ANALYSIS OF SLEEP-WAKE TRANSITION DYNAMICS
BY STOCHASTIC MEAN FIELD MODEL AND
METASTABLE STATES

DISSERTATION

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ABSTRACT

Sleep-wake dynamics change across postnatal development: wake bout durations of newborn rats are exponentially distributed but become power law distributed within three weeks, in contrast to sleep bout durations that exhibit exponential distributions at all ages. To gain insight into network processes that may underlie sleep-wake transitions in these animals, a stochastic model is constructed, consisting of two mutually-inhibitory Erdos-Renyi or scale-free random graphs in which each node represents a neuron that can be in one of three possible states (excited, basal and inhibited) with distinct firing rates. The two graphs alternate between high and low levels of activity with durations that can have either an exponential or a power law distribution depending on parameters. The dynamics are investigated by numerical and analytical studies of stochastic differential equations related to the model, including the four-dimensional stochastic mean field equations. The full model is approximated by one dimensional reflected Brownian motion which has a power law distribution of bout durations on an intermediate regime, and the approximation is supported by large deviation theory and the penalization method.
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CHAPTER 1
INTRODUCTION

1.1 Background on sleep-wake neuronal network

More than 40 million people in the U.S. have a chronic sleep disorder, and 62% of American adults experience a sleep problem a few nights per week [6]. The mechanism of sleep-wake regulation has been a hot issue among researchers, and the study of sleep problems has a long history. In 1916, Baron Constantin von Economo, a Viennese neurologist, began to see patients with damaged regions of the brain that regulate sleep and wakefulness. He was able to identify the areas of the brain in which lesions caused specific alterations of wake-sleep regulation. In recent decades, key components of the wake-sleep-regulatory system were investigated, and were found at the sites he discovered [3].

Some recent experiments show surprising statistical properties of the durations of sleep and wake episodes. The experimental data in rats show that sleep bout durations exhibit exponential distributions at all ages, but wake bout durations exhibit exponential distributions at younger ages while a power law distribution emerges in adults (see Figure 1.1) [1]. A power law distribution has probability density function of the form $f(t) \approx C t^{-\alpha}$ for $t > 0$ and its graph in log-log coordinates is a straight line, since

$$\ln(f(t)) = -\alpha \ln t + C_1.$$  \hspace{1cm} (1.1)
On the other hand, the probability density function of an exponential distribution is $f(t) = \lambda e^{-\lambda t}$ for $t > 0$, and it is plotted as a straight line on semi-log coordinates, since

$$\ln(f(t)) = -\lambda t + \lambda_1.$$  \hspace{1cm} (1.2)

Furthermore, exponential distributions of sleep durations and power law distributions of wake durations have been reported for other species; mice, cat and human [2]. A comparison of several species shows that the exponents of the power law distributions are almost identical, while the time constants of exponential distributions differ (see Figure 1.2).
Figure 1.1: Plots of sleep (A) and wake (B) bout durations for rats at day 2 and day 21. The left plots were constructed using semi-log coordinates; straight lines on these plots indicate that the data follow an exponential distribution. The right plots were constructed using log-log coordinates; straight lines on these plots indicate that the data follow a power law distribution [1]. Sleep bout durations show exponential distributions at all ages, but wake bout durations show the power law distribution at day 21 in contrast to the exponential distribution at day 2.
Figure 1.2: Plots of distributions of wake and sleep episode durations for mice, rats, cats, and humans. (a) Log-log plot of the distributions of the durations of wake episodes. All distributions form parallel straight lines, indicating that wake episode durations for all species closely follow a power law with almost identical exponents. (b) Semi-log plot of distributions of sleep durations. All distributions form straight lines with different slopes, indicating that all species follow an exponential distribution with a different value of characteristic time constants [2].
Sleep and wake are opposing behavioral states regulated by the collective behavior of a large number of neurons, and understanding transitions between these states demands analysis on multiple scales. Additional experiments have identified opposing networks of neurons - comprised respectively of wake- or sleep-active cells, and competition between these two networks may govern transitions between sleep and wakefulness [3]. That is, in the wake state, arousal promoting neurons are active and inhibit sleep promoting neurons and vice versa in the sleep state (see Figure 1.3).

These neuronal networks are modeled by two mutually inhibiting random graphs (eg. Erdős-Rényi or scale-free) where each node represents a neuron which fires as a Poisson process. Some analysis of the two state random network model is given in Dr. Olmez’s thesis [7]. There he also presents an improved method for detecting power law and bounded power law distributions. He applies the new method to bout duration data of rats (from Mark Blumberg, University of Iowa), concluding that these are better fit by bounded power law distributions (having a power law on an intermediate regime) rather than power law distributions. This model is well developed to show alternation between high and low levels of sleep-wake activity with dominant durations that can have a power-law distribution over an intermediate time interval followed by an exponential distribution determined by parameters in the model (see Figure 2.4). While the model is useful for exploring network effects, it is computationally expensive to simulate - an expense that grows quickly with network size. In fact, he only considered networks of size \( N = 100 \), but the number of neurons governing sleep-wake regulation is much greater than 100. To resolve this limitation, here the stochastic mean field model is constructed based on the random network model. This stochastic mean field model cannot reflect the structure of the graph, but his observation supports the conclusion that structure of the graph does not much affect the dominant time distribution. The stochastic mean field model can be
represented as a solution of four dimensional stochastic equations and the change of
the network size $N$ is done by the change of coefficients of the equations. It is also
possible to consider the limiting process as $N \to \infty$.

Mathematically and biologically, exponentially-distributed bout durations are both
common and easy to produce in a model. It is known that exponential distribution
is due to the memoryless property of the process.

**Definition 1.1.** A random variable $X$ possesses the memoryless property if $\mathbb{P}\{X > 0\} = 1$, and for every $x \geq 0$ and $t \geq 0$, it satisfies that
\[
\mathbb{P}\{X > t + x\} = \mathbb{P}\{X > x\}\mathbb{P}\{X > t\}.
\] (1.3)

Therefore, one of the goals of this thesis will be to understand what features of
this model give the source of memory when the distribution is nonexponential.
Figure 1.3: A schematic drawing showing some key components of the ascending arousal system (left) and the key projections of sleep regulating component, ventrolateral preoptic nucleus (VLPO) to the main components of the ascending arousal system (right) [3]. The arousal-promoting nuclei work to promote wakefulness through general activation in the cerebral cortex. Sleep-promoting nuclei direct inhibitory connections to all of the arousal-promoting nuclei, promoting sleep and turning off arousal systems during non-REM sleep [4].

The main goal of this thesis is to investigate how interactions between these networks of neurons can give rise to the observed dynamics such as power law distribution of dominant times, its universal exponent, and what the differences of the power law intervals for the different classes of random graph models (eg. Erdös-Rényi or scale-free) with different degree distributions result from. These will investigated via multiple levels of stochastic mean field models, focusing first on the symmetric case where the two clusters have same degree distributions and same size. The analysis
suggests that the mutually-inhibitory interactions between the two networks act to restrict the dynamics to a submanifold on which the competition can be approximated by reflected Brownian motion.

1.2 Outline of the thesis

In this thesis, sleep-wake transition dynamics and its bout distributions are analyzed numerically and analytically in various approaches. In Sections 2.1 and 2.2, the main goal and results of this project are overviewed and preliminaries on random graph theory are given. In Section 2.3 the mutually inhibitory sleep and wake random network model is constructed, and its stochastic dynamics are described as a Markov process. In Section 2.4, this network model is converted to the stochastic mean field model, where the populations of neurons solve the stochastic meal field equation of the form $dN = \mu(N)dt + B(N)dW$. Detailed calculations to get the drift vector $\mu$ and diffusion matrix $B$ will be given. Investigating the deterministic part of the mean field equation $dN = \mu(N)dt$ shows that this system has bistable states where the states correspond to the sleep dominant state and the wake dominant state. In Section 2.5, numerical analysis of the stochastic meal field model gives insight how to study it analytically. It is observed that the power law distribution of the bout durations appears in the intermediate region, and the trajectories fall within a linear band. Therefore, the full model is approximated by one dimensional reflected Brownian motion on a finite interval and the distribution of its first passage time is studied in Section 2.6. For larger networks, the width of the trajectory band narrows. This phenomenon is studied by large deviation theory which says that when the network size $N$ is large, the trajectories are close to the most probable path in exponentially small probability in $\frac{1}{N}$. The full model has bistable states, and it is approximated by one dimensional reflected Brownian motion in Chapter 2. In Chapter 3, it will be
analytically proved that reflected Brownian motion is a limiting process of a certain sequence of bistable potential processes by the penalization method and uniqueness of solutions of Skorokhod problems. In Chapters 4 and 5, the conclusion, ongoing and future work will be discussed.
CHAPTER 2
STOCHASTIC MEAN FIELD MODEL

2.1 Overview

Biophysical studies show that sleep and wake promoting neural networks mutually inhibit each other. In this chapter, the dynamics of neural networks will be modeled by two competing random graphs in which each node represents a neuron that can be in one of three different states (exited, basal and inhibited). They excite each other intra-cluster and mutually inhibit inter-cluster. This stochastic network model shows that there are transitions between sleep and wake dominant states, and power law distributions of dominant times (see Fig. 2.4). This network model will be converted to a continuous stochastic mean field model which shows similar behavior to the network model, but it prominently reduces the computational cost for large networks, and it is easier to change the key parameters such as the network size or the mean degrees of the graphs to see the effect of parameters on the dynamics. To get insight into the mechanism, the stochastic mean field model is analyzed numerically, and it will be shown that the power law distribution of bout durations appears in the intermediate regime which depends on the combination of graphs in different types, size, and degree distributions. Based on these observations, its mechanism will be
investigated by two different approaches: approximation by reflected Brownian motion and large deviation theory. Basic backgrounds on graph theory for the stochastic model will be given in the following section.

2.2 Preliminary review on random graphs

Neuronal networks can be modeled by random graphs where each node represents a neuron, and nodes are connected by edges representing synaptic connections. Here, two classes of random graphs: Erdős-Rényi and scale-free random graphs will be considered, since both of them are commonly used to model neural networks, but have qualitatively different properties.

The Erdős-Rényi random graph was presented by Erdős and Rényi in 1950s-1960s, and the properties of such networks have been studied analytically [8, 9]. This graph has been used to model neuronal networks ubiquitously [10, 11], since its properties are relatively well-known, and it is simple to construct: only two parameters are required to set up this model. An Erdős-Rényi random graph is denoted by $G(N,p_N)$ for its two parameters: $N$, the number of nodes and $p_N$, the probability an edge exists between any pair of nodes. The degree distribution $p_d(k)$ which is the expected fraction of nodes in the network with degree $k$ is binomial such that

$$p_d(k) = \binom{N-1}{k} p_N^k (1 - p_N)^{N-1-k}.$$  \hfill (2.1)

If $Np_N=$constant as $N \to \infty$,

$$\lim_{N \to \infty} p_d(k) = \frac{(Np_N)^k e^{-Np_N}}{k!},$$  \hfill (2.2)
that is, the degree distribution is a Poisson distribution for large \( n \) and \( np = \text{constant} \), then the mean and variance of the degree distribution are

\[
E(\text{deg}) = \sum_{k=1}^{N-1} kp_d(k) = (N-1)p_N, \quad \text{and}
\]

\[
\text{Var}(\text{deg}) = E(k^2) - (N-1)p_N^2 = (N-1)p_N(1-p_N). \tag{2.4}
\]

Scale-free random graphs were introduced in two papers, one by Watts and Strogatz on small-world networks [12], and one by Barabási and Albert on scale-free networks [13]. In this model, there are some nodes with an extremely large number of edges, called hubs, and the degree distribution follows a power law. This has been used to model real world systems including transportation networks, the World Wide Web, the actors collaboration network in movie databases, scientific co-authorship and citation networks and also biological systems such as neural networks [11, 14].

In scale-free networks, the degree distribution \( p_d(k) \) exhibits a power law tail with an exponent \( \gamma \) which is taking a value typically between 2 and 3. That is,

\[
p_d(k) = Ck^{-\gamma}, \quad \text{for} \quad C = 1/\sum_{k=k_{\text{min}}}^{N-1} k^{-\gamma}, \tag{2.5}
\]

where \( k_{\text{min}} \) is the minimum possible degree.

Then, expected number of nodes of degree \( k \) and variance are

\[
E(\text{deg}) = \sum_{k=k_{\text{min}}}^{N} Ck^{-\gamma+1}, \quad \text{and}
\]

\[
\text{Var}(\text{deg}) = E(k^2) - (N-1)p_N^2 = \sum_{k=k_{\text{min}}}^{N} Ck^{-\gamma+2} - E(k^2)^2. \tag{2.7}
\]

This graph is called scale-free for its scale invariance property. Scaling the degree \( k \) by a constant \( \alpha \) changes only a proportionate scaling of the function itself such as

\[
p_d(\alpha k) = C'(\alpha k)^{-\gamma} = \alpha^{-\gamma}p_d(k) \propto p_d(k). \tag{2.8}
\]
Figure 2.1 illustrates the clustering for each graph. It is remarkable that for the scale-free graph, there are some nodes which have much larger degree than the others. The difference in degree distributions between these two graphs is shown in Figure 2.2. These two classes of random graphs will be considered to model the wake and sleep promoting neuronal networks, and it will be studied how the graph type affects the dynamics of the model.

Figure 2.1: Examples of ER random graph (left) and scale-free network (right) [5]. The red colored nodes in the scale-free network are hubs which have many more edges than most of the other nodes.
Figure 2.2: Plots of degree distributions of an ER random graph (left) and a scale-free random graph (right), where $k$ denotes the number of degree and $N(k)$ denotes the number of nodes of degree $k$. For large network size, the degree distribution of ER random graphs follows a Poisson distribution, but it follows a power law distribution for scale-free random graphs.

2.3 Model description

2.3.1 Basic structure of the model

Each sleep and wake neuronal network is modeled by a random graph (e.g. Erdös-Rényi or scale-free), and the two networks are connected by inhibitory edges. In each cluster, a node represents a neuron that has three possible states (excited (E), basal (B), inhibited (I)) and fires according to a Poisson process with the rate $\lambda_E$, $\lambda_B$, or $\lambda_I$ corresponding to its state. When a neuron fires, it excites its neighbors in the same cluster, but inhibits those in the opposing cluster. When an I neuron or a B neuron receives excitatory stimulus via intra-cluster edges, its status changes
to B or E respectively in the order I → B → E (it cannot directly change from I to E). Conversely, when an E neuron or a B neuron receives inhibitory stimulus via inter-cluster edges, its status converts to B or I in the order E → B → I. Moreover, E or I neurons revert to basal state spontaneously with rates $\gamma_E$ and $\gamma_I$, respectively. A schematic diagram of this model is shown in Figure 2.3.

![Figure 2.3: Model consists of two mutually inhibitory and internally excitatory randomly wired clusters. Nodes correspond to neurons that have three different states; exited, basal and inhibited. The neurons that are at states E, B, I are assumed to fire at rates $\lambda_E > \lambda_B > \lambda_I$, respectively and the ones that are at states E and I are assumed to relax to B at rates $\gamma_E$ and $\gamma_I$.](image)

2.3.2 Stochastic process dynamics

Let $S$ and $W$ represent the sleep and wake random graphs respectively. Assume that the degree distribution between two clusters is binomial with the probability $q_N$
that an inter-cluster edge exists. Then, $d_I$ and $v_I$, the average and the variance of inhibitory degrees respectively, are given by

$$d_I = Nq_N, \quad \text{and}$$  

$$v_I = Nq_N(1 - q_N).$$  

(2.9)  

(2.10)

Let $N^j$ and $N^j_i$ denote the size of $j$ cluster and the number of $i$ state nodes in the $j$ cluster for $i = E, B, I$ and $j = W, S$ respectively (i.e. $N^j = N^j_E + N^j_B + N^j_I$), and let $d^j_i$ and $v^j_i$ denote the average and the variance of degrees in the $j$ cluster for $j = S, W$. Since the size of each network is fixed, the four collective variables $N^W_I$, $N^W_E$, $N^S_I$, and $N^S_E$ will be considered, and then $N^W_B = N^W - N^W_E - N^W_I$ and $N^S_B = N^S - N^S_E - N^S_I$.

Let $N = [N^W_I, N^W_E, N^S_I, N^S_E]^T$ be a population vector, and $\Delta N$ be the change of $N$ in a small time interval $\Delta t$. The possible changes for the populations in the time interval $\Delta t$ are due to relaxation and firing of neurons. Since $E$ and $I$ neurons are spontaneously converted to $B$ by rates $\gamma_E$ and $\gamma_I$ respectively, $N^j_i$ decreases by one with probability $\sim N^j_i \gamma_i \Delta t$ for $i = E, I$ and $j = W, S$. When a neuron fires, several neurons who receive excitatory or inhibitory stimulus change their states at the same time. It is assumed that the edges are rerandomized after firing a neuron. Suppose that a neuron in the $W$ cluster fires and it has $m_1$ I and $m_2$ B neighbors in the same cluster and $k_1$ B and $k_2$ E neighbors in the S cluster. Then, $\Delta N = [\Delta N^W_I, \Delta N^W_E, \Delta N^S_I, \Delta N^S_E]^T = [-m_1, m_2, k_1, -k_2]^T$ with the probability $\sim P_W(m_1, m_2)Q_S(k_1, k_2)\lambda_W \Delta t$, where

$$P_W(m_1, m_2) = \text{Prob( losing } m_1 \text{ I's & gaining } m_2 \text{ E's in W cluster | a W neuron fires)}$$

$$Q_S(k_1, k_2) = \text{Prob( gaining } k_1 \text{ I's & losing } k_2 \text{ E's in S cluster | a W neuron fires),}$$
and $\tilde{\lambda}_W$ is the mean firing rate in W cluster such that
\[
\tilde{\lambda}_W = \lambda_I N_I^W + \lambda_E N_E^W + \lambda_B (N^W - N_I^W - N_E^W). \tag{2.11}
\]

Let $p_d^E(k)$ and $p_d^I(k)$ be excitatory and inhibitory degree distributions respectively. In this mean field model, it is assumed that a neuron on any state in the wake (or sleep) fires with the mean firing rate $\tilde{\lambda}_W$ (or $\tilde{\lambda}_W$). For non-negative integers $x_1$ and $x_2$ and for $j = W, S$, $P_j(x_1, x_2)$ and $Q_j(x_1, x_2)$ can be calculated such as
\[
P_j(x_1, x_2) = \sum_{k=x_1+x_2}^{N^j} \frac{p_d^E(k)}{N^j} \left( \binom{N^j - 1}{x_1} \binom{N^j_k}{x_2} \binom{N^j_k}{k-x_1-x_2} \right)
\]
\[
+ \sum_{k=x_1+x_2}^{N^j} \frac{p_d^I(k)}{N^j} \left( \binom{N^j - 1}{x_1} \binom{N^j_k}{x_2} \binom{N^j_k}{k-x_1-x_2} \right)
\]
\[
= \sum_{k=x_1+x_2}^{N^j} \frac{p_d^E(k)}{N^j} \left( \binom{N^j - 1}{x_1} \binom{N^j_k}{x_2} \binom{N^j_k}{k-x_1-x_2} \right)
\]
\[
= \sum_{k=x_1+x_2}^{N^j} \frac{p_d^E(k)}{N^j} \left( \binom{N^j - 1}{x_1} \binom{N^j_k}{x_2} \binom{N^j_k}{k-x_1-x_2} \right), \tag{2.12}
\]

since $N\left( \binom{N-1}{k} \right) = (N-k)\left( \binom{N}{k} \right)$. Similarly,
\[
Q_j(x_1, x_2) = \sum_{k=x_1+x_2}^{N^j} \frac{p_d^I(k)}{N^j} \left( \binom{N^j - 1}{x_1} \binom{N^j_k}{x_2} \binom{N^j_k}{k-x_1-x_2} \right). \tag{2.13}
\]

The rate parameters are chosen as
\[r_I = 0.002, \ \gamma_E = 0.005, \ \lambda_I = 0.001, \ \lambda_B = 0.003, \ \lambda_E = 0.016,\]

to have bistable states in the mean field equations, which will be discussed in detail in Section 2.4.5. All possible changes for the populations in the time interval $\Delta t$ are given in Table 2.1.
In Figure 2.4, a realization of time series of $N_E^W$ and $N_E^S$ is given. If $N_E^W >> N_E^S$, the wake promoting neurons are dominant, so the process is in the wake state, and vice versa if $N_E^W << N_E^S$. However, the distribution of bout durations depends on the definition of sleep and wake active durations, it will be defined rigorously in Section 2.5. Figure 2.4 supports that the model is plausible to show transitions between sleep and wake states and the power law distribution appears in the intermediate regime.
\[
\Delta \mathbf{N} = [-1, 0, 0, 0]^T, \quad \Delta \mathbf{N} = [0, -1, 0, 0]^T, \quad \Delta \mathbf{N} = [0, 0, -1, 0]^T, \quad \Delta \mathbf{N} = [0, 0, 0, -1]^T,
\]

where

\[
P_W(m_1, m_2) = \text{Prob( losing } m_1 \text{ I's & gaining } m_2 \text{ E's in W cluster | a W neuron fires)}
\]

\[
P_S(k_1, k_2) = \text{Prob( losing } k_1 \text{ I's & gaining } k_2 \text{ E's in S cluster | a S neuron fires)}
\]

\[
Q_W(m_1, m_2) = \text{Prob( gaining } m_1 \text{ I's & losing } m_2 \text{ E's in W cluster | a S neuron fires)}
\]

\[
Q_S(k_1, k_2) = \text{Prob( gaining } k_1 \text{ I's & losing } k_2 \text{ E's in S cluster | a W neuron fires)}
\]

and

\[
\bar{\lambda}_W = \lambda_I N_I^W + \lambda_E N_E^W + \lambda_B (N - N_I^W - N_E^W),
\]

\[
\bar{\lambda}_S = \lambda_I N_I^S + \lambda_E N_E^S + \lambda_B (N - N_I^S - N_E^S),
\]

are mean firing rates in each cluster.

Table 2.1: Possible changes in each population with the corresponding probabilities
Figure 2.4: (Left) A realization of time series of $\frac{N_{E}^{W}}{N_{W}}$ (blue) and $\frac{N_{E}^{S}}{N_{S}}$ (green), where $W$ and $S$ are both ER random graphs of excitatory mean degrees $d_{E} = 3.8$, and of inhibitory mean degree $d_{I} = 3.5$. This shows transitions between the wake dominant state and the sleep dominant state, where it is sleep state if $N_{E}^{W} << N_{E}^{S}$ and in the wake state if $N_{E}^{W} >> N_{E}^{S}$. (Right) Log-log plot of the distribution of sleep bout durations (blue) of ER random graph model with the size $N = 100$, the mean degree in the wake cluster=3, and the mean degree in the sleep cluster=4.2, and the approximated power law distribution (red). The power law distribution is detected on the interval $[60, 700]$ with the exponent $-1.3032$. These figures are given by Dr. Olmez.

2.4 Stochastic mean field equations

2.4.1 Backgrounds on stochastic processes

In this section, some background on stochastic processes will be introduced to construct the stochastic mean field equations for the sleep-wake transition model.
Definition 2.1 ([15]). A one-dimensional Wiener process (standard Brownian motion), is a stochastic process \( \{W_t\}_{t \geq 0} \) satisfying the following conditions:

1. Independent increments: For all \( t > s \geq 0 \), the increment \( W_t - W_s \) is independent of \( \{W_u\}_{0 \leq u \leq s} \);

2. Gaussian increments: For all \( t > s \geq 0 \), the increment \( W_t - W_s \) is normally distributed with zero mean and variance \( t - s \).

3. Continuity of paths: \( \{W_t\}_{t \geq 0} \) are continuous functions of \( t \).

Then it is well-known that the standard Brownian motion satisfies the following properties:

- If \( W_0 = x \), then \( W_t \) has the \( \mathcal{N}(x, t) \) normal distribution with mean \( x \) and variance \( t \). That is,
  \[
  \mathbb{P}_x(W_t \in (a, b)) = \int_a^b p_t(x, y)dy, \tag{2.16}
  \]
  where \( p_t(x, y) \) is the transition probability density of Brownian motion such that
  \[
  p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}. \tag{2.17}
  \]

- Markov property: \( \{W_t\}_{t \geq 0} \) is a Markov process, i.e.,
  \[
  \mathbb{P}(W_{t+s} \in A|W_u, u \leq t) = \mathbb{P}(W_{t+s} \in A|W_t) \text{ a.s.} \tag{2.18}
  \]
  for any Borel set \( A \) and all \( t, s \geq 0 \).

- Scaling property: For any \( c > 0 \), \( \{cW_{t/c^2}\}_{t \geq 0} \) is a Wiener process.
Definition 2.2. Let \( \mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\} \subset \mathcal{F} \) be the \( \sigma \)-algebra generated by all events of the form \( \{a \leq W_s < b\} \), for \( 0 \leq s \leq t, a < b \). Then \( \mathcal{F}_s \subset \mathcal{F}_t \), whenever \( s \leq t \). The family \( \{\mathcal{F}_t\}_{t \geq 0} \) is called the canonical filtration generated by the Brownian motion.

Definition 2.3. A random variable \( \tau : \Omega \to [0, \infty) \) is called a stopping time with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) if \( \tau^{-1}([0,t]) := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \) for all \( t > 0 \).

Remark For an open set \( A \supset W_0 \), the first exit time of the standard Brownian motion \( W_t \)

\[
\tau_A = \inf\{t > 0 : W_t \notin A\} \tag{2.19}
\]

is a stopping time.

The sleep and wake bout durations will be defined as first passage times of stochastic process over some sleep- and wake- active domain in section 2.5.

In Table 2.2, notions which will be frequently used to set up the stochastic mean field model are given.
<table>
<thead>
<tr>
<th>$N^i_j$</th>
<th>the number of $i$ state nodes in $j$ cluster for $i = E, B, I$ and $j = S, W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = [N^W_I, N^W_E, N^S_I, N^S_E]^T$</td>
<td>a population vector</td>
</tr>
<tr>
<td>$N^j_B = N - N^j_I - N^j_E$</td>
<td>the number of Basal neurons in $j$ cluster for $j = S, W$, where $N$ is the size of a cluster.</td>
</tr>
<tr>
<td>$p^E_d(k)$ ($p^I_d(k)$)</td>
<td>the excitatory (inhibitory) degree distribution</td>
</tr>
<tr>
<td>$d_E$ ($d_I$)</td>
<td>the average of excitatory (inhibitory) degree</td>
</tr>
<tr>
<td>$v_E$ ($v_I$)</td>
<td>the variance of excitatory (inhibitory) degree</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>the firing rate of $i$ neuron for $i = E, B, I$.</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>the relaxation rate of $i$ neuron for $i = E, I$ to $B$ neuron.</td>
</tr>
<tr>
<td>$\bar{\lambda}_j$</td>
<td>the mean firing rate in $j$ cluster for $j = S, W$</td>
</tr>
<tr>
<td>$\bar{\lambda}_W = \lambda_I N^W_I \lambda_E N^W_E + \lambda_B N^W_B$,</td>
<td></td>
</tr>
<tr>
<td>$\bar{\lambda}_S = \lambda_I N^S_I + \lambda_E N^S_E + \lambda_B N^S_B$.</td>
<td></td>
</tr>
</tbody>
</table>

| Table 2.2: Notations |

### 2.4.2 Construction of the stochastic mean field equations

In this section, it will be describe how to convert the discrete network model into the continuous stochastic mean field model by using master equations for the probability that the population is $N$ at time $t$ and Fokker-Planck equations. Based on the stochastic process dynamics in Table 2.1, define the vector $\mu(N)$ and the four by four symmetric positive definite covariance matrix $V(N)$ such that

$$
\mu(t, N) = E[\Delta N]/\Delta t, 
$$

(2.20)
\[ V(t, N) = E[\Delta N(\Delta N)^T] \Delta t. \] (2.21)

Let \( p(t, N) = p(t, N_I^W, N_E^W, N_I^S, N_E^S) \) be the probability that \( N = [N_1, N_2, N_3, N_4]^T = [N_I^W, N_E^W, N_I^S, N_E^S]^T \) at time \( t \), then the master equation

\[
p(t + \Delta t, N) = p(t, N) + \Delta t \sum_{i=1}^{4} p(t, N + e_i) \gamma_i(N + e_i) + \Delta t \sum_{m_1, m_2, k_1, k_2} p(t, N - [-m_1, m_2, k_1, k_2]^T) w_{m_1, m_2, k_1, k_2}(t, N - [-m_1, m_2, k_1, k_2]^T) \]

\[
+ \Delta t \sum_{m_1, m_2, k_1, k_2} p(t, N - [m_1, -m_2, -k_1, k_2]^T) s_{m_1, m_2, k_1, k_2}(t, N - [m_1, -m_2, -k_1, k_2]^T) \]

\[
+ \Delta t p(t, N) p_0(t, N), \] (2.22)

where

\[
\gamma_1(N) = N_I^W r_I, \quad \gamma_2(N) = N_E^W \gamma_E, \]

\[
\gamma_3(N) = N_I^S r_I, \quad \gamma_4(N) = N_E^S \gamma_E, \] (2.23)

\[
w_{m_1, m_2, k_1, k_2}(t, N) = P_W(m_1, m_2) Q_S(k_1, k_2) \bar{\lambda}_W, \] (2.24)

\[
s_{m_1, m_2, k_1, k_2}(t, N) = Q_W(m_1, m_2) P_S(k_1, k_2) \bar{\lambda}_S, \] (2.25)

\[
p_0(t, N) = 1 - \left( \sum_{i=1}^{4} \gamma_i(t, N) + \sum_{m_1, m_2, k_1, k_2} w_{m_1, m_2, k_1, k_2}(t, N) + \sum_{m_1, m_2, k_1, k_2} s_{m_1, m_2, k_1, k_2}(t, N) \right). \] (2.26)

Expand out each term in Taylor series about the point \((t, N)\), then

\[
p(t, N + e_i) \gamma_i(t, N + e_i) \approx p(t, N) \gamma_i(t, N) + \frac{\partial (p \gamma_i)}{\partial N_i} + \frac{1}{2} \frac{\partial^2 (p \gamma_i)}{\partial N_i^2}; \] (2.27)
\[ p(t, N - [m_1, m_2, k_1, -k_2]^T)w_{m_1,m_2,k_1,k_2}(t, N - [m_1, m_2, k_1, -k_2]^T) \]
\[ \approx p(t, N)w_{m_1,m_2,k_1,k_2}(t, N) + m_1 \frac{\partial (pw_{m_1,m_2,k_1,k_2})}{\partial N_1} - m_2 \frac{\partial (pw_{m_1,m_2,k_1,k_2})}{\partial N_2} \]
\[ - k_1 \frac{\partial (pw_{m_1,m_2,k_1,k_2})}{\partial N_3} + k_2 \frac{\partial (pw_{m_1,m_2,k_1,k_2})}{\partial N_4} \]
\[ + \frac{1}{2} \left( m_1^2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_1^2} + m_2^2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_2^2} \right) \]
\[ + k_1^2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_3^2} + k_2^2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_4^2} \]
\[ - 2m_1m_2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_1 \partial N_2} - 2k_1k_2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_3 \partial N_4} \]
\[ + 2 \sum_{i=1,2} \left( (-1)^i m_i k_1 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_i \partial N_3} + (-1)^{i+1} m_i k_2 \frac{\partial^2 (pw_{m_1,m_2,k_1,k_2})}{\partial N_i \partial N_4} \right) \right), \]
(2.28)

and

\[ p(t, N - [m_1, -m_2, -k_1, k_2]^T)s_{m_1,m_2,k_1,k_2}(t, N - [m_1, -m_2, -k_1, k_2]^T) \]
\[ \approx p(t, N)s_{m_1,m_2,k_1,k_2}(t, N) - m_1 \frac{\partial (ps_{m_1,m_2,k_1,k_2})}{\partial N_1} + m_2 \frac{\partial (ps_{m_1,m_2,k_1,k_2})}{\partial N_2} \]
\[ + k_1 \frac{\partial (ps_{m_1,m_2,k_1,k_2})}{\partial N_3} - k_2 \frac{\partial (ps_{m_1,m_2,k_1,k_2})}{\partial N_4} \]
\[ + \frac{1}{2} \left( m_1^2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_1^2} + m_2^2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_2^2} \right) \]
\[ + k_1^2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_3^2} + k_2^2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_4^2} \]
\[ - 2m_1m_2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_1 \partial N_2} - 2k_1k_2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_3 \partial N_4} \]
\[ + 2 \sum_{i=1,2} \left( (-1)^i m_i k_1 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_i \partial N_3} + (-1)^{i+1} m_i k_2 \frac{\partial^2 (ps_{m_1,m_2,k_1,k_2})}{\partial N_i \partial N_4} \right) \right). \]
(2.29)

Substitute these expressions into (2.22), then \( p(t, N) \) approximately solves the Fokker-Planck equation,

\[ \frac{\partial p(t, N)}{\partial t} = - \sum_{i=1}^{4} \frac{\partial p(t, N) \mu_i(t, N)}{\partial N_i} + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 p(t, N) V_{ij}(t, N)}{\partial N_i \partial N_j}. \]
(2.30)
It is known that the probability distribution $p(t, N)$ that satisfies (2.30) is identical to the distribution of solutions to the stochastic differential equation system

$$dN = \mu(t, N)dt + B(t, N)dW,$$

(2.31)

where $\mu(t, N) = E[\Delta N]/\Delta t$ is the drift vector, $B(t, N)$ is the diffusion matrix which is the square root of the covariance matrix $V(t, N) = E[\Delta N(\Delta N)^T]\Delta t$, and $W = [W_1, W_2, W_3, W_4]^T$ is a vector of Wiener processes [16]. Since the system is autonomous, let $\mu(t, N) = \mu(N)$, and $B(t, N) = B(N)$. The system of equations (2.31) is called the stochastic mean field equations for the sleep-wake network model described above. In the following sections, detailed calculations to get $\mu$ and $B$ will be provided.

### 2.4.3 Probabilities of changes of the population vector $N$.

Recall that, in Table 2.1,

$$P_W(x_1, x_2) = \text{Prob}(\text{losing } x_1 \text{ I's} \ & \text{gaining } x_2 \text{ E's in W cluster} \mid \text{a W neuron fires})$$,

$$P_S(x_1, x_2) = \text{Prob}(\text{losing } x_1 \text{ I's} \ & \text{gaining } x_2 \text{ E's in S cluster} \mid \text{a S neuron fires})$$,

$$Q_W(x_1, x_2) = \text{Prob}(\text{gaining } x_1 \text{ I's} \ & \text{losing } x_2 \text{ E's in W cluster} \mid \text{a S neuron fires})$$,

$$Q_S(x_1, x_2) = \text{Prob}(\text{gaining } x_1 \text{ I's} \ & \text{losing } x_2 \text{ E's in S cluster} \mid \text{a W neuron fires})$$.

In Section 2.3.2, it was shown that, for non-negative integers $x_1$ and $x_2$ and $j = W, S$,

$$P_j(x_1, x_2) = \sum_{k=x_1+x_2}^{N_j-1} p_d^E(k) \frac{N_j!}{x_1!x_2!(N_j-k-x_1-x_2)!} \frac{N_j^E}{k},$$

(2.32)
and
\[ Q_j(x_1, x_2) = \sum_{k=x_1+x_2}^{N_j} p'_d(k) \binom{N_j}{x_1} \binom{N_j}{x_2} \binom{N_j}{k-x_1-x_2}. \] (2.33)

Then following propositions will be used to compute the expected numbers of neurons in each state who change their states when a neuron in a cluster fires.

**Proposition 2.1.** For non-negative integers \( x_1, x_2, \) and for \( j = S, W, \)
\[
\sum_{x_1, x_2} x_1 P_j(x_1, x_2) = d_E \frac{N_j^j}{N}, \] (2.34)
\[
\sum_{x_1, x_2} x_2 P_j(x_1, x_2) = d_E \frac{N_j^j}{N}, \] (2.35)
\[
\sum_{x_1, x_2} x_1 Q_j(x_1, x_2) = d_I \frac{N_j^j}{N}, \] (2.36)
\[
\sum_{x_1, x_2} x_2 Q_j(x_1, x_2) = d_I \frac{N_j^j}{N}. \] (2.37)

**Proof.** It is enough to show the equation (2.34) holds for \( j = W, \) since others can be obtained by the same argument.

\[
\sum_{x_1, x_2} x_1 P_W(x_1, x_2) \quad (2.38)
\]
\[
= \sum_{x_1, x_2} x_1 \sum_{k=x_1+x_2}^{N-1} p_d^E(k) \binom{N_W}{x_1} \binom{N_W}{x_2} \binom{N_W}{k-x_1-x_2} \binom{N}{k} \] (2.39)
\[
= \sum_{k=0}^{N-1} \sum_{x_1, x_2 \atop x_1 + x_2 \leq k} p_d^E(k) x_1 \binom{N_W}{x_1} \binom{N_W}{x_2} \binom{N_W}{k-x_1-x_2} \binom{N}{k} \] (2.40)
\[
= \sum_{k=0}^{N-1} \sum_{x_1, x_2 \atop x_1 + x_2 \leq k} p_d^E(k) x_1 \binom{N_W}{x_1} \binom{N_W}{x_2} \binom{N_W}{k-x_1-x_2} \binom{N}{k} \frac{N_W}{k} \frac{N_W}{k-1} \binom{N-1}{k-1} \] (2.41)
\[
= \frac{N_W}{N} \sum_{k=0}^{N-1} k p_d^E(k) \sum_{x_1, x_2 \atop x_1 + x_2 \leq k} x_1 \binom{N_W-1}{x_1-1} \binom{N_W}{x_2} \binom{N_W}{k-x_1-x_2} \binom{N-1}{k-1} \] (2.42)
\[
= \frac{N_W}{N} \sum_{k=0}^{N-1} k p_d^E(k) = d_E \frac{N_W}{N}, \] (2.43)
Proposition 2.2. For non-negative integers $x_1$, $x_2$, and for $j = S, W$,

\[
\sum_{x_1, x_2} x_1^2 P_j(x_1, x_2) = \frac{N_j^j(N_j^j - 1)}{N(N - 1)} (v_E + d_E^2 - d_E) + d_E \frac{N_j^j}{N},
\tag{2.45}
\]

\[
\sum_{x_1, x_2} x_2^2 P_j(x_1, x_2) = \frac{N_j^j(N_j^j - 1)}{N(N - 1)} (v_E + d_E^2 - d_E) + d_E \frac{N_j^j}{N},
\tag{2.46}
\]

\[
\sum_{x_1, x_2} x_1^2 Q_j(x_1, x_2) = \frac{N_j^j(N_j^j - 1)}{N(N - 1)} (v_I + d_I^2 - d_I) + d_I \frac{N_j^j}{N},
\tag{2.47}
\]

\[
\sum_{x_1, x_2} x_2^2 Q_j(x_1, x_2) = \frac{N_j^j(N_j^j - 1)}{N(N - 1)} (v_I + d_I^2 - d_I) + d_I \frac{N_j^j}{N}.
\tag{2.48}
\]

Proof. For $j = W$, Eq. (2.45) can be proved as

\[
\sum_{x_1, x_2} x_1^2 P_W(x_1, x_2) = \sum_{x_1, x_2} x_1^2 \sum_{k=x_1+x_2}^{N-1} p_d^E(k) \binom{N_j^W}{x_1} \binom{N_j^W}{x_2} \binom{N_j^W}{k-x_1-x_2}
\tag{2.49}
\]

\[
= \sum_{k=0}^{N-1} \sum_{x_1+x_2 \leq k} p_d^E(k) \frac{(x_1(x_1 - 1) + x_2)(N_j^W)}{x_1} \binom{N_j^W}{x_2} \binom{N_j^W}{k-x_1-x_2}
\tag{2.50}
\]

\[
= \sum_{k=0}^{N-1} \sum_{x_1+x_2 \leq k} p_d^E(k) \frac{x_1 \binom{N_j^W}{x_1} \binom{N_j^W}{x_2} \binom{N_j^W}{k-x_1-x_2}}{N} \binom{N}{k}
\tag{2.51}
\]

\[
+ \sum_{k=0}^{N-1} \sum_{x_1+x_2 \leq k} p_d^E(k) \frac{x_1 \binom{N_j^W}{x_1} \binom{N_j^W}{x_2} \binom{N_j^W}{k-x_1-x_2}}{N} \binom{N}{k}
\tag{2.52}
\]

\[
= \sum_{k=0}^{N-1} \sum_{x_1+x_2 \leq k} p_d^E(k) \frac{(N_j^W)(N_j^W - 1)}{N(N-1)} \binom{N_j^W}{x_1-2} \binom{N_j^W}{x_2} \binom{N_j^W}{k-x_1-x_2}
\tag{2.53}
\]

\[
+ \sum_{k=0}^{N-1} \sum_{x_1+x_2 \leq k} p_d^E(k) \frac{x_1 \binom{N_j^W}{x_1} \binom{N_j^W}{x_2} \binom{N_j^W}{k-x_1-x_2}}{N} \binom{N}{k} \tag{2.54}
\]
Proposition 2.3. For non-negative integers \(x_1, x_2\), and for \(j = S, W\),

\[
\sum_{x_1, x_2} x_1 x_2 P_j(x_1, x_2) = \frac{N^j_I N^j_B}{N(N-1)} (v_E + d_E^2 - d_E),
\]

\[
\sum_{x_1, x_2} x_1 x_2 Q_j(x_1, x_2) = \frac{N^j_B N^W_E}{N(N-1)} (v_I + d_I^2 - d_I).
\]

Proof. For \(j = W\), Eq. (2.59) can be proved as

\[
\sum_{x_1, x_2} x_1 x_2 P_W(x_1, x_2)
= \sum_{x_1, x_2} x_1 x_2 \sum_{k=0}^{N-1} p^E_d(k) \left( \binom{N^W_I}{x_1} \binom{N^W_B}{x_2} \binom{N^W_E}{k-x_1-x_2} \right) \left( \frac{N}{k(k-1)} \right) \left( \frac{N(N-1)}{k-2} \right)
= \sum_{k=0}^{N-1} \sum_{x_1, x_2 \leq k} p^E_d(k) \binom{N^W_I}{x_1} \binom{N^W_B}{x_2} \binom{N^W_E}{k-x_1-x_2} \left( \frac{N}{k(k-1)} \right) \left( \frac{N(N-1)}{k-2} \right)
= \sum_{k=0}^{N-1} \sum_{x_1, x_2 \leq k} p^E_d(k) N^W_I N^W_B \left( \binom{N^W_I-1}{x_1-1} \binom{N^W_B-1}{x_2-1} \binom{N^W_E}{k-x_1-x_2} \right) \left( \frac{N(N-1)}{k(k-1)} \right) \left( \frac{N(N-1)}{k-2} \right)
\]
\[
\begin{align*}
N_W^{W} 
&= \frac{N_W^{W} N_W^{B}}{N(N-1)} \left( \sum_{k=0}^{N-1} k(k-1)p_E^E(k) \sum_{x_1, x_2} \frac{(N_{x_1-1}^{W})}{(N_{x_2-1}^{W})} \left( \frac{N_{k-x_1-x_2}^{W}}{k-2} \right) \right) \\
&= \frac{N_W^{W} N_W^{B}}{N(N-1)} (v_E + d_E^2 - d_E).
\end{align*}
\]

\[(2.65)\]

\[
\begin{align*}
2.4.4 \quad \text{The drift vector } \mu(N)
\end{align*}
\]

The drift vector \( \mu(N) \) is obtained by averaging over all the possible changes of populations in a small time change \( \Delta t \) and dividing by \( \Delta t \).

Let the \( \mu(N) = (\mu_1(N), \mu_2(N), \mu_3(N), \mu_4(N))^T \), then the first component function \( \mu_1(N) \) can be calculated as

\[
\begin{align*}
\mu_1(N) &= \frac{E[\Delta N_I^W]}{\Delta t} \\
&= -N_I^W r_I - \sum_{m_1, m_2 \atop k_1, k_2} m_1 P_W(m_1, m_2) Q_S(k_1, k_2) \lambda_W \\
&\quad + \sum_{m_1, m_2 \atop k_1, k_2} m_1 Q_W(m_1, m_2) P_S(k_1, k_2) \lambda_S \\
&= -N_I^W r_I - d_E \frac{N_I^W}{N} \lambda_W + d_I \frac{N_B^W}{N} \lambda_S,
\end{align*}
\]

by Proposition 2.1. Similarly, other component functions are obtained such as

\[
\begin{align*}
\mu_2(N) &= \frac{E[\Delta N_E^W]}{\Delta t} \\
&= -N_E^W \gamma_E + \sum_{m_1, m_2 \atop k_1, k_2} m_2 P_W(m_1, m_2) Q_S(k_1, k_2) \lambda_W \\
&\quad - \sum_{m_1, m_2 \atop k_1, k_2} m_2 Q_W(m_1, m_2) P_S(k_1, k_2) \lambda_S \\
&= -N_E^W \gamma_E + d_E \frac{N_B^W}{N} \lambda_W - d_I \frac{N_B^W}{N} \lambda_S,
\end{align*}
\]

\[(2.68)\]
\[ \mu_3(N) = \frac{E[\Delta N_i^S]}{\Delta t} \]

\[ = -N_i^S r_I + \sum_{m_1,m_2, k_1,k_2} k_1 P_W(m_1, m_2) Q_S(k_1, k_2) \bar{\lambda}_W \]

\[ - \sum_{m_1,m_2, k_1,k_2} k_1 Q_W(m_1, m_2) P_S(k_1, k_2) \bar{\lambda}_S \]

\[ = -N_i^S r_I + d_I N_b N^S \bar{\lambda}_W - d_E N_f N^S \bar{\lambda}_S, \]  

(2.69)

and

\[ \mu_4(N) = \frac{E[\Delta N_i^E]}{\Delta t} \]

\[ = -N_e^S \gamma_E - \sum_{m_1,m_2, k_1,k_2} k_2 P_W(m_1, m_2) Q_S(k_1, k_2) \bar{\lambda}_W \]

\[ + \sum_{m_1,m_2, k_1,k_2} k_2 Q_W(m_1, m_2) P_S(k_1, k_2) \bar{\lambda}_S \]

\[ = -N_e^S \gamma_E - d_I N_b N^S \bar{\lambda}_W + d_E N_B N^S \bar{\lambda}_S. \]  

(2.70)

Analysis of the deterministic part of the stochastic mean field equation \( dX_t = \mu(X_t) \) gives insight into the behavior of the process. In this model, the deterministic part has bistable steady states that correspond to the sleep dominant and wake dominant states, respectively, and solutions the deterministic equations approach to one of the steady state depending on their initial positions as time goes to infinity. The diffusion term works for the process to move between these two steady states. The scales of transitions depend on the strength of the diffusion term. In the following section, the bistability of the deterministic part will be investigated using bifurcation theory.

**2.4.5 Analysis of the bistable states of the deterministic part**

In this section, it is assumed that the model is symmetric, since the symmetric mean field model will be mainly considered. Asymmetric cases will be discussed later in
Section 2.5. That is, assume that both clusters are drawn from the same random graph model (Erdős-Rényi (ER) or scale-free(SF)) of the same size $N$, and have the same degree distribution. Then, the mean excitatory and inhibitory degrees $d_E$ and $d_I$ will be treated as the key parameters for the bifurcation study, and other parameters are fixed such as

$$r_I = 0.002, \gamma_E = 0.005, \lambda_I = 0.001, \lambda_B = 0.003, \lambda_E = 0.016.$$ 

Those rate constants have been chosen to get the bistability of the the deterministic part.

The bifurcation diagram of $d_I$ vs. $N^W_E$ for a fixed $d_E$ (Figure 2.5) which is numerically calculated by using computational program XPPAUT [17] shows that there exist two subcritical pitchfork bifurcation points at $d_I = 3.892$, $d_I = 43.25$, and two saddle node bifurcation points at $d_I = 3.784$, $d_I = 44.3$. There is a bistable regime between the saddle node bifurcation points. One of the stable steady states at which $N^W_E >> N^S_E$ corresponds to the wake dominant state, and another stable steady state at which $N^W_E << N^S_E$ corresponds to the sleep dominant state. Moreover, continuation of the subcritical pitchfork bifurcation points with respect to $d_I$ and $d_E$ computed by using AUTO [18] shows that the bistable window is proportional to $d_E$ and $d_I$. On the left bifurcation point, $d_I \approx 0.55d_E$ and on the right one, $d_I \approx 5.9d_E$ (see Figure 2.6).
Figure 2.5: Bifurcation diagram for $N_E^W$ vs $d_I$. The black solid line represents stable steady states and the red dashed line represents unstable steady states. Subcritical pitchfork (PF) bifurcation points occur at $d_I = 3.892$ and $d_I = 43.25$ for $N = 100$, $d_E = 7$, and saddle node (SN) bifurcation points occur at $d_I = 3.784$, $d_I = 44.3$.

Figure 2.6: Continuation of subcritical pitchfork bifurcation points for $d_E$ vs $d_I$. 
2.4.6 The diffusion matrix $B(N)$

The diffusion matrix $B(N)$ is the square root of the covariance matrix

$$V(N) = \begin{bmatrix} v_{11}(N) & v_{12}(N) & v_{13}(N) & v_{14}(N) \\ v_{21}(N) & v_{22}(N) & v_{23}(N) & v_{24}(N) \\ v_{31}(N) & v_{32}(N) & v_{33}(N) & v_{34}(N) \\ v_{41}(N) & v_{42}(N) & v_{43}(N) & v_{44}(N) \end{bmatrix}. $$

Since the model is symmetric, the covariance matrix $V$ is symmetric. Therefore, it is sufficient to calculate $v_{ij}$ for $i \geq j$.

$$v_{11}(N) = \frac{E[(\Delta N^W_I)^2]}{\Delta t} = N_I^W r_I + \sum_{m_1,m_2,k_1,k_2} m_1^2 p_W(m_1,m_2) Q_S(k_1,k_2) \bar{\lambda}_W$$

$$+ \sum_{m_1,m_2,k_1,k_2} m_1^2 q_W(m_1,m_2) P_S(k_1,k_2) \bar{\lambda}_S$$

$$= N_I^W r_I + \left( \frac{N_I^W (N_I^W - 1)}{N(N-1)} \left(v_E + d_E^2 - d_E\right) + d_E \frac{N_I^W}{N} \right) \bar{\lambda}_W$$

$$+ \left( \frac{N_B^W (N_B^W - 1)}{N(N-1)} \left(v_I + d_I^2 - d_I\right) + d_I \frac{N_B^W}{N} \right) \bar{\lambda}_S, \tag{2.71}$$

by proposition 2.2. Similarly,

$$v_{22}(N) = \frac{E[(\Delta N^W_E)^2]}{\Delta t} = N_E^W \gamma_E + \sum_{m_1,m_2,k_1,k_2} m_2^2 p_W(m_1,m_2) Q_S(k_1,k_2) \bar{\lambda}_W$$

$$+ \sum_{m_1,m_2,k_1,k_2} m_2^2 q_W(m_1,m_2) P_S(k_1,k_2) \bar{\lambda}_S$$

$$= N_E^W \gamma_E + \left( \frac{N_B^W (N_B^W - 1)}{N(N-1)} \left(v_E + d_E^2 - d_E\right) + d_E \frac{N_B^W}{N} \right) \bar{\lambda}_W$$

$$+ \left( \frac{N_E^W (N_E^W - 1)}{N(N-1)} \left(v_I + d_I^2 - d_I\right) + d_I \frac{N_E^W}{N} \right) \bar{\lambda}_S, \tag{2.72}$$

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\[ v_{33}(N) = \frac{E[(\Delta N_i^W)^2]}{\Delta t} \]
\[ = N_i^S r_I + \sum_{m_1,m_2} k_i^2 P_W(m_1,m_2) Q_S(k_1,k_2) \bar{\lambda}_W \]
\[ + \sum_{m_1,m_2} k_i^2 Q_W(m_1,m_2) P_S(k_1,k_2) \bar{\lambda}_S \]
\[ = N_i^S r_I + \left( N_i^S (N_i^S - 1) \left( v_E + d_E^2 - d_E \right) + d_E N_i^S \right) \bar{\lambda}_S \]
\[ + \left( \frac{N_i^S (N_i^S - 1)}{N(N - 1)} (v_I + d_I^2 - d_I) + d_I N_i^S \right) \bar{\lambda}_W, \]
(2.73)

and

\[ v_{44}(N) = \frac{E[(\Delta N_i^E)^2]}{\Delta t} \]
\[ = N_E^S r_E + \sum_{m_1,m_2} k_i^2 P_W(m_1,m_2) Q_S(k_1,k_2) \bar{\lambda}_W \]
\[ + \sum_{m_1,m_2} k_i^2 Q_W(m_1,m_2) P_S(k_1,k_2) \bar{\lambda}_S \]
\[ = N_E^S r_E + \left( N_E^S (N_E^S - 1) \left( v_E + d_E^2 - d_E \right) + d_E N_E^S \right) \bar{\lambda}_S \]
\[ + \left( \frac{N_E^S (N_E^S - 1)}{N(N - 1)} (v_I + d_I^2 - d_I) + d_I N_E^S \right) \bar{\lambda}_W, \]
(2.74)

\[ v_{12}(N) = v_{21}(N) = \frac{E[(\Delta N_i^W \Delta N_i^E)]}{\Delta t} \]
\[ = - \sum_{m_1,m_2} m_1 m_2 P_W(m_1,m_2) Q_S(k_1,k_2) \bar{\lambda}_W \]
\[ - \sum_{m_1,m_2} m_1 m_2 Q_W(m_1,m_2) P_S(k_1,k_2) \bar{\lambda}_S \]
\[ = \frac{N_i^W N_i^W}{N(N - 1)} (v_E + d_E^2 - d_E) \bar{\lambda}_W - \frac{N_i^W N_i^W}{N(N - 1)} (v_I + d_I^2 - d_I) \bar{\lambda}_S, \]
(2.75)
and
\[
v_{34}(N) = v_{43}(N) = \frac{E[\Delta N_I^S \Delta N_W^S]}{\Delta t} \\
- \sum_{m_1, m_2 \atop k_1, k_2} k_1 k_2 P_W(m_1, m_2) Q_S(k_1, k_2) \tilde{\lambda}_W \\
- \sum_{m_1, m_2 \atop k_1, k_2} k_1 k_2 Q_W(m_1, m_2) P_S(k_1, k_2) \tilde{\lambda}_S \\
= \frac{N^S B^S N^S}{N(N-1)} (v_I + d_I^2 - d_I) \tilde{\lambda}_S - \frac{N^S W^S}{N(N-1)} (v_E + d_E^2 - d_E) \tilde{\lambda}_W,
\]
by proposition 2.3.

\[
v_{13}(N) = v_{31}(N) = \frac{E[\Delta N_I^W \Delta N_I^S]}{\Delta t} \\
= - \sum_{m_1, m_2 \atop k_1, k_2} m_1 k_1 P_W(m_1, m_2) Q_S(k_1, k_2) \tilde{\lambda}_W \\
- \sum_{m_1, m_2 \atop k_1, k_2} m_1 k_1 Q_W(m_1, m_2) P_S(k_1, k_2) \tilde{\lambda}_S \\
= -d_E d_I \frac{N^W B^W N^S}{N^2} \tilde{\lambda}_W - d_E d_I \frac{N^W W^S}{N^2} \tilde{\lambda}_S,
\]

\[
v_{14}(N) = v_{41}(N) = \frac{E[\Delta N_I^W \Delta N_W^S]}{\Delta t} \\
= \sum_{m_1, m_2 \atop k_1, k_2} m_1 k_2 P_W(m_1, m_2) Q_S(k_1, k_2) \tilde{\lambda}_W \\
+ \sum_{m_1, m_2 \atop k_1, k_2} m_1 k_2 Q_W(m_1, m_2) P_S(k_1, k_2) \tilde{\lambda}_S \\
= d_E d_I \frac{N^W B^W N^S}{N^2} \tilde{\lambda}_W + d_E d_I \frac{N^W W^S}{N^2} \tilde{\lambda}_S,
\]

\[
v_{23}(N) = v_{32}(N) = \frac{E[\Delta N_I^E \Delta N_I^S]}{\Delta t} \\
= \sum_{m_1, m_2 \atop k_1, k_2} m_2 k_1 P_W(m_1, m_2) Q_S(k_1, k_2) \tilde{\lambda}_W \\
+ \sum_{m_1, m_2 \atop k_1, k_2} m_2 k_1 Q_W(m_1, m_2) P_S(k_1, k_2) \tilde{\lambda}_S \\
= d_E d_I \frac{N^W B^W N^S}{N^2} \tilde{\lambda}_W + d_E d_I \frac{N^W E^S}{N^2} \tilde{\lambda}_S,
\]

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and

\[ v_{24}(N) = v_{42}(N) = \frac{E[\Delta N_E^W \Delta N_E^S]}{\Delta t} \]

\[ = - \sum_{m_1,m_2 \atop k_1,k_2} m_2 k_2 P_W(m_1, m_2) Q_S(k_1, k_2) \hat{\lambda}_W \]

\[ - \sum_{m_1,m_2 \atop k_1,k_2} m_2 k_2 Q_W(m_1, m_2) P_S(k_1, k_2) \hat{\lambda}_S \]

\[ = - d_E d_I \frac{N_B^W N_I^S}{N^2} \hat{\lambda}_W - d_E d_I \frac{N_E^W N_B^S}{N^2} \hat{\lambda}_S, \]

by proposition 2.1.

The deterministic part \( \mu(N) = (\mu_1(N), \mu_2(N), \mu_3(N), \mu_4(N))^T \) of the stochastic mean field equations is the following:

\[ \mu_1(N) = - N_I^W r_I - d_E \frac{N_I^W}{N} \hat{\lambda}_W + d_I (N - N_I^W - N_E^W) \hat{\lambda}_S, \]

\[ \mu_2(N) = - N_E^W \gamma_E + d_E \frac{(N - N_I^W - N_E^W)}{N} \hat{\lambda}_W - d_I \frac{N_E^W}{N} \hat{\lambda}_S \]

\[ \mu_3(N) = - N_I^S r_I + d_I \frac{(N - N_I^S - N_E^S)}{N} \hat{\lambda}_W - d_E \frac{N_I^S}{N} \hat{\lambda}_S \]

\[ \mu_4(N) = - N_E^S \gamma_E - d_I \frac{N_E^S}{N} \hat{\lambda}_W + d_E \frac{(N - N_I^S - N_E^S)}{N} \hat{\lambda}_S. \]

A sample solution of the stochastic mean field equations is given in Figure 2.7.

In Figure 2.8, projections of trajectories on the \((N_I^W, N_E^S)\) plane are given for each ER or SF graph, and inhibitory mean degree \(d_I\) for a fixed network size \(N = 100\) and excitatory mean degree \(d_E = 7\). By symmetry of the model, at the unstable steady state (more precisely which is a saddle point), \(N_I^W = N_E^S\) and \(N_I^W = N_I^S\).

For a fixed \(d_E\), as \(d_I\) increase, \(N_E^W\) value at the saddle point decreases (see Figure 2.5), and the projection of the unstable steady state on \((N_I^W, N_E^S)\) plane moves towards the origin. A trajectory that moves between two stable steady states passes through a neighborhood of the saddle point. Therefore, the region covered by the trajectory is more curved for larger \(d_I\). Moreover, the trajectory band is broader for SF networks than ER networks, since the variance of the degree distribution of
a SF network is greater than the variance of excitatory degrees for an ER network. The variance is a critical parameter in the diffusion term, since the maximum of positive eigenvalues of the diffusion matrix is greater if the variance is greater. For instance, for an ER graph $G(100, 0.7)$, (that is, $N = 100$, $p_N$, the probability that an edge exist is 0.7), $d_E = 6.93$ and $v_E = 6.45$, but for a SF graphs with the scaling exponent $\gamma = 2.38$ and $k_{\text{min}} = 3$, $d_E = 6.99$ but $v_E = 77.04$. That is, for the ER and SF graphs with the similar mean degree, diffusion coefficient of the SF graph is greater than the ER graph. The difference between the variance of ER/SF graphs increases as $d_E$ increases (see 2.9). For the combination of parameters $(d_E, d_I)$ near the left subcritical pitchfork bifurcation points where $d_I \approx 0.55d_E$, the trajectory band projected onto the $(N^W_E, N^W_E)$ plane is almost linear since at the left bifurcation point, $N^W_E = N^S_E \approx \frac{N}{2})$. The process will be approximated by one dimensional reflected Brownian motion (see Section 2.6 for details) by the choice of the pair of $(d_E, d_I)$ near the left bifurcation point.

![Figure 2.7: Time series of sample solution for $N^W_E$ (blue) and $N^S_E$ (green) for ER random graphs with $d_E = 7$ and $d_I = 3.85$.](image)

Figure 2.7: Time series of sample solution for $N^W_E$ (blue) and $N^S_E$ (green) for ER random graphs with $d_E = 7$ and $d_I = 3.85$. 

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Figure 2.8: Projections of sample trajectories on \((N^W_E, N^S_E)\) plane for each network (ER or SF) and mean inhibitory degree. Black dots and red dots represent stable and unstable steady states of deterministic part respectively.
2.5 Trajectory study and numerical analysis

In this section, it will be assumed that the model is symmetric, since this is the simplest to analyze and to approximate by one dimensional well-known stochastic models. Asymmetric cases of the random network models were studied in [7], and asymmetric cases in the stochastic mean field model will be discussed in Chapter 5.

Analysis of numerical simulations of the stochastic mean field model can be used to gain insights to investigate the dynamics of the system. A bout duration will be defined as the escape time of the process from sleep and wake active domains. The distribution of bout durations for various combinations of the graph type, network
size and mean degrees will be obtained numerically to see the appearance of power law
distribution in some time interval. It will be investigated what the differences of the
power law intervals and exponents for the different classes of random graph models
(eg. Erdős-Rényi or scale-free) with different degree distributions result from. And
then, studies on the combination of the drift and diffusion parts of the system help
to understand the behavior of trajectories, especially linear shaped trajectory band.
Findings from this section will give the motivation for the analysis in the Sections
2.6, 2.7 by reflected Brownian motions and large deviation theory.

2.5.1 Bout distributions

There are several ways to define bout durations, and the simplest one is to define a
wake bout from the time at which $N_E^S < N_E^W$ to the time at which $N_E^S > N_E^W$ and
vice versa for a sleep bout. By this definition, numerous small recrossing events are
counted as bouts. However, it does not make sense for those very short bouts to be
considered as sleep and wake durations. (see Figure 2.10).
Figure 2.10: (Top) A realization of time series of $N_E^W$ and $N_E^S$. (Bottom) Plot of bout durations defined simply by the time from $N_E^S < N_E^W$ to $N_E^S > N_E^W$ and vice versa. The intervals of values 0 and 1 correspond to sleep and wake bouts respectively. This shows too frequent transitions between sleep and wake dominant states.

To avoid this situation, sleep/wake domains on the $(N_E^S, N_E^W)$ plane will be defined by using the numerically computed stationary distribution of the process. The stationary distribution is bimodal with the two peaks corresponding to a large number of excited nodes in one cluster and small number of excited nodes in the other cluster in the $(N_E^S, N_E^W)$ plane. (See Figure 2.11). Define wake- and sleep-active domains as the two disconnected components of the largest disconnected super-level set of the probability density function. (Figure 2.12). Let $D^W(D^S) \subset \mathbb{R}^2$ be the wake (sleep) active domain which includes the peak where $N_E^W > N_E^S$ (or $N_E^W < N_E^S$).
Figure 2.11: Simulated probability density function of \((N_E^S, N_E^W)\)

Figure 2.12: Wake-active and sleep-active domains defined as the largest non-overlapping level set of the p.d.f in Figure 2.11; Blue: the wake active domain \((D^W)\). Green: the sleep active domain \((D^S)\). Black curve: a trajectory corresponding to the \(B^2\) type wake bout, defined below. Red curve: a transition bout from \(D^W\) to \(D^S\).
Since the network is symmetric, only the distribution of wake bouts will be considered. Let \( \tilde{N}(t) \) be the projection of \( N(t) \) on the \( (N^S_E, N^W_E) \) plane. Assume that for \( t_0 > 0, \tilde{N}(t_0) \in D^S \). Let

\[
\begin{align*}
    t_{W_1} &:= \inf\{ t > t_0 : \tilde{N}(t) \in D^W \}, \\
    t_{S_1} &:= \sup\{ t < t_{W_1} : \tilde{N}(t) \in D^S \}, \\
    t_{S_2} &:= \inf\{ t > t_{W_1} : \tilde{N}(t) \in D^S \}, \\
    t_{W_2} &:= \sup\{ t < t_{S_2} : \tilde{N}(t) \in D^W \}.
\end{align*}
\]

(2.82) (2.83) (2.84) (2.85)

Then, wake bouts can be defined in two ways such that

\[
\begin{align*}
    B^1 &= t_{W_2} - t_{W_1} \\
    B^2 &= B^1 + T_{W \rightarrow S} = t_{S_2} - t_{W_1}
\end{align*}
\]

(2.86) (2.87)

where \( T_{S \rightarrow W} = t_{W_1} - t_{S_1} \) is called the transition bout from \( D^S \) to \( D^W \). Then a \( B^1 \)-bout can be interpreted as the time interval from the first time at which the trajectory leaving the sleep domain enters the wake domain to the last time when it departs the wake domain before entering the sleep domain, and \( B^2 \)-bouts can be described as the time interval from from the first time at which the trajectory leaving the sleep domain enters the wake domain to the next time entering sleep domain.

To see effect of degree distribution and network size on the bout distribution, five graphs in different combinations of network size, graph type and mean degrees of graphs are considered: \( ER(100, 7), ER(100, 10), ER(1000, 20), SF(100, 7) \) and \( SF(100, 10) \), where \( ER(N, d_E) \) (or \( SF(N, d_E) \)) represents that each cluster is an Erdös-Rényi (or scale-free) random graph of the size \( N \), the mean excitatory degree \( d_E \) and the mean inhibitory degree \( d_I = 0.55d_E \).
Figure 2.13: Log-log plot of distributions of $B^1$- and $B^2$- bouts of ER(100,7)

Figure 2.14: Log-log plot of distributions of $B^1$- and $B^2$- bouts of ER(1000,20)
Wake bout distributions of ER(100,7) and ER(1000,20) from one realization are shown in Figure 2.13, and 2.14 respectively. It is observed that a power law distribution appears over the intermediate range of time, and it is followed by an exponential distribution. The interval and the exponent of the power law distribution are confirmed by the power law detection method shown in Chapter A. Moreover, the expected values of scale constant, lower and upper cut off of the power law regime for the bout duration from 10 realizations of $10^4$ wake bouts in each network are given in Tables 2.3, 2.4 and 2.5. For the ER(100,7) model, it takes about 8 hours to get one realization of $10^4$ wake bouts. The ten realizations with different data sizes 2000, 5000, and 10000 were investigated. Let $\alpha_i^k$ be the estimated exponent of the realization $i = 1 \cdots 10$, and $k = 2000, 5000, 10000$, and let

$$\hat{\alpha}^k = \frac{1}{10} \sum_{i=1}^{10} \alpha_i^k.$$  \hfill(2.88)

The data size $k$ is chosen such that

$$\max_i \left| \alpha_i^k - \hat{\alpha}^k \right| < 0.1.$$  \hfill(2.89)

For each $k = 2000, 5000, 10000$,

$$\max_i \left| \alpha_i^{2000} - \hat{\alpha}^{2000} \right| = 0.4416,$$

$$\max_i \left| \alpha_i^{5000} - \hat{\alpha}^{5000} \right| = 0.2304,$$

$$\max_i \left| \alpha_i^{10000} - \hat{\alpha}^{10000} \right| = 0.0783,$$

so the realizations of data size $10^4$ have been used. This was tested only for the ER(100,7) model, and applied for all types of models.
<table>
<thead>
<tr>
<th>Type</th>
<th>ER(100,7)</th>
<th>ER(100,10)</th>
<th>ER(1000,20)</th>
<th>SF(100,7)</th>
<th>SF(100,10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^1$</td>
<td>0.9914</td>
<td>0.9669</td>
<td>1.2394</td>
<td>0.7211</td>
<td>0.7347</td>
</tr>
<tr>
<td>$B^2$</td>
<td>1.2879</td>
<td>1.2500</td>
<td>1.5081</td>
<td>1.1644</td>
<td>1.0160</td>
</tr>
</tbody>
</table>

Table 2.3: Exponent on the power law distribution

<table>
<thead>
<tr>
<th>Type</th>
<th>ER(100,7)</th>
<th>ER(100,10)</th>
<th>ER(1000,20)</th>
<th>SF(100,7)</th>
<th>SF(100,10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^1$</td>
<td>9.9143e-01</td>
<td>9.6692e-01</td>
<td>1.2394e+00</td>
<td>7.2111e-01</td>
<td>7.3467e-01</td>
</tr>
<tr>
<td>$B^2$</td>
<td>3.2714e+01</td>
<td>3.2500e+01</td>
<td>1.2906e+01</td>
<td>3.4089e+01</td>
<td>2.4580e+01</td>
</tr>
</tbody>
</table>

Table 2.4: Lower cutoff $t_{\text{min}}$ for the power law regime

<table>
<thead>
<tr>
<th>Type</th>
<th>ER(100,7)</th>
<th>ER(100,10)</th>
<th>ER(1000,20)</th>
<th>SF(100,7)</th>
<th>SF(100,10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^1$</td>
<td>2.8390e+02</td>
<td>2.8128e+02</td>
<td>1.1069e+02</td>
<td>1.0473e+02</td>
<td>1.0637e+02</td>
</tr>
<tr>
<td>$B^2$</td>
<td>5.4606e+02</td>
<td>4.2827e+02</td>
<td>2.8853e+02</td>
<td>4.2319e+02</td>
<td>3.8311e+02</td>
</tr>
</tbody>
</table>

Table 2.5: Upper cutoff $t_{\text{max}}$ for the power law regime
Table 2.6: The ratio $t_{max}/t_{min}$ for the power law regime

<table>
<thead>
<tr>
<th>Type</th>
<th>ER(100,7)</th>
<th>ER(100,10)</th>
<th>ER(1000,20)</th>
<th>SF(100,7)</th>
<th>SF(100,10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^1$</td>
<td>2.8635e+02</td>
<td>2.9090e+02</td>
<td>8.9309e+01</td>
<td>1.4523e+02</td>
<td>1.4479e+02</td>
</tr>
<tr>
<td>$B^2$</td>
<td>1.6692e+01</td>
<td>1.3178e+01</td>
<td>2.2356e+01</td>
<td>1.2414e+01</td>
<td>1.5586e+01</td>
</tr>
</tbody>
</table>

First of all, it is remarkable that the exponent of SF(100,7)$\approx$SF(100,10) $<\approx$ ER(100,7) $\approx$ ER(100,10)$<\approx$ER(1000,20), which is related to the width of the swath in $(N^W_E, N^S_E)$ plane covered by trajectories (See Figure 2.15 (a) (b)). It is shown that the mean degree in each network does not much affect the width of the trajectory band but the bands are more narrow for networks with greater size and ER graphs than SF graphs (see Figure 2.15). That is because, a SF graph has greater variance of excitatory degrees than a ER graph, which affects the diffusion part. For instance, the variance of excitatory degree $v_E = 6.51$ of ER(100, 7), but $v_E = 77.04$ of SF(100, 7). Moreover, when the system is normalized by the network size $N$, the diffusion part is proportional to $\frac{1}{\sqrt{N}}$. Figures 2.16, 2.17 show that as $N$ increases the trajectory bands narrow and the exponent approaches to $-1.5$ which is the exponent of the power law distribution for the reflected Brownian motion (details will be given in section 2.6). We investigated this behavior for large $N$ by large deviation theory in section 2.7.
Figure 2.15: Plots of sample trajectories on the $(N^W_E, N^S_E)$ plane of (a) ER(100,7) and ER(100,10) and (b) SF(100,7) and SF(100,10), which shows that the mean degree does not much affect the width of the trajectory band. (c) Trajectories of the model SF(100,7) (red), ER(100,7) (blue) and ER(1000,20) (black) are super imposed $(N^W_E/N, N^S_E/N)$ plane.
Figure 2.16: Sample trajectories in the \((N_E^S, N_E^W)\) plane for several network sizes \(N\). The trajectory band narrows as \(N\) increases.

Figure 2.17: Exponents of the power law distribution of \(B^1\)-bouts for each network size.
For the same type graph with the same size, greater mean degree speed up the transition between two domains. That is, when ER(100,7) and ER(100,10) (or SF(100,7) and SF(100,10)) are compared it is observed that the exponent are robust but lower and upper cutoffs are smaller for the networks with smaller mean degrees. It is also shown that $B^2$-bouts (bouts with the transition part) have greater exponents than $B^1$-bouts in all types of networks in Table 2.3.

2.5.2 Study on the drift and diffusion coefficients

Recall the stochastic mean field equation,

$$dN = \mu(N)dt + B(N)dW,$$  \hspace{1cm} (2.90)

where $\mu$ is referred to a drift vector, and $B$ is called a diffusion matrix. This concentration of trajectories on a band can be partially understood by a comparison of the drift and diffusion of the process. The band of trajectories can be explained by combination of the drift and the diffusion parts of the system. Figure 2.18 is obtained by averaging the vectors $\mu$ over all $N_W^I, N_S^I$, and plotted on the $(N_E^W, N_E^S)$ plane. Since the drift vectors point towards the band, it pushes a trajectory to the band and let it be restricted in the region. In Figure 2.19, there are plots of magnitude of drift vector, maximum of eigenvalues of diffusion matrix, their ratio and the Péclet numbers which is a dimensionless number relevant in the effect of diffusion of a system defined as

$$\text{Pécllet number} = \frac{\text{length scale} \cdot \| \text{drift} \|}{\frac{1}{2} \text{max(eigenvalues of diffusion tensor)}^2},$$  \hspace{1cm} (2.91)

for the ER(100,7). Small Péclet numbers on a region represent that the diffusion is dominant on the region. In Figure 2.19 (a), it is observed that magnitudes of drift vectors are smaller within the band, and Figure 2.19 (d) shows that the Péclet numbers are smallest within the band. That is, the effect of diffusion is stronger than
the drift in that region. Therefore, within the band, the dominant diffusion implies a source of memory that is involved in the creation of bounded power-law distributed regions in the distributions of bout durations. By drift vectors towards the band, trajectories are trapped and stay in the band, where the diffusion is dominating. Therefore, the process can be approximated by one dimension stochastic process restricted on a finite interval.

A Brownian motion with two reflecting boundaries is the natural one dimensional model to describe power law distribution of the bout durations which will be given in section 2.6. After that, the width of band for large $N$ will be discussed in section 2.7 by large deviation theory.

Figure 2.18: Plots of averaged drift vectors on the $(N_E^W, N_E^S)$ plane.
Figure 2.19: Plots of (a) the magnitude of drift vectors, (b) the maximum of eigenvalues of diffusion matrix, (c) their ratio for magnitude of drift vectors to the maximum of eigenvalues of diffusion matrix, and (d) Péclet numbers defined in (2.19).
2.6 First passage time of reflected Brownian motion

There are several ways to define a reflected brownian $Y_t$ on the half line $[0, \infty)$ with reflecting boundary $0$, but the simplest definition is that

$$Y_t = |W_t|, \text{ for } Y_0 > 0,$$

(2.92)

where $W_t$ is the traditional Brownian motion. Here, a reflected brownian motion defined on finite interval $[-M, M]$ with reflecting barriers at $-M$ and $M$ will be considered as a simple model to understand power law behavior of the full model. Sleep and wake active domains of the network model correspond to $[-M, -x_0]$ and $[x_0, M]$, respectively. Then, negative bout $\tau_-$ and positive bout $\tau_+$ for $Z_t$ can be defined similarly to wake and sleep bouts including the transition bout in the stochastic mean field model such that

$$\tau_+ := \inf \{ t : Z_t = -x_0 \mid Z_0 = x_0 \},$$

(2.93)

$$\tau_- := \inf \{ t : Z_t = x_0 \mid Z_0 = -x_0 \}.$$

(2.94)

By symmetry of Brownian motion, $\tau_+$ can be viewed as

$$\tau_+ = \inf \{ t : W_t = -x_0 \text{ or } W_t = 2M + x_0 \mid W_0 = x_0 \}$$

(2.95)

(See Figure 2.20).
Figure 2.20: Blue: a sample positive bout of two sided RBM for $M = 5$, $x_0 = 1$ by the definition (2.93). Red: a sample positive bout by the definition (2.95).

Let

$$\hat{\tau}_+ = \inf \{ t : W_t = 0 \text{ or } W_t = 2M + 2x_0 | W_0 = 2x_0 \},$$

(2.96)

then

$$\tau_+ =_d \hat{\tau}_+, \text{ since } W_t - W_0 \sim N(0, t)$$

(2.97)

Let $f_{T_0 \wedge T_a}^x$ be the probability density function (pdf) of

$$T_0 \wedge T_a = \min \{ t : W_t = 0 \text{ or } W_t = a | W_0 = x \}.$$  

(2.98)

Then, for $x = 2x_0$ and $a = 2M + 2x_0$,

$$T_0 \wedge T_a = \hat{\tau}_+.$$  

(2.99)
In [19] (p.99), the probability density function of $T_0 \wedge T_a$ is that

$$f_{T_0 \wedge T_a}(t) = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} \left[ (2na + x) \exp\left(-\frac{(2na + x)^2}{2t}\right) 
+ (2na + a - x) \exp\left(-\frac{(2na + a - x)^2}{2t}\right) \right].$$

(2.100)

The distribution of first exit time for a $d$-dimensional system $X_t$ which solves

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

(2.101)

can be obtained by the backward Kolmogorov equation [20]. Let $G$ be the infinitesimal generator of $X \in \mathbb{R}^d$ which has the form

$$Gf(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f(x)}{\partial x_i}$$

(2.102)

where $a = \sigma\sigma^T$ and $f \in C^2_K(\mathbb{R}^d)$, the space of twice continuously differentiable functions on $\mathbb{R}^d$ with compact support. Let $D$ be an open bounded Borel subset of $\mathbb{R}^d$ with boundary $\partial D$, and let $\tau_D$ be the first exit time of $X_t$ from the domain $D$

$$\tau_D = \inf\{t : X_t \notin D\}.$$

(2.103)

Then, $u(t,x)$, the probability that $X$ with $X_0 = x$ did not exit the domain $D$ before $t$, i.e.

$$u(t,x) = \mathbb{P}_x(\tau_D \geq t)$$

(2.104)

is a solution of the backward Kolmogorov equation

$$\frac{\partial u}{\partial t}(t,x) = Gu(t,x) \quad \text{on} \ (t,x) \in \mathbb{R}^+ \times D$$

(2.105)

$$u(0,x) = 1, \quad x \in D$$

(2.106)

$$u(t,x) = 0, \quad x \in \partial D, \ t > 0,$$

(2.107)

and the pdf of the exit time is given by $-\frac{\partial u}{\partial t}(t,x)$. 56
Therefore, the backward Kolmogorov equation associated to the process \( W_t \) is that

\[
  u_t = \frac{1}{2} u_{xx},
\]

(2.108)

\[
  u(x, t) = 1 \quad \text{for} \quad x \in (0, a),
\]

(2.109)

\[
  u(0, t) = 0, \quad u(0, a) = 0 \quad \text{for} \quad t > 0,
\]

(2.110)

since \( \sigma = 1, b = 0 \) in eq. (2.101), and the pdf of \( T_0 \land T_a \) is that

\[
  f_{T_0 \land T_a}^x(t) = -\frac{\partial u}{\partial t}(u, x).
\]

(2.111)

Thus, by solving the partial differential equation, another form of pdf is obtained as

\[
  f_{T_0 \land T_a}^x(t) = \sum_{n=1}^{\infty} \frac{n\pi(1 + (-1)^{(n+1)})}{a^2} \exp\left(-\frac{(n\pi)^2}{2a^2 t}\right) \sin\left(\frac{n\pi x}{a}\right).
\]

(2.112)

Assume that \( a >> x \). In equation (2.100), the 0th term

\[
  \frac{1}{\sqrt{2\pi t^3}} \left[ x \exp\left(-\frac{x^2}{2t}\right) + (a - x) \exp\left(-\frac{(a - x)^2}{2t}\right) \right]
\]

(2.113)

is dominant for \( t < C_1 a^2 \) for some \( C_1 \), since

\[
  \exp\left(-\frac{(2na + a - x)^2}{2t}\right) \leq \exp\left(-\frac{(2na + x)^2}{2t}\right) \leq \exp\left(-\frac{2n}{C_1}\right) < 10^{-8}.
\]

(2.114)

for all \( n \geq 1 \), if \( C_1 < 0.1 \). Moreover in eq. (2.113),

\[
  \exp\left(-\frac{x^2}{2t}\right) >> \exp\left(-\frac{(a - x)^2}{2t}\right)
\]

(2.115)

for \( a >> x \) and \( t < C_1 a^2 \). Therefore,

\[
  f_{T_0 \land T_a}^x(t) \sim \frac{1}{\sqrt{2\pi t^3}} x \exp\left(-\frac{x^2}{2t}\right), \text{if } t < C_1 a^2.
\]

(2.116)

In addition, if \( t > C_2 x^2 \) for some \( C_2 \), then

\[
  \exp\left(-\frac{x^2}{2t}\right) \sim 1.
\]

(2.117)
For instance,
\[ \exp\left(-\frac{x^2}{2t}\right) \in (0.98, 1), \text{ if } t > 25x^2, \tag{2.118} \]
and
\[ \exp\left(-\frac{x^2}{2t}\right) \in (0.99, 1), \text{ if } t > 50x^2. \tag{2.119} \]
That is,
\[ f_x(t) \sim \frac{1}{\sqrt{2\pi t^3}} x, \tag{2.120} \]
a power law distribution of the exponent -1.5, if \( C_2x^2 < t < C_1a^2 \) for some \( C_1 \) and \( C_2 \).

In eq. (2.112), for \( t > C_3a^2 \) for some \( C_3 \), the first term
\[ \frac{2\pi}{a^2} \exp\left(-\frac{\pi^2}{4a^2} t\right) \sin\left(\frac{\pi x}{a}\right), \tag{2.121} \]
is dominant, since the second term is zero, and
\[ \exp\left(-\frac{(n\pi)^2}{2a^2} t\right) < 2 \times 10^{-4}, \tag{2.122} \]
if \( n \geq 3 \) and \( t > \frac{a^2}{5} \). In summary,
\[ f_x(t) \sim \begin{cases} \frac{1}{\sqrt{2\pi t^3}} x \exp\left(-\frac{x^2}{2t}\right), & \text{if } t < C_1a^2 \\ \frac{1}{\sqrt{2\pi t^3}} x, & \text{if } C_3x^2 < t < C_2a^2 \\ \frac{2\pi}{a^2} \exp\left(-\frac{\pi^2}{4a^2} t\right) \sin\left(\frac{\pi x}{a}\right), & \text{if } t > C_3a^2 \end{cases} \tag{2.123} \]
for some \( C_1, C_2 \) and \( C_3 \). These show that if \( x < < a \), in the intermediate regime, the first passage time follows power law distribution of universal scaling parameter \(-3/2\), but for large \( t \), it follows exponential distribution (see Figure 2.21) and its time constant depends on the reflecting boundary \( M \). Moreover, the lower cutoff and upper cutoff of the power law distribution depend only \( x_0 \) and \( M \) respectively.
Figure 2.21: The probability density function of the first passage time of reflected Brownian motion

2.7 Large deviation theory and minimum action method

Large-deviation theory is used to study the exponential asymptotic of the probability of rare events such as transitions between two wells of double well potential processes. The most probable path (MPP) is obtained by minimizing a so-called action functional over the set of all possible sample paths. This theory also informs
the distribution of random transition times between attractors of the deterministic system [15].

In section 2.5, it is shown that as the network size \((N)\) increases, the band of trajectories narrows (Figure 2.16). By large deviation theory, this can be understood as for large \(N\), the trajectories moves near the most probable path from sleep domain steady states to wake domain steady state of the deterministic part (and vice versa).

In this section, large deviation theory will be introduced briefly and then numerical results computing MPPs will be shown and compare those to sample trajectories for models with various size \(N\).

Consider a random process \(X_\varepsilon(t)\) in \(\mathbb{R}^n\) that solves the stochastic differential equation
\[
\frac{dX_\varepsilon}{dt} = b(X_\varepsilon) + \sqrt{\varepsilon}dW,
\]
\(X_\varepsilon(0) = x_0\).

Let \(C[0,T]\) the set of \(\mathbb{R}^n\)-valued continuous functions on the interval \([0,T]\). For \(\phi, \varphi \in C[0,T]\), define the metric
\[
\rho_T(\phi, \varphi) = \sup_{0 \leq t \leq T} |\phi(t) - \varphi(t)|.
\]

For \(\phi \in C[0,T]\), define an action functional associated with (2.126) as
\[
S_T[\phi] = \frac{1}{2} \int_0^T |\sigma^{-1}(\phi' - \phi)|^2 dt
\]
if the integral is finite, otherwise \(S_T[\phi] = \infty\).

The Wentzell-Freidlin theory [21] gives that:

\textbf{Theorem 2.1.} For any \(\delta > 0\), \(\gamma > 0\), and \(K > 0\) there exists an \(\varepsilon_0 > 0\) such that
\[
\mathbb{P}(\rho_T(X_\varepsilon, \phi) < \delta) \geq \exp \left\{ -\frac{1}{\varepsilon} (S_T[\phi] + \gamma) \right\}
\]
for \(\varepsilon < \varepsilon_0\)

where \(T > 0\) and \(\phi \in C[0,T]\) are such that \(\phi(0) = x_0\) and \(T + S_T[\phi] \leq K\).
Theorem 2.2. For any \( \delta > 0, \gamma > 0 \) and \( s_0 > 0 \) there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( s < s_0 \),

\[
\begin{align*}
\mathbb{P}(\rho_T(X_\varepsilon, \Phi_T(s)) \geq \delta) & \leq \exp \left\{ -\frac{1}{\varepsilon} (s - \gamma) \right\}, \\
\end{align*}
\]  

(2.130)

where

\[
\Phi_T(s) = \{ \phi \in C[0,T], \phi(0) = x_0, S_T[\phi] \leq s \} \text{ for } s > 0
\]  

(2.131)

and

\[
\rho_T(X_\varepsilon, \Phi_T(s)) = \sup_{\phi \in \Phi_T(s)} \rho_T(X_\varepsilon, \phi).
\]  

(2.132)

Assume that the deterministic part of (2.126) is bistable, and let \( a, b \in \mathbb{R}^n \) denote its stable steady states (or attractors). Consider paths from \( a \) to \( b \). These Wentzell-Freidlin estimates say that the probability that \( X_\varepsilon \) stays in a \( \delta \)-neighborhood of a path \( \phi \) is

\[
\mathbb{P}(\rho_T(X_\varepsilon, \phi) < \delta) \approx \exp \left\{ -\frac{1}{\varepsilon} S_T[\phi] \right\}.
\]  

(2.133)

Let \( B_\delta(b) \) be \( \delta \)-neighborhood of \( b \). Then the probability that a particle starting at \( a \) arrives the neighborhood of \( b \) within a time \( T \),

\[
P_T := \mathbb{P}(\min\{t > 0 : X_\varepsilon(t) \in B_\delta(b)\} < T | X_\varepsilon(0) = a),
\]  

(2.134)

can be estimated such as

\[
\lim_{\varepsilon \to 0} \varepsilon \ln P_T = -\min_{\phi : \phi(0) = a, \phi(T) = b} S_T[\phi].
\]  

(2.135)

That is, \( P_T \) has the maximum when \( S_T[\phi] \) has the minimum. Therefore, if the minimum action trajectory from \( a \) to \( b \) exists and is unique, it is interpreted as the most probable path (MPP) from \( a \) to \( b \) the minimizer of \( S_T[\phi] \) is the most probable path in : the one whose cost is minimum. Moreover, the probability that the process moves by another path is exponentially smaller in \( \varepsilon \).
Recall the stochastic mean field equations,

\[ d\mathbf{N} = \mu(\mathbf{N})dt + \mathbf{B}(\mathbf{N})d\mathbf{W}, \]  

(2.136)

where the drift vector \( \mu(\mathbf{N}) \) and the diffusion matrices \( \mathbf{B}(\mathbf{N}) \) are given in section 2.4.4 and 2.4.6. Rewrite this equation in terms of the normalized population \( \mathbf{X} = \left[ \frac{N^W_I}{N}, \frac{N^W_E}{N}, \frac{N^S_I}{N}, \frac{N^S_E}{N} \right]^T \), then

\[ d\mathbf{X} = \mu(\mathbf{X})dt + \sqrt{\varepsilon}\mathbf{B}(\mathbf{X})d\mathbf{W} \]  

(2.137)

where \( \varepsilon = 1/N \). Then the action functional is defined by

\[ S_T(U) = \int_0^T L(\dot{U}, U) dt \]  

(2.138)

where \( L(\dot{U}, U) = \frac{1}{2} < \dot{U} - b(U), a^{-1}(U)(\dot{U} - b(U)) > \). Then the minimum action path (MPP) from sleep dominant peak to wake dominant peak, which minimizes \( S_T(U) \) are numerically calculated by using minimum action method given in [22]. For details about this method, see section B. Recall that \( ER(N, d_E) \) represents a model on which both clusters are ER type graphs with network size \( N \) and intra cluster mean degree \( d_E \). Sample trajectories superimposed on the MPP’s for \( ER(100,7), \) \( ER(200,14), \) \( ER(500,25) \) and \( ER(1000,20) \) are given in Figure 2.22- 2.25 respectively. These show that for larger \( N \), trajectories tend to be close to the MPP as discussed above (see also Figure 2.26 to compare trajectories in each network).
Figure 2.22: Four sample trajectories (blue) imposed on the MPP (red) for ER(100,7)
Figure 2.23: Four sample trajectories (blue) imposed on the MPP (red) for ER(200,14)
Figure 2.24: Four sample trajectories (blue) imposed on the MPP (red) for ER(500,25)
Figure 2.25: Four sample trajectories (blue) imposed on the MPP (red) for ER(1000,20)
Figure 2.26: Sample trajectories (blue) imposed on the MPP (red) for each ER(100,7) ER(200,14) ER(500,25) and ER(1000,20). It is observed that if \( N \) is larger, sample trajectories are closer to the MMP.

In Figure 2.17, it was shown that the exponent of the power law distribution increases and approaches to -1.5 which is the exponent of the power law distribution of reflected Brownian motions as mentioned in Section 2.6. This can be explained that as \( N \) increase, the process behaves more like a one dimensional Brownian motion. This intuition could be supported by In Figure 2.27, where the locations on the boundary
of the $D^W$, $D^S$ domains at which the trajectories enter or leave the domain to go to the other domain are plotted in dots for ER(100, 7) and for ER(1000, 20). For ER(100, 7), the dots are broadly streamed on the boundary of each domain, while for ER(1000, 20), the dots huddle in the center. This means that for large $N$, trajectories go into and out of a domain via a small neighborhood the point which is most close to the other domain, and the wake (or sleep) $B^2$-bout durations of this process can be approximated by the positive (or negative) bouts of the reflected Brownian motion defined on a finite interval.

Figure 2.27: Locations on the boundaries of the $D^W$, $D^S$ domains at which the trajectories enter or leave the domain to go to the other domain are plotted in dots for ER(100, 7) (left) and for ER(1000, 20) (right).

Large deviation theory is applied to a small perturbed system for the deterministic equations. However, for the scale-free graph, the variance of excitatory degrees ($v_E$) is pretty much larger than the one for the ER graph, which gives stronger diffusion
of the process. Figure 2.28 shows that \( v_E \) vs \( d_E \) for the ER graph and SF graphs respectively, and it is observed that \( v_E \) increases as \( d_E \) increases for a SF (see Figure 2.29).

![Plot of variance of degrees vs mean degrees for ER (blue) and SF (red) graphs with the size \( N = 1000 \).](image)

Figure 2.28: Plot of variance of degrees vs mean degrees for ER (blue) and SF (red) graphs with the size \( N = 1000 \).
Figure 2.29: Plot of trajectories for SF(1000,20) (blue), SF(1000,15) (red) and SF(1000,10) (black), where their variances are $v_E = 2537.3$, $v_E = 1202.6$ and $v_E = 249.1$ respectively.
CHAPTER 3

PENALIZATION METHOD

The stochastic mean field model for the sleep-wake transition dynamics is a bistable process, and it was approximated by a reflected Brownian motion to understand power law behavior of bout durations in section 2.6. However, traditional double-well potential models under the influence of white noise show exponential distributions of the escape times from the wells [23]. Therefore, it was interesting to ask whether a family of double well potential processes could converge to RBM, and they have an power law distribution of the escape times from the wells in the intermediate time interval, as the stochastic differential equations for the network model suggested. In this chapter, it will be analytically proved that the limiting process of a specified sequence of bistable potential processes, where the potential between the wells is almost zero, and steep walls of the wells prevents the process from exiting a bounded interval, is a reflected Brownian motion with two sided barrier by using the penalization method.

It has been studied for the reflected diffusion processes as a limiting process of solutions of stochastic differential equations with penalization terms that satisfy some assumptions [24, 25, 26, 27, 28]. However, previous studies considered only the linear penalization term, so here the global Lipschitz condition was the critical factor to prove the results. It is proved that a limiting process of the specified family of non-linear bistable potential processes is a reflected Brownian motion without global
Lipchitz condition. The following theorem is the most recent result for a process with a linear penalization term.

**Theorem 3.1 ([28]).** Suppose that there is a unique strong solution $X$ of a stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + K_t, \quad t \in \mathbb{R}^+$$

(3.1)

with reflecting boundary condition on a convex domain $D$. Here $x_0 \in \bar{D} = D \cup \partial D$, $X$ is a reflecting process on $\bar{D}$, $K$ is a bounded variation process with variation $|K|$ increasing only, when $X_t \in \partial D$, $W$ is a $d$-dimensional standard Wiener process on a filtered probability space $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}, \mathbb{P}^n)$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions. Suppose that for $n \in \mathbb{N}$, we are given measurable coefficients $\sigma_n : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b_n : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and a standard Wiener process $W_n$, and assume that there exists a solution $X^n$ of the following SDE

$$X^n_t = x_0 + \int_0^t \sigma_n(s, X^n_s) dW^n_s + \int_0^t b_n(s, X^n_s) ds + K^n_t, \quad t \in \mathbb{R}^+$$

(3.2)

where $K^n_t = -n \int_0^t (X^n_s - \Pi(X^n_s)) ds$ which is called a penalization term since it forces $X^n_t$ to stay close to $\bar{D}$, $\Pi(x)$ is the projection of $x$ on $\bar{D}$. $\sigma_n$ and $b_n$ satisfy that

$$||\sigma_n(t, x)||^2 + |b_n(t, x)|^2 \leq C(1 + |x|^2), (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad n \in \mathbb{N}$$

(3.3)

for some $C > 0$. If $W_n \xrightarrow{p} W$ in $C(\mathbb{R}^+, \mathbb{R}^d)$ and (3.1) is pathwise unique then $X^n \xrightarrow{p} X$ in $C(\mathbb{R}^+, \mathbb{R}^d)$.

Here, a one dimensional bistable potential process $X_n(t)$ will be considered, which solves

$$X_n(t) = X_0 + \int_0^t \mu_n(X_n(s)) ds + \sigma W(t),$$

(3.4)

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where $W$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, and for a fixed $k \in \mathbb{N}$ and $n > k$,

\[
\mu_n(x) = -v'_n(x),
\]

(3.5)

\[
v_n(x) = \alpha_n(x^{2n} - M^{2(n-k)}x^{2k}),
\]

(3.6)

$\alpha_n$ satisfies that

\[
\alpha_n M^{2n} \to 0 \text{ and } n\alpha_n M^{2n} \to \infty, \text{ as } n \to \infty,
\]

(3.7)

so that

\[
\lim_{n \to \infty} \mu_n(x) = \begin{cases} 
-\infty & \text{for } x \geq M, \\
0 & \text{for } -M < x < M, \\
\infty & \text{for } x \leq -M.
\end{cases}
\]

(3.8)

Let $\xi_n$ and $-\xi_n$ be locations of minima of the potential function $v_n(x)$ (i.e. $\mu(\pm \xi_n) = 0$), then

\[
\xi_n = \left(\frac{k}{n}\right) \frac{1}{2(n-k)} M \to M, \text{ as } n \to \infty.
\]

(3.9)

Let $D_n$ denote an open interval $D_n = (-\xi_n, \xi_n)$, and let’s assume that $X_0 \in D_n$. Rewrite $X_n$ in the form

\[
X_n(t) = X_0 + Y_n(t) + K_n(t),
\]

(3.10)

where

\[
Y_n(t) = \sigma W(t) + V_n(t),
\]

(3.11)

\[
V_n(t) = \int_0^t 1_{D_n}(X_n(s)) \mu_n(X_n(s)) ds,
\]

(3.12)

and

\[
K_n(t) = \int_0^t 1_{D_n^c}(X_n(s)) \mu_n(X_n(s)) ds,
\]

(3.13)
then $K_n(t)$ is the non-linear penalization term for this process. The goal of this chapter is to prove that the sequence of $X_n$ converges to the two sided RBM $Z$ in probability as $n \to \infty$ when $X_0 = Z_0 \in (-M, M)$, where two sided RBM $Z$ can be written as the following [29].

Let $Z(t)$ be a reflected Brownian motion defined on $[-M, M]$ with two reflecting barriers, then it can be represented as

$$Z(t) = Z_0 + \sigma W(t) + L(t) - U(t), \quad (3.14)$$

where $W$ is a standard Brownian motion with $W(0) = 0$, $L$ and $U$ are continuous functions that increase only when $Z = -M$ or $Z = M$, respectively. Then, $U$ and $L$ can be written as

$$L(t) = -\inf_{s \leq t} (Z_0 + \sigma W(s) - U(s) + M), \quad (3.15)$$

$$U(t) = -\inf_{s \leq t} (Z_0 + \sigma W(s) + L(s) - M). \quad (3.16)$$

These $L$ and $U$ called local times of the reflected Brownian motion at $-M$ and $M$ respectively. In the following section, backgrounds on probability necessary to prove the lemmas and theorems will be given.

### 3.1 Preliminaries

A martingale is a stochastic process $M_t$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is *conditionally constant*, that is, its expected value for any $s > t$ is the same as its current state at $t$ of prediction. The standard Brownian motion is one of the Martingale processes.

**Definition 3.1** (Martingales [30]). A *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of $\sigma$-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t. \quad (3.17)$$
A $d$-dimensional stochastic process $\{X_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t \geq 0}$ (and with respect to $\mathbb{P}$) if

1. $X_t$ is $\mathcal{M}_t$-measurable for all $t$,
2. $E[|X_t|] < \infty$ for all $t$, and
3. $E[X_s|\mathcal{M}_t] = X_t$ for all $s \geq t$.

**Remark** The standard Brownian motion $W_t$ in $\mathbb{R}^d$ is a martingale with respect to the $\sigma$-algebras $\mathcal{F}_t$ generated by $\{W_s; s \leq t\}$, since

$$E[|W_t|^2] \leq E[|W_t|^2] = |W_0|^2 + nt < \infty$$

(3.18)

and if $s \geq t$,

$$E[W_s|\mathcal{F}_t] = E[W_s - W_t + W_t|\mathcal{F}_t] = E[W_s - W_t|\mathcal{F}_t] + E[W_t|\mathcal{F}_t] = 0 + W_t = W_t.$$

(3.19)

**Definition 3.2** (Semimartingale [31]). A process $X$ is called a continuous semimartingale if it can be represented as

$$X_t = M_t + A_t, \quad t \geq 0,$$

(3.20)

where $M$ is a continuous local martingale and $A$ is a continuous bounded variation process satisfying $A_0 = 0$. If $X, Y$ are continuous semimartingales,

$$[X, Y]_t = [M, N]_t,$$

(3.21)

where $M$ and $N$ are local martingale parts of $X$ and $Y$ respectively.

**Definition 3.3.** Let $|X|_t$ denote the total variation of $X$ on $[0, t]$, then

$$[X]_t = \sup_P \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|,$$

(3.22)

where the supremum runs over the set of all partitions $P = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. 

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Definition 3.4. Let \([X]_t\) denote the quadratic variation of \(X\) on \([0, t]\), then
\[
[X]_t = \lim_{||P|| \to 0} \sum_{i=1}^{n} (X(t_i) - X(t_{i-1}))^2,
\]
(3.23)
where \(P\) runs over all partitions \(P = \{0 = t_0 < t_1 < \cdots < t_n = t\}\), and
\[
||P|| = \max_{1 \leq i < k} (t_i - t_{i-1}).
\]
(3.24)

Definition 3.5. Let \([X, Y]_t\) denote the covariation of two process \(X\) and \(Y\) on \([0, t]\), then
\[
[X, Y]_t = \lim_{||P|| \to 0} \sum_{i=1}^{n} (X(t_i) - X(t_{i-1}))(Y(t_i) - Y(t_{i-1})).
\]
(3.25)

Total variation and quadratic variation of stochastic processes has important applications in stochastic calculus such as integration by parts for stochastic integrations, martingales, and Burkholder-Davis-Gundy inequalities. One example of finite quadratic variation processes is the Brownian motion, such that \([W]_t = t\) in probability.

A reflected Brownian motion can be viewed as a solution of Skorokhod problem which was introduced by Anatoliy Skorokhod in [32, 33].

Definition 3.6 (Skorokhod problem[34]). Let \(D\) be an open connected domain in \(\mathbb{R}^d\). Define the set \(N_x\) of inward normal unit vectors at \(x \in \partial D\). Let \(Y\) be an \(\{\mathcal{F}_t\}\)-adapted process with continuous trajectories. We say that a pair \((X, K)\) of \(\{\mathcal{F}_t\}\)-adapted processes is a solution of the Skorokhod problem associated with \(Y\) if

- \(X = Y + K\),

- \(X\) is \(\bar{D}\) -valued,

- \(K\) is a process with locally bounded variation such that \(K_0 = 0\) and
\[
K_t = \int_0^t n_s d|K|_s, \quad |K|_t = \int_0^t 1_{\{X_s \in \partial D\}} d|K|_s, \quad t \in \mathbb{R}^+. \tag{3.26}
\]
where \(n_s \in \mathcal{N}_{X_s}\), if \(X_s \in \partial D\), and \(|K|_t\) is the total variation of \(K\) on\([0, t]\).
3.2 Main results

The following lemmas are used to prove that the sequence of bistable potential processes $X_n$ converges to the two sided RBM $Z$ in probability as $n \to \infty$ when $X_0 = Z_0 \in (-M,M)$. To prove convergence of $X_n$ to $Z$, it will be shown that a sequence of solutions of Skorokhod problems $\Pi(X_n)$ converges to $Z$ using uniqueness of solutions of Skorokhod problems, and then it will be prove that $X_n$ is very close to $\Pi(X_n)$.

**Lemma 3.1.** Let $\{X_n\}_{t \geq 0}$ be a family of stochastic processes which solve (3.4). For a fixed $T > 0$,

$$\sup_n E[\sup_{t \leq T} |X_n(t)|^p] < \infty \quad (3.27)$$

for all $p \geq 1$.

**Proof.** To prove this lemma, the arguments in the proof of theorems in [26] are used.

Recall that

$$X_n(t) = X_0 + Y_n(t) + K_n(t), \quad (3.28)$$

where

$$Y_n(t) = \sigma W(t) + V_n(t), \quad (3.29)$$

$$V_n(t) = \int_0^t 1_{D_n}(X_n(s))\mu_n(X_n(s))ds, \quad (3.30)$$

and

$$K_n(t) = \int_0^t 1_{D_n^c}(X_n(s))\mu_n(X_n(s))ds, \quad (3.31)$$
for \( \mu_n \) given in (3.5) and an open interval \( D_n = (-\xi_n, \xi_n) \), \( \xi_n \) defined in (3.9). Then, \( K_n \) and \( V_n \) are of locally bounded variation, since they can be represented as differences of increasing processes. That is,

\[
V_n(t) = V_n^+(t) - V_n^-(t) \\
= \int_0^t 1_{[0,\xi_n]}(X_n(s))\mu_n(X_n(s))ds - \int_0^t 1_{(-\xi,0]}(X_n(s))(\mu_n(X_n(s)))ds,
\]

(3.32)

and

\[
K_n(t) = K_n^+(t) - K_n^-(t) \\
= \int_0^t 1_{(-\infty,\xi_n]}(X_n(s))\mu_n(X_n(s))ds - \int_0^t 1_{[\xi,\infty)}(X_n(s))(\mu_n(X_n(s)))ds,
\]

(3.33)

since \( \mu_n(x) \geq 0 \) on \( [0,\xi_n] \cup (-\infty, \xi_n] \) and \( \mu_n(x) \leq 0 \) on \( (-\xi, 0] \cup [\xi, \infty) \). Note that \( Y_n = \sigma W + V_n \) is a semimartingale, since \( W \) is an \((\mathcal{F}_t)\) adapted local martingale and \( V_n \) is an \((\mathcal{F}_t)\) process with locally bounded variation with \( W(0) = V_n(0) = 0 \). By the argument in [35, Lemma 2.2],

\[
|X_n(t) - X_0|^2 \leq |Y_n(t)|^2 + 2 \int_0^t (Y_n(t) - Y_n(s))dK_n(s),
\]

(3.34)
since

\[ |X_n(t) - X_0|^2 = |Y_n(t)|^2 + 2Y_n(t)K_n(t) + |K_n(t)|^2 \]
\[ = |Y_n(t)|^2 + 2 \int_0^t Y_n(t) dK_n(s) + 2 \int_0^t K_n(s) dK_n(s) \]
\[ (\because [K_n]_t = 0) \]  
\[ = |Y_n(t)|^2 + 2 \int_0^t (Y_n(t) - Y_n(s))dK_n(s) \]
\[ + 2 \int_0^t Y_n(s) dK_n(s) + 2 \int_0^t K_n(s) dK_n(s) \]
\[ = |Y_n(t)|^2 + 2 \int_0^t (Y_n(t) - Y_n(s))dK_n(s) \]
\[ + 2 \int_0^t (X_n(s) - X_0)dK_n(s) \]
\[ \leq |Y_n(t)|^2 + 2 \int_0^t (Y_n(t) - Y_n(s))dK_n(s). \]  

(3.35)

Here,

\[ (X_n(s) - X_0)dK_n(s) = (X_n(s) - X_0)1_{\{X_n(s) > \xi_n\}}\mu_n(X_n(s))ds \]
\[ + (X_n(s) - X_0)1_{\{X_n(s) < -\xi_n\}}\mu_n(X_n(s))ds \leq 0, \]  

(3.37)

for all \( s \in \mathbb{R}^+ \).
By the stochastic integration by parts (Proposition C.1),

\[
2 \int_0^t (Y_n(t) - Y_n(s))dK_n(s)
= 2 \int_0^t \left(K_n(s)dy_n(s) - [(Y_n(t) - Y_n), K_n]_t\right) \text{ a.s.}
= 2 \int_0^t (X_n(s) - X_0 - Y_n(s))dy_n(s) + [Y_n, K_n]_t
= 2\sigma \int_0^t (X_n(s) - X_0)dW(s) + 2 \int_0^t (X_n(s) - X_0)dV_n(s)
- 2 \int_0^t Y_n(s)dy_n(s) + [Y_n, K_n]_t
= 2\sigma \int_0^t (X_n(s) - X_0)dW(s) + 2 \int_0^t (X_n(s) - X_0)dV_n(s)
- |Y_n(t)|^2 + [Y_n]_t + [Y_n, K_n]_t,
\]

(3.38)

where \([Y_n, K_n]_t = 0\), since \(Y_n\) is semimartingale and \(K_n\) is an adapted locally bounded variation process, and by (3.21). Thus, by (3.34) and (3.38),

\[
|X_n(t) - X_0|^2 \leq 2\sigma \int_0^t (X_n(s) - X_0)dW(s) + 2 \int_0^t (X_n(s) - X_0)dV_n(s) + [Y_n]_t.
\]

(3.39)

For every stopping time \(\tau\) and \(p \in \mathbb{N}\), there exists \(c_1 = c_1(p, \tau)\) such that

\[
E \sup_{t \leq \tau} |X_n(t) - X_0|^{2p}
\leq c_1 \left( E \sup_{t \leq \tau} \left| \sigma \int_0^t (X_n(s) - X_0)dW(s) \right|^p \right)
+ E \sup_{t \leq \tau} \left| \int_0^t (X_n(s) - X_0)dV_n(s) \right|^p + E[Y_n]_\tau^p.
\]

(3.40)

By Schwarz inequality and Burkholder-Davis-Gundy inequality (Therem C.1), for some \(c_2 > 0\),

\[
E \left[ \sup_{t \leq \tau} \left| \sigma \int_0^t (X_n(s) - X_0)dW(s) \right|^p \right]
\leq \sigma^p \left( E \sup_{t \leq \tau} |X_n(t) - X_0|^{2p} \right)^{1/2} \left( E \sup_{t \leq \tau} |W(t)|^{2p} \right)^{1/2}
\leq c_2 \left( E \sup_{s \leq \tau} |X_n(s) - X_0|^{2p} \right)^{1/2} \left( E[W]_\tau^p \right)^{1/2},
\]

(3.41)
and

\[
E \left[ \sup_{t \leq \tau} \left| \int_0^t (X_n(s) - X_0) dV_n(s) \right|^p \right] \leq \left( E \sup_{t \leq \tau} |X_n(s) - X_0|^{2p} \right)^{1/2} \left( E|V_n|^2 p \right)^{1/2}.
\]

(3.42)

Since

\[
\]

(3.43)

for some \(c_3\) and \(c_4\) depending on \(p, \tau\) and \(\sigma\),

\[
E \sup_{t \leq \tau} |X_n(t) - X_0|^{2p} \\
\leq c_3 \left( \left( E \sup_{s \leq \tau} |X_n(s) - X_0|^{2p} \right)^{1/2} \left( (E[W]^p)^{1/2} + (E|V_n|^2)^{1/2} \right) \right) \\
+ c_4(E[W]^p + E|V_n|^2 p).
\]

(3.44)

Let

\[
x = \left( E \sup_{s \leq \tau} |X_n(s) - X_0|^{2p} \right)^{1/2} > 0,
\]

(3.45)

\[
d_1 = \frac{c_3}{2} \left( (E[W]^p)^{1/2} + (E|V_n|^2)^{1/2} \right), \quad \text{and}
\]

(3.46)

\[
d_2 = c_4(E[W]^p + E|V|^2 p),
\]

(3.47)

then

\[
x^2 \leq 2d_1 x + d_2,
\]

\[
(x - d_1)^2 \leq d_1^2 + d_2,
\]

\[
x \leq d_1 + \sqrt{d_1^2 + d_2}, \quad \text{and}
\]

\[
x^2 \leq 2d_1^2 + 2(d_1^2 + d_2).
\]

(3.48)
Since \( d_1^2 \leq c_5 d_2 \) for some \( c_5 \), there exist \( C_p = C_p(\tau, \sigma) \) such that
\[
E \sup_{t \leq \tau} |X_n(t) - X_0|^{2p} \leq C_p \left[ E [W]_\tau^p + E |V_n|_\tau^{2p} \right]. \tag{3.49}
\]

Let
\[
b_n(x) = 1_{D_n}(x)\mu_n(x), \tag{3.50}
\]
then since \( \{|b_n(x)|\} \) uniformly converges to 0, for any \( \tau, E |V_n|_\tau^{2p} \to 0 \) as \( n \to \infty \).

Therefore, for every \( p \in \mathbb{N} \) and every \( T \in \mathbb{R}^+ \),
\[
\sup_n E \sup_{t \leq T} |X_n(t)|^{2p} < \infty.
\]

\[\square\]

**Lemma 3.2.** Let
\[
X_n(t) = X_0 + \sigma W(t) + \int_0^t b_n(X_n(s))ds \tag{3.51}
\]
for \( b_n \) given in (3.50), and let \( \Pi_n(x) \) be a projection function on \([-\xi_n, \xi_n]\) such that
\[
\Pi_n(x) = \begin{cases} 
  x & \text{if } -\xi_n \leq x \leq \xi_n, \\
  \xi_n & \text{if } x > \xi_n, \\
  -\xi_n & \text{if } x < -\xi_n.
\end{cases} \tag{3.52}
\]

Then \((\Pi_n(X_n), K_n)\) is a solution of the Skorokhod problem associated with
\[
\tilde{X}_n = \tilde{X}_n - (X_n - \Pi_n(X_n)) \tag{3.53}
\]
on \( D_n = [-\xi_n, \xi_n] \).

**Proof.** Since \( X_n = \tilde{X}_n + K_n \),
\[
\Pi_n(X_n) = \Pi_n(X_n) - X_n + \tilde{X}_n + K_n = \tilde{X}_n + K_n, \tag{3.54}
\]
and \( \Pi_n(X_n) \) is \( \tilde{D}_n \)-valued.
Since the inward unit normal vectors is that

\[ n_s = \begin{cases} 
1 & \text{if } \Pi_n(X_n(s)) = -\xi_n \text{ (or } X_n(s) < \xi_n), \\
-1 & \text{if } \Pi_n(X_n(s)) = \xi_n \text{ (or } X_n(s) > \xi_n), 
\end{cases} \tag{3.55} \]

\[ \mu_n(x) > 0 \text{ on } (-\infty, -\xi_n) \text{ and } \mu_n(x) < 0 \text{ on } (\xi_n, \infty), \]

\[ \int_0^t n_s d|K_n|(s) \]

\[ = \int_0^t n_s \mathbf{1}_{\{X_n(s) < -\xi_n\}}|\mu_n(X_n(s))| + n_s \mathbf{1}_{\{X_n(s) > \xi_n\}}|\mu_n(X_n(s))|ds \]

\[ = \int_0^t \mathbf{1}_{\{X_n(s) < -\xi_n\}}\mu_n(X_n(s)) + \mathbf{1}_{\{X_n(s) > \xi_n\}}\mu_n(X_n(s))ds = K_n(t), \tag{3.56} \]

and

\[ \int_0^t \mathbf{1}_{\{\Pi_n(X_n(s)) \in \partial D_n\}} d|K_n|(s) \]

\[ = \int_0^t \mathbf{1}_{\{X_n(s) \in \partial D_n\}} \left( \mathbf{1}_{\{X_n(s) \in D_n^+\}} |\mu_n(X_n(s))| \right) ds = |K_n|_{\partial D}. \tag{3.57} \]

**Lemma 3.3.** For a every \( T > 0, \ p \geq 1, \)

\[ E \left[ \sup_{t \leq T} |X_n(t) - \Pi_n(X_n(t))|^p \right] \to 0 \text{ as } n \to \infty. \tag{3.58} \]

**Proof.** Since

\[ |X_n(t) - \Pi_n(X_n(t))| \]

\[ = \mathbf{1}_{\{X_n(t) \in \partial D_n\}}(-X_n(t) - \xi_n) + \mathbf{1}_{\{X_n(t) \in D_n^+\}}(X_n(t) - \xi_n) \]

\[ = \mathbf{1}_{\{X_n(t) < -\xi_n\}}(-X_n(t) - \xi_n) + \mathbf{1}_{\{X_n(t) > \xi_n\}}(X_n(t) - \xi_n), \tag{3.59} \]

it is enough to show that

\[ E \left[ \sup_{t \leq T} (\mathbf{1}_{\{X_n(t) > \xi_n\}}(X_n(t) - \xi_n))^p \right] \to 0 \text{ as } n \to \infty. \tag{3.60} \]
Let \( \{M_n\} \) be a decreasing sequence converging to \( M \) as \( n \to \infty \). For some \( C_p > 0 \),

\[
E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - \xi_n) \right|^p \right] \\
\leq C_p \left( E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] + E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(M_n - \xi_n) \right|^p \right] \right) \\
\leq C_p (E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] + |M_n - \xi_n|^p) \\
\leq C_p (E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] 1_{(\sup_{t \leq T} X_n(t) > M_n)} + E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] 1_{(\sup_{t \leq T} X_n(t) < M_n)} \right) + |M_n - \xi_n|^p \right). \tag{3.61}
\]

Since \( M_n \to M \) and \( \xi_n \to M \) as \( n \to \infty \)

\[
E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] 1_{(\sup_{t \leq T} X_n(t) < M_n)} \right] \\
\leq E \left[ |\xi_n - M_n|^p 1_{(\sup_{t \leq T} X_n(t) < M_n)} \right] \leq |\xi_n - M_n|^p \to 0 \tag{3.62}
\]

Hence we need to show that

\[
E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] 1_{(\sup_{t \leq T} X_n(t) > M_n)} \right] \to 0. \tag{3.63}
\]

Recall Young’s inequality:

\[
ab \leq \frac{a^2}{2} + \frac{b^2}{2 \varepsilon}, \text{ for any } \varepsilon > 0. \tag{3.64}
\]

For any \( \delta_n > 0 \),

\[
E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^p \right] 1_{(\sup_{t \leq T} X_n(t) > M_n)} \right] \\
\leq \frac{\delta_n}{2} E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^{2p} \right] + \frac{1}{2\delta_n} P \left( \sup_{t \leq T} X_n(t) > M_n \right) \tag{3.65}
\]

By Lemma 3.1, for all \( q \geq 1 \)

\[
\sup_n E \left[ \sup_{t \leq T} |X_n(t)|^q \right] < \infty, \tag{3.66}
\]

therefore

\[
\sup_n E \left[ \sup_{t \leq T} \left| 1_{(X_n(t) > \xi_n)}(X_n(t) - M_n) \right|^{2p} \right] \leq \sup_n E \left[ \sup_{t \leq T} |X_n(t) - M_n|^{2p} \right] < \infty. \tag{3.67}
\]
By Lemma 3.4,

\[ P \left( \sup_{t \leq T} X_n(t) > M_n \right) \to 0, \]  

(3.68)

If \( \{\delta_n\} \) is chosen such that \( \delta_n \to 0 \), then

\[ \frac{1}{\delta_n} P \left( \sup_{t \leq T} X_n(t) > M_n \right) \to 0 \text{ as } n \to \infty, \]  

(3.69)

and (3.65), (3.66) implies (3.63).

Lemma 3.4. Let \( \{M_n\} \) be a decreasing sequence converging to \( M \) as \( n \to \infty \). Then

\[ P \left( \sup_{t \leq T} |X_n(t)| > M_n \right) \to 0. \]  

(3.70)

Proof. Recall that \( \mu_n(x) = \alpha_n(2kM^{2(n-k)}x^{2k-1} - 2nx^{2n-1}) \). Let \( r_n(x) \) be a tangent line of \( \mu_n(x) \) at \( x = M_n \), then

\[ r_n(x) = -a_n(x - b_n) \]  

(3.71)

where

\[ a_n = -\mu'_n(M_n) = O(\alpha_n n^2 M_n^{2n-2}) > 0, \]  

(3.72)

\[ b_n = (M_n - \mu_n(M_n)/\mu'_n(M_n)) = O(1). \]  

(3.73)

Let \( g(x) = r_n(x) - \mu_n(x) \), then \( g'(x) = 0 \) only when \( x = \pm M_n \) and \( g'(x) < 0 \) on \((-M_n, M_n)\). Since

\[ g'(x) = r'_n(x) - \mu'_n(x) = -a_n(x - b_n) = \mu'_n(M_n) - \mu'_n(x) \]

\[ = \alpha_n(2k(2k - 1)M^{2n-2k}M_n^{2k-2} - 2n(2n - 1)M_n^{2n-2}) \]

\[ - \alpha_n(2k(2k - 1)M^{2n-2k}x^{2k-2} - 2n(2n - 1)x^{2n-2}), \]

\[ = \alpha_n(M_n^2 - x^2) \left[ 2k(2k - 1)M^{2n-2k} \sum_{i=0}^{k-2} x^{2i} M_n^{2(k-2-i)} \right. \]

\[ - 2n(2n - 1) \sum_{j=0}^{n-2} x^{2j} M_n^{2(n-2-i)} \].

(3.74)
it is sufficient to show that
\[
2k(2k - 1)M^{2n-2k} \sum_{i=0}^{k-2} x^{2i} M_n^{2(k-2-i)} - 2n(2n - 1) \sum_{j=0}^{n-2} x^{2j} M_n^{2(n-2-j)} < 0 \tag{3.75}
\]
on \((-\infty, \infty)\). Then, since \(k < n\) and \(0 < M < M_n\),
\[
2k(2k - 1)M^{2n-2k} \sum_{i=0}^{k-2} x^{2i} M_n^{2(k-2-i)} - 2n(2n - 1) \sum_{j=0}^{n-2} x^{2j} M_n^{2(n-2-j)}
\]
\[
< 2n(2n - 1)M_n^{2n-2k} \sum_{i=0}^{k-2} x^{2i} M_n^{2(k-2-i)} - 2n(2n - 1) \sum_{j=0}^{n-2} x^{2j} M_n^{2(n-2-i)}
\]
\[
= 2n(2n - 1) \sum_{i=0}^{k-2} x^{2i} M_n^{2(k-2-i)} - 2n(2n - 1) \sum_{j=0}^{n-2} x^{2j} M_n^{2(n-2-i)}
\]
\[
= -2n(2n - 1) \sum_{j=k-1}^{n-2} x^{2j} M_n^{2(n-2-i)} \leq 0 \tag{3.76}
\]
Moreover, \(g(x)\) has a local minimum and local maximum at \(x = -M_n\) and \(x = M_n\) such that \(g(-M_n) < 0\) and \(g(M_n) = 0\) respectively, since \(g'(\pm M_n) = 0\), \(g''(-M_n) < 0\) and \(g''(M_n) > 0\). Since \(g(x) \to \pm \infty\) as \(x \to \pm \infty\), there exists \(\tilde{M}_n > M\) such that
\[
m_n(x) \leq r_n(x) \quad \text{for } x \geq -\tilde{M}_n. \tag{3.77}
\]
Let \(l_n(x)\) be a tangent line of \(\mu_n(x)\) at \(x = -M_n\). Then, by the symmetry of \(\mu_n(x)\),
\[
\mu_n(x) \geq l_n(x) \quad \text{for } x \leq \tilde{M}_n. \tag{3.78}
\]
Let \(Y^r_n(t)\) and \(Y^l_n(t)\) be solutions of stochastic differential equations
\[
Y^r_n(t) = X_0 + \int_0^t r_n(Y^r_n(s))ds + \sigma W(t), \tag{3.79}
\]
and
\[
Y^l_n(t) = X_0 + \int_0^t l_n(Y^l_n(s))ds + \sigma W(t), \tag{3.80}
\]
respectively, then by Lemma C.1,
\[
Y^r_n(t) = b_n + e^{-a_n t}(X_0 - b_n) + \sigma \int_0^t e^{-a_n(t-u)} dW_u. \tag{3.81}
\]
By Bernstein-type inequality (Lemma C.3),

\[
\begin{align*}
\mathbb{P}\{ \sup_{0 \leq t \leq T} Y^r_n(t) > M_n \} & \leq \mathbb{P}\{ \sup_{0 \leq t \leq T} \sigma \int_0^t e^{-a_n(t-u)} \, dW_u > M_n - \max_{0 \leq t \leq T} (b_n + e^{-a_n t} (Y^r_n(0) - b_n)) \} \\
& \leq \exp \left\{ - \frac{C_n^2}{2\Phi(T)} \right\},
\end{align*}
\] (3.82)

where \( C_n = \left( M_n - \max_{0 \leq t \leq T} (b_n + e^{-a_n t} (Y^r_n(0) - b_n)) \right) / \sigma = \mathcal{O}(1) \), and

\[
\Phi(t) = \int_0^t e^{-2a_n(t-u)} \, du = \frac{1}{2a_n} (1 - e^{-2a_n t}).
\] (3.83)

Thus

\[
\mathbb{P}\{ \sup_{0 \leq t \leq T} Y^r_n(t) > M_n \} \leq \exp(-\kappa a_n)
\] (3.84)

for some \( \kappa = \mathcal{O}(1) > 0 \). Since \( a_n = \mathcal{O}(\alpha_n n^2 M_n^{2n-2}) \),

\[
\mathbb{P}\{ \sup_{0 \leq t \leq T} Y^r_n(t) > M_n \} \to 0 \text{ as } n \to \infty.
\] (3.85)

By the same argument,

\[
\mathbb{P}\{ \inf_{0 \leq t \leq T} Y^l_n(t) < -M_n \} \to 0 \text{ as } n \to \infty.
\] (3.86)

By the comparison lemma (Lemma C.2),

\[
\begin{align*}
\mathbb{P}\{ \sup_{0 \leq t \leq T} |X_n(t)| > M_n \} & \leq \mathbb{P}\{ \sup_{0 \leq t \leq T} Y^r_n(t) > M_n \} + \mathbb{P}\{ \inf_{0 \leq t \leq T} Y^l_n(t) < -M_n \} \to \infty,
\end{align*}
\] (3.87)

as \( n \to \infty \). \( \square \)

**Lemma 3.5.** Let \( \bar{X}_n(t) = X_0 + \sigma W(t) + \int_0^t b_n(X_n(s)) \, ds \) as in Lemma 3.2, then

\[
\bar{X}_n = \bar{X}_n + \Pi_n(X_n) - X_n \underset{p}{\to} X_0 + \sigma W = \bar{X}
\] (3.88)

in \( \mathcal{C}([0,T], \mathbb{R}) \) with the uniform metric \( |X - Y|_T := \sup_{t \leq T} |X(t) - Y(t)| \).
Proof. For every $\varepsilon > 0$, $T > 0$,

$$
\lim_{n \to \infty} \mathbb{P}(|\bar{X}_n - \bar{X}|_T \geq \varepsilon) \leq \lim_{n \to \infty} \mathbb{P}\left(\sup_{t \leq T} \int_0^t |b_n(X_n(s))|ds \geq \varepsilon\right)
$$

\leq \lim_{n \to \infty} \mathbb{P}(|c_n| \geq \varepsilon) = 0,
\tag{3.89}
$$

where

$$
c_n = \max_{x \in \mathbb{R}} |b_n(x)| = \left| \mu_n \left( M \left( \frac{k(2k - 1)}{n(2n - 1)} \right)^{\frac{1}{2(n-k)}} \right) \right|
$$

$$
= 2\alpha_n M^{2n-1} \left( \frac{k(2k - 1)}{n(2n - 1)} \right)^{\frac{k-1}{2(n-k)}} \left[ 1 - \frac{k(2k - 1)}{n(2n - 1)} \right] \to 0,
\tag{3.90}
$$

since by (3.7),

$$\alpha_n M^{2n} \to 0 \text{ as } n \to \infty.\tag{3.91}$$

Thus,

$$\bar{X}_n \xrightarrow{p} \bar{X} = X_0 + \sigma W.\tag{3.92}$$

For any $t < T$, and for all $p \geq 1$, by Lemma 3.3,

$$
E \left[ |X_n(t) - \Pi_n(X_n(t))|^p \right] \leq E \left[ \sup_{t \leq T} |X_n(t) - \Pi_n(X_n(t))|^p \right] \to 0.
\tag{3.93}
$$

Since convergence in $p$-th mean implies convergence in probability,

$$|X_n - \Pi_n(X_n)| \xrightarrow{p} 0.\tag{3.94}$$

\hfill \blacksquare

Lemma 3.6. Let $Z$ be a reflected Brownian motion defined on $[-M, M]$ with two reflecting boundaries at $M$ and $-M$ and $Z(0) = X_0$, which can be written as $Z(t) = X_0 + \sigma W(t) + L(t) - U(t)$ where $L$ and $U$ are local times of $Z$ at $-M$ and $M$ as defined in (3.15), (3.16) respectively. Let $\Pi_n$ be the projection function on $[-\xi_n, \xi_n]$ given in (3.52) and let $K_n$ be the penalization term of $X_n$ of the form in (3.13). Then

$$
(\Pi_n(X_n), K_n) \xrightarrow{p} (Z, L - U) \text{ as } n \to \infty,
\tag{3.95}
$$

88
in $\mathcal{C}([0,T],\mathbb{R})$ with the uniform metric $|X - Y|_T := \sup_{t \leq T} |X(t) - Y(t)|$.

**Proof.** Let $D = (-M,M)$. By Lemma 3.2, $(\Pi_n(X_n), K_n)$ is a solution of the Skorokhod problem associated with $\tilde{X}_n$ on $D_n$ (i.e., $\Pi_n(X_n) = \tilde{X}_n + K_n$).

Let $\tilde{X}_n = H_n + G_n$ where $G_n(t) = \sigma W(t) + \int_0^t b_n(X_n(s)) ds$ and $H_n(t) = X_0 + \Pi_n(X_n) - X_n$, then by Lemma C.2, $G_n$ satisfies the condition (UT) (See Definition C.1), since $G_n(0) = 0$, $\sigma W$ is a square integrable martingale, $V_n = \int_0^t b_n(X_n(s)) ds$ is of locally bounded variation, and

$$
\sup_{n \in \mathbb{N}} \{ E[|\sigma W|_q + |V_n|_q^2] \} < \infty, \quad q \in \mathbb{R}^+. 
$$

(3.96)

Since $H_n(t) = X_0 + \Pi_n(X_n) - X_n \xrightarrow{p} X_0$ in $\mathcal{C}([0,T],\mathbb{R})$ (by eq. (3.94) in Lemma 3.5), and $G_n(t) \xrightarrow{p} \sigma W$ in $\mathcal{C}([0,T],\mathbb{R})$ by uniform convergence of $b_n(x)$ to 0,

$$(H_n, G_n, \tilde{X}_n) \xrightarrow{p} (X_0, W, X_0 + \sigma W) \text{ in } \mathcal{C}([0,T],\mathbb{R}).$$

(3.97)

Thus, by Theorem C.2,

$$(H_n, G_n, \tilde{X}_n, \Pi_n(X_n), K_n) \xrightarrow{p} (X_0, W, X_0 + \sigma W, X, K), \text{ in } \mathcal{C}([0,T],\mathbb{R})$$

(3.98)

where $(X, K)$ is a solution of the Skorokhod problem associated with $X_0 + \sigma W$ on $D$. However, for the two sided RBM $Z$ on $[-M,M]$ with $Z(0) = X_0$, $(Z, L-U)$ is also a solution of the Skorokhod problem associated with $X_0 + \sigma W$ on $D$. By uniqueness of solutions of Skorokhod problems, $Z = X$ a.s. Thus,

$$
\Pi_n(X_n) \xrightarrow{p} Z \text{ in } \mathcal{C}([0,T],\mathbb{R}) \text{ as } n \to \infty.
$$

(3.99)

\[ \square \]

**Theorem 3.2.** Let $\{X_n\}$ be a sequence of one-dimensional bistable processes defined as above, and $Z$ be a two sided reflected Brownian motion on $[-M,M]$. Then for a fixed $T > 0$,

$$
X_n \xrightarrow{p} Z \text{ in } \mathcal{C}([0,T],\mathbb{R}), \text{ as } n \to \infty.
$$

(3.100)
Proof. By Lemma 3.6,
\[
\Pi_n(X_n) \xrightarrow{p} Z \text{ in } C([0,T],\mathbb{R}) \text{ as } n \to \infty, \tag{3.101}
\]
and by Lemma 3.5, \(|\Pi_n(X_n) - X_n| \xrightarrow{p} 0\). Therefore,
\[
X_n = \Pi_n(X_n) + (X_n - \Pi_n(X_n)) \xrightarrow{p} Z \text{ in } C([0,T],\mathbb{R}). \tag{3.102}
\]

Corollary 3.1. Let \(c\) be a constant in \((-M,M)\), and let
\[
\tau_{X_n} = \inf \{t : X_n(t) \geq c \mid X_n(0) = X_0 < c\}, \tag{3.103}
\]
and
\[
\tau_Z = \inf \{t : Z(t) \geq c \mid Z(0) = X_0 < c\}, \tag{3.104}
\]
then
\[
\tau_{X_n} \xrightarrow{p} \tau_Z \text{ as } n \to \infty. \tag{3.105}
\]

Proof. Note that
\[
\mathbb{P}(\tau_{X_n} \leq t) = \mathbb{P}(\sup_{0 \leq s \leq t} X_n(s) \geq c),
\]
\[
\mathbb{P}(\tau_Z \leq t) = \mathbb{P}(\sup_{0 \leq s \leq t} Z(s) \geq c). \tag{3.106}
\]
Since the supremum of a continuous process is a continuous mapping in \(C([0,\infty),\mathbb{R})\), \(S_{X_n}(t) := \sup_{0 \leq s \leq t} X_n(s)\) converges to \(S_Z(t) := \sup_{0 \leq s \leq t} Z(s)\) in probability, and in distribution by the continuous mapping theorem. For any \(\varepsilon > 0\),
\[
\mathbb{P}(\tau_Z \leq t + \varepsilon) = \mathbb{P}(S_Z(t + \varepsilon) \geq c) \geq \limsup_n \mathbb{P}(S_{X_n}(t) \geq c) = \limsup_n \mathbb{P}(\tau_{X_n} \leq t), \tag{3.107}
\]
\[
\mathbb{P}(\tau_Z \leq t - \varepsilon) = \mathbb{P}(S_Z(t - \varepsilon) \geq c) \leq \liminf_n \mathbb{P}(S_{X_n}(t) \geq c) = \liminf_n \mathbb{P}(\tau_{X_n} \leq t). \tag{3.108}
\]
Biophysical experiments show that sleep-wake dynamics change across postnatal development: wake bout durations of newborn rats are exponentially distributed but become power law distributed within three weeks, in contrast to sleep bout durations that exhibit exponential distributions at all ages [1, 2]. Mathematically and biologically, exponentially-distributed bout durations are both common and easy to produce in a model. The variation of the time constant for sleep bouts is also consistent with scaling theories concerning species size [2]. The more challenging aspect is to identify mechanisms on competing network models that produce power law bout distributions, and moreover to account for the conserved exponent in the power law. An exponential distribution is due to the memoryless property of the process, so if the power law distribution appears in some time interval, some where the process acts like a form of memory.

Since the sleep and wake promoting neuronal networks in the brain are inhibiting each other [3], a model was constructed comprising two competing clusters, where each cluster is modeled by a random graph such as an Erdös-Rényi or a scale-free, and each node fires like a Poisson processes with the firing rate depending on its state (excited, basal or inhibited). It is assumed that when a neuron fires, it excites connected neighbors in the same cluster, and it inhibits connected nodes in the other
cluster. It is also assumed that excited and inhibited nodes spontaneously relax to the basal state with some rate.

The deterministic mean field equation with respect to the population in each state shows that this system is bistable for some parameter regime, so parameters were chosen to get bistability. The network model was converted to a four-dimensional stochastic mean field equation. Let \( p(t, N) \) be the probability that the population vector is \( N \) at time \( t \), then it satisfies the master equation

\[
p(t + \Delta t, N) = p(t, N) + \Delta t \sum_{m_1, m_2, k_1, k_2} p(t, N - [m_1, m_2, k_1, k_2]) \mathbb{P}(\Delta N = [m_1, m_2, k_1, k_2]).
\]

(4.1)

Expanding each term in Taylor series about the point \((t, N)\), then it is approximated by a Fokker-Planck equation (2.30). The probability distribution \( p(t, N) \) that satisfies (2.30) is identical to the distribution of solutions for the stochastic differential equation system

\[
dN = \mu(t, N)dt + B(t, N)dW,
\]

(4.2)

The advantages of the stochastic mean field model is that it shows similar behavior to the network model but it reduces computational cost and it is easier to change key parameters such as the network size or mean degree in a random graph. By numerical simulation of this model, it is detected that the bout durations exhibit power law distribution in the intermediate regime of times, and the scaling exponent and power law intervals depend on the choice of network type and network size, but not much on the mean degree. It is also observed that the projection of the trajectories on \((N^W_E, N^R_E)\) plane are trapped in a linear band as the drift vectors of the system and diffuse on it much like a Brownian motion, and as the network size increases, the width of band narrows.
One dimensional reflected Brownian motion $B_t$ which is defined on a finite interval $[-M, M]$ was considered to show the existence of a power law distribution and the robustness of the scaling constant. It has been shown that the probability density function $f^x(t)$ of the first exit time of reflected Brownian motion $\tau = \min\{t : B_t < -x \mid B_0 = x\}$ satisfies

$$f^x(t) \sim \frac{1}{\sqrt{2\pi t^3}} x \exp\left(-\frac{x^2}{2t}\right), \text{ if } t < C_1 a^2$$

(4.3)

$$f^x(t) \sim \frac{1}{\sqrt{2\pi t^3}} x, \text{ if } C_3 a^2 < t < C_2 a^2$$

(4.4)

$$f^x(t) \sim \frac{2\pi}{a^2} \exp\left(-\frac{\pi^2}{2a^2} t\right) \sin\left(\frac{\pi x}{a}\right), \text{ if } t > C_3 a^2,$$

(4.5)

for some $C_1, C_2,$ and $C_3$ if $a = 2M + 2x$ and $M >> x$.

A linear shaped band of trajectories can be described by large deviation theory. It is said that the minimizer of the action functional is called the most probable path (MPP) for the transition from the wake dominant steady state to the sleep dominant steady state of the deterministic part and the probability that the process moves by another path from the MPP is exponentially smaller in $1/N$. The most probable math is computed numerically by minimum action method introduced in [22], and checked that for large $N$, sample paths are close to the MPP.

In Chapter 2, the full model is approximated by one dimensional reflected Brownian motion, however it was originally a bistable process. Therefore, in Chapter 3, a family of one dimensional bistable potential processes $\{X_n\}$ was introduced and it is proved that the limiting process of the bistable potential process is the reflected Brownian motion $Z$ by using penalization method, where penalization term is the one that forces the process to stay close to the given interval. To prove convergence of $X_n$ to $Z$, it was shown that a sequence of solutions of Skorokhod problems, $\Pi(X_n)$ converges to the RBM using uniqueness of solutions of Skorokhod problems, and then
it was proved that $X_n$ is very close to $\Pi(X_n)$. Finally, it was proved that the distribution of the bout duration for $X_n$ converges to the distribution of the bout duration for $Z$ by the continuous mapping theorem.

The dynamics on the sleep-wake competing network is modeled by a four dimensional stochastic mean field model, which has bistable deterministic part, and the projection of the trajectories on the $(N^W_E, N^S_E)$ plane has a linear band shape with narrow width for large $N$. Therefore, it can be approximated by a one dimensional bistable process which is close to the reflected Brownian motion in probability. Analysis of reflected Brownian motion shows the robustness of the exponent of the power law distribution and the estimation of the power law regime.

These analyses suggest that the mutually-inhibitory interactions between the two networks act to restrict the dynamics to a linear band. Within the band, the dominant diffusion implies a source of memory that is involved in the creation of bounded power-law distributed regions in the distributions of bout durations. For large networks, the process can be approximated by reflected Brownian motion which has the robust power law exponent -1.5. Thus the observed power law would ultimately be due to the universality and criticality of Brownian motion.
CHAPTER 5
ONGOING AND FUTURE WORK

5.1 Asymmetric stochastic mean field equation

The project described above has natural extensions; in the sleep-wake transition dynamics, it remains to construct an asymmetric cluster model wherein dominance times of one cluster are exponentially distributed while the other cluster retains a power law distribution of bout durations. The idea is to set up a random network model with two clusters in a different combination of graph types, network sizes, and degree distributions. The ultimate goal is to construct a developing network model in which both clusters begin with exponential distributions, but one cluster develops a power law distribution as the model evolves and matures, mimicking the changes observed in developing mammals. Modeling with growing random networks is also considered, but it is more difficult to construct than the fixed sized networks. It is expected that these modification of models gives the change of distribution of bout durations.

5.2 Power law distribution for metastable processes

Another ongoing project is to investigate the power law distribution of general metastable processes. Metastable processes models are arises in various fields (e.g. chemical reactions [36], climate change [37], biological systems [38] including the sleep-wake
transition system) where the distribution of escape times from a domain containing a metastable state plays a central role, the importance of the choice of domain to get plausible results is discussed in [39, 40].

It is known that the escape times follow an exponential distribution for a traditional bistable potential process with a small diffusion term by the large deviation theorem. However, in experimental data, a power law distribution is sometimes observed. A natural question arises such that if the power law distribution is originated from its own structure or it is a result from trivial recrossing due to too close domains. To understand the relation between the power law and the structure of the system, a family of double-well potential processes is considered, where the curvature at the local maximum between two wells and the depth of the well is independently controlled to see the effect of those factors on the burst distributions.

Let $X_t$ be a bistable potential process which solves

$$dX_t = -V'_{\kappa,d}(X_t)dt + \sigma dW$$

where the potential function $V_{\kappa,d}(x)$ has the form

$$V_{\kappa,d}(x) = ax^6 + bx^4 + cx^2$$

satisfying

1. locations of minima of $V : \pm m$. (i.e. $V'(m) = 0, c < 0$),

2. Minimum of $V : V(\pm m) = -d$ (i.e. $d$: well depth),

3. Curvature of $V$ at $0 : \kappa$.

Then, the coefficients $a$, $b$, and $c$ can be written in term of $\kappa$, $d$ and $m$ such that

$$a = -\frac{\kappa}{2m^4} + \frac{2d}{m^6}, \quad b = \frac{\kappa}{m^2} - \frac{3d}{m^4}, \quad c = -\frac{\kappa}{2}$$

(5.3)
Figure 5.1: Plots of potential functions $V_{\kappa,d}$ for $d=0.3$, $\kappa = 0.1, 1, 1.8$ respectively. Black dashed lines represent the boundaries $x_0 = \pm 0.1$

Define a bout by

$$\tau_{\kappa,d}^{x_0} := \min\{t : X_t < -x_0 | X(0) = x_0\}, \text{ for a fixed } x_0 > 0.$$  \hspace{1cm} (5.4)

The reliability function $R_{\kappa,d}(t, x)$ is the probability that $\tau_{\kappa,d}^{x_0}$ is greater than $t$ assuming initial populations $X_0 = x_0$. That is,

$$R_{\kappa,d}^{x_0}(t, x) = \mathbb{P}(\tau_{\kappa,d} \geq t | X_{n,k}(0) = x). \hspace{1cm} (5.5)$$

$R_{\kappa,d}^{x_0}(t, y)$ satisfies the backward Kolmogorov equation:

$$\frac{\partial R_{\kappa,d}}{\partial t} = -V_{\kappa,d}' \frac{\partial R_{\kappa,d}}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 R_{\kappa,d}}{\partial y^2} \hspace{1cm} (5.6)$$
with
\[ R_{\kappa,d}(0, y) = 1 \text{ for } y \in [-x, \infty) \]
\[ R_{\kappa,d}(t, -x) = 0 \]
\[ R_{\kappa,d}(t, \infty) = 1. \] (5.7)

Then, the probability density of \( \tau_{n,k}^{x_0} \) is given by
\[ -\frac{\partial R_{\kappa,d}(t, x_0)}{\partial t}. \]

The above partial differential equation is computed numerically using MATLAB and the power law regime of the density function is estimated by the following algorithm.

Let \( \{t_1, \ldots, t_n\} \) be a partition of time, and \( f_{n,k}(t) \) be the probability density function of \( \tau_{n,k}. \)

Let
\[ \alpha_{i,j} = \frac{\log(f_{n,k}(\log(t_j))) - \log(f_{n,k}(\log(t_i)))}{\log(t_j) - \log(t_i)}, \] (5.8)

which is estimated exponent of the power law distribution on the interval \((t_i, t_j)\). For \( i < j \), compute
\[ \xi(i, j) = \frac{|\alpha_{i,j} - \alpha_{i,\frac{i+j}{2}}| + |\alpha_{i,j} - \alpha_{\frac{i+j}{2},j}|}{2\alpha_{i,j}}, \] (5.9)

which is called a stability error on the interval \((t_i, t_j)\). And then, find the longest interval \((t_{\min}, t_{\max}) = (t_i, t_j)\) such that \( \xi(i, j) < \theta \), for some given threshold \( \theta \). It is assumed that it has power law distribution on \((t_{\min}, t_{\max})\) with the correcting exponent \( \alpha_{i,j} \) on the interval by (5.8). To confirm the existence of the power law, consider \( t_1 = t_{\min}, t_2, \ldots, t_n = t_{\max} \), a logarithm partition of \([t_{\min}, t_{\max}]\), and let \( \alpha_{ij} = \text{exponent on } [t_i, t_j] \text{ for } i < j \). Compute the maximum deviation \( d = \max_{i,j}(\alpha_{ij} - \min_{i,j}(\alpha_{ij})) \), and accept the power law on \([t_{\min}, t_{\max}]\) if \( d < \xi \) for a give threshold \( \xi \). It is subtle to choose suitable thresholds, but via numerous numerical simulations, \( \theta = 0.1 \) and \( \xi = 0.1\alpha \) were chosen.

By numerical simulations, we get insight that the lower cutoff of the power law distribution \( t_{\min} \) does not depend on the depth of the well, but depends only on the
curvature $\kappa$ at the local maximum and $x_0$. It is also observed that $t^{\kappa,x_0}_{\text{min}}$ of this bistable potential process is very close to the lower cutoff $t^{RBM,x_0}_{\text{min}}$ for the Brownian motion (BM). (See Figure 5.2, 5.3.) It can be explained by that the process behaves much like a standard Brownian on the flat-ish region of the potential function near the local maximum.
Figure 5.2: Numerically detected \( t_{\text{min}} \) for each \( x_0 \) and \( \kappa \). Green: \( t_{\text{min}} \) of RBM.

Figure 5.3: Numerically detected \( t_{\text{min}} \) for each \( x_0 \) and \( d \) with a fixed curvature \( \kappa = 1.5 \).
Claim Let \( V_\kappa(x) \) be a potential function of the curvature \( \kappa \) at 0 and \( V_\kappa(0) = 0 \) defined as above. For small \( \varepsilon > 0 \), there exists \( \delta \) such that \( |t_{RBM,x_0}^{\kappa,x_0} - t_{min}^\kappa| < \varepsilon \) if \( |V_\kappa(x_0)| < \delta \). This conjecture is tested by numerical simulations (see Figure 5.4).

![Figure 5.4: Left: Plots of pairs \((x_0, \kappa)\) on which \(|V_\kappa(x_0)| > C_1 = 8e - 4\) Right: Plots of pairs \((x_0, \kappa)\) on which \(|V_\kappa(x_0)| > C_2 = 7e - 3\)](image)

The following is the idea to prove this claim: Consider an interval \( D = (-x_1, x_1) \) on which the potential function \( V_\kappa(x) \) is almost flat. Let \( X_t \) be a solution of

\[
dX_t = -V_\kappa'(X_t)dt + \sigma dW_t, \quad X_0 = x_0 < x_1.
\]

Choose \( t_1 \) for which

\( X_t \in D \) with high probability, if \( t < t_1 \).

Note that \( t_1 \) would be the lower bound for the upper cutoff \( t_{max} \). Define a stopping time

\[
\tau_{x_0}^{\kappa} = \min\{t > 0 : X_t \notin D | X_0 = x_0 \}.
\]
First of all, prove that \( X_t \approx W_t \) in probability for \( t < t_1 \) if \( \tau > t_1 \), and then define stopped processes \( \tilde{X}_t \) and \( \tilde{W}_t \) corresponding to \( X_t \) and \( W_t \) respectively such that

\[
\tilde{X}_t = X_{\min(t,t_1)}, \quad \tilde{W}_t = W_{\min(t,t_1)}.
\] (5.10)

And then, show that \( \tilde{X}_t \approx \tilde{W}_t \) in probability for all \( t \). Finally, it will be proved that \( t_{\min}^{RBM,x_0} \approx t_{\min}^{\kappa,x_0} \) by the same argument used in the penalization method.

It is also tried to find \( t_{\max} \) as a function of \( \kappa, d, \) and \( x_0 \), but numerical simulations show that changes of \( x_0 \) does not effect on \( t_{\max} \) like that \( t_{\max} \) of RBM is only related not to the \( x_0 \) but the reflecting boundary \( M \).

The main goal of this work is to set up a criterion to distinguish the power law distribution of the exit times due to the structure of the process from the garbage due to trivial recrossing events in terms of the potential function of the process. It is expected that the proofs of the above claims give the relation between the structure of the potential function and the power law distribution on a bounded interval. This could be applied to the metastable models in various fields to study its power behavior of escape times.
BIBLIOGRAPHY


Appendix A

POWER LAW ESTIMATION

The following power law estimation algorithm is the summary of the method given in [41]. Three steps for responsible power law fitting are that

1. estimating parameters of the power law fit such as \( x_{\text{min}} \) and \( x_{\text{max}} \), cutoffs of the power law regime and the exponent \( \alpha \) by Kolmogorov-Smirnov (KS) method,

2. model validation by calculating the goodness-of-fit between the fit and the data, and performing \( p \)-value hypothesis test (plausability), and

3. model comparison against alternative distributions (defensibility against other distributions).

A.1 Detecting parameters by KS method

The following fitting procedure is referred to as the KS method. Suppose that a data set \( X \) and two distinct data points \( x_1 < x_2 \) are given. The KS metric \( \rho_X^{KS} \) for a bounded power law fit to data set \( X \) on the interval \( I = [x_1, x_2] \) is defined as:

\[
\rho_X^{KS}(x_1, x_2) = \sup_{y \in I} |F_X^I(y) - F^{I, \alpha(I)}(y)|,
\]  

(A.1)

where \( \alpha(I) \) is the estimated exponent of power law fit to data on interval \( I \), \( F_X^I \) is the empirical cumulative distribution function (cdf) for the data in \( I \), and \( F^{I, \alpha(I)} \) is the
theoretical cdf of a power law distribution on the data in $I$ with scaling parameter $\alpha(I)$.

Then, the penalized KS metric for bounded power law fits is defined as:

$$\rho^p_{KS}(x_1, x_2) = \rho^K_{S}(x_1, x_2) + d \log \left( \frac{x_1}{x_2} \right),$$  \hspace{1cm} (A.2)

where $d > 0$ is a fixed penalty coefficient.

The bounded power law distribution on the given interval $I = [x_{\min}, x_{\max}]$ has the probability density function $p^{I,\alpha}(x) = C_{\alpha} x^{-\alpha}$ on $I$ where $C_{\alpha} = \frac{-\alpha + 1}{x_{\max}^{\alpha + 1} - x_{\min}^{\alpha + 1}}$. Then, the scaling parameter $\hat{\alpha}$ maximizes the following logarithmic likelihood density:

$$L = n \log C_{\alpha} - \alpha \sum_{i=1}^{n} C_{\alpha} x_i^{-\alpha}.$$  \hspace{1cm} (A.3)

Th power law interval $I = [x_{\min}, x_{\max}]$ will be obtained by iterative penalized KS method with penalty coefficient as following:

1. Find an initial interval $[\hat{x}_{\min}(1), \hat{x}_{\max}(1)]$ which minimizes the KS metric such as

$$I_1 = [\hat{x}_{\min}(1), \hat{x}_{\max}(1)] = \arg \min_{x_1, x_2 \in L_m(X), \frac{x_2}{x_1} > l} \rho^K_{S}(x_1, x_2),$$  \hspace{1cm} (A.4)

where $L_m(X)$ is a subset of $X$ to reduce cost and $l$ is a reasonable threshold, and find the scaling parameter $\hat{\alpha}$ on $I_1$. Estimate the $p$-value of the obtained power law fit, and if the $p$-value < 0.1, the bounded power law hypothesis for the data set $X$ is rejected and end the algorithm.

2. Set two penalized KS metrics for the penalty coefficient: $d_{\text{good}} = 10^{-5}$ and $d_{\text{bad}} = 10^{-1}$. And then, inductively, find $[\hat{x}_{\min}, \hat{x}_{\max}]$ such that

$$\hat{x}_{\min}(k + 1) = \arg \min_{x_1 \in L_m(X), x_1 < \frac{\hat{x}_{\max}(k)}{\hat{x}_{\max}(k)}} \rho^p_{KS}(x_1, \hat{x}_{\max}(k)),$$  \hspace{1cm} (A.5)
\[ \hat{x}_{\text{max}}(k + 1) = \arg \min_{x_2 \in L_m(X), x_2 < l x_{\text{min}}(k+1)} \text{pKS}_{X}^{}(\hat{x}_{\text{min}}(k + 1), x_2) \]  

(A.6)

for \( k = 1, 2, \cdots, K \), for the corresponding metric.

3. (a) Take \( d = \sqrt{d_{\text{good}} d_{\text{bad}}} \).

(b) Fit a bounded power law to \( X \) by using iterative penalized KS penalty coefficient \( d \).

(c) If the resulting bounded power law fit is validated, update \( I, \hat{\alpha} \) and then set \( d_{\text{good}} := d \); otherwise set \( d_{\text{bad}} := d \). Repeat this step until intervals converge to an interval on which the power law fit is validated.

(d) If \( |d_{\text{bad}} - d_{\text{good}}| < d_{\text{tol}} = 10^{-4} \), break out of the loop.

**A.2 Fitting validation**

In order to validate a power law fit obtained by the KS method, generating many semi-parametric bootstrap samples and fit a power law to each of these samples using the fitting procedure described above. This gives a power law fit to each bootstrap sample and KS metric value for each power law fit. The \( p \)-value estimation of the power law hypothesis is the fraction of times that these KS metric values are greater than the KS metric value of the fit from first step, which is accepted if the \( p \)-value \( p > 0.1 \).

**A.3 Model comparison**

Compare power law distribution against other alternative distributions such as exponential distribution or log-normal distribution to see if a distribution is a better fit or the data is insufficient to distinguish one fit from the other. The likelihood ratio test [42] is used to compare models for the best fit.
Appendix B

MINIMUM ACTION METHOD

Consider a stochastic differential equation

\[ dX = b(X)dt + \epsilon \sigma(X)dW, \]  \hspace{1cm} (B.1)

then for \( U \in C_{[0,T]} \), the action functional associated with (B.1) is defined as

\[ S_T(U) = \int_0^T P(t)dt, \]  \hspace{1cm} (B.2)

for

\[ P(t) := L(\dot{U}, U) = \frac{1}{2} < \dot{U} - b(U), a^{-1}(U)(\dot{U} - b(U)) >, \]  \hspace{1cm} (B.3)

if the integral is finite. Otherwise, set \( S_T(U) = \infty \).

Minimum action method (MAM) is a numerical method to find that minimum action path \( \phi \) that minimize the action functional \( S_T(\phi) \) given in [22].

**Step 1. Discretization of the action functional:** Discretize the time domain \([0, T]\) with a mesh with sizes \( \Delta t = T/n \), and let

\[ t_j := j \Delta t, \quad t_{j+\frac{1}{2}} := (j + \frac{1}{2}) \Delta t, \quad \text{for} \quad j = 0, 1, \ldots, n. \]  \hspace{1cm} (B.4)

Use the midpoint rule to compute the integral \( \int_0^T P(t)dt \), that is,

\[ S_T(U) = \Delta t \sum_{0}^{n-1} P_{j+\frac{1}{2}}, \]  \hspace{1cm} (B.5)
where
\[
P_{j + \frac{1}{2}} = \frac{1}{2} \left( U_{j+1} - U_j \right) - b \left( \frac{U_{j+1} + U_j}{2} \right),
\]
\[
a \left( \frac{U_{j+1} + U_j}{2} \right) \left( \frac{U_{j+1} - U_j}{\Delta t} - b \left( \frac{U_{j+1} + U_j}{2} \right) \right).
\]

**Step 2. Minimization of the discretization action functional by L-BFGS:**

L-BFGS (limited memory Broyden-Fletcher-Goldfarb-Shanno algorithm) is one of the most efficient quasi-Newton methods for large-scale problems [43]. Let \( S_1 \) and \( S_2 \) be two stable fixed points (those are corresponding to wake and sleep dominant states respectively in the sleep-wake model) and \( U_1 \) be the saddle point of \( b(x) \) in (B.1). Let \( \phi_0 \) be a initial path which is a union of line segments from \( S_1 \) to \( U_1 \) and \( U_1 \) to \( S_2 \), and let \( \phi_k \) be the current iterate. Then, the next one is defined as

\[
\phi_{k+1} = \phi_{k+1} + \alpha_k p_k
\]

where \( p_k = -H_k \Delta f_k, f_k = S_T(\phi_k) \). \( H_k \) is an approximation of the inverse of the Hessian of \( f(x) \) at \( x_k \), and \( \alpha_k \) is a step length satisfying the Wolfe condition:

\[
\begin{aligned}
f_{k+1} &\leq f_k + c_1 \alpha_k \Delta f_k^T p_k \\
\Delta f_{k+1}^T p_k &\geq c_2 \Delta f_k^T p_k
\end{aligned}
\]

with \( 0 < c_1 < c_2 < 1 \). Here, \( H_k \) is defined recursively by

\[
H_{k+1} = (I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho_k s_k s_k^T
\]

where

\[
s_k = \phi_{k+1} - \phi_k, \quad y_k = \Delta f_{k+1} - f_k, \quad \rho_k = \frac{1}{y_k^T s_k}.
\]

The initial Hessian approximation \( H_0 \) is chosen as the identity matrix scaled by

\[
\gamma_k = \frac{s_k^T y_k}{y_k^T y_k}.
\]
The gradient of $S_T(U)$ is necessary to use L-BFGS method in the step 2. Let

$$
\Delta U_j := \frac{U_{j+1} - U_j}{\Delta t}, \quad \bar{U}_j := \frac{U_{j+1} + U_j}{2}
$$

then

$$
\frac{\partial S_T}{\partial U_j^k} = \frac{1}{2} \Delta t \left[ \left\langle -\frac{e_k}{\Delta t} - \frac{1}{2} \partial_k b (\bar{U}_j) , a^{-1}(\bar{U}_j) \left( \Delta U_j - b (\bar{U}_j) \right) \right\rangle 
+ \left\langle \Delta U_j - b (\bar{U}_j) , \frac{1}{2} \partial_k (a^{-1}) (\bar{U}_j) \left( \Delta U_j - b (\bar{U}_j) \right) \right\rangle 
+ \left\langle \Delta U_j - b (\bar{U}_j) , a^{-1}(\bar{U}_j) \left( -\frac{e_k}{\Delta t} - \frac{1}{2} \partial_k b (\bar{U}_j) \right) \right\rangle 
+ \left\langle \frac{e_k}{\Delta t} - \frac{1}{2} \partial_k b (\bar{U}_{j-1}) , a^{-1}(\bar{U}_{j-1}) \left( \Delta U_{j-1} - b (\bar{U}_{j-1}) \right) \right\rangle 
+ \left\langle \Delta U_{j-1} - b (\bar{U}_{j-1}) , \frac{1}{2} \partial_k (a^{-1}) (\bar{U}_{j-1}) \left( \Delta U_{j-1} - b (\bar{U}_{j-1}) \right) \right\rangle 
+ \left\langle \Delta U_{j-1} - b (\bar{U}_{j-1}) , a^{-1}(\bar{U}_{j-1}) \left( \frac{e_k}{\Delta t} - \frac{1}{2} \partial_k b (\bar{U}_{j-1}) \right) \right\rangle \right] ,
$$

where $U_j = [U^1_j, U^2_j, U^3_j, U^4_j]^T$.
Appendix C

PENALIZATION METHOD

**Proposition C.1** (Stochastic integration by parts). Let $X$ and $Y$ be (continuous) semimartingales. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X dY + \int_0^t Y dX + [X,Y]_t \text{ a.s. for } t \geq 0. \quad (C.1)$$

**Theorem C.1** (Burkholder-Davis-Gundy inequality [44]). For any $1 \leq p < \infty$ there exist positive constants $c_p, C_p$ such that, for all local martingales $X$ with $X_0 = 0$ and stopping times $\tau$, the following inequality holds.

$$c_p \mathbb{E} \left[ [X]_{\tau}^{p/2} \right] \leq \mathbb{E} \left[ (X^*_\tau)^p \right] \leq C_p \mathbb{E} \left[ [X]_{\tau}^{p/2} \right], \quad (C.2)$$

where $[X]$ denotes the quadratic variation of a process $X$, and $X^*_\tau \equiv \sup_{s \leq \tau} |X_s|$ is its maximum process.

Furthermore, for continuous local martingales, this statement holds for all $0 < p < \infty$.

**Lemma C.1** (Solutions of Ornstein-Uhlenbeck equations). The Ornstein-Uhlenbeck process is a stochastic process which solves

$$dX_t = -\kappa(X_t - \theta) \, dt + \sigma \, dW_t, \quad (C.3)$$

where $W_t$ is a standard Brownian motion, and $\kappa > 0$, $\theta$, and $\sigma > 0$ are constants. Then,

$$X_t = \theta + e^{-\kappa(t-s)}(X_s - \theta) + \sigma \int_s^t e^{-\kappa(t-u)} \, dW_u. \quad (C.4)$$
Proof. Let \( Y_t = X_t - \theta \), then \( Y_t \) satisfies the SDE:

\[
dY_t = -\kappa Y_t \, dt + \sigma \, dW_t.
\]  

(C.5)

In SDE (C.5), the process \( Y_t \) is seen to have a drift towards the value zero, at an exponential rate \( \kappa \). This motivates the change of variables

\[
Y_t = e^{-\kappa t} Z_t \quad \Leftrightarrow \quad Z_t = e^{\kappa t} Y_t ,
\]

(C.6)

which should remove the drift. A calculation with the product rule for Itô integrals shows that this is so:

\[
dZ_t = \kappa e^{\kappa t} Y_t \, dt + e^{\kappa t} \, dY_t \\
= \kappa e^{\kappa t} Y_t \, dt + e^{\kappa t} (-\kappa Y_t \, dt + \sigma \, dW_t) \\
= 0 \, dt + \sigma e^{\kappa t} \, dW_t.
\]

(C.7)

The solution for \( Z_t \) is immediately obtained by Itô-integrating both sides from \( s \) to \( t \):

\[
Z_t = Z_s + \sigma \int_s^t e^{\kappa u} \, dW_u.
\]

(C.8)

Reversing the changes of variables, we have:

\[
Y_t = e^{-\kappa t} Z_t = e^{-\kappa (t-s)} Y_s + \sigma e^{-\kappa t} \int_s^t e^{\kappa u} \, dW_u ,
\]

(C.9)

and

\[
X_t = Y_t + \theta = \theta + e^{-\kappa (t-s)} (X_s - \theta) + \sigma \int_s^t e^{-\kappa (t-u)} \, dW_u .
\]

(C.10)

\[ \square \]

Lemma C.2 (Comparison lemma [15]). Let

\[
x^{(i)}_t = x^{(i)}_0 + \int_0^t f_i(x^{(i)}_s, s) \, ds + \int_0^t F(x^{(i)}_s, s) \, dW_s , \quad i = 1, 2,
\]

(C.11)

denote the solutions of two one-dimensional stochastic differential equations, where we assume that the drift and diffusion coefficients are Lipschitz continuous uniformly in \( t \). If both,
• the initial conditions are almost surely ordered: \( x^{(1)}_0 \leq x^{(2)}_0 \) \( \mathbb{P} \)-almost surely,

• the drift coefficients are ordered: \( f_1(x,t) \leq f_2(x,t) \) for all \((x,t)\),

then

\[
\mathbb{P}\{x^{(1)}_t \leq x^{(2)}_t \ \forall t\} = 1. \quad (C.12)
\]

**Lemma C.3** (Bernstein-type inequality [15]). Let \( \phi(u) \) be a Borel-measurable deterministic function such that

\[
\Phi(t) = \int_0^t \phi(u)^2 du \quad (C.13)
\]

exists. Then

\[
\mathbb{P}\left\{ \sup_{0 \leq s \leq t} \int_0^s \phi(u) dW u \geq \delta \right\} \leq \exp \left\{ -\frac{\delta^2}{2\Phi(t)} \right\}. \quad (C.14)
\]

**Definition C.1** (UT condition [45, 27]). For every \( q \in \mathbb{R}^+ \) the family of random variables

\[
\left\{ \int_{[0,q]} U^n_s dZ^n_s, n \in \mathbb{N}, U^n \in \mathcal{U}_q^n \right\} \quad (C.15)
\]

is tight in \( \mathbb{R} \), where \( \mathcal{U}_q^n \) is the class of discrete predictable processes of the form

\[
U^n_s = U^n_0 + \sum_{i=0}^k U^n_{t_i} 1_{t_i < s \leq t_{i+1}} \quad (C.16)
\]

such that \( 0 = t_0 < t_1 < \cdots < t_k = q \) and every \( U^n_i \) is \( \mathcal{F}_t^n \) measurable, \( |U^n_i| \leq 1 \) for every \( i \in \mathbb{N} \cup \{0\} \), \( n, k \in \mathbb{N} \).

**Proposition C.2** ([27]). If a sequence \( \{Y_n\} \) is of the form \( Y^n_t = Y^n_0 + M^n_t + V^n_t \), where \( M^n \) is a square integrable martingale and \( V_n \) is a process with bounded variation such that

\[
\sup_{n \in \mathbb{N}} \{ |Y^n_0| + E([M^n]_q + |V^n|_q^2) \} < \infty, \quad q \in \mathbb{R}^+, \quad (C.17)
\]

where \([M^n] \) is the quadratic variation of \( M^n \), then \( \{Y_n\} \) satisfies (UT) condition.
Theorem C.2 ([46]). Let $D_n$, $D$ be convex domains, and $\{Y_n\}$ be a sequence of processes such that $Y_0^n \in D$. Let $\{(X^n, K^n)\}$ be a sequence of solutions of the Skorokhod problem associated with $\{Y^n\}$ on $\{D^n\}$. If for some sequences of processes $\{H^n\}$, $\{G^n\}$

$$
(H^n, G^n, Y^n) \xrightarrow{\mathcal{D}} (H, G, Y) \text{ in } \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \quad \text{(C.18)}
$$

(respectively)

$$
(H^n, G^n, Y^n) \xrightarrow{\mathcal{P}} (H, G, Y) \text{ in } \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d}), \quad \text{(C.19)}
$$

and $|\Delta Y| < r_0$ ( $\Delta Y_t := Y_t - Y_{t-}$) then

$$
(H^n, G^n, X^n, K^n, Y^n) \xrightarrow{\mathcal{D}} (H, G, X, K, Y) \text{ in } \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{dd}) \quad \text{(C.20)}
$$

(respectively)

$$
(H^n, G^n, X^n, K^n, Y^n) \xrightarrow{\mathcal{P}} (H, G, X, K, Y), \text{ in } \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{dd}), \quad \text{(C.21)}
$$

where $(X, K)$ is a solution of the Skorokhod problem associated with the process $Y$. 

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