides an adequate estimation of such variable \( N_{tc} \) so that it can be used for control design.
4.2.2 Local Model Predictive Control Development

The MPC controller is then designed based on the linearized model (3.5) in each divided region, to achieve the feedback control of the air-path system while handling a set of states and inputs constraints. The overall model considered for the MPC control design is given by:

\[
\begin{bmatrix}
    x(k+1) \\
u(k+1) \\
w(k+1) \\
r(k+1) \\
d(k+1)
\end{bmatrix}
= \begin{bmatrix}
    A_{\varnothing} & B_{\varnothing} & P_{\varnothing} & 0 & 0 \\
    0 & I & 0 & 0 & 0 \\
    0 & 0 & I & 0 & 0 \\
    0 & 0 & 0 & I & 0 \\
    0 & 0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
u(k) \\
w(k) \\
r(k) \\
d(k)
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    I \\
    0 \\
    0 \\
    0
\end{bmatrix}
\begin{bmatrix}
    \Delta u(k) \\
\end{bmatrix}
\]

where the augmented state \( \tilde{x}(k) = [x(k), u(k), w(k), r(k), d(k)]^T \), the control \( \tilde{u}(k) = u(k+1) - u(k) \). For the implementation, the exogenous signal \( w \in \mathbb{R}^s \), the reference output variables \( r \in \mathbb{R}^m \) and the model error \( d \in \mathbb{R}^m \) are all assumed to be constant over the horizon and will be updated at each sampling time. A Kalman Filter observer is designed based on the augmented model to estimate the state \( x \) and the error \( d \) while suppressing the measurement noise. The Kalman Filter is based on the following equations:

\[
\begin{bmatrix}
    \dot{x}(k+1) \\
\dot{d}(k+1)
\end{bmatrix}
= \begin{bmatrix}
    A_{\varnothing} & 0 \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    \dot{x}(k) \\
\dot{d}(k)
\end{bmatrix}
+ \begin{bmatrix}
    B_{\varnothing} \\
    0
\end{bmatrix}
\begin{bmatrix}
    u(k) \\
w(k)
\end{bmatrix}
+ \begin{bmatrix}
    P_{\varnothing} \\
0
\end{bmatrix}
\begin{bmatrix}
    \Delta u(k) \\
\end{bmatrix}
\]

+ \begin{bmatrix}
    L_x \\
    L_d
\end{bmatrix}
\begin{bmatrix}
    y(k) - C_{\varnothing} \dot{x}(k) - D_{\varnothing} u(k) - C_d \dot{d}(k)
\end{bmatrix}
\]

\[
(4.3)
\]
where \( L_x, L_d \) are the optimal Kalman gains, \( C_d \) is the output disturbance matrix and \( v_x(k) \) and \( v_d(k) \) are the state and output measurement noises.

The tracking problem can be recast as a constrained minimization of the output errors. The constraints are considered as the physical limits of the actuator positions and the safe operation for the engine system, for instance, the maximum Surge Index. As a result, Equation (2.24) is linearized and expressed as a function of the augmented states, so that the limits on the SI could be included in the state constraints.

The performance index for the receding horizon optimization is considered as a quadratic cost function that is minimized at each sampling time \( k \):

\[
\min J(k) = \frac{1}{2} \sum_{i=1}^{H_e-1} \| \tilde{e}_{k+i} \|^2_Q + \frac{1}{2} \sum_{i=0}^{H_u-1} \| \tilde{u}_{k+i} \|^2_R + \frac{1}{2} \| \tilde{e}_{k+H_e} \|^2_P
\]

s.t. \( \tilde{x}(k + 1) = \tilde{A} \tilde{x}(k) + \tilde{B} \tilde{u}(k) \)

\[
\tilde{e}(k) = \tilde{C} \tilde{x}(k),
\]

\[
u_{min} \leq \tilde{u}_{k+i} \leq \nu_{max}, \quad i = 0, 1, ..., H_u - 1
\]

\[
x_{min} \leq \tilde{x}_{k+i} \leq x_{max}, \quad i = 0, 1, ..., H_e
\]

where \( \| v \|_M^2 = v^T M v \), \( \tilde{u}_{k+i}, i = 0, 1, \cdots, H_u - 1 \) are the optimization vectors and \( \tilde{e}_{k+i}, i = 0, 1, \cdots, H_e \) are the predicted errors. \( H_e \) and \( H_u \) denote the prediction horizon for tracking error and control inputs, respectively.

To reduce the computational complexity, \( H_e \) is usually chosen such that \( H_e > H_u \) and \( \Delta u(k) \equiv 0 \) for \( k = H_u, H_u + 1, ..., H_e \) [53]. The design parameters \( Q = Q^T, R = R^T \) are positive definite weighting matrices, and \( P = P^T \) is the terminal penalty matrix, which is normally designed to stabilize the system. The inequality equations represent the constraints. The optimization solved at each time step results in an optimal trajectory \( \tilde{u}_{k+i}^*, i = 0, 1, \cdots, H_u - 1 \), over the prediction horizon, while only the first part \( \tilde{u}_k^* \) of the optimal input signal is adopted into the controller.
4.2.3 Quadratic Programming

The MPC solves the optimization problem (4.13) at each time step by predicting $H_e$ steps ahead the evolution of state trajectory. Considering that the dynamic optimization (4.13) usually requires heavy computational load, the finite horizon optimization is reformulated as a static quadratic programming, for which an explicit solution can be derived and directly implemented [111]. By defining the vector:

$$U = [\tilde{u}_k^T, \tilde{u}_{k+1}^T, \cdots, \tilde{u}_{k+H_u-1}^T]^T$$ (4.5)

and substituting $\tilde{x}(k + i) = \tilde{A}^i\tilde{x}(k) + \sum_{j=0}^{i-1}\tilde{A}^j\tilde{B}\tilde{u}(k + i - j - 1)$, the system states within the prediction horizon can be represented as a function of initial state $\tilde{x}(k)$ and all the control vectors $U$.

$$X = \tilde{A}\tilde{x}_0 + \tilde{B}U, \quad E = \tilde{C}\tilde{A}\tilde{x}_0 + \tilde{C}\tilde{B}U$$ (4.6)

Where $\bar{A}, \bar{B}, \bar{C}$ can be easily derived from the system matrices:

$$\bar{A} = \begin{bmatrix} \tilde{A} \\ \tilde{A}^2 \\ \vdots \\ \tilde{A}^{H_u} \\ \tilde{A}^{H_e} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \tilde{B} & 0 & \cdots & 0 \\ \tilde{A}\tilde{B} & \tilde{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}^{H_u-1}\tilde{B} & \tilde{A}^{H_u-2}\tilde{B} & \cdots & \tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}^{H_e-1}\tilde{B} & \tilde{A}^{H_e-2}\tilde{B} & \cdots & \tilde{A}^{H_e-H_u}\tilde{B} \end{bmatrix}$$

$$\bar{C} = \text{diag} \begin{bmatrix} \tilde{C} & \tilde{C} & \cdots & \tilde{C} \end{bmatrix}_{H_e}$$

To integrate the cost function at each sampling time together, the weighting matrices should be rewritten in a compact form:

$$\bar{Q} = \text{diag} \begin{bmatrix} Q & Q & \cdots & Q \end{bmatrix}_{H_e}, \quad \bar{R} = \text{diag} \begin{bmatrix} R & R & \cdots & R \end{bmatrix}_{H_u}$$ (4.7)
Note that the terminal penalty for $\tilde{x}_{k+H_e}$ is different from the first $H_e - 1$ states, thus the last element in $\bar{Q}$ is represented by matrix $P$. As a result, the optimization can be written as a standard form of quadratic programming:

$$
\min_{U} \quad J(U, \xi_k) = \left\{ \frac{1}{2} U^T H U + \xi_k^T F U \right\} + \frac{1}{2} \xi_k^T Y \xi_k \\
\text{s.t.} \quad G U \leq W + E \xi_k
$$

(4.8)

The matrices $H, F, Y$ can be expressed as the form of:

$$
H = (\bar{C} \bar{B})^T \bar{Q} \bar{C} \bar{B} + \bar{R}, \quad F = (\bar{C} \bar{A})^T \bar{Q} \bar{C} \bar{B}, \quad Y = (\bar{C} \bar{A})^T \bar{Q} \bar{C} \bar{A}
$$

(4.9)

where $\xi_k = [\hat{x}(k), u(k-1), w(k), r(k), d(k)]^T$ represents the initial value of the augmented state vector at the sampling step and the matrices $G, W, E$ can be easily obtained from the plant model and the design matrices. At each sampling time step, the initial state vectors $\xi_k$ are obtained either from measurement or estimation, and fed into the optimizer as the initial value for the finite horizon prediction. Thus, the QP problem (4.15) can be solved iteratively online using search methods. The solution gives the optimal value $U^*$ over the horizon, and only the first vector $\Delta u^*(k)$ is implemented as the controller input.

### 4.2.4 Simulation Results

The overall control strategy was applied to the nonlinear Diesel engine air-path system model and verified for different transient operations. The feed-forward control and the reference maps for the performance variables, derived from the steady-state optimization, are scheduled as functions of engine speed and fuel flow rate. The feedback controller, designed starting from the linearized models to track the desired performance variables, was then integrated with the feed-forward control and implemented into the full-order nonlinear model.
The saturation limits for the actuators were considered as $u_{VGT} = [20\%, 100\%]$, $u_{EGR} = [0\%, 50\%]$ and $u_{VGC} = [30\%, 100\%]$. The sampling time was set as 20ms, and the prediction horizon chosen to be $H_u = 2, H_e = 8$, as a result of the trade-off between computational complexity and controller performance. The weighting matrices were tuned as $Q = \text{diag}(1, 15, 0.5)$, $R = \text{diag}(50, 50, 50)$ and $P = \text{diag}(80, 80, 80)$.

A first case study was conducted considering steady-state engine operating conditions, where the sequence of engine torque increases from 263Nm to 313Nm at 3s and falls back to 213Nm at 6s, and then the engine speed rises from 1500r/min to 1560r/min at 3s and drops to 1440r/min at 6s. A random white noise was added to the outputs ($p_{IM}$ and $\dot{m}_c$) to simulate the effects of measurement noise from the engine sensors.

The simulation results for the regulated output variables and actuator positions are shown in Figure 4.4a and Figure 4.4b, respectively. The figures compare the performance of the standalone feed-forward control, to the complete feed-forward and feedback (MPC) controller.

Figure 4.4a shows that the controlled outputs asymptotically converge to the desired set-points. Due to the interpolation error in the feed-forward control maps, the feed-forward control introduces a steady-state offset from the desired values at 3-6s and 6-9s, which is compensated by the feedback control. The performance of the closed-loop system indicates not only a fast response to step commands, but also that the controller is robust to both model uncertainties and measurement noise.

A second case study was then evaluated to provide results for a more complex transient scenario, specifically considering part of the acceleration profile shown in Figure 3.5. The simulation results are shown in Figure 4.5a and Figure 4.5b.

Note that the desired trajectory for the outputs that results from the steady-state reference maps, is systematically different from the model outputs in presence of the
feed-forward controller. This mismatch is due to the limitations of a control design based on static optimization. In particular, the desired references for turbocharger shaft speed and boost pressure are considerably lower than the results obtained by the feed-forward controller during the gear shift transients.

On the other hand, the feedback control is effective at tracking the desired boost pressure and turbocharger speed traces, resulting in a faster recovery after the gear shifts (specifically at 5s and 9s) and into a reduced turbo lag.

Figure 4.5b shows the actuator positions commanded by the feedback controller. In particular, the improved performance during the gear shift transients is mainly
achieved by nearly closing the EGR valve to increase the amount of flow that can be utilized by the turbine, hence providing a faster boost pressure recovery. On the other hand, closing the EGR valve during the load transient may lead to a significant reduction of the in-cylinder EGR fraction. This, in combination with the reduced boost pressure, may result into an increase of soot formation.

To this extent, Figure 4.6 shows the EGR ratio and the engine air-fuel ratio resulting from the feed-forward and feedback controls. Both controllers are able to
Figure 4.6: Performance of EGR Ratio and Air-fuel Ratio Under Gear Shifting Transients.

maintain the EGR ratio above a lower limit during the gear shift transient, with more evident excursions on the low side for the case of the feedback control.

The simulation results obtained for the second case study (harsh acceleration with gear shift) show a different system behavior. In particular, low EGR fraction is recirculated to retain torque performance and the soot formation can be considered more critical than the NO\textsubscript{x} formation. In absence of an emission model (not considered in this work), the engine air-fuel ratio could be used as an indicator for the formation of soot, particularly when the tip-in transient that follows the gear shifting event causes a sudden drop of the air mass flow rate. This phenomenon is evident in Figure 4.5a around 6s and 10s, corresponding to peaks in the vehicle acceleration rate.

As Figure 4.6 shows, the feedback controller is more effective in preventing the engine AFR from dropping during the tip-in transients. The inclusion of the turbocharger speed in the objective function of the MPC controller allows for the boost
pressure and air mass flow rate to recover faster than the feed-forward case, leading to a prompt response of the air-path system.

4.3 MPC-based Coordinated Performance Optimization

4.3.1 Problem Statement

Although the above MPC design in conventional way shows reasonable results, the solution is hardly appealing. Since the additional output trajectory of turbocharger shaft speed is derived from interpolating the steady state maps, the actual operating conditions can deviate considerably from the desired trajectory in transient conditions, despite the controller action. This is due to the characteristic stiff behavior of the system, which includes the fast dynamics of the manifold pressures and the slow dynamics of the turbocharger speed, The phenomenon, which has considerably detrimental effects on the operations of the system in transient conditions, leads to the so-called turbo lag issue.

Since the VGC affects directly the compressor flow rate and output power, the additional degree of freedom introduced by this actuator can be exploited in the control design to optimize the trajectory of the compressor operating point. In particular, for a given engine operating point specified by the pair \((p_{IM}^*, \dot{m}_c^*)\), by varying the VGC opening, the compressor characteristic maps are shifted, allowing to maximize the compressor efficiency. There are mainly two benefits in improving the compressor efficiency. First, an increased compressor efficiency is typically characterized by higher power absorbed by the compressor, which allows to further improve engine efficiency and quick turbocharger shaft speed recovery during transient. Secondly, it is important to notice that maximizing the compressor efficiency ensures stable operation of the compressor since the surge line is pushed away from the operating
Figure 4.7: Proposed Model Predictive Control Scheme for Air-path System.

point. This is due to the fact that the efficiency increases moving towards the center of compressor characteristic map, as shown in Figure 4.8.

As a result, on top of the traditional tracking of the output variables \((p_{IM}^*, m_c^*)\) problem, the extra degree of freedom provided by the VGC is exploited in this work to optimize the compressor efficiency. It is important to notice that the idea of pursuing a combined control objective, tracking and efficiency optimization, is supported by the system analysis shown in the previous section. In fact, although the VGC has relatively low control authority on the output variables \((p_{IM}, m_c)\), its control authority with respect to the compressor performance variables \((SI, N_{tc}, \eta_c)\) is high.

4.3.2 Steady State Characterization

To be able to optimize the compressor efficiency by shaping the characteristic maps, it is crucial to obtain a proper quantification of the peak efficiency region as a function of the inputs and/or state variables. To this extent, for each VGC opening, the operating point with maximum efficiency has been identified in the compressor map as shown in Figure 4.8. The curve that represents the maximum efficiency value for each given VGC, \(\eta_c^*(u_{VGC}) : [0, 100] \rightarrow \mathbb{R}\), is shown in Figure 4.9. As a result,
at each operating condition, the optimization of the compressor efficiency can be performed by selecting the VGC opening whose maximum efficiency point is closer to the current operating condition. Although this is an indirect way to optimize the compressor efficiency, the methodology (as shown later) is very straightforward to implement and allows, at the same time, to achieve results that are very close to the optimal solution.

It is important to notice that each maximum efficiency point, \( \eta^*_c(u_{VGCl}) \) for a given
$u_{VGC, i} \in [0, 100]$, identifies a $(\dot{m}_{corr,i}, \beta_i)$ pair in the compressor characteristic map. For example, the red dot in Figure 4.8, which corresponds to the maximum efficiency point for a VGC opening of 100%, $\eta^*_c(100) = 74\%$, identifies the pair $(\dot{m}_{corr,i}, \beta_i) = (0.17, 1.55)$.

As a result, two functions $\eta^*_c,\dot{m}_{corr}(u_{VGC}) : [0, 100] \rightarrow \mathbb{R}$ and $\eta^*_c,\beta(u_{VGC}) : [0, 100] \rightarrow \mathbb{R}$ can be defined so that

$$
\dot{m}_{corr,i} = \eta^*_c,\dot{m}_{corr}(u_{VGC,i}) \\
\beta_i = \eta^*_c,\beta(u_{VGC,i}).
$$

(4.10)

Figure 4.9: Values of Maximum Efficiency at Different VGC Positions.

Figure 4.10: Compressor Efficiency for the Desired Output $y_r$ at Operating Point 6 from Sweeping the VGC (Triangle) and Steady State Characterization of Minimum Distance (Star).
Those two functions are of particular interest because, given an operating point \((\dot{m}_{\text{corr}}^*, \beta^*)\), the VGC opening to achieve optimal efficiency, \(u_{VGC}^*\), is chosen by minimizing the Euclidean distance between the point \((\dot{m}_{\text{corr}}^*, \beta^*)\) and \((\eta_{c,m_{\text{corr}}}^*(u_{VGC}^*), \eta_{c,\beta}^*(u_{VGC}^*))\). That is, \(u_{VGC}^*\) is chosen to minimize
\[
\sqrt{(\dot{m}_{\text{corr}}^* - \eta_{c,m_{\text{corr}}}^*(u_{VGC}^*))^2 + (\beta^* - \eta_{c,\beta}^*(u_{VGC}^*))^2}
\] (4.11)

As an example of how the control approach operates in optimizing the compressor efficiency, Figure 4.8 and Figure 4.10 show the optimization results for the operating point 6 in Table 3.1. In Figure 4.8 the desired output variable \(y_r\) is represented as a red triangle, and the peak efficiency curve is represented by the black dashed line. It is possible to see that \((\eta_{c,m_{\text{corr}}}^*(70\%), \eta_{c,\beta}^*(70\%))\), represented as a blue star, is the point in the peak efficiency curve with minimum distance from \(y_r\). As a result, the control algorithm would choose \(u_{VGC}^* = 70\%\) as the optimal VGC opening.

It is important to notice how the choice of \(u_{VGC}^* = 70\%\) also ensures stable operation of the system. In Figure 4.8 it is in fact possible to see how \(y_r\) is far away from the surge line corresponding to \(u_{VGC}^* = 70\%\). On the other hand, the compressor would be surging at low openings of the VGC, where the efficiency is much lower. This, again, is due to the fact that the efficiency increases when moving towards the center of the compressor characteristic map.

In order to verify the effectiveness of the proposed methodology, the values of compressor efficiency for the given desired output \(y_r = [p_{IM}^*, \dot{m}_c^*]^T\), were also obtained by sweeping the VGC positions from 0% to 100% and are shown in Figure 4.10 (red triangles). As it can be easily seen, although the proposed methodology optimizes indirectly the compressor efficiency, the result obtained is very close to what would be the optimal solution.

In this study, a piecewise linear regression method is used to calibrate the two
functions $\eta_{c,m}^*(u_{VGC})$ and $\eta_{c,\beta}^*(u_{VGC})$. The results of the fitting are shown in Figure 4.11. Note that the corresponding values of the compressor outlet pressure and mass flow rate can be obtained based on the inlet conditions. In particular, let

$$y_{0,t}^*(u_{VGC}) := [p_{IM,0}, \dot{m}_{c,0}]^T = [p_a \cdot \eta_{c,\beta}^*(u_{VGC}), \sqrt{T_a/T_{ref}} \cdot \eta_{c,m}^*(u_{VGC})]^T$$  \hspace{1cm} (4.12)

where $T_{ref}(p_{ref})$ represents the reference compressor inlet temperature (pressure), and $T_a(p_a)$ represents the ambient temperature (pressure), respectively.

### 4.3.3 Coordinated Transient Optimization with MPC

The control problem, which considers desired output tracking and optimization of the compressor performance during transient, is cast as a constrained receding horizon...
optimization. The cost function to be minimized includes both the tracking error and the distance from the compressor operating point to the peak efficiency curve. As a result, the control algorithm aims at minimizing at each sampling time $k$:

$$
\min J(k) = \frac{1}{2} \sum_{i=1}^{H_e} \left\{ \| y_{i|k} - y_{ri|k} \|^2_Q + \| y_{i|k} - y_{rc}(u_{VG C}) \|^2_P \right\} + \frac{1}{2} \sum_{i=0}^{H_u-1} \| \Delta u_{i|k} \|^2_R 
$$  \hspace{1cm} (4.13a)

Subject to:

$$
x_{i+1|k} = A_e x_{i|k} + B_e u_{i|k} + E_e w_{i|k}, \quad x_{0|k} = \hat{x}(k)
$$  \hspace{1cm} (4.13b)

$$
d_{i+1|k} = d_{i|k}, \quad d_{0|k} = \hat{d}(k), \quad i = 0, ..., H_e - 1
$$

$$
y_{i|k} = C_e x_{i|k} + D_e u_{i|k} + d_{i|k}, \quad i = 0, ..., H_e
$$  \hspace{1cm} (4.13c)

$$
u_{i|k} = u_{i-1|k} + \Delta u_{i|k}, \quad i = 0, ..., H_u - 1
$$

$$
\Delta u_{i|k} \equiv 0, \quad i = H_u, ..., H_e - 1
$$  \hspace{1cm} (4.13d)

$$
w_{i+1|k} = w_{i|k}, \quad w_{0|k} = w(k), \quad i = 0, ..., H_e - 1
$$

$$
y_{ri+1|k} = y_{ri|k}, \quad y_{r0|k} = y_r(k), \quad i = 0, ..., H_e - 1
$$  \hspace{1cm} (4.13e)

$$
u_{\min} \leq u_{i|k} \leq u_{\max}, \quad i = 0, ..., H_e - 1
$$

$$
\Delta u_{\min} \leq \Delta u_{i|k} \leq \Delta u_{\max}, \quad i = 0, ..., H_u - 1
$$  \hspace{1cm} (4.13f)

where $\|v\|^2_M := v^T M v$ and $(\cdot)_{i|k}$ denotes the value of the vector at predicted step $i$ starting from current time $k$. $H_e$ and $H_u$ denote the prediction horizon for output and control inputs, respectively. Note that the second item in the cost function (4.13a) represents the minimization of the Euclidean norm from the compressor operating point to the peak efficiency curve. The weighting matrices $Q = Q^T$, $P = P^T$, $R = R^T > 0$ are tunable parameters. $\Delta u(k) := u(k+1) - u(k)$, defined as the changing rate of control inputs, is set as optimization variables. The inequality equations (4.13e)
and (4.13f) represent the system constraints. The constraints include the physical limits of the actuator positions and the safe operation for the engine system, for instance, the maximum boost pressure and turbocharger shaft speed. For the sake of simplicity, the exogenous signal $w$, reference output variables $y_r \in \mathbb{R}^r$ and model mismatches $d \in \mathbb{R}^d$ are assumed to be constant over the horizon and will be updated at each sampling time, as shown in (4.13d).

To reduce the computational complexity, $H_e$ is chosen such that $H_e > H_u$ and $\Delta u_{i|k} \equiv 0$ for $i = H_u, ..., H_e - 1$. The optimization solved at each time step results in an optimal trajectory $\Delta u_{i|k}^*$, $i = 0, ..., H_u - 1$, over the prediction horizon, and only the first part $\Delta u_{0|k}^*$ of the optimal input signal is adopted into the controller. In the standard MPC set-up, the control region $\varrho$ is not changed over the prediction region. As indicated in [53], if switches between the system dynamics over the horizon $H_e$ are considered, problem (4.13) becomes a mixed integer quadratic program (MIQP) program whose explicit solution requires more floating point operations for its evaluations and more memory for its storage.

Note that the Kalman Filter observer (4.3) is designed to estimate the unknown state $x$ and the mismatch $d$ based on the augmented model. Note that the Kalman Filter observer design relies on the linear models in different regions, therefore, 16 observers need to be designed. The observer is switched once the new MPC controller is activated. In order to smoothen the estimation during the switches between different models, the estimation results at last time step are fed into the new observer after switches are detected, to initialize the observer.

### 4.3.4 Softly Switched Mechanism

In order to make the linear MPC work over the entire engine operating range, different linearized models are used in each divided regions. The conventional way to design
the linear MPC controller is attained by switching the controllers every time the operating condition crosses the boundary of the region \([52, 53]\), which considers no intermediate transition. Although such hard switch is simple to be implemented, it may not give satisfactory switching performances because the control law switches instantaneously, possibly causing some impulsive behaviors, such as chattering, high overshoot and even actuator failure if the control objectives of the two MPC in 2 regions greatly differ \([115]\).

In this study, a softly switching mechanism \([115]\) is applied to make the switching process smooth. In particular, when the operating condition transitions to a different region, instead of shifting the control law abruptly, the optimization considers a convex combination of the old (before the switch) and new (after the switch) cost functions by using time varying weights. In this way, the switching process constitutes a finite number of small hard switches, and provides smooth performances during the transition process. The softly switching mechanism, hence, is achieved based on the redefinition of the cost function:

\[
\min J(k) = \frac{1}{2} \sum_{i=1}^{H_e} \alpha_{1,k} \left\{ \|y_{\varphi_1,i|k} - y_{\varphi_1|k}\|_Q^{2} + \|y_{\varphi_1,i|k} - y_{\eta_1}(u_{VGC})\|_P^{2} \right\} \\
+ \frac{1}{2} \sum_{i=1}^{H_e} \alpha_{2,k} \left\{ \|y_{\varphi_2,i|k} - r_{i|k}\|_Q^{2} + \|y_{\varphi_2,i|k} - y_{\eta_2}(x_{VGC})\|_P^{2} \right\} \\
+ \frac{1}{2} \sum_{i=1}^{H_u-1} \|\Delta u_{i|k}\|_{R_{\varphi_1} + \alpha_{2,k} R_{\varphi_2}}^{2} \tag{4.14a}
\]

Subject to:

\[
x_{\varphi_1,i+1|k} = \tilde{A}_{\varphi_1} x_{\varphi_1,i|k} + \tilde{B}_{\varphi_1} u_{i|k} + \tilde{E}_{\varphi_1} w_{i|k}, \\
d_{\varphi_1,i+1|k} = d_{\varphi_1,i|k}, \quad i = 0, ..., H_e - 1 \\
x_{\varphi_1,0|k} = \hat{x}_{\varphi_1}(k), \quad d_{\varphi_1,0|k} = \hat{d}_{\varphi_1}(k), \\
y_{\varphi_1,i|k} = \tilde{C}_{\varphi_1} x_{\varphi_1,i|k} + \tilde{D}_{\varphi_1} u_{i|k} + d_{\varphi_1,i|k}, \quad i = 0, ..., H_e \tag{4.14b}
\]
\[ x_{q_2,i+1|k} = \bar{A}_{q_2} x_{q_2,i|k} + \bar{B}_{q_2} u_i|k| + \bar{E}_{q_2} w_i|k|, \]
\[ d_{q_2,i+1|k} = d_{q_2,i|k|}, \quad i = 0, ..., H_e - 1 \]
\[ x_{q_2,0|k} = \hat{x}_{q_2}(k), \quad d_{q_2,0|k} = \hat{d}_{q_2}(k), \]
\[ y_{q_2,i|k} = \bar{C}_{q_2} x_{q_2,i|k} + \bar{D}_{q_2} u_i|k| + d_{q_2,i|k|}, \quad i = 0, ..., H_e \]
\[ x_{\min} \leq x_{q_1,i|k} \leq x_{\max}, \quad \hat{x}_{q_1}(k) \quad \hat{d}_{q_1}(k) \]
\[ x_{\min} \leq x_{q_2,i|k} \leq x_{\max}, \quad \hat{x}_{q_2}(k) \quad \hat{d}_{q_2}(k) \]

and the constraints (4.13c), (4.13d), (4.13e).

where \( \alpha_{2,k} = 1 - \alpha_{1,k} \) is a time-varying weighting factor, which is assumed to be constant over the prediction horizon and is updated at each sampling step. The switching mechanism begins when the engine operating condition crosses the boundary of the designed regions. During the switching process, the prediction is accomplished using both the old (w.r.t. region \( \rho_1 \)) and new (w.r.t. region \( \rho_2 \)) models simultaneously and the corresponding observed states \((\hat{x}_{q_1}(k), \hat{d}_{q_1}(k))\) and \((\hat{x}_{q_2}(k), \hat{d}_{q_2}(k))\) are used to initialize the prediction. The weighting value \( \alpha_{2,k} \) increases linearly from 0 to 1 along time axis during the switching process. When \( \alpha_{2,k} = 1 \), the combined MPC becomes the new MPC and the softly switching process terminates. The switching window is a tunable parameter, which determines how fast the switching process is transformed.

### 4.3.5 Quadratic Programming

The model predictive control solves the optimization problem (4.13) at each time step by predicting \( H_e \) steps ahead the evolution of state trajectory. By substituting

\[ x_{i|k} = \bar{A}^i x_{0|k} + \sum_{j=0}^{i-1} \bar{A}^j \bar{B} u_{i-j-1|k}, \]

the system states at any time step can be represented as a function of initial state \( x_{0|k} \) and all the control vectors \( u_{i|k}, \quad i = 0, ..., H_u - 1 \). Thus, the finite horizon optimization problem (4.14) can be reformulated as a quadratic
For the proposed switching mechanism, the problem (4.14) results into the sum of the two quadratic program problems, where the coefficient matrices will depend on the time-varying parameters $\alpha_{1,k}, \alpha_{2,k}$:

$$\min_U J(k) = \frac{1}{2} U^T [\alpha_{1,k} H_{\varrho_1} + \alpha_{2,k} H_{\varrho_2}] U$$

$$+ [\alpha_{1,k} \xi_{\varrho_1}^T(k) F_{\varrho_1} + \alpha_{2,k} \xi_{\varrho_2}^T(k) F_{\varrho_2}] U$$

Subject to:

$$GU \leq W + E \begin{bmatrix} \xi_{\varrho_1}(k) \\ \xi_{\varrho_2}(k) \end{bmatrix}$$

(4.15)

where $\xi_{\varrho}(k) = [\hat{x}_o(k), u(k-1), w(k), r(k), \hat{d}(k)]^T$ for $\varrho = \{1, 2\}$ represents the initial value of the augmented state vector that is obtained from measurement or estimation at the sampling $k$, and the matrices $H_{\varrho}, F_{\varrho}$ and $G, W, E$ can be easily obtained from the plant model and the design matrices $Q_{\varrho}, R_{\varrho}, P_{\varrho}$. The readers can refer to [111] for more details. The optimization variable $U := [\Delta u_{0|k}, \cdots, \Delta u_{H_u-1|k}]^T$ denotes the control inputs over the prediction horizon. Since $\alpha_{1,k} H_{\varrho_1} + \alpha_{2,k} H_{\varrho_2} > 0$, the optimization problem (4.15) is convex with respect to $U$. Thus, the QP problem (4.15) can be solved iteratively online using gradient-based search methods. The solution gives the optimal value $U^*$ over the horizon, and only the first vector $\Delta u_{0|k}^*$ is implemented as the controller input.

### 4.3.6 Simulation Results

The proposed control strategy has been tested in simulation using the full nonlinear model of the system. In this section, a case study that compares the performance of the MPC control algorithm with a Proportional-Integral-Derivative (PID) controller, similar to the one implemented in the production engine ECU, is presented to verify the effectiveness of the proposed design.
Simulation Setup

The nonlinear 5-state engine model developed and validated in [22] is used as the plant model to evaluate the proposed controller. For benchmarking purposes, a decentralized PID control was implemented and tuned based on [40], where the EGR valve and VGT positions are used to track the desired air mass flow rate and intake manifold pressure, respectively. In addition, a decoupled PID controller was designed for the VGC as an anti-surge control, to ensure that the compressor is operated within a certain percentage (10% in this study) away from the surge limit.

The simulations are performed in MATLAB/Simulink, where the quadratic programming in (4.15) is solved online using the “active-set” algorithm. The tuning parameters for the two different controllers are listed as follows:

1. Sampling time: $T_s = 0.02s$;
2. Prediction horizon: $H_u = 2$, $H_e = 10$;
3. Weight matrices: $Q = \text{diag}(\{100, 75\})$, $R = \text{diag}(\{2, 2, 1\})$, $P = \text{diag}(\{6, 6\})$;
4. Saturation limits: $u_{VGT} \in [20\%, 100\%], u_{EGR} \in [0\%, 100\%], u_{VGC} \in [0\%, 100\%]$;
5. Switching window: $T_w = 0.2s$;
6. PID parameters:

$$
\begin{align*}
K_{PVGT} &= 0.2, \quad &K_{IVGT} &= 0.15; \\
K_{PEGR} &= 0.045, \quad &K_{IEGR} &= 0.02; \\
K_{PVGC} &= 3, \quad &K_{IVGC} &= 2.
\end{align*}
$$

The value of the weighting factor $\alpha_{2,k}$ for the softly switching mechanism linearly increases from 0 to 1 once the switching signal is detected, and will be kept constant after the switching window $T_w$. 

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Results on an Acceleration Profile

A case study is presented here to compare the performance of the PID control with the proposed MPC controller. In particular, it is noted that at low to medium engine operating condition where the corresponding compressor operating points squeeze in the lower-left region, the optimization of distance simply saturates the VGC actuator at 100% to achieve high efficiency and rotational speed at the same time. Therefore, the proposed methodology will benefit more on medium to high engine speed and torque conditions where the corresponding compressor operating conditions spread out the operating range. Therefore, an harsh acceleration profile, shown in Figure 4.12, with several gear shifts was considered in this study. Moreover, measurement noise is not considered in this case study.

Figure 4.12: Exogenous Inputs of Fuel Flow Rate and Engine Speed for a Harsh Acceleration.
Note that the reference trajectory for the outputs that results from the steady-state reference maps in the “original” architecture, is maintained unchanged for both PID and MPC controllers. The objective of tracking the reference trajectory of air mass flow and boost pressure is conventionally defined [16] to indirectly control the air-fuel ratio and EGR fraction that allows to reduce the emissions. On the other hand, the new objective integrated in the MPC design to reshape the peak efficiency region of the compressor operating points, is mainly considered for the use of VGC system to improve the efficiency of the overall system and mitigate the turbo lag issues.

In Figure 4.13a, the output reference trajectories and the output trajectories for the PID and the Proposed MPC controller are compared. It can be seen that very

![Graph](image)

Figure 4.13: (a) Comparison of the Tracking Output Variables for the Case Study; (b) Comparison of the Engine Performance Variables for the Case Study.
little difference in the output tracking can be observed between the PID and MPC, with the exception that the MPC produces a slightly higher overshoot of intake manifold pressure between the 1st to 2nd gear shift at 1.5s. This occurs because for those operating conditions the VGC position is nearly saturated at 100%, as shown in Figure 4.14b. The pre-swirl imposed by the VGC actuator to the flow at the

Figure 4.14: (a) Comparison of Compressor Performance Variables for the Case Study; (b) VGT, EGR and VGC Positions.
impeller eye reduces the compressor work, thus increasing the rotational speed and the pressure ratio.

Figure 4.14a and Figure 4.13b compare the performance variables for the compressor ($\eta_c, N_{tc}$, $SI$) and the engine ($z_{EGR}$, $PMEP$), respectively, where $PMEP = p_{EM} - p_{IM}$ denotes the pumping losses. The MPC controller presents two clear advantages with respect to the PID baseline controller. First, it generally improves the compressor efficiency, as indicated in Figure 4.13b, while processing the same air flow rate $\dot{m}_c$ and maintaining the same boost pressure $p_{IM}$. The improved efficiency allows for reducing the power requirement of the compressor and, ultimately, the turbine work and the back pressure at the exhaust manifold. Note that the EGR fraction at the intake manifold is barely penalized, and confirmed in Figure 4.13b. Furthermore, the improved efficiency implies that the compressor operating points are moved away from the surge line, as shown in Figure 4.14a.

In addition to the efficiency improvement, the MPC controller leads to a significant improvement of the turbocharger transient response. Figure 4.14a shows that the loss in turbocharger speed consequent to a gear shift is remarkably reduced compared to the baseline controller, which allows for a faster recovery during the following acceleration. Considering the 2nd to 3rd gear shifting transient at 5s as an example, increasing the turbocharger shaft speed from $60 kr/min$ to $80 kr/min$ requires 1.9s for the PID controller, while only 1.2s for the MPC controller. Thus, the inclusion of the compressor efficiency in the objective function allows improving the turbocharger response during the gear shift transients. While the engine emissions and aftertreatment system have not considered in this study, it is well known that a faster response of the air-path system during engine load transients is beneficial to reducing particulate matter emissions.
Finally, Figure 4.14b, shows the actuator positions during the acceleration transient. The MPC algorithm generally commands higher VGC positions than the PID baseline, to maintain high compressor efficiency rather than to just prevent surge. Moreover, the VGT and EGR valve positions are also slightly increased from the baseline, to counteract the effect that the VGC brings to the system.

The slight reduction of the engine pumping losses due to the coordinated VGT-EGR-VGC control leads to a marginal improvement of the engine indicated efficiency, without penalizing the EGR ratio and ultimately the emissions. In addition, the improved transient response during the gear shift transients is mainly achieved by the coordination of VGC and EGR, namely increasing the VGC position to accelerate the turbocharger shaft and closing the EGR valve to increase the exhaust flow rate through the turbine, hence providing a faster boost pressure recovery.

4.4 Summary

Applications of Model Predictive Control to the air-path system are investigated in this chapter. In addition to the conventional MPC tracking control that relies on definition of an additional output variable, a different framework of MPC design is proposed to deal with the over-actuation in the system, which features a novel approach for performance optimization that improves the compressor efficiency and stability. To reduce the oscillations when the linear MPC controller is switched, a softly switching mechanism is investigated to achieve a smooth transition to different engine operating regions. This was achieved by combining the objective functions of the old and new local controllers with time varying weighting factors.

A simulation study was conducted considering a vehicle acceleration profile with
gear shifts, testing the transient response of the VGT-EGR-VGC system. The analysis conducted verifies the benefits of the variable geometry compressor and of the coordinated control strategy on the performance of the turbocharger and engine.

In particular, the controller optimizes the compressor operating point, avoiding the onset of surge during the rapid load transients occurring at each gear shift. Furthermore, a systematic improvement of the compressor efficiency is achieved, resulting into reduced pumping losses and improved engine indicated efficiency without penalizing the EGR ratio and engine emissions. Finally, the controller coordinates the three actuators during the gear shifts to reduce the turbo lag phenomenon. This is achieved by increasing the deviation of the inlet flow to the compressor (through the VGC actuator), to reduce the power consumption and accelerate the turbocharger shaft, and simultaneously closing the EGR valve to increase the exhaust flow rate through the turbine and providing a faster boost pressure recovery.
One solution to resolve the input redundancy, as performed in the previous chapter and of common use in the literature [43,61], is to define additional controlled outputs to make the system square. Nevertheless, this solution is hardly appealing, since it completely forgoes the flexibility to achieve a desirable transient performance (in this case, turbo lag minimization for drivability) through the enhanced configurations. Moreover, it requires installing additional sensors to measure the auxiliary output, and optimizing experimentally the corresponding reference targets, which is a costly and time consuming proposition for the automotive industry.

This chapter is devoted to specifically address these issues, by developing a systematic methodology for the control of over-actuated systems, to achieve simultaneous control and optimization for over-actuated systems applicable to engine air-path control. To start, the geometry of over-actuated systems is first exploited, leading to the distinction between strongly and weakly input redundant systems, formalized in geometric terms. While strong input redundancy in the input space yields identical state and output trajectory, weakly input-redundant systems enjoy the property that redundancy in the input space induces a redundancy in the state space, in the sense that a family of input/state trajectories exists, which are all compatible with a given output reference trajectory. As a result, the steady-state behavior of weakly
input-redundant systems can in principle be selected to reproduce a given reference trajectory while satisfying additional performance criteria, for instance optimizing a given cost function and fulfilling the constraints.

The concept of an inverse model allocation [116] is adopted in the framework of the linear output regulation problem, where the redundancy in the inverse system is characterized using geometric techniques, allowing dynamic allocation of state and input reference trajectory independent from the design of the stabilizer. In particular, this study focuses on the design of model predictive allocator to achieve constraint satisfaction and asymptotic evolution of the trajectories to a pre-computed steady-state target. The objective of our study is the analysis of the stability and feasibility properties of the proposed schemes and the characterization of suitable sufficient conditions for stability. The results in this chapter concerning the structure of over-actuated linear systems will serve as a theoretical foundation for the subsequent chapters.

The chapter is organized as follows: In section 5.1, we briefly overview the geometry of input redundant systems, followed by the definition of strong input redundancy and weak input redundancy. In section 5.2, the concept of inverse model allocation is presented in the framework of the linear output regulation problem, together with the salient aspects of the synthesis of the extended reference model that are important for the development of the results. The design of the model predictive allocation module is described in Section 5.3, whereas the stability and feasibility analysis under the assumption of perfect tracking is presented.
5.1 Overview of Linear Input Redundant Systems

With the general terminology of over-actuated systems, we denote control systems characterized by a larger number of independent control inputs than regulated outputs. To deal with system constraints in the framework of model predictive control, it is more convenient to adopt here a formulation in the discrete-time domain, while the original continuous-time formulation of the main results has been presented in [116].

Consider a discrete time LTI system described as

$$x(t+1) = Ax(t) + Bu(t), \quad z(t) = Cx(t)$$

(5.1)

with $t \in \mathbb{N}_+ := \{0, 1, \cdots \}$, state $x \in \mathcal{X} \cong \mathbb{R}^n$, control input $u \in \mathcal{U} \cong \mathbb{R}^m$ and regulated output $z \in \mathcal{Z} \cong \mathbb{R}^p$. System (5.1) is said to be over-actuated whenever $m > p$ and rank $B \geq p$.

The most apparent limitation of the state of the art in control allocation [64,67,70], particularly detrimental for our purposes, is the inability of dealing with systems (5.1) having vector fields satisfying the condition

$$\text{rank } B > p$$

(5.2)

where $p$ denotes the number of independent regulated outputs. This distinction was recognized early on by Zaccarian [82], where a taxonomy of over-actuated linear systems was proposed on the basis of the distinction between the null space of the control input matrix (yielding strong redundancy) versus the null space of the plant transfer function matrix at steady state, i.e., the multivariable DC gain of the plant model (yielding weak redundancy), as formalized in the next definition.

Definition 1.

- The system (5.1) is strongly input redundant from $u$ if it satisfies $\ker B \neq 0$;
• The system $(5.1)$ is weakly input redundant from $u$ to $y$ if $P := \lim_{z \to 1} (C(zI - A)^{-1}B)$ is a finite matrix which satisfies $\ker P \neq 0$.

In principle, over-actuated systems may consist of both types of redundancy according to Definition 1. To avoid trivialities and overlap, we henceforth focus on systems that are strongly input redundant when $m > \text{rank} B = p$, and systems that are weakly input redundant when $m = \text{rank} B > p$. Moreover, the following additional assumptions are made on system $(5.1)$:

**Assumption 1.**

1. The plant model $(5.1)$ is right-invertible, i.e., the output $z$ is functionally controllable from the input $u$;

2. The pair $(A, B)$ is stabilizable.

For the case of strongly input redundant systems, while the inverse of the system is not uniquely defined in the input space, the state trajectories of the inverse system are uniquely determined by the initial conditions, as shown in Figure 5.1a. The regulation

![Figure 5.1: Input Versus Input/State Redundancy for Over-actuated Systems.](image-url)
task is achieved by designing a control law that specifies a virtual control in a reduced-dimensional input space to produce the desired reference output. The distribution of the control effort among the redundant actuators is then accomplished by a static control allocation algorithm. In particular, the rank condition $m > \text{rank } B = p$ implies that $B$ has a non-empty nullspace of dimension $m - p$, in which $u$ can vary without affecting the state dynamics. This leads to the definition of conventional control allocation policy, realized by factorizing the input matrix $B/\ker B$. The geometric characterization of strongly input redundant systems is shown in Figure 5.2a. Let $\ker B$ denote the kernel of the input map $B$, and $P_u$ denote the canonical projection from $\mathcal{U}$ to the factor space $\mathcal{U}/\ker B$. By Theorem 1 (in Appendix) it is well known that there is a unique linear map $\bar{B}$ such that the diagram in Figure 5.2a commutes, i.e.,

$$B = \bar{B}P_u$$

where $P_u$ and $\bar{B}$ are epic and monic, respectively. This gives the alternative system

Figure 5.2: Commutative Diagrams for Over-actuated Systems.
\[ x(t + 1) = Ax(t) + \bar{B}v(t), \quad z(t) = Cx(t) \]
\[ v(t) = P_u u(t) \]

(5.4)

where \( v \in \mathbb{R}^p \) is interpreted as a *virtual control input*, that specifies the control effort, produced by a square and right-invertible “reduced-order” system, namely \((C, A, \bar{B})\).

Recall that as a result of right-invertibility, the *functional reproducibility problem* for (5.4) is solvable (uniquely) provided that the reference trajectory, \( z_{\text{ref}}(\cdot) \), is sufficiently smooth. This amounts in finding initial conditions \( x^0_{\text{ref}} \in \mathcal{X} \) and all input function \( v_{\text{ref}}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^p \) such that

\[ x_{\text{ref}}(t + 1) = Ax_{\text{ref}}(t) + \bar{B}v_{\text{ref}}(t), \quad x_{\text{ref}}(0) = x^0_{\text{ref}} \]
\[ z_{\text{ref}}(t) = Cx_{\text{ref}}(t) \]

(5.5)

Since the pair \((C, A, \bar{B})\) is square, it follows from the diagram in Figure 5.2a that the trajectories of the inverse system for the “reduced-order” system (5.5) are uniquely determined by the selection of the initial condition \( x^0_{\text{ref}} \). Then, the possibility of control allocation in strongly input redundant system lies in assignment of the trajectories of the actuator, \( u(t) \), that give the desired control \( v_{\text{ref}}(t) \). This can be typically achieved through the left inverse of the static map \( P_u \), which is not unique, since \( \emptyset \neq \ker B \subset \ker P_u \).

On the contrary, if the plant is weakly input redundant, the trajectories of the inverse system are not uniquely determined by the initial conditions, hence there exist *independently controllable state trajectories* (obtained from different inputs) that are all compatible with a given output reference. Consequently, any control allocation policy will inevitably interact with the system states and affect the system dynamics.

This concept is illustrated in Figure 5.1b, showing the fundamental difference between
the two classes of systems: In strongly input redundant systems, the trajectory \( x_{\text{ref}}(t) \) corresponding to \( z_{\text{ref}}(t) \) remains fixed, in spite of redundancy in the selection of \( u_{\text{ref}}(t) \), whereas in weakly input redundant systems the steady-state trajectories \( x_{\text{ref}}(t) \) and \( u_{\text{ref}}(t) \) are both nonunique.

The characterization of input/state redundancy for weakly input redundant system can also be accomplished in geometric terms, as shown in Figure 5.2b [116]. Let \( \mathcal{R}^* \) denote the maximal controllability subspace contained in \( \ker C \), and let \( \mathbb{F}(\mathcal{R}^*) \) be the set of friends of \( \mathcal{R}^* \) (see Definition 7 in Appendix). Let \( \rho := \dim \mathcal{R}^* > 0 \), which holds due to weak input redundancy (see Theorem 3 in Appendix). Define \( \mathcal{V} := B^{-1} \mathcal{R}^* \); it is known that \( \dim \mathcal{V} = m - p \) (see Theorem 3 in Appendix). Accordingly, define \( A_F := A + BF \), \( \Phi := A_F|\mathcal{R}^* \) as the domain restriction of \( A_F \) to \( \mathcal{R}^* \), and \( \bar{A}_F \) as the induced map on \( \mathcal{X}/\mathcal{R}^* \) by \( A \). Let \( \bar{B}_F : \mathcal{V} \rightarrow \mathcal{X} \) denote the domain restriction of the mapping \( B \) to \( \mathcal{V} \), \( T : \mathcal{V} \rightarrow \mathcal{U} \) be the insertion map of \( \mathcal{V} \) to \( \mathcal{U} \), \( \Sigma : \mathcal{R}^* \rightarrow \mathcal{X} \) denote the insertion map of \( \mathcal{R}^* \) to \( \mathcal{X} \), and \( \Xi : \mathcal{V} \rightarrow \mathcal{R}^* \) be the codomain restriction of the mapping \( \bar{B}_F \) to \( \mathcal{R}^* \). It is known that such \( \Xi \) exists because \( \text{im} \bar{B}_F \subset \mathcal{R}^* \) and \( \text{im} \Sigma = \mathcal{R}^* \). Furthermore, let \( P_\gamma \) denote the canonical projection of \( \mathcal{U} \) to \( \mathcal{U}/\mathcal{V} \), and \( P_\pi \) denote the canonical projection of \( \mathcal{X} \) to \( \mathcal{X}/\mathcal{R}^* \) respectively, as shown in Figure 5.2b. Let \( \bar{A}_F \) be the induced map of \( A_F \) on \( \mathcal{X}/\mathcal{R}^* \), and let \( \bar{B}_{\|} := P_\pi B \) and \( \bar{B} \) such that \( \bar{B}_{\|} = \bar{B} P_\gamma \). Such \( \bar{B} \) exists since \( \mathcal{V} \subset \ker \bar{B}_{\|} \). As a result, there exist two “reduced-order” systems: the triplet \((\bar{C}, \bar{A}_F, \bar{B})\) is a square and right-invertible system that uniquely solves the regulation problem within the factor space \( \mathcal{X}/\mathcal{R}^* \); the triplet \((0, \Phi, \Xi)\) generates the redundant trajectories evolving on \( \mathcal{R}^* \) that do not affect the output trajectories.

To translate this geometric characterization in coordinates, let \( \mathcal{X} := \mathcal{R}^* \oplus \mathcal{I} \) and \( \mathcal{U} := \mathcal{V} \oplus \mathcal{V}_c \), where \( \mathcal{I}, \mathcal{V}_c \) are arbitrary complementary subspaces of \( \mathcal{R}^* \) and \( \mathcal{V} \), respectively. Let \( T \in \mathbb{R}^{n \times n} \) and \( G \in \mathbb{R}^{m \times m} \) be matrices whose columns yield
corresponding basis for $\mathcal{X}$ and $\mathcal{U}$, adapted to $\mathcal{R}^*$ and $\mathcal{T}$, and $\mathcal{V}$ and $\mathcal{V}_c$, respectively. Applying the regular feedback transformation $u = Fx + G\bar{u}$, and the coordinate change $\bar{x} = T^{-1}x$, one obtains the alternative system description

$$\bar{x}(t + 1) = A_T\bar{x}(t) + B_G\bar{u}(t), \quad z(t) = C_T\bar{x}(t)$$  (5.6)

where $A_T = T^{-1}(A + BF)T$, $B_G = T^{-1}BG$, $C_T = CT$, having the matrix representation

$$A_T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B_G = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \quad C_T = \begin{bmatrix} 0 & C_2 \end{bmatrix}$$  (5.7)

It is known that $(A_{11}, B_{11})$ is controllable [Ch.5 [117]], and $(A_{22}, B_{22})$ is stabilizable by Assumption 1. Defining $\bar{x} := (\bar{x}_1, \bar{x}_2)$ and $\bar{u} := (\bar{u}_1, \bar{u}_2)$, one obtains the representation of the two “reduced-order” system depicted in Figure 5.2b as follows

$$\bar{x}_1(t + 1) = A_{11}\bar{x}_1(t) + A_{12}\bar{x}_2(t) + B_{11}\bar{u}_1(t) + B_{12}\bar{u}_2(t)$$
$$\bar{x}_2(t + 1) = A_{22}\bar{x}_2(t) + B_{22}\bar{u}_2(t)$$
$$z(t) = C_2\bar{x}_2(t)$$  (5.8)

It is clear now that the input/state redundancy corresponds to the forced trajectories of $(\bar{x}_1, \bar{u}_1)$ that evolves on $\mathcal{R}^*$, and that the output trajectory is assigned irrespectively of the selection of $\bar{x}_1(0), \bar{u}_1(\cdot) \in \mathcal{V}$ and $\bar{x}_1(\cdot) \in \mathcal{R}^*$. Consequently, the functional reproducibility problem for weakly input redundant system resorts to finding initial conditions $\bar{x}_{2,\text{ref}}^0 \in \mathcal{T}$ and input functions $\bar{u}_{2,\text{ref}}(\cdot): \mathbb{R} \to \mathcal{V}_c$ such that

$$\bar{x}_{2,\text{ref}}(t + 1) = A_{22}\bar{x}_{2,\text{ref}}(t) + B_{22}\bar{u}_{2,\text{ref}}(t), \quad \bar{x}_{2,\text{ref}}(0) = \bar{x}_{2,\text{ref}}^0$$
$$z_{\text{ref}}(t) = C_2\bar{x}_{2,\text{ref}}(t)$$  (5.9)

Since the triplet $(C_2, A_{22}, B_{22})$ is square and right-invertible, it follows from the diagram in Figure 5.2b that the trajectories of the inverse system in (5.9) are uniquely determined by the selection of the initial condition $\bar{x}_{2,\text{ref}}^0$. Moreover, the invariant
factors of the “reduced-order” system \((C_2, A_{22}, B_{22})\) are preserved from those of the original system \((C, A, B)\) (see Theorem 4 in Appendix). The possibility of performing control allocation for weakly input redundant system is left to the assignment of the controlled trajectories \(\bar{x}_1(\cdot)\) via the choice of \(\bar{x}_1(0)\) and \(\bar{u}_1(\cdot)\), which produces identical output trajectory.

To understand how the system decoupling in (5.8) could be utilized to achieve better performance for weakly input redundant systems, the geometric characterization of input/state redundancy has then been exploited in the framework of output regulation problem, leading to the definition of an inverse model allocation that will be specialized in the next section.

5.2 Principle of Inverse Model Allocation for Linear Output Regulation

Consider the following linear full-information version of the discrete linear output regulation problem

\[
\begin{align*}
    w(t + 1) &= Sw(t) \\
    x(t + 1) &= Ax(t) + Bu(t) + B_e w(t) \\
    y(t) &= Cx(t), \quad z(t) = Dx(t) \\
    r(t) &= C_e w(t), \quad e(t) = y(t) - r(t) \tag{5.10}
\end{align*}
\]

with state space of the exosystem \(\mathcal{W} \cong \mathbb{R}^q\), state space of the plant \(\mathcal{X} \cong \mathbb{R}^n\), input space \(\mathcal{U} \cong \mathbb{R}^m\), regulated output space \(\mathcal{Y} \cong \mathbb{R}^p\), and auxiliary performance output space \(\mathcal{Z} \cong \mathbb{R}^s\). The vector \((w, x) \in \mathcal{W} \oplus \mathcal{X}\) collects the signals available for feedback. Without substantial loss of generality, this study focuses on weakly input redundant systems. Accordingly, it is assumed that \(m > p\) and, rank \(B = m\),
rank $C = p$ to avoid trivialities, unnecessary complications and overlap with previous results. Moreover, in addition to Assumption 1, we make the following simplifying, yet reasonable assumption:

**Assumption 2.**

1. The matrix $S$ is semi-simple, with $\text{spec } S \cap \mathbb{C}^+ = \emptyset$, where $\mathbb{C}^+ := \{\lambda \in \mathbb{C} : |\lambda| > 1\}$.

2. The set of invariant zeros of the triplet $(C, A, B)$ and the spectrum of $S$ are disjoint;

It is well known [118, 119] that the regulator problem is solvable if and only if there exist mappings $\Pi : \mathcal{W} \rightarrow \mathcal{X}$, $\Gamma : \mathcal{W} \rightarrow \mathcal{U}$ satisfying Francis Equations

$$
\Pi S = A\Pi + B\Gamma + B_e,
$$
$$
C\Pi = C_e
$$

(5.11)

Solvability of (5.11) is guaranteed by the Assumption that the set of invariant zeros of the triplet $(C, A, B)$ and the spectrum of $S$ are disjoint. The pair $(\Pi, \Gamma)$ yields a controlled invariant subspace $S := \{(x, w) : x = \Pi w\}$ for the augmented system (5.10), which is invariant under the control law $u = \Gamma w$, and where the regulated error vanishes identically [119].

The full-information output regulation problem amounts to the design of a controller, that processes the signal $(w(t), x(t))$ and produces a control $u(t)$ providing internal stability and asymptotic error regulation. In the standard framework of
output regulation, such a controller is comprised of a servomechanism and a stabilizer [120,121]. A dynamic (or static) state-feedback stabilizer is typically given in the form

\[
\begin{align*}
x_c(t+1) &= A_c x_c(t) + B_c u_c(t) \\
y_c(t) &= C_c x_c(t) + D_c u_c(t)
\end{align*}
\tag{5.12}
\]

with state \(x_c \in \mathcal{X}_c \cong \mathbb{R}^{n_c}\), input \(u_c \in \mathcal{U}_c \cong \mathbb{R}^n\) and output \(y_c \in \mathcal{Y}_c \cong \mathbb{R}^m\), such that the closed-loop matrix

\[
A_a = \begin{bmatrix} A_c & B_c \\ BC_c & A + BD_c \end{bmatrix}
\tag{5.13}
\]

resulting by setting \(u_c = x\) and \(u = y_c\) satisfies \(\text{spec } A_a \subset \mathbb{C}^-\), where \(\mathbb{C}^- := \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}\). The design of (5.12) can be accomplished, for instance, via an optimal \(\mathbb{H}_2\) or \(\mathbb{H}_\infty\) design for the triplet \(((C; D), A, B)\), thus, it is out of the scope of this research, and assumed to be given a priori.

Provided that the stabilizer is pre-designed, the regulation problem obviously reduces to the selection of the servomechanism. In the basic setup of regulator synthesis, the servomechanism is merely an inverse of the plant model. For over-actuated systems, the concept of inverse model allocation has been proposed in [116], with the objective of treating systematically the redundancy in the inverse model and the capability of optimizing certain performance criteria. To shape the reference trajectory in the state and input spaces while preserving output invariance, we resort to the method introduced in [116], which considers a dynamical system that realizes an inverse of the plant, termed as an extended reference model.

**Definition 2.** A dynamical system of the form

\[
\begin{align*}
w(t+1) &= Sw(t) \\
x_r(t) &= \Pi w(t) + \Sigma \xi(t) \\
\xi(t+1) &= \Phi \xi(t) + \Xi \nu(t) \\
u_r(t) &= \Gamma w(t) + \Psi \xi(t) + \Upsilon \nu(t)
\end{align*}
\tag{5.14}
\]
with additional state $\xi \in \mathbb{R}^{n_{\xi}}$ and additional input $v \in \mathbb{R}^{n_{v}}$, is said to be an extended reference model for the system (5.10) if, for any initial condition $(w(0), \xi(0)) = (w_0, \xi_0)$ and any $v(\cdot) \in L_{\infty}$ the signals $x_r(\cdot), u_r(\cdot)$ are bounded and satisfy

$$x_r(t + 1) = Ax_r(t) + Bu_r(t) + B_e w(t),$$

$$r(t) = C x^r(t)$$

(5.15)

where $w(t) = S^t w(0)$.

The geometric characterization of weak input redundancy presented in the previous section leads directly to an algorithm for the synthesis of the extended reference model.

**Proposition 1.** [116]

Let $(\Pi, \Gamma)$ denote an arbitrary solution of the regulator equation (5.11), which exists due to the standing assumptions [119]. Let $\mathcal{R}^* \subset \mathcal{X}$ denote the maximal controllability subspace contained in $\ker C$, and let $\mathcal{F}(\mathcal{R}^*)$ be the set of friends of $\mathcal{R}^*$

---

1In fact, it has been shown [84] that by the assumption of weak input redundancy, the regulator equation has infinitely many solutions $(\Pi, \Gamma)$.
(see Definition 7 in Appendix). Select, arbitrarily, a symmetric set $\Omega_\rho \subset \mathbb{C}^-$ of complex numbers, and let $F \in \mathbb{F}(R^*)$ be such that $\text{spec} A_F|\mathbb{R}^* = \Omega_\rho$, where $A_F := A + BF$. Moreover, let $\Upsilon, \Xi, \Sigma$ be defined as shown in the commutative diagram in Figure 5.2b, let $\Psi : R^* \to \mathcal{U}$ be defined as $\Psi = F \Sigma$, and let $\Phi := A_F|\mathbb{R}^*$. With this assignment, system (5.14) is an extended reference model for (5.10), with $X_{\xi} \cong R^*$ and $\mathcal{U}_c \cong \mathcal{V}$.

**Proof.** Following the commutative diagram in Figure 5.2b, one obtains that

\[
x_r(t + 1) = HSw(t) + \Sigma \Phi \xi(t) + \Sigma \Xi v(t)
\]
\[
= AHw(t) + B\Gamma w(t) + B_\epsilon w(t) + A_F \Sigma \xi(t) + B\Upsilon v(t)
\]
\[
= A(Hw(t) + \Sigma \xi(t)) + B(\Gamma w(t) + F \Sigma \xi(t) + \Upsilon v(t)) + B_\epsilon w(t)
\]
\[
= Ax_r(t) + Bu_r(t) + B_\epsilon w(t)
\]

(5.16)

where $Cx_r(t) = CHw(t) = C_\epsilon w(t)$. Boundedness of $(x_r(\cdot), u_r(\cdot))$ for all $w(0) \in \mathcal{W}$ and $v(\cdot) \in L_\infty$ follows from Assumption 2 and by internal stability of the $\xi-$dynamics.

The reason behind the adopted terminology lies in the fact that system (5.14) allows for redundancy in the selection of the reference trajectory $(x_r(\cdot), u_r(\cdot))$ that generates the desired output $C_\epsilon w(\cdot)$ for (5.10) irrespectively of the controlled trajectory $(\xi(\cdot), v(\cdot))$. On-line optimization of cost functionals can thus be accomplished by using continuous gradient-based or piecewise constant selection in a receding-horizon framework of the update law for $\xi(\cdot)$ or the control $v(\cdot)$, without modification of the desired output trajectory. The manipulation of the free trajectory $(\xi(\cdot), v(\cdot))$ will be performed in the sequel by means of receding-horizon optimization.

With the stabilizer (5.12) designed at hand, the design of a full-information regulator is accomplished by letting $u_c = x - x_r$ and $u = y_c + u_r$ (see Figure 5.3). As a
matter of fact, changing coordinates as $x \mapsto \tilde{x} := x - x_r$ one obtains the closed-loop system

\[
\begin{align*}
\xi(t+1) &= \Phi \xi(t) + \Xi v(t) \\
w(t+1) &= Sw(t) \\
x_c(t+1) &= A_c x_c(t) + B_c \tilde{x}(t) \\
\tilde{x}(t+1) &= B C_c x_c(t) + (A + B D_c) \tilde{x}(t) \\
e(t) &= C \tilde{x}(t)
\end{align*}
\] (5.17)

which – by inspection – satisfies the control objectives of asymptotic regulation of $e(t)$ and boundedness of all state trajectories for bounded $v(t)$. Note that the trajectory $\tilde{x}(\cdot)$ depends only on the initial conditions $\tilde{x}(0) = x(0) - \Pi w(0) + \Sigma \xi(0)$ and $x_c(0)$, hence the external input $v$ does not have any influence on the error trajectory, $e(\cdot)$. This is easily seen from the following statement

\[
C x(t) = C \tilde{x}(t) + C x_r(t)
= \begin{pmatrix} 0 & C \end{pmatrix} A_d^t \begin{pmatrix} x_c(0) & \tilde{x}(0) \end{pmatrix}^T + C e S^t w(0)
\] (5.18)

On the other hand, the actual state trajectory $x(\cdot)$ can be modified by the free input $v(\cdot)$ via the assignment of $x_r(\cdot)$. Consequently, the extended reference model (5.14) explicitly decouples the controller for dynamic allocation of the trajectories of the inverse model of the plant, from the controller used to steer the trajectories of the closed loop system to the desired output.
5.2.1 A Numerical Example

To demonstrate the previous findings with an illustrative example, consider a non-square LTI system given by

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

\[
S = \begin{bmatrix} 0.99 & 0.16 \\ -0.16 & 0.99 \end{bmatrix}, \quad B_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_e = \begin{bmatrix} 1 & 0 \end{bmatrix}
\] (5.19)

where \( m = \text{rank} \ B > p \), hence, the system is weakly input redundant. A stabilizing control is first designed such that the eigenvalues of the closed-loop matrix \((A + BK)\) are assigned at \(0.9 \pm 0.1i\). As explained in previous section, \( F \in \mathcal{F}(\mathbb{R}^*) \) is chosen that places the eigenvalue of \( A_F|\mathbb{R}^* \) at 0.75. Moreover, a particular solution of the regulator equation (5.11) is given as

\[
\Pi = \begin{bmatrix} 1 & 0 \\ -0.57 & 0.73 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -0.52 & -0.04 \\ 0.15 & -1.08 \end{bmatrix}
\] (5.20)

Figure 5.4 shows the system trajectories by applying Francis and Wonham’s regulator [119]. As it can be seen, the solution of the regulator equation (5.11) \( x_r(t) = \Pi w(t), u_r(t) = \Pi w(t) \) constructs an invariant subspace (black solid-lines) that corresponds to the desired output. On the other hand, the stabilizer forces the system to converge to the invariant subspace, and hence, the error vanishes asymptotically, as shown in Figure 5.5.

Figure 5.6 shows the results by applying the extended servomechanism. Likewise, the black solid-lines represent the particular solution \( x_r(t) = \Pi w(t), u_r(t) = \Pi w(t) \). However, different from Figure 5.4, two additional reference trajectories (dashed-lines) are also obtained, by assigning \( \xi(0) = 0 \) and two bounded inputs \( v(t) \) for the extended reference model (2), i.e. \( v(t) = 3 + \sin(t) \) (green dashed-lines) and...
Figure 5.4: Reference/Actual State (Left) and Input (Right) Trajectories of Francis-Wonham’s Regulator for the Numerical Example.

Figure 5.5: Error Trajectories of Francis-Wonham’s Regulator for the Numerical Example.
Figure 5.6: Reference State (Left) and Input (Right) Trajectories of the Extended Servomechanism for the Numerical Example.

Figure 5.7: Error Trajectories of the Extended Servomechanism for the Numerical Example.
\[ v(t) = -3 + 2 \sin(0.5t - \frac{\pi}{6}) \] (purple dashed-lines). It can be readily seen that, due to the effective decoupling of the redundancy in the system from the output regulation, the perturbation of the state and input trajectories, following the proposed extended reference model, does not affect the output (being the first state as shown in Figure 5.6) and the error (as shown in Figure 5.7). Then, the control allocation amounts in selecting optimal reference trajectories \( (x_r, u_r) \) corresponding to certain performance criteria.

### 5.2.2 Strategies for Dynamic Linear Reference Allocation

According to the architecture depicted in Figure 5.3, the selection of the trajectory \( \xi(\cdot) \), which is the available degree of freedom to reshape the reference \( x_r(\cdot), u_r(\cdot) \), is accomplished by setting appropriately the initial condition \( \xi(0) \) in the extended servomechanism and by controlling the evolution of \( \xi(\cdot) \) via the control \( v(\cdot) \). The reference allocation module in Figure 5.3 has precisely the goal of generating the control \( v(\cdot) \) according to certain performance criteria. Two different allocation policies are proposed here, which take typical quadratic cost functions into consideration.

#### A. Optimization on reference state and input trajectory

A first application of the extended reference generator considers selecting the trajectory \( \xi(\cdot) \) to optimize the quadratic functional of the input and state references

\[
J(z_r(t), u_r(t)) = \sum_{k=t}^{t+N} \frac{1}{2} z_r^T(k) W_z z_r(k) + \frac{1}{2} u_r^T(k) W_u u_r(k) \tag{5.21}
\]

where \( W_x \in \mathbb{R}^{n \times n} \) and \( W_u \in \mathbb{R}^{m \times m} \) are symmetric and positive-definite weighting matrices, and \( N \geq +\infty \), denotes the optimization horizon. Note that (5.21) can be more conveniently written as

\[
J(x_r(t), u_r(t)) = \sum_{k=t}^{t+N} \frac{1}{2} x_r^T(k) W_x x_r(k) + \frac{1}{2} u_r^T(k) W_u u_r(k) \tag{5.22}
\]
where \( W_x := D^T W_z D \) is, in general, only positive semi-definite. It is important to keep in mind that the reference input and state trajectory has been parameterized from the extended servomechanism

\[
x_r(t) = \Pi w(t) + \Sigma \xi(t),
\]
\[
u_r(t) = \Gamma w(t) + \Psi \xi(t) + \Upsilon v(t),
\]
\[
z_r(t) = D (\Pi w(t) + \Sigma \xi(t))
\]

(5.23)

where the optimal trajectory \( \xi^*(\cdot) \) must be a trajectory of

\[
\xi^*(t + 1) = \Phi \xi^*(t) + \Xi v^*(t)
\]

(5.24)

for all \( t \geq 0 \) forced by an optimal control law \( v^*(\cdot) \). The allocation policy is accomplished by setting \( \xi(0) = \xi_0 \) and \( v(t) = v^*(t), \ t \geq 0 \). Clearly, this strategy only calls for a feed-forward action, hence it does not affect stability of the closed-loop system.

**B. Optimization on feedback state and input trajectories**

A second application considers the optimization of the quadratic functional of the actual input and state trajectory

\[
J(x(t), u(t)) = \sum_{k=t}^{t+N} \frac{1}{2} x^T(k) W_x x(k) + \frac{1}{2} u^T(k) W_u u(k)
\]

(5.25)

where \( W_x, W_u \) are weighting matrices defined in the same way as above. Applying the change of coordinates \( x \to \tilde{x} = x - x_r \), one obtains

\[
x(t) = \tilde{x}(t) + \Pi w(t) + \Sigma \xi(t)
\]
\[
u(t) = C_c x_c(t) + D_c \tilde{x}(t) + \Gamma w(t) + \Psi \xi(t) + \Upsilon v(t)
\]
\[
z(t) = D (\tilde{x}(t) + \Pi w(t) + \Sigma \xi(t))
\]

(5.26)
The allocation policy is achieved in a similar manner from the forced trajectory (5.24) by setting $\xi(0) = \xi_0$ and $v(t) = v_*(t)$. However, note that this policy does employ feedback $(\tilde{x}, x_c)$ from the plant, as shown in Figure 5.3. Notwithstanding the fact that the stability of the allocation module and the compensated plant can be solved independently, the stability of the overall interconnected loop needs to be analyzed.

5.3 Model Predictive Control Allocation

In previous sections, weak input redundancy has been exploited for inverse model allocation in the output regulation problem for over-actuated systems. The synthesis of extended reference model (5.17) satisfies the control objectives – in particular, asymptotic regulation of $e(t) – \text{irrespective of the action of the external input } v(\cdot)$, as long as $v(\cdot) \in \ell_{\infty}$. It is also noticed that since the augmented plant (with state $\tilde{x}_a$) is autonomous, its trajectories do not depend on $v(\cdot)$; rather, $\tilde{x}_a(t), t \geq 0$, depends uniquely on the initial condition $\tilde{x}_a(0) = (x(0) - x_r(0)) \oplus x_c(0)$, hence on the selection of $\xi(0)$. Therefore, the problem of allocating the reference trajectories is formulated independently from the regulation problem and can be cast as a dynamic optimization problem involving suitable performance criteria and system constraints. However, if one lets the cost function of the said optimization problem depend on the trajectory $\tilde{x}_a(\cdot)$ and the overall control $u(\cdot)$ (as proposed Strategy B in Section 5.2.2), as opposed to depend solely on $x_r(\cdot)$ and $u_r(\cdot)$ in feed-forward fashion (as proposed Strategy A in Section 5.2.2), then stability of the overall interconnection becomes an issue.

In this section, a dynamic control allocation scheme is proposed, which focuses on the design of model predictive allocator to achieve performance optimization and constraint satisfaction. The proposed control scheme, as shown in Figure 5.3, is to find a full-information plug-in regulator and a suitable interconnection structure such
that all trajectories of the closed-loop system are bounded and satisfy \( \lim_{t \to \infty} e(t) = 0 \), while exploiting – in a sense to be defined – the over-actuated characteristic of the dynamics.

5.3.1 Problem Formulation: Model Predictive Allocation

The design of the allocator module should be accomplished in such a way that suitable properties are enforced in the already compensated system. In particular, it is desirable that, if possible, states and input trajectories of the plant model remain confined within a polyhedron of the form

\[
\mathcal{P} \triangleq \{ u \in U, \ x \in \mathcal{X}, \ w \in \mathcal{W} \}
\]  

(5.27)

where \( U \subset \mathcal{U}, \ \mathcal{X} \subset \mathcal{X}^\circ \) and \( \mathcal{W} \subset \mathcal{W} \) are given compact sets. Correspondingly, appropriate constraining sets for the reference trajectories, \( \mathcal{X}_r \subset \text{int} \mathcal{X}, \ \mathcal{U}_r \subset \text{int} \mathcal{U} \) are defined such that the constraints \( x_r(t) \in \mathcal{X}_r, u_r(t) \in \mathcal{U}_r \) shall be enforced by the allocator for all \( t \geq 0 \). It is required \( \mathcal{W} \) to be a forward invariant set under the flow of the exosystem, which prompts the following assumption, ensuring that \( (\xi, v) = (0, 0) \) lies in the feasible set \( \mathcal{X}_r \times \mathcal{U}_r \):

**Assumption 3.** The solution \( (\Pi, \Gamma) \) of the regulator equation satisfies \( \Pi \mathcal{W} \subset \mathcal{X}_r, \ \Gamma \mathcal{W} \subset \mathcal{U}_r \).

In addition, we focus on the case of asymptotically constant references signals, that is, such that for any \( w_0 \in \mathcal{W}, \) there exists \( w_* \in \mathcal{W} \) such that \( \lim_{t \to \infty} w(t) = w_* \). Clearly, the spectrum of \( S \) consists in this case of eigenvalues within the unit circle and simple eigenvalues on the unit circle itself.

According to the architecture proposed in Section 5.3, also shown in Figure 5.3, once the closed loop system (5.17) is designed to be internally stable, the error \( e \) is asymptotically regulated, and its trajectory simply depends on the initial conditions.
\( \tilde{x}(0) \) and \( x_c(0) \). As a result, the external signals \((\xi(t), v(t))\) do not have any influence on the error. On the other hand, on-line selection of the reference trajectories \((x_r(t), u_r(t))\) is accomplished by controlling the evolution of \(\xi(t)\) via the assignment of \(v(t)\). Therefore, the problem of allocating the reference trajectories can be effectively separated from the design of the regulator and formulated independently as a solution of a dynamic optimization problem involving suitable performance criteria and system constraints.

A convenient and widely used method to deal with constrained dynamic optimization is model predictive control (MPC) [122]. The model predictive paradigm is employed here for the design of the allocation module in Figure 5.3, with the goal of enforcing constraints satisfaction and optimization of a cost functional penalizing the auxiliary performance output, \(z\), and the control input. Note that MPC is not used here for the design of the stabilizer (5.12), which is assumed to be given.

The strategy adopted for the synthesis of the allocator proceeds as follows: A static optimization procedure (commonly referred to as target calculation) is initially performed to determine an optimal feasible solution in the family of equilibria of the system that are compatible with a given setpoint. Then, a model predictive control allocation policy is derived on the basis of the controllable dynamical system in (5.14), which shapes the unobservable trajectory of the closed-loop system evolving on the subspace \( \mathcal{R}^\ast \) of \( \mathcal{X} \).

### 5.3.2 Target Calculation

Target calculation is a standard procedure when solving tracking problems in the MPC framework [114,123]. For a given constant exosystem signal \( w(t) \equiv w_\ast \), an optimal solution shall be determined among all equilibrium pairs \((x_\ast, u_\ast)\) that yields
the desired output set-point $Qw_*$. For example, target calculation has been investigated in the context of offset-free tracking for linear systems under the presence of disturbances and model uncertainties in [114] and in model predictive control allocation in [70]. The cost function considered at this stage applies a quadratic penalty to the auxiliary performance output and the actuator efforts, compatibly with the constraints. In particular, the target calculation problem is cast as the solution of the following quadratic program: Given $w_* \in W$, determine

$$
(\xi^*, v^*) = \arg\min_{\xi^*, v^*} \{ z^T W_z z_* + u^T W_u u_* \}
$$

subject to:

1. $(I - \Phi)^{-1} \xi^* - \Xi v_* = 0$
2. $x_* = \Pi w_* + \Sigma \xi_* \in \mathcal{X}_t,$
3. $u_* = \Gamma w_* + \Psi \xi_* + \Upsilon v_* \in \mathcal{U}_t$

where $W_z, W_u$ are positive definite matrices, whereas $W_x := D^T W_z D$ is, in general, positive semi-definite. Note that the existence of admissible solutions of $(\xi^*, v^*)$ for the optimization problem (5.28) for any $w_* \in W$ is guaranteed by Assumption 3.

### 5.3.3 Problem Formulation: Stability and Feasibility

For notational convenience, the dependence on the temporal variable $t$ on signals will be henceforth denoted by a subscript, that is, we let $\zeta_t := \zeta(t)$ denote the value of the signal $\zeta(\cdot)$ at time $t$. Applying the change of coordinates $\tilde{w} := w - w_*$, $\tilde{\xi} := \xi - \xi_*$ and $\tilde{v} := v - v_*$, the extended reference model (5.14) is rewritten as the system

$$
\begin{align*}
\tilde{w}_{t+1} &= S \tilde{w}_t, \\
\tilde{\xi}_{t+1} &= \Phi \tilde{\xi}_t + \Xi \tilde{v}_t \\
x_{r_t} &= x_* + \Pi \tilde{w}_t + \Sigma \tilde{\xi}_t, \\
u_{r_t} &= u_* + \Gamma \tilde{w}_t + \Psi \tilde{\xi}_t + \Upsilon \tilde{v}_t
\end{align*}
$$

(5.29)
which has an equilibrium at the origin \((\tilde{w}, \tilde{\xi}, \tilde{v}) = (0, 0, 0)\). Consequently, the deviation from the optimal equilibrium (in the state and input spaces of the augmented plant) reads as

\[
x_t - x_\star = \tilde{x}_t + \Pi \tilde{w}_t + \Sigma \tilde{\xi}_t
\]

\[
u_t - u_\star = C_c x_c_t + D_c \tilde{x}_t + F \tilde{w}_t + \Psi \tilde{\xi}_t + \Upsilon \tilde{v}_t
\]

(5.30)

It is worth noting that, if the control policy \(\tilde{v}(\cdot)\) (to be determined) for the controllable subsystem of (5.29) is such that the trajectory \(\tilde{\xi}_t\) is defined for all \(t \geq 0\), then \(\lim_{t \to \infty} (\tilde{x}_a_t, \tilde{w}_t) = (0, 0)\) in (5.30). This is confirmed by the asymptotically stable and autonomous dynamics of \(\tilde{x}_a\) and \(\tilde{w}\), seen in (5.17) and (5.29) respectively. Hence, boundedness of all trajectories and convergence to the optimal equilibrium is guaranteed if the allocation policy stabilizes the controllable subsystem of (5.29).

With the previous discussion at hand, the cost function for the dynamic reference allocator is selected to penalize the deviations from the optimal equilibrium

\[
\min_{\bar{\nu}_t} J(\tilde{x}_a_t, \tilde{w}_t, \tilde{\xi}_t, \tilde{v}_t) = \sum_{k=0}^{\infty} \left\{ (x_{k|t} - x_\star)^T W_x (x_{k|t} - x_\star) + (u_{k|t} - u_\star)^T W_u (u_{k|t} - u_\star) \right\}
\]

subject to:

\[
\tilde{w}_{k+1|t} = S \tilde{w}_{k|t},
\]

\[
\tilde{x}_{a_{k+1|t}} = A_a \tilde{x}_{a_{k|t}},
\]

\[
\tilde{\xi}_{k+1|t} = \Phi \tilde{\xi}_{k|t} + \Xi \tilde{v}_{k|t},
\]

\[
x_{k|t} \in \mathcal{X}, \ u_{k|t} \in \mathcal{U}, \ \forall k = 0, 1, 2, ... \]

(5.31)

where \((\cdot)_{k|t}\) denotes prediction at time \(t + k\) from sampling time \(t\). Applying the control sequence \(\bar{\nu}_t := \{\bar{v}_{0|t}, \ \bar{v}_{1|t}, \ ...\}\) over the predicted horizon and solving the optimization at each sampling time, one obtains the optimal control law

\[
\bar{v}_t = \bar{v}_{0|t}, \quad \forall \ t \geq 0
\]

(5.32)

A practical way to solve the dynamic optimization (5.31) is to apply a finite horizon
receding optimization, imposed with a terminal cost and terminal constraint set. In particular, the cost function (5.31) is replaced by

\[ J(\tilde{x}_a, \tilde{w}, \tilde{\xi}, \tilde{v}) = \sum_{k=0}^{N-1} \left\{ (x_{k|t} - x_*)^T W_x (x_{k|t} - x_*) + (u_{k|t} - u_*)^T W_u (u_{k|t} - u_*) \right\} + V_f(\tilde{\xi}_{N|t}) \]  \hspace{1cm} (5.33)

with terminal cost \( V_f(\tilde{\xi}_{N|t}) := \tilde{\xi}_{N|t}^T P_f \tilde{\xi}_{N|t} \), where \( P_f \) is a suitably defined positive definite matrix. The terminal constraint set will be introduced in the sequel. Efficient online optimization algorithms, such as quadratic programming and multi-parametric programming, are available for real time implementation of a solver [111]. Following the definition of dynamic reference allocator given in Definition 2, the control policy (5.32) must satisfy the following requirements:

a) **Nominal Stability (0-GAS)**: When \( \tilde{x}_a \equiv 0 \) and \( \tilde{w} \equiv 0 \), the controlled system

\[ \tilde{\xi}_{t+1} = \Phi \tilde{\xi}_t + \Xi \tilde{v}_{0|t}, \quad \tilde{\xi}_0 = \xi_0 - \xi^* \]  \hspace{1cm} (5.34)

has an asymptotically stable equilibrium at the origin. Note that this corresponds to Strategy A in Section 5.2.2.

b) **Converging Input / Converging State (CICS)**: There exists a set \( \mathcal{X}_{\tilde{\xi}} \) such that for all \( \tilde{\xi}_0 \in \mathcal{X}_{\tilde{\xi}} \) and for any converging feasible trajectory \((\tilde{x}_a, \tilde{w}_t)\), \( \tilde{\xi}_t \) is a converging signal. Since the allocation employs the feedback \( \tilde{x}_a \), this corresponds to Strategy B in Section 5.2.2.

In the specific context considered in this work, the following requirement is added:

c) **Feasibility**: For any converging feasible \((\tilde{x}_a, \tilde{w}_t)\), the constraints are satisfied for the closed loop system, namely, \( x_t \in \mathcal{X}, u_t \in \mathcal{U}, \forall t > 0 \).
Remark 1. In studying nominal stability, since $\tilde{x}_a \equiv 0$, it is implicitly assumed that the augmented plant is initialized at the same initial condition as the reference trajectory. Therefore, feasibility of the closed loop system follows trivially from feasibility of the reference trajectory.

5.3.4 Nominal Stability

To prove nominal stability, let $\tilde{x}_a \equiv 0$ and $\tilde{w} \equiv 0$ in (5.30) to obtain $x - x_* = \Sigma \tilde{\xi}$ and $u - u_* = \Psi \tilde{\xi} + \Upsilon \tilde{v}$. As a result, the receding horizon optimization (5.33) becomes

$$
\min J_a(\tilde{\xi}_t, \tilde{v}_t) = V_f(\tilde{\xi}_{N|t}) + \sum_{k=0}^{N-1} \left\{ \left( \Sigma \tilde{\xi}_{k|t} \right)^T W_x \left( \Sigma \tilde{\xi}_{k|t} \right) + \left( \Psi \tilde{\xi}_{k|t} + \Upsilon \tilde{v}_{k|t} \right)^T W_u \left( \Psi \tilde{\xi}_{k|t} + \Upsilon \tilde{v}_{k|t} \right) \right\}
$$

subject to $\tilde{\xi}_{k+1|t} = \Phi \tilde{\xi}_{k|t} + \Xi \tilde{v}_{k|t}$, $x_* + \Sigma \tilde{\xi}_{k|t} \in \mathcal{X}$,

$$
u_* + \Psi \tilde{\xi}_{k|t} + \Upsilon \tilde{v}_{k|t} \in \mathcal{U}, \quad \tilde{\xi}_{N|t} \in \mathcal{E}_{af}(\xi_*)
$$

for all $k = 0, 1, ..., N - 1$ (5.35)

where the terminal constraint set $\mathcal{E}_{af}(\xi^*)$, centered at $\xi^*$, is imposed to ensure feasibility. In equation (5.35), $N \in \mathbb{N}$ is the length of the prediction horizon and the input sequence $\tilde{v}_t = \left\{ \tilde{v}_{0|t} \ldots \tilde{v}_{N-1|t} \right\}$ is the optimization variable.

The following proposition establishes condition for nominal stability and feasibility of the steady-state trajectory. Since feasibility and stability of finite-horizon constrained optimal control have been widely studied [122], only a sketch of the proof, which borrows heavily from [124], will be given.

Proposition 2. Consider the system (5.29) and the control law given by the solution of the optimization problem (5.35). Let $\Upsilon^\dagger$ denote a left inverse of $\Upsilon$, which exists because the insertion map $\Upsilon : \mathcal{V} \to \mathcal{W}$ is monic. Assume that:

1. The pair $(D\Sigma, \Phi - \Xi \Upsilon^\dagger \Psi)$ is detectable.
2. $\tilde{\xi}_0 \in \mathcal{E}_0$, where $\mathcal{E}_0$ denotes the set of initial states $\tilde{\xi}_0$ for which the optimization (5.35) is feasible.

3. $\mathcal{E}_{af}$ is chosen as the maximal positive invariant set of the system $\tilde{\xi}_{t+1} = \Phi \tilde{\xi}_t$ subject to constraints in (5.35).

4. $P_f$ is selected as the positive definite solution of $-P_f + \Sigma^T W_x \Sigma + \Psi^T W_u \Psi + \Phi^T P_f \Phi \preceq 0$, which exists as $\text{spec}\Phi \in \mathbb{C}^-$ by assumption.

Then the following results hold:

(a) The problem (5.35) is persistently feasible [124].

(b) The origin $\tilde{\xi} = 0$ of the controlled subsystem in (5.29) is asymptotically stable, with domain of attraction that contains $\mathcal{E}_0$.

Proof. Persistent feasibility follows directly from the assumptions by application of standard arguments, for instance [124, Thm. 13.2]. To prove asymptotic stability of the origin, we select the Lyapunov function candidate to be the value function (5.35) at its optimal value $V(\tilde{\xi}_t) := J_a(\tilde{\xi}_t, \tilde{v}_t^*)$. Let $\tilde{v}_t^* := \left\{ \tilde{v}_{0|t}^*, \tilde{v}_{1|t}^*, \ldots, \tilde{v}_{N-1|t}^* \right\}$ denote the optimal control sequence and $\left\{ \tilde{\xi}_{0|t}, \tilde{\xi}_{1|t}, \ldots, \tilde{\xi}_{N|t} \right\}$ be the predicted state vector derived from (5.35) at sampling time $t$. At sampling time $t + 1$, applying $\tilde{v}_t = \tilde{v}_{0|t}^*$, one obtains $\tilde{\xi}_{t+1} = \tilde{\xi}_{1|t}$. Considering a non-optimal control sequence $\tilde{v}_{t+1} := \left\{ \tilde{v}_{1|t}^*, \ldots, \tilde{v}_{N-1|t}^*, \tilde{v} \right\}$ at time $t + 1$, one obtains

$$V(\tilde{\xi}_{t+1}) - V(\tilde{\xi}_t) \leq J_a(\tilde{\xi}_{t+1}, \tilde{v}_{t+1}) - J_a(\tilde{\xi}_t, \tilde{v}_t^*)$$

$$= \left( \Sigma \tilde{\xi}_{N|t} \right)^T W_x \left( \Sigma \tilde{\xi}_{N|t} \right) + \left( \Psi \tilde{\xi}_{N|t} + \Upsilon \tilde{v} \right)^T W_u \left( \Psi \tilde{\xi}_{N|t} + \Upsilon \tilde{v} \right)$$

$$+ \left( \Phi \tilde{\xi}_{N|t} + \Xi \tilde{v} \right)^T P_f \left( \Phi \tilde{\xi}_{N|t} + \Xi \tilde{v} \right) - \tilde{\xi}_{N|t}^T P_f \tilde{\xi}_{N|t}$$

$$- \left( \Sigma \tilde{\xi}_t \right)^T W_x \left( \Sigma \tilde{\xi}_t \right) - \left( \Psi \tilde{\xi}_t + \Upsilon \tilde{v}_t \right)^T W_u \left( \Psi \tilde{\xi}_t + \Upsilon \tilde{v}_t \right)$$

(5.36)
Since system (5.29) is open-loop stable, the control can be turned off when $\tilde{\xi}_t$ evolves into the positively invariant set $E_{af}$ after $N$ steps [122]. As a result, applying input $\tilde{v} = 0$ that is feasible in $E_{af}$, assumption 4) implies that

$$\Delta V := V(\tilde{\xi}_{t+1}) - V(\tilde{\xi}_t) \leq -\left( \Sigma \tilde{\xi}_t \right)^T W_x \left( \Sigma \tilde{\xi}_t \right) - \left( \Psi \tilde{\xi}_t + \mathcal{R} \tilde{v}_t \right)^T W_u \left( \Psi \tilde{\xi}_t + \mathcal{R} \tilde{v}_t \right) \leq 0 \quad (5.37)$$

As a result, the function $V(\tilde{\xi}_t)$ is a weak Lyapunov function, and trajectory $\tilde{\xi}_t$ is bounded. By LaSalle’s Invariance Principle [125], trajectories converge to the largest invariant set $M$ contained in the set $\mathcal{T} := \{ \tilde{\xi} : D\Sigma \tilde{\xi} = 0, \Psi \tilde{\xi} + \mathcal{R} \tilde{v} = 0 \}$. In particular,

$$\lim_{t \to \infty} D\Sigma \tilde{\xi}_t = 0, \quad \lim_{t \to \infty} \Psi \tilde{\xi}_t + \mathcal{R} \tilde{v}_t = 0 \quad (5.38)$$

and, for any initial condition $\tilde{\xi}_0 \in M$, the trajectories of system (5.34) satisfy

$$\tilde{\xi}_{t+1} = (\Phi - \Xi \mathcal{R}^\dagger \Psi) \tilde{\xi}_t, \quad D\Sigma \tilde{\xi}_t = 0 \quad (5.39)$$

which is the zero dynamics of system (5.34) under the optimal control policy, with respect to the output $\zeta = D\Sigma \tilde{\xi}$. Since the pair $(D\Sigma, \Phi - \Xi \mathcal{R}^\dagger \Psi)$ is detectable by assumption, convergence of $D\Sigma \tilde{\xi}_t$ implies convergence $\tilde{\xi}_t$ to zero. Since $\mathcal{R}$ is monic, one obtains $\lim_{t \to \infty} \tilde{v}_t = 0$ as well.

A crucial ingredient of Proposition 2 is detectability of the pair $(D\Sigma, \Phi - \Xi \mathcal{R}^\dagger \Psi)$, a condition that is expressed in terms of the realization of the extended reference model, and that one must check for all left inverses $\mathcal{R}^\dagger$. A more preferable detectability condition would be expressed in terms of the realization of the original plant model, as this latter can be checked a priori. To this end, we resort to the geometric characterization of the state/input redundancy given in Section 5.3, which leads to the characterization of a sufficient condition for detectability, summarized in the next Proposition.
Proposition 3. Let $\mathcal{S} := \mathcal{R}^* + \langle A | \text{im} B \rangle$, and let $\mathcal{U} := \mathcal{V} \oplus \mathcal{V}_c$, where $\mathcal{V}_c$ denotes an arbitrary complementary subspace of $\mathcal{V}$. Let $Q : \mathcal{U} \to \mathcal{U}$ be the natural projection on $\mathcal{V}_c$ along $\mathcal{V}$. Let $F \in \mathbb{F}(\mathcal{R}^*)$ be chosen such that the map induced on $\mathcal{S}/\mathcal{R}^*$ by $A_F$ has all eigenvalues in $\mathbb{C}^-$, and define $A_{QF} := A + BQF$. If the subsystem $(D|\mathcal{R}^*, A_{QF}|\mathcal{R}^*)$ is detectable, then the pair $(D\Sigma, \Phi - \Xi \Upsilon^\dagger \Psi)$, obtained from the corresponding synthesis of the extended reference model, is detectable.

The complete proof of the Proposition 3 requires some background in geometric control theory. Since it is not highly pertained to the subsequent discussion, the proof has been placed in Appendix A.3 for the convenience of the reader, together with an introduction to the fundamentals of geometric control theory.

5.3.5 Incorporating the Transient Behavior

Consider now the case of $(\tilde{x}_a, \tilde{w}) \neq 0$, so that the allocator incorporates the information regarding the transient behavior of the augmented plant. Since the trajectories $(\tilde{x}_a, \tilde{w})$ are governed by autonomous convergent dynamics, boundedness of the state $\tilde{\xi}$ is guaranteed. However, the transient behavior may produce a violation of the constraints, therefore feasibility becomes the primary concern in this case. It is worth noting that, since the stabilizer has not been defined with the objective of constraint satisfaction in mind, the only result that can be expected at this stage is the existence of feasible solution from specified sets of initial conditions. The integration in the stabilizer of mechanisms to prevent constraint violation is the subject of future investigation.

To simplify the notation, define $\eta := (\tilde{w}, \tilde{x}_a, \tilde{\xi})$; then the closed loop system (5.17) is written as

$$\eta_{t+1} = \Theta \eta_t + \Delta \tilde{v}_t \tag{5.40}$$
where \( \Theta \) and \( \Delta \) are given by:

\[
\Theta := \begin{bmatrix}
S & 0 & 0 \\
0 & A_a & 0 \\
0 & 0 & \Phi
\end{bmatrix}, \quad \Delta := \begin{bmatrix}
0 \\
0 \\
\Xi
\end{bmatrix}
\] (5.41)

It is readily seen that only the modes associated to \( \tilde{\xi} \) are controllable from \( \tilde{v} \), and that (5.40) is Lyapunov stable. In this scenario, the optimization problem (5.31) becomes:

\[
\min_{\tilde{v}_t} J_k(\eta_t, \tilde{v}_t) = \sum_{k=0}^{N-1} \left\{ (x_{k|t} - x_\star)^T W_x (x_{k|t} - x_\star) + (u_{k|t} - u_\star)^T W_u (u_{k|t} - u_\star) \right\} + \tilde{\xi}_{N|t}^T P_f \tilde{\xi}_{N|t}
\]

subject to:

\[
\eta_{k+1|t} = \Theta \eta_{k|t} + \Delta \tilde{v}_{k|t}
\]

\[
x_{k|t} \in \mathcal{X}, \quad u_{k|t} \in \mathcal{U}, \quad \tilde{\xi}_{N|t} \in \mathcal{E}_{bf}(\tilde{w}_{N|t}, \tilde{x}_a_{N|t}, \xi_\star)
\]

for all \( k = 0, 1, ..., N - 1 \) (5.42)

Note that the terminal penalty and the terminal constraint set are considered only with respect to \( \tilde{\xi} \) in equation (5.42). This is because adding any terminal cost or constraint for the uncontrollable modes will produce no effect.

The following proposition describes the feasibility and asymptotic behavior of the proposed control scheme (5.42).

**Proposition 4.** Consider the closed loop system (5.40) and the control law given by (5.42), and let \( \mathcal{O}_\infty(\Theta) \) denote the maximal positive invariant set for the autonomous system \( \eta_{t+1} = \Theta \eta_t \) subject to the constraints in (5.42). Let the same assumption 1) and 4) from proposition 2 hold. In addition:

1. Let the terminal set be chosen that

\[
\mathcal{E}_{bf}(\tilde{w}, \tilde{x}_a, \xi_\star) \triangleq \{ \tilde{\xi} \in \mathbb{R}^\rho : \exists (\tilde{w}, \tilde{x}_a) \in \mathcal{W} \oplus \mathcal{X}_a, \text{ s.t. } \eta \in \mathcal{O}_\infty(\Theta) \}
\]

Note that the terminal penalty and the terminal constraint set are considered only with respect to \( \tilde{\xi} \) in equation (5.42). This is because adding any terminal cost or constraint for the uncontrollable modes will produce no effect.
2. Suppose the initial condition satisfies \( \eta_0 \in K_N(O_\infty(\Theta)) \), where \( K_N(O_\infty(\Theta)) \) denotes the maximal \( N \)-step stabilizable set, (see Ch. 11, [124]).

Under these assumptions, the receding optimization (5.42) is persistently feasible and \( \tilde{\xi}_t \) converges to the origin.

Proof. Assumption 2) indicates that there exists a control sequence \( \tilde{v}_0 := \{ \tilde{v}_{0|0}, \tilde{v}_{1|0}, \ldots, \tilde{v}_{N-1|0} \} \) such that the optimization (5.42) is feasible at time \( t = 0 \). Moreover, combining assumption 1) and assumption 2) it follows that \( E_{bf} \neq \emptyset \), \( \forall t \geq 0 \), and that the terminal state \( \eta_{N|t} \in O_\infty(\Theta) \), which is a control invariant set for the closed loop system (5.40). As a result, the optimization problem is persistently feasible, (Thm. 13.2, [124]). To prove stability, remember that the system (5.40) is open-loop stable, and that \( \tilde{v} = 0 \) is a feasible input in \( E_{bf} \). Therefore, applying \( \tilde{v} = 0 \) and by assumption 4) from proposition 2, the monotonicity of the value function holds for (5.42). As a result, it follows that

\[
\lim_{t \to \infty} D(\tilde{x}_t + \Pi \tilde{w}_t + \Sigma \tilde{\xi}_t) = 0, \\
\lim_{t \to \infty} C_c x_{ct} + D_c \tilde{x}_t + \Gamma \tilde{w}_t + \Psi \tilde{\xi}_t + \Upsilon \tilde{v}_t = 0
\]

Convergence of \( (\tilde{\xi}_t, \tilde{v}_t) \) follows by convergence of \( (\tilde{x}_a, \tilde{w}) \) and the results in section 5.3.4.

\[ \square \]

Remark 2. Differently from the nominal case, since the initial condition follows \( x_0 = \tilde{x}_0 + x* + \Pi \tilde{w}_0 + \Sigma \tilde{\xi}_0 \), the choice of \( \tilde{\xi}_0 \) poses the opportunity to assign partial of the initial condition \( \tilde{x}_0 \), corresponding to the specified subspace \( R* \). In fact, denoting \( \text{Proj}_{\tilde{w},\tilde{x}_a}(O_\infty) \) the projection of \( O_\infty \) onto the subspace \( \mathcal{W} \oplus \mathcal{X}_a \), and \( \Omega_{\tilde{\xi}_0} := \{ (\tilde{w}, \tilde{x}_a) : (\tilde{w}, \tilde{x}_a, \tilde{\xi}_0) \in O_\infty \} \), it follows that \( \Omega_{\tilde{\xi}_0} \subset \cup_{\tilde{\xi}_0} \text{Proj}_{\tilde{w},\tilde{x}_a}(O_\infty) \). Note that \( \Omega_{\tilde{\xi}_0} \) represents the admissible set of \( (\tilde{w}, \tilde{x}_a) \) for the conventional design method with absence of the extended reference model. Consequently, the assignment of \( \tilde{\xi}_0 \) allows
to enlarge the domain of admissible exogenous signal $\tilde{w}_0$ and state $\tilde{x}_{a_0}$. Thus, the initial feasibility assumption of the receding horizon optimization in Proposition 4 is not overly restrictive, and may be achieved by an appropriate choice of $\tilde{\xi}_0$.

5.4 Summary

Input redundancy has been investigated in this study using a geometric characterization of over-actuated systems, in which weakly input redundant systems yield the existence of multiple trajectories in the state space yielding a given reference output. An extended reference model has then been presented for the synthesis of a regulator that achieves dynamic allocation of a family of input and state reference trajectories, independently from the solution of the associated output regulation problem.

The incorporation of constraints in the definition of the performance objective of the predictive inverse-model allocation scheme has been addressed as well. The main results highlight the potential benefits of the proposed schemes in terms of modularity (the inverse model allocator can be designed independently of the stabilizing controller) and flexibility (independent selection of the initial conditions on a certain subspace may enlarge the feasibility domain.) Coordinate-free sufficient conditions for stability of the allocation policy have been formulated.

Simulation studies based on the proposed methodology will be investigated for the control of air-path system in the following chapter.
In this chapter, the regulator and model predictive allocator proposed in the previous chapter are applied to the control of the Diesel engine air-path system equipped with VGT-EGR-VGC actuators. Differently from conventional applications of MPC in constrained optimal control of air-path systems in Diesel engines [52, 53], in this approach receding horizon optimization is implemented as a means to shape the steady state trajectory \( (x_r(t), u_r(t)) \) by optimizing a functional with respect to the free variable \( \xi(t) \), whereas the actual stabilizing control policy does not resort to MPC techniques.

A preliminary simulation study on the Diesel engine air-path system confirms the theoretical findings that the allocation is effectively separated from the output regulation, and that the constraints violation can be avoided by modifying the reference trajectories in the input and state space which yield the same output. Moreover, comparing to a conventional MPC tracking control design, the proposed methodology contributes to the improvements of overall system performance and the reduction of computation load and calibration efforts.

The chapter is organized as follows: in Section 6.1, a preliminary application
of the methodology is implemented to the Diesel engine air-path system. A detailed
discussion of control development process and simulation results are presented. Then,
a comparative study between the proposed MPC allocator and the conventional MPC
tracking control design is conducted in Section 6.2, which shows the effectiveness of
the methodology. Finally, conclusions are given in Section 6.3.

6.1 Preliminary Application

To demonstrate the benefits of the proposed regulator, the methodology has been
implemented to the control of the air-path system of Diesel engine, depicted in Fig-
ure 2.1, considered as a preliminary example of a technological application. As a
first step towards a more comprehensive control design, the linearized model (3.5)
has been adopted for both control design and verification. The linearized model in
Chapter 3 is reformulated in the framework of output regulation theory. To address
the direct feed-through term appearing at the output equation with respect to the
actuator VGC, the linearized model is augmented with an integrator at the control
input $u_{VGC}$, yielding a system of the form

$$
\begin{align*}
    w(t + 1) &= Sw(t) \\
    x(t + 1) &= Ax(t) + Bu(t) + Pw(t) \\
    y(t) &= Cx(t), \quad z(t) = Dx(t) \\
    r(t) &= Qw(t), \quad e(t) = y(t) - r(t)
\end{align*}
$$

(6.1)

where the augmented state vector $x \in \mathcal{X} \cong \mathbb{R}^4$, the control input $u \in \mathcal{U} \cong \mathbb{R}^3$ and
the regulated output $y \in \mathcal{Y} \cong \mathbb{R}^2$ are respectively given by

$$
\begin{align*}
    x &= [p_{IM}, p_{EM}, N_{tc}, u_{VGC}]^T \\
    u &= [u_{EGR}, u_{VGT}, v_{VGC}]^T \\
    y &= [p_{IM}, \dot{m}_c]^T
\end{align*}
$$

(6.2)
and $e \in \mathcal{E} \cong \mathbb{R}^2$ denotes the regulated error. The discrete-time system matrices $A, B, C, D$, linearized at the operating point 4 in Table 3.1, read as

$$A = \begin{bmatrix}
0.38 & 0.23 & 0.82 & -0.048 \\
0.38 & 0.33 & 0.68 & -0.047 \\
0.094 & 0.08 & 0.20 & 0.077 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad \quad B = \begin{bmatrix}
0.18 & -0.024 & -0.0012 \\
-1.98 & -0.467 & -0.0006 \\
0.025 & -0.058 & 0.0012 \\
0 & 0 & 0 & 0.02
\end{bmatrix}$$

$$C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-5.68 & 0 & 21.72 & -1.82
\end{bmatrix} \quad \quad D = \begin{bmatrix}
-1.18 & 0 & 3.71 & -0.28 \\
0 & 0 & 1 & 0 \\
2.01 & 0 & -6.52 & 0.32
\end{bmatrix} \quad (6.3)$$

It must be noted that (with a mild abuse of notation) all signals in the model (6.1) represent deviations from the nominal operating condition. In the following sections, notation difference will be omitted distinguishing explicitly variables representing deviations from steady state from those representing actual values, as this will be clear from the context.

A block diagram of the proposed model predictive allocator for Diesel engine air-path system is shown in Figure 6.1. The allocator is implemented as an add-on

---

Figure 6.1: Block Diagram of the Model Predictive Allocator for Air-path System.
module (represented by the block in green in the figure), which shapes the reference state trajectories \((p_{IM}^r, p_{EM}^r, N_{tc}^r)\) and the input trajectories \(u_{VGT}^r, u_{EGR}^r, u_{VGC}^r\) given the output reference pair \((p_{IM}^*, \dot{m}_c^*)\). It is worth noting that a feedback tracking (stabilizing) control, calibrated and embedded in the engine control unit, is assumed to be available, independently from the allocation module. Therefore, comparing to the aforementioned design schemes in Figure 4.3 and Figure 4.7, it can be readily seen that the proposed methodology features a reconfigurable structure that allows one to perform system optimization without the need to modify the existing calibrated controller.

The simulation study considered here is relative to the transient control of the Diesel engine behavior during vehicle tip-in (acceleration) and tip-out (deceleration)
transients. The profiles of the engine operating conditions, namely engine speed $N_{eng}$ and fuel flow rate $\dot{m}_{fuel}$, are shown in Figure 6.2a. During the tip-in transient (from 5s to 10s), the fuel injection increases instantaneously due to the driver’s torque demand, while the engine speed rises relatively slowly because of the crankshaft dynamics. The tip-out transient (from 15s to 20s) behaves similarly. As in Figure 6.2b, where the operating points are represented in the compressor map, such transients may cause surge of the compressor, due to the characteristic stiff behavior of the turbocharger system operating at low flow rate-high boosting conditions.

To model the transient behavior of $N_{eng}$ and $\dot{m}_{fuel}$, the exogenous system model in the first equation in (6.1) has been identified as a second-order system for $N_{eng}$ and a first-order system for $\dot{m}_{fuel}$ as shown in Figure 6.2a.

The steady-state optimization procedure conducted in Chapter 3 is applied to determine the optimal set-point $(p_{IM}^*, \dot{m}_{c}^*)$ for the output variables in correspondence of the engine setpoint $(N_{eng}^*, \dot{m}_{fuel}^*)$. The method in question aims at minimizing the brake-specific fuel consumption (BSFC) while not compromising the emissions variables, AFR and EGR ratio, through design of experiments at each engine operating point. The time-varying reference for the regulated output, $r(t) = [p_{IM}^*(t), \dot{m}_{c}^*(t)]$, is generated by the exosystem from the linearization of the aforementioned static feed-forward map. Finally, an auxiliary performance output to be used in the dynamic steady-state optimization is selected as $z = [\eta_c, N_{tc}, SI] \in \mathcal{X} \cong \mathbb{R}^3$. The auxiliary performance variables focus primarily on the compressor operating point, specifically considering the turbocharger shaft speed and the compressor efficiency. As already shown in Chapter 3, $N_{tc}$ and $\eta_c$ are directly related to the turbo lag phenomenon, as well as to the engine fuel consumption. The Surge Index (SI) is considered as an additional performance metric to evaluate the stability of the compressor.
6.1.1 Problem Formulation and Control Development

The control objective is formulated as tracking \((p_{l,M}^*, \dot{m}_c^*)\) while achieving optimization of the performance variables (compressor efficiency, turbocharger speed, and surge index constraints). The objective function of the allocation algorithm is defined so that the auxiliary performance variables are maximized without overly penalizing the control inputs, which corresponds to the selection of \((x_r(t), u_r(t))\) as a minimizer of the following cost functional:

\[
J(x(t), u(t)) = \sum_{k=t}^{t+N} \left[ \frac{1}{2} \left( 1 - \frac{z(k)}{z_{\max}} \right)^T W_z \left( 1 - \frac{z(k)}{z_{\max}} \right) + \frac{1}{2} u(k)^T W_u u(k) \right]
\]

(6.4)

where \(1\) denotes a column vector with unitary elements, \(W_z \in \mathbb{R}^{3 \times 3}\) and \(W_u \in \mathbb{R}^{3 \times 3}\) are symmetric and positive definite weighting matrices, and \(N\) is the prediction horizon. The state variables and the control inputs are subject to saturation limits defined as follows:

\[
x \in X := \{ x : x_{\min} \leq x \leq x_{\max} \}
\]

\[
u \in U := \{ u : u_{\min} \leq u \leq u_{\max} \}
\]

\[
z \in Z := \{ z : z_{\min} \leq z \leq z_{\max} \}
\]

where \(Z\) primarily consists of the constraints on \(SI\). As discussed in the previous chapter, the system trajectory \(x(t), z(t)\) and \(u(t)\) in the prediction model is expressed in terms of the reference trajectory outlined in Equation (5.26) as

\[
x(t) = \bar{x}(t) + \Pi w(t) + \Sigma \xi(t)
\]

\[
u(t) = C_c x_c(t) + D_c \bar{x}(t) + \Gamma w(t) + \Psi \xi(t) + \Upsilon v(t)
\]

\[
z(t) = D (\bar{x}(t) + \Pi w(t) + \Sigma \xi(t))
\]

where the forced trajectory \(\xi(t)\) satisfies:

\[
\xi(t + 1) = \Phi \xi(t) + \Xi v(t)
\]
A target computation procedure is performed, given the steady state value $w_\star$, to determine the steady state values $(\xi_\star, v_\star)$ for the extended reference model by solving the optimization problem

$$(\xi_\star, v_\star) = \text{argmin}_{\xi_\star, v_\star} \left\{ \frac{1}{2} \left( 1 - \frac{z_\star}{z_{\text{max}}} \right)^T W_z \left( 1 - \frac{z_\star}{z_{\text{max}}} \right) + \frac{1}{2} u_\star^T W_u u_\star \right\}$$

subject to:

$$(I - \Phi)^{-1} \xi_\star - \Xi v_\star = 0$$

$$z_\star = D(\Pi w_\star + \Sigma \xi_\star) \in \mathcal{Z}_c,$$

$$x_\star = \Pi w_\star + \Sigma \xi_\star \in \mathcal{X}_t,$$

$$u_\star = \Gamma w_\star + \Psi \xi_\star + \Upsilon v_\star \in \mathcal{U}_c$$

and by determining the maximal positive invariant sets $\mathcal{O}_\infty$. Then, after defining $\tilde{w}(t) := w(t) - w_\star$, $\tilde{\xi}(t) := \xi(t) - \xi_\star$ and $\tilde{x}(t) := x(t) - x_\star$, a receding horizon optimization is performed online to calculate the dynamic allocation policy at each sampling instant $t$.

$$\min_{\tilde{v}_t} J(\tilde{x}_a, \tilde{w}_t, \tilde{\xi}_t, \tilde{v}_t) = \sum_{k=t}^{t+N} \left\{ \frac{1}{2} \left( x_{k|t} - x_\star \right)^T W_x \left( x_{k|t} - x_\star \right) \right\} + \tilde{\xi}_{N|t}^T P_f \tilde{\xi}_{N|t}$$

subject to:

$$\tilde{w}_{k+1|t} = S \tilde{w}_{k|t}, \quad \tilde{\xi}_{k+1|t} = \Phi \tilde{\xi}_{k|t} + \Xi \tilde{v}_{k|t},$$

$$x_{c,k+1|t} = A_c x_{c,k|t} + B_c \tilde{x}_{k|t}, \quad \tilde{x}_{k+1|t} = BC_c x_{c,k|t} + (A + BD_c) \tilde{x}_{k|t}$$

$$x_{k|t} \in \mathcal{X}, \quad u_{k|t} \in \mathcal{U}, \quad z_{k|t} \in \mathcal{Z}, \quad \forall k = 0, 1, 2, ..., N - 1$$

$$(\tilde{w}(N), \tilde{x}(N), x_c(N), \xi(N)) \in \mathcal{O}_\infty$$

where $W_x := D^T W_x D \in \mathbb{R}^{4 \times 4}$ is positive semi-definite, and $\tilde{v}_t := \{ \tilde{v}_{0|t}, \tilde{v}_{1|t}, \ldots \}$. Then, the optimal solution of (6.6) yields the allocation policy

$$v(t) := v_\star + \tilde{v}_{0|t}$$

where $\tilde{v}_t := \text{argmin} J(\tilde{x}_a, \tilde{w}_t, \tilde{\xi}_t, \tilde{v}_t)$ is the optimal control law over the predicted horizon.
6.1.2 Simulation Results

In this section, the model predictive allocation has been implemented to the Diesel engine air-path system model. In particular, the computation of the maximal positive invariant set $\mathcal{O}_\infty$ is performed using the MPT Toolbox [126], and the receding horizon optimization is implemented in MATLAB/Simulink.

The constraints considered for the linearized system include saturation limits for actuators, $u_{VGT}, u_{EGR} \in [-20\%, 20\%], u_{VGC} \in [-30\%, 30\%]$, and state $p_{IM} \in [-60kPa, 60kPa], p_{EM} \in [-75kPa, 75kPa], N_{tc} \in [-30kr/min, 30kr/min]$, representing the permissible deviation from the nominal value. A more significant hard constraint for the system to be considered, representing the safe operation of the compressor from surge and choke limit, is defined as the Surge Index ($SI$), with limits [5\%, 95\%]. The control objective is to track the desired set-point of air mass flow

Figure 6.3: Maximal Output Admissible Set for Exosystem (a) $N_{eng}, \dot{m}_{fuel}$ and (b) $p_{IM}, p_{EM}$: Projection of $\Omega_0$ and $\mathcal{O}_\infty$ onto the Subspace $\mathcal{W}$ and $\mathcal{X}',$ Shifted to the Target Values.
rate \( \dot{m}_c^* \) and intake manifold pressure \( p_{IM}^* \), and to improve the compressor efficiency \( \eta_c \) and shaft speed \( N_{tc} \) while avoiding compressor surge.

Figure 6.3a and Figure 6.3b show the computed maximal feasible sets for the exogenous signals \( N_{eng}, \dot{m}_{fuel} \) and state \( p_{IM}, p_{EM} \) respectively, obtained from a stabilizing control that places the closed-loop system poles at \( 0.65 \pm 0.25i \), \( 0.5 \pm 0.45i \). Note that these sets, which represent the projection of \( \Omega_0, O_\infty \) onto the associated subspaces, are shifted by adding the target values. It is clear that \( \Omega_0 \subset O_\infty \), confirming the advantage of the proposed regulator over the conventional design, as suggested in Remark 2 in chapter 5.

A standard linear quadratic regulator has been used as the stabilizing compensator, and a prediction horizon \( N = 5 \) with sampling time equal to \( 0.02s \) has been adopted. The implementation of the optimization strategy is outlined in Algorithm 1. The results of the proposed regulator design and allocation strategies are shown in Figure 6.4 to Figure 6.5. In this study, the performance of the proposed \( MPC-allocator \) is compared with a \( SS-Allocator \) obtained by simply applying the steady state optimal allocation policy \( v(t) \equiv v^* \) in (5.28) resulting in the steady state pair \( x_r(t) = \Pi w(t) + \Sigma \xi(t), u_r(t) = \Gamma w(t) + \Psi \xi(t) + \Upsilon v^* \), and a \( Baseline Regulator \) obtained by selecting \( \xi(0) \equiv 0 \) and \( v(t) \equiv 0 \) resulting in the steady state pair \( x_r(t) = \Pi w(t), u_r(t) = \Gamma w(t) \).

In Figure 6.4, the output reference trajectories and the actual trajectories obtained with the different regulators are compared. It can be seen that the trajectories \( p_{IM}, \dot{m}_c \) converge asymptotically to the reference trajectories with negligible differences between the different strategies.

Figure 6.5 shows a comparison of the actuator positions and auxiliary performance variables, where the performance improvements achieved by the \( MPC-allocator \) can be readily observed. In particular, compared to the \( Baseline Regulator \), the proposed
Algorithm 1 (Allocation Policy Implementation)

1. **Initialization**: at sampling time $t = 0$, $w(0) = w_0$, $x(0) = x_0$ and $\xi(0) = \xi_0$;

2. **Target Calculation**: at sampling time $t \geq 0$, provided the full information of $w(t), x(t), \xi(t)$, perform target calculation to obtain $(\xi_*, v_*, x_*, u_*)$, the maximal positive invariant set $O_\infty$ and $N$-step stabilizable set $K_N(O_\infty)$;

3. **Prediction Feasibility**: write $\tilde{w}(t) = w(t) - w_*, \tilde{\xi}(t) = \xi(t) - \xi_*$, and $\tilde{x}(t) = x(t) - \Pi w(t) - \Sigma \xi(t)$, and perform check
   
   if $(\tilde{w}(t), \tilde{\xi}(t), \tilde{x}(t)) \notin K_N(O_\infty)$
   
   $\tilde{\xi}(t) := \arg\min_{\xi_0(t)} \|\tilde{\xi}_0(t) - \tilde{\xi}(t)\|^2$
   
   s.t. $(\tilde{w}(t), \tilde{\xi}_0(t), \tilde{x}(t)) \notin K_N(O_\infty)$
   
   else continue;

4. **Receding Horizon Optimization**: The optimal allocation policy is calculated $\tilde{v}_i^*$ from (6.6);

5. **Reference Signals**: The reference state and input $(x_r(t), u_r(t))$ are calculated based on the extended reference model (see Definition (2)).

---

$^1$ Target Calculation is performed offline and implemented online as maps to reduce computational load.

$^2$ The re-initialization also considers the augmented state VGC position $u_{VGC}$ in the example.
Figure 6.4: Comparison of Regulated Outputs, Intake Manifold Pressure (Left) and Air Mass Flow Rate (Right) Among Different Regulators on Linear Model.

Figure 6.5: Comparison of Actuator Positions (Left) and Auxiliary Performance Variables (Right) Among Different Regulators on Linear Model.
approach improves the compressor efficiency and turbocharger shaft speed without affecting the output tracking performance. This is mainly produced by a substantial actuation on the VGC, due to the fact that, among the three actuators, the VGC has the highest control authority on the auxiliary performance variables $\eta_c, N_{tc}$ and $SI$. In particular, increasing the VGC position results in an improvement of all performance variables. It is also noteworthy that, although small differences are observed, the VGT and EGR valve positions converge to different steady state values, which is primarily needed to compensate the deviations of the regulated variables caused by the VGC actuation.

It is worth noting that, compared to the $SS$-Allocator, the $MPC$-Allocator is capable to maintain the system transient response within the limits. For instance, the

![Graph](image)

Figure 6.6: Comparison of State Variables of Extended Reference Model Between $SS$-Allocator and $MPC$ Allocator.
EGR valve is saturated at 0% at 20s, while the SI is kept within the maximum variation range at 15s, as shown in Figure 6.5. This is mainly achieved, by inspection, through the re-initialization of internal dynamics $\xi(t)$ and VGC position $u_{VGC}$ at those instants, which partly modifies the state $\tilde{x}(t)$ and makes the constrained predicted optimization problem shift back to the feasible set. Figure 6.6 clearly shows the effect of re-initializing $\xi(t)$ at time 15s, where a considerable jump $\xi$ occurs, but does not cause a significant impulsive behavior in the system responses. Moreover, it can be noticed that the settling time for $\xi(t)$ is smaller for the MPC-Allocator than for the SS-Allocator, as shown in Figure 6.6.

Finally, the designed regulators have been applied to the nonlinear engine model, developed and calibrated in Chapter 2. To provide the state information to the design of the regulator, an observer of the form

$$\dot{\hat{x}}(t + 1) = A\hat{x}(t) + Bu(t) + Pw(t) + L(y(t) - C\hat{x}(t))$$  (6.8)

is implemented, where $y(t)$ represents the vector $[p_{IM}, \dot{m}_c]^T$ obtained from the output of the nonlinear plant model, and $L$ is the observer gain matrix. By comparing the state equation in Equation (6.1) and observer dynamics in Equation (6.8), the mismatches at the initial condition $x(0) - \hat{x}(0)$ and the error term $L(y(t) - C\hat{x}(t))$ act as unknown disturbances to the model predictive allocator, which could possibly affect the stability and feasibility of the closed loop system. Since this study does not cover the topic of robust stability and feasibility with modeling uncertainties, which represents a significant challenge for future work, the observer (6.8) is designed such that $\|x(0) - \hat{x}(0)\| < \varepsilon_1$ and $\|L(y(t) - C\hat{x}(t))\| < \varepsilon_2$ for arbitrarily, but fixed, values of $\varepsilon_1$ and $\varepsilon_2$.

The simulation results performed on the nonlinear model are shown in Figure 6.7 and Figure 6.8. It can be readily seen that the overall system performances match well with what is observed from the linear simulation, although it is not surprising
Figure 6.7: Comparison of Regulated Outputs, Intake Manifold Pressure (Left) and Air Mass Flow Rate (Right), Among Different Regulators on Nonlinear Engine Model.

Figure 6.8: Comparison of Actuator Positions (Left) and Auxiliary Performance Variables (Right) Among Different Regulators on Nonlinear Engine Model.
to observe the occurrence of a steady state tracking error, as seen in Figure 6.7. In particular, the SS-Allocator and MPC-Allocator present relatively larger errors on $p_{EM}$ and $\dot{m}_c$ than the Baseline Regulator. This is because with the allocators, the substantial VGC actuation pushed the trajectories of the system away from the nominal operating condition used in the linearization as shown in Figure 6.8, thus leading to increased model mismatches. However, as shown in Figure 6.8, the $\eta_c, N_{tc}$ performance is still improved and the SI constraint is satisfied for the MPC-Allocator (compared with the results obtained from the linear simulation, shown in Figure 6.5).

6.2 Comparison to Conventional Model Predictive Tracking Control Design

A comparative simulation study is conducted on the basis of the Diesel engine air-path system model, to show the performance improvements of the proposed regulator design compared to the conventional model predictive tracking controller. The model predictive tracking control is designed with a hierarchical scheme: a steady state target calculation is first performed to track the output references while achieving simultaneous optimization of performance variables, resulting in the definition of the cost functional as follows:

$$\begin{align*}
(x_*, u_*) &= \text{argmin}_{x_*, u_*} \left\{ \frac{1}{2} (y_* - r)^T W_y (y_* - r) + \frac{1}{2} (z_* - z_{max})^T W_z (z_* - z_{max}) + \frac{1}{2} u_*^T W_u u_* \right\} \\
\text{subject to:} & \quad x_* = (I - A)^{-1} Bu_* + (I - A)^{-1} P w \in X_e \\
& \quad y_* = C x_* + D u_* , \quad z_* = C_z x_* + D_z u_* \in Z_e , \quad u_* \in U_e ,
\end{align*}$$

(6.9)

where $W_y, W_z, W_u$ are positive definite weighting matrices. Moreover, the formulation of the problem considers the 3-state model, with $x = [p_{IM}, p_{EM}, N_{tc}]^T$ and $u = [u_{VGT}, u_{EGR}, u_{VGC}]^T$. Then, in the lower level of the hierarchy, a model predictive
control is designed to track the computed state references at each sampling time $t$, where the control law is derived by solving the finite horizon optimization problem

$$\min_{u_t} J(x_t, w_t, u_t) = \sum_{k=t}^{t+N} \left\{ \frac{1}{2} (x_{k|t} - x_*)^T W_x (x_{k|t} - x_*) + \frac{1}{2} (u_{k|t} - u_*)^T W_u (u_{k|t} - u_*) \right\}$$

$$+ \frac{1}{2} (x_{k|t} - x_*)^T W_f (x_{k|t} - x_*)$$

subject to:

$$w_{k+1|t} = Sw_{k|t},$$

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + Pw_{k|t},$$

$$x_{k|t} \in \mathcal{X}, \ u_{k|t} \in \mathcal{U}, \ z_{k|t} \in \mathcal{Z}, \ \forall k = 0, 1, 2, ..., N - 1$$

$$x_{N|t} \in \mathcal{X}_f$$

where $u_t = \{u_0|t, ..., u_{N-1|t}\}$ denotes the optimization variable, and $W_f, \mathcal{X}_f$ denotes the terminal weight and terminal constraint set, respectively.

It is clear that the steady state solution in (6.9) will be affected by the selection of the matrices $W_y, W_z$, depending on the relative weighting factors on the output tracking and performance optimization. Therefore, to investigate the effect of choice of weighting matrices in the model predictive tracking control, a comparative study is conducted for different case studies, in which the values of $W_y, W_z, W_u$ are summarized in Table 6.1. In this Table, diag($a, b$) denotes a diagonal matrix with elements $a, b$ on

<table>
<thead>
<tr>
<th>Point</th>
<th>$W_y$</th>
<th>$W_z$</th>
<th>$W_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>diag(10, 10)</td>
<td>diag(0, 0)</td>
<td>diag(1, 1)</td>
</tr>
<tr>
<td>Case 2</td>
<td>diag(10, 10)</td>
<td>diag(5, 5)</td>
<td>diag(1, 1)</td>
</tr>
<tr>
<td>Case 3</td>
<td>diag(10, 10)</td>
<td>diag(10, 10)</td>
<td>diag(1, 1)</td>
</tr>
</tbody>
</table>
Table 6.2: Summary of Receding Optimization Problems for MPC-Tracking and MPC-Allocator

<table>
<thead>
<tr>
<th>Performance Metrics</th>
<th>MPC-Tracking</th>
<th>MPC-Allocator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prediction Horizon</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>NO. of States</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>NO. of Optimization Variables</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>Computation Time</td>
<td>20ms</td>
<td>8ms</td>
</tr>
</tbody>
</table>

the matrix diagonal. It can be seen that from Case 1 to Case 3, the weight on the performance variables \( W_z \) gradually increases, while \( W_y \) and \( W_z \) are kept constant.

A comparison between MPC-Tracking and the MPC-Allocator is conducted first by considering the complexity of solving the receding horizon optimization problem. In particular, the evaluation is performed based on MATLAB/Simulink and MPT Toolbox [126]. For the same prediction horizon, the MPC-Allocator has a longer number of states but much smaller number of optimization variables, resulting in less than half of the computation time required by the MPC-Tracking control.

Then, different control strategies have been applied to the linear and nonlinear plant models, and their performances are quantitatively evaluated, where the mean square tracking errors and performance variables are shown in Figure 6.9. To better appreciate the variations, the performance variables \( \eta_c, N_{tc} \) of the MPC-Tracking control have been normalized with respect to that of the MPC-Allocator based on the following formula

\[
\text{Performance Level} = \frac{z_{\text{Track}} - z_{\text{Allocat}}}{z_{\text{Allocat}}} \tag{6.11}
\]

where \( z_{\text{Track}} \) and \( z_{\text{Allocat}} \) denote the performance variables for MPC-Tracking control.
and MPC-Allocator, respectively. Figure 6.9 shows that MPC-Tracking control at Case 1 has equally tracking errors (same for $p_{IM}$ and slightly higher for $m_c$), but significantly lower performances on $\eta_c$ and $N_{tc}$.
Figure 6.11: Comparison of Combined Root Mean Square Tracking Error and Average Performance Level Between MPC-Allocator and MPC-Tracking Control Performed on Linear (Left) and Nonlinear (Right) Engine Model.

As the weight on $W_z$ increases, the performance variables for the MPC-Tracking control improve (they even outperform the MPC-Allocator at Case 3), but the tracking error becomes also significantly worse. The comparison on the nonlinear plant model, represented in Figure 6.10, shows the similar results. Figure 6.11, where averaged tracking errors and performance levels are shown as dots, clearly shows the trade-off between tracking and performance optimization for different MPC-Tracking control case studies. However, in addition to the significant reduction of calibration efforts of tuning the weighting factors, the MPC-Allocator offers an extra opportunity to further improve the overall performance comparing to the MPC tracking control, by moving the operating point to the bottom right.

Finally, Figure 6.12 to Figure 6.14 illustrate a comparison of transient simulation results on the nonlinear engine model between the MPC-Allocator and the
MPC-Tracking control for three case studies, confirming the results summarized in Figure 6.10. From these figures, an additional point to be noticed is that the MPC-Tracking control tends to keep the EGR valve closed (see Figure 6.13 and Figure 6.14), as the weight $W_z$ increases. Although the performance variables, such as $N_{tc}$, could be somehow improved by lowering the EGR flow rate, it is detrimental for the engine system due to significantly low EGR ratio and high back pressure. On the other hand, the MPC-Allocator primarily uses the VGC actuator, which takes good advantages of the decoupling nature of VGC on the performance variables $\eta_c, N_{tc}$ from the regulated outputs $p_{IM}, \dot{m}_c$.

### 6.3 Summary

To summarize, this chapter focuses on the application of methodology proposed in previous chapter on Diesel engine air-path systems. The study confirms the theoretical findings, where the output regulation problem is effectively separated from the allocation of the input and state trajectories that optimizes the performance variables. Moreover, stability and feasibility of model predictive allocation prevent the compressor from surge during abrupt load transient. A comparative study between the MPC allocator and the conventional MPC tracking control, which qualitatively and quantitatively evaluates the performance improvements, as well as the overall reduction in control design and calibration costs, confirms the effectiveness of the proposed methodology.

This discussion also highlights the unsolved challenge of robust stability and feasibility property when implementing the control design on the nonlinear system model, which represents a significant research topic for future work.
Figure 6.12: Comparison of MPC Tracking Control Case 1 and Proposed MPC Allocation.
Figure 6.13: Comparison of MPC Tracking Control Case 2 and Proposed MPC Allocation.
(a) Intake Manifold Pressure and Air Mass Flow Rate.

(b) Control Inputs (Left) and Performance Variables (Right).

Figure 6.14: Comparison of MPC Tracking Control Case 3 and Proposed MPC Allocation.
CHAPTER 7
CONCLUSIONS AND FUTURE WORK

7.1 Conclusions

The gap between the technology improvement in multi-actuated engine systems and advancement of control algorithms is striking in the automotive industry. Due to limitations in conventional control design methods, the extra-degrees of freedom available for optimizing engine performance are nowadays not fully exploited. Multi-objective optimization is often limited to steady-state calibration, and control design optimization is usually performed only on low order square systems. By ignoring the system dynamics in its entirety, the design and calibration of the air-path control system is only suboptimal, hence innovative methodologies are needed to exploit the potential benefits from the available, redundant actuation.

This dissertation has been dedicated to the investigation of various methodologies for control and optimization of over-actuated systems. Starting from more conventional approaches, a novel, comprehensive and systematic methodology was presented, which exploits the geometry of over-actuated systems. An inverse model allocation was proposed for linearized models of air-path systems in Diesel engines, where the problem of allocating the reference trajectories was formulated independently from the regulation problem and casted as a dynamic optimization problem involving suitable performance criteria. Then, a receding horizon optimization within the model
predictive control framework was proposed to solve the dynamic optimization problem while achieving the constraint satisfaction. Sufficient conditions for stability and recursive feasibility have been established for the MPC allocation.

In addition to pursuing an advancement of the state of the art in control theory, the study also focused on the application of the proposed methodology that – apart from its intended generality – is conceived with engine air-path systems in mind. In particular, the Diesel engine air-path system, equipped with VGT-EGR-VGC actuators, was used in simulation as a test bench for verification of all the proposed control methodologies. Starting from the validated model of the engine air-path system, a VGC system model was integrated to the conventional VGT-EGR system configuration, to study its effects on fuel economy, emissions, as well as on the stability of the compressor. Then, a model-based system analysis and optimization was conducted, which served as the ground work to understand the control authorities of the different actuators, as well as the dynamic property of the system. Based on the results of the system analysis, applications of conventional Model Predictive Control design methods were investigated as feedback control strategies to achieve better transient performance: one relies on definition of an additional output variable to make the system square; the other features a novel framework in which the cost function incorporates the performance optimization (compressor efficiency and stability), together with the minimization of tracking errors to explore the benefit of over-actuation in the system.

Finally, a comparative study was conducted between the conventional MPC techniques and the proposed MPC allocator. A thorough analysis with simulation results on the nonlinear engine air-path model was offered to support the theoretical findings, which showcased the benefits of the proposed over-actuated control. Since the optimization is geometrically decoupled from the regulation problem, the tracking
performance is not affected by the optimization. Unlike the conventional methods in which different weighting factors lead to trade-off between tracking and optimization, the proposed method requires much less calibration work but yields better performances.

7.2 Future Work

The current study has presented a proof of concept for advances in control theory and engineering applications. The work has successfully shown the effectiveness, both theoretically and through simulation results, of the proposed methodology on simultaneous control and optimization of over-actuated systems. However, the analysis presented in this dissertation (especially in Chapter 5 and Chapter 6), while providing fruitful results, also leads to the possibility of improvement for future related studies. To this extent, the following section focuses on examining opportunities for advancing the current research, presenting open research topics and challenges.

7.2.1 Robust Output-feedback Regulators for Over-actuated Systems

In spite of the fact that preliminary simulation results verify the effectiveness of the proposed inverse model allocation on a linearized model, substantial work needs to be dedicated in the future considering the fact that various uncertainties, such as parameter variations, unmodeled dynamics and disturbances, exist when implemented on an actual engine system, for which proving stability, feasibility and optimality will be significantly more challenging. A robust regulator should be designed for uncertain
plant models of the form

\[
\begin{align*}
    w(t + 1) &= Sw(t) \\
    x(t + 1) &= A(\mu)x(t) + B(\mu)u(t) + P(\mu)w(t) \\
    e(t) &= C(\mu)x(t) - Q(\mu)w(t)
\end{align*}
\] (7.1)

where the unknown parameters \( \mu \) is assumed to range on a prescribed set \( \mathcal{P} \), such that the following objectives are satisfied:

(a) robust optimality or sub-optimality of reference trajectory \((x_r(t), u_r(t))\) is achieved by minimization of the defined cost functional under the presence of uncertain parameters;

(b) the equilibrium \((x_\star, x_c, \xi_\star)\) of the unforced closed-loop system (5.17) is asymptotically stable;

(c) the forced closed-loop system satisfies that \( \lim_{t \to +\infty} e(t) = 0 \) for any initial condition \((x(0), x_c(0), \xi(0)) \in X_0. \)

(d) the closed loop trajectory satisfies the constraints under the presence of uncertainties such that \( x(t) \in \mathcal{X}, u(t) \in \mathcal{U}, z(t) \in \mathcal{Z}, t \geq 0 \) for any initial condition \((\tilde{x}(0), x_c(0), \xi(0)) \in X_0 \) and for any \( \mu \in \mathcal{P}. \)

While the task (b) and (c) has been traditionally termed as \textit{structurally stability} and can been resolved by the celebrated “internal model principle”, the procedure for incorporating the parametric uncertainties in the allocator and achieving desired robust optimal performances as well as constraint fulfillment will be a critical challenge in a future study.
7.2.2 Over-actuated Control of Linear Parameter Varying Systems

The regulator design and validation up to present relies on a linearized model around a fixed engine operating point \((N_{eng}^*, m_{fuel}^*)\). Although this approach does work properly in a certain region around this operating point in this study, performance will degrade eventually due to the fact that the highly nonlinear engine dynamics vary significantly with respect to speed and torque. In order to enlarge the applicable engine operating range for the methodology, nonlinear control design techniques should be investigated in future studies.

While the extension of the proposed methodology to nonlinear systems will require significant amount of research, a forseeable and promising advancement of the methodology can be obtained by resorting to Linear Parameter Varying (LPV) systems, which could capture the nonlinear dynamic behavior of the plant while building on a consolidated body of the knowledge from linear design methods. A possible LPV formulation of the problem reads as follows:

\[
\begin{align*}
    w(t + 1) &= S(\theta)w(t) \\
    x(t + 1) &= A(\theta)x(t) + B(\theta)u(t) + P(\theta)w(t) \\
    e(t) &= C(\theta)x(t) - Q(\theta)w(t)
\end{align*}
\]  
(7.2)

where the time-varying parameter vector \(\theta(t)\) is assumed to be measured or can be estimated online. While design of parameter dependent stabilizers for LPV systems can be addressed using available techniques [127,128], the geometry of LPV systems still needs considerable research efforts to characterize the input and state redundancy for over-actuated LPV systems.
7.2.3 Integration with Reference Governor for Constrained Control

The system input/state constraints are dealt with in this dissertation within the framework of model predictive control, where recursive feasibility is guaranteed, for any initial condition $w(0), \xi(0), x(0) \in \mathcal{X}_0$, by a forced terminal constraint $w(N), \xi(N), x(N) \in \mathcal{X}_f$ embedded in the receding horizon optimization problem. In light of the fact that the allocation policy $v(t)$ assigns the internal states of the extended reference model $\xi(t)$ that ultimately affect the reference state and input trajectories $(x_r(t), u_r(t))$, the constraint is fulfilled intrinsically by modifying the references in the state and input space.

The proposed methodology can be certainly related to “reference/command governor” theory [129, 130], where an add-on module of a nonlinear low-pass filter is integrated to the loop to adjust the reference/command signals to enforce system constraints. In fact, adopting a reference governor that manages directly the state and input reference trajectories provides a larger number of degrees of freedom to deal with constraints. In particular, for weakly input redundant systems, state/input references corresponding to subspace $\mathcal{R}^*$ can be reshaped to meet constraint requirements, yielding no effect on the regulated output. On the other hand, when it becomes necessary, state/input references corresponding to the complementary subspace $\mathcal{T}$ could also be altered, but this will inevitably alter the regulated output. Therefore, it would be of great interest to look for a hierarchical reference governor structure that meets the constraint requirements while minimally affecting output regulation.

7.2.4 Experimental Validation of the Approach on Diesel Engine System

A final phase of the proposed work plan will be the implementation and experimental validation of the proposed methodology on a prototype controller of Diesel engine system. In that case, more complicated transient profile must be considered to evaluate
the performance improvements, in terms of fuel economy, turbo lag and compressor stability via suitably tuning the weighting matrices of the cost functional, and compare with the results from production ECU controller. Moreover, since computational load and storage requirement are of critical importance for real-time implementation, efficient online optimization algorithm, such as multi-parametric quadratic programming, should be exploited to compute off-line the optimal allocation policy and realize it as maps in a prototype, real-time controller.
Appendix A

GEOMETRIC CONTROL THEORY

This appendix is intended to provide supplementary materials to the reader, who might not be familiar with geometric control theory, to support the main concepts encountered in this dissertation. For complete treatment on linear geometric control theory, the reader is referred to [117,131]. In particular, the fundamentals introduced in the sequel is mainly adopted from Chapter.0 in [117].

A.1 Notation and Background

Definition 3. Let $C : \mathcal{X} \to \mathcal{Y}$ and let $\mathcal{V} \subset \mathcal{X}$ be a subspace with insertion map $V : \mathcal{V} \to \mathcal{X}$. The domain restriction of $C$ to $\mathcal{V}$ is the map $C|\mathcal{V} : \mathcal{V} \to \mathcal{Y}$ defined by $C|\mathcal{V} := CV$.

Definition 4. Let $C : \mathcal{X} \to \mathcal{Y}$ and let $\mathcal{W} \subset \mathcal{Y}$ be a subspace with insertion map $W : \mathcal{W} \to \mathcal{Y}$, and $\text{im } C \subset \mathcal{W} \subset \mathcal{Y}$. The codomain restriction of $C$ to $\mathcal{W}$ is the map $\mathcal{W}|C : \mathcal{X} \to \mathcal{W}$ defined by $W(\mathcal{W}|C) := C$.

Definition 5. If $A\mathcal{V} \subset \mathcal{V}$, the restriction to the invariant subspace $\mathcal{V} \subset \mathcal{X}$ of the endomorphism $A : \mathcal{X} \to \mathcal{X}$ is the map $A|\mathcal{V} : \mathcal{V} \to \mathcal{V}$ that satisfies $V(A|\mathcal{V}) = AV$, where $V$ is the insertion map of $\mathcal{V}$ in $\mathcal{X}$.

Definition 6. Assume $A\mathcal{V} \subset \mathcal{V}$. Let $\tilde{\mathcal{X}} := \mathcal{X}/\mathcal{V}$ and denote $P : \mathcal{X} \to \tilde{\mathcal{X}}$ the
canonical projection. Then there is a unique map $\bar{A} : \bar{\mathcal{X}} \to \bar{\mathcal{X}}$ such that $\bar{A}P = PA$, where the map $\bar{A}$ is called the map induced on $\mathcal{X} / V$ by $A$.

![Figure A.1: Commutative Diagrams for Induced Maps](image)

**Theorem 1.**

Let $\mathcal{X}, \mathcal{Y}$ be linear vector spaces, and let $C : \mathcal{X} \to \mathcal{Y}$ be a linear map. Let $V$ be a subspace of $\mathcal{X}$ with the property that $V \subseteq \ker C$. Then, there is a unique linear transformation $\bar{C} : \mathcal{X} / V \to \mathcal{Y}$ such that the diagram in Figure A.2 commutes, i.e., $\bar{C}P = C$.

![Figure A.2: Commutative Diagram](image)

**Definition 7.** Let $A : \mathcal{X} \to \mathcal{X}$ and $B : \mathcal{U} \to \mathcal{X}$. A subspace $V \subset \mathcal{X}$ is called control-invariant subspace of the pair $(A, B)$ if there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that $(A + BF)V \subseteq V$, where $F$ is termed as a friend of $V$. 
Definition 8. Let \( A : \mathcal{X} \rightarrow \mathcal{X} \) and \( B : \mathcal{U} \rightarrow \mathcal{X} \). A subspace \( \mathcal{R} \subset \mathcal{X} \) is called controllability subspace of the pair \((A, B)\) if there exists a map \( F : \mathcal{X} \rightarrow \mathcal{U} \) and \( G : \mathcal{U} \rightarrow \mathcal{U} \) such that \( \mathcal{R} = \langle A + BF \mid \text{im}(BG) \rangle \).

Definition 9. Let \( A : \mathcal{X} \rightarrow \mathcal{X} \) and \( C : \mathcal{X} \rightarrow \mathcal{Y} \). A subspace \( \mathcal{S} \subset \mathcal{X} \) is called conditioned invariant subspace of the pair \((C, A)\) if there exists a map \( G : \mathcal{Y} \rightarrow \mathcal{X} \) such that \( (A + GC)\mathcal{S} \subset \mathcal{S} \).

Theorem 2. Suppose \( \mathcal{V} \) is a control-invariant subspace of the pair \((A, B)\), and \( F_0 \in \mathcal{F}(\mathcal{V}) \) be an arbitrary friend of \( \mathcal{V} \). Let \( F : \mathcal{X} \rightarrow \mathcal{U} \), then \( F \in \mathcal{F}(\mathcal{V}) \) if and only if \( (F_0 - F)\mathcal{V} \subset B^{-1}\mathcal{V} \).

A.2 Proof of the Main Results in Section 5.1

Theorem 3.

For the triplet \((C, A, B)\) that is assumed to be right-invertible and weakly input redundant, that is, \( \text{rank } B = m, \text{rank } C = p \), and \( m > p \). Let \( \mathcal{R}^* \) be the maximal controllability subspace of pair \((A, B)\) contained in ker \( C \), and let \( \mathcal{V} := B^{-1}\mathcal{R}^* \), where \( B^{-1} \) denote the inverse image of \( B \). Then, it follows that

1. \( \dim \mathcal{R}^* > 0 \);
2. \( \dim \mathcal{V} = m - p \).

Proof. Let \( \mathcal{V}^* \) be the maximal controlled invariant subspace contained in ker \( C \), and let \( \mathcal{S}^* \) denote the minimal conditioned invariant subspace containing im \( B \) for the triplet \((C, A, B)\). Then the following relation holds

\[ \mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^* \]  \hspace{1cm} (A.1)

This identity was first proved in [132]. Since the triplet \((C, A, B)\) is not left-invertible, it follows that \( \mathcal{V}^* \cap \text{im } B \neq \{0\} \); this, in turn, implies that \( \mathcal{R}^* \neq \{0\} \).
To show the second claim, let $\rho = \dim \mathcal{R}^*$, and $\kappa = \dim \mathcal{R}^* \cap \text{im } B$. Select the same coordinate system used to derive Equation 5.6: Let $\mathcal{X} := \mathcal{R}^* \oplus \mathcal{T}$ and $\mathcal{U} := \mathcal{V} \oplus \mathcal{V}_c$, where $\mathcal{T}, \mathcal{V}_c$ are arbitrary complementary subspaces of $\mathcal{R}^*, \mathcal{V}$ respectively. Let $T \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}^{m \times m}$ denote corresponding adapted basis on $\mathcal{X}$ and $\mathcal{U}$, such that their columns span $\mathcal{R}^*, \mathcal{T}$ and $\mathcal{V}, \mathcal{V}_c$ respectively. Applying the regular feedback transformation $u = Fx + G\bar{u}$, the coordinate change $\bar{x} = T^{-1}x$, one obtains the transformed system (5.6) having the matrix representation

$$A_T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B_G = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \quad C_T = \begin{bmatrix} 0 & C_2 \end{bmatrix}$$

where, in particular, $A_{11} \in \mathbb{R}^{\rho \times \rho}, A_{22} \in \mathbb{R}^{(n-\rho) \times (n-\rho)}$ and $B_{11} \in \mathbb{R}^{\rho \times \kappa}, B_{12} \in \mathbb{R}^{(n-\rho) \times (m-\kappa)}$. Recall that, by definition, the pair $(A_{11}, B_{11})$ is controllable, and by assumption, that the pair $(A_{22}, B_{22})$ is stabilizable. The second claim is proved if we show that the triplet $(C_2, A_{22}, B_{22})$ is square. To prove this, we write the system as

$$\begin{bmatrix} A_T - sI & B_G \\ C_T & 0 \end{bmatrix} = \begin{bmatrix} A_{11} - sI & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} - sI & 0 & B_{22} \\ 0 & C_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} - sI & B_{11} & A_{12} & B_{12} \\ 0 & 0 & A_{22} - sI & B_{22} \\ 0 & 0 & C_2 & 0 \end{bmatrix}$$

(A.2)

Note that pre- and post-multiplication of nonsingular matrices or rearranging the columns does not change the rank of matrices. Thus, the obtained matrix in (A.2) must still have rank $n + p$ as a polynomial matrix. This implies that the block

$$\begin{bmatrix} A_{22} - sI & B_{22} \\ C_2 & 0 \end{bmatrix}$$

is full row rank, so that $B_{22}$ must have at least $p$ independent columns. The fact that $B_{22}$ has exactly $p$ columns is shown as follows: Suppose that $B_{22}$ has more than
Then, the triplet \((C_2, A_{22}, B_{22})\) is not left-invertible. As a result, by the first claim, the maximal controllability subspace contained in \(\ker C_2\) for the triplet \((C_2, A_{22}, B_{22})\) has nonzero dimension, which contradicts the maximality of \(\mathcal{R}^\star\) for \((C, A, B)\).

\[\text{Theorem 4.} \ [133],\]

Let \(\mathcal{R}^\star\) be the maximal controllability subspace of pair \((A, B)\) contained in \(\ker C\), and let \(F \in \mathbb{F}(\mathcal{R})\) be an arbitrary friend of \(\mathcal{R}^\star\). Define \(A_F := A + BF\), \(\bar{A}_F := A_F|\mathcal{X}/\mathcal{R}^\star\), and let \(\bar{C} : \mathcal{X}/\mathcal{R}^\star \to \mathcal{X}^\prime\), \(\bar{B} : \mathcal{U}/(B^{-1}\mathcal{R}^\star) \to \mathcal{X}/\mathcal{R}^\star\) be defined as shown in Figure 5.2b. Let \(\mathcal{Z}(C, A_F, B)\) and \(\mathcal{Z}(\bar{C}, \bar{A}_F, \bar{B})\) denote the set of invariant zeros for the triplets \((C, A_F, B)\) and \((\bar{C}, \bar{A}_F, \bar{B})\), respectively. Then, it follows that \(\mathcal{Z}(C, A_F, B) \equiv \mathcal{Z}(\bar{C}, \bar{A}_F, \bar{B})\).

\[\text{Proof.}\]

Applying the same change of coordinates and feedback transformation used to derive Equation 5.6, one obtains the following realization of the system:

\[A_T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B_G = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \quad C_T = \begin{bmatrix} 0 & C_2 \end{bmatrix}\]

To look at the invariant factors of the original system \(\mathcal{Z}(C, A_F, B)\) and the “reduced-order” system \(\mathcal{Z}(\bar{C}, \bar{A}_F, \bar{B})\), define

\[M(s) := \begin{bmatrix} A_T - sI & B_G \\ C_T & 0 \end{bmatrix}, \quad N(s) := \begin{bmatrix} A_{22} - sI & B_{22} \\ 0 & C_2 \end{bmatrix}\]
By rearranging the columns of $M(s)$, it follows that

$$M(s) = \begin{bmatrix} A_T - sI & B_G \\ C_T & 0 \end{bmatrix} = \begin{bmatrix} A_{11} - sI & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} - sI & 0 & B_{22} \\ 0 & C_{12} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} - sI & B_{11} \\ 0 & 0 & A_{12} & B_{12} \\ 0 & 0 & A_{22} - sI & B_{22} \\ 0 & C_{12} & 0 & 0 \end{bmatrix}$$

(A.3)

Since $(A_{11}, B_{11})$ is controllable, the invariant factors of $[A_{11} - sI \ B_{11}]$ are unitary. Therefore, the invariant factors of $M(s)$ are the same as those of $N(s)$. 

**A.3 A Sufficient Condition for Detectability in Section 5.3**

To prove Proposition 3, a series of intermediate results is first established in the form of technical lemmas.

**Lemma 1.** The pair $(D\Sigma, \Phi - \Xi \Upsilon^\dagger \Psi)$ is detectable if the pair $(D, A + B(I - \Upsilon \Upsilon^\dagger)F)$ is detectable.

*Proof.* The lemma is proved by contradiction. Assume that the pair $(D, A + B(I - \Upsilon \Upsilon^\dagger)F)$ is detectable but the pair $(D\Sigma, \Phi - \Xi \Upsilon^\dagger \Psi)$ is not. Then, there exists $\lambda \in \text{spec}(\Phi - \Xi \Upsilon^\dagger \Psi) \cap \mathbb{C}^+$ and $\xi \in \mathbb{C}^p$, $\xi \neq 0$ such that

$$(\Phi - \Xi \Upsilon^\dagger \Psi)\xi = \lambda\xi, \quad D\Sigma\xi = 0$$

Since the insertion map $\Sigma : \mathscr{R}^* \to \mathscr{X}$ is monic, by applying $\Sigma$ on both sides, one simply obtains

$$(\Sigma \Phi - \Sigma \Xi \Upsilon^\dagger \Psi)\xi = \lambda\Sigma\xi$$
Recalling the synthesis of the extended reference model, the above equation readily implies:

\[(A + BF) \Sigma \xi - B \mathcal{Y} \mathcal{Y}^\dagger F \Sigma \xi = \lambda \Sigma \xi\]

\[\Rightarrow (A + B(I - \mathcal{Y} \mathcal{Y}^\dagger)F) \Sigma \xi = \lambda \Sigma \xi\]

As a result, there exists \(\lambda \in \text{spec} \left( A + B(I - \mathcal{Y} \mathcal{Y}^\dagger)F \right) \cap \mathbb{C}^+\) and \(x = \Sigma \xi \in \mathbb{C}^n, x \neq 0\) such that

\[(A + B(I - \mathcal{Y} \mathcal{Y}^\dagger)F) x = \lambda x, \quad Dx = 0\]

which contradicts the assumption. \(\square\)

Let \(\mathcal{U}\) be expressed as \(\mathcal{U} = \mathcal{V} \oplus \mathcal{V}_c\), where \(\mathcal{V} = B^{-1}\mathbb{R}^*\), and \(\mathcal{V}_c\) denotes an arbitrary complementary subspace of \(\mathcal{V}\). Denote by \(Q : \mathcal{U} \to \mathcal{U}\) the projection on \(\mathcal{V}_c\) along \(\mathcal{V}\), and consider, alongside the plant model \(\mathcal{P} = \{C, A, B\}\), the squared plant model by \(\mathcal{P}_{sq} := \{C, A, BQ\}\), where the \(m - p\) directions in the input space on \(\mathcal{V}\) have been erased.

**Lemma 2.**

(i) The subspace \(\mathbb{R}^*\) is a controlled-invariant subspace for \(\mathcal{P}_{sq}\) and each \(F \in \mathcal{F}(\mathbb{R}^*)\), friend of \(\mathbb{R}^*\) for \(\mathcal{P}\), is also a friend of \(\mathbb{R}^*\) for \(\mathcal{P}_{sq}\).

(ii) Define \(A_{QF} := A + BQF\). For \(\mathcal{P}_{sq}\) the spectrum of the restriction of \(A_{QF}\) to \(\mathbb{R}^*\), \(\text{spec}(A_{QF}|_{\mathbb{R}^*})\), is fixed for any \(F \in \mathcal{F}(\mathbb{R}^*)\).

**Proof.** Let \(F \in \mathcal{F}(\mathbb{R}^*)\). Since \((F - QF)\mathbb{R}^* = (I - Q)F\mathbb{R}^* \subset \text{im} (I - Q) = B^{-1}\mathbb{R}^*\), then by Theorem 2, it follows that \(QF \in \mathcal{F}(\mathbb{R}^*)\). Hence, \((A + BQF)\mathbb{R}^* \subset \mathbb{R}^*\), and the first claim is proven. To prove the second claim, let \(F_i \in \mathcal{F}(\mathbb{R}^*), i = 1, 2,\) and \(A_{QF_i}\) denote the restriction of \(A + BQF_i\) to \(\mathbb{R}^*\). Again by Theorem 2, it follows \((F_1 - F_2)\mathbb{R}^* \subset B^{-1}\mathbb{R}^*\). Then, for any \(\bar{x} \in \mathbb{R}^*\), one obtains

\[\Sigma (\bar{A}_{QF_1} - \bar{A}_{QF_2}) \bar{x} = (A + BQF_1) \Sigma \bar{x} - (A + BQF_2) \Sigma \bar{x}\]

\[= BQ(F_1 - F_2) \Sigma \bar{x}\]

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Since $\Sigma \bar{x} \in \mathcal{R}^*$, it follows that $(F_1 - F_2) \Sigma \bar{x} \in B^{-1} \mathcal{R}^*$. Remember that $\ker Q = B^{-1} \mathcal{R}^*$, then it follows that $\Sigma(\bar{A}_{QF_1} - \bar{A}_{QF_2}) \bar{x} = 0$, $\forall \bar{x} \in \mathcal{R}^*$. Moreover, the insertion map $\Sigma$ being monic implies that $\bar{A}_{QF_1} = \bar{A}_{QF_2}$, hence the map $A_{QF} | \mathcal{R}^*$, is independent of $F$ for $F \in \mathcal{F}(\mathcal{R}^*)$. □

Fix now a basis adapted to the chain $0 \subset \mathcal{R}^* \subset \mathcal{I} \subset \mathcal{X}$, where $\mathcal{I} := \mathcal{R}^* + \langle A | \im B \rangle$, and a basis simultaneously adapted to $\mathcal{V}$ and $\mathcal{V}_c$ for $\mathcal{U}$. In these sets of coordinates, the maps $A$, $B$ and $BQ$ have representation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & 0 \end{bmatrix}, \quad BQ = \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

being

$$\mathcal{R}^* = \im \begin{bmatrix} I_p \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{V} = \im \begin{bmatrix} I_{m-p} \\ 0 \end{bmatrix}, \quad \mathcal{V}_c = \im \begin{bmatrix} 0 \\ I_p \end{bmatrix}$$

The fact that $\mathcal{R}^*$ is a controlled-invariant subspace for $\mathcal{P}$ implies that $\im A_{22} \subset \im B_{22}$, hence the structure of any $F \in \mathcal{F}(\mathcal{R}^*)$ is the following

$$F = \begin{bmatrix} F_{11} & * & * \\ F_{21}^* & F_{22} & * \end{bmatrix}$$

where $*$ denotes unimportant entries, $F_{21}^*$ is such that

$$B_{22} F_{21}^* = -A_{21} \quad (A.4)$$

and $F_{11}$, $F_{22}$ assign eigenvalues to the controllable pairs $(A_{11}, B_{11})$ and $(A_{22}, B_{22})$, respectively. Clearly, for any $F \in \mathcal{F}(\mathcal{R}^*)$

$$A + BQF = \begin{bmatrix} A_{11} + B_{12} F_{21}^* & A_{12} + B_{12} F_{22} & * \\ 0 & A_{22} + B_{22} F_{22} & * \\ 0 & 0 & A_{33} \end{bmatrix}$$
where $A_{33}$ represents the induced map of $A_{QF}$ to $\mathcal{X}/\mathcal{I}$, which is fixed for any $F \in \mathbb{F}(\mathbb{R}^*)$ and $A_{QF}|\mathcal{X}/\mathcal{I} = A|\mathcal{X}/\mathcal{I}$. We now see immediately that $(A + BQF)\mathbb{R}^* \subset \mathbb{R}^*$ for any $F \in \mathbb{F}(\mathbb{R}^*)$. Also, the second claim implies that under the restriction (A.4) on $F_{21}^*$, the map $A_{11} + B_{12}F_{21}^*$ is fixed for any $F \in \mathbb{F}(\mathbb{R}^*)$.

**Remark 3.** For the “squared” triplet $(C, A, BQ)$, the controllability subspace contained in $\ker C$ is $\emptyset$, and $\mathbb{R}^*$ is a controlled-invariant subspace. In particular, $A_{11} + B_{12}F_{21}^*$ represents the induced map $A_{QF}|(\mathbb{R}^*/\emptyset)$, for which the spectrum is part of the set of the invariant zeros for $P_{sq}$, and is independent of the choice of $F \in \mathbb{F}(\mathbb{R}^*)$, by [134].

**Lemma 3.** Let $F \in \mathbb{F}(\mathbb{R}^*)$ be selected such that $\text{spec}(A_F|\mathcal{I}/\mathbb{R}^*) \subset \mathbb{C}^-$. Denote by $D|\mathbb{R}^*$ the domain restriction of the performance output map $D : \mathcal{X} \to \mathcal{Z}$ to $\mathbb{R}^*$. Then, the pair $(D, A + B(I - YY^\dagger)F)$ is detectable if and only if the pair $(D|\mathbb{R}^*, A_{QF}|\mathbb{R}^*)$ is detectable.

**Proof.** Since $Y : \mathcal{V} \to \mathcal{U}$ is the insertion map from $\mathcal{V}$ to $\mathcal{U}$, the map $YY^\dagger : \mathcal{U} \to \mathcal{U}$ is the projection on $\mathcal{V}$ along $\mathcal{V}_c$. As a result, $Q = I - YY^\dagger$. Denoting by $D = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}$ the representation of $D$ in the basis adapted to $\mathbb{R}^*$ used in the proof of Lemma 2, detectability of the pair $(D, A + B(I - YY^\dagger)F)$ is equivalent to the condition

$$\text{rank} \begin{bmatrix} A_{11} + B_{12}F_{21}^* - \lambda I & A_{12} + B_{12}F_{22} & * \\ 0 & A_{22} + B_{22}F_{22} - \lambda I & * \\ 0 & 0 & A_{33} - \lambda I \end{bmatrix} = n + s$$

for all $\lambda \in \text{spec}(A + BQF) \cap \mathbb{C}^+$. Since by assumption $\text{spec}(A_{22} + B_{22}F_{22}) = \ldots$
spec($A_F|^\mathcal{I}/\mathcal{R}^\star$) $\subset \mathbb{C}^-$, and \( \text{spec}(A_{33}) = \text{spec}(A|^\mathcal{X}/\mathcal{I}) \subset \mathbb{C}^- \) by Assumption 1, the above condition becomes

\[
\text{rank} \begin{bmatrix} A_{11} + B_{12}F_{21}^\star - \lambda I \\ D_1 \end{bmatrix} = \rho + s
\]

for all \( \lambda \in \text{spec} \left( A_{11} + B_{12}F_{21}^\star \right) \cap \mathbb{C}^+ \). The result follows from the fact that \( A_{11} + B_{12}F_{21}^\star \) and \( D_1 \) are respectively the coordinate representations of \( A_Q|^\mathcal{R}^\star \) and \( D|^\mathcal{R}^\star \).

Combining the above lemmas, one obtains the proof of Proposition 3. \qed
BIBLIOGRAPHY


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