ON THE NONVANISHING OF CENTRAL L-VALUES ASSOCIATED TO HECKE EIGENFORMS

DISSERTATION

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ABSTRACT

In this dissertation we establish new asymptotic formulas for moments of central \( L \)-values associated to Hecke eigenforms in the weight aspect. To do this we exploit the Shimura correspondence between integral weight and half integral Hecke eigenforms. Our work is based upon Kohnen’s explicit isomorphism and Waldspurger’s identity between the space of modular forms of weight \( k \) level 1 and a subspace (often referred to as Kohnen’s space) of the cusp forms of weight \( k + 1/2 \) and level 4.
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CHAPTER 1
INTRODUCTION

The introduction gives an overview of automorphic L-functions and the tools exploited in this thesis. We then present our new results on the moments of central L-values in the weight aspect.

1.1 Modular forms

The theory of half-integral weight forms is mostly scattered throughout various papers. However, a very good primer can be found in “Introduction to Elliptic Curves and Modular Forms” [10]. The first two sections of this dissertation expound on the more notable differences of half-integral weight forms from the integral weight case omitting the details of their proof.

The theory of modular forms begins with the modular group $\Gamma = SL_2^+(\mathbb{Z})$ acting on the upper half plane $\mathbb{H} := \{ z \in \mathbb{C} : \Im(z) > 0 \}$ via fractional linear transformations.

$$\alpha z := \frac{az + b}{cz + d} \quad \text{where} \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

(Fractional linear transformation)

In this paper we will use the subgroups $\Gamma_0(N) \subset \Gamma$ defined by

1
\[ \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\} \]

Note that \( \Gamma = \Gamma_0(1) \). For each group \( \Gamma_0(N) \) there exists a (non-unique) fundamental domain \( F \) satisfying the following properties

- \( F \) is an open subset in \( \mathbb{H} \)
- The closure \( \overline{F} \) of \( F \) in \( \mathbb{C} \) intersects every orbit of \( \Gamma_0(N) \)
- No two points of \( F \) are \( \Gamma \)-equivalent

The fundamental domain \( F \) is not compact and it’s closure will intersect the boundary of \( \mathbb{H} \) which is \( \partial \mathbb{H} = \mathbb{R} \cup \{ \infty \} \). These points are called the cusps of the group \( \Gamma_0(N) \) and they fall into equivalence classes under the group action if we extend its definition to the real numbers by the same action and to infinity by

\[
\alpha \infty = \frac{a}{c} \text{ if } c \neq 0 \quad \text{and} \quad \alpha \infty = 0 \text{ if } c = 0 \quad (1.1.1)
\]

A holomorphic function \( f : \mathbb{H} \rightarrow \mathbb{C} \) is called a modular function of level \( N \) if there exists another function \( j : \Gamma_0(N) \times \mathbb{H} \rightarrow \mathbb{C} \) which is holomorphic in the second variable and

\[
f(\alpha z) = j(\alpha, z)f(z) \quad \text{for all } \alpha \in \Gamma_0(N) \quad (1.1.2)
\]

The new function \( j(\alpha, z) \) is known as the factor of automorphy for \( f \). It necessarily satisfies a consistency property

\[
j(\alpha \beta, z) = j(\alpha, \beta z)j(\alpha, z) \quad \text{for all } \alpha, \beta \in \Gamma_0(N) \quad (1.1.3)
\]
If $f$ is holomorphic at the cusps of $\Gamma_0(N)$ (i.e. $\lim_{z \to z_0} f(z)$ exists for all cusps $z_0$) $f$ is called a **modular form**, and if additionally, $f$ vanishes at its cusps it is called a **cusp form**.

The theory of half integral weight modular forms is deeply tied to quadratic characters. The set of all quadratic characters can be represented as
$$
\chi_D(x) = \left(\frac{D}{x}\right)
$$
where $D$ is a fundamental discriminant. Additionally, we define the arithmetic function $\epsilon_x = \left(\frac{-4}{x}\right)^{1/2}$ using the principle branch of the squareroot. While the character $\chi_{-4}(x)$ is multiplicative, the function $\epsilon_x$ isn’t because $\epsilon_3^2 = i$ but $\epsilon_3\epsilon_3 = i^2 = -1$. Instead we have the factorization formula:

$$
\epsilon_{ab} = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right)\epsilon_a\epsilon_b
$$

(1.1.4)

**Definition 1.** Let $k$ and $N$ be positive integers.

A modular form with group $\Gamma_0(N)$ and factor of automorphy $j(\alpha, z) = (cz + d)^k$ is called an **integral weight modular form** of weight $k$ and level $N$.

Suppose additionally that $4|N$. A modular form with group $\Gamma_0(N)$ and factor of automorphy $j(\alpha, z) = \left(\frac{c}{d}\right)\epsilon_d^{-2k-1}(cz + d)^{k+1/2}$ is called a **half-integral weight modular form** of weight $k + 1/2$ and level $N$.

Such functions form a finite dimensional vector space over $\mathbb{C}$ and even have a Fourier expansion $f(z) = \sum_{n \geq 0} a(n)e(nz)$ (the first coefficient vanishes when $f$ is a cusp form). The space of modular forms of weight $\eta = k$ or $k + 1/2$ and level $N$ is denoted by $M_{\eta}(N)$ and the cusp forms are denoted by $S_{\eta}(N)$. The fact that any two cusp forms vanish at their cusps allows us to define their **Petersson inner product**
\[ \langle F, G \rangle := \frac{1}{[\Gamma_0(1) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}} F(z) \overline{G(z)} y^{k-2} \, dx \, dy \quad F, G \in S_k(N) \]

\[ \langle f, g \rangle := \frac{1}{[\Gamma_0(4) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^{k-3/2} \, dx \, dy \quad f, g \in S_{k+1/2}(N) \]

**Integral weight modular forms**

Whereas number theory is particularly interested in the modular group, a natural question is how to extend this idea to a larger group. In the integral weight case we use the group \( G = \text{GL}_2^+(\mathbb{Q}) \). For \( f \in M_k(N) \) and \( \alpha \in \Gamma_0(N) \) we define the **slash operator**

\[
 f(z) \big| [\alpha]_k := (\det \alpha)^{k/2} (cz + d)^{-k} f(\alpha z)
\]

(1.1.5)

This is a group action of \( G \) on the space of modular forms \( M_k(N) \), which extends to the group algebra of \( G \) over the scalar field \( \mathbb{C} \). With this group algebra one develops the Hecke operators \( T_k(n), n \geq 1 \). This paper only considers the full modular group, and the properties of Hecke operators are given in the following proposition.

**Proposition 1.**

1. The Hecke operators act on the spaces \( M_k(1) \) and \( S_k(1) \)

\[ T_k(n) : M_k(1) \rightarrow M_k(1) \]

\[ T_k(n) : S_k(1) \rightarrow S_k(1) \]
2. The algebra of \( \{ T(n^2) : n \geq 1 \} \) is commutative and is generated by \( \{ T(p) : p \text{ prime} \} \) and are Hermitian with respect to the Petersson inner product.

\[
T_k(m)T_k(n) = \sum_{d \mid (m,n)} d^{k-1} T\left( \frac{mn}{d^2} \right) \quad m, n \geq 1
\]

\[
T_k(mn) = \sum_{d \mid (m,n)} \mu(d) d^{k-1} T\left( \frac{m}{d} \right) T\left( \frac{n}{d} \right) \quad m, n \geq 1
\]

\[
T(p^{v+1}) = T(p)T(p^v) - p^{k-1}T(p^{v-1}) \quad p \text{ prime}
\]

\[
\langle T_k(n)f, g \rangle = \langle f, T_k(n)g \rangle \quad \text{for all } f, g \in S_k(1)
\]

3. There exists an orthogonal basis of forms in \( S_k(1) \) which are simultaneously eigenforms of every Hecke operator. These forms can be normalized so that their Fourier coefficients are precisely the eigenvalues of the Hecke operators. Specifically, we can choose an orthogonal basis \( B_k \) of \( S_k(1) \) such that for any \( f \in B_k \),

\[
f(z) = \sum_{n \geq 1} a(n)e(nz) \quad T_k(n)f = a(n)f
\]

For any \( f(z) = \sum a(n)e(nz) \in B_k \) we define its \( L \)-function as

\[
L(f, s) := \sum_{n \geq 1} \frac{a(n)}{n^s}
\]

the above properties show that this function has an Euler product

\[
L(f, s) = \prod_p \left( 1 - a(p)p^{-s} + p^{k-1-2s} \right)^{-1}
\]

**Half-integral weight modular forms**

We still want to work in the larger group \( G \); however, a branch must be choosen for the square root in the factor of automorphy. For the group \( \Gamma_0(4) \) our choice
of \( j(\alpha, z) = \left( \frac{c}{d} \right) \epsilon_d^{-2k-1}(cz + d)^{k+1/2} \) resolved this problem, but this function is not defined for the larger group \( G \). The solution is to define a four sheeted-covering of \( GL_2^+(\mathbb{Q}) \) containing all the possible roots.

**Definition 2.** (Shimura, [21]) Let \( T \subset \mathbb{C} \) denote the group of fourth roots of unity. We now define \( G_{k+1/2} \) to be the set of all ordered pairs \( (\alpha, \phi(z)) \), where \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}) \) and \( \phi(z) \) is a holomorphic function on \( \mathbb{H} \) such that

\[
\phi(z)^2 = t \frac{cz + d}{\sqrt{\det \alpha}} \quad \text{for some} \ t \in \{-1, 1\}.
\]

\( G_{k+1/2} \) becomes a group with multiplication defined by \( (\alpha, \phi(z))(\beta, \psi(z)) = (\alpha \beta, \phi(\beta z)\psi(z)) \).

This gives us the exact sequence

\[
1 \longrightarrow T \overset{t \mapsto (1, t)}{\longrightarrow} G \overset{P : (\alpha, \phi(z)) \mapsto \alpha}{\longrightarrow} GL_2^+(\mathbb{Q}) \longrightarrow 1
\]

For \( \xi = (\alpha, \phi(z)) \in G_{k+1/2} \) and any integer \( k \), we define the **slash operator** analogously to (1.1.5) for \( f \in M_{k+1/2}(N) \)

\[
f(z)[\xi]_{k+1/2} := f(\alpha z)\phi(z)^{-2k-1}
\]

(1.1.7)

giving us a group action that extends to the group algebra of \( G_{k+1/2} \).

For an integer \( 4|N \), the group \( \Gamma_0(N) \) has an isomorphic copy in \( G \) under correspondence \( \alpha \leftrightarrow (\alpha, J(\alpha, z)) \), where \( J(\alpha, z) = \left( \frac{c}{d} \right) \epsilon_d^{-1}(cz + d)^{1/2} \). By lifting an element \( \alpha \) of the group \( \Gamma_0(N) \) to \( G \) the slash operator \( f(z)[\alpha]_{k+1/2} \) is well defined.
So we use $\tilde{\Gamma}_0(4) := P^{-1}(\Gamma_0(4))$ rather than $SL_2^+(\mathbb{Z})$ as our starting point. The group $G_{k+1/2}$ is used to construct the Hecke operators $T_{k+1/2}(n)$ which have the properties:

**Proposition 2.**

1. *The Hecke operators act on the spaces $M_{k+1/2}(4)$ and $S_{k+1/2}(4)$*

   $$T_{k+1/2}(n) : M_{k+1/2}(4) \rightarrow M_{k+1/2}(4)$$
   $$T_{k+1/2}(n) : S_{k+1/2}(4) \rightarrow S_{k+1/2}(4)$$

2. *If $n$ is not a perfect square then $T_{k+1/2}(n) = 0$. 
   
3. *The algebra of $\{T_{k+1/2}(n^2) : n \geq 1\}$ is commutative, is generated by $\{T_{k+1/2}(p^2) : p$ prime$\}$ and the Hecke operators $T_{k+1/2}(n^2)$ are Hermitian with respect to the Petersson inner product.*

   $$T_{k+1/2}(m^2)T_{k+1/2}(n^2) = \sum_{d|(m^2,n^2)} d^{k-1}T_{k+1/2}\left(\frac{m^2n^2}{d^2}\right) \quad m, n \geq 1$$

   $$T_{k+1/2}(m^2n^2) = \sum_{d|(m^2,n^2)} \mu(d)d^{k-1}T_{k+1/2}\left(\frac{m^2}{d}\right)T_{k+1/2}\left(\frac{n^2}{d}\right) \quad m, n \geq 1$$

   $$T_{k+1/2}(p^{2v+2}) = T_{k+1/2}(p^2)T_{k+1/2}(p^{2v}) - p^{2k-2}T_{k+1/2}(p^{2v-2}) \quad p$ prime$,$

   $$\langle T_{k+1/2}(n^2)f, g \rangle = \langle f, T_{k+1/2}(n^2)g \rangle \quad \text{for all } f, g \in S_{k+1/2}(1)$$
4. For any prime \( p \), \( T_{k+1/2}(p^2) \left( \sum a(n)e^{2\pi inz} \right) = \\
\sum \left( a(p^n) + \left( \frac{(-1)^kn}{p} \right) p^{k-1}a(n) + p^{2k-1}a(n/p^2) \right) e^{2\pi inz} \\
\) (1.1.8)

where \( a(n/p^2) = 0 \) if \( n/p^2 \) is not an integer.

The second statement marks a serious defect of the Hecke theory for half integral weight forms in that we are missing almost all of the Hecke polynomials leading to a limited version of an Euler product.

**Proposition 3. (Shimura [21])** Let \( f \in M_{k+1/2}(N) \) be an eigenform of all the Hecke operators \( T(p^2) \) with eigenvalues \( \lambda_p \) and \( m \) is not divisible by any square prime to \( N \). Then

\[
\sum_{n=1}^{\infty} a(mn^2)n^{-s} = a(m) \prod_p \frac{1 - \left( \frac{(-1)^km}{p} \right) p^{k-1-s}}{1 - \lambda_pp^{-s} + p^{k-2-2s}} \\
\) (1.1.9)

### 1.2 Kohnen’s space and the theory of newforms

The big break in constructing \( L \)-functions from half integral weight forms came with Shimura’s paper [21] where it was shown that integral weight \( L \)-functions could be constructed from half integral weight forms.

**Shimura Correspondence. [21]** Suppose that \( f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi inz} \in S_{k+1/2} \) is an eigenform for \( T_{k+1/2}(p^2) \) for all primes \( p \) with corresponding eigenvalue \( \lambda_p : T_{k+1/2}(p^2)f = \lambda_p f \). Define a function \( g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi inz} \) by the formal identity

\[
\sum_{n=1}^{\infty} b_n n^{-s} = \prod_p \frac{1}{1 - \lambda_pp^{-s} + p^{2k-1-2s}} \\
\) (1.2.1)
Then \( g \in M_{2k}(N') \) for some integer \( 4|N' \). If \( k \geq 5 \), then \( g \) is a cusp form.

Furthermore, (1.2.1) is equivalent to:

1. \( b_1 = 1 \)

2. \( b_p = \lambda_p \) for all primes \( p \)

3. \( b_{\lambda^\nu} = \lambda_p b_{\lambda^{\nu-1}} - p^{k-2}b_{\lambda^{\nu-2}} \) for \( \nu \geq 2 \)

4. \( b_{mn} = b_mb_n \) if \( m \) and \( n \) are relatively prime.

Because the Hecke operators are commutative and Hermitian, we can choose a basis of simultaneous eigenforms for \( S_{k+1/2}(4) \). Therefore, Shimura’s correspondence defines a linear transformation from \( S_{k+1/2}(4) \) into \( S_{2k}(1) \) and, therefore, an endomorphism of the Hecke algebras. Kohnen showed (1980 [11], 1982 [13]) that there is a useful subspace of the cusp forms where the correspondence is in fact an isomorphism of Hecke algebras. Define \( S_{k+1/2}^+(4) \) as the subspace of \( S_{k+1/2}(4) \) consisting of modular forms whose Fourier coefficients are supported only on the integers satisfying \((-1)^k n \equiv 0, 1 \) (mod 4).

In addition to the Hecke operators we define:

\[
 f(z)|U_4 = \frac{1}{4} \sum_{v(4)} f \left( \frac{z + v}{4} \right) \quad \text{(Degenerate Hecke operator)}
\]

\[
 f(z)|W_4 = (-2iz)^{-k-1/2}f \left( -\frac{1}{4z} \right) \quad \text{(Fricke involution)}
\]

**Proposition 4. (Kohnen, [11])**

1. \( U_4 \) and \( W_4 \) leave \( S_{k+1/2} \) stable

---

\(^1\)It has since been shown that one can always take \( N' = N/2 \) (Niwa 1975 [20])
2. The operator $Q := U_4 W_4$ is Hermitian:

$$\langle f | Q | g \rangle = \langle f, g | Q \rangle$$ for all $f, g \in S_{k+1/2}(4)$

3. The operator $Q$ satisfies the quadratic equation $(Q - \alpha)(Q - \beta) = 0$ where

$$\alpha = \chi_8(2k + 1)2^k$$ and $$\beta = -\frac{1}{2} \alpha.$$ Its $\alpha$-eigenspace is $S_{k+1/2}^+(4)$

The third part gives us an orthogonal projection.

$$\text{proj} := \frac{Q - \beta}{\alpha - \beta} : S_{k+1/2}(4) \mapsto S_{k+1/2}^+(4)$$

(1.2.2)

With this we define the Hecke operators on $S_{k+1/2}^+(4)$ by

$$T_{k+1/2}^+(p^2) = \begin{cases} \text{proj} \circ T_{k+1/2}(p^2) & \text{if } p \neq 2 \\ \frac{3}{2} \text{proj} \circ T_{k+1/2}(p^2) & \text{if } p = 2 \end{cases}$$

(1.2.3)

Proposition 5. Let $f = \sum a(n)q^n$ be in $S_{k+1/2}^+(4)$. Then

$$f | T_{k+1/2}^+(p^2) = \sum_{n \geq 1, (-1)^k n = 0, 1(4)} \left( a(p^2 n) + \left( \frac{(-1)^k n}{p} \right) p^{k-1} a(n) + p^{2k-1} a(n/p^2) \right) q^n$$

and the $T_{k+1/2}^+(p^2)$ generate a commutative $\mathbb{C}$-algebra of Hermitean operators \( \{ T_{k+1/2}^+(n^2) : n \geq 1 \} \) which preserve $S_{k+1/2}^+(4)$.

Proposition 6. (Kohnen, [11])

There exist orthogonal bases $F_k$ and $B_k$ of $S_{k+1/2}^+(4)$ and $S_{2k}(1)$ respectively with the properties:

1. The functions of $F_k$ and $B_k$ are common eigenfunctions of the Hecke operators $T_{k+1/2}^+(n^2)$ and $T_k(n)$ respectively.
2. Suppose that \( f \in S_{k+1/2}^+(4) \) and its eigenvalues are given by \( f|T_{k+1/2}^+(n^2) = \lambda_n f \).

Then there exists \( F \in S_{2k}(1) \) such that \( F|T_{2k}(n) = \lambda_n F \) for all \( n \geq 1 \).

3. The Fourier expansions of \( f \) and \( F \) are related as follows: if \( f = \sum_{n \geq 1} a(n)q^n \) and \( F = \sum_{n \geq 1} A(n)q^n \), and if \( D \) is a fundamental discriminant (i.e. \( D \) equals 1 or is the discriminant of a quadratic field) such that \((-1)^k D > 0\), then

\[
a(|D|)A(n) = \sum_{d | n} \chi_D(d)d^{k-1}a(|D|n^2/d^2) \quad (1.2.4)
\]

Alternatively, this identity can express the ratio of two Fourier coefficients of \( f \) which differ by a perfect square in terms of the Fourier coefficients of \( F \):

\[
a(|D|n^2) = a(|D|) \sum_{d | n} \mu(d)\chi_D(d)d^{k-1}A(n/d)
\]

4. If \( D \) is as in 3 and \( (D, k) \neq (1, 0) \), the map \( \mathcal{L}_{D,k}^+ \) defined by

\[
\sum_{n \geq 1} b(n)q^n \rightarrow \sum_{n \geq 1} \left( \sum_{d | n} \left( \frac{D}{d} \right) d^{k-1}b \left( \frac{n^2}{d^2}|D| \right) \right) q^n \quad (1.2.5)
\]

maps \( S_{k+1/2}^+(4) \) to \( S_{2k}(1) \) and commutes with the Hecke operators. There exists a linear combination in the \( \mathcal{L}_{D,k}^+ \) which is an isomorphism.

### 1.3 L-functions and Waldspurger’s identity

For any form \( F \in S_{2k}(1) \) fundamental discriminant \( D \) such that \((-1)^k D > 0\) and character \( \chi_D \) we define
\[ L(F \otimes \chi_D, s) := \sum_{n \geq 1} \frac{\chi_D(n) A(n)}{n^s} \quad \text{\textit{(L-function of } F \text{, twisted by } \chi_D)} \]

\[ \Lambda(F \otimes \chi_D, s) := \left( \frac{D_0}{2\pi} \right)^s \Gamma(s) L(F \otimes \chi_D, s) \quad \text{\textit{(Completed } L\text{-function)}} \]

where \( D_0 \) is the modulus of the character \( \chi_D \). The completed \( L \)-functions satisfies the functional equation \( \Lambda(F \otimes \chi_D, 2k - s) = \Lambda(F \otimes \chi_D, s) \) for all \( F \in \mathcal{B}_k \), which is used construct the analytic continuation \( L(F \otimes \chi_D, s) \).

**Lemma 1.** If \( k \geq 1 \) is odd then \( L(F \otimes \chi_D, k) = 0 \). Suppose \( k \geq 2 \) is even and \( F \in \mathcal{B}_k \). Then \( D_0 = D \) and

\[ L(F \otimes \chi_D, k) = 2 \sum_{n \geq 1} \frac{\lambda_F(n)}{n^{1/2}} W_k \left( \frac{2\pi n}{D} \right) \quad \text{\textit{(1.3.1)}} \]

where

\[ W_k(y) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) y^{-u} du \]

and \( K(u) \) is an odd meromorphic function, bounded in vertical strips and containing a single pole of residue 1 at \( u = 0 \).

Similarly, for a form \( f \in S_{k+1/2}(4) \)

\[ L(f, s) = \sum_{n \geq 1} \frac{a_f(n)}{n^s} \quad \text{\textit{(L-function of } f \text{)}} \]

\[ \Lambda(f, s) = \pi^{-s} \Gamma(s) L(f, s) \quad \text{\textit{(Completed } L\text{-function)}} \]
In the integral weight case one uses the Fricke involution, which preserves eigenforms to derive the functional equation. In the half-integral weight case this is not an option. The functional equation is not quite as nice in this case because the involution operator $W_4$ does not necessarily preserve the forms in $\mathcal{F}_k$. However, by proposition 4, $f(z)|W_4 = \frac{1}{\alpha} f(z)|U_4 = \frac{1}{\alpha} \sum_{n \geq 1} a(4n)e(nz)$ when $f \in \mathcal{F}_k$ that gives the functional equation

$$\pi^{-s} \Gamma(s)L(f, s) = \pi^{s-\nu} \Gamma(\nu - s) \chi_8(2k + 1) 2^{-k} L(f|U_4, \nu - s)$$  \hspace{1cm} (1.3.2)

**Proposition 7.** Let $\nu = k + 1/2$ and $f \in \mathcal{F}_k$. Then

$$L(f, \nu/2) = \sum_{n \geq 1} \frac{a_f(n)}{n^{\nu/2}} \mathcal{W}_{\nu/2} \left( \frac{\pi n}{X} \right) + \chi_2(2k + 1) \sqrt{2} \sum_{n \geq 1} \frac{a_f(4n)}{(4n)^{\nu/2}} \mathcal{V}_{\nu/2}(\pi n X)$$  \hspace{1cm} (1.3.3)

$$L^2(f, \nu/2) = \sum_{m,n \geq 1} \frac{a_f(n)a_f(m)}{(mn)^{\nu/2}} V_{\nu/2} \left( \frac{\pi^2 mn}{X} \right) + 2 \sum_{m,n \geq 1} \frac{a_f(4n)a_f(4m)}{(4n)^{\nu/2}(4m)^{\nu/2}} V_{\nu/2}(\pi^2 mn X)$$  \hspace{1cm} (1.3.4)

where

$$\mathcal{W}_{\nu/2}(y) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} K(u)y^{-u} du$$  \hspace{1cm} (1.3.5)

$$\mathcal{V}_{\nu/2}(y) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u)y^{-u} du$$  \hspace{1cm} (1.3.6)

Here $X > 0$ and $K(u)$ is the same function from lemma 1.

When dealing only with half-integral weight forms, we will use $\nu = k + 1/2$ rather than $k$ and lower case letters for the modular forms and their coefficients (conversely we use capital letters for integral weight modular forms and their coefficients). There are a number of ways to normalize Fourier coefficients. For integral weight Hecke forms (i.e. forms $F \in \mathcal{B}_k$) we have the **Hecke normalization** $\lambda_F(n) = A_F(n)/A_F(1)$
where \( \lambda_F(n) \) is the n-th Hecke eigenvalue of \( F \). In addition there are the normalizations \( \tilde{A}_F(n) = A_F(n) n^{\frac{k}{2}} \), \( \tilde{\alpha}_f(n) = a_f(n) n^{\frac{k}{2}} \), moving the central values of the \( L \)-functions to \( 1/2 \) and there is the Petersson normalization \( \omega_{f,f} \) and \( \omega_{F,F} \) where

\[
\omega_f = \frac{1}{4} \frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}} \frac{1}{\langle f, f \rangle} \quad \text{and} \quad \omega_F = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{1}{\langle F, F \rangle} \tag{1.3.7}
\]

The next proposition was first discovered by Waldspurger [25].

**Proposition 8.** (Kohnen, [12]) Suppose that \( D \) is a fundamental discriminant such that \( (-1)^k D > 0 \). Then

\[
2\omega_f \tilde{\alpha}_f^2(|D|) = \omega_F L(F \otimes \chi_D, k) \tag{1.3.8}
\]

We will make use of this result in the proof of theorem 2. Note that this connects the Ramanujan-Petersson conjecture for half integral weight forms with a version of the Lindelöf hypothesis for integral weight forms:

\[
\left\{ \omega_f \tilde{\alpha}_f^2(|D|) \ll |D|^\epsilon \right\} \iff \left\{ \omega_F L(F \otimes \chi_D, k) \ll |D|^\epsilon \right\}
\]

### 1.4 The trace formula and Kloosterman sums

For positive integers \( m, n \) and \( c \) we define the Kloosterman sums

**Definition 3.**

\[
S(m, n; c) := \sum_{x(c)}^* e \left( \frac{mx + nx}{c} \right) \\
T(m, n; c) := \sum_{x(c)} \left( \frac{x}{c} \right) e \left( \frac{mx + nx}{c} \right) \\
K_n(m, n; 4c) := \sum_{x(4c)} e_x^c \left( \frac{4c}{x} \right) e \left( \frac{mx + nx}{4c} \right)
\]

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We will write $K(m, n; 4c)$ in place of $K_1(m, n; 4c)$, as this will occur with some frequency. The second sum is called a Salíe sum, and it will prove invaluable in theorem 2 as the third sum can be factored with the following lemma.

**Lemma 2.** [6] Suppose $\kappa$ is odd and let $4c = ab$ with $a \equiv 0(4)$ and $(a, b) = 1$. We then have

$$K_\kappa(m, n; 4c) = K_{b+\kappa-1}(m\bar{b}, n\bar{b}; a)T(m\bar{a}, n\bar{a}; b)$$

where $\bar{a}$ and $\bar{b}$ stand for $a\bar{a} \equiv 1(b)$ and $b\bar{b} \equiv 1(a)$.

In particular if $b \equiv 1(4)$ then $K(m, n; 4c) = K(m\bar{b}, n\bar{b}; a)T(m\bar{a}, n\bar{a}; b)$. If $(2n, q) = 1$ then we have the useful formula for the Salíe sum (lemma 4.9 [7])

$$T(m, n; q) = G(n, q) \sum_{y^2 \equiv mn(q)} e\left(\frac{2y}{q}\right)$$

In the case where the modulus of the Salíe sum is an odd prime power $q^\beta$ and $mn = a^2$, then the congruence $y^2 \equiv a^2(q^\beta)$ has only two solutions, $y = a$ and $y = -a$ so

$$T(m, n; q^\beta) = \epsilon q^{1/2}\left\{ e\left(\frac{2a}{q^\beta}\right) + e\left(\frac{-2a}{q^\beta}\right) \right\} \quad (1.4.1)$$

Kloosterman sums appear in the Petersson trace formulas. The classical trace formula is

$$\sum_{F \in B_k} \omega_F A_F(n) A_F(m) = \delta_{mn} + 2\pi i^{-k} \sum_{c \geq 1} c^{-1} S(m, n; c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

**Lemma 3.** (This formula is easily derived from proposition 4 [14]) Suppose that $(-1)^km, (-1)^kn \equiv 0, 1 \mod (4)$.

$$\sum_{f \in F_k} \omega_f a_f(m)a_f(n) = \delta_{mn} + \sum_{c \geq 1} H_{c,k} K_{2k+1}(m, n; 4c) J_{k-1/2}\left(\frac{\pi}{c} \sqrt{mn}\right) \quad (1.4.2)$$
where

\[ H_{c,k} = \chi_8(2k + 1)(1 - i^{2k+1})(1 + \chi_4(c))\frac{\pi\sqrt{2}}{4c} \]

Both these formulas will hold for any orthogonal basis, although one is usually interested in the Hecke basis.

### 1.5 Statement of results

The goal of this section is to state the main results in this dissertation. Theorem 1 establishes a sharp asymptotic formula for the second moment of central \(L\)-values attached to half-integral weight Hecke forms in the weight aspect.

**Theorem 1.** Let \(F_k\) be the orthogonal basis of \(S_{k+1/2}^+\) given in proposition 6. For a fixed constant \(3 < A\) and \(3 < \sigma < A\) and arbitrary \(\epsilon > 0\) we have

(a) For \((-1)^k m \equiv 0, 1(4)\) such that \(m \leq \nu^{1-\epsilon}\)

\[
\sum_{f \in \mathcal{F}_k} \omega_f L(f, \nu/2) \tilde{a}_f(m) = \begin{cases}
  m^{-1/2}(1 + \chi_8(2k + 1)\sqrt{2}) + O(\nu^{1-\epsilon}) & \text{if } (-1)^k m \equiv 0(4) \\
  m^{-1/2} + O(\nu^{1-\epsilon}) & \text{if } (-1)^k m \equiv 1(4)
\end{cases}
\]

(b)

\[
\sum_{f \in \mathcal{F}_k} \omega_f L^2(f, \nu/2) = \alpha_k \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} + \beta_k + O(\nu^{-\sigma})
\]

where
\[ \alpha_k = 2 + \frac{\sqrt{2}}{16} \chi_8(2k + 1) \]

\[ \beta_k = 2\gamma + (-1)^k p \frac{\log(2\pi^4)}{2} + \frac{\sqrt{2}}{16} \chi_8(2k + 1) \left\{ 3\gamma + (-1)^k p \frac{\pi}{2} - \log(p^2/2) \right\} \]

**Remark:** This result is actually true for any orthogonal basis of \( S_{k+1/2}^+(4) \).

Next, we evaluate asymptotically the second moment of twisted central L-values attached to newforms of integral weights, which improve substantially the previous work by [17]. Moreover this approach is different from [17] and based on Waldspurger’s formula. Recall that for any prime \( p \equiv 1(4) \), \( \chi_p \) defines a character modulo \( p \). By lemma 1, \( L(F \otimes \chi_p, k) = 0 \) if \( k \) is odd. Therefore, we consider only even integers in theorem 2.

**Theorem 2.** Let \( \epsilon > 0 \) be an arbitrary positive number. For positive even integers \( k \) and primes \( p \equiv 1(4) \) such that \( p \leq k^{1/2} \) we have

\[ \sum_{F \in B_k} \omega_F L^2(F \otimes \chi_p, k) = 4 \left( 1 - \frac{1}{p} \right) \log \left( \frac{e^{\gamma} kp}{2\pi} \right) + O_\epsilon(p^{1+\epsilon}k^{-1/2} \log(k)) \]

**Corollary 1.** For \( 1 \leq p \leq k^{1/2-\epsilon} \) we have

\[ \# \{ F \in B_k : L(f \otimes \chi_p, k) \neq 0 \} \gg \epsilon \frac{k}{\log^2(k)} \]

**Proof.** Using the Cauchy-Schwartz inequality we have

\[ \left( \sum_{F \in B_k} \omega_F L(F \otimes \chi_p, k) \right)^2 \leq \left( \sum_{F \in B_k} \omega_F \right) \left( \sum_{F \in B_k} \omega_F L^2(F \otimes \chi_p, k) \right) \]
Following the proof from [17] we use the formulas:

\[
\sum_{F \in B_k} \omega_F L(F \otimes \chi_p, k) = 2 + O(k^{-1}) \quad \text{and} \quad \omega_F \ll \frac{\log(k)}{k}
\]

along with the theorem to obtain

\[
\frac{k}{\log^2(k)} \ll \sum_{F \in B_k \atop L(F \otimes \chi_p, k) \neq 0} 1
\]

\[\square\]

**Corollary 2.** For \(1 \leq p \leq k^{1/2-\epsilon}\) we have

\[
\#\{f \in \mathcal{F}_k : L(f, k/2 + 1/4)L(F \otimes \chi_p, k) \neq 0\} \gg \frac{k}{p^{1/2}\log^4(k)}
\]

*In particular we have*

\[
\#\{f \in \mathcal{F}_k : L(f, \nu/2) \neq 0\} \gg \frac{k}{\log^4(k)}
\]

**Proof.** Using Cauchy’s inequality and proposition 8

\[
\left| \sum_{f \in \mathcal{F}_k} \omega_f L(f, \nu/2) \tilde{a}_f(p) \right| \leq \left( \sum_{f \in \mathcal{F}_k} \omega_f L^2(f, \nu/2) \right)^{1/2} \left( \sum_{f \in \mathcal{F}_k \atop L(f, \nu/2) \neq 0} \omega_f \tilde{a}_f^2(p) \right)^{1/2}
\]

\[
\ll \left( \sum_{f \in \mathcal{F}_k} \omega_f L^2(f, \nu/2) \right)^{1/2} \left( \sum_{f \in \mathcal{F}_k \atop L(f, \nu/2) \neq 0} \omega_F L(F \otimes \chi_p, k) \right)^{1/2}
\]

\[
\leq \left( \sum_{f \in \mathcal{F}_k} \omega_f L^2(f, \nu/2) \right)^{1/2} \left( \sum_{f \in \mathcal{F}_k} \omega_F L^2(F \otimes \chi_p, k) \right)^{1/4} \left( \sum_{f \in \mathcal{F}_k \atop L(f, \nu/2) \neq 0 \atop L(F \otimes \chi_p, k) \neq 0} \omega_F \right)^{1/4}
\]

Using theorem 2 we have for \(p \leq k^{1/2-\epsilon}\)
\[
\# \{ f \in \mathcal{F}_k : L(f, \nu/2) L(F \otimes \chi_p, k) \neq 0 \} \gg \epsilon \frac{k}{p^{1/2} \log^4(k)}
\]
CHAPTER 2
THE ESTERMANN SERIES

The first half of this paper studies the central values $L(f, \nu/2)$ for $f \in S^{+}_{k+1/2}(4)$. Using proposition 7 and applying the trace formula (1.4.2), $L(f, \nu/2)$ is represented as a sum of two types of series: the diagonal terms given by the Kronecker delta symbol and the off-diagonal terms containing the Kloosterman sums and the Bessel function. Within the off-diagonal terms we run into a special kind of series first studied by Theodor Estermann [2]:

$$E(s, x/c) = \sum_{n \geq 1} a(n)e\left(\frac{nx}{c}\right)n^{-s}$$

where $a(n)$ is some periodic function.

2.1 Analytic continuation

**Lemma 4.** Let $c \geq 1$ and $(x, 4c) = 1$. For $1 < \Re(s)$ define the following:

$$E_k(s, x/4c) = (4c)^s \sum_{n \geq 1} \frac{e\left(\frac{nx}{4c}\right)}{\left(-1\right)^{n\equiv 0,1(4)}} n^{-s}$$  \hspace{1cm} (2.1.1)

The function $E_k(s, x/4c)$ can be extended to a meromorphic function when $c = 1$ and an entire function when $c > 1$. Its Laurent expansion is
\[ E_k(s, x/4c) = \frac{A_{-1}(x/4c)}{s - 1} + A_0(x/4c) + \cdots \] (2.1.2)

where \( A_{-1}(x/4c) = 1 + e\left(\frac{(-1)^k x}{4}ight) \) if \( c = 1 \), vanishes for all \( c > 1 \) and
\[ A_0(x/4c) = -\sum_{0 < \alpha \leq 4c} e\left(\frac{\alpha x}{4c}\right) \psi(\alpha/4c) \]

Here \( \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \) is the digamma function. \( E_k(s, x/4c) \) has the functional equation, valid for all \( \Re(s) > 1 \): \( E_k(1 - s, x/4c) = \)
\[ \frac{\Gamma(s)}{(2\pi)^s} \sum_{0 < \alpha \leq 4c} e\left(\frac{\alpha x}{4c}\right) \left[ e\left(\frac{-s}{4}\right) \sum_{n=1}^{\infty} e\left(\frac{n\alpha}{4c}\right) n^{-s} + e\left(\frac{s}{4}\right) \sum_{n=1}^{\infty} e\left(\frac{-n\alpha}{4c}\right) n^{-s}\right] \] (2.1.3)

For any \( M_1 \leq 0 \) and \( M_2 \geq 1 \), \( E_k(s, x/4c) \) satisfies a convexity bound in the vertical strip \( M_1 \leq \sigma \leq M_2 \)
\[ E_k(s, x/4c) \ll_{M_1,M_2,c,\epsilon} (1 + |t|)^{2\alpha(\sigma) + \epsilon} \] (2.1.4)

where the implied constant does not depend on \( c \) for \( M_1 \leq \sigma \leq 0 \) and the function \( \alpha(\sigma) \) is given by
\[ \alpha(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq 0 \\ 1/2 - \sigma/2 & \text{if } 0 \leq \sigma \leq 1 \\ 1/2 - \sigma & \text{if } -M \leq \sigma \leq 0 \end{cases} \]

Proof. Define the following,
\[ E(s, a, x/4c) := (4c)^s \sum_{n \geq 1 \atop n \equiv a(4)} e\left(\frac{nx}{4c}\right) n^{-s} \]
so that \( E_k(s, x/4c) = E(s, 0, x/4c) + E(s, (-1)^k, x/4c) \). We extend \( E(s, a, x/4c) \) to the complex plane by representing it in terms of the Hurwitz zeta function \( \zeta(s, z) = \sum_{m \geq 0} (m + z)^{-s} \) defined for \( \Re(s) > 1 \) and \( \Re(z) > 0 \). This series can be extended to a meromorphic function defined for all \( s \neq 1 \) with the Laurent expansion at \( s = 1 \)

\[
\zeta(s, z) = \frac{1}{s - 1} - \psi(z) + \cdots
\]  

(2.1.5)

For \( 0 \leq z \leq 1 \) and \( \Re(s) > 1 \) there is a functional equation:

\[
\zeta(1 - s, z) = \frac{\Gamma(s)}{(2\pi)^s} \left[ e\left(-\frac{s}{4}\right) \sum_{m=1}^\infty e(mz) m^{-s} + e\left(\frac{s}{4}\right) \sum_{m=1}^\infty e(-mz) m^{-s} \right]
\]  

(2.1.6)

The function \( E(s, a, x/4c) \) can be written as a sum of Hurwitz zeta functions

\[
E(s, a, x/4c) = \sum_{0 < \alpha \leq 4c, \alpha \equiv 1(4)} e\left(\frac{\alpha x}{4c}\right) \zeta(s, \alpha/4c)
\]

giving us the Laurent expansion (2.1.2) and the functional equation (2.1.3).

Let \( M > 0 \) and \( \epsilon > 0 \) be arbitrary. Then \( E_k(\sigma + it, x/4c) \ll_{\epsilon, c} |t|^\epsilon \) for \( 0 \leq \sigma \) and by Stirling’s formula and the functional equation (2.1.3), \( E_k(\sigma + it, x/4c) \ll_{M, \epsilon} (1 + |t|)^{1-2\sigma + \epsilon} \) for \( -M \leq \sigma \leq 0 \). By the Phragmen-Lindelöf principle this gives us the bound \( E_k(\sigma + it, x/4c) \ll_{\epsilon, c} (1 + |t|)^{1-\sigma + \epsilon} \) for \( 0 < \sigma < 1 \).

\[\square\]

**Lemma 5.** Let \( c, n \geq 1, (4x, c) = 1 \) and \( p \equiv 1(4) \) a prime. For \( 1 < \Re(s) \) define the following:

\[
E_p(s, x/4c) := \sum_{n \geq 1} e\left(\frac{pn^2x}{4c}\right) \chi_p(n)n^{-s}
\]  

(2.1.7)

Put \( \delta = (c, p) \), \( p \overline{p} \equiv 1(4c) \), \( 4c \overline{c} \equiv 1(p) \) and define:
The series $E_p(s, x/4c)$ can be extended to a meromorphic function with single simple pole at $s = 1$ with residue

$$
\text{Res}_{s=1} E_p(s, x/4c) = [4c, p]^{-1} \sum_{a(4c) \equiv b(p)} e\left(a \frac{2p \chi_p}{4c}\right) \chi_p(b)
$$

$E_p(s, x/4c)$ has the functional equation, valid for all $\Re(s) > 1$:

$$
E_p(1 - s, x/4c) = [4c, p]^{-s} \sum_{a(4c)} e\left(a \frac{2p \chi_p}{4c}\right) \sum_{b(4c) \equiv a(b)} \chi_p(b)
$$

$$
\times \frac{\Gamma(s)}{(2\pi)^s} \left[ e\left(-s/4\right) \sum_{n \geq 1} e(n \lambda_{a,b}) n^{-s} + e(s/4) \sum_{n \geq 1} e(-n \lambda_{a,b}) n^{-s} \right]
$$

For any $M_1 \leq 0$ and $M_2 \geq 1$, $E_p(s, x/4c)$ satisfies a convexity bound in the vertical strip $M_1 \leq \sigma \leq M_2$

$$
E_p(s, x/4c) \ll_{M_1, M_2, c, \epsilon} (1 + |t|)^{\alpha(\sigma) + \epsilon}
$$

where $\alpha(\sigma)$ is the same function given in the previous lemma.
Proof.

\[ E_p(s, x/4c) := \sum_{n \geq 1} e\left(\frac{pn^2x}{4c}\right) \chi_p(n)n^{-w} \]
\[ = \sum_{a(4c)} e\left(\frac{a^2px}{4c}\right) \sum_{b(p)} \sum_{\substack{n \equiv a(4c) \bmod p \equiv b(\delta)}} n^{-w} \]

Using the Chinese remainder theorem the congruences \( n \equiv a(4c) \) and \( n \equiv b(p) \) are solvable if and only if \( b \equiv a(\delta) \) in which case the solutions can be written as \( n = m[4c, p] + \lambda_{a,b}[4c, p] \) for \( m \geq 0 \) so the series can be written as a sum of Hurwitz zeta functions

\[ E_p(s, x/4c) = [4c, p]^{-s} \sum_{a(4c)} e\left(\frac{a^2px}{4c}\right) \sum_{\substack{b(p) \bmod p \equiv a(\delta)}} \chi_p(b) \zeta(s, \lambda_{a,b}) \]

It can therefore be extended to the complex plane using the functional equation (see the properties of Hurwitz zeta functions above)

\[ E_p(1 - s, x/4c) = [4c, p]^{s-1} \sum_{a(4c)} e\left(\frac{a^2px}{4c}\right) \sum_{\substack{b(p) \bmod p \equiv a(\delta)}} \chi_p(b) \]
\[ \times \frac{\Gamma(s)}{(2\pi)^s} \left[ e(-s/4) \sum_{n \geq 1} e(n\lambda_{a,b})n^{-s} + e(s/4) \sum_{n \geq 1} e(-n\lambda_{a,b})n^{-s} \right] \]

with a possible simple pole at \( s = 1 \) given by

\[ \text{Res}_{s=1} E_p(s, x/4c) = [4c, p]^{-1} \sum_{a(4c)} e\left(\frac{a^2px}{4c}\right) \chi_p(b) \]
Proof of the convexity bound is routine and basically the same as in the previous
lemma.

\[ \square \]

### 2.2 Contribution from the residues

The functions \( E_k(s, x/4c) \) will show up in pairs leading to double poles. Therefore,
knowing its logarithmic derivative will be useful.

**Lemma 6.**
\[ \frac{d}{ds} \log \left( E_k(s, x/4c) \right) = -\frac{1}{s - 1} + \frac{A_0(x/4c)}{A_{-1}(x/4c)} + \cdots \] (2.2.1)

**Proof.** The expansion can be computed as follows

\[
\frac{d}{ds} \log \left( E_k(s, x/4c) \right) = \frac{d}{ds} \log \left( \frac{(s - 1)E_k(s, x/4c)}{A_{-1}(x/4c)} \right) - \frac{1}{s - 1} \\
= \frac{d}{ds} \log \left( 1 + \frac{A_0(x/4c)}{A_{-1}(x/4c)}(s - 1) + \cdots \right) - \frac{1}{s - 1}
\]

and with the Taylor expansion for \( \log(1 + y) \) this becomes

\[
\frac{d}{ds} \log \left( E_k(s, x/4c) \right) = -\frac{1}{s - 1} + \frac{A_0(x/4c)}{A_{-1}(x/4c)} + \cdots
\]

\[ \square \]

The next two lemmas are fundamental to the proof of the second theorem. By
twisting the exponential sums with multiplicative characters we obtain sharper results
than one would with the classical Kloosterman sums.
Lemma 7. Put $\delta = (p, 4c)$ and let $p \equiv 1(4)$. Then

$$\sum_{a(4c)} K(a^2 p, p; 4c) \sum_{b(p) \equiv a(\delta)} \chi_p(b) = 0$$

Lemma 8. Then the sum

$$\sum_{a(4c)} K(a^2 p, p; 4c) \sum_{b(p) \equiv a(\delta)} \chi_p(b) e(\pm n\lambda_{a,b})$$

vanishes if $p|c$, $p|n$ or $n$ is odd. If $(nc, p) = 1$ then

$$\sum_{a(4c)} K(a^2 p, p; 4c) \sum_{b(p) \equiv a(\delta)} \chi_p(b) e(\pm 2n\lambda_{a,b}) \ll p^{1/2}c^{1/2}(c, p^4 \equiv n^2)$$

We will need a few facts about quadratic exponential sums. For non-zero integers, $b > 0$, and $(a, b) = 1$ we define

$$G(a, b) := \sum_{x(b)} e\left(\frac{ax^2}{b}\right)$$

(2.2.2)

Its properties are outlined in [18] p. 85-87. We will use

$$G(a, bc) = G(ab, c)G(ac, b) \quad \text{if } (b, c) = (a, bc) = 1$$

$$G(1, b) = \epsilon_b b^{1/2} \left(\frac{a}{b}\right) \quad \text{if } b \text{ is odd}$$

$$G(a, b) = \epsilon_a \left(\frac{-b}{a}\right) G(1, b) \quad \text{if } b \text{ is a power of } 2$$

(2.2.3)

$$G(1, b) = (1 + i)b^{1/2} \quad \text{if } b \equiv 0(4)$$

$$G(1, b) = 0 \quad \text{if } b \equiv 2(4)$$

Combining these properties we have

$$G(a, c) = (1 + i)c^{1/2} \epsilon_a \left(\frac{-c}{a}\right) \quad \text{if } 4|c \text{ and } (a, c) = 1$$

(2.2.4)

More generally there is the exponential sum
\[ G(a, b; c) := \sum_{x(c)} e\left(\frac{ax^2 + bx}{c}\right) \]  

(2.2.5)

Its properties are outlined in lemma 5.4.5 [5]. If \((a, 4c) = 1\) we have

\[ G(a, b; 4c) = \begin{cases} 
    e\left(-\frac{\overline{a}(b/2)^2}{4}\right)G(a, 4c) & \text{if } b \text{ is even} \\
    0 & \text{if } b \text{ is odd}
\end{cases} \]  

(2.2.6)

where \(a\overline{a} \equiv 1(4c)\).

\textbf{Proof}. Proof of lemma 7

We will write \(cp^\alpha\) where \((c, p) = 1\) in place of \(4c\) in order to simplify the notation used in the proof. Denote the sum in the lemma by

\[ S := \sum_{a(cp^\alpha)} K(a^2p, p; cp^\alpha) \sum_{\substack{b(p) \equiv \alpha(\delta) \mod{p} \atop b(p)}} \chi_p(b) \]

If \(\alpha = 0\) then \(\delta = 1\) and we have \(S = \sum_{a(c)} K(a^2p, p; c) \sum_{b(p)} \left(\frac{p}{b}\right) = 0\). If \(\alpha = 1\) we factor the Kloosterman sum by applying (2)

\[ K(a^2p, p; cp) = K(a^2, 1; c)T(a^2p\overline{c}, p\overline{c}; p) = K(a^2, 1; c)\sum_{x(p)} \left(\frac{x}{p}\right) = 0 \]

Suppose now that \(\alpha \geq 2\). As above we factor the Kloosterman sum then split the outer sum \(a(cp^\alpha)\) to \(a_1(p^\alpha)\) and \(a_2(c)\).
\[ S = \sum_{a_1(p^\alpha)} \left( \frac{p}{a} \right) K(a^2 p\bar{p}, p\bar{p}; c) T(a^2 p\bar{c}, p\bar{c}; p^\alpha) \]

\[
= \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) K((a_1 + a_2 p^\alpha)^2 p\bar{p}, p\bar{p}; c) T(a_1^2 p\bar{c}, p\bar{c}; p^\alpha) \]

\[
= \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2 p\bar{c}, p\bar{c}; p^\alpha) \sum_{a_2(c)} K(a_2^2 p^{\alpha+1}, p\bar{p}; c) \]

where the last line is obtained by absorbing \( p^{2\alpha} \) into the square. Now we repeat the process on \( K(a_2^2 p^{\alpha+1}, p\bar{p}; c) \) with the factorization \( c = qr \), where \( r \) is the even part of \( c \) using the notation \( \tilde{q} \equiv 1(r) \) and \( \tilde{r} \equiv 1(q) \).

\[
S = \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2 p\bar{c}, p\bar{c}; p^\alpha) \sum_{a_2(c)} K_q(a_2^2 p^{\alpha+1}, p\bar{p}; q\tilde{q}; r) T(a_2^2 p^{\alpha+1}, p\bar{p}; q\tilde{r}; q) \]

\[
= \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2 p\bar{c}, p\bar{c}; p^\alpha) \times \sum_{a_2(q)} \sum_{a_3(r)} K_q((a_2 + a_3 q)^2 p^{\alpha+1}, p\bar{p}; q\tilde{q}; r) T(a_2^2 p^{\alpha+1}, p\bar{p}; q\tilde{r}; q) \]

\[
= \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2 p\bar{c}, p\bar{c}; p^\alpha) \sum_{a_2(q)} \sum_{a_3(r)} K_q(a_2^2 p^{\alpha+1}, p\bar{p}; q\tilde{q}; r) T(a_2^2 p^{\alpha+1}, p\bar{p}; q\tilde{r}; q) \]

For the second sum, we open up \( K \) to obtain a quadratic Gauss sum, using the same tilde notation to represent the inverse of \( x \) with respect to \( r \).
\[
\sum_{a_3(r)} K_q \left( a_3^2 p^{\alpha+1} q, \overline{p}^{\alpha} \tilde{q}; r \right)
\]

\[
= \sum_{a_3(r)} \sum_{x(r)} \epsilon_q \left( \frac{r}{x} \right) e \left( \frac{a_3^2 p^{\alpha+1} q \tilde{x} + d^2 \overline{p}^{\alpha-1} \tilde{q} x}{r} \right)
\]

\[
= \sum_{x(r)} \epsilon_q \left( \frac{r}{x} \right) G(p^{\alpha+1} q \tilde{x}, r) e \left( \frac{\overline{p}^{\alpha-1} \tilde{q} x}{r} \right)
\]

Using (2.2.4) this is equal to

\[
(1 + i) r^{1/2} \sum_{x(r)} \epsilon_q \left( \frac{r}{x} \right) \epsilon_{p^{\alpha+1} q \tilde{x}} \left( \frac{-r}{p^{\alpha+1} q \tilde{x}} \right) e \left( \frac{\overline{p}^{\alpha-1} \tilde{q} x}{r} \right)
\]

Since \( p \equiv 1(4) \), \( \epsilon_q \epsilon_{p^{\alpha+1} q \tilde{x}} = \epsilon_q \left( \frac{-1}{x} \right) \) and \( \left( \frac{-r}{p^{\alpha+1} q \tilde{x}} \right) = \left( \frac{-1}{q \tilde{x}} \right) \left( \frac{r}{p^{\alpha+1} q x} \right) \) so we have

\[
\sum_{a_3(r)} K_q \left( a_3^2 p^{\alpha+1} q, \overline{p}^{\alpha} \tilde{q}; r \right)
\]

\[
= (1 + i) r^{1/2} \epsilon_q \left( \frac{-1}{q} \right) \left( \frac{r}{p^{\alpha+1} q} \right) \sum_{x(r)} e \left( \frac{\overline{p}^{\alpha-1} \tilde{q} x}{r} \right)
\]

\[
= (1 + i) r^{1/2} \epsilon_q \left( \frac{-1}{q} \right) \left( \frac{r}{p^{\alpha+1} q} \right) \mu(r) = 0
\]

since \( (\overline{p}^{\alpha-1} \tilde{q}, r) = 1 \) and \( r \) is not squarefree.
Proof. Proof of lemma 8 Using Weil’s bound for Kloosterman sums one obtains the bound

$$\sum_{\alpha(4c)} K(a^2 p, p; 4c) \sum_{b(p) \equiv \alpha(\delta)} \chi_p(b)e(n\lambda_{a,b}) \ll c^{3/2} p^{1/2} \tau(c)(c, p)^{1/2}$$

which would produce a result close to what was obtained in [17]. However, our intent is to exploit the cancellation using (2.2.3) and (2.2.6). As with lemma 7, we will write $cp^\alpha$ where $(c, p) = 1$ in place of $4c$ and denote the sum by

$$S := \sum_{\alpha(cp^\alpha)} K(a^2p, p; cp^\alpha) \sum_{b(p) \equiv \alpha(\delta)} \chi_p(b)e(n\lambda_{a,b})$$

First we will show that $S = 0$ when $\alpha \geq 1$. Recall that when $\alpha = 1$, $K(a^2p, p; cp) = 0$, so we can assume that $\alpha \geq 2$. In the case $\alpha \geq 1$ we have $\lambda_{a,b} = \frac{a}{cp^\alpha}$. Applying (2) and splitting the sum over $cp^\alpha$ gives the following.

$$S = \sum_{\alpha(cp^\alpha)} K(a^2p, p; cp^\alpha) \sum_{b(p) \equiv \alpha(\delta)} \chi_p(b)e(\pm n\lambda_{a,b})$$

$$= \sum_{\alpha_1(p^\alpha)} \left( \frac{p}{a_1} \right) K((a_1 + a_2 p^\alpha)^2 p^\alpha, p^\alpha; c) T(a_1^2 p^\alpha, p^\alpha; p^\alpha) e \left( \pm \frac{n(a_1 + a_2 p^\alpha)}{cp^\alpha} \right)$$

Applying the reciprocity formula $\frac{c}{p^\alpha} + \frac{p^\alpha}{c} \equiv \frac{1}{cp^\alpha} \mod 1$ so
\[ S = \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2p\bar{c},p\bar{c};p^\alpha) e \left( \pm \frac{na_1\bar{c}}{p^\alpha} \right) \]

\[ \times \sum_{a_2(c)} K((a_1p^\alpha + a_2)^2p^{\alpha+1},p\bar{p};c)e \left( \pm \frac{n(a_1p^\alpha + a_2)}{c} \right) \]

\[ = \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2p\bar{c},p\bar{c};p^\alpha) e \left( \pm \frac{na_1\bar{c}}{p^\alpha} \right) \sum_{a_2(c)} K(a_1^2p^{\alpha+1},p\bar{p};c)e \left( \pm \frac{a_2n}{c} \right) \]

Using (1.4.1) the left half of this ultimately vanishes.

\[ \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2p\bar{c},p\bar{c};p^\alpha) e \left( \pm \frac{na_1\bar{c}}{p^\alpha} \right) \]

\[ = p \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) T(a_1^2\bar{c},p^{\alpha-1}) e \left( \pm \frac{na_1\bar{c}}{p^\alpha} \right) \]

\[ = p^{3/2} \left( \frac{c}{p^{\alpha-1}} \right) \sum_{a_1(p^\alpha)} \left( \frac{p}{a_1} \right) \left\{ e \left( \frac{2a_1\bar{c}}{p^{\alpha-1}} \right) + e \left( -\frac{2a_1\bar{c}}{p^{\alpha-1}} \right) \right\} e \left( \pm \frac{na_1\bar{c}}{p^\alpha} \right) \]

Suppose that \( A \) is a positive integer. Using lemmas 3.1 and 3.2 from [9],

\[ \sum_{a(p^\alpha)} \chi_p(a)e \left( \frac{aA}{p^\alpha} \right) = \mu(p^{\alpha-1})\chi_p(p^{\alpha-1})\tau(\chi_p) \sum_{d|(A,p^{\alpha-1})} d\chi_p(\alpha)\mu \left( \frac{p^{\alpha-1}}{d} \right) \]

which vanishes if \( \alpha > 1 \). We get a similar result if \( A \leq -1 \) and the sum clearly vanishes if \( A = 0 \). In conclusion \( S = 0 \) when \( \alpha \geq 1 \).

In the case \( \alpha = 0 \), \( e(\pm n\lambda_{a,b}) = e \left( \pm \frac{na\bar{p}}{c} \right) e \left( \pm \frac{na\bar{c}}{p} \right) \) and
\[
\sum_{a(c)} K(a^2 p, p; c) \sum_{b(p)} \chi_p(b) e(\pm n\lambda_{a,b}) \\
= p^{1/2} \left( \frac{p}{nc} \right) \sum_{a(c)} K(a^2 p, p; c) e \left( \pm \frac{n\alpha p}{c} \right) \\
= p^{1/2} \left( \frac{p}{nc} \right) \sum_{x(c)} \epsilon_x \left( \frac{c}{x} \right) e \left( \frac{px}{c} \right) G(p\overline{\tau}, \pm n\overline{p}; c)
\]

Using (2.2.6) this vanishes if \( n \) is odd. If \( n \) is even it equals

\[
p^{1/2} \left( \frac{p}{nc} \right) \sum_{x(c)} \epsilon_x \left( \frac{c}{x} \right) e \left( \frac{px}{c} \right) G(p\overline{\tau}, c)
\]

\[
= (1 + i) p^{1/2} c^{1/2} \left( \frac{p}{nc} \right) \sum_{x(c)} \epsilon_x \left( \frac{c}{x} \right) e \left( \frac{x(p \mp \overline{p}^3 n^2/4)}{c} \right) \epsilon_{p\overline{\tau}} \left( -c \right)
\]

\[
= (1 + i) p^{1/2} c^{1/2} \left( \frac{p}{n} \right) \sum_{x(c)} \epsilon \left( \frac{x(p \mp \overline{p}^3 n^2/4)}{c} \right)
\]

\[
= (1 + i) p^{1/2} c^{1/2} \left( \frac{p}{n} \right) \sum_{x(c)} \epsilon \left( \frac{x(p^4 \mp n^2/4)}{c} \right)
\]

\[
\ll p^{1/2} c^{1/2} (c, p^4 \mp n^2/4)
\]
CHAPTER 3
HARMONIC AVERAGES OF MOMENTS

3.1 Proof of theorem 1

The first result is concerned only with forms of weight \( \nu = k + 1/2 \), so whenever asymptotics are concerned they will be phrased in terms of \( \nu \). With the more delicate terms, for instance character sums, the use of \( k \) will be clearer. As a result most formulas will be expressed in terms of both \( \nu \) and \( k \).

**Theorem 1.** Let \( \mathcal{F}_k \) be the orthogonal basis of \( S^+_{k+1/2}(4) \) given in proposition 6. For a fixed constant \( 3 < A \) and \( 3 < \sigma < A \) and \( \epsilon > 0 \) we have

(a) For \( (-1)^k m \equiv 0, 1(4) \) such that \( m \leq \nu^{1-\epsilon} \)

\[
\sum_{f \in \mathcal{F}_k} \omega_f L(f, \nu/2) \tilde{a}_f(m) = \begin{cases} 
m^{-1/2}(1 + \chi_S(2k + 1)\sqrt{2}) + O_\epsilon(\nu^{-1}) & \text{if } (-1)^k m \equiv 0(4) \\
m^{-1/2} + O_\epsilon(\nu^{-1}) & \text{if } (-1)^k m \equiv 1(4) \end{cases}
\]

(b) \[
\sum_{f \in \mathcal{F}_k} \omega_f L^2(f, \nu/2) = \alpha_k \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} + \beta_k + O(\nu^{-\sigma})
\]
where

\[ \alpha_k = 2 + \frac{\sqrt{2}}{16} \chi_8(2k + 1) \]

\[ \beta_k = 2\gamma + (-1)^k \frac{\pi}{2} - \frac{\log(2\pi^4)}{2} + \frac{\sqrt{2}}{16} \chi_8(2k + 1) \left\{ 3\gamma + (-1)^k \frac{\pi}{2} - \log(\pi^2/2) \right\} \]

**Proof.** Proof of part (a)

Using corollary 3, proposition 7 with \( X = 2, K(u) = e^{u^2} u^{-1} \) and the fact that the coefficients \( a_f(n) \) are supported only for \((-1)^k n \equiv 0, 1(4)\), we have

\[ \omega_f L(f, \nu/2)\bar{a}_f(m) = \sum_{n \geq 1} \bar{a}_f(n)\bar{a}_f(m)W_{\nu/2}(\pi n/2)n^{-1/2} + \chi_8(2k + 1)\sqrt{2} \sum_{n \geq 1} \bar{a}_f(4n)\bar{a}_f(m)W_{\nu/2}(2\pi n)n^{-1/2} \]

\[ \sum_{f \in \mathcal{F}_k} \omega_f \sum_{n \geq 1} \bar{a}_f(n)\bar{a}_f(m)W_{\nu/2}(\pi n/2)n^{-1/2} = S + R \]

\[ \chi_8(2k + 1)\sqrt{2} \sum_{f \in \mathcal{F}_k} \omega_f \sum_{n \geq 1} \bar{a}_f(4n)\bar{a}_f(m)W_{\nu/2}(2\pi n)n^{-1/2} = S + R \]

where

\[ S = m^{-1/2}W_{\nu/2}(\pi m/2) \]

\[ S = \begin{cases} m^{-1/2} \chi_8(2k + 1)\sqrt{2}W_{\nu/2}(\pi m/2) & \text{if } (-1)^k m \equiv 0(4) \\ 0 & \text{if } (-1)^k m \equiv 1(4) \end{cases} \]
and

\[ R = \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \left( \frac{4c}{x} \right) \epsilon_x^{2k+1} \sum_{n \geq 1} \sum_{n \equiv 0, 1(4)}^{\infty} e \left( \frac{m x + n \bar{x}}{4c} \right) \frac{W_{\nu/2}(\pi n/2)}{\sqrt{n}} J_{\nu-1} \left( \frac{\pi}{c} \sqrt{mn} \right) \]

\[ \mathcal{R} = \chi (2k+1) \sqrt{2} \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \left( \frac{4c}{x} \right) \epsilon_x^{2k+1} \sum_{n \geq 1} e \left( \frac{m x + 4n \bar{x}}{4c} \right) \frac{W_{\nu/2}(2\pi n)}{\sqrt{4n}} J_{\nu-1} \left( \frac{2\pi}{c} \sqrt{mn} \right) \]

**Evaluation of S**

\[ S = m^{-1/2} \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} K(u) \left( \frac{2}{m\pi} \right)^u \, du \]

The exponential decay of \( K(u) \) along with Stirling’s approximation allows us to move the line of integration to \( \Re(u) = -\sigma \). As long as \( 2A < \nu \), we will pass a single pole at \( u = 0 \) which gives us \( S = S_1 + S_2 \) where

\[ S_1 = m^{-1/2} \frac{1}{2\pi i} \int_{(-\sigma)} \frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} K(u) \left( \frac{2}{m\pi} \right)^u \, du \]

\[ S_2 = m^{-1/2} \text{Res}_{u=0} \frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} K(u) \left( \frac{2}{m\pi} \right)^u \]

\[ S_2 = m^{-1/2} \text{ and using lemma 9, } S_1 \ll \sigma^{-1/2} \nu^{-\sigma}. \text{ By choosing } \sigma = \frac{\epsilon + 2}{2\epsilon + 1} \text{ we obtain the desired result.} \]

**Evaluation of S.**

\[ S = \begin{cases} 
  m^{-1/2} \chi (2k+1) \sqrt{2} \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} \left( \frac{2}{m\pi} \right)^u \, du & \text{if } (-1)^km \equiv 0(4) \\
  0 & \text{if } (-1)^km \equiv 1(4) 
\end{cases} \]
The exponential decay of \( K(u) \) along with Stirling’s approximation allows us to move the line of integration to \( \Re(u) = -\sigma \). As long as \( 2A < \nu \) we will pass a single pole at \( u = 0 \) which gives us \( S = S_1 + S_2 \) where

\[
S_1 = \begin{cases} 
  m^{-1/2} \chi_8(2k+1) \sqrt{2} \frac{1}{2\pi i} \int_{(-\sigma)} \frac{\Gamma(\nu/2+u)}{\Gamma(\nu/2)} \left( \frac{2}{m\pi} \right)^u du & \text{if } (-1)^k m \equiv 0(4) \\
  0 & \text{if } (-1)^k m \equiv 1(4) 
\end{cases}
\]

\[
S_2 = \begin{cases} 
  m^{-1/2} \chi_8(2k+1) \sqrt{2} \text{Res}_{u=0} \frac{\Gamma(\nu/2+u)}{\Gamma(\nu/2)} \left( \frac{2}{m\pi} \right)^u du & \text{if } (-1)^k m \equiv 0(4) \\
  0 & \text{if } (-1)^k m \equiv 1(4) 
\end{cases}
\]

By choosing \( \sigma = \frac{\epsilon + 2}{2\epsilon + 1} \) we obtain the desired result:

\[
S = \begin{cases} 
  m^{-1/2} \chi_8(2k+1) \sqrt{2} + O(\nu^{-1}) & \text{if } (-1)^k m \equiv 0(4) \\
  0 & \text{if } (-1)^k m \equiv 1(4) 
\end{cases}
\]

**Evaluation of \( R \)** Using Weil’s bound \(|K_{2k+1}(m, n; 4c)| \leq c^{1/2} \tau(c)m^{1/2} \) so we have

\[
R \ll m \sum_{n,c \geq 1} c^{-1/2} \tau(c)n^{-1/2} \left| W_{\nu/2}(\pi n/2)J_{\nu-1} \left( \frac{\pi}{c} \sqrt{mn} \right) \right|
\]

For \( \sqrt{mn}/c \gg \nu \) it is necessary that \( n \gg \nu^2/m \gg \nu^{1+\epsilon} \). Using the bounds \( J_{\nu-1}(x) \ll 1 \) and \( W_{\nu/2}(y) \ll (\nu/y)^{\eta} \). Therefore, the sum over such \( c \) and \( n \) is
\[
\ll m \sum_{n \gg \nu^{1+\epsilon}} n^{-1/2} (\nu/n)^\eta \sum_{1 \leq c \leq \sqrt{mn}/\nu} c^{-1/2} \tau(c) \\
\ll m^{3/4} \nu^{-1/2} \log(\nu) \sum_{n \gg \nu^{1+\epsilon}} n^{-1/4} (\nu/n)^\eta \ll \nu^{1/4} \sum_{n \gg k^{1+\epsilon}} n^{-1/4} (\nu/n)^\eta
\]

Now we choose \( \eta = 11/4(1 + \epsilon^{-1}) \) so that \( n^{-1/4} (\nu/n)^\eta \leq n^{-3} \) giving us the bound \( \ll \nu^{-1} \).

Considering the bound \( J_{\nu-1}(x) \ll \left( \frac{ex}{2x} \right)^{\nu-1} \), we can choose a positive constant \( C \) such that \( J_{\nu-1}\left( \frac{\pi}{c} \sqrt{mn} \right) \ll 2^{-\nu} \sqrt{mn}/c \) for \( \sqrt{mn}/c \leq C\nu \). Using the bound \( W_{\nu/2}(y) \ll (\nu/y)^2 \), we get the sum over such \( n, c \) is

\[
\ll \epsilon m 2^{-\nu} \sum_{n \geq 1} (\nu/n)^\eta \sum_{c \leq C\sqrt{mn}/\nu} c^{-1/2} \tau(c) \ll \nu^{-1}
\]

**Evaluation of \( \mathcal{R} \)** An argument similar to that for \( R \) gives us \( \mathcal{R} \ll \nu^{-1} \).

**Proof.** Proof of part (b)

Using lemma 3, proposition 7 with \( X = 4 \), \( K(u) = e^{u^2} u^{-1} \) and the fact that the coefficients \( a_f(n) \) are supported only for \((-1)^k n \equiv 0, 1(4)\).

\[
\sum_{f \in \mathcal{F}_k} \omega_f \sum_{m,n \geq 1} \frac{a_f(n)a_f(m)}{n^{\nu/2}m^{\nu/2}} V_{\nu}(mn/4) = S + R
\]

and

\[
\sum_{f \in \mathcal{F}_k} \omega_f \sum_{m,n \geq 1} \frac{2a_f(4n)a_f(4m)}{(4n)^{\nu/2}(4m)^{\nu/2}} V_{\nu}(4mn) = S + \mathcal{R}
\]
where \( S = \sum_{m \geq 1} \frac{\mathcal{V}_\nu(m^2/4)m^{-1}}{(-1)^k m \equiv 0, 1(4)} \) and \( S = \frac{1}{2} \sum_{m \geq 1} \mathcal{V}_\nu(4m^2)m^{-1} \)

\[
R = \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \chi(x) e^{2k+1} \sum_{m,n \geq 1} e \left( \frac{mx + nx}{4c} \right) \frac{V_\nu(mn/4)}{\sqrt{mn}} J_{\nu-1} \left( \frac{\pi}{c} \sqrt{mn} \right)
\]

\[
\mathcal{R} = \frac{1}{2} \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \chi(x) e^{2k+1} \sum_{m,n \geq 1} e \left( \frac{mx + nx}{c} \right) \frac{V_\nu(4mn)}{\sqrt{mn}} J_{\nu-1} \left( \frac{4\pi}{c} \sqrt{mn} \right)
\]

We will show that

\[
S = \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(2\pi) + \gamma + (-1)^k \frac{\pi}{2} + \frac{3 \log(2)}{2} + O(\nu^{-\sigma})
\]

\[
S = \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(2\pi) + \gamma + O(\nu^{-\sigma})
\]

\[
R = \frac{\sqrt{2}}{16} \chi(2k + 1) \left\{ \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(\pi/4) + 2\gamma + (-1)^k \frac{\pi}{2} \right\} + O(\nu^{-\sigma})
\]

\[
\mathcal{R} = \frac{\sqrt{2}}{16} \chi(2k + 1) \left\{ \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(2\pi) + \gamma \right\} + O(\nu^{-\sigma})
\]

**Evaluation of \( S \)**

\[
S = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \pi^{-2u} \left[ 4^{-u-1} \zeta(2u+1) + 4^{-u-1} \zeta(2u+1, 1/2 + (-1)^k/4) \right] du
\]
The exponential decay of $K(u)$ along with Stirling’s approximation allows us to move the line of integration to $\Re(u) = -\sigma/2$. Along the way we pass a double pole at $u = 0$, which gives us $S = S_1 + S_2$ where

$$S_1 = \frac{1}{2\pi i} \int_{(-\sigma/2)} \frac{\Gamma^2(l/2 + u)}{\Gamma^2(l/2)} K(u) \pi^{-2u} \left[ 4^{-u-1}\zeta(2u+1)+4^{-u-1}\zeta(2u+1, 1/2 - (-1)^k/4) \right] du$$

$$S_2 = 4^{-1} \text{Res}_{u=0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u)(2\pi)^{-2u} \left[ \zeta(2u + 1) + \zeta(2u + 1, 1/2 - (-1)^k/4) \right]$$

By lemma 9, $S_1 \ll \nu^{-\sigma}$. Using the Laurent expansion for $\zeta(s, z)$ given in equation (2.1.5) and the special values $\psi(1/4) = -\gamma - \frac{\pi}{2} - 3\log(2)$ and $\psi(3/4) = -\gamma + \frac{\pi}{2} - 3\log(2)$ ([3], 8.366)

$$\frac{\zeta'(2u + 1)}{\zeta(2u + 1)} = -\frac{1}{2u} + \gamma + \cdots \quad (3.1.1)$$

and

$$\frac{\zeta'(2u + 1, \frac{1}{2} - (-1)^k/4)}{\zeta(2u + 1, \frac{1}{2} - (-1)^k/4)} = -\frac{1}{2u} + \gamma + (-1)^k \frac{\pi}{2} + 3\log(2) + \cdots \quad (3.1.2)$$

The part of the residue containing $\zeta(2u + 1)$ evaluates as:
\[4^{-1} \text{Res}_{u=0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u)(2\pi)^{-2u} \zeta(2u + 1)
\]
\[= 4^{-1} \lim_{u \to 0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1)
\]
\[\times \left[ 2 \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} + \frac{K'(u)}{K(u)} + \frac{2}{u} - 2 \log(2\pi) + 2 \frac{\zeta'(2u + 1)}{\zeta(2u + 1)} \right]
\]
\[= 4^{-1} \lim_{u \to 0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1)
\]
\[\times \left[ 2 \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} + 2u - 2 \log(2\pi) + 2\gamma \right]
\]
\[= \frac{1}{2} \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \frac{\log(2\pi)}{2} + \frac{\gamma}{2}
\]

The part of the residue containing \(\zeta(2u + 1, 1/2 - (-1)^k/4)\) evaluates as:

\[4^{-1} \text{Res}_{u=0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1, 1/2 - (-1)^k/4)
\]
\[= 4^{-1} \lim_{u \to 0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1, 1/2 - (-1)^k/4)
\]
\[\times \left[ 2 \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} + \frac{K'(u)}{K(u)} + \frac{2}{u} - 2 \log(2\pi) + 2 \frac{\zeta'(2u + 1, 1/2 - (-1)^k/4)}{\zeta(2u + 1, 1/2 - (-1)^k/4)} \right]
\]
\[= 4^{-1} \lim_{u \to 0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1, 1/2 - (-1)^k/4)
\]
\[\times \left[ 2 \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} + 2u - 2 \log(2\pi) + 2\gamma + (-1)^k \pi + 6 \log(2) \right]
\]
\[= \frac{1}{2} \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \frac{\log(2\pi)}{2} + \frac{\gamma}{2} + \frac{(-1)^k \pi}{2} + \frac{3 \log(2)}{2}
\]

So \(S = \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} + \gamma - \log(2\pi) + (-1)^k \pi + \frac{3 \log(2)}{2} + O_\sigma(\nu^{-\sigma})\)
Evaluation of $S$.

\[
S = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \pi^{-2u} 4^{-u-1/2} \zeta(2u + 1) du
\]

The exponential decay of $K(u)$ along with Stirling’s approximation allows us to move the line of integration to $\Re(u) = -\sigma/2$. Along the way we pass a double pole at $u = 0$ which gives us $S = S_1 + S_2$ where

\[
S_1 = \frac{1}{2\pi i} \int_{(-\sigma/2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \pi^{-2u} 4^{-u-1/2} \zeta(2u + 1) du
\]

\[
S_2 = \text{Res}_{u=0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \pi^{-2u} 4^{-u-1/2} \zeta(2u + 1)
\]

Using lemma 9, $S_1 \ll \nu^{-\sigma}$. Evaluating the residue

\[
S_2 = \frac{1}{2} \text{Res}_{u=0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u)(2\pi)^{-2u} \zeta(2u + 1)
\]

\[
= \frac{1}{2} \lim_{2u \to 0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1)
\]

\[
\times \left[ 2 \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} + \frac{K'(u)}{K(u)} + \frac{2}{u} - 2 \log(2\pi) + 2 \frac{\zeta'(2u + 1)}{\zeta(2u + 1)} \right]
\]

\[
= \frac{1}{2} \lim_{2u \to 0} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} u^2 K(u)(2\pi)^{-2u} \zeta(2u + 1)
\]

\[
\times \left[ 2 \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} + 2u - 2 \log(2\pi) + 2 \gamma \right]
\]

\[
= \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(2\pi) + \gamma
\]

We get $S = \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(2\pi) + \gamma + O(\nu^{-\sigma})$
Evaluation of $R$. Using the Mellin representations

\[ J_{\nu-1}(x) = \frac{1}{2\pi i} \int_{(-1)} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} 2^s x^{-s-1} ds \]

and

\[ V_{\nu/2}(y) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) y^{-u} du \]

the first remainder term is given by

\[ R = \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \chi_4(c) \epsilon_x^{2k+1} \sum_{m,n \geq 1} e \left( \frac{mx + nx}{4c} \right) \frac{V_{\nu/2}(mn/4)}{\sqrt{mn}} J_{\nu-1} \left( \frac{\pi}{c} \sqrt{mn} \right) \]

\[ = \frac{1}{16\pi} \left( \frac{1}{2\pi i} \right)^2 \sum_{c \geq 1} c^{-1} H_{c,k} \sum_{x(4c)} \chi_4(c) \epsilon_x^{2k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} \frac{K(u)}{(2\pi c)^{2u}} \times \int_{(-1)} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E_k(1 + u + s/2, x/4c) E_k(1 + u + s/2, x/4c) ds du \]

Recall that

\[ H_{c,k} = \chi_s(2k + 1)(1 - i^{2k+1})(1 + \chi_4(c)) \frac{\pi \sqrt{2}}{4c} \]  \hspace{1cm} (3.1.3)

Using lemma 4, we are able to move the line of integration for the inner integral from $\Re(s) = -1$ to $\Re(s) = -7$ passing a double pole when $c = 1$ at $s = -2u$ so that

\[ R = R_1 + R_2 \] where
\[ R_1 = \frac{\sqrt{2}}{32} \left( \frac{1}{2\pi i} \right)^2 \chi_8(2k + 1)(1 - i^{2k+1}) \sum_{x(4)} \chi_4(x) e_x^{2k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \frac{1}{(2\pi)^{2u}} \]
\[
\times \text{Res}_{s=-2u} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E_k(1 + u + s/2, x/4) E_k(1 + u + s/2, \pi/4) ds du
\]

\[ R_2 = \frac{1}{16\pi} \left( \frac{1}{2\pi i} \right)^2 \sum_{c \geq 1} c^{-1} H_{c,k} \sum_{x(4c)} \chi_4c(x) e_x^{2k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \frac{1}{(2\pi e)^{2u}} \]
\[
\times \int_{(-\pi)} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E_k(1 + u + s/2, x/4c) E_k(1 + u + s/2, \pi/4c) ds du
\]

**Evaluation of \( R_1 \)** To evaluate the residue we multiply the expression by \((s + 2u)^2\) and take the derivative using logarithmic differentiation to handle the abundance of terms.

\[ R_1 = \frac{\sqrt{2}}{32} \chi_8(2k + 1)(1 - i^{2k+1}) \sum_{x(4)} \chi_4(x) e_x^{2k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \frac{1}{(2\pi)^{2u}} \]
\[
\times \text{Res}_{s=-2u} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E_k(1 + u + s/2, x/4) E_k(1 + u + s/2, \pi/4) ds du
\]

\[
= \frac{\sqrt{2}}{32} \chi_8(2k + 1)(1 - i^{2k+1}) \sum_{x(4)} \chi_4(x) e_x^{2k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \frac{1}{(2\pi)^{2u}} \]
\[
\times \lim_{s \to -2u} (s + 2u)^2 \left( \frac{1}{2\pi} \right)^s \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} E_k(1 + u + s/2, x/4) E_k(1 + u + s/2, \pi/4) \]
\[
\times \left\{ \frac{1}{2} \frac{\Gamma'(\nu/2 + s/2)}{\Gamma(\nu/2 + s/2)} + \frac{1}{2} \frac{\Gamma'(\nu/2 - s/2)}{\Gamma(\nu/2 - s/2)} - \log(2\pi) + \frac{A_0(x/4)}{A_{-1}(x/4)} + \frac{A_0(\pi/4)}{A_{-1}(\pi/4)} \right\} du
\]
\[
\sqrt{\frac{2}{32}} \chi_8(2k + 1)(1 - i^{2k+1}) \frac{1}{2\pi i} \int (2) \frac{\Gamma(\nu/2 + u)\Gamma(\nu/2 - u)}{\Gamma^2(\nu/2)} K(u)
\]

\[
\times \sum_{x(4)} \chi_4(x)e_x^{2k+1} A_{-1}(x/4)A_{-1}(\bar{x}/4)
\]

\[
\times \left\{ \frac{1}{2} \frac{\Gamma'(\nu/2 - u)}{\Gamma(\nu/2 - u)} + \frac{1}{2} \frac{\Gamma'(\nu/2 + u)}{\Gamma(\nu/2 + u)} - \log(2\pi) + \frac{A_0(x/4)}{A_{-1}(x/4)} + \frac{A_0(\bar{x}/4)}{A_{-1}(\bar{x}/4)} \right\} du
\]

\(A_{-1}(x/4) = A_{-1}(\bar{x}/4)\) and \(A_0(x/4) = A_0(\bar{x}/4)\) because \(x \equiv \bar{x} \pmod 4\), and a quick calculation shows that \(\sum_{x(4)} \chi_4(x)e_x^{2k+1} A_{-1}(x/4)A_{-1}(\bar{x}/4) = 2(1 + i^{2k+1})\). For the remaining part we get

\[
A_0(x/4)A_{-1}(\bar{x}/4) + A_0(\bar{x}/4)A_{-1}(x/4) =
\]

\[
-2 \left\{ \sum_{0<\alpha\leq4 \ (\equiv 0,1 \pmod 4)} e\left(\frac{\alpha x}{4}\right) \psi(\alpha/4) + \sum_{0<\alpha\leq4 \ (\equiv 0,1 \pmod 4)} e\left(\frac{(\alpha - 1)k + \alpha x}{4}\right) \psi(\alpha/4) \right\}
\]

The function \(e_x^{2k+1}\) can be expanded as \(e_x^{2k+1} = \frac{1 + i^{2k+1}}{2} \left( \frac{4}{x} \right) + \frac{1 - i^{2k+1}}{2} \left( -\frac{4}{x} \right)\).

This gives us the Gauss sums

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\[
\sum_{0<\alpha\leq 4} \sum_{x(4)} \chi_4(x)e^{2k+1} e\left(\frac{\alpha x}{4}\right) \psi(\alpha/4)
\]
\[
= \sum_{0<\alpha\leq 4} \frac{1+i^{2k+1}}{2} \tau(x)\chi_4(\alpha)\psi(\alpha/4) + \frac{1-i^{2k+1}}{2} \tau(x)\chi_{-4}(\alpha)\psi(\alpha/4)
\]
\[
= \frac{1-i^{2k+1}}{2} \tau(x) \sum_{0<\alpha\leq 4} \chi_{-4}(\alpha)\psi(\alpha/4)
\]
\[
= (1+i^{2k+1}) \psi\left(\frac{2-(-1)^k}{4}\right)
\]
\[
= -(1+i^{2k+1})(\gamma + (-1)^k \pi/2 + 3 \log(2))
\]

and

\[
\sum_{0<\alpha\leq 4} \sum_{x(4)} \chi_4(x)e^{2k+1} e\left(\frac{((-1)^k + \alpha) x}{4}\right) \psi(\alpha/4)
\]
\[
= \sum_{0<\alpha\leq 4} \frac{1+i^{2k+1}}{2} \tau(x)\chi_4((-1)^k + \alpha) + \frac{1-i^{2k+1}}{2} \tau(x)\chi_{-4}((-1)^k + \alpha)\psi(\alpha/4)
\]
\[
= \frac{1-i^{2k+1}}{2} \tau(x) \sum_{0<\alpha\leq 4} \chi_{-4}((-1)^k + \alpha)\psi(\alpha/4)
\]
\[
= (1+i^{2k+1}) \psi(1)
\]
\[
= -(1+i^{2k+1}) \gamma
\]

Therefore the sum
\[
\sum_{x(4)} \chi_4(x) e_x^{2k+1} \left( A_0(x/4) A_{-1}(\bar{x}/4) + A_0(\bar{x}/4) A_{-1}(x/4) \right) \\
= 2(1 + i^{2k+1})(2\gamma + (-1)^k \pi/2 + 3 \log(2))
\]

We conclude that

\[
R_1 = \frac{\sqrt{2}}{8} \chi_8(2k + 1) \frac{1}{2\pi i} \int \frac{\Gamma(\nu/2 + u) \Gamma(\nu/2 - u)}{\Gamma^2(\nu/2)} K(u) \\
\times \left\{ \frac{1}{2} \frac{\Gamma'(-\nu/2 - u)}{\Gamma(-\nu/2 - u)} + \frac{1}{2} \frac{\Gamma'(-\nu/2 + u)}{\Gamma(-\nu/2 + u)} - \log(2\pi) + 2\gamma + (-1)^k \pi/2 + 3 \log(2) \right\} du
\]

The integrand is odd so by the residue theorem

\[
R_1 = \frac{\sqrt{2}}{16} \chi_8(2k + 1) \left\{ \frac{\Gamma'(-\nu/2)}{\Gamma(-\nu/2)} - \log(\pi/4) + 2\gamma + (-1)^k \pi/2 \right\}
\]

**Evaluation of \( R_2 \)**

We make the change of variable \( z = -(s + 2u) \) in the inner integral to get

\[
R_2 = \frac{1}{16\pi} \left( \frac{1}{2\pi i} \right)^2 \sum_{c \geq 1} c^{-1} H_{c,k} \sum_{x(4c)} \chi_{4c}(x) e_x^{2k+1} \int \frac{\Gamma^2(\nu/2 + u) K(u)}{\Gamma^2(\nu/2)} \frac{1}{c^2u} \\
\times \int (2\pi)^z E_k(1 - z/2, x/4c) E_k(1 - z/2, \bar{x}/4c) dsdu
\]

Using the functional equation (2.1.3) for \( E_k \) gives us \( R_2 = R_2^+ + R_2^- \) where

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\[ R_2^- = F \sum_{c,m,n \geq 1} c^{-1} H_{c,k} \sum_{0 < a, b \leq 4c} \frac{(-1)^k a^0 b^0, 1(4)}{(-1)^k a^0 b^0, 1(4)} K_{2k+1}(a, b; 4c) \int_{(2)} \frac{\Gamma^2(\nu/2 + u) K(u)}{\Gamma^2(\nu/2)} \frac{c^{2u}}{e^{2u}} \]

\[ \times \left\{ \int_{(3)} \frac{\Gamma(\nu/2 - u - z/2)}{\Gamma(\nu/2 + u + z/2)} \Gamma^2 \left( \frac{z}{2} \right) e \left( -\frac{z}{4} \right) e \left( \frac{na + mb}{4c} \right) (mn)^{-z/2} \right. \]

\[ + \frac{\Gamma(\nu/2 - u - z/2)}{\Gamma(\nu/2 + u + z/2)} \Gamma^2 \left( \frac{z}{2} \right) e \left( \frac{z}{4} \right) e \left( \frac{-na - mb}{4c} \right) (mn)^{-z/2} \right\} dz \] 

\[ \text{and} \]

\[ R_2^+ = F_1 \sum_{c,m,n \geq 1} c^{-1} H_{c,k} \sum_{0 < a, b \leq 4c} \frac{(-1)^k a^0 b^0, 1(4)}{(-1)^k a^0 b^0, 1(4)} K_{2k+1}(a, b; 4c) \int_{(2)} \frac{\Gamma^2(\nu/2 + u) K(u)}{\Gamma^2(\nu/2)} \frac{c^{2u}}{e^{2u}} \]

\[ \times \left\{ \int_{(3)} \frac{\Gamma(\nu/2 - u - z/2)}{\Gamma(\nu/2 + u + z/2)} \Gamma^2 \left( \frac{z}{2} \right) e \left( \frac{nb - ma}{4c} \right) (mn)^{-z/2} \right. \]

\[ + \frac{\Gamma(\nu/2 - u - z/2)}{\Gamma(\nu/2 + u + z/2)} \Gamma^2 \left( \frac{z}{2} \right) e \left( \frac{na - mb}{4c} \right) (mn)^{-z/2} \right\} dz \] 

Where \( F_1 \) is some constant.
Evaluation of $R_2^-$ Using lemma 11

$$R_2^- = F_2 \sum_{c,m,n \geq 1} c^{-1} H_{c,k} \sum_{0 < a,b \leq 4c} \frac{K_{2k+1}(a,b;4c)}{\Gamma^2(\nu/2 + u) \Gamma^2(\nu/2)} \frac{K(u)}{c^{2u}}$$

$$\times \left\{ e \left( \frac{-na - mb}{4c} \right) \int_0^\infty J_{\nu-1}(x)(iJ_0(x\sqrt{mn}) - Y_0(x\sqrt{mn})) \left( \frac{x}{2} \right)^{-2u} \, dx 
- e \left( \frac{na + mb}{4c} \right) \int_0^\infty J_{\nu-1}(x)(iJ_0(x\sqrt{mn}) + Y_0(x\sqrt{mn})) \left( \frac{x}{2} \right)^{-2u} \, dx \right\} du$$

Using Weil's bound $K_\kappa(m,n;c) \leq c^{1/2} \tau(c)(m,n,c)^{1/2}$ we get

$$\sum_{c \geq 1} c^{-2-2\sigma} \sum_{0 < a,b \leq 4c} |K_{2k+1}(a,b;4c)| \ll 1 \quad (3.1.4)$$

for $\sigma > 0$. When $m = n = 1$ lemma 12 (b) applies for sufficiently large $\nu$. Moving the line of integration to $\Re(u) = \sigma$ and applying (3.1.4) we have the bound

$$\ll \int_{(\sigma)} \left| \frac{\cos(\pi u)\Gamma^2(\nu/2 - u)K(u)}{\Gamma^2(\nu/2)} \right| \, du$$

$$+ \int_{(\sigma)} \left| \frac{\Gamma(2u)\Gamma^2(\nu/2 - u)\sin(\pi(\nu/2 - u))}{\Gamma^2(\nu/2)} K(u) \right| \, du$$

With Stirling's formula $\Gamma(2u)\sin(\pi(\nu/2 - u)) \ll 1$ and by lemma 9, $\frac{\Gamma^2(\nu/2 - u)}{\Gamma^2(\nu/2)} \ll \nu^{-\sigma}$ so the contribution from $m = n = 1$ has the bound $\ll \nu^{-\sigma}$.

For $m, n > 1$ lemma 12 (a) applies for large $\nu$. Moving the line of integration to $\Re(u) = 1$ and applying (3.1.4) we have the bound
\[ \ll \sum_{m,n>1} \int_{(1)} \left| \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \right| \times \left\{ \left| \int_0^\infty J_{\nu-1}(x)(iJ_0(x\sqrt{mn}) - Y_0(x\sqrt{mn}))x^{-2u} \, dx \right| \\
+ \left| \int_0^\infty J_{\nu-1}(x)(iJ_0(x\sqrt{mn}) + Y_0(x\sqrt{mn}))x^{-2u} \, dx \right| \right\} \, |du| \ll \nu^2 \sum_{m,n>1} (mn)^{\frac{1-\nu}{2}} \ll \nu^{2-\nu} \]

**Evaluation of** \( R_2^+ \) **Move the lines of integration to** \( \Re(u) = 1 \) and \( \Re(z) = \sigma \) **and denote the imaginary parts of** \( z \) **and** \( u \) **by** \( t_1 \) **and** \( t_2 \) **respectively. By Stirling’s formula**

\[ \Gamma^2 \left( \frac{z}{2} \right) \ll |t_1|^{2\sigma-1}e^{-\pi|t_1|} \]

Applying (3.1.4) as well as lemmas 9 and 10 we get the bound

\[ R_2^+ \ll \int_{-\infty}^{\infty} \nu^2 e^{-t_2^2} \int_{-\infty}^{\infty} (\nu/2 + |t_2 + t_1/2|)^{-2-\sigma} |t_1|^{2\sigma-1}e^{-\pi|t_1|} \, dt_1 \, dt_2 \ll \nu^{-\sigma} \]

**Evaluation of** \( \mathcal{R} \)

**Using the Mellin representations**

\[ J_{\nu-1}(x) = \frac{1}{2\pi i} \int_{(-1)} \frac{\Gamma((\nu + s)/2)}{\Gamma((\nu - s)/2)} 2^s x^{-s-1} \, ds \]  

(3.1.5)

and

\[ V_{\nu}(y) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} K(u) \pi^{-2u} y^{-u} \, du \]  

(3.1.6)

gives us
\[ R = \frac{1}{2} \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \chi_{4c}(x) e^{k+1} \sum_{m,n \geq 1} e \left( \frac{m x + n \overline{x}}{c} \right) \frac{V_{\nu}(4mn)}{\sqrt{mn}} J_{\nu-1} \left( \frac{4\pi}{c} \sqrt{mn} \right) \]

\[ = \frac{1}{8\pi} \left( \frac{1}{2\pi i} \right)^2 \sum_{c \geq 1} H_{c,k} c^{-1} \sum_{x(4c)} \chi_{4c}(x) e^{k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} \frac{K(u)}{(2\pi c)^{2u}} \times \int_{(s = -2u)} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E(1 + s/2 + u, 0, x/4c) E(1 + s/2 + u, 0, \pi/4c) ds du \]

Recall that

\[ H_{c,k} = \chi_8(2k + 1)(1 - i^{2k+1})(1 + \chi_4(c)) \frac{\pi \sqrt{2}}{4c} \tag{3.1.7} \]

We move the line of integration for the inner integral from \( \Re(s) = -1 \) to \( \Re(s) = -7 \) passing a double pole at \( s = -2u \) so that \( R = R_1 + R_2 \) where

\[ R_1 = \frac{1}{8\pi} \left( \frac{1}{2\pi i} \right)^2 \sum_{c \geq 1} H_{c,k} c^{-1} \sum_{x(4c)} \chi_{4c}(x) e^{k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} \frac{K(u)}{(2\pi c)^{2u}} \times \text{Res}_{s = -2u} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E(1 + s/2 + u, 0, x/4c) E(1 + s/2 + u, 0, \pi/4c) ds du \]

\[ R_2 = \frac{1}{8\pi} \left( \frac{1}{2\pi i} \right)^2 \sum_{c \geq 1} H_{c,k} c^{-1} \sum_{x(4c)} \chi_{4c}(x) e^{k+1} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} \frac{K(u)}{(2\pi c)^{2u}} \times \int_{(s = -7)} \frac{\Gamma(\nu/2 + s/2)}{\Gamma(\nu/2 - s/2)} \left( \frac{1}{2\pi} \right)^s E(1 + s/2 + u, 0, x/4c) E(1 + s/2 + u, 0, \pi/4c) ds du \]
Evaluation of $\mathcal{R}_1$

$$\mathcal{R}_1 = \frac{\sqrt{2}\chi_s(2k+1)(1-i^{2k+1})}{16} \sum_{x(4)} \left( \frac{4}{x} \right) \epsilon_x^{2k+1} \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} \frac{K(u)}{(2\pi)^{2u}}$$

$$\times \text{Res}_{s=-2u} \frac{\Gamma((\nu+s)/2)}{\Gamma((\nu-s)/2)} \left( \frac{1}{2\pi} \right)^s \zeta^2(1+u+s/2) du$$

Using (3.1.1), corollary 6 and evaluating the sum

$$\sum_{x(4)} \left( \frac{4}{x} \right) \epsilon_x^{2k+1} = 1 + i^{2k+1}$$

The integrand is odd so by moving the integral to $\Re(u) = -2$ and evaluating the residue we get that

$$\mathcal{R}_1 = \frac{\sqrt{2}\chi_s(2k+1)}{8} \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma^2(\nu/2 + u)}{\Gamma^2(\nu/2)} \frac{K(u)}{(2\pi)^{2u}}$$

$$\times \text{Res}_{s=-2u} \frac{\Gamma((\nu+s)/2)}{\Gamma((\nu-s)/2)} \left( \frac{1}{2\pi} \right)^s (s+2u)^2 \zeta^2(1+u+s/2)$$

$$\times \left\{ \frac{1}{2} \left( \frac{\nu}{2} + \frac{s}{2} \right) + \frac{1}{2} \left( \frac{\nu}{2} - \frac{s}{2} \right) - \log(2\pi) + \frac{2}{s+2u} + \frac{\zeta'(1+u+s/2)}{\zeta(1+u+s/2)} \right\} du$$

$$= \frac{\sqrt{2}\chi_s(2k+1)}{8} \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma \left( \frac{\nu}{2} + u \right) \Gamma \left( \frac{\nu}{2} - u \right)}{\Gamma^2(\nu/2)} K(u)$$

$$\times \left\{ \frac{1}{2} \left( \frac{\nu}{2} - u \right) + \frac{1}{2} \left( \frac{\nu}{2} + u \right) - \log(2\pi) + \gamma \right\} du$$

$$\mathcal{R}_1 = \frac{\sqrt{2}}{16} \chi_s(2k+1) \left\{ \frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} - \log(2\pi) + \gamma \right\}$$
Evaluation of $R_2$ This is similar to the evaluation of $R_2$ and yields the same result $R_2 \ll \nu^{-\sigma}$.

3.2 Proof of theorem 2

Theorem 2. Let $\epsilon > 0$ be an arbitrary positive number. For $p \leq k^{1/2}$ we have

$$
\sum_{F \in B_k} \omega_F L^2(F \otimes \chi_p, k) = 4 \left(1 - \frac{1}{p}\right) \log \left(\frac{e^{\gamma k p}}{2\pi}\right) + O_\epsilon(p^{1+\epsilon} k^{-1/2} \log(k))
$$

Proof. Using (1.2.4), proposition 8, and the approximate functional equation (1.3.1) with $K(u) = e^{u^2} u^{-1}$

$$
\frac{1}{4} \omega_F L^2(F \otimes \chi_p, k)
$$

$$
= \frac{1}{2} \omega_f L(F \otimes \chi_p, k) \tilde{a}_f^2(p)
$$

$$
= \omega_f \sum_{n \geq 1} \frac{\chi_p(n)}{n^{1/2}} W_k \left(\frac{2\pi n}{p}\right) \tilde{A}_F(n) \tilde{a}_f^2(p)
$$

$$
= \omega_f \sum_{n \geq 1} \frac{\chi_p(n)}{n^{1/2}} W_k \left(\frac{2\pi n}{p}\right) \sum_{d|n} \frac{\chi_p(d)}{d^{1/2}} \tilde{a}_f \left(\frac{p n^2}{d^2}\right)
$$

$$
= \omega_f \sum_{n,m \geq 1} \frac{\chi_p(mn)}{(nm)^{1/2}} W_k \left(\frac{2\pi mn}{p}\right) \frac{\chi_p(m)}{m^{1/2}} \tilde{a}_f (pn^2)
$$

$$
= \sum_{n \geq 1} \omega_f \tilde{a}_f(p) \tilde{a}_f(pn^2) \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(k+u)}{\Gamma(k)} K(u) \left(\frac{p}{2\pi}\right)^u \zeta_p(1+u) \frac{\chi_p(n)}{n^{u+1/2}} du
$$

We apply the trace formula (3) to this and get

$$
\frac{1}{4} \sum_{F \in B_k} \omega_F L^2(F \otimes \chi_p, k) = S + R
$$

where

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\[S = \frac{1}{2\pi i} \int_{-1}^{2} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) du\]

\[R = \frac{1}{2\pi i} \int_{-1}^{2} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) du \]

\[\times \sum_{\substack{n \geq 1 \\
c \geq 1}} \frac{\chi_p(n)}{n^{u+1/2}} H_{c,k} K(pn^2, p; 4c) J_{k-1/2} \left( \frac{pn}{c} \right) du\]

and \(H_{c,k} = \chi_8(2k + 1)(1 - i)(1 + \chi_4(c)) \frac{\pi \sqrt{2}}{4c}\)

The exponential decay of \(K(u)\) along with Stirling’s approximation allows us to move the line of integration from \(\Re(u) = 2\) to \(-1\). Along the way we pass a double pole at \(u = 0\) and get \(S = S_1 + S_2\) where

\[S_1 = \text{Res}_{u=0} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u)\]

\[S_2 = \frac{1}{2\pi i} \int_{-1}^{2} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) du\]

To evaluate \(S_1\), we compute the following Laurent series.

\[\frac{\Gamma(k + u)}{\Gamma(k)} = 1 + u \frac{\Gamma'(k)}{\Gamma(k)} + \cdots\]

\[\left( \frac{p}{2\pi} \right)^u = 1 + u \log \left( \frac{p}{2\pi} \right) + \cdots\]

\[K(u) = u^{-1} + \cdots\]

\[\zeta_p(1 + u) = \left( 1 - p^{-1} \right) \left( 1 + u \frac{\log(p)}{p - 1} + \cdots \right) \left( u^{-1} + \gamma + \cdots \right)\]
Using these expansions we calculate the residue in $S_1$ giving us

$$S_1 = (1 - p^{-1}) \left( \frac{\Gamma'(k)}{\Gamma(k)} + \gamma + \log \left( \frac{p}{2\pi} \right) + \frac{\log(p)}{p - 1} \right)$$

With the choice of $\Re(u) = -1$, we have the bounds $\zeta_p(1 + u) \ll (1 + |t|)^{1/2}$ and $\frac{\Gamma(k + u)}{\Gamma(k)} \ll k^{-1}$ so $S_2 \ll p^{-1/2}k^{-1}$. Using the asymptotic formula $\frac{\Gamma'(k)}{\Gamma(k)} = \log(k) + O(k^{-1})$

$$S = (1 - p^{-1}) \log \left( \frac{e^\gamma kp}{2\pi} \right) + O(k^{-1} + \log(p)p^{-1})$$

Using the Mellin representation formula for the Bessel function given in (3.3.2)

$$J_{k-1/2}(x) = \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left( \frac{k}{2} + \frac{1}{4} + \frac{s}{2} \right)}{\Gamma\left( \frac{k}{2} + \frac{1}{4} - \frac{s}{2} \right)} 2^s x^{-s-1} ds$$

and opening up the Kloosterman sum $K(m, n; c) = \sum_{x(c)} \left( \frac{c}{x} \right) \epsilon_x e \left( \frac{mx + nx}{c} \right)$, the remainder term becomes

$$R = \frac{1}{2(2\pi i)^2} \int_{(2)} \int_{(-1)} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) \sum_{n,c \geq 1} \frac{\chi_p(n)}{n^{u+1/2}} H_{c,k}$$

$$\times \sum_{x(4c)} \epsilon_x \left( \frac{4c}{x} \right) e \left( \frac{pn^2 x + px}{4c} \right) \frac{\Gamma\left( \frac{k}{2} + \frac{1}{4} + \frac{s}{2} \right)}{\Gamma\left( \frac{k}{2} + \frac{1}{4} - \frac{s}{2} \right)} \left( \frac{2c}{\pi pn} \right)^{s+1} dsdu$$
\[
\frac{1}{2(2\pi i)^2} \int_{(2)} \int_{(-1)} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) \sum_{c \geq 1} H_{c,k} \\
\times \sum_{x(4c)} \epsilon_x \left( \frac{4c}{x} \right) e^{-\frac{p(x)}{4c}} \frac{\Gamma\left( \frac{k}{2} + \frac{1}{4} + \frac{s}{2} \right)}{\Gamma\left( \frac{s}{2} + \frac{1}{4} - \frac{s}{2} \right)} \left( \frac{2c}{\pi p} \right)^{s+1} E_p(s + u + 3/2, x/4c)dsdu
\]

By moving the line of integration for the inner integral from \(-1\) to \(-7\) we pass a possible pole at \(s = -u - 1/2\) and get

\[
R = R_1 + R_2\]

where

\[
R_1 = \frac{1}{8(2\pi i)^2} \int_{(2)} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) \frac{\Gamma\left( \frac{k}{2} - \frac{u}{2} \right)}{\Gamma\left( \frac{k}{2} + \frac{u}{2} + \frac{1}{2} \right)} \\
\times \left( \frac{2}{\pi p} \right)^{1/2-u} \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \left( \frac{4c}{x} \right) e^{2k+1} e^{-\frac{p(x)}{4c}} \sum_{a(c)} b(p) e^{-\frac{b(d)}{a(d)}} K(a^2 p, p; 4c)du
\]

\[
R_2 = \frac{1}{2(2\pi i)^2} \int_{(2)} \frac{\Gamma(k + u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1 + u) \\
\times \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \left( \frac{4c}{x} \right) e^{2k+1} e^{-\frac{p(x)}{4c}} \sum_{s \geq 1} \left( \frac{2c}{\pi p} \right)^{s+1} E_p(s + u + 3/2, x/4c)dsdu
\]

Remarkably, the first term \(R_1\) vanishes by lemma 7. To evaluate \(R_2\) we make the substitution \(s + u + 3/2 = 1 - z\) in the inner integral to get
\[ R_2 = \frac{1}{2(2\pi i)^2} \int_{(2)} \frac{\Gamma(k+u)}{\Gamma(k)} K(u) \left( \frac{p}{2\pi} \right)^u \zeta_p(1+u) \]

\[ \times \sum_{c \geq 1} H_{c,k} \sum_{x(4c)} \left( \frac{4c}{x} \right)^{2k+1} e^{2k+1} \left( \frac{p\pi}{4c} \right) \]

\[ \times \int_{(9/2)} \frac{\Gamma\left( \frac{k}{2} - \frac{z}{2} - \frac{u}{2} \right)}{\Gamma\left( \frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2} \right)} \left( \frac{2c}{\pi p} \right)^{1/2-z-u} E_p(1-z,x/4c) dsdu \]

\[ = F_1 \int_{(2)} \frac{\Gamma(k+u)}{\Gamma(k)} K(u) \left( \frac{p}{4} \right)^{2u-1/2} \zeta_p(1+u) \]

\[ \times \sum_{c \geq 1} H_{c,k} \sum_{a(4c)} K(a^2p,p;4c)[4c,p]^{-1} e^{1/2-u} \]

\[ \times \sum_{b(p)} \chi_p(b) \int_{(9/2)} \left\{ \frac{\Gamma\left( \frac{k}{2} - \frac{z}{2} - \frac{u}{2} \right)}{\Gamma\left( \frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2} \right)} \Gamma(z) e(n\lambda_{a,b}) e\left( -\frac{z}{4} \right) \left( \frac{4cn}{p[4c,p]} \right)^{-z} \right\} \]

\[ dzdu \]

\[ = F_2 \int_{(2)} \frac{\Gamma(k+u)}{\Gamma(k)} K(u) \left( \frac{p}{4} \right)^{2u-1/2} \zeta_p(1+u) \]

\[ \times \sum_{c \geq 1} H_{c,k} \sum_{a(4c)} K(a^2p,p;4c)[4c,p]^{-1} e^{1/2-u} \]

\[ \times \left\{ \sum_{b(p)} \chi_p(b) e(n\lambda_{a,b}) \int_0^\infty J_{k-1/2}(x) e^{-ixy/2} \left( \frac{x}{2} \right)^{-u-1/2} dx \right\} \]

\[ + \sum_{b(p)} \chi_p(b) e(-n\lambda_{a,b}) \int_0^\infty J_{k-1/2}(x) e^{ixy/2} \left( \frac{x}{2} \right)^{-u-1/2} dx \right\} \]

\[ du \]

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where \( y = \frac{4cn}{p[4c,p]} \), and \( F_1, F_2 \) are constants which do not depend on \( k \) or \( p \). It is given by lemma 8 that we need only consider even \( n \) and \((cn,p) = 1 \). Moving the line of integration to \( \Re(u) = \epsilon/4 \) we have

\[
R_2 \ll p^{\epsilon/2-3/2} \sum_{n \geq 1} \sum_{c \geq 1} c^{-3/2-\epsilon/4} \left\{ \int_{(\epsilon/4)} (k + u) \frac{\Gamma(k+u)}{\Gamma(k)} K(u) \zeta_p(1+u) \sum_{a(4c)} K(a^2p, p; 4c) \chi_p(b) e(2n\lambda_{a,b}) \right. \\
\times \left. \int_{0}^{\infty} J_{k-1/2}(x) e^{-ixn/p^2} \left( \frac{x}{2} \right)^{-u-1/2} dx \right| \\
+ \int_{(\epsilon/4)} (k + u) \frac{\Gamma(k+u)}{\Gamma(k)} K(u) \zeta_p(1+u) \sum_{a(4c)} K(a^2p, p; 4c) \chi_p(b) e(-2n\lambda_{a,b}) \right| \\
\times \left. \int_{0}^{\infty} J_{k-1/2}(x) e^{ixn/p^2} \left( \frac{x}{2} \right)^{-u-1/2} dx \right\} \left| du \right|
\]

Now we split the sum according to lemma 14. Since \((n,p) = 1 \), we are left with \( n < p^2 \) and \( n > p^2 \). Using lemma 8 and the bound \( \frac{\Gamma(k+u)}{\Gamma(k)} \ll k^{\epsilon/4} \) from lemma 9, we have that the first sum is

\[
\ll p^{\epsilon/2-1} k^{-1/2} \log(k) \sum_{c \geq 1} c^{-1-\epsilon/4} \sum_{n < p^2} \{(c, p^4 - n^2) + (c, p^4 + n^2)\}(1 - n/p^2)^{-1/2} \\
\ll p^{\epsilon/2-1} k^{-1/2} \log(k) \sum_{n < p^2} (1 - n/p^2)^{-1/2} \sum_{d|p^4 \pm n^2} d \sum_{c \geq 1} (cd)^{-1-\epsilon/4} \\
\ll p^{\epsilon/2-1} k^{-1/2} \log(k) \sum_{n < p^2} (1 - n/p^2)^{-1/2} \tau(p^4 \pm n^2) \\
\ll p^{\epsilon/2-1} k^{-1/2} \log(k) \sum_{n < p^2} (1 - n/p^2)^{-1/2} \tau\tau(p^4 \pm n^2) \ll p^{\epsilon+1} k^{-1/2} \log(k)
\]

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For \( n > p^2 \) we leave \( \Re(u) = 2 \). Using (8), lemma 14 (a) and the bound \( \frac{\Gamma(k + u)}{\Gamma(k)} \ll k^2 \) along with the exponential decay of \( K(u) \) (to ensure convergence), we get that the sum over \( n > p^2 \) is bounded by

\[
\ll p^3 \sum_{n > p^2} (n/p^2)^{2-k} (e/4)^{k/2} (1 - p^2 / n^2)^{-1}
\]

Because \( (1 - p^2 / n^2)^{-1} \leq p^4 \) for \( n > p^2 \), we get

\[
\ll p^7 (1 + 1/p^2)^{2-k} (e/4)^{k/2} \ll p^7 (e/4)^{k/2},
\]

which is sufficient to complete the proof.

\[
\square
\]

### 3.3 Integral transforms and bounds

In this section we prove the various bounds which allow us to control growth in the weight aspect.

**Lemma 9.** Let \( \Re(u) = \sigma \) and \( A > 1/2 \) be a fixed constant. For all sufficiently large \( \nu \) and \( -A < \sigma < A \)

\[
\frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} \ll \nu^\sigma
\]

**Proof.** Stirling’s approximation gives
\[
\frac{\Gamma(\nu/2 + u)}{\Gamma(\nu/2)} \sim \left(\frac{\nu/2 + u - 1}{\nu/2 - 1}\right)^{u+\nu/2-1/2} (\nu/2 - 1)^u
\]

As \(\nu \to \infty\)

\[
\left(\frac{u + \nu/2 - 1}{\nu/2 - 1}\right)^{u+\nu/2-1/2} \to e^u \quad \text{which gives us the result.}
\]

\[\square\]

Lemma 10. [19] Let \(s = \sigma + it\) and \(A > 1/2\) be a fixed constant. For all sufficiently large \(\nu\) (\(\nu \geq \nu_0(A)\)) and \(0 \leq \sigma < A\), we have

\[
\frac{\Gamma(\nu/2 - s)}{\Gamma(\nu/2 + s)} \ll (\nu/2 + |t|)^{-2\sigma}
\]

Lemma 11. [19] Let \(\nu > 2\) be any integer and \(y > 0\). Suppose that \(0 < \Re(u) < \nu/2 - 2\). Then

\[
\frac{1}{2\pi i} \int_{(3)} \frac{\Gamma(\nu/2 - u - z/2)}{\Gamma(\nu/2 + u + z/2)} \Gamma^2 \left(\frac{z}{2}\right) e\left(\frac{z}{4}\right) y^{-z} dz
\]

\[
= 2\pi \int_0^\infty J_{\nu-1}(x)(iJ_0(yx) - Y_0(yx)) \left(\frac{x}{2}\right)^{-2u} dx
\]

and

\[
\frac{1}{2\pi i} \int_{(3)} \frac{\Gamma(\nu/2 - u - z/2)}{\Gamma(\nu/2 + u + z/2)} \Gamma^2 \left(\frac{z}{2}\right) e\left(\frac{-z}{4}\right) y^{-z} dz
\]

\[
= -2\pi \int_0^\infty J_{\nu-1}(x)(iJ_0(yx) + Y_0(yx)) \left(\frac{x}{2}\right)^{-2u} dx
\]

Lemma 12. [19] Let \(s = \sigma + it\) and \(1/2 < A\) be a fixed constant. For all sufficiently large \(\nu\) (\(\nu \geq \nu_0(A)\)) and \(1/2 \leq \sigma \leq A\),

(a) If \(1 < y\) then
\[
\int_0^\infty J_{\nu-1}(x)J_0(yx)x^{-s}dx \ll e^{\pi|t|/2} \frac{y^{\sigma-\nu}}{1-y^{-2}},
\]

\[
\int_0^\infty J_{\nu-1}(x)Y_0(yx)x^{-s}dx \ll e^{\pi|t|/2} \frac{y^{\sigma-\nu}}{1-y^{-2}},
\]

(b) \[3\] 6.574 (2)

\[
\int_0^\infty J_{\nu-1}(x)J_0(x)x^{-2u}dx = \frac{\Gamma(2u)\Gamma^2(\nu/2 - u)\sin(\pi(\nu/2 - u))}{\pi 2^{3u}\Gamma^2(\nu/2 + u)}
\]

and [1] 6.8, (36)

\[
\int_0^\infty J_{\nu-1}(x)Y_0(x)x^{-2u}dx = -2^{-2u-1}\cos(\pi u)\frac{\Gamma^2(\nu/2 - u)}{\Gamma^2(\nu/2 + u)}
\]

Lemma 13. For \(0 < \Re(u) < k - 9/2\)

\[
\int_{(9/2)} \frac{\Gamma\left(\frac{k}{2} - \frac{z}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2}\right)} \Gamma(z)e\left(\pm\frac{z}{4}\right) y^{-z}dz = \int_0^\infty J_{k-1/2}(x) e^{\pm ixy/2} \left(\frac{x}{2}\right)^{-u-1/2}dx
\]

Proof. The integrals convergence is given by Stirling’s formula

\[
\frac{\Gamma\left(\frac{k}{2} - \frac{z}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2}\right)} \Gamma(z)e\left(\pm\frac{z}{4}\right) y^{-z}dz = \int_0^\infty J_{k-1/2}(x) e^{\pm ixy/2} \left(\frac{x}{2}\right)^{-u-1/2} dx
\]

Using the inverse Mellin transforms [1] 7.3 (6) and (7), we have

\[
\frac{1}{2\pi i} \int_{(1/2)} \Gamma(z)e\left(\pm\frac{z}{4}\right) y^{-z}dz = e^{\pm iy}
\]

(3.3.1)

Furthermore, the Bessel function has the Mellin transform [1] 6.8 (1)

\[
\frac{2^{s-1}\Gamma\left(\frac{k}{2} - \frac{1}{4} + \frac{s}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{3}{4} - \frac{s}{2}\right)} = \int_0^\infty J_{k-1/2}(x)x^{s-1}dx \quad (-k + 1/2 < \Re(s) < 1/2)
\]

and after a change of variables
\[
\frac{\Gamma\left(\frac{k}{2} - \frac{z}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2}\right)} = \int_0^\infty J_{k-1/2}(x) \left(\frac{x}{2}\right)^{-z-u-1/2} \, dx \tag{3.3.2}
\]

for \(0 < \Re(z + u) < k\). Moving the line of integration to \(\Re(z) = \frac{1}{2}\), we are able to exchange the order of integration and use (3.3.1)

\[
\frac{1}{2\pi i} \int_{(9/2)} \frac{\Gamma\left(\frac{k}{2} - \frac{z}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2}\right)} \Gamma(z) e \left(\pm \frac{z}{4}\right) y^{-z} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{(1/2)} \frac{\Gamma\left(\frac{k}{2} - \frac{z}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{z}{2} + \frac{u}{2} + \frac{1}{2}\right)} \Gamma(z) e \left(\pm \frac{z}{4}\right) y^{-z} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{(1/2)} \int_0^\infty J_{k-1/2}(x) \left(\frac{x}{2}\right)^{-z-u-1/2} \Gamma(z) e \left(\pm \frac{z}{4}\right) y^{-z} \, dx \, dz
\]
\[
= \int_0^\infty J_{k-1/2}(x) \left(\frac{x}{2}\right)^{-u-1/2} \frac{1}{2\pi i} \int_{(1/2)} \Gamma(z) e \left(\pm \frac{z}{4}\right) \left(\frac{xy}{2}\right)^{-z} \, dz \, dx
\]
\[
= \int_0^\infty J_{k-1/2}(x) e^{\pm iy/2} \left(\frac{x}{2}\right)^{-u-1/2} \, dx
\]

\(\square\)

**Lemma 14.** Let \(u = \sigma + it\) and \(A > 0\) be a fixed constant. For all sufficiently large \(k\), and \(0 < \sigma < A\),

(a) if \(y > 1\), then

\[
\int_0^\infty J_{k-1/2}(x) e^{\pm iy} x^{-u-1/2} \, dx \ll e^{2\pi \rho|t|/2} \left(\frac{y}{2}\right)^{\sigma-k} \left(\frac{e}{4}\right)^{k/2} \frac{1}{1 - y^4}
\]

(b) if \(k^{-1} < y < 1 - k^{-1}\), then

\[
\int_0^\infty J_{k-1/2}(x) e^{\pm iy} x^{-u-1/2} \, dx \ll (1 + |t|)(1 - y)^{-1/4} k^{-\sigma-1/2} \log(k)
\]
Proof. (a) For $-1/2 < \sigma < k - 1$, we have the formulae 1.12.13 and 2.12.11 [1], which together give

$$
\int_0^\infty J_{k-1/2}(x) e^{\pm ixy/2} x^{-u-1/2} \, dx
$$

$$
= \frac{(y/2)^{u-k} e^{\pm 2\pi i(k-u)} \Gamma(k-u)}{2^{k-1/2} \Gamma(k+1/2)} F \left( \frac{k-u}{2}, \frac{k-u+1}{2}; k+1/2; (2/y)^2 \right)
$$

$$
\ll e^{2\pi |t|} \left( \frac{y}{2} \right)^{\sigma-k} \sum_{r=0}^{\infty} \frac{2^{-k-2r} \Gamma(k-\sigma+2r)}{\Gamma(k+1/2+r)r!} \left( \frac{2}{y} \right)^{2r}
$$

Using Stirling’s formula we have

$$
\frac{\Gamma(k-\sigma+2r)}{\Gamma(k+1/2+r)r!} \leq \frac{(k+2r)^{k+2r}}{(k+r)^{k+r}} e^{-r}
$$

$$
\leq \left( 1 + \frac{r}{k+r} \right)^{k+r} \left( 1 + \frac{k}{2r} \right)^r 2^{2r} e^{-r} \leq 2^{2r} e^{\frac{k}{2}}
$$

so

$$
\int_0^\infty J_{k-1/2}(x) e^{\pm ixy/2} x^{-u-1/2} \, dx \ll e^{2\pi |t|} \left( \frac{y}{2} \right)^{\sigma-k} \left( \frac{e}{4} \right)^{k/2} \frac{1}{1 - \frac{4}{y^2}}
$$

For part (b) we split the range of integration as follows:

$$
\int_0^\infty = \int_0^{k/4} + \sum_{K=2^{r-2}k} \int_K^{2K}
$$

and set $I_K = \int_K^{2K}$.
For $0 \leq x \leq \nu/4$ we use the bound $J_{k-1/2}(x) \ll \left(\frac{e x}{2k}\right)^{k-1/2}$ finding that the contribution is

$$\int_0^{\nu/4} J_{k-1/2}(x) e^{\pm i x y/2} x^{-u-1/2} dx \ll k^{-\sigma+1/2} \left(\frac{e}{8}\right)^k$$

(3.3.3)

The function $J_{k-1/2}(x)$ can be written as a sum of a Fourier transform and a Laplace transform ([3] 8.411.13) $J_{k-1/2}(x) = A_k(x) + B_k(x)$, where

$$A_k(x) = \frac{1}{\pi} \int_0^{\pi} \cos \left((k - 1/2)\theta - x \sin(\theta)\right) d\theta$$

and

$$B_k(x) = \frac{1}{\pi} \int_0^\infty e^{-(k-1/2)\theta - x \sinh(\theta)} d\theta$$

After making some adjustments we have

$$A_k(x) = \frac{1}{\pi} \int_0^{\pi} \cos \left((k - 1/2)\theta - x \sin(\theta)\right) d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi/2} \Re \left\{ e^{-ix\sin(\theta)} \left[ e^{i(k-1/2)\theta} + e^{-i(k-1/2)(\pi-\theta)} \right] \right\} d\theta$$

$$= \frac{1}{\pi} \int_0^{1} \Re \left\{ e^{-ix\theta} f_k(\theta) \right\} (1 - \theta^2)^{-1/2} d\theta$$

where $f_k(\theta) = e^{i(k-1/2)\arcsin(\theta)} + e^{-i(k-1/2)(\pi-\arcsin(\theta))}$. Applying integration by parts or bounding trivially:
\[
\int_{K}^{2K} A_k(x)e^{\pm ixy} x^{-u-1/2} dx \\
= \frac{1}{\pi} \int_{0}^{1} \Re \left\{ e^{-ix\theta} f_k(\theta) \right\} (1 - \theta^2)^{-1/2} \int_{K}^{2K} e^{ix(u+y)} x^{-u-1/2} dxd\theta \\
\ll (1 + |t|) K^{-\sigma-1/2} \int_{0}^{1} (1 - \theta)^{-1/2} \min\{|\theta - y|^{-1}, K\} d\theta
\]

We split the range of integration around the singularity \( \theta = y \) using the two points \( \theta = (1 - K^{-2})y \) and \( \theta = (1 + K^{-2})y \). In the range \([0, (1 - K^{-2})y]\), we use the bound \(|\theta - y|^{-1}\) to get:

\[
\int_{0}^{(1-K^{-2})y} (1 - \theta)^{-1/2} \min\{|\theta - y|^{-1}, K\} d\theta \ll (1 - y + y/K^2)^{-1/2} \int_{0}^{(1-K^{-2})y} (y - \theta)^{-1} d\theta \\
\ll (1 - y)^{-1/2} \log(K)
\]

In the range \([(1 - K^{-2})y, (1 + K^{-2})y]\) we use the trivial bound \(K\) to get:

\[
\int_{(1-K^{-2})y}^{(1+K^{-2})y} (1 - \theta)^{-1/2} K d\theta \ll K^{-1}
\]

In the range \([(1 - K^{-2})y, 1]\) we use the bound \(|\theta - y|^{-1}\) to get:
\begin{align*}
&\int_{(1+K^{-2})y}^{1} (1 - \theta)^{-1/2}(\theta - y)^{-1} \, d\theta \\
&= (1 - y)^{-1/2} \log \left( \frac{(1 - y)^{1/2} - (1 - \theta)^{1/2}}{(1 - y)^{1/2} + (1 - \theta)^{1/2}} \right) \bigg|_{\theta=(1+K^{-2})y}^{\theta=1} \ll (1 - y)^{-1/2} \log(K)
\end{align*}

The sum of these estimates gives us

\begin{align*}
&\int_{K}^{2K} A_k(x) e^{\pm ixy} x^{-u-1/2} \, dx \ll (1 + |t|) (1 - y)^{-1/2} K^{-\sigma-1/2} \log(K) \quad (3.3.4)
\end{align*}

because by assumption $k^{-1} \leq y \leq 1 - k^{-1}$, and therefore, $(1 - y)^{-1/2} \leq k^{1/2} \ll K^{1/2}$.

\begin{align*}
&\int_{K}^{2K} B_k(x) e^{\pm ixy} x^{-u-1/2} \, dx \\
&= \frac{1}{\pi} \int_{0}^{\infty} e^{-(k-1/2)\theta} \int_{K}^{2K} e^{x\sinh(\theta)} e^{\pm ixy} x^{-u-1/2} \, dx \, d\theta \\
&\ll K^{1/2-\sigma} \int_{0}^{\infty} e^{-(k-1/2)\theta} e^{-K\sinh(\theta)} \, d\theta \\
&\ll K^{1/2-\sigma} \int_{0}^{\infty} e^{-K\theta} \, d\theta \ll K^{-1/2-\sigma}
\end{align*}
BIBLIOGRAPHY


