On the Farrell-Jones Isomorphism Conjecture

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Abstract

In this thesis, we study three different aspects of the Farrell-Jones Conjecture (FJC). The first is the study of the conjecture for groups admitting nice but not necessarily proper actions on CAT(0)-spaces (stabilizers can be infinite). It is a natural question that if the point stabilizers of the action satisfy the conjecture, whether the original group satisfies the conjecture. For this, we introduce the notion of hyperdiscrete group actions. Every proper action is hyperdiscrete. There are many other interesting examples. It turns out this new notion of group actions mostly fit into the framework for proving FJC developed by A. Bartels, W. Lück and H. Reich. The second is the study of inheritance properties of the conjecture. We study the problem that if a group has a subgroup of finite index satisfying the conjecture, whether the group itself satisfies the conjecture. We reduce the problem to a special case and results obtained for this special case strongly suggests the rationalized conjecture is invariant under commensuration. The third part of this thesis is a joint work with J. Lafont and S. Prassidis. We study the Farrell Nil-groups associated to a virtually cyclic group, which is the obstruction to reduce the family of virtually cyclic groups used in FJC to the family of finite groups. We indeed study the more general Farrell Nil-groups associated to a finite order automorphism of a ring \( R \). We show that any such Farrell Nil-group is either trivial, or infinitely generated (as an abelian group). Building on this first result, we then show that any finite group that occurs in such
a Farrell Nil-group occurs with infinite multiplicity. If the original finite group is a
direct summand, then the countably infinite sum of the finite subgroup also appears
as a direct summand. We use this to deduce a structure theorem for countable Far-
rell Nil-groups with finite exponent. Finally, as an application, we show that if $V$ is
any virtually cyclic group, then the associated Farrell or Waldhausen Nil-groups can
always be expressed as a countably infinite sum of copies of a finite group, provided
they have finite exponent (which is always the case in dimension 0).
Dedicated to my parents
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Bibliography
1.1 Why the Farrell-Jones Conjecture

The most important invariant of a topological space $X$ is its fundamental group $\pi_1(X)$. However, certain geometric and topological problems force one not to consider the fundamental group directly, but to consider some mathematical objects produced out of it. The most important ones of such are various *algebraic $K$-groups* $K_i(\mathbb{Z}[\pi_1(X)])$, and *algebraic $L$-groups* $L_i(\mathbb{Z}[\pi_1(X)])$, $i \in \mathbb{Z}$, associated to the integral group ring $\mathbb{Z}[\pi_1(X)]$ over the fundamental group.

In general, one can define algebraic $K$-groups of an arbitrary small additive category and algebraic $L$-groups of an arbitrary small additive category with *involution*, meaning there is an additive functor from the category to itself with square the identity. For a ring $R$ (with involution), the corresponding small additive category (with involution) is the category of finitely generated free $R$-modules. In the special case when $R = \mathbb{Z}[\pi_1(X)]$, the involution is usually given by $\Sigma a_g g \mapsto \Sigma \epsilon(g)a_g g^{-1}$, where $\epsilon : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$ is the “orientation homomorphism” of $X$. Precise definitions of these groups can be found in [64][72][62]. We will however only need some of their basic properties, which are reviewed in Appendix B, by viewing them as functors from the category of small additive categories to the category of abelian groups.
The groups $K_i(\mathbb{Z}[\pi_1(X)])$, $L_i(\mathbb{Z}[\pi_1(X)])$ contain important information about the underlying space $X$ and are crucial in many applications.

1.1.1 Some Geometric Recognition Problems

Among all kinds of topological spaces, CW-complexes and manifolds are of the most importance. It is an important question to characterize spaces having the homotopy type of a finite CW-complex, a compact topological manifold, a compact smooth manifold, etc. And for two manifolds, it is of fundamental importance to recognize whether they are the same in an appropriate sense. It turns out, in order to have a satisfactory theory for solving such geometric problems, one should follow a general pattern, which can roughly be summarized by

Geometric Problem = Weaker Geometric Assumptions + Algebraic Obstruction

and the algebraic obstruction here usually lies in $K_i(\mathbb{Z}[\pi_1(X)])$, $L_i(\mathbb{Z}[\pi_1(X)])$, $i \in \mathbb{Z}$ or their relatives.

**Wall’s finiteness obstruction:** The problem here is to characterize spaces having the homotopy type of a finite CW-complex. A natural weaker geometric assumption for a space $X$ to have the homotopy type of a finite CW-complex is that it must be finitely dominated, meaning there is a finite CW-complex $K$ and maps $\iota : X \to K$, $r : K \to X$, so that $r \circ \iota$ is homotopic to $1_X$. This weaker geometric assumption allows one to construct an algebraic obstruction $w(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$—Wall’s finiteness obstruction, and $X$ has the homotopy type of a finite CW-complex if and only if $w(X)$ vanishes [67][68]. Here $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is the reduced projective class group of the ring $\mathbb{Z}[\pi_1(X)]$, i.e. it is the cokernel of the map $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\pi_1(X)])$ induced by the obvious ring homomorphism $\mathbb{Z} \to \mathbb{Z}[\pi_1(X)]$.  

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**Surgery obstruction:** The next natural question is when a finite CW-complex has the homotopy type of a compact (smooth) manifold. A natural weaker geometric assumption on the space $X$ is it must be a Poincaré complex, roughly, this means its homology and cohomology groups satisfy Poincaré duality. For a precise definition, see [69, Section 2][52, Section 3.1]. Another feature of a manifold is it has a (stable) tangent (normal) bundle. This leads to the other natural geometric assumption on $X$: the existence of a *degree one normal map*, i.e. there exists a vector bundle $\xi : E(\xi) \to X$ over $X$ and a continuous map $f : M \to X$ from a closed manifold $M$ to $X$ so that $f$ sends the fundamental class of $M$ to the fundamental class of $X$ and the pullback bundle $f^*(\xi)$ is stably isomorphic to the tangent bundle of $M$. These two weaker geometric assumptions on $X$ allow one to construct a *surgery obstruction* $o(X) \in L_n(\mathbb{Z}[\pi_1(X)])$ with the property that $X$ is homotopy equivalent to a manifold within the normal bordism class of $M$ if and only if $o(X)$ vanishes provided $n \geq 5$, where $n = \dim(X)$. See [69][52] for more details of the theory.

**The s-cobordism theorem and Whitehead torsion:** In high dimensions, namely in dimensions greater than 4, the s-cobordism theorem is crucial in deciding whether two closed smooth manifolds are diffeomorphic. An *h-cobordism* between two closed manifolds $M$ and $N$ is a compact manifold $W$ with $\partial W = M \amalg N$, so that the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ are homotopy equivalences. Diffeomorphic manifolds are h-cobordant. Hence a natural assumption for $M$ to be diffeomorphic to $N$ is the existence of an h-cobordism $W$ between them. Now the natural question is whether $W$ is trivial, i.e. diffeomorphic to $M \times [0,1]$. This is decided by an algebraic obstruction $\tau(W, M) \in Wh(\pi_1(M))$–the *Whitehead torsion* of the homotopy equivalence $M \leftrightarrow W$, where $Wh(\pi_1(M)) = K_1(\mathbb{Z}[\pi_1(M)])/\pm \pi_1(M)_{ab}$ is the *Whitehead group* of
$\pi_1(M)$. The $s$-cobordism theorem, which is due to Barden, Mazur and Stallings, says $W$ is trivial iff $\tau(W,M)$ vanishes, provided $\dim(M) \geq 5$. See [52] for a proof.

One sees from the above examples that $K_i(\mathbb{Z}[\pi_1(X)]), L_i(\mathbb{Z}[\pi_1(X)])$ play a crucial role in geometric topology. However these groups are in general extremely hard to compute. Nevertheless, we will see in Section 1.2 that the Farrell-Jones conjecture provides a way towards the understanding and computation of these groups.

1.1.2 Borel and Novikov Conjectures

Borel conjecture and Novikov conjecture are the remaining two most famous unsolved conjectures in manifold topology. Using surgery theory, one can show that the Farrell-Jones conjecture implies these two conjectures. We briefly recall the two conjectures here and refer the interested readers to [54, Section 2.2.6], [48, Section 23] for discussions of the connections between these conjectures.

**Borel Conjecture:** Let $M$ and $N$ be two closed aspherical manifolds. If $\pi_1(M) \cong \pi_1(N)$, then any homotopy equivalence between $M$ and $N$ is homotopic to a homeomorphism.

**Novikov Conjecture:** Let $M$ be a closed oriented smooth manifold and $G$ be any group. Then the higher signatures of $M$ at any cohomology class $u \in H^*(BG; \mathbb{Q})$ is homotopy invariant relative to any continuous map $f : M \to BG$: for any orientation preserving homotopy equivalence $g : N \to M$ of closed oriented smooth manifolds, we have the equality of higher signatures:

\[ < L(N) \cup (f \circ g)^*(u), [N] > = < L(M) \cup f^*(u), [M] > \]

where $L(M) \in H^*(M), L(N) \in H^*(N)$ are the $L$-classes of $M$ and $N$ respectively, and $[M], [N]$ are their fundamental classes, compatible with the chosen orientations.
1.1.3 Other Applications

The Farrell-Jones Conjecture also implies a number of other conjectures in geometry, topology and algebra. For example, it implies Bass-, Serre-, and Kaplansky-conjectures. For a more detailed discussion, see [54], [6, Section 1.2].

1.2 Formulation of the Conjecture

We start by some discussion on the heuristic idea about the formulation of the conjecture in Section 1.2.1 and then formulate the conjecture in its most general form in Section 1.2.2.

1.2.1 Heuristic Idea

Let $R$ be an associative ring with unit and $G$ be any group. One wants to understand various algebraic $K$- and $L$-groups, $K_i(R[G]), L_i(R[G]), i \in \mathbb{Z}$, of the group ring $R[G]$ (the case when $R = \mathbb{Z}$ and $G = \pi_1(X)$ is of particular importance in geometric topology, as one sees from the previous section). Let us focus on the discussion of $K$-groups for simplicity (the discussion for $L$-groups is completely parallel). A first guess together with some evidence suggest that $K_i(R[G])$ is approximately a combination of the $K$-theory of $R$ and the homology of $G$, i.e.

$$K_i(R[G]) \approx K_i(R) + H_i(G), i \in \mathbb{Z}$$

This leads to the consideration of the generalized homology theory $H_*(-; K_R)$ determined by the non-connective algebraic $K$-theoretic ring spectrum $K_R$ (see Appendix B). This is a generalized homology theory (satisfying the Eilenberg-Steenrod axioms) whose coefficient groups are given by $H_*(pt, K_R) = \pi_i(K_R) = K_i(R), i \in \mathbb{Z}$. See Appendix A for more detail about spectrum and construction of generalized homology.
theory. One then expects \(H_*(BG; K_R)\) computes \(K_*(R[G])\), where \(BG\) is the classifying space for \(G\). A special case of the Farrell-Jones conjecture says that this should be the case when \(G\) is torsion free and \(R\) is regular, namely it is Noetherian and every finitely generated left \(R\)-module has a finite dimensional resolution by finitely generated projective left \(R\)-modules. Every principal ideal domain is regular. In particular, \(Z\) is regular.

However when \(G\) is not torsion free or \(R\) is not regular, this is not true. A simple counterexample is given by \(G = Z\) and \(R\) non-regular. In this case, a simple calculation shows that \(H_i(BZ; K_R) = H_i(S^1; K_R) = K_i(R) \oplus K_{i-1}(R)\), but by the Bass-Heller-Swan formula, \(K_i(R[Z]) = K_i(R) \oplus K_{i-1}(R) \oplus NK_i(R) \oplus NK_{i-1}(R), i \in \mathbb{Z}\), where \(NK_i(R)\) is the Bass Nil-group of the ring \(R\), which can well be non-zero if \(R\) is non-regular (see Section 1.4.3 for definitions of Nil-groups). Hence, in general, \(H_*(BG; K_R)\) misses some part of \(K_*(R[G])\) due to the existence of Nil-groups.

**Remark 1.2.1.** We also see from this example that why one has to use the non-connective ring spectrum: \(K_*(R[Z])\) depends not only on \(K_*(R)\), but also on the \(K\)-group one dimension below, due to the Bass-Heller-Swan formula. A similar reason shows one has to use the \(L\)-theory spectrum with *decoration* \(-\infty\) in the \(L\)-theoretic Farrell-Jones conjecture, due to the Shaneson Splitting [65]. A quick and nice introduction to algebraic \(L\)-groups and the issue of decorations can be found in [48, Section 17]. A discussion on the role of decorations and Shaneson Splittings in the Farrell-Jones conjecture can be found in [48, Section 21].

Now more evidence [25]–[35] suggests that \(K_i(R[G])\) should indeed be a combination of the homology of \(G\) and the \(K\)-theory of \(R[V]\), where \(V < G\) ranges over the virtually cyclic subgroups of \(G\) (a virtually cyclic group is a group containing a cyclic
subgroup of finite index), i.e.

\[ K_i(R[G]) \approx \sum_{V < G \text{ virtually cyclic}} K_i(R[V]) + H_i(G), i \in \mathbb{Z} \]

Then the idea is to promote the generalized homology theory \( H_*(-; K_R) \) to an equivariant generalized homology theory \( H^*_*(-; K_R) \) with coefficient in \( K_R \): for each group \( G \), \( H^*_G(-; K_R) \) is a \( G \)-equivariant generalized homology theory (satisfying the equivariant version of the Eilenberg-Steenrod axioms, see more detail in Appendix A), so that the coefficients are given by \( H^*_G(G/H; K_R) = K_*(R[H]) \) for any coset space \( G/H \). In particular, at a point, we have \( H^*_G(pt; K_R) = K_*(R[G]) \). Moreover, one should have \( H^*_G(EG; K_R) \cong H_*(BG; K_R) \). Now one expects \( H^*_G(E_{VC}G; K_R) \) computes the \( K \)-theory of the group ring \( R[G] \) and the Farrell-Jones conjecture predicts this is the case. Here \( E_{VC}G \) is the classifying space of the group \( G \) relative to the family of virtually cyclic subgroups of \( G \). See the following section for a definition.

### 1.2.2 Davis-Lück and Bartels-Reich Constructions

The conjecture was firstly formulated by Farrell and Jones in their famous paper [36] using the notion of homology theories with coefficient in twisted and stratified \( \Omega \)-spectra, see [60] for more detail about this notion. Davis and Lück [19] reformulated the conjecture using the language of spaces and spectra over a category, which they developed in the same paper. Bartels and Reich [13] then, using the Davis-Lück construction, extended the conjecture to allow for coefficients in any additive category.

We now briefly describe their constructions. Let \( G \) be a discrete group and \( \text{Or}G \) the associated orbit category, whose objects are homogeneous \( G \)-sets and morphisms are \( G \)-equivariant maps. For any spectrum over the category \( \text{Or}G \), i.e. any functor \( E: \text{Or}G \to \text{Sp} \), where \( \text{Sp} \) is the category of spectra, Davis and Lück [19] constructed
a $G$-equivariant homology theory $H^G_*(-; E)$, see Appendix A for more details about the construction. Now for any small additive category $\mathcal{A}$ with a right $G$-action (as covariant additive functors), Bartels and Reich [13, Definition 2.1] constructed a functor $F_\mathcal{A}: \text{Or} G \to \mathcal{A}$, from the orbit category to the category of small additive categories. Then the non-connective $K$-theory spectrum functor $K^{-\infty}: \mathcal{A} \to \text{Sp}$, constructed by Pedersen and Weibel in [56], composed with $F_\mathcal{A}$, gives a functor $K_\mathcal{A}: \text{Or} G \to \text{Sp}$. When $\mathcal{A}$ is an additive $G$-category with involution, then a similar construction using the $L$-theory spectrum functor (with decoration $-\infty$) $L^{-\infty}: \mathcal{A}^* \to \text{Sp}$, constructed by Ranicki [62], gives another functor $L_\mathcal{A}: \text{Or} G \to \text{Sp}$. The Davis-Lück construction then applies to these two functors and gives two $G$-equivariant homology theories $H^G_*(-; K_\mathcal{A}), H^G_*(-; L_\mathcal{A})$. Basic properties of the two functors $K^{-\infty}: \mathcal{A} \to \text{Sp}$, $L^{-\infty}: \mathcal{A}^* \to \text{Sp}$ are recalled in Appendix B.

For $\mathcal{F}$, a family of subgroups of $G$ which is closed under taking subgroups and conjugations, denote by $E_{\mathcal{F}} G$ the classifying space for $G$ relative to the family $\mathcal{F}$. It is characterized, up to $G$-equivariant homotopy equivalence, by the universal property that, for any $G$-CW complex $X$ with isotropies in $\mathcal{F}$, there is a $G$-equivariant map $X \to E_{\mathcal{F}} G$, which is unique up to $G$-equivariant homotopy. See [53] for a survey on this subject.

**Farrell-Jones Conjecture.** For any group $G$, any additive category (with involution) $\mathcal{A}$ with a right $G$-action and $n \in \mathbb{Z}$, the $K$-theoretic assembly map

$$A^K_{\mathcal{VC}} : H^G_n(E_{\mathcal{VC}} G; K_\mathcal{A}) \to H^G_n(pt; K_\mathcal{A})$$

and the $L$-theoretic assembly map

$$A^L_{\mathcal{VC}} : H^G_n(E_{\mathcal{VC}} G; L_\mathcal{A}) \to H^G_n(pt; L_\mathcal{A})$$

(1.2.1)
induced by the obvious map $E_{VC}G \to pt$, are isomorphisms.

**Remark 1.2.2.** When $\mathcal{A}$ is the additive category (with involution) of finitely generated free $R$-modules of a ring $R$ (with involution) with trivial $G$-action, then the right hand side of the assembly maps 1.2.1, 1.2.2 become the familiar algebraic $K$- and $L$-groups (with decoration $-\infty$) of the group ring, namely $K_n(R[G])$ and $L^{<\infty}_n(R[G])$. The advantage of allowing coefficients in any additive category lies in several places. Firstly, it has better inheritance properties. For example, if the conjecture holds for a group $G$ with coefficients in any additive category, then it holds for any of its subgroups with coefficients in any additive category, see [13, Section 4][6, Section 1.3] for more inheritance properties of the conjecture. Secondly, it contains the case of the conjecture with twisted coefficient ring [13, Example 2.6]. Thirdly, it implies the fibered Farrell-Jones conjecture, see [13, Remark 4.4].

**Remark 1.2.3.** The right hand side of the assembly maps 1.2.1, 1.2.2 are generally very hard to compute, in contrast, there are many tools available for the computation of the left hand side of the assembly maps, since $H^G_\ast(-;K_R), H^G_\ast(-;L_R)$ are homology theories. For example, there are Atiyah-Hirzebruch spectral sequences available for these homology theories. Therefore the conjecture provides a way for the computation of various $K$- and $L$-groups. Most importantly, the assembly maps 1.2.1, 1.2.2 naturally fit into the framework of surgery theory, so that it implies the famous Novikov and Borel conjectures.

If $E_{VC}G$ is replaced by $E_{\mathcal{F}}G$ in the assembly maps 1.2.1,1.2.2, then one gets the Farrell-Jones conjecture for the group $G$ relative to the family of subgroups $\mathcal{F}$. The importance of this lies in the following transitivity principle (see [36, Theorem A.10][54, Theorem 65]):
Transitivity Principle 1.2.4. If $A^K_F$ is an isomorphism and every group in $\mathcal{F}$ satisfies the $K$-theoretic Farrell-Jones conjecture 1.2.1, then $G$ satisfies the $K$-theoretic Farrell-Jones conjecture. The same holds for the $L$-theoretic Farrell-Jones conjecture.

The conjecture has been proved for some important classes of groups including Gromov hyperbolic groups [11][8], CAT(0)-groups [8][70], cocompact lattices in virtually connected Lie groups [6], virtually polycyclic groups [6], virtually solvable groups [71], etc. However it still remains open for many important classes of groups, including mapping class groups, amenable groups, Thompson groups, $Out(F_n)$, etc.

1.3 Methods for Proving the Conjecture

Bartels, Lück and Reich developed a general framework, scattered in various papers, including [7][8][10][11] etc, for proving the conjecture, with some of their ideas very much influenced by the works of Farrell, Hsiang and Jones [23][24][27][28][29] etc. This section is devoted to an outline of their framework.

1.3.1 Identifying the Homotopy Fiber: A Controlled Algebra Approach

The main idea is to firstly interpret the assembly maps 1.2.1,1.2.2 as maps of homotopy groups induced by certain maps between certain spectra, and then to prove the homotopy fibers of the maps are homotopically trivial. This is achieved by a controlled algebra approach to the conjecture, which was firstly done in [5] with coefficient in a ring $R$ and then was generalized to coefficients in any additive category in [13]. Let us now briefly describe the construction here.

Let $\mathcal{A}$ be a (small) additive category with right $G$-action and $X$ be a left $G$-space. The additive category $\mathcal{C}(X;\mathcal{A})$ of geometric modules over $X$ with coefficient in $\mathcal{A}$ is
defined as follows: objects are functions $A : X \to \text{Ob}(A)$ with \textit{locally finite support}, i.e. $\text{supp} A = \{ x \in X | A_x \neq 0 \}$ is a locally finite subset of $X$, meaning every point in $X$ has an open neighborhood whose intersection with $\text{supp} A$ is finite. An object will usually be denoted by $A = (A_x)_{x \in X}$. A morphism $\phi : A \to B$ is a matrix of morphisms $(\phi_{y,x} : A_x \to B_y)_{(x,y) \in X \times X}$ such that there are only finitely many nonzero entries in each row and each column. Compositions of morphisms are given by matrix multiplications. There is a right $G$-action on $\mathcal{C}(X; A)$ which is given by

$$(g^* A)_x = g^*(A_{gx}), \quad (g^* \phi)_{y,x} = g^*(\phi_{gy,gx})$$

and the fixed category is denoted by $\mathcal{C}^G(X; A)$.

For any object $A$ and morphism $\phi$ in $\mathcal{C}(X; A)$, their supports are the following sets:

$$\text{supp} A = \{ x \in X | A_x \neq 0 \}, \quad \text{supp} \phi = \{ (x,y) \in X \times X | \phi_{y,x} \neq 0 \}$$

In order to get interesting subcategories of $\mathcal{C}(X; A)$, one can prescribe certain support conditions on objects and morphisms. The convenient language for this purpose is coarse structures on spaces introduced in [45]. We recall the definition here (with a slight modification for convenience).

**Definition 1.3.1.** A \textit{coarse structure} $(\mathcal{E}, \mathcal{F})$ on a set $X$ is a collection $\mathcal{E}$ of subsets of $X \times X$ and a collection $\mathcal{F}$ of subsets of $X$ satisfying the following properties:

1. If $E', E'' \in \mathcal{E}$, then $E' \cup E'' \subset E$ for some $E \in \mathcal{E}$;
2. If $E', E'' \in \mathcal{E}$, then $E' \circ E'' = \{ (x,y) \in X \times X | \exists z \in X \text{ s.t.} (x,z) \in E' \text{ and } (z,y) \in E'' \} \subset E$ for some $E \in \mathcal{E}$;
3. The diagonal $\Delta = \{ (x,x) | x \in X \}$ is contained in some $E \in \mathcal{E}$;
4. If $F', F'' \in \mathcal{F}$, then $F' \cup F'' \subset F$ for some $F \in \mathcal{F}$.
If there is a $G$-action on the set $X$, we then require every member in $\mathcal{E}$ and $\mathcal{F}$ to be $G$-invariant, where $G$ acts on $X \times X$ diagonally. If $p : Y \to X$ is a $G$-equivariant map, then the pullback $((p \times p)^{-1}\mathcal{E}, p^{-1}\mathcal{F})$ is a coarse structure on $Y$.

Now if $(\mathcal{E}, \mathcal{F})$ is a coarse structure on a $G$-space $X$, one can define a subcategory $\mathcal{C}(X, \mathcal{E}, \mathcal{F}; \mathcal{A})$ of $\mathcal{C}(X; \mathcal{A})$, with object and morphism supports contained in members of $\mathcal{F}$ and $\mathcal{E}$ respectively, in addition to the finiteness conditions on them. $G$ acts on this additive subcategory and the fixed subcategory is denoted by $\mathcal{C}^G(X, \mathcal{E}, \mathcal{F}; \mathcal{A})$. The pair $(\mathcal{E}, \mathcal{F})$ are also usually referred to as control conditions on morphisms and objects. We will omit $\mathcal{A}$ from the notation if everything is clear.

The control condition on morphisms used to construct a model for the Farrell-Jones conjecture is the equivariant continuously controlled condition introduced in [5], which is a generalization to the equivariant setting of the continuously controlled condition introduced in [2]. We recall the definitions here.

**Definition 1.3.2.** (Equivariant continuous control) Let $X$ be a topological space with a $G$-action by homeomorphisms. A subset $E \subset (X \times [1, \infty))^2$ is called equivariantly continuously controlled if the following holds:

1. For every $x \in X$ and every $G_x$-invariant open neighborhood $U$ of $(x, \infty)$ in $X \times [1, \infty]$, there exists a $G_x$-invariant neighborhood $V \subset U$ of $(x, \infty)$ such that $U^c \times V \cap E = \emptyset$, where $U^c$ is the complement of $U$ in $X \times [1, \infty]$ and $G$-acts on $X \times [1, \infty]$ diagonally with trivial action on $[1, \infty]$;

2. There exists $\alpha > 0$, depending on $E$, such that if $(x, t) \times (x', t') \in E$, then $|t - t'| < \alpha$;

3. $E$ is symmetric, i.e. if $(p, q) \in E$, then $(q, p) \in E$;

4. $E$ is invariant under the diagonal action of $G$. 

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The collection of equivariantly continuously controlled subsets of \((X \times [1, \infty))^2\) will be denoted by \(E^X_{Gcc}\) and it satisfies the conditions (1)-(3) in definition 1.3.1.

**Definition 1.3.3.** For any \(G\)-space \(X\) and additive category \(\mathcal{A}\) with right \(G\)-action, one defines the following categories:

1. \(\mathcal{O}(X) = \mathcal{C}(G \times X \times [1, \infty), (p \times p)^{-1}E^X_{Gcc}, q^{-1}F_{Gc})\), where \(p : G \times X \times [1, \infty) \to X \times [1, \infty)\) and \(q : G \times X \times [1, \infty) \to G \times X\) are projections and \(F_{Gc}\) consists of \(G\)-cocompact subsets of \(G \times X\). We will omit \((p \times p)^{-1}\) and \(q^{-1}\) from the notation later on;

2. \(\mathcal{T}(X)\) is the full subcategory of \(\mathcal{O}(X)\) consisting of those objects \(A\) with the following property: there exists \(n > 1\) such that if \(A_{(g,x,t)} \neq 0\), then \(t < n\);

3. \(\mathcal{D}(X)\) is the quotient category of \(\mathcal{O}(X)\) by the full subcategory \(\mathcal{T}(X)\): it has the same objects as \(\mathcal{O}(X)\), and any morphism from \(A\) to \(B\) in \(\mathcal{D}(X)\) is represented by a morphism \(\phi : A \to B\) in \(\mathcal{O}(X)\), with two morphisms \(\phi, \psi : A \to B\) identified if their difference \(\phi - \psi\) factors through an object in \(\mathcal{O}(X)\), whose support is contained in \(G \times X \times [1, n]\) for some finite \(n\);

4. Every subgroup \(H < G\) acts on these three categories and the fixed categories will be denoted by \(\mathcal{O}_G^H(X), \mathcal{T}_G^H(X)\) and \(\mathcal{D}_G^H(X)\) respectively. For \(H = G\), we will omit the subscript \(G\) from the definition.

**Remark 1.3.4.** Note \(\mathcal{O}_G^H(X)\) is different from \(\mathcal{O}_G^H(X)\). By definition, \(\mathcal{O}_G^H(X) = \mathcal{C}^H(H \times X \times [1, \infty), (p \times p)^{-1}E^X_{Hcc}, q^{-1}F_{Hc})\), here the \(H\)-action on \(X\) and \(\mathcal{A}\) are induced from the \(G\)-action on them. But \(\mathcal{O}_G^H(X) = \mathcal{C}^H(G \times X \times [1, \infty), (p \times p)^{-1}E^X_{Gcc}, q^{-1}F_{Gc})\).

These constructions define functors from the category of \(G\)-CW complexes to the category of additive categories. The importance of these categories lies in the following facts:
**Fact 1.3.5.** (1) The sequence

\[ \mathcal{T}^G(X) \to \mathcal{O}^G(X) \to \mathcal{D}^G(X) \]

is a Karoubi filtration, hence gives rise to a fibration sequence

\[ \mathbb{K}^{-\infty}(\mathcal{T}^G(X)) \to \mathbb{K}^{-\infty}(\mathcal{O}^G(X)) \to \mathbb{K}^{-\infty}(\mathcal{D}^G(X)) \]

of spectra after applying the non-connective K-theory, and therefore a long exact sequence on K-groups;

(2) The additive category \( \mathcal{O}^G(pt) \) has trivial K-groups;

(3) The functor \( \pi_* : \mathcal{T}^G(X) \to \mathcal{T}^G(pt) \) induced by the obvious map \( \pi : X \to \{pt\} \) is an equivalence of categories, hence induces isomorphisms on K-groups;

(4) There is a natural isomorphism between the two functors \( H_*^G(\_, \mathbb{K}_A) \) and \( K_{*+1}(\mathcal{D}^G(\_)) := \pi_{i+1}(\mathbb{K}^{-\infty}(\mathcal{D}^G(X))) \) from the category of \( G \)-CW complexes to the category of graded abelian groups. In particular, the map

\[ K_{*+1}(\mathcal{D}^G(\text{VC}_G)) \to K_{*+1}(\mathcal{D}^G(pt)) \quad (1.3.1) \]

is a model for the assembly map 1.2.1.

For information about Karoubi filtrations, see [15]. Fact (2) can be proved by an Eilenberg swindle argument. Fact (3) can be checked directly. Fact (4) is firstly proven in [5] for additive categories of finitely generated free modules over rings, and proven for arbitrary coefficients in [13]. One has the same constructions and results for L-theory and details can be found in [8]. The above facts implies the following:

**Theorem 1.3.6.** For any \( G \)-CW complex, \( \mathbb{K}^{-\infty}(\mathcal{O}^G(X)) \) is the homotopy fiber of the map \( \mathbb{K}^{-\infty}(\mathcal{D}^G(X)) \to \mathbb{K}^{-\infty}(\mathcal{D}^G(pt)) \). Hence the assembly map 1.2.1 is an isomorphism if and only if \( K_*(\mathcal{O}^G(\text{VC}_G)) = 0 \). The rationalized assembly map (by
tensoring with \( \mathbb{Q} \) over \( \mathbb{Z} \) is an isomorphism if and only if \( K_*(\mathcal{O}^G(E_{\mathcal{F}G})) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \).

The same result holds for the \( L \)-theoretic assembly map 1.2.2.

**Proof.** Proof for the \( K \)-theory part is essentially [11, Proposition 3.8] and the same argument applies to \( L \)-theory. See also [8, Theorem 5.2]. For convenience, we outline the proof here. Consider the following commutative diagram of long exact sequences of abelian groups

\[
\begin{array}{cccccccccc}
\cdots & \longrightarrow & K_i(T^G(X)) & \longrightarrow & K_i(O^G(X)) & \longrightarrow & K_i(D^G(X)) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & A & & \\
\cdots & \longrightarrow & K_i(T^G(pt)) & \longrightarrow & K_i(O^G(pt)) & \longrightarrow & K_i(D^G(pt)) & \longrightarrow & \cdots
\end{array}
\]

By Fact 1.3.5 (2), \( K_i(O^G(pt)) = 0 \), hence \( K_i(D^G(pt)) \cong K_{i-1}(T^G(pt)) \). But by Fact 1.3.5 (3), \( K_{i-1}(T^G(pt)) \cong K_{i-1}(T^G(X)) \), therefore \( K_i(D^G(pt)) \cong K_{i-1}(T^G(X)) \). We thus get a long exact sequence

\[
\begin{array}{cccccccccc}
\cdots & \longrightarrow & K_i(O^G(X)) & \longrightarrow & K_i(D^G(X)) & \longrightarrow & K_i(D^G(pt)) & \longrightarrow & \cdots
\end{array}
\]

which implies \( K^{-\infty}(O^G(X)) \) is the homotopy fiber of the map \( K^{-\infty}(D^G(X)) \rightarrow K^{-\infty}(D^G(pt)) \).

Because of this theorem, the category \( O^G(X) \) is usually referred to as the obstruction category. Of course, if \( K_*(\mathcal{O}^G(E_{\mathcal{F}G})) = 0 \) for some family \( \mathcal{F} \), then the Farrell-Jones conjecture holds for \( G \) relative to \( \mathcal{F} \).

### 1.3.2 General Strategy for Proving Homotopy Triviality of the Homotopy Fiber

In this subsection, we briefly outline a general strategy for proving \( K_*(\mathcal{O}^G(E_{\mathcal{F}G})) = 0 \). More details can be found in the references mentioned at the beginning of this section.
General Strategy 1.3.7. (1) For any metric space $Z$ with an isometric $G$-action, introduce $Z$ into the obstruction category, so to get $\mathcal{O}^G(E_F G, Z)$, where $\mathcal{O}^G(E_F G, Z)$ is the additive category of certain geometric modules over $G \times E_F G \times [1, \infty) \times Z$ with coefficient in the fixed additive category $\mathcal{A}$. This construction is functorial in $Z$ in an appropriate sense. In particular, when $Z$ is a point, we get back the original obstruction category $\mathcal{O}^G(E_F G)$;

(2) For any cycle $c \in K_i(\mathcal{O}^G(E_F G, Z))$, assign a “size” to it in the $Z$ direction, so that when $Z$ is nice, a $G$-simplicial complex with stabilizers in $\mathcal{F}$ for example, and the size of $c$ is very small, we get $c = 0$;

(3) Find a “nice” metric space $Y$ so that $G$ acts “nicely”. For example, $Y$ is a CAT(0)-space with a proper and cocompact isometric $G$-action.

(4) Produce a simplicial complex $\Sigma$ with a simplicial $G$-action out of the $G$-space $Y$ together with a contracting $G$-map $F: Y \to \Sigma$, where $\Sigma$ is endowed with a certain metric and contracting means decreasing distance in certain sense. $\Sigma$ will usually be the nerve of certain $G$-equivariant open cover of $Y$;

(5) Form a commutative diagram

$$
\begin{array}{ccc}
K_*(\mathcal{O}^G(E_F G, Y)) & \xrightarrow{F_*} & K_*(\mathcal{O}^G(E_F G, \Sigma)) \\
\downarrow{p_*} & & \downarrow{q_*} \\
K_*(\mathcal{O}^G(E_F G)) & & \\
\end{array}
$$

where $p_*, q_*, F_*$ are induced by $p: Y \to pt$, $q: \Sigma \to pt$ and $F: Y \to \Sigma$ respectively.

(6) For any cycle $c \in K_*(\mathcal{O}^G(E_F G))$, construct a transfer map $\tau_c: \mathcal{O}^G(E_F G) \to \mathcal{O}^G(E_F G, Y)$, so that the diagram commutes and after applying the contracting map $F$, the size of $\tau_c(c)$ becomes very small, which forces $F_*(\tau_c(c)) = 0$ and proves $c = 0$.

A similar strategy also applies to prove the $L$-theoretic Farrell-Jones conjecture.
1.3.3 Axiomatic Formulations Running the General Strategy

In this subsection, we record some axiomatic formulations of conditions on groups which run the general strategy described in the previous section.

**Transfer Reducibility.** In [8], Bartels and Lück gave an axiomatic formulation for groups to satisfy the $K$-theoretic Farrell-Jones conjecture up to dimension one and the $L$-theoretic Farrell-Jones conjecture in all dimensions. This is the notion of *transfer reducibility* of a group relative a family of subgroups (an earlier version of this notion had already appeared in [11], in which the authors proved the $K$-theoretic Farrell-Jones conjecture for hyperbolic groups). It had already been proven that CAT(0)-groups in [7] and Hyperbolic groups (essentially in [10]) are transfer reducible relative to the family of virtually cyclic subgroups, hence the Farrell-Jones conjecture (up to dimension 1) holds for these groups. Wegner [70] generalized the concept of transfer reducibility to the notion of *strong transfer reducibility* and showed that groups that are strongly transfer reducible relative to a family of subgroups satisfy the $K$-theoretic Farrell-Jones conjecture in all dimensions relative to this family of subgroups. He thus proved the $K$-theoretic Farrell-Jones conjecture in all dimensions for CAT(0)-groups by proving CAT(0)-groups are strongly transfer reducible relative to the family of virtually cyclic subgroups. We now recall the definition of strongly transfer reducibility which will be needed later. To do this, we have to first recall several related notions.

**Definition 1.3.8.** ([70, Definition 2.7]) A *strong homotopy action* of a group $G$ on a space $X$ is a continuous map

$$
\Psi : \prod_{j=0}^{\infty} ((G \times [0,1])^j \times G \times X) \to X
$$

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with the following properties:

(i) \( \Psi(\cdots, g_l, 0, g_{l-1}, \cdots) = \Psi(\cdots, g_l, \Psi(g_{l-1}, \cdots)) \)

(ii) \( \Psi(\cdots, g_l, 1, g_{l-1}, \cdots) = \Psi(\cdots, g_l g_{l-1}, \cdots) \)

(iii) \( \Psi(e, t_j, g_{j-1}, \cdots) = \Psi(g_{j-1}, \cdots) \)

(iv) \( \Psi(\cdots, t_l, e, t_{l-1}, \cdots) = \Psi(\cdots, t_l t_{l-1}, \cdots) \)

(v) \( \Psi(\cdots, t_1, e, x) = \Psi(\cdots, x) \)

(vi) \( \Psi(e, x) = x \)

where \( e \in G \) is the identity element.

**Definition 1.3.9.** ([70, Definition 2.3]) Let \( \Psi \) be a strong homotopy action of \( G \) on a metric space \((X, d_X)\). Let \( S \subseteq G \) be a finite symmetric subset containing the trivial element \( e \in G \). Let \( k \in \mathbb{N} \) be a natural number.

(i) For \( g \in G \), we define \( F_g(\Psi, S, k) \subseteq \text{map}(X, X) \) by

\[
F_g(\Psi, S, k) := \{ \Psi(g_k, t_k, \cdots, g_0, ?) : X \to X | g_k \in S, t_i \in [0, 1], g_k g_{k-1} \cdots g_0 = g \}
\]

(ii) For \( (g, x) \in G \times X \) we define \( S_{\Psi, S, k}^1(g, x) \subseteq G \times X \) as the subset consisting of all \( (h, y) \in G \times X \) with the following property: there are \( a, b \in S \), \( f \in F_a(\Psi, S, k) \) and \( \tilde{f} \in F_b(\Psi, S, k) \) such that \( f(x) = \tilde{f}(y) \) and \( h = ga^{-1}b \). For \( n \geq 2 \) we set

\[
S_{\Psi, S, k}^n(g, x) := \{ S_{\Psi, S, k}^1(h, y) | (h, y) \in S_{\Psi, S, k}^{n-1}(g, x) \}
\]

We will also need the following:

**Definition 1.3.10.** ([8, Definition 1.5]) Let \( X \) be a metric space and \( N \in \mathbb{N} \) be a natural number. \( X \) is said to be **controlled \( N \)-dominated** if for every \( \epsilon > 0 \), there is a finite \( CW \)-complex \( K \) of dimension at most \( N \), maps \( i : X \to K, r : K \to X \) and a homotopy \( H \) between \( r \circ i \) and \( 1_X \), so that for every \( x \in X \), the diameter of \( \{ H(x, t) | t \in [0, 1] \} \) is at most \( \epsilon \).
**Definition 1.3.11.** ([70, Definition 3.1]) Let $\mathcal{F}$ be a family of subgroups of the group $G$. $G$ is called strongly transfer reducible relative to $\mathcal{F}$ if there exists a natural number $N \in \mathbb{N}$ with the following property: for every finite symmetric subset $S \subseteq G$ containing the trivial element $e \in G$ and every natural numbers $k, n \in \mathbb{N}$, there are

(i) a compact contractible controlled $N$-dominated metric space $X$;

(ii) a strong homotopy $G$-action $\Psi$ on $X$;

(iii) a cover $\mathcal{U}$ of $G \times X$ by open subsets;

such that

(i) $\mathcal{U}$ is an open $\mathcal{F}$-cover, meaning $\mathcal{U}$ is $G$-equivariant, either $gU = U$ or $gU \cap U = \emptyset$, and $G_U \in \mathcal{F}, \forall U \in \mathcal{U}, g \in G$, with respect to the action of $G$ on $G \times X$ by $g(h, x) = (gh, x)$;

(ii) $\dim(\mathcal{U}) \leq N$, where $\dim(\mathcal{U})$ denotes the covering dimension;

(iii) for every $(g, x) \in G \times X$ there exists $U \in \mathcal{U}$ with $S^U_{\Psi, S, k}(g, x) \subseteq U$.

**Theorem 1.3.12.** ([8, Theorem 1.1] [70, Theorem 1.1]) Suppose a group $G$ is strongly transfer reducible with respect to a family of subgroups $\mathcal{F}$. Let $\mathcal{F}_2$ be the family of subgroups consisting of $V < G$ so that there is $F \in \mathcal{F}$ with $[V : F] \leq 2$, then:

(i) The $K$-theoretic Farrell-Jones conjecture 1.2.1 holds for $G$ relative to $\mathcal{F}$;

(ii) The $L$-theoretic Farrell-Jones conjecture 1.2.2 holds for $G$ relative to $\mathcal{F}_2$;

In particular, if $\mathcal{F} = \mathcal{V} \mathcal{C}$, then $\mathcal{F}_2 = \mathcal{V} \mathcal{C}$ and the Farrell-Jones conjecture holds for $G$.

**Remark 1.3.13.** As already mentioned, the proof of the above theorem runs the General Strategy 1.3.7.

**Farrell-Hsiang Method.** Bartels and Lück introduced the notion of Farrell-Hsiang Group relative to a family of subgroups in [9], which is motivated by the work of Farrell
and Hsiang [23]. They showed in [9] that if $G$ is a Farrell-Hsiang group relative to a family of subgroups $\mathcal{F}$, then the Farrell-Jones conjecture holds for $G$ relative to $\mathcal{F}$ with coefficient in any additive category. The proof also follows the General Strategy 1.3.7. This notion is then used by Bartels, Lück and Farrell to prove the conjecture for cocompact lattices in virtually connected Lie groups in [6], and by Farrell and Wu to prove the conjecture for solvable Baumslag-Solitar groups in [39]. The notion of Farrell-Hsiang group was then generalized by Wegner to generalized Farrell-Hsiang group to prove the conjecture for virtually solvable groups in [71]. Although this notion will not be used in our treatment, we record the definition here for convenience.

**Definition 1.3.14.** ([9, Definition 1.1]) Let $\mathcal{F}$ be a family of subgroups of a finitely generated group $G$. $G$ is called a Farrell-Hsiang group relative to $\mathcal{F}$ if the following holds for some fixed word metric $d_G$ on $G$:

There exists a natural number $N$ so that for every natural number $n$ there is a surjective homomorphism $\alpha_n : G \to F_n$ with $F_n$ a finite group such that the following condition is satisfied. For any hyperelementary subgroup $H$ of $F_n$ we set $\overline{H} := \alpha_n^{-1}(H)$ and require that there exists a simplicial complex $E_H$ of dimension at most $N$ with a cell preserving simplicial $\overline{H}$-action whose stabilizers belong to $\mathcal{F}$, and an $\overline{H}$-equivariant map $f_H : G \to E_H$ such that $d_G(g, h) < n$ implies $d_{E_H}^1(f_H(g), f_H(h)) < 1/n$ for all $g, h \in G$, where $d_{E_H}^1$ is the $l^1$-metric on $E_H$.

A finite group $H$ is said to be hyperelementary if it can be written as an extension

$1 \to C \to H \to P \to 1$, where $C$ is cyclic and $P$ is a $p$-group for some prime $p$.

**Theorem 1.3.15.** ([9, Theorem 1.2]) If $G$ is a Farrell-Hsiang group relative to a family of subgroups $\mathcal{F}$, then the Farrell-Jones conjecture holds for $G$ relative to $\mathcal{F}$ with coefficient in any additive category.
1.4 Main Results

The results of this thesis are threefold, corresponding to three different aspects of the Farrell-Jones conjecture, which are treated in the following three chapters respectively. In Chapter 2, we study the conjecture for groups admitting nice but not necessarily proper actions on CAT(0)-spaces, i.e. stabilizers can be infinite (the case when the stabilizers are finite, i.e. CAT(0)-groups, has already been proved [8][70]). A natural question is if the point stabilizers of a group action satisfy the conjecture, whether the original group satisfies the conjecture. For this, we introduce a new notion of group action which generalizes proper action. We call it hyperdiscrete group action and there are many interesting examples besides proper actions. It turns out this new notion of group actions mostly fits into the framework for proving the conjecture developed by Bartels, Lück and Reich, which was outlined in the previous section. In Chapter 3, we study inheritance properties of the conjecture. It is known that if a group satisfies the conjecture, then any of its subgroup also satisfies the conjecture. We study the other direction: if a group has a subgroup of finite index satisfying the conjecture, whether the group itself satisfies the conjecture. We reduce the problem to a special case and results obtained for this special case strongly suggests the rational Farrell-Jones conjecture is inherited by over-groups of finite index. In Chapter 4, we study Farrell Nil-groups associated to a virtually cyclic group, which are the obstructions to reducing the family of virtually cyclic groups used in the Farrell-Jones conjecture to the family of finite groups. These groups are mysterious and very little about them was previously known. We however, for the first time, obtain some structure theorems for these groups. Results obtained in this chapter are joint work with Jean Lafont and Stratos Prassidis.
1.4.1 Main Results of Chapter 2

The first part of this chapter is the introduction and study of the notion of hyper-discrete group action (Definition 2.1.1). It is a generalization of proper action. While hyperdiscrete actions share many nice properties of proper actions, the class of groups admitting such actions is considerably larger than the class of groups admitting proper actions. This is guaranteed by the main theorem of this first part:

Theorem 2.1.9. Let $G$ acts on a proper CAT(0)-space $X$ by isometries. If the action is discrete and uniformly locally finite (Definition 2.1.8), then it is hyperdiscrete.

This theorem provides many interesting examples of hyperdiscrete actions besides examples coming from proper actions. For example, if one starts with a finite graph of groups, whose edge groups are of finite index in the corresponding vertex groups, then the action of the fundamental group of this graph of groups on the corresponding Bass-Serre tree is hyperdiscrete. This also applies to the more general notion of complex of groups, see for example [14]. See also Corollary 2.1.10, Corollary 2.1.11.

The second part of this chapter is the application of hyperdiscrete action to the study of Farrell-Jones conjecture. The main theorem is:

Theorem 2.5.1. Let $X$ be a finite dimensional proper CAT(0)-space. Suppose $G$ admits an action on $X$ by isometries with compact quotient and the induced action of $G$ is discrete on $FS(X)$, hyperdiscrete and locally finite on $FS(X) - FS(X)^R$. Then

(1) The $K$-theoretic Farrell-Jones conjecture holds for $G$ relative to the family of subgroups $\mathcal{F}$;

(2) The $L$-theoretic Farrell-Jones conjecture holds for $G$ relative to the family of subgroups $\mathcal{F}_2$,
where $\mathcal{F}$ is the family of subgroups of $G$ generated by subgroups $F < G$ admitting a short exact sequence $1 \rightarrow G_x \rightarrow F \rightarrow \mathbb{Z} \rightarrow 1$ for some $x \in X$ and $\mathcal{F}_2$ is the family of subgroups consisting of $V < G$ so that there is $F \in \mathcal{F}$ with $[V : F] \leq 2$.

See Definition 2.2.1 for the notions of the flow space $FS(X)$ and its subspace $FS(X)^R$ associated to a metric space $X$.

An important corollary of this theorem is the following, which partially answers the question: if the point stabilizers of a group action satisfy the conjecture, whether the original group satisfies the conjecture.

**Corollary 2.5.2.** With the above assumptions, if in addition every point stabilizer of the $G$-action on $X$ is virtually solvable, then both of the $K$-and $L$-theoretic Farrell-Jones conjecture holds for $G$.

**Remark 1.4.1.** Of course, one cares about when the induced action of $G$ on $FS(X)$ satisfy the assumption of Theorem 2.5.1. Certainly, when the $G$-action on $X$ is proper, then the induced action on $FS(X)$ is proper, hence hyperdiscrete and locally finite. This reproves the conjecture for CAT(0)-groups. If $G$ surjects onto $H$ and $H$ acts on $X$ properly, then the induced action on $FS(X)$ is hyperdiscrete and locally finite, and the above theorem applies. If in addition the kernel of $G \rightarrow H$ is virtually solvable, then by the above corollary, $G$ satisfies the conjecture. However this can also be deduced by some inheritance properties of Farrell-Jones conjecture, see [6, Theorem 1.8]. In order to obtain more interesting examples, we have to weaken the assumption on the group action of $G$ on $X$. The most natural assumption is the action being discrete and uniformly locally finite. Quite nice result has been obtained under this assumption, see Proposition 2.2.3, see also Remark 2.2.4.
1.4.2 Main Results of Chapter 3

The Farrell-Jones conjecture has some nice inheritance properties, for example, if a group satisfies the conjecture, then any subgroup of the group also satisfies the conjecture [13]. Conversely, one could ask if a group has a subgroup of finite index satisfies the conjecture, whether the group itself satisfies the conjecture. This turns out to be a hard question. In Chapter 3, we first reduce the problem to a special case:

Lemma 3.1.1 The following statements are equivalent:

(1) Let $H < G$ be a subgroup of finite index. If $H$ satisfies the Farrell-Jones conjecture with coefficient in any additive category, then $G$ satisfies the conjecture with coefficient in any additive category;

(2) Let $G = N \rtimes F$ be the semi-direct product of a normal subgroup $N$ and a finite subgroup $F$. If $N$ satisfies the Farrell-Jones conjecture with coefficient in any additive category, then $G$ satisfies the conjecture with coefficient in any additive category.

We then use the controlled algebra approach to the study of the problem for the special case $G = N \rtimes F$. The main theorem we obtain is the following (recall the notations from Definition 1.3.3):

Theorem 3.3.1 Let $G = N \rtimes F$ with $F$ a finite group of order $n$ and $\mathcal{A}$ be a $G$-additive category (with involution). Then for any $G$-CW complex $X$, there is an additive functor $\Omega : \mathcal{O}^G(X; \mathcal{A}) \to \mathcal{O}^G(X; \mathcal{A})$ with the following properties:

(1) $\Omega$ factors through $\mathcal{O}^N(X; \mathcal{A})$: there exist a restriction functor $\tau : \mathcal{O}^G(X; \mathcal{A}) \to \mathcal{O}^N(X; \mathcal{A})$ and an induction functor $\alpha : \mathcal{O}^N(X; \mathcal{A}) \to \mathcal{O}^G(X; \mathcal{A})$ so that $\Omega = \alpha \circ \tau$;

(2) $\Omega$ extends to a functor $\overline{\Omega} : \mathcal{O}^N_G(X; \mathcal{A}) \to \mathcal{O}^N_G(X; \mathcal{A})$, where $\mathcal{O}^N_G(X; \mathcal{A})$ is defined in Definition 1.3.3 (4). It is an additive category (with involution) containing $\mathcal{O}^G(X; \mathcal{A})$ as a subcategory, see also Remark 1.3.4;
(3) The extended functor \( \Omega \) is naturally isomorphic to the diagonal functor \( \Delta^n \): 
\[ \mathcal{O}^N_G(X; A) \to \mathcal{O}^N_G(X; A), \quad \Delta^n(A) = A^n \text{ on any object } A \text{ and } \Delta^n(\phi) = \phi^n \text{ on any morphism } \phi; \]

(4) On the level of \( K \)-groups, \( \Omega_* = n \times : K_i(\mathcal{O}^N_G(X; A)) \to K_i(\mathcal{O}^N_G(X; A)) \) is multiplication by \( n \) for all \( i \in \mathbb{Z} \). Therefore we have the following commutative diagram

\[ \begin{array}{ccc}
K_i(\mathcal{O}^N_G(X; A)) & \xrightarrow{\Omega_*} & K_i(\mathcal{O}^G(X; A)) \\
\downarrow \text{inc}_* & & \downarrow \text{inc}_* \\
K_i(\mathcal{O}^N_G(X; A)) & \xrightarrow{\Pi_* = n \times} & K_i(\mathcal{O}^N_G(X; A))
\end{array} \]

Remark 1.4.2. Part (4) of the above theorem is due to the additivity theorem for algebraic \( K \)-theory [59]. I could not find an additivity theorem for algebraic \( L \)-theory in the literature. If such a theorem can be proved, then part (4) of Theorem 3.3.1 would also hold for algebraic \( L \)-theory.

Remark 1.4.3. Theorem 3.3.1 is related to the original inheritance problem in the following way: if \( G = N \times F \), then the classifying space \( E_{VC}G \), viewed as an \( N \)-space, is also a model for the classifying space \( E_{VC}N \). If \( N \) satisfies \( K \)-theoretic Farrell-Jones conjecture, then by Theorem 1.3.6, \( K_i(\mathcal{O}^N(E_{VC}G; A)) = 0, \forall i \in \mathbb{Z} \). Hence, taking \( X = E_{VC}G \), \( \Omega_* = 0 \) in the above commutative diagram. On the other hand, the bottom commutative square in Theorem 3.3.1 strongly suggests \( \Omega_* = n \times \). If this is true, then \( K_i(\mathcal{O}^G(E_{VC}G; A)) \otimes \mathbb{Q} = 0 \) and this implies the rational Farrell-Jones conjecture holds for \( G \). Similar argument applies to \( L \)-theoretic Farrell-Jones conjecture if there is an additivity theorem for algebraic \( L \)-theory.
1.4.3 Main Results of Chapter 4

For computations, it’s important to know whether one can reduce the family from virtually cyclic groups \( \mathcal{VC} \) to the family of finite groups \( \mathcal{FIN} \) in the Farrell-Jones conjecture. According to the Transitivity Principle 1.2.4, the obstruction lies in the relative assembly map \( H^*_V(E_{\mathcal{FIN}}V \to H^*_V(pt) \) induced by \( E_{\mathcal{FIN}}V \to pt \), where \( V \) is a virtually infinite cyclic group. While this map is always split injective [3], it is not surjective in general. In this Chapter, we give a structure for the cokernel of the \( K \)-theoretic relative assembly map with coefficient in \( \mathbb{Z} \):

**Theorem 4.5.1** For any virtually cyclic group \( V \), there exists a finite abelian group \( H \) with the property that there is an isomorphism:

\[
\bigoplus_{\infty} H \cong \text{CoKer} \left( H^*_0(E_{\mathcal{FIN}}V; K_{\mathbb{Z}}) \to K_0(\mathbb{Z}[V]) \right)
\]

The same result holds in dimension \( n \) whenever \( \text{CoKer} \left( H^*_n(E_{\mathcal{FIN}}V; K_{\mathbb{Z}}) \to K_n(\mathbb{Z}[V]) \right) \) has finite exponent.

This theorem actually follows from our more general study of the structure of Farrell Nil-groups associated to a ring and an automorphism of the ring of finite order. We now recall the related notions. For a ring \( R \) and an automorphism \( \alpha : R \to R \), one can form the twisted polynomial ring \( R_\alpha[t] \), which as an additive group coincides with the polynomial ring \( R[t] \), but with product given by \((rt^i)(st^j) = r\alpha^{-i}(s)t^{i+j}\).

There is a natural augmentation map \( \varepsilon : R_\alpha[t] \to R \) induced by setting \( \varepsilon(t) = 0 \). For \( i \in \mathbb{Z} \), the *Farrell twisted Nil-groups* \( NK_i(R, \alpha) := \ker(\varepsilon_*) \) are defined to be the kernels of the induced \( K \)-theory map \( \varepsilon_* : K_i(R_\alpha[t]) \to K_i(R) \). This induced map is split injective, hence \( NK_i(R, \alpha) \) can be viewed as a direct summand in \( K_i(R_\alpha[t]) \). In the special case where the automorphism \( \alpha \) is the identity, the ring \( R_\alpha[t] \) is just the
ordinary polynomial ring $R[t]$, and the Farrell twisted Nil reduces to the ordinary Bass Nil-groups, which we just denote by $NK_i(R)$. We establish the following:

**Theorem 4.2.1.** Let $R$ be a ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. Then $NK_i(R, \alpha)$ is either trivial, or infinitely generated as an abelian group.

The proof of this result relies heavily on a method developed by Farrell [21], who first showed in 1977 that the lower Bass Nil-groups $NK_*(R)$ with $* \leq 1$, are always either trivial, or infinitely generated. This result was subsequently extended to the higher Bass Nil-groups $NK_*(R)$ with $* \geq 1$ by Prasolov [57] (see also van der Kallen [47]). For Farrell’s twisted Nils, when the automorphism $\alpha$ has finite order, Grunewald [42] and Ramos [61] independently established the corresponding result for $NK_*(R, \alpha)$ when $* \leq 1$. All these papers used the same basic idea, which we call Farrell’s Lemma. We exploit the same idea, and establish our own version of Farrell’s Lemma 4.2.2.

Next we refine somewhat the information we have on these Farrell Nils, by focusing on the finite subgroups arising as direct summands. In section 4.3, we establish:

**Theorem 4.3.1** Let $R$ be a ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $H \leq NK_i(R, \alpha)$ is a finite subgroup, then $\bigoplus_\infty H$ also appears as a subgroup of $NK_i(R, \alpha)$. Moreover, if $H$ is a direct summand in $NK_i(R, \alpha)$, then so is $\bigoplus_\infty H$.

In the statement above, $\bigoplus_\infty H$ denotes the direct sum of countably infinitely many copies of $H$. Theorem 4.3.1 together with some group theoretic facts enable us to deduce a structure theorem for certain Farrell Nil-groups. In section 4.4, we prove:
Theorem 4.4.1 Let $R$ be a countable ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $NK_i(R, \alpha)$ has finite exponent, then there exists a finite abelian group $H$, so that $NK_i(R, \alpha) \cong \bigoplus_{\infty} H$.

A straightforward corollary of Theorem 4.4.1 is the following:

Corollary 4.4.2. Let $G$ be a finite group, $\alpha \in \text{Aut}(G)$. Then there exists a finite abelian group $H$, whose exponent divides some power of $|G|$, with the property that $NK_0(\mathbb{Z}G, \alpha) \cong \bigoplus_{\infty} H$.

The same result in the above corollary possibly holds in dimensions other than zero. For a discussion, see Remark 4.4.3, Remark 4.4.4.
Chapter 2: Hyperdiscrete Action and the Farrell-Jones Conjecture

This chapter is devoted to the study of the notion of hyperdiscrete group action and its applications in the study of the Farrell-Jones conjecture for groups admitting “nice” but not necessary proper actions on CAT(0)-spaces. It is organized as follows. In Section 2.1, we introduce the notion of hyperdiscrete group action, which is a generalization of proper action. We study such group actions and find conditions for a group action to be hyperdiscrete. A large class of group actions, which are not proper, turn out to be hyperdiscrete, this is Theorem 2.1.9. In Section 2.2, we recall the definition of the flow space $FS(X)$ associated to a metric space $X$ introduced by Bartels and Lück [7]. We then study the induced group action on the flow space if there is a certain group action on $X$ when $X$ is CAT(0). In Section 2.3, we study flow lines in $FS(X)$ with bounded $G$-period when $X$ is CAT(0) with a certain isometric $G$-action and prove Theorem 2.3.1 which is an analog of [7, Theorem 4.2] in our setting. This theorem is one of the three ingredients in proving strong transfer reducibility 1.3.11. In Section 2.4, we study a dynamical system $(FS, \Phi, G)$ with weaker assumptions than the one introduced in [7, Convention 5.1]. We prove Theorem 2.4.3 in our weaker setting which is an analog of [7, Theorem 5.6]. This theorem is the second ingredient in proving strong transfer reducibility. In the final
Section 2.5, we prove our main theorem Theorem 2.5.1 by proving strong transfer reducibility using the results above together with the third ingredient proved by Wegner [70].

2.1 Hyperdiscrete Group Actions

In this section, we introduce and study the notion of hyperdiscrete group action. This is a generalization of proper group action and it captures the intuition that compact subsets are moved apart. For simplicity, we only consider isometric group actions on metric spaces, but it can certainly be defined for general group actions.

Definition 2.1.1. Let \((X, d)\) be a metric space. A group \(G\) acting on \(X\) by isometries is said to be proper if the set \(\{g | g \in G \text{ s.t. } gK \cap L \neq \emptyset\}\) is finite for any compact subsets \(K, L \subseteq X\) and hyperdiscrete if the set \(\{gK | g \in G, gK \cap L \neq \emptyset\}\) of subsets of \(X\) is finite. It is said to be discrete if every orbit is discrete.

The term hyperdiscrete will be justified soon. The following lemma is very useful and will be used many times later.

Lemma 2.1.2. Let \((X, d)\) be a metric space, \(G\) acts on \(X\) by isometries, and assume the action is discrete. Let \((x_n)\) and \((g_n)\) be sequences in \(X\) and \(G\) respectively. If \(x_n \to x\) and \(g_n x_n \to x'\). Then there exists \(N \in \mathbb{N}\), s.t. when \(n \geq N\), \(g_n x = x'\).

Proof. Since every orbit is discrete, there exists \(\delta > 0\), s.t. \(B_\delta(x) \cap G \cdot x = x\). For this \(\delta\), there exists \(N\), s.t. when \(n \geq N\), \(d(x_n, x) < \frac{\delta}{4}\) and \(d(g_n x_n, x') < \frac{\delta}{4}\). Hence \(d(g_n x, x') \leq d(g_n x, g_n x_n) + d(g_n x_n, x') < \frac{\delta}{2}\), and thus \(d(g_N^{-1} g_n x, x) \leq d(g_N^{-1} g_n x, g_N^{-1} x') + d(g_N^{-1} x', x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta\) for \(n \geq N\). This implies \(g_n x = g_N x, n \geq N\).

Note the above argument shows \(g_n x\) can be arbitrary close to \(x'\), and \(g_n x\) becomes constant when \(n\) is large enough, hence \(g_n x = x', n \geq N\). \(\square\)
Corollary 2.1.3. Let $G$ acts on $(X,d)$ by isometries. Suppose the action is discrete, then

(i) For any compact subset $K \subseteq X$, the set $G \cdot K := \bigcup_{g \in G} gK$ is closed in $X$, in particular every orbit is closed;

(ii) There exists for any compact subsets $K,L \subseteq X$, an $\epsilon > 0$ such that for every $g \in G$ with $gK \cap L = \emptyset$, we have $gK^\epsilon \cap L^\epsilon = \emptyset$, where $K^\epsilon := \{x \in X \mid \exists y \in K \text{ s.t. } d(x,y) < \epsilon\}$ is the $\epsilon$-neighborhood of $K$ in $X$.

Proof. (i) Let $(g_n),(x_n)$ be sequences in $G$ and $K$ respectively so that $g_n x_n \to x$. Since $K$ is compact, by passing to a subsequence, we may assume $x_n \to x' \in K$. Hence by Lemma 2.1.2, $g_n x' = x$ when $n$ large enough, thus $x \in G \cdot K$ which implies $G \cdot K$ is a closed subset of $X$.

(ii) We will show there exists $\delta > 0$ s.t for all $g \in G$, $gK^\delta \cap L = \emptyset$ whenever $gK \cap L = \emptyset$, then $\epsilon = \frac{\delta}{2}$ will suffice. Suppose such $\delta$ does not exist, then we can find sequences $g_n \in G$, $x_n \in K$ and $y_n \in L$ with $g_n K \cap L = \emptyset$ and $d_X(g_n x_n, y_n) \to 0$. Since $K,L$ are compact, by passing to subsequences, we may assume $x_n \to x \in K$ and $y_n \to y \in L$, thus $g_n x_n \to y$. Now Lemma 2.1.2 applies again and we get $g_n x = y$ when $n$ large enough which contradicts the fact that $g_n K \cap L = \emptyset$.

Recall that if $(X,d_X)$ is a metric space, the Hausdorff distance between closed subsets of $X$ defines a metric on the space of closed subsets of $X$. Let $C(X)$ denote the subspace consisting of compact subsets of $X$, which is called the hyperspace of $X$. Now if $X$ is in addition compact, then $C(X)$ is compact [14, Lemma 5.31], hence complete. If $A \subseteq X$ is a subset, then there are two natural metrics on $C(A)$ which are easily seen to be the same. If $f : (X,d_X) \to (Y,d_Y)$ is a continuous map between metric spaces, then the induced map on hyperspaces is continuous. This can be easily
deduced from [14, Lemma 5.32]. Now if $G$ acts by isometries on $X$, then the induced action on $C(X)$ is also by isometries. The following proposition justifies the term hyperdiscrete.

**Lemma 2.1.4.** Let $G$ acts on $(X, d_X)$ by isometries

(i) If the action is hyperdiscrete, then it is discrete;

(ii) If the action is hyperdiscrete, then the induced action on the hyperspace $C(X)$ is discrete;

(iii) If $X$ is proper, then the action on $X$ is hyperdiscrete if and only if the induced action on the hyperspace $C(X)$ is discrete.

**Proof.** (i) Suppose not, then one can find $g_n x \to x$ with $g_m x \neq g_n x \neq x$ when $m \neq n$. Now let $K = \{g_1 x, g_2 x, \ldots, g_n x, \ldots, x\}$, then $K$ is compact and $g_n K \cap K \neq \emptyset$ for each $n$, hence $\{g_1 K, g_2 K, \ldots, g_n K, \ldots\}$ is a finite set since the action is hyperdiscrete, this together with the fact that $g_n x$ is the unique limit point of $g_n K$ implies $\{g_1 x, g_2 x, \ldots, g_n x, \ldots\}$ is a finite set which contradicts to the fact that $g_n x$ are all distinct.

(ii) Suppose the induced action on $C(X)$ is not discrete, then there exists a sequence $(g_n)$ in $G$ and a compact subset $K \subseteq X$ s.t. $g_n K, K$ are all distinct and $g_n K \to K$ in the Hausdorff distance. Since $\{gK | g \in G, gK \cap K \neq \emptyset\}$ is a finite set, we may assume $g_n K \cap K = \emptyset$ for all $n$ by deleting a finite number of elements in the sequence $(g_n K)$. Now for any $x \in K$, $\{g_n x\}_{n \in \mathbb{N}}$ is a discrete and closed subset of $X$ since $G \cdot x$ is, hence $\epsilon := \inf_n d(g_n x, K) > 0$. However $g_n K \to K$ implies $d(g_n x, K) < \epsilon$ when $n$ is large enough. We thus get a contradiction.

(iii) Because of part (ii), we only have to prove if the induced action on $C(X)$ is discrete, then the action on $X$ is hyperdiscrete. Suppose not, then there exists a
compact subset \( K \subseteq X \) s.t. the set \( A = \{ gK | g \in G, gK \cap K \neq \emptyset \} \) is infinite. Note that every element of \( A \) is contained in a large fixed closed ball which by assumption is compact. Hence \( A \subseteq C(X) \) has a limit point since \( A \) is contained in a compact subset of \( C(X) \). Note however that every orbit of a compact set in \( C(X) \) has no limit point since by assumption and Corollary 2.1.3 (i), every orbit of a compact subset in \( C(X) \) is discrete and closed. We thus get a contradiction and complete the proof. 

Now we give another equivalent characterization of hyperdiscrete action.

**Proposition 2.1.5.** Let \( G \) acts by isometries on \( X \), then the followings are equivalent:

(i) The action is hyperdiscrete;

(ii) If we have sequences \( g_n, x_n \) with the property that \( x_n \to x \) and \( g_n x_n \to x' \), then there exists a subsequence \( g_{n_i} \) s.t. the set \( \{ g_{n_i} M \} \) is finite for any compact subset \( M \subseteq X \) containing \( x \).

*Proof.* (i)\( \Rightarrow \) (ii): Let \( K = \{ x_n, x \}_{n \in \mathbb{N}} \) and \( L = \{ g_n x_n, x' \}_{n \in \mathbb{N}} \). Then they are compact. Note \( g_n K \cap L \neq \emptyset \) for each \( n \), hence \( \{ g_n K \}_{n \in \mathbb{N}} \) is a finite set. Observe that \( g_n x \) is the unique limit point of \( g_n K \), hence \( \{ g_n x \}_{n \in \mathbb{N}} \) is a finite set. Thus there is a subsequence \( (g_{n_i}) \) s.t. \( g_{n_i} x = x_0 \) is constant. Now for any compact \( M \) containing \( x \), we have \( g_{n_i} M \cap \{ x_0 \} = x_0 \) is nonempty for all \( i \), thus the set \( \{ g_{n_i} M \}_{i \in \mathbb{N}} \) is finite.

(ii)\( \Rightarrow \) (i): Suppose there exist compact subsets \( K \) and \( L \) s.t. \( \{ gK | g \in G, gK \cap L \neq \emptyset \} \) is infinite. Then there are sequences \( x_n \in K \) and \( g_n \in G \) such that \( g_n x_n \in L, \forall n \in \mathbb{N} \) and \( g_m K \neq g_n K \) if \( m \neq n \). By passing to subsequences, we may assume \( x_n \to x \) and \( g_n x_n \to x' \), hence there exists a subsequence \( g_{n_i} \) s.t. \( \{ g_{n_i} M \}_{i \in \mathbb{N}} \) is finite for any compact \( M \) containing \( x \). Note \( K \) contains \( x \), thus \( \{ g_{n_i} K \}_{i \in \mathbb{N}} \) is finite which contradicts to the fact that \( g_m K \neq g_n K \) if \( m \neq n \). \( \square \)

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Remark 2.1.6. The notion of hyperdiscrete action can be defined more generally for group actions by homeomorphisms and the above proposition still holds in the more general setting since the proof doesn’t use any properties of metric spaces and isometric group actions.

Of course proper group actions are hyperdiscrete, but there are hyperdiscrete actions which are not proper. If $G$ acts properly on $X$, and $H$ acts on $X$ via an epimorphism $H \twoheadrightarrow G$, then the induced action of $H$ on $X$ is hyperdiscrete. To find other interesting examples, we will formulate some sufficient conditions for group actions to be hyperdiscrete.

Recall from [14, Chapter II.2, Proposition 2.7] that there is a unique center for every bounded subset of a complete CAT(0)-space. In the following lemma we show that the center of a compact subset varies continuously as the compact subset varies. Recall that $C(X)$, the hyperspace of all compact subsets of a metric $X$, is a metric space under Hausdorff distance.

Lemma 2.1.7. Let $(X, d_X)$ be a complete CAT(0)-space and $(C(X), d_H)$ be its hyperspace. Then the map $c : C(X) \to X$ which maps every compact subset to its center is continuous.

Proof. Let $(K_n)$ be a sequence in $C(X)$ and $K_n \to K$ in $C(X)$. We want to show $c(K_n) \to c(K)$ in $X$. Let $r_n \geq 0$ be the smallest number s.t. $K_n \subseteq \overline{B}_{r_n}(c(K_n))$, $r \geq 0$ be the smallest number s.t. $K \subseteq \overline{B}_r(c(K))$ and $s_n \geq 0$ be the smallest number s.t. $K \subseteq \overline{B}_{s_n}(c(K_n))$. So $r$ is the radius of $K$ and $r_n$ is the radius of $K_n$ for each $n$. We first show $s_n \to r$. Note since $K$ is compact, there is $x_n \in K$ s.t. $s_n = d_X(c(K_n), x_n)$. For $x_n$, we can find $y_n \in K_n$ s.t. $d_X(x_n, y_n) \leq d_H(K_n, K)$. Now we have $r \leq s_n = d_X(c(K_n), x_n) \leq d(c(K_n), y_n) + d(y_n, x_n) \leq r_n + d_H(K_n, K)$. 

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Similarly one has
\[ r_n \leq r + d_H(K_n, K) \]
Thus we have
\[ r \leq s_n \leq r_n + d_H(K_n, K) \leq r + 2d_H(K_n, K) \]
Let \( n \to \infty \), one gets \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} r_n = r \).

Now we have the situation that \((c(K_n))\) is a sequence in \( X \) with the property that \( K \subseteq \overline{B}_{s_n}(c(K_n)) \) and \( s_n \) converges to \( r \) which is the radius of \( K \). Then the argument in the proof of \([14, \text{Chapter II.2, Proposition 2.7}]\) shows that \( c(K_n) \) converges to \( c(K) \) which implies \( c \) is continuous. This completes the proof. \( \square \)

Next we introduce a finiteness condition on a group action which is needed for our treatment.

**Definition 2.1.8.** Let \( G \) acts on a metric space \( X \) by isometries

(i) The action is said to be **locally finite** if for any \( x \in X \), there exists some open ball \( B_r(x) \) centered at \( x \) of radius \( r > 0 \) such that \( G_x \), the stabilizer of \( x \), acts on \( B_r(x) \) as a finite group: the image of the obvious map \( \iota_{x,r} : G_x \to \text{Isom}(B_r(x)) \) is finite, where \( \text{Isom}(B_r(x)) \) denotes the isometry group of \( B_r(x) \) (of course \( G_x \) itself doesn’t have to be finite);

(ii) The action is said to be **uniformly locally finite** if for any \( x \in X \), any \( r > 0 \), \( G_x \) acts on \( B_r(x) \) as a finite group;

(iii) If the group action is locally finite, then the **local order** at \( x \in X \) of the group action is defined to be \( k^G_{X,x} = \inf_{r>0} \{ \| \iota_{x,r} \| \mid \iota_{x,r} : G_x \to \text{Isom}(B_r(x)) \} \). The group action is said to have **bounded local order** if \( \sup_{x \in X} \{ k^G_{X,x} \} \) is finite and this number will be denoted by \( k^G_X \).
Theorem 2.1.9. Let $G$ acts on a proper CAT(0)-space $X$ by isometries. If the action is discrete and uniformly locally finite, then the action is hyperdiscrete.

Proof. It suffices to show the induced action on the hyperspace is discrete since it is equivalent to hyperdiscreteness when $X$ is proper. Suppose not, then there exists a sequence $(g_n)$ of elements of $G$ and a compact subset $K$ of $X$ s.t. $g_nK$’s are distinct and $g_nK \rightarrow K$ in the Hausdorff distance. Note that every compact subset of a CAT(0)-space has a unique center, hence $K$ has a unique center, call it $x_0$. According to Lemma 2.1.7, we have $g_nx_0 \rightarrow x_0$. However since the action is discrete, there exists $N > 0$ s.t when $n \geq N$, $g_nx_0 = x_0$, so without loss of generality, we may assume $g_n \in G_{x_0}$, $\forall n > 0$. Now since $g_nK \rightarrow K$ in the Hausdorff distance, there exists $r > 0$ s.t. $g_nK \subseteq B_r(x_0), \forall n > 0$. But the action is uniformly locally finite, and $g_n \in G_{x_0}, \forall n > 0$, the set $\{g_nK \mid n > 0, n \in \mathbb{N}\}$ must be finite which contradicts the fact that $g_nK$’s are distinct.

Corollary 2.1.10. Let $(G, \Gamma)$ be a graph of groups. Suppose each edge groups are of finite indices in their corresponding vertex groups. Then the action of the fundamental group of the graph of groups on the corresponding Bass-Serre tree is hyperdiscrete.

Proof. According to the finiteness assumption on the graph of groups, the Bass-Serre tree is locally finite and hence proper. It is also obvious that the action is discrete and uniformly locally finite, hence hyperdiscrete by Proposition 2.1.9.

It is actually easy to see that we have the more general

Corollary 2.1.11. Let $X$ be a locally finite CAT(0)-polyhedral complex with finitely many shapes as defined in [14]. Suppose $G$ acts on $X$ by isometric automorphisms, then the action is hyperdiscrete.
One notes that proper actions behave nicely under proper maps: if we have an equivariant continuous proper surjective map, then one of the source and target actions being proper forces the other one to be proper. However hyperdiscrete actions do not enjoy such property. Nevertheless one can prove the following, which is sufficient for our purpose.

**Proposition 2.1.12.** Let $X$ and $Y$ be proper CAT(0)-metric spaces with isometric $G$-actions. Suppose both of the actions are discrete and uniformly locally finite, then the diagonal action of $G$ on $X \times Y$ is discrete and uniformly locally finite, hence hyperdiscrete. If in addition $k^G_X, k^G_Y < \infty$, then $k^G_{X \times Y} < \infty$.

*Proof.* It is easy to see the diagonal action is discrete. To see the diagonal action is uniformly locally finite, let $B_r(x, y)$ be a ball in $X \times Y$ centered at $(x, y) \in X \times Y$ of radius $r > 0$. We have $B_r(x, y) \subseteq B_r(x) \times B_r(y)$ and $G_{(x,y)} = G_x \cap G_y$. Note that since the $G$-actions on $X$ and $Y$ are uniformly locally finite, $G_{(x,y)} = G_x \cap G_y$ acts as a finite group on $B_r(x) \times B_r(y)$, hence it acts as a finite group on $B_r(x, y)$. Now if both $X$ and $Y$ are CAT(0), then $X \times Y$ is CAT(0), hence by Proposition 2.1.9, the diagonal action is hyperdiscrete.

Finally we note that $k^G_{X \times Y, (x,y)} \leq k^G_{X,x} \times k^G_{Y,y}$, hence $k^G_{X \times Y} \leq k^G_X \times k^G_Y$ and the last claim follows. This completes the proof. \qed

### 2.2 The Flow Space Associated to a CAT(0)-Space

In this section, we first recall the definition of the flow space associated to a metric space introduced by A. Bartels and W. Lück [7]. A number of facts concerning the flow space when the underlying space is CAT(0) will be taken from [7]. We then study the induced group action on the flow space if there is a group action on the
underlying space. We are particularly interested in when the induced action has nice separating properties.

**Definition 2.2.1.** Let \((X, d_X)\) be a metric space

(i) A *generalized geodesic* in \(X\) is a continuous map \(c : \mathbb{R} \to X\) with the property that there exist \(c_-, c_+ \in \mathbb{R} = [-\infty, \infty]\) satisfying

\[
c_- \leq c_+, \ c_- \neq \infty, \ c_+ \neq -\infty
\]

so that \(c\) is locally constant outside the interval \(I_c := (c_-, c_+)\) and restricts to an isometry on \(I_c\);

(ii) The *flow space* associated to the metric space \((X, d_X)\) is the metric space \((FS(X), d_{FS(X)})\) with an \(\mathbb{R}\)-action, where

\[
FS(X) := \{ c : \mathbb{R} \to X \mid c \text{ is a generalized geodesic} \}
\]

\[
d_{FS(X)}(c, d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}} \, dt
\]

and the \(\mathbb{R}\)-action is given by:

\[
\Phi : \mathbb{R} \times FS(X) \to FS(X), \ \Phi_\tau(c)(t) = c(t + \tau), \text{for } c \in FS(X), t, \tau \in \mathbb{R}
\]

(iii) If a group \(G\) acts by isometries on \(X\), then the induced action on \(FS(X)\) is also by isometries and commutes with the \(\mathbb{R}\)-action. In this way, we obtain a system \((FS(X), \Phi, G)\). Define \(FS(X)_f = \{ c \in FS(X) \mid -\infty < c_- \leq c_+ < \infty \}\), it is invariant under the \(\mathbb{R}\)-action and the \(G\)-action, hence gives rise to a subsystem \((FS(X)_f, \Phi, G)\);

(iv) If \(G\) acts on \(X\) by isometries, then for any \(c \in FS(X)\), its \(G\)-period is defined to be the number \(\text{per}^G_\phi(c) = \inf\{t \mid t > 0, \exists g \in G \text{ with } \Phi_t(c) = gc\} \in \mathbb{R}\).
[0, \infty]$, where the infimum over the empty set is defined to be $\infty$. For $\gamma > 0$, let $FS(X)_{> \gamma} = \{ c \in FS(X) \mid \text{per}_G^c > \gamma \}$ and $FS(X)_{\leq \gamma}$ be its complement. Note that $FS(X)_{\leq 0} = FS(X)_{\leq 0}$.

In the following proposition, we collect all the properties concerning $FS(X)$ that will be needed later when $X$ is CAT(0), proofs can be found in [7].

**Proposition 2.2.2.** Let $X$ be a proper CAT(0) space, then

(i) $(FS(X), d_{FS(X)})$ is a proper metric space, hence locally compact;

(ii) The flow is uniformly continuous: for any $\tau, \epsilon > 0$, there exists $\delta > 0$ such that

$$d_{FS(X)}(c_1, c_2) \leq \delta \text{ and } t \in [-\tau, \tau] \Rightarrow d_{FS(X)}(\Phi_t(c_1), \Phi_t(c_2)) \leq \epsilon$$

(iii) $d_X(c(t), d(t)) \leq e^{||t||}d_{FS(X)}(c, d) + 2$, for any $c, d \in FS(X)$ and $t \in \mathbb{R}$;

(iv) For any $t_0 \in \mathbb{R}$, the evaluation map $ev_{t_0} : FS(X) \to X, c \mapsto c(t_0)$ is uniformly continuous and proper;

(v) The map $E_f : FS(X)_f - FS(X)_R \to \mathbb{R} \times X \times X$ defined by $E_f(c) = (c_-, c(c_-), c(c_+))$ is a homeomorphism onto its image $\{(r, x, y) \in \mathbb{R} \times X \times X \mid x \neq y \}$;

(vi) Assume that the covering dimension of $X$ satisfies $\text{dim}(X) \leq N$, then the covering dimension $\text{dim}(FS(X) - FS(X)_{\leq 0}) \leq 3N + 2$;

(vii) $FS(X) - FS(X)_{\leq 0}$ is locally connected;

Now we will investigate the induced group action on the flow space and its flow subspaces if there is a group action on the original space. We are particularly interested in when the induced action is hyperdiscrete. As noted earlier, if $G$ acts properly on $X$, or if $G$ surjects onto a group which acts properly on $X$, then since any evaluation map $FS(X) \to X$ is $G$-equivariant and surjective, it acts hyperdiscretely on
$FS(X)$. However, hyperdiscreteness on $X$ cannot ensure hyperdiscreteness on $FS(X)$ in general, it may even fail to be discrete. Nevertheless, we have the following:

**Proposition 2.2.3.** Let $G$ acts by isometries on a proper CAT(0)-space $X$. If the action is discrete and uniformly locally finite, then the induced $G$-action is hyperdiscrete and locally finite on $FS(X)_f - FS(X)^R$, and is discrete on $FS_f(X)$. If in addition the $G$-action on $X$ has bounded local order, i.e. $k^G_X < \infty$, then $k^G_{FS(X)_f - FS(X)^R} < \infty$.

**Proof.** Consider $E_f : FS(X)_f - FS(X)^R \to \mathbb{R} \times X \times X$ as defined in Proposition 2.2.2 (v). It is a homeomorphism onto the image. There is an obvious $G$-action on the target space with trivial action on the $\mathbb{R}$ factor, and $E_f$ is $G$-equivariant. Endow the target space with the standard product metric, then the $G$-action on it is by isometries.

By Proposition 2.1.12, the $G$-actions on $\mathbb{R} \times X \times X$ and $X \times X$ are both hyperdiscrete and uniformly locally finite. Hence the $G$-action on $FS(X)_f - FS(X)^R$ is hyperdiscrete and locally finite due to the embedding $E_f$. Note that we cannot conclude the action on $FS(X)_f - FS(X)^R$ is uniformly locally finite because $E_f$ does not map bounded subsets to bounded subsets.

To show the induced action is discrete on $FS_f(X)$, we choose any $c \in FS_f(X)$. Now if $c \in FS(X)_f - FS(X)^R$, then Proposition 2.2.3 and Lemma 2.1.4 shows the orbit of $c$ is discrete; if $c \in FS(X)^R$, then $g_n c \to c$, implies $g_n c(0) \to c(0)$ since the evaluation map is continuous. Since the action on $X$ is discrete, Lemma 2.1.2 applies and $g_n$ stabilizes $c(0)$ eventually, hence $g_n c = c$ when $n$ large enough. This shows the orbit of $c \in FS(X)^R$ is discrete.

Finally, if $k^G_X < \infty$, then by Proposition 2.1.12, $k^G_{\mathbb{R} \times X \times X} < \infty$, and we get $k^G_{FS(X)_f - FS(X)^R} < \infty$ due to the embedding $E_f$. $\square$
Remark 2.2.4. According to the above proposition, one sees that the induced $G$-action on the subflow space $FS(X)_f - FS(X)^R$ inherits nice separating properties if the original $G$-action has nice separating property. However this is in general not true for the subflow space $FS(X) - FS(X)^R$, mainly because of the existence of infinite geodesics. It is crucial however, in view of our main theorem (Theorem 2.5.1 below), that the induced action on $FS(X) - FS(X)^R$ is hyperdiscrete. It seems not a good option to impose other conditions in addition to the assumptions in the above proposition to ensure the induced action on $FS(X) - FS(X)^R$ is hyperdiscrete, otherwise we will lose many interesting examples we want to include. A better way is possibly to modify the metric on the flow space so that under the same assumptions as in Proposition 2.2.3, the induced action on $FS(X) - FS(X)^R$ is hyperdiscrete. This might be done by taking the angular metric between infinite geodesics into account. This is a direction for future research.

2.3 Orbits with $G$-Period Bounded Above

Looking at the definition of strongly transfer reducibility (see Definition 1.3.11), one has to construct an open cover for $G \times X$ with certain properties. If $G$ acts on a CAT(0)-space $X$, then for any base point $x_0 \in X$, there is a transfer map

$$\iota: G \times X \rightarrow FS(X)$$

which sends $(g, x) \in G \times X$ to the generalized geodesic $c$ joining $gx_0$ and $gx$ with $c(0) = gx_0$. So one possible way to construct an open cover of the source space is by constructing and then pulling back an open cover of the target space. Now any number $\gamma > 0$ splits the flow space $FS(X)$ into two parts: $FS(X)_{\leq \gamma}$, the subspace consisting of points with $G$-periods bounded by $\gamma$ and $FS(X)_{>\gamma}$, the subspace consisting of
points with $G$-period bounded below by $\gamma$. Elements in $FS(X)_{\leq \gamma}$ with positive $G$-period have nice geometric feature—they are axis of some hyperbolic isometries while elements in $FS(X)_{> \gamma}$ is adapted to the use of the notion of box (see Definition 2.4.5). Hence the construction of an open cover of $FS(X)$ naturally falls into the study of these two subspaces separately.

This section is devoted to the study of $FS(X)_{\leq \gamma}$ and is parallel to [7, Section 4]. We prove Theorem 2.3.1 which is an analog of Theorem 4.2 in that paper, but instead of requiring the group action to be proper, we weaken the action to be discrete on both of the original space and the flow space associated to it.

**Theorem 2.3.1.** Let $X$ be a CAT(0)-space which is proper and of finite covering dimension. Let $G$ acts on $X$ by isometries which gives rise to a system $(FS(X), \Phi, G)$. Suppose the action is discrete on $X$ so that the induced action on $FS(X)$ is also discrete, then for any compact subset $K \subseteq X$, there is a natural number $M$, so that for any $\gamma > 0$, there exists a collection $\mathcal{V}$ of open subsets of the flow space $FS(X)$ satisfying:

(i) Each element $V \in \mathcal{V}$ is an open $\mathcal{F}$-subset of the $G$-space $FS(X)$ (the family $\mathcal{F}$ is the family in the main theorem);

(ii) $\mathcal{V}$ is $G$-invariant;

(iii) $G \setminus \mathcal{V}$ is finite;

(iv) $\text{dim} \mathcal{V} \leq M$;

(v) There is $\epsilon$ with the following property: for $c \in FS(X)_{\leq \gamma}$ such that $c(t) \in G \cdot K$ for some $t \in \mathbb{R}$, there is $V \in \mathcal{V}$ such that $\Phi_{[-\gamma, \gamma]}(c)' \subseteq V$.

The proof of the theorem follows the scheme of [7, Section 4] and the rest of this section is devoted to it. For basic properties of isometries of CAT(0)-spaces, see [14].
Note first that \( c \in FS(X) \) has G-period \( 0 < \text{per}_G^G(c) < \infty \) if and only if \( c \) is a biinfinite geodesic which is an axis of some hyperbolic element \( g \in G \) and has G-period 0 if and only if \( c \in FS(X)\mathbb{R} \), i.e. a constant map.

**Lemma 2.3.2.** Let \( c : \mathbb{R} \to X \) be a geodesic line. Put

\[
G_{\Phi_R(c)} = \{ g \in G \mid g(\Phi_R(c)) = \Phi_R(c) \} = \{ g \in G \mid \exists \tau \in \mathbb{R} \text{ with } gc(t) = c(t + \tau), \forall t \in \mathbb{R} \}
\]

Then \( G_{\Phi_R(c)} \in \mathcal{F} \).

**Proof.** This is an analog of Lemma 4.4 of [7] and the proof is similar. Identifying \( c(\mathbb{R}) \) with \( \mathbb{R} \), then the image of the obvious map \( G_{\Phi_R(c)} \to \text{Isom}(\mathbb{R}) \) is a discrete subgroup since the action has discrete orbit. Also noting that the action is by translations, hence the image is either trivial or isomorphic to \( \mathbb{Z} \). The kernel of the obvious map is obviously a subgroup of some isotropy subgroup, hence \( G_{\Phi_R(c)} \in \mathcal{F} \). \( \Box \)

Following [7], we introduce:

**Notation 2.3.3.** Let \( \gamma > 0 \) and \( K \subseteq X \) compact be as in Theorem 4.1

(i) Let \( G_{\leq \gamma}^{\text{hyp}} \subseteq G \) be the set of all hyperbolic \( g \in G \) of translation length \( l(g) \leq \gamma \) such that some axis of \( g \) intersects \( G \cdot K \).

There is an equivalence relation \( \sim \) on \( G_{\leq \gamma}^{\text{hyp}} \) for which \( g \sim g' \) if and only if their axes are parallel. Put \( A_{\leq \gamma} := G_{\leq \gamma}^{\text{hyp}} / \sim \). The conjugation action of \( G \) on \( G \) restricts to \( G_{\leq \gamma}^{\text{hyp}} \) and descends to an action of \( G \) on \( A_{\leq \gamma} \). For \( a \in A_{\leq \gamma} \), set \( G_a := \{ g \in G \mid g \cdot a = a \} \).

(ii) For \( a \in A_{\leq \gamma} \), let \( FS_a \subseteq FS(X) \) denote the subspace of \( FS(X) \) that consists of all geodesics \( c \) which are axis for some \( g \in a \) and intersect \( G \cdot K \). Define

\[
p_a : FS_a \to X, \quad c \mapsto c(0)
\]
Let \( Y_a := FS_a/\Phi \) be the quotient of \( FS_a \) by the action of the flow \( \Phi \) and

\[
q_a : FS_a \to Y_a
\]

be the canonical projection. The action of \( G \) on \( FS(X) \) restricts to an action of \( G_a \) on \( FS_a \) and this action descends to an action on \( Y_a \) so that \( q_a \) is \( G_a \)-equivariant.

**Lemma 2.3.4.** There is a compact subset \( K_\gamma \subseteq X \) such that \( c(0) \in G \cdot K_\gamma \) for all \( c \in \bigcup_{a \in A_{\leq \gamma}} FS_a \).

**Proof.** This is Lemma 4.8 of [7]

The next lemma is the key to the rest of the constructions. It was proved in [7] using the fact that the \( G \)-action on \( FS(X) \) is proper. However it can still be proved under the weaker assumption that the action of \( G \) on \( FS(X) \) is discrete. The key to the proof is to use Lemma 2.1.2.

**Lemma 2.3.5.** Let \( \gamma > 0 \) and \((c_n)\) be a sequence in \( \bigcup_{a \in A_{\leq \gamma}} FS_a \) that converges to \( c \in FS(X) \). Then there are \( g \in G_{\leq \gamma}^{hyp} \) and an infinite subset \( I \subseteq \mathbb{N} \) such that \( c \) and all \( c_i, i \in I \) are axes for \( g \) and intersect \( G \cdot K \). Hence \( \bigcup_{a \in A_{\leq \gamma}} FS_a \) is a closed subset of \( FS(X) \).

**Proof.** There are \( g_n \in G_{\leq \gamma}^{hyp} \) and \( t_n \in [0, \gamma] \) such that \( g_n c_n = \Phi_{t_n}(c_n) \). By passing to a subsequence, we may assume \( t_n \to t_0 \in [0, \gamma] \). Then \( g_n c_n = \Phi_{t_n}(c_n) \to \Phi_{t_0}(c) \) by continuity of the flow. Thus we have a situation as in Lemma 2.1.2 and can conclude that \( g_n c = \Phi_{t_0}(c) \) for \( n \) large enough. Since \( g_n \) is hyperbolic, this means \( c \) is an axis for all \( g_n \) when \( n \) is large enough.

To show \( c \) intersects \( G \cdot K \), note by definition, for each \( n \), there is \( s_n \in \mathbb{R} \) so that \( c_n(s_n) \in G \cdot K \). Since \( g_n c_n(s) = c_n(s + t_n) \) and \( t_n \in [0, \gamma] \), we can arrange
\[ s_n \in [0, \gamma], \forall n \geq 0. \] By passing to a subsequence, we may assume \( s_n \to s_0 \in [0, \gamma] \), then \( c_n(s_n) \to c(s_0) \). Now by Corollary 2.1.3 (i), \( G \cdot K \) is closed in \( X \), hence \( c(s_0) \in G \cdot K \).

The proof of the following several lemmas and propositions are almost exactly the same as the proof of the corresponding statements in [7]. The key is the use of Lemma 2.3.5 and Lemma 2.1.2. We will omit most of the proofs, as interested readers can find them in [7, section 4].

**Lemma 2.3.6.** Let \( \gamma > 0 \). Then
\[
\begin{align*}
(i) & \text{ } G \setminus A_{\leq \gamma} \text{ is finite; } \\
(ii) & \text{ } G \cdot FS_a \subseteq FS(X) \text{ is closed for all } a \in A_{\leq \gamma}; \text{ there is } K_a \subseteq G \cdot FS_a \text{ compact such that } G \cdot K_a = G \cdot FS_a; \\
(iii) & \text{ } \text{There is } \epsilon > 0 \text{ such that } d_{FS}(FS_a, FS_b) > \epsilon \text{ for all } a \neq b \in A_{\leq \gamma}; \\
(iv) & \text{ } \text{Consider } c \in FS(X)_{\leq \gamma} - FS(X)^R \text{ such that } c(t) \in K \text{ for some } t \in \mathbb{R}, \text{ then } \\
& \text{ } c \in FS_a \text{ for some } a \in A_{\leq \gamma}.
\end{align*}
\]

*Proof.* (i) Exactly the same as [7, Lemma 4.10 (i)];
(ii) Exactly the same as [7, Lemma 4.10 (ii)];
(iii) Exactly the same as [7, Lemma 4.10 (iii)];
(iv) The main idea is the same as [7, Lemma 4.10 (iv)], here one uses Lemma 2.1.2 instead of [7, Lemma 4.6].

**Proposition 2.3.7.** Let \( \gamma > 0 \) and \( a \in A_{\leq \gamma} \). Then
\[
\begin{align*}
(i) & \text{ } p_a : FS_a \to X \text{ is an isometric embedding with closed image; } \\
(ii) & \text{ } \text{there is a } G_a \text{-invariant metric } d_a \text{ on } Y_a \text{ that generates the topology; with this } \\
& \text{ } \text{metric } Y_a \text{ is a proper metric space; }
\end{align*}
\]
(iii) there is $\tau_a : FS_a \to \mathbb{R}$, such that $c \mapsto (q_a(c), \tau_a(c))$ defines an isometry $FS_a \to Y_a \times \mathbb{R}$ which is compatible with the flow, i.e., $\tau_a(\Phi_t(c)) = \tau_a(c) + t, \forall t \in \mathbb{R}, c \in FS_a$.

(iv) for $y \in Y_a$, $G_y = \{g \in G_a \mid gy = y\}$ lies in the family $\mathcal{F}$ and the action of $G_a$ on $Y_a$ is discrete.

Proof. (i) Exactly the same as [7, Proposition 4.11(i)];

(ii) Exactly the same as [7, Proposition 4.11(ii)];

(iii) Exactly the same as [7, Proposition 4.11(iii)];

(iv) For the first part, use Lemma 2.3.2. Note the family of subgroups is $\mathcal{F}$ instead of $\mathcal{VC}$. The second part almost goes exactly the same as [7, Proposition 4.11(iv)], but one has to use Lemma 2.1.2 instead of [7, Lemma 4.6].

Lemma 2.3.8. We have $\dim(G_a \setminus Y_a) \leq \dim(X)$.

Proof. Exactly the same as [7, Lemma 4.12].

Proposition 2.3.9. Let $\gamma > 0$ and $a \in A_{\leq \gamma}$. There is an open $\mathcal{F}$-cover $\mathcal{V}_a$ of $Y_a$ such that

(i) $\dim(\mathcal{V}_a) \leq \dim(X)$;

(ii) $\mathcal{V}_a$ is $G_a$-invariant;

(iii) $G_a \setminus \mathcal{V}_a$ is finite.

Proof. Exactly the same as [7, Proposition 4.13] except replacing $\mathcal{VC}$ by $\mathcal{F}$.

Lemma 2.3.10. ([7, Lemma 4.14]) Let $(Z,d_Z)$ be a metric space with an action of a group $H$ by isometries. Let $A$ be an $H$-invariant subspace. For $\emptyset \neq U \subsetneq A$, we define

$$Z(U) := \{z \in Z \mid d_Z(z,U) < d_Z(z,A \setminus U)\}$$

and set $Z(A) = Z, Z(\emptyset) = \emptyset$. Then for $U, V \subseteq A$,
(i) $Z(U)$ is open in $Z$;

(ii) $Z(U \cap V) = Z(U) \cap Z(V)$;

(iii) $Z(U) \cap A = U$ holds if and only if $U$ is open in $A$;

(iv) for all $g \in H$ we have $Z(gU) = gZ(U)$.

Let $\mathcal{E}$ denote the family of subgroups of $G$ generated by the stabilizers of the $G$-action on $X$.

**Lemma 2.3.11.** There exists $\epsilon_\mathbb{R} > 0$ and a $G$-invariant cofinite collection $\mathcal{V}_\mathbb{R}$ of open $\mathcal{E}$-subsets of $FS(X)$ such that

(i) $\dim(\mathcal{V}_\mathbb{R}) < \infty$;

(ii) if the image of $c \in FS(X)^\mathbb{R}$ in $X$ intersects $G \cdot K$, then there is $U \in \mathcal{V}_\mathbb{R}$ such that $B_{\epsilon_\mathbb{R}}(\Phi_\mathbb{R}(c)) \subseteq U$.

**Proof.** This is [7, Lemma 4.15] with the family of finite subgroups $\mathcal{F}_\mathbb{N}$ replaced by the family of subgroups $\mathcal{E}$. For the proof, just note that the discreteness of the $G$-action on $FS(X)$ guarantees the existence of an $\mathcal{E}$-neighborhood for every point in $FS(X)$, then the proof of [7, Lemma 4.15] carries over here. $\square$

**Proof of Theorem 2.3.1.** With all the lemmas and propositions above in our setting, the proof goes exactly the same as the proof of [7, Theorem 4.2]. For the convenience of the reader, we outline the construction of the collection of open subsets $\mathcal{V}$ in the theorem and the verification work can be found in [7].

We know by Lemma 2.3.6(i), there is a finite subset $R \subseteq A_{\leq \gamma}$ of complete representatives of the $G$-action on $A_{\leq \gamma}$. For each $a \in R$, there is an open $\mathcal{F}$-cover $\mathcal{V}_a$ of $Y_a$ as in Proposition 2.3.9. Pull back this open cover via $q_a$ to obtain an open $\mathcal{F}$-cover $\mathcal{W}_a$ of $FS_a$, then set $\mathcal{W}_{ga} = g\mathcal{W}_a$. Note that elements in $\mathcal{W}_a$ are only open
in $FS_a$, and may not be open in $FS(X)$. To resolve this, first use Lemma 2.3.6(iii) to find $\delta > 0$ neighborhood $U_a \subseteq FS(X)$ of $FS_a$ for each $a$ with the property that $U_a \cap U_b = \emptyset$ whenever $a \neq b \in A_{\leq \gamma}$. Note that $FS_a$ and $U_a$ are $G_a$-subsets of $FS(X)$. Now within each $G_a$-subset $U_a$, apply the construction of Lemma 2.3.10 to elements in $W_a$ with respect to the $G_a$-invariant subspace $FS_a \subseteq U_a$. This gives a collection of open $F$-subsets $U_a$ of $FS(X)$ for each $a$ in $A_{\leq \gamma}$. Then $V = V_{\mathbb{R}} \cup (\cup_{a \in A_{\leq \gamma}} W_a)$ is the desired collection of open subsets, where $V_{\mathbb{R}}$ is as in Lemma 2.3.11.

\[\Box\]

### 2.4 Orbits with $G$-Period Bounded Below

This section is devoted to the study of orbits with $G$-period bounded below by some positive number. In analog to [7] and [10], the study is in a more general setting rather than just the flow space associated to a CAT(0)-space. We introduce a dynamical system $(FS, \Phi, G)$ which is a generalization of the one introduced by A. Bartels, W. Lück and H. Reich in [7] [10] as a tool to attack the Farrell-Jones Conjecture. In their setup, the group action is required to be proper, we weaken it to hyperdiscrete and locally finite. Also, in their setup, the orders of the finite subgroups of the group are assumed to be uniformly bounded, however this assumption can be removed by the work of A. Mole and H. Rüping [55]. It turns out that an analog of Theorem 5.6 of [7] (Theorem 2.4.3 below) can still be proved.

**General Setup 2.4.1.** Let $(FS, \Phi, G)$ denote the following system

(i) $(FS, d_{FS})$ is a locally compact metric space with an isometric $G$-action;

(ii) $FS$ is also a flow space $\Phi : \mathbb{R} \times FS \to FS$, which is uniformly continuous (see its definition in Proposition 2.2.2 (ii));

(iii) The $G$-action commutes with the $\mathbb{R}$-action;

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(iv) The $G$-action on $FS - FS^R$ is hyperdiscrete and locally finite;
(v) $FS - FS^R$ has finite covering dimension: $d = \dim(FS - FS^R) < \infty$;
(vi) $FS - FS^R$ is locally connected.

**Definition 2.4.2.** Let $(FS, \Phi, G)$ be as in above

(i) Let $\mathcal{E}$ denote the family of subgroups of $G$ generated by stabilizers of the $G$-action on $FS$ and $\mathcal{F}$ denote the family defined in the same way as in the main theorem Theorem 2.5.1;

(ii) Let $B \subseteq FS$, the stabilizer of $B$ is defined to be $G_B = \{g \in G \mid gB = B\}$.

Recall a subset $B \subseteq FS$ is called a $C$-subset for a family $C$ of subgroups of $G$ if $G_B \in C$ and for any $g \in G$, either $gB = B$ or $gB \cap B = \emptyset$;

(iii) As in Definition 2.2.1 (iv), for any $x \in FS$, its $G$-period is the number $\text{per}_G^x(x) = \inf\{t \mid t > 0, \exists g \in G \text{ with } \Phi_t(x) = gx\} \in [0, \infty]$. Let $FS_{>\gamma} = \{x \in FS \mid \text{per}_G^x(x) > \gamma\}$ and $FS_{\leq \gamma}$ be its complement. We have $FS^R = FS_{\leq 0}$.

Now we are ready to state the main theorem of this section. It is an analog of Theorem 5.6 of [7], whose proof is essentially contained in [10].

**Theorem 2.4.3.** Let $(FS, \Phi, G)$ be a system as in General Setup 2.4.1. Then there exists $N = N(d) \in \mathbb{N}$ such that the following holds: For any $\alpha > 0$, there is $\gamma > 0$ such that for any compact subset $K \subseteq FS_{>\gamma}$, there is a collection $\mathcal{V}$ of open $\mathcal{E}$-subsets of $FS$ such that:

(i) $\mathcal{V}$ is $G$-invariant: $g \in G, V \in \mathcal{V} \Rightarrow gV \in \mathcal{V}$;

(ii) $\dim \mathcal{V} \leq N$;

(iii) $G \setminus \mathcal{V}$ is finite;

(iii) for every $c \in G \cdot K$, there is $V \in \mathcal{V}$ such that $\Phi_{[-\alpha,\alpha]}(c) \subseteq V$. 

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Remark 2.4.4. In Theorem 5.6 of [7], the elements in $\mathcal{V}$ are stated to be $\mathcal{VC}$-subsets, however, they are actually $\mathcal{FLN}$-subsets according to the proof. So here we state them to be $\mathcal{E}$-subsets, not $\mathcal{F}$-subsets.

The proof of Theorem 2.4.3 will follow the scheme and techniques in [10] developed by A. Bartels, W. Lück and H. Reich, and will occupy the rest of this section. The validity of this theorem in our setting is mainly because analogs of Lemma 2.14, Lemma 2.16 and Lemma 2.17 in [10] can still be proved in our setting, and the general position results obtained in section 3 of [10] are only used locally, hence can be freely imported here, since our action is assumed to be locally finite with bounded local orders. Therefore an analog of Proposition 4.1 of [10] is still true. We start by recalling all the necessary definitions and constructions from [10].

Definition 2.4.5. An (equivariant) box $B$ for the system $(FS, \Phi, G)$ is a subset $B \subseteq FS$ with the following properties:

(i) $B$ is a compact $\mathcal{E}$-subset;

(ii) There exists a real number $l_B > 0$, called the length of the box, with the property that for every $x \in B$ there exist real numbers $a_-(x) \leq 0 \leq a_+(x)$ and $\epsilon(x) > 0$ satisfying

$$l_B = a_+(x) - a_-(x)$$

$$\Phi_t(x) \in B, \text{ for } t \in [a_-(x), a_+(x)]$$

$$\Phi_t(x) \not\in B, \text{ for } t \in (a_-(x) - \epsilon(x), a_-(x)) \cup (a_+(x), a_+(x) + \epsilon(x)).$$

All basic properties of a box proved in [10] carry over to here since they do not depend on the properness of the action which is assumed there. We list them here, and proofs can be found in [10]. Firstly let us introduce some terms and notations.
**Definition 2.4.6.** Let $B \subseteq FS$ be a box:

(i) $B^o$ and $\partial B$ denote its interior and boundary in $FS$ respectively;

(ii) The set $S_B := \{ x \in B \mid a_+(x) + a_-(x) = 0 \}$ is called the central slice of $B$. $S_B$ is closed, hence compact;

(iii) The sets $\partial_\pm B := \{ x \in B \mid a_\pm(x) = 0 \}$ are called the top and bottom of $B$ and $\partial_\pm B^o := \{ \Phi_{a_\pm(x)}(x) \mid x \in S_B \cap B^o \}$ are the open top and open bottom;

(iv) Define $\pi_B : B \to S_B$ by $\pi_B(x) = \Phi_{\frac{a_+(x) + a_-(x)}{2}}(x)$;

(v) For a closed nonempty $\mathcal{E}$-subset $V \subseteq S_B$ and $a, b \in \mathbb{R}$ with $-\frac{l_B}{2} \leq a < b \leq \frac{l_B}{2}$, define $B(V; a, b) := \Phi_{[a,b]}(V) := \{ \Phi_t(v) \mid v \in V, t \in [a,b] \}$, when $a = -b$, we will abbreviate the notation to be $B(V; 2b)$;

(vi) A subset $S \subseteq B$ is called transversal to the flow with respect to the box $B$ if $S \cap \Phi_{[-l_B/2, l_B/2]}(x)$ consists of at most one point for every $x \in S_B$.

**Proposition 2.4.7.** With the above notations, we have

(i) $a_\pm(gx) = a_\pm(x), \forall g \in G_B, x \in B$. $B^o, S_B, \partial_\pm B, \partial_\pm B^o$ are all $\mathcal{E}$-subsets of $FS$ unless any of them is empty and we have

$$G_B = G_{B^o} = G_{S_B} = G_{\partial_\pm B} = G_{\partial_\pm B^o}$$

(ii) The maps $a_\pm : B \to \mathbb{R}, \ x \mapsto a_\pm(x)$ are continuous;

(iii) The map $\mu_B : S_B \times [-\frac{l_B}{2}, \frac{l_B}{2}] \to B, \ (x, t) \mapsto \Phi_t(x)$ is a $G_B$-equivariant homeomorphism with the obvious $G_B$ action on the source space;

(iv) The space $S_B \cap B^o$ is locally connected;

(v) There exists $\epsilon_B > 0$ depending only on $B$ such that the numbers $\epsilon(x)$ can be chosen so that $\epsilon(x) > \epsilon_B, \forall x \in B$;

(vi) $B(V; a, b)$ defined in Definition 2.4.6 (v) is a box.
Lemma 2.13 of [10] uses the properness of the group action, however an analog still holds when the action is hyperdiscrete:

**Lemma 2.4.8.** Let $B, C$ be boxes with $B \subseteq C$. Then we can find for every $x \in S_B$ a closed neighborhood $U \subseteq S_B$ of $x$ (relative to $S_B$) satisfying:

(i) $U$ is a $G_x$-invariant $E$-subset of $S_B$;

(ii) $U$ is transversal to the flow with respect to $C$.

**Proof.** Let $\tau_{S_B} : S_B \to [-l_C/2, l_C/2]$ be the continuous map which is the restriction of the projection $C \to [-l_C/2, l_C/2]$ via $\mu_C$, which is defined in Proposition 2.4.7 (iii). Since the map is continuous, we can choose a closed neighborhood $U_1 \subseteq S_B$ of $x$ such that $|\tau_{S_B}(u) - \tau_{S_B}(x)| < l_B/2, \forall u \in U_1$. Since the action is hyperdiscrete and by isometries, there exists a closed neighborhood $U_2 \subseteq S_B$ of $x$ s.t. $gU_2 \cap U_2 \neq \emptyset \Rightarrow g \in G_x$. Note $U_1, U_2$ are compact since $S_B$ is compact. Let $U = \bigcap_{g \in G_x} g(U_1 \cap U_2) \subseteq S_B$. Note that it is indeed a finite intersection since $\{g(U_1 \cap U_2) \mid g \in G, g(U_1 \cap U_2) \cap (U_1 \cap U_2) \neq \emptyset\}$ is a finite set because the action is hyperdiscrete. Thus $U$ is a closed neighborhood of $x \in S_B$. Note $U$ is $G_x$-invariant. By the choice of $U_2$, $U$ is an $E$-subset of $S_B$. The proof that $U$ is transversal to the flow with respect to $C$ goes exactly as in the proof of [10, Lemma 2.13] and will be omitted. \(\Box\)

**Lemma 2.4.9.** For every $x \in FS - FS^R$, there exists a nonequivariant box whose interior contains $x$.

**Proof.** This is Lemma 2.11 of [10] \(\Box\)

In the following lemma, we prove the existence of boxes. We follow the general strategy for the proof of [10, Lemma 2.14] in which stabilizers being finite is used. However certain places can be modified to obtain the same result in our setting.
Lemma 2.4.10. For every \( x \in FS - FS^R \), there exists a box \( B \) satisfying

(i) \( G_B = G_{S_B} = G_x \);

(ii) \( x \in S_B \cap B^c \);

(iii) \( S_B \) is connected.

Proof. According to Lemma 2.4.9, there exists a nonequivariant box \( C \) with \( x \in C^c \). Since the action is hyperdiscrete and by isometries, there exists a small closed ball \( U \) of \( x \) such that \( U \) is an \( \mathcal{E} \)-subset with \( G_U = G_x \). Without loss of generality, we may assume \( C \subseteq U \), \( x \in S_C \cap C^c \), and \( \Phi_t(s) \notin C^c \), \( \forall s \in S_C \), \( t \in [-l_C, -l_C/2] \cup [l_C/2, l_C] \), otherwise we may perform the construction defined in Definition 2.4.6 (v). By intersecting a small closed ball centered at \( x \) with \( S_C \), one obtains a closed neighborhood \( S_0 \subseteq S_C \cap C^c \) of \( x \in S_C \cap C^c \) such that \( gS_0 \subseteq C^c \), \( \forall g \in G_x \). Without loss of generality, we may assume \( gS_0, \forall g \in G_x \), is transversal to the flow with respect to the (nonequivariant) box \( C \), otherwise the nonequivariant version of Lemma 2.4.8 together with the fact \( \{gS_0 \mid g \in G_x \} \) is a finite set (since the action is hyperdiscrete, or one could use the fact that the action is locally finite) will find a new \( S_0 \) for us with the desired property.

Now Define \( S_1 = \bigcap_{g \in G_x} S_0 \cap \pi_C(gS_0) \), where \( \pi_C : C \to S_C \) is the retraction onto the central slice defined in Definition 2.4.6 (iv). Note that it is indeed a finite intersection since \( \{gS_0 \mid g \in G, gS_0 \cap S_0 \neq \emptyset \} \) is a finite set because the action is hyperdiscrete. Hence \( S_1 \subseteq S_0 \) is a compact neighborhood of \( x \in S_0 \). Therefore by construction and transversality, for every \( g \in G_x \) and \( s \in S_1 \), there exists a unique \( \tau_g(s) \in (-l_C/2, l_C/2) \) and a unique \( s' \in S_0 \) s.t. \( \Phi_{\tau_g(s)}(s) = gs' \in gS_0 \). The function \( \tau_g(s) \) is continuous since it is the restriction to \( S_1 \) of the continuous function

\[
\tau_g : S_1 \to (-l/2, l/2), \quad s \mapsto -p_C(g^{-1}s)
\]
where $p_C : C \to [-l/2,l/2]$ the the projection via $\mu_C$, recall $\mu_C$ is defined in Proposition 2.4.7 (iii).

**Claim 1:** For any $g \in G_x$ and $s \in S_1$, let $s' \in S_0$ be the unique element s.t. $\Phi_{\tau_g^{-1}(s)}(s) = g^{-1}s'$, then $s' \in S_1$ and for any $h \in G_x$, $\tau_{gh}(s') = \tau_h(s) - \tau_{g^{-1}}(s)$.

**Proof:** $\Phi_{\tau_h(s)-\tau_{g^{-1}}(s)}(s') = \Phi_{\tau_h(s)}(s') = \Phi_{\tau_h(s)}(g) = g\Phi_{\tau_h(s)}(s) = ghS'' \in ghS_0$ for a unique $s'' \in S_0$. Now since $ghS_0 \subseteq C^\circ$ and $\tau_h(s) - \tau_{g^{-1}}(s) \in (-l_C,l_C)$, we conclude $\tau_h(s) - \tau_{g^{-1}}(s) \in (-l_C/2,l_C/2)$, because at the beginning $C$ was chosen such that $\Phi_{\tau}(s) \not\in C^\circ, \forall s \in S_C, t \in [-l_C,-l_C/2] \cup [l_C/2,l_C]$. Hence $s' \in S_1$ since $h \in G_x$ is arbitrary. Also since $\Phi_{\tau_{gh}(s')}(s') \in ghS_0$, we must have $\tau_{gh}(s') = \tau_h(s) - \tau_{g^{-1}}(s)$ by transversality. This proves the first claim.

**Claim 2:** If $g, h \in G_x$ such that $gS_0 = hS_0$, then $\tau_g(s) = \tau_h(s), \forall s \in S_1$

**Proof:** By definition, there exist $s_1, s_2 \in S_0$ such that $\Phi_{\tau_g(s)}(s) = gs_1 \in gS_0$ and $\Phi_{\tau_h(s)}(s) = hs_2 \in hS_0$. However $gS_0 = hS_0$ and it is transversal to the flow line with respect to $C$, one must have $\tau_g(s) = \tau_h(s)$.

Note that $G_x$ acts on the finite set $\{gS_0 \mid g \in G_x\}$, let $H < G_x$ be the stabilizer of $S_0$ under the $G_x$ action, then $[G_x : H] < \infty$, and we denote this number by $n$. Claim 2 says that if $gH = hH$, then $\tau_g = \tau_h$ as functions on $S_1$. Choose one representative $g_i, i = 1, 2, \ldots, n$ from each coset, and define the following function which is independent of the choice of representatives.

$$\tau : S_1 \to (-l_C/2,l_C/2), \quad \tau(s) = \frac{1}{n} \sum_{i=1}^{n} \tau_{g_i}(s)$$

Now let $S_2 = \{\Phi_{\tau(s)}(s) \mid s \in S_1\}$, we want to show $S_2$ is $G_x$-invariant. So pick $g \in G_x$ and $u = \Phi_{\tau(s)}(s) \in S_2, s \in S_1$. Let $s' \in S_0$ be the unique element s.t.

$$\Phi_{\tau_g^{-1}(s)}(s) = g^{-1}s'$$
by Claim 1, $s' \in S_1$. We will show $gu = \Phi_{\tau(s')}(s') \in S_2$. Firstly

$$
\tau(s') = \frac{1}{n} \sum_{i=1}^{n} \tau_{g_i}(s')
= \frac{1}{n} \sum_{i=1}^{n} \tau_{gg_i}(s') \quad \text{by Claim 2}
= \frac{1}{n} \sum_{i=1}^{n} (\tau_{g_i}(s) - \tau_{g^{-1}}(s)) \quad \text{by Claim 1}
= \tau(s) - \tau_{g^{-1}}(s)
$$

Hence

$$
gu = g\Phi_{\tau(s)}(s) = \Phi_{\tau(s') + \tau_{g^{-1}}(s)}(gs) = \Phi_{\tau(s')}(g\Phi_{\tau_{g^{-1}}(s)}(s))
= \Phi_{\tau(s')}(gg^{-1}s') = \Phi_{\tau(s')}(s') \in S_2
$$

This proves $S_2$ is $G_x$-invariant.

Finally choose the connected component of $x$ in $S_2$ which we call $S_3$ and define $B := \Phi_{[-\delta/2, \delta/2]}(S_3)$ for some small $\delta > 0$ s.t. $B \subset C^\circ$. One easily checks, as in the proof of Lemma 2.14 of [10], that $B$ is a box as required. This completes the proof.

Note that in Theorem 2.4.3, on one hand, open subsets in $\mathcal{V}$ are required to be very long in the flow direction, on the other hand, there is a uniform bound on the covering dimension. In order to control the dimension of the cover, the orders of finite subgroups of $G$ are assumed to be uniformly bounded in [10]. Though this is no longer true in our case, things are still under control. Recall in General Setup 2.4.1, the $G$-action on $FS - FS^\mathbb{R}$ is assumed to be locally finite and has bounded local orders, hence for every $x \in FS - FS^\mathbb{R}$, there exists a small ball $B_{r_x}(x)$ so that the order of $\iota_{x,r_x}(G_x) \subseteq \text{Isom}(B_{r_x}(x))$ is bounded by $k_{FS}^G$. Thus if we can arrange central
slices to be contained in such small balls, which actually can be easily done, we are still in good shape. The following two lemmas, which are analogs of Lemma 2.16 and Lemma 2.17 of [10] respectively, show such long and thin boxes exist. Properness of the action is used in their constructions, however hyperdiscreteness also suffices. In the proofs, we will write down the constructions of the boxes very carefully so as to see how hyperdiscreteness helps everything go through, but the checking work (that they are boxes with desired properties) will be omitted since they are exactly the same as in the proofs of [10].

From now on, we will, for every $x \in FS - FS^R$, fix a ball $B_{r_x}(x)$ mentioned above.

**Lemma 2.4.11.** For every $x \in FS - FS^R$ and $0 < l < \text{per}_\Phi(x)$, there exists a box $C$ satisfying

(i) $l_C = l$;

(ii) $G_C = G_x$;

(iii) $x \in SC \cap C^o$;

(iv) $SC$ is connected;

(v) $SC \subset B_{r_x}(x)$.

**Proof.** Let $x \in FS - FS^R$ and $B$ be a box constructed as in Lemma 2.4.10. By restricting to a small appropriate subset of $S_B$, we may assume $S_B \subset B_{r_x}(x)$, $l_B < l$ and $l_B + \epsilon_B < \text{per}_\Phi(x)$, where $\epsilon_B$ is as in Proposition 2.4.7 (v)

As in the proof of Lemma 2.16 of [10], one easily shows

$$\Phi_{[-l/2, l/2]}(x) \cap g\Phi_{[-l/2, l/2]}(x) \neq \emptyset \Rightarrow g \in G_x$$

Hence for any $g \notin G_x$, by Corollary 2.1.3(ii), there exists $\epsilon > 0$ such that

$$\Phi_{[-l/2, l/2]}(x)^c \cap g\Phi_{[-l/2, l/2]}(x)^c = \emptyset$$
By uniform continuity of the flow, there exists a closed ball $V_x^1$ centered at $x$ such that $\Phi_{[-l/2, t/2]}(V_x^1) \subseteq \Phi_{[-l/2, t/2]}(x)^\epsilon$, hence

$$\Phi_{[-l/2, t/2]}(V_x^1) \cap g\Phi_{[-l/2, t/2]}(V_x^1) \neq \emptyset \Rightarrow g \in G_x$$

We conclude from $l + \epsilon_B < \text{per}_G(x)$ that

$$\Phi_{[-l/2 + l_B, t/2]}(x) \cap \Phi_{[-l/2 - \epsilon_B, -l/2]}(x) = \emptyset$$

$$\Phi_{[-l/2, t/2 - l_B]}(x) \cap \Phi_{[t/2, t/2 + \epsilon_B]}(V_x^2) = \emptyset$$

By continuity of the flow and compactness, we can find another closed ball $V_x^2$ centered at $x$ such that

$$\Phi_{[-l/2 + l_B, t/2]}(V_x^2) \cap \Phi_{[-l/2 - \epsilon_B, -l/2]}(V_x^2) = \emptyset$$

$$\Phi_{[-l/2, t/2 - l_B]}(V_x^2) \cap \Phi_{[t/2, t/2 + \epsilon_B]}(V_x^2) = \emptyset$$

Put $V_x = V_x^1 \cap V_x^2$. Then $V_x$ is a closed ball centered at $x$, hence $G_x$-invariant. By construction, it has the following properties:

1. $\Phi_{[-l/2, t/2]}(V_x)$ is an $\mathcal{E}$-subset of the $G$-space $FS$;
2. $G_{\Phi_{[-l/2, t/2]}(V_x)} = G_x$;
3. $\Phi_{[-l/2 + l_B, t/2]}(V_x) \cap \Phi_{[-l/2 - \epsilon_B, -l/2]}(V_x) = \emptyset$;
4. $\Phi_{[-l/2, t/2 - l_B]}(V_x) \cap \Phi_{[t/2, t/2 + \epsilon_B]}(V_x) = \emptyset$.

Now let $T$ be the component of $S_B \cap B^0 \cap V_x^0$ and $\overline{T}$ be its closure in $S_B$, and put $C = \Phi_{[-l/2, t/2]}(\overline{T})$. One checks that $C$ is a box having all the required properties. Details can be found in [10, Lemma 2.16].

**Lemma 2.4.12.** Let $a, b, c > 0$ be real numbers satisfying $c > a + 2b$ and $K \subseteq FS > a + 2b + 2c$ a cocompact $G$-invariant subset. Then there exist a $G$-set $\Lambda$ and for every $\lambda \in \Lambda$, a point $x_\lambda$, boxes $A_\lambda, B_\lambda, C_\lambda$ with $x_\lambda \in A_\lambda \subseteq B_\lambda \subseteq C_\lambda$ such that:
(i) \( \Lambda \) is \( G \)-cofinite;
(ii) We have \( l_{A,\lambda} = a \), \( l_{B,\lambda} = a + 2b \), \( l_{C,\lambda} = a + 2b + 2c \);
(iii) \( S_{C,\lambda} \) is connected;
(iv) \( S_{A,\lambda} \subseteq S_{B,\lambda} \subseteq S_{C,\lambda} \subseteq B_{r_x}(x) \);
(v) \( A,\lambda \subseteq B,\lambda \) and \( B,\lambda \subseteq C,\lambda \);
(vi) \( K \subseteq \bigcup_{\lambda \in \Lambda} A,\lambda \);
(vii) \( gA,\lambda = A,gg \), \( gB,\lambda = B,gg \) and \( gC,\lambda = C,gg \), \( \forall g \in G, \lambda \in \Lambda \);
(viii) If \( B,\lambda \cap B,\lambda' \neq \emptyset \), then \( B,\lambda \subseteq C,\lambda' \) and \( S_{B,\lambda} \) is transversal to the flow with respect to \( C,\lambda' \).

Proof. For every \( x \in FS_{a+2b+2c} \), let \( C_x \) be a box of length \( a + 2b + 2c \) as constructed in Lemma 2.4.11. Choose a small closed ball \( T_x \) in \( S_{C_x} \cap C_x \) centered at \( x \). Clearly \( T_x \) is \( G_x \)-invariant. It can be arranged that \( C_x \) is connected. For all \( x \in FS_{a+2b+2c} \), obviously \( K \subseteq \bigcup_{x \in K} C_x(T_x; a) \), where we use the notation from Definition 2.4.6 (v). Since \( K \) is cocompact and \( G \)-invariant, there is a cofinite \( G \)-subset \( I \subseteq K \) satisfying

\[
K \subseteq \bigcup_{x \in I} C_x(T_x; a)
\]

Claim 1: Fixing \( x \in I \), for any \( y \in T_x \), there exists a closed \( G_y \)-invariant small ball \( V_y \subseteq T_x \) centered at \( y \) such that for all \( z \in I \)

\[
C_x(\{y\}; a + 2b) \cap C_z(T_z; a + 2b) = \emptyset \Rightarrow C_x(V_y; a + 2b) \cap C_z(T_z; a + 2b) = \emptyset
\]

Proof. Let \( \epsilon > 0 \) be such that for all \( g \in G \),

\[
C_x(\{y\}; a + 2b)^\epsilon \cap gC_z(T_z; a + 2b) = \emptyset
\]

whenever \( C_x(\{y\}; a + 2b) \cap gC_z(T_z; a + 2b) = \emptyset \). Now since \( I \) is \( G \)-cofinite, one sees that there exists \( \epsilon > 0 \) such
that for all \( z \in I \)

\[
C_x(\{y\}; a + 2b) \cap C_z(T_z; a + 2b) = \emptyset \Rightarrow C_x(y; a + 2b)^\circ \cap C_z(T_z; a + 2b) = \emptyset
\]

Hence by uniform continuity of the flow, one can find \( V_y \) as desired.

For every \( y \in T_x \), define \( I_y = \{ z \in I \mid C_x(\{y\}; a + 2b) \cap C_z(T_z; a + 2b) \neq \emptyset \} \)

Claim 2: \( I_y \) is finite.

Proof. Suppose not, then the set \( I'_y = \{ z \in I \mid C_x(\{y\}; a + 2b) \cap C_z(T_z; a + 2b) \neq \emptyset \} \)

is infinite. Now since \( I \) is \( G \)-cofinite, \( I'_y \) must contain infinitely many elements from

one orbit, hence there exists \( z_0 \in I \), such that the set \( \{ gz_0 \in I \mid C_x(\{y\}; a + 2b) \cap gC_{z_0}(T_{z_0}; a + 2b) \neq \emptyset, g \in G \} \)

is infinite, hence the set \( \{ gz_0(T_{z_0}; a + 2b) \mid C_x(\{y\}; a + 2b) \cap gC_{z_0}(T_{z_0}; a + 2b) \neq \emptyset, g \in G \} \)

is infinite which contradicts the fact that the \( G \)-action on \( FS \) is hyperdiscrete. This proves claim 2.

Note for \( z \in I_y \), we have \( C_x(\{y\}; a + 2b) \subseteq C_z^\circ \), since \( a + 2b < c \) and \( l_C = a + 2b + 2c \).

Thus we can find for \( z \in I_y \), a closed \( G_y \)-invariant small ball \( U_y(z) \subseteq T_x \) of \( y \) so that

\( C_x(U_y(z); a + 2b) \) is contained in \( C_z^\circ \) because \( C_x(\{y\}; a + 2b) \) is compact and the

flow is continuous. By applying Lemma 2.4.8 if necessary, we may assume \( U_y(z) \) is

transversal to the flow with respect to \( C_z \).

For each \( y \in T_x \), put \( V_y = V_y \cap \bigcap_{z \in I_y} U_y(z) \) and choose a closed \( G_y \)-invariant

small ball \( W_y \subseteq T_x \) centered at \( y \) such that \( W_y \subseteq U_y^\circ \). Since \( T_x \) is compact, we

can find \( y(x)_1, y(x)_2, \cdots, y(x)_{n(x)} \) in \( T_x \) so that \( T_x = \bigcup_{i=1}^{n(x)} W_{y(x)_i} \). We can arrange

\( W_{gy} = gW_y, n(x) = n(gx) \) and \( y(gx)_i = gy(x)_i, \forall g \in G \).

Finally define \( \Lambda = \{ y(x)_i \mid x \in I, i = 1, 2, \cdots, n(x) \} \) and for \( \lambda \in \Lambda \), define

\( C_\lambda = C_x, B_\lambda = C_x(U_y(x)_i; a + 2b) \) and \( A_\lambda = C_x(W_{y(x)_i}; a) \). One checks as in the proof

of [10, Lemma 2.17], these boxes have all the required properties. This completes the proof. \( \square \)
The next major step in [10] is the proof of their Proposition 4.1. An analog can still be proved in our setting. We will firstly state the analog and then explain why it still holds.

**Proposition 2.4.13.** There exists a number \( M = M(k^G_{FS}, d) \) depending only on \( k^G_{FS} \) and \( d \) with the following properties:

For every \( \alpha, \epsilon \in \mathbb{R} \) with \( 0 < \epsilon < \alpha \), there exists \( \gamma = \gamma(\alpha, \epsilon, M) > 0 \) such that for every cocompact \( G \)-invariant subset \( K \) of \( FS_{> \gamma} \), there exists a collection \( \mathcal{D} \) of boxes satisfying:

(i) \( K \subseteq \bigcup_{D \in \mathcal{D}} \Phi_{(-\epsilon, \epsilon)}(D^\circ) \);

(ii) For every \( x \in FS \) which lies on the open bottom or open top of a box in \( \mathcal{D} \), the set \( \Phi_{[-\alpha, -\epsilon] \cup [\epsilon, \alpha]}(x) \) does not intersect the open bottom or open top of a box in \( \mathcal{D} \);

(iii) For every \( x \in FS \), there is no box \( D \in \mathcal{D} \) such that \( \Phi_{[0, \alpha]}(x) \) intersects both the open bottom and open top of \( D \);

(iv) The covering dimension of the collection \( \{ D^\circ \mid D \in \mathcal{D} \} \) is less than or equal to \( M \);

(v) For \( g \in G, D \in \mathcal{D} \), we have \( gD \in \mathcal{D} \);

(vi) There exists a finite subset \( \mathcal{D}_0 \subseteq \mathcal{D} \) such that for every \( D \in \mathcal{D} \), there exists \( g \in G \) with \( gD \in \mathcal{D}_0 \);

(vii) \( \Phi_{[-\alpha, -\epsilon, \alpha, \epsilon]}(D) \) is an \( \mathcal{E} \)-subset of \( FS \) for all \( D \in \mathcal{D} \).

The difference between Proposition 2.4.13 above and Proposition 4.1 of [10] are only in two places: firstly, the number \( M \) here depends on \( k^G_{FS} \) and \( d \). In [10], this number depends on \( k_G \) and \( d \), where the two \( d \)'s have the same meaning, while their \( k_G \), which is assumed to be finite, is the supremum of the orders of all finite subgroups of \( G \). Clearly, \( k^G_{FS} \) is a natural replacement for \( k_G \), because we replace properness by
locally finiteness. The second difference is in part (vii), we replace their $\mathcal{FIN}$ (the family of finite subgroups) by the family $\mathcal{E}$. Recall $\mathcal{E}$ is the family of subgroups of $G$ generated by stabilizers.

Let us sketch the main idea of the proof of [10, Proposition 4.1]. The collection $\mathcal{D}$ is produced inductively out of the boxes $A_\lambda, B_\lambda, C_\lambda$ constructed in Lemma 2.4.11. Let $N := |G\backslash \Lambda|$, the number of $G$-orbits. It is shown inductively for $r = 0, 1, 2, \ldots, N$ that for any subset $\Xi \subseteq \Lambda$ consisting of $r$ $G$-orbits, there exists, roughly speaking, a collection of boxes satisfying those properties listed in the proposition. The inductive argument takes place in the interior of $C_\lambda$ and depends on the fact that $G_{C_\lambda} = G_{S_{C_\lambda}} = G_{x_\lambda}$ is finite and also on the fact that the orders of stabilizers are uniformly bounded (which is guaranteed by the assumption $k_G < \infty$). These properties enable one to obtain some general position property for the collections of boxes, so that the covering dimension is under control (Proposition 3.2, 3.3 of [10]). Those arguments still go through in our case because they are local and we have constructed thin enough boxes so that $S_{C_\lambda} \subseteq B_{r_{x_\lambda}} (x_\lambda)$ and hence $G_{C_\lambda} = G_{S_{C_\lambda}}$ acts as a finite group on $C_\lambda$, and there is also a uniform bound (which is $k_{FS}$) on these local orders. Hyperdiscreteness of the action has to be used in several places, but it is used in the same way as in the proof of Lemma 2.4.10 and Lemma 2.4.11: roughly speaking, only finitely many compact subsets in the same orbit can squeeze in a small place. Having such observations, it is not hard to see Proposition 2.4.13 holds here. We will not present the proof here, since it goes exactly as the proof of [10, Proposition 4.1] and it is very long and technical. Interested readers may consult [10].

Now the proof of Theorem 2.4.3 goes exactly as the argument on Page 1848 of [10]. For completeness, we carry over the proof here.
Proof of Theorem 2.4.3. Let $M = M(k^G_{FS}, d)$ be as in Proposition 2.4.13 and $N = 2M + 1$. For any $\alpha > 0$, let $\epsilon > 0$ with $0 < \epsilon < \alpha$ and $\gamma = \gamma(M, 4\alpha, \epsilon)$ be the number with respect to $(4\alpha, \epsilon)$ in Proposition 2.4.13. Now for any compact $K \subseteq FS_{>\gamma}$, let $\mathcal{D}$ be the collection of boxes in Proposition 2.4.13. Put $\mathcal{V} = \{\Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D^0) \mid D \in \mathcal{D}\}$. Clearly, $\mathcal{V}$ satisfies properties (i), (iii), (iv) in Theorem 2.4.3. We show that $\dim \mathcal{V} \leq N = 2M + 1$. Suppose not, then there exist distinct elements $D_1, D_2, \ldots, D_{2M+3}$ of $\mathcal{D}$ such that $\bigcap_{k=1}^{2M+3} \Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D^0_k) \neq \emptyset$. Let $x$ be a point in the intersection. Note $x \in \Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D^0_k)$ implies $\Phi_{2\alpha}(x) \in D^0_k$ or $\Phi_{-2\alpha}(x) \in D^0_k$; otherwise, since $\epsilon < \alpha$, there will be a point $y \in FS$ with $\Phi_{[0,4\alpha]}(y)$ intersecting both the open bottom and open top of $D_k$ which violates Proposition 2.4.13 (iii). Hence we can find $M + 2$ pairwise distinct elements among $D^0_1, D^0_2, \ldots, D^0_{2M+3}$ whose intersection is nonempty which contradicts to Proposition 2.4.13 (iv). This completes the proof. \qed

2.5 Main Theorem

**Theorem 2.5.1.** Let $X$ be a finite dimensional proper CAT(0)-space. Suppose $G$ admits an action on $X$ by isometries with compact quotient and the induced action of $G$ is discrete on $FS(X)$, hyperdiscrete and locally finite on $FS(X) - FS(X)^G$. Then

(1) The $K$-theoretic Farrell-Jones conjecture holds for $G$ relative to the family of subgroups $\mathcal{F}$;

(2) The $L$-theoretic Farrell-Jones conjecture holds for $G$ relative to the family of subgroups $\mathcal{F}_2$,

where $\mathcal{F}$ is the family of subgroups of $G$ generated by subgroups $F < G$ admitting a short exact sequence $1 \to G_x \to F \to \mathbb{Z} \to 1$ for some $x \in X$ and $\mathcal{F}_2$ is the family of subgroups consisting of $V < G$ so that there is $F \in \mathcal{F}$ with $[V : F] \leq 2$. 

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Proof. We show the group $G$ is strongly transfer reducible relative to the family $F$, then by Theorem 1.3.12, we obtain the main theorem.

Now our assumptions together with Proposition 2.2.2 show the assumptions of General setup 2.4.1 are satisfied, hence Theorem 2.4.3 holds. Again according to the assumption Theorem 2.3.1 holds. Now because the action of $G$ on $X$ is assumed to be cocompact, there is compact $K_0 \subseteq X$ so that $G \cdot K_0 = X$, fix $K_0$, choose $M > 0$ to be as in Theorem 2.3.1 with respect to this fixed choice. Then the argument in [7, Section 6.3] shows that $FS(X)$ admits long $F$-covers at infinity and periodic flow lines, i.e. there is $M > 0$ such that for every $\gamma > 0$ there is a collection $\mathcal{V}$ of open $F$-subsets of $FS(X)$ and $\epsilon > 0$ satisfying:

(i) $\mathcal{V}$ is $G$-invariant;

(ii) $\dim \mathcal{V} \leq M$;

(iii) there is a compact subset $K \subseteq FS(X)$ such that

$FS(X)_{\leq \gamma} \cap G \cdot K = \emptyset$

$\cdot$ for $z \in FS(X) - G \cdot K$ there is $V \in \mathcal{V}$ s.t. $B_\epsilon(\Phi_{[-\gamma,\gamma]}(z)) \subseteq V$

This fact together with Theorem 2.4.3 imply $FS(X)$ admitting long thin covers [7, Theorem 5.7], i.e. there is $\hat{N} \in \mathbb{N}$ such that for every $\alpha > 0$ there exists an open $F$-cover $U$ of $FS(X)$ of dimension at most $\hat{N}$ and $\epsilon > 0$ such that the following holds:

(i) For every $c \in FS(X)$, there is $U \in U$ such that $B_\epsilon(\Phi_{[-\alpha,\alpha]}(c)) \subseteq U$;

(ii) $G \setminus U$ is finite.

Set $N := \max\{\hat{N}, 2\dim(X) + 1\}$.

Now for any $x_0 \in X$ and $R > 0$, the closed ball $\overline{B}_R(x_0) \subseteq X$ is a compact contractible controlled $(2\cdot \dim(X)+1)$-dominated metric space by [7, Lemma 6.2]. Projection along geodesics starting at $x_0$ gives rise to a deformation retraction $H^R$ from
$X$ to $\overline{B}_R(x_0)$ and this in turn gives rise to a strong homotopy action $\Psi^R$ of $G$ on $\overline{B}_R(x_0)$, see [70, Remark 2.2].

Now let $S \subseteq G$ and $k, n \in \mathbb{N}$ be as in Definition 1.3.11 and fix a base point $x_0 \in X$. According to [70, Lemma 3.5] (which is true for general group actions on CAT(0)-spaces), there exists $\alpha(S, k, n) > 0$ with the following property:

For all $\epsilon > 0$, there are $R, T > 0$ such that for every $(g, x) \in G \times \overline{B}_R(x_0)$ and $(h, y) \in S^n_{\Psi^R}(g, x)$ there is $\tau \in [-\alpha, \alpha]$ with

$$d_{FS(X)}(\Phi_T \circ \iota(g, x), \Phi_{T+\tau} \circ \iota(h, y)) \leq \epsilon$$

For the $\alpha$ above, there is $\epsilon > 0$ and a long thin cover $\mathcal{U}$ of $FS(X)$ with certain properties mentioned earlier. For this $\epsilon > 0$, let $R, T > 0$ be as in above and we choose $\overline{B}_R(x_0)$ to be our compact contractible controlled $N$-dominated space. Consider the transfer map

$$\iota : G \times \overline{B}_R(x_0) \to FS(X), \quad (g, x) \mapsto c_{g x_0, g x}$$

where $c_{g x_0, g x}$ is the unique generalized geodesic with $(c_{g x_0, g x})_- = 0, c(-\infty) = g x_0$, and $c(\infty) = g x$. Note that $\iota$ is $G$-equivariant, then the pullback $(\Phi_T \circ \iota)^{-1}(\mathcal{U})$ gives rise to an open $\mathcal{F}$-cover $\mathcal{V}$ of $G \times \overline{B}_R(x_0) \to FS(X)$ and a similar argument as in the proof of [7, Lemma 5.12, Theorem 5.11] shows that $\mathcal{V}$ has the properties required in the definition of strongly transfer reducibility. Hence $G$ is strongly transfer reducible relative to $\mathcal{F}$. This completes the proof.

As a corollary, we have:

**Corollary 2.5.2.** With the above assumptions, if in addition every point stabilizer of the $G$-action on $X$ is virtually solvable, then both of the $K$-and $L$-theoretic Farrell-Jones conjecture holds for $G$. 

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Proof. If every point stabilizer is virtually solvable, then the families \( \mathcal{F} \) and \( \mathcal{F}_2 \) appeared in Theorem 2.5.1 consist of virtually solvable groups, which satisfy the Farrell-Jones conjecture by the work of Wegner [71]. Hence by Transitivity Principle 1.2.4, the Farrell-Jones conjecture holds for \( G \).
Chapter 3: Passage to Overgroups of Finite Index

As already mentioned in Chapter 1 (see Remark 1.2.2), the Farrell-Jones conjecture has some nice inheritance properties, for example, it is inherited by subgroups. This section studies the opposite direction: whether the conjecture is inherited by over-groups of finite index. If this is true, then the conjecture is commensurably invariant. More generally, one could ask if the conjecture is quasi-isometric invariant.

Remark 3.0.3. One thing to note is that there is a variant of the conjecture which is the Farrell-Jones conjecture with wreath product. One says $G$ satisfies Farrell-Jones conjecture with wreath product if $G \wr F$ satisfies the Farrell-Jones conjecture for any finite group $F$. Recall $G \wr F = G^{(|F|)} \rtimes F$, where $F$ acts by translation on the coordinates. This version of the conjecture is closed by passing to overgroups of finite index and it is known that CAT(0)-groups, hyperbolic groups and $GL_n(R)$ with $R$ a ring whose underlying abelian group is finitely generated, satisfy Farrell-Jones conjecture with wreath product [12]. Nevertheless, it’s still unknown whether the usual version of the conjecture is closed under passing to overgroups of finite index.

We use the controlled algebra approach to the Farrell-Jones conjecture, as described in Section 1.3.1, to study this problem. Let $H < G$ be a subgroup of finite index and suppose the conjecture holds for $H$. Then according to Theorem 1.3.6, $K_*(\mathcal{O}^H(E_{WC} H)) = 0$. Now the idea is to construct a restriction functor
\[ \tau : \mathcal{O}^G(E_{VC}G) \to \mathcal{O}^H(E_{VC}H) \] and an induction functor \( \alpha : \mathcal{O}^H(E_{VC}H) \to \mathcal{O}^G(E_{VC}G) \), then study their composition. Ideally, one would expect, on the level of \( K \)-groups, the composition is multiplication by \( n \), where \( n \) is the index of \( H \) in \( G \) (this phenomena happens in group homology). If this is true, then the rational Farrell-Jones conjecture is inherited by over-groups of finite index. Our main result strongly suggests this is the case.

This chapter is organized as follows. In Section 3.1, we reduce the problem to a special case, which is indeed needed to prove our main theorem. In Section 3.2, we construct, for every \( G \)-CW complex \( X \) (which is automatically an \( H \)-CW complex), an induction functor \( \alpha : \mathcal{O}^H(X) \to \mathcal{O}^G(X) \) and a restriction functor \( \tau : \mathcal{O}^G(X) \to \mathcal{O}^H(X) \). The induction functor always exists, even if the subgroup is not of finite index, but the restriction functor exists only when the subgroup is of finite index. In the case when \( X = E_{VC}G \), we can also view \( X \) as a model for \( E_{VC}H \) and we are in the situation described in the previous paragraph. In Section 3.3, we study the composition of these two functors and prove the main theorem.

### 3.1 Reduction to a Special Case

**Lemma 3.1.1.** The following statements are equivalent:

1. Let \( H < G \) be a subgroup of finite index. If \( H \) satisfies the Farrell-Jones conjecture with coefficient in any additive category, then \( G \) satisfies the conjecture with coefficient in any additive category;

2. Let \( G = N \rtimes F \) be the semi-direct product of a normal subgroup \( N \) and a finite subgroup \( F \). If \( N \) satisfies the Farrell-Jones conjecture with coefficient in any additive category, then \( G \) satisfies the conjecture with coefficient in any additive category.
Proof. The only nontrivial direction is (2) ⇒ (1). So suppose (2) is true. Let $H < G$ be as in (1). Consider the translation action of $G$ on $G/H$, it induces a group homomorphism $G \to Aut(G/H)$, denote its kernel by $N$. It is a normal subgroup of $G$ and $N < H$. Note $[G : N]$ is finite since $Aut(G/H)$ is finite. By assumption, $H$ satisfies FJC with coefficient in any additive category, hence $N$ satisfies FJC with coefficient in any additive category. Now let $F = G/N$ and consider the wreath product $N \wr F = N[^F] \rtimes F$. Since $N$ satisfies FJC, it follows $N[^F]$ satisfies FJC since FJC is closed under taking direct sum, see [8, Lemma 2.3]. Hence by assumption $N \wr F$ satisfies FJC. Therefore $G$ satisfies FJC because $G$ injects into $N \wr F$, see for example [38]. This completes the proof.

3.2 Induction and Restriction

If $\phi : H \to G$ is a group homomorphism, then the $H$-equivariant and $G$-equivariant homology theories used in the Farrell-Jones conjecture 1.2.1, 1.2.2 are related by some induction structure, i.e. $\phi$ induces a homomorphism $\phi_* : H_*^H(X) \to H_*^G(\phi_*X)$ for any $H$-space $X$, where $\phi_*X = G \times_H X$ (we omit the coefficient here for simplicity). This is useful in studying inheritance properties of the conjecture, see [4]. One advantage of the controlled algebra approach to the conjecture is that, while it allows us to construct an induction map, it also enables us to construct a restriction map if $H$ is a finite index subgroup of $G$, both on the level of obstruction category.

3.2.1 Induction

Let $H < G$ be groups (not necessary of finite index or normal), $X$ be a $G$-CW complex and $\mathcal{A}$ be an additive category with right $G$-action. So $H$ inherits actions on both $X$ and $\mathcal{A}$. The identity of $G$ is denoted by $e$. Define the induction functor.
\( \alpha_X \) (we will omit \( X \) from the subscript if everything is clear) as follows:

\[
\alpha : \mathcal{O}^H(X;A) \to \mathcal{O}^G(X;A)
\]

**On objects:**

\[
(\alpha A)_{(g,x,t)} = (g^{-1})^*(A_{(e,g^{-1}x,t)})
\]

for any object \( A \) in \( \mathcal{O}^H(X) \) and point \((g,x,t) \in G \times X \times [1, \infty)\)

**On morphisms:**

\[
(\alpha \phi)_{(g',x',t'),(g,x,t)} = \begin{cases} 
0 & \text{if } g^{-1}g' \notin H \\
(g^{-1})^*(\phi_{(g^{-1}g',g^{-1}x',t'),(e,g^{-1}x,t)}) & \text{if } g^{-1}g' \in H
\end{cases}
\]

for any morphism \( \phi \) in \( \mathcal{O}^H(X) \) and \((g,x,t), (g',x',t') \in G \times X \times [1, \infty).\)

There are many things to check (we will omit the \( t \) component since it is irrelevant).

1. **\( \alpha A \) is \( G \)-invariant:** for any \( l \in G \), we have

\[
[l^*(\alpha A)]_{(g,x)} = l^*[l(g,x)] = (g^{-1}l^{-1})^*[A_{(e,g^{-1}x)}] = (g^{-1})^*(A_{(e,g^{-1}x)}) = (\alpha A)_{(g,x)}
\]

hence \( l^*(\alpha A) = \alpha A \);

2. **\( \alpha \phi \) is \( G \)-invariant:** for any \( l \in G \), we have, if \( g^{-1}g' \in H \),

\[
[l^*(\alpha \phi)]_{(g',x',g,x)} = l^*[l(g',x')l(g,x)] = (g^{-1}l^{-1})^*[\phi_{(g^{-1}g',g^{-1}x'),(e,g^{-1}x)}] = (\alpha \phi)_{(g',x'),(g,x)}
\]

one also sees from the above equalities that if \( g^{-1}g' \notin H \), then both sides are 0, hence

\( l^*(\alpha \phi) = \alpha \phi \);

3. **\( \alpha(id) = id \):**

\[
[\alpha(id)]_{(g',x'),(g,x)} = \begin{cases} 
0 & \text{if } g^{-1}g' \notin H \\
(g^{-1})^*[id_{(g^{-1}g',g^{-1}x'),(e,g^{-1}x)}] & \text{if } g^{-1}g' \in H
\end{cases} = \begin{cases} 
0 & \text{if } (g,x) \neq (g',x') \\
id_{(g',x'),(g,x)} & \text{if } (g,x) = (g',x')
\end{cases}
\]
(4) $\alpha(\phi \circ \psi) = (\alpha\phi) \circ (\alpha\psi)$: we only consider the case when $g^{-1}g' \in H$. Now on one hand,

$$[\alpha(\phi \circ \psi)](g',x'),(g,x) = (g^{-1})^*[(\phi \circ \psi)(g^{-1}g',g^{-1}x'),(e,g^{-1}x)]$$

$$= (g^{-1})^*[\phi(g^{-1}g',g^{-1}x') \circ \psi(h,y),(e,g^{-1}x)] \quad (3.3)$$

Note we are taking sum over $(h,y) \in H \times X$.

On the other hand

$$[(\alpha\phi) \circ (\alpha\psi)](g',x'),(g,x) = (\alpha\phi)(g',x') \circ (\alpha\psi)(g,x)$$

Note $l$ is ranging over $G$ right now, however, only those terms with $g^{-1}l \in H$ are nonzero, hence we may let $l = gh$ and the right hand side of the above equality becomes

$$(\alpha\phi)(g',x') \circ (\alpha\psi)(gh,y),(g,x)$$

$$= (g^{-1})^*[\phi(g^{-1}g',g^{-1}x'),(h,g^{-1}y)] \circ (g^{-1})^*[\psi(h,g^{-1}y),(e,g^{-1}x)]$$

$$= (g^{-1})^*\{[\phi(g^{-1}g',g^{-1}x'),(h,g^{-1}y)] \circ [\psi(h,g^{-1}y),(e,g^{-1}x)]\} \quad (3.4)$$

Comparing (3.3) and (3.4) we get $\alpha(\phi \circ \psi) = (\alpha\phi) \circ (\alpha\psi)$;

(5) Additivity: one easily verifies $\alpha(A \oplus B) = (\alpha A) \oplus (\alpha B)$;

(6) Object support: for any object $A$ in $\mathcal{O}^H(X)$, by definition, $(g,x,t) \in \text{supp}(\alpha A)$ implies $(e,g^{-1}x,t) \in \text{supp}A$. Elements of this form must be contained in a compact subset $K \subset X$ by the support condition on $A$, hence $\text{supp}(\alpha A) \subset G \cdot K$ which is as required;

(7) Morphism support: for any morphism $\phi$ in $\mathcal{O}^H(X)$, there exists $E \in \mathcal{E}^X_{Hcc}$ such that the projection of $\text{supp}\phi$ in $(X \times [1, \infty))^2$ is contained in $E$. By definition of $\alpha\phi$ and the fact that every object support is $G$-cocompact, there exists a compact subset
$K \subset X$ such that the projection of $\text{supp}(\alpha \phi)$ in $(X \times [1, \infty))^2$ is contained in $E' = \{(x, t) \times (x', t') \mid \exists g, g' \in G \text{ s.t. } g^{-1}g' \in H, g^{-1}x, g'^{-1}x' \in K, (g^{-1}x, t) \times (g^{-1}x', t') \in E\}$.

We show $E' \in E^X_{Gcc}$ which implies $\alpha$ respects the morphism support condition.

(i) $E'$ is symmetric: suppose $(x, t) \times (x', t') \in E'$, then we have $g, g' \in G$ with the properties in the definition of $E'$. Now since $E$ is symmetric, $(g^{-1}x', t') \times (g^{-1}x, t) \in E$ and since $E$ is $H$-invariant and $g^{-1}g' \in H$, we get $(g'^{-1}x', t') \times (g^{-1}x, t) \in E$. This shows $g', g$ fulfill the properties for the pair $(x', t') \times (x, t)$ in the definition of $E'$.

Hence $E'$ is symmetric;

(ii) $E'$ is $G$-invariant: suppose $(x, t) \times (x', t') \in E'$ and $l \in G$. Let $g, g'$ as before.

One easily verifies that $lg, lg'$ send $(lx, t) \times (lx', t')$ into $E'$, hence $E'$ is $G$-invariant;

(iii) Bounded control in the $t$ direction: this is clear since $E$ is controlled;

(iv) $E'$ is $G$-equivariantly continuously controlled: suppose not, then there exists $x_0 \in X$ and $G_{x_0}$-invariant open neighborhood $W$ of $(x_0, \infty] \subset X \times [1, \infty]$ such that for all open set $V \subset U$, we have

$$(W^c \times V) \cap E' \neq \emptyset$$

We may assume $W = U \times (r, \infty]$ for some open $G_{x_0}$-invariant neighborhood $U$ of $x_0$ in $X$ by the following argument: we first can find an open neighborhood $U_{x_0}$ of $x_0$ in $X$ and $r > 0$ so that $U_{x_0} \times (r, \infty] \subset W$. We then have for any $g \in G_{x_0}$, $gU_{x_0} \times (r, \infty] \subset W$ since $W$ is $G_{x_0}$-invariant. Hence $(\bigcup_{g \in G_{x_0}} gU_{x_0}) \times (r, \infty] \subset W$. Then $U = \bigcup_{g \in G_{x_0}} gU_{x_0}$ is as desired. Note $U \times (r, \infty] \subset W$ implies $((U \times (r, \infty))^c \times V) \cap E' \neq \emptyset$

Now since $X$ is a $G$-CW complex, the Slice theorem (see [5]) applies, so that we can find a descending sequence $\{V^k\}_{k \in \mathbb{N}}$ of open neighborhoods of $x_0$ with the following property:

(a) Each $V^k$ is $G_{x_0}$-invariant;
(b) \( gV^k \cap V^k = \emptyset \) if \( g \notin G_{x_0} \);

(c) \( \bigcap_{k \geq 1} G \cdot V^k = G \cdot x_0 \)

We may assume \( V^k \subset U \) and hence

\[
[(U \times (r, \infty))^c \times (V^k \times (k + r, \infty))] \cap E' \neq \emptyset
\]

thus we can find a sequence

\[
(x_k, t_k) \times (x'_k, t'_k) \in [(U \times (r, \infty))^c \times (V^k \times (k + r, \infty))] \cap E' \quad (3.5)
\]

By the definition of \( E' \), there exist \( g_k, g'_k \) such that

\[
g_k^{-1}g'_k \in H, \ g_k^{-1}x_k, g'_k^{-1}x'_k \in K, \ (g_k^{-1}x_k, t_k) \times (g'_k^{-1}x'_k, t'_k) \in E \quad (3.6)
\]

Since \( E \) is \( H \)-invariant and \( g_k^{-1}g'_k \in H \), the above also implies

\[
(g_k^{-1}x_k, t_k) \times (g'_k^{-1}x'_k, t'_k) \in E \quad (3.7)
\]

Now since \( K \) is compact, by passing to a subsequence, we may assume \( g'_k^{-1}x'_k \to y \). Now since \( x'_k \in V^k \), we have for every \( n \in \mathbb{N} \) and all \( k > n \), \( g'_k^{-1}x'_k \in G \cdot V^k \subset G \cdot V^n \), this implies \( y \in G \cdot V^n \) for all \( n \), thus \( y \in \bigcap_{n \geq 1} G \cdot V^n = G \cdot x_0 \). Hence there is \( g \in G \) so that \( y = gx_0 \). Therefore \( g'_k^{-1}x'_k \to y = gx_0 \), so \( g^{-1}g'_k^{-1}x'_k \to x_0 \in V^1 \). Thus when \( k \) is large enough \( g^{-1}g'_k^{-1} \in G_{x_0} \). Note that \( t'_k \to \infty \), hence by (3.7) and the control condition in the \( t \) direction, we have \( t_k \to \infty \). Now by (3.5), when \( k \) is large enough, \( x_k \notin U \), hence \( g^{-1}g'_k^{-1}x_k \notin U \) since \( g^{-1}g'_k^{-1} \in G_{x_0} \) and \( U \) is \( G_{x_0} \)-invariant. Thus when \( k \) is large enough, \( g'_k^{-1}x_k \notin gU \).

Now consider \( gU \), it is \( G_y \)-invariant since \( y = gx_0 \). Hence \( H_y \)-invariant. Now because \( E \) is \( H \)-equivariantly continuously controlled, we can find an \( H_x \)-invariant open neighborhood \( W \) of \( y = gx_0 \) and \( N > 1 \) such that

\[
[(gU \times [r, \infty])^c \times (W \times [N, \infty])] \cap E = \emptyset
\]
However, on the one hand, \((g_k'^{-1}x_k, t_k) \times (g_k'^{-1}x'_k, t'_k) \in E\) for all \(k\) by (3.7), while on the other hand, when \(k\) is large enough, we showed \(g_k'^{-1}x_k \notin gU, g_k'^{-1}x'_k \to y \in W\) and \(t'_k \to \infty\), so we also have \((g_k'^{-1}x_k, t_k) \times (g_k'^{-1}x'_k, t'_k) \in (gU \times [r, \infty])^c \times (W \times [N, \infty])\). We thus get a contradiction, which completes the proof.

We now finished all the verification work, so \(\alpha\) defines a genuine additive functor. \(\alpha\) of course also induces a functor \(D^H(X; A) \to D^G(X; A)\), but we will not need it.

### 3.2.2 Restriction

Let us now turn to the construction of the restriction functor. It actually has already been used in [5] in the special case when the coefficient is a ring. However for our purpose, we need a more explicit form of it. We will construct the restriction in two steps: we first note that there is a natural inclusion of additive categories (the control condition will be checked later):

\[
\iota : \mathcal{O}^G(X) = \mathcal{C}^G(G \times X \times [1, \infty), \mathcal{E}_{GCC}^X, \mathcal{F}_{GCC}) \to \mathcal{C}^H(G \times X \times [1, \infty), \mathcal{E}_{HCC}^X, \mathcal{F}_{HCC})
\]

and then by summing over the fiber \(G/H\) of the bundle \(G \cong H \times G/H \to H\) we get

\[
s : \mathcal{C}^H(G \times X \times [1, \infty), \mathcal{E}_{HCC}^X, \mathcal{F}_{HCC}) \to \mathcal{C}^H(H \times X \times [1, \infty), \mathcal{E}_{HCC}^X, \mathcal{F}_{HCC}) = \mathcal{O}^H(X)
\]

The restriction map \(\tau : \mathcal{O}^G(X) \to \mathcal{O}^H(X)\) will then be the composition of them. To achieve this, we will have to assume \(H < G\) is of finite index, say \(n\) and choose a complete set of representatives \(\{g_1, g_2, \cdots, g_n\}\) for the left action of \(H\) on \(G\) so that \(G \cong H \times \{g_1, g_2, \cdots, g_n\}\) as left \(H\)-spaces (of course with discrete topology)

To see the inclusion \(\iota\) is well-defined, we have to show it respects the support conditions. Clearly it respects the object support condition since \(H < G\) is of finite index. It respects the morphism support condition because of the following lemma:
Lemma 3.2.1. Suppose \([G : H] < \infty\), then \(E^X_{Gcc} \subseteq E^X_{Hcc}\)

Proof. The only nontrivial part is (1) of definition 1.3.2. Let \(E \in E^X_{Gcc}\) and \(U\) be an \(H_x\)-invariant open neighborhood of \((x, \infty)\) in \(X \times [1, \infty]\). Let \(W = \bigcap_{g \in G_x} gU\), this is an open set since \([G_x : H_x] \leq [G : H] < \infty\) and \(U\) is \(H_x\)-invariant. Obviously \(W\) is \(G_x\)-invariant, so since \(E \in E^X_{Gcc}\), there is an \(G_x\)-invariant open neighborhood \(V\) of \((x, \infty)\) in \(X \times [1, \infty]\) such that \((W^c \times V) \cap E = \emptyset\), therefore \((U^c \times V) \cap E = \emptyset\), this completes the proof.

Let us now turn to the construction of the map \(s\). Recall that we have an identification of left \(H\)-spaces: \(G \cong H \times \{g_1, g_2, \cdots, g_n\}\). It induces an identification of left \(H\)-spaces:

\[
G \times X \times [1, \infty) \cong H \times \{g_1, g_2, \cdots, g_n\} \times X \times [1, \infty)
\]

Now the map \(s\) is induced by the \(H\)-equivariant projection:

\[
p : H \times \{g_1, g_2, \cdots, g_n\} \times X \times [1, \infty) \to H \times X \times [1, \infty)
\]

It respects the support conditions. Let us write out the functor explicitly:

On objects: \((sA)_{(h,x,t)} = \bigoplus_{i=1}^n A_{(h g_i, x, t)}\)

On morphisms: for \(\phi : A \to B\), \(s\phi : sA \to sB\) is the unique map determined by the universal properties of direct sum and product and hence given by a matrix of morphisms on pairs of points:

\[
(s\phi)_{(h', x', t'), (h, x, t)} = \bigoplus_{j=1}^n A_{(h g_j, x, t)} \to \bigoplus_{i=1}^n B_{(h' g_i, x', t')}
\]

The restriction map \(\tau\) is then given by the same formula since \(\iota\) is an inclusion.

It’s not hard to see that the functor \(s\) actually induces an equivalence of categories.
Actually when $H < G$ is of finite index, the projection $G \times X \to X$ is a resolution of the $H$-space $X$ and one can use any resolution to define the obstruction category, they are all equivalent. For the definition of resolution and more details, see [5]

### 3.3 Main Theorem

This section is devoted to the proof of the following main theorem of this chapter.

**Theorem 3.3.1.** Let $G = N \rtimes F$ with $F$ a finite group of order $n$ and $A$ be a $G$-additive category (with involution). Then for any $G$-CW complex $X$, there is an additive functor $\Omega : \mathcal{O}^G(X; A) \to \mathcal{O}^G(X; A)$ with the following properties:

1. $\Omega$ factors through $\mathcal{O}^N(X; A)$: there exist a restriction functor $\tau : \mathcal{O}^G(X; A) \to \mathcal{O}^N(X; A)$ and an induction functor $\alpha : \mathcal{O}^N(X; A) \to \mathcal{O}^G(X; A)$ so that $\Omega = \alpha \circ \tau$;

2. $\Omega$ extends to a functor $\overline{\Omega} : \mathcal{O}^N_G(X; A) \to \mathcal{O}^N_G(X; A)$, where $\mathcal{O}^N_G(X; A)$ is defined in Definition 1.3.3 (4). It is an additive category (with involution) containing $\mathcal{O}^G(X; A)$ as a subcategory, see also Remark 1.3.4;

3. The extended functor $\overline{\Omega}$ is naturally isomorphic to the diagonal functor $\Delta^n : \mathcal{O}^N_G(X; A) \to \mathcal{O}^N_G(X; A)$, $\Delta^n(A) = A^n$ on any object $A$ and $\Delta^n(\phi) = \phi^n$ on any morphism $\phi$;

4. On the level of $K$-groups, $\overline{\Omega}_* = n \times : K_*(\mathcal{O}^N_G(X; A)) \to K_*(\mathcal{O}^N_G(X; A))$ is multiplication by $n$ for all $i \in \mathbb{Z}$. Therefore we have the following commutative diagram

\[
\begin{array}{ccc}
K_i(\mathcal{O}^N(X; A)) & \xrightarrow{\tau_*} & K_i(\mathcal{O}^G(X; A)) \\
\Omega_* & \to & \alpha_* \\
\text{inc}_* & \downarrow & \text{inc}_* \\
K_i(\mathcal{O}^N_G(X; A)) & \xrightarrow{\overline{\Omega}_* = n \times} & K_i(\mathcal{O}^N_G(X; A))
\end{array}
\]
The rest of this section is devoted to the proof of this theorem. We already have:

The induction map $\alpha$:

$$\alpha : \mathcal{O}^H(X) \to \mathcal{O}^G(X)$$

On objects: $(\alpha A)(g,x,t) = (g^{-1})^*(A_{(e,g^{-1}x,t)})$

On morphisms:

$$(\alpha \phi)(g',x',t'),(g,x,t) = \begin{cases} 0 & \text{if } g^{-1}g' \notin H \\ (g^{-1})^*(\phi(g^{-1}g',g^{-1}x',t'),(e,g^{-1}x,t)) & \text{if } g^{-1}g' \in H \end{cases}$$

The restriction map $\tau$:

$$\tau : \mathcal{O}^G(X) \to \mathcal{O}^H(X)$$

On objects: $(\tau A)(h,x,t) = \bigoplus_{i=1}^n A_{(h g_i,x,t)}$

On morphisms: $(\tau \phi)(h',x',t'),(h,x,t) = (\phi(h'g_i,x',t'),(h g_i,x,t))_{i,j=1}^n$

We now study their composition:

Lemma 3.3.2. The composition $\alpha \circ \tau : \mathcal{O}^G(X) \to \mathcal{O}^G(X)$ is given by:

On objects: $[(\alpha \circ \tau)(A)](g,x,t) = \bigoplus_{i=1}^n A_{(g g_i,x,t)}$

On morphisms:

$$[(\alpha \circ \tau)(\phi)](g',x',t'),(g,x,t) = \begin{cases} 0 & \text{if } g^{-1}g' \notin H \\ (\phi(g'g_i,x',t'),(g g_i,x,t))_{i,j=1}^n & \text{if } g^{-1}g' \in H \end{cases}$$

Proof. On objects, we have

$$[(\alpha \circ \tau)(A)](g,x,t) = [\alpha(\tau A)](g,x,t)$$

$$= (g^{-1})^*[(\tau A)_{(e,g^{-1}x,t)}]$$

$$= (g^{-1})^*\bigoplus_{i=1}^n A_{(g_i,g^{-1}x,t)}$$

$$= \bigoplus_{i=1}^n (g^{-1})^*[A_{(g_i,g^{-1}x,t)}]$$

$$= \bigoplus_{i=1}^n A_{(g g_i,x,t)}$$
and on morphisms, we have

\[
[(\alpha \circ \tau)(\phi)]_{(g',x',t'),(g,x,t)} = [\alpha(\tau\phi)]_{(g',x',t'),(g,x,t)}
\]

\[
= \begin{cases} 
0 & \text{if } g^{-1}g' \notin H \\
(g^{-1})^i[(\tau\phi)_{(g^{-1}g',g^{-1}x',t'),(e,g^{-1}x,t)}] & \text{if } g^{-1}g' \in H
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } g^{-1}g' \notin H \\
(g^{-1})^i[(\phi_{(g^{-1}g',g^{-1}x',t'),(g,g^{-1}x,t)})_{i,j=1}^n] & \text{if } g^{-1}g' \in H
\end{cases}
\]

This completes the proof. 

From now on, the composition \( \alpha \circ \tau \) will be denoted by \( \Omega \) and we specialize to the case that \( G \) is of the form \( G = N \rtimes F \), a semi-direct product of a normal subgroup \( N \triangleleft G \) with a finite subgroup \( F < G \). Note that this is of no loss of generality according to Lemma 3.1.1. Now in the above construction, we can let \( H = N \) and choose the complete representatives for the coset \( G/N \) to be \( F \), which is a finite subgroup. Set \( |F| = n \).

**Notation 3.3.3.** Given an object \( A \) in an additive category \( \mathcal{A} \), let \( A^n \) denote the direct sum of \( n \) copies of \( A \), we also use the notation \( \bigoplus_{a \in F} A \) for \( A^n \) to indicate that the direct sum is indexed by \( F \). Let \( \{A_a\}_{a \in F}, \{B_b\}_{b \in F} \) be objects indexed by \( F \). For any morphism \( f : A_c \to B_d \) with \( c,d \in F \), let \( M(c,d ; f) : \sum_{a \in F} A_a \to \sum_{b \in F} B_b \) be the morphism with \((d,c)\)-th component \( f \) and other component zero. If we interpret it as a matrix indexed by \( F \), then the only nonzero entry is \((c,d)\)-th entry, which is \( f \). It’s easy to see that when composable, \( M(a_1, b_1 ; f)M(a_2, b_2 ; g) = 0 \) if \( b_1 \neq a_2 \) and \( M(a_1, b_1 ; f) \circ M(a_2, b_2 ; g) = M(a_1, b_2 ; f \circ g) \) if \( b_1 = a_2 \), where \( a_1, a_2, b_1, b_2 \in F \).

**Definition 3.3.4.** For any object \( A \in \mathcal{O}^G(X) \), we define morphisms:
(1) \( \eta^A : \Omega(A) \to A^n \)

\[
\eta^A_{(g',x',t'),(g,x,t)} : \Omega(A)_{(g,x,t)} = \bigoplus_{a \in F} A_{(ga,x,t)} \to A^n_{(g',x',t')} = \bigoplus_{b \in F} A_{(g'b,x',t')}
\]

\[
= \begin{cases} 
0 & \text{if } (ga, x, t) \neq (g', x', t'), \forall a \in F \\
M(c, c; 1) & \text{if } (gc, x, t) = (g', x', t') \text{ for some } c \in F
\end{cases}
\]

(2) \( \xi^A : A^n \to \Omega(A) \)

\[
\xi^A_{(g',x',t'),(g,x,t)} : A^n_{(g,x,t)} = \bigoplus_{a \in F} A_{(g,x,t)} \to \Delta(A)_{(g',x',t')} = \bigoplus_{b \in F} A_{(g'b,x',t')}
\]

\[
= \begin{cases} 
0 & \text{if } (g, x, t) \neq (g'b, x', t') \forall b \in F \\
M(c, c; 1) & \text{if } (g, x, t) = (g'c, x', t') \text{ for some } c \in F
\end{cases}
\]

Lemma 3.3.5. Both \( \eta^A \) and \( \xi^A \) are \( G \)-invariant for any object \( A \in \mathcal{O}^G(X) \), their supports are contained in \( \mathcal{E}_{Gcc}^X \) and \( \eta^A \circ \xi^A = \text{id}_{A^n} \) and \( \xi^A \circ \eta^A = \text{id}_{\Omega(A)} \), hence they are isomorphisms in \( \mathcal{O}^G(X) \).

Proof. These two morphisms are obviously \( G \)-invariant by definition. The projections of their supports to \( (X \times [1, \infty))^2 \) are easily seen to be the diagonal, hence in \( \mathcal{E}_{Gcc}^X \).

Now \( (\eta^A \xi^A)_{(g',x',t'),(g,x,t)} = 0 \) if \( (x', t') \neq (x, t) \) and when \( (x', t') = (x, t) \), we have

\[
(\eta^A \xi^A)_{(g',x,x'),(g,x,t)} = \sum_{(g'',x'',t'')} \eta^A_{(g',x,x'),(g'',x'',t'')} \circ \xi^A_{(g'',x'',t''),(g,x,t)}
\]

\[
= \sum_{g'' \in G} \eta^A_{(g',x,x'),(g'',x,x')} \circ \xi^A_{(g'',x,x'),(g,x,t)}
\]

\[
= \sum_{a \in F} \eta^A_{(g',x,x'),(ga^{-1},x,t)} \circ \xi^A_{(ga^{-1},x,t),(g,x,t)}
\]

The right hand side can be nonzero only when \( g' = gc \) for some \( c \in F \). In this case, the right hand side becomes

\[
\sum_{a \in F} \eta^A_{(gc,x,x'),(ga^{-1},x,t)} \circ \xi^A_{(ga^{-1},x,t),(g,x,t)}
\]

\[
= \sum_{a \in F} \eta^A_{(ga^{-1},ac),x,t),(ga^{-1},x,t)} \circ \xi^A_{(ga^{-1},x,t),(g,x,t)}
\]

\[
= \sum_{a \in F} M(ac, ac; 1) M(a, a; 1)
\]

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This is nonzero only when \( c = 1 \), that is \( g = g' \), in this case it is equal to

\[
\sum_{a \in F} M(a, a; 1)M(a, a; 1) = \sum_{a \in F} M(a, a; 1) = id
\]

This proves \( \eta^A \circ \xi^A = id_{A^n} \). Similar argument shows the other identity holds. This completes the proof. \( \square \)

Therefore the functor \( \Omega \) is naturally isomorphic to its conjugation by \( \eta^A \), i.e. the functor \( \Theta : \mathcal{O}^G(X) \to \mathcal{O}^G(X) \) defined by:

**On objects:** \( \Theta(A) = A^n \)

**On morphisms:** \( \Theta(\phi) = \eta^B \circ \Omega(\phi) \circ \xi^A \) for \( \phi : A \to B \).

Let us compute more explicitly for \( \Theta(\phi) \), we have

\[
\Theta(\phi)(g',x',t'),(g,x,t) = \sum_{(g'' ,x'',t'')} [\eta^B \circ \Omega(\phi)](g',x',t'),(g'' ,x'',t'') \circ \xi^A(g'' ,x'',t''),(g,x,t)
\]

\[
= \sum_{g'' \in G} \sum_{(g'' ,x'',t'')} \eta^B(\phi)(g',x',t'),(g'' ,x'',t'') \circ \Omega(\phi)(g'' ,x'',t''),(ga^{-1},x,t) \circ \xi^A(ga^{-1},x,t),(g,x,t)
\]

\[
= \sum_{(g',x',t')} \sum_{g'' \in G} \eta^B(\phi)(g',x',t') \circ \Omega(\phi)(g'' ,x',t''),(ga^{-1},x,t) \circ \xi^A(ga^{-1},x,t),(g,x,t)
\]

\[
= \sum_{(g'',b^{-1},x',t')} \sum_{a \in F} \Omega(\phi)(g'b^{-1},x',t'),(ga^{-1},x,t) \circ \xi^A(ga^{-1},x,t),(g,x,t)
\]

\[
= \sum_{(g',b^{-1},x',t')} \sum_{a \in F} M(b, b; 1) \circ \Omega(\phi)(g'b^{-1},x',t'),(ga^{-1},x,t) \circ M(a, a; 1)
\]

By Lemma 3.3.2, the above expression is zero if \((ga^{-1})^{-1}g'b^{-1} \notin N\), and if it is in \( N \), then there exists \( k \in N \) so that \( ag^{-1}g'b^{-1} = k \). There are unique \( l \in N, c \in F \) so that \( g^{-1}g' = lc \), plug in, we get \( alcb^{-1} = k \). Since \( N \) is normal in \( G \), there is \( m \in N \) so
that \( al = ma \) and we get that \( mabc^{-1} = k \), hence we must have \( m = k \) and \( ac = b \).

Note \( c \) is determined when \( g, g' \) are given. Now the above sum becomes

\[
\sum_{a \in F} M(ac, ac; 1) \circ \Omega(\phi)(g'c^{-1}a^{-1}, x', t') \circ \Omega(g^{-1}, x, t) \circ M(a, a; 1)
\]

The second equality above is just by a simple matrix multiplication. Let us organize the above argument into the following proposition.

**Proposition 3.3.6.** The functor \( \Omega : \mathcal{O}^G(X) \to \mathcal{O}^G(X) \) is naturally isomorphic to the functor \( \Theta : \mathcal{O}^G(X) \to \mathcal{O}^G(X) \) which is given by

- **on objects:** \( \Theta(A) = A^n \)
- **on morphisms:** \( \Theta(\phi : A \to B) : A^n \to B^n \)

\[
\Theta(\phi)(g', x', t'), (g, x, t) = \sum_{a \in F} M(ac, a; \phi(g', x', t'), (g, x, t))
\]

where \( c \in F \) is uniquely determined by \( g^{-1}g' = lc, l \in N, c \in F \).

To go further, we need to recall the additivity theorem in \( K \)-theory. Let \( A \) be an additive category. We endow it with the exact structure by proclaiming sequences isomorphic to 0 \( \to A \to A \oplus B \to B \to 0 \) to be exact. Now let \( F' \to F \to F'' : A \to B \) be a sequence of additive functors and natural transformations between them. It is said to be **short exact** if for any object \( A \in A \), the sequence 0 \( \to F'(A) \to F(A) \to F''(A) \to 0 \) is short exact.

**Theorem 3.3.7.** (Additivity Theorem) Suppose \( F' \to F \to F'' : A \to B \) is a short exact sequence of additive functors between additive categories. Then \( F_* = F'_* + F''_* : K_i(A) \to K_i(B), \forall i \in \mathbb{N} \).
Proof of the additivity theorem can be found, for example in [59]. As an easy corollary of the additivity theorem, one has

**Corollary 3.3.8.** (1) If \( F, F' : \mathcal{A} \to \mathcal{B} \) are naturally isomorphic, then \( F_* = F'_* : K_i(\mathcal{A}) \to K_i(\mathcal{B}), \forall i \in \mathbb{N} \).

(2) Let \( \Delta^n : \mathcal{A} \to \mathcal{A} \) be the functor defined by \( \Delta^n(A) = A^n \) and \( \Delta^n(\phi : A \to B) = \phi^n : A^n \to B^n \), the diagonal functor, then \( \Delta_*^n(x) = nx, \forall x \in K_i(\mathcal{A}), i \in \mathbb{N} \).

**Remark 3.3.9.** One should be a little bit careful about \( K_0 \). The \( K \)-groups in the Farrell-Jones conjecture use the non-connective spectrum associated to additive categories while the \( K \)-groups in the above theorem and corollary are the Quillen \( K \)-groups. They agree in positive degrees, however in degree 0, they are related by \( \pi_0(K^{-\infty}(\mathcal{A})) = K_0(P(\mathcal{A})) \), where \( P(\mathcal{A}) \) is the idempotent completion of \( \mathcal{A} \).

**Proposition 3.3.10.** With all the previous notations, we have

(1) The functors \( \Omega, \Theta : \mathcal{O}^G(X) \to \mathcal{O}^G(X) \) extend to functors \( \overline{\Omega}, \overline{\Theta} : \mathcal{O}^N_G(X) \to \mathcal{O}^N_G(X) \) with the same formulas as given in Lemma 3.3.2 and Lemma 3.3.6. The extended functors are still naturally isomorphic. Recall \( \mathcal{O}^N_G(X) = C^N(G \times X \times [1, \infty), (p \times p)^{-1}E_{\text{Gcc}}; q^{-1}F_{Gc}); \)

(2) The functor \( \overline{\Theta} \) is naturally isomorphic to the diagonal functor \( \Delta^n \). Hence on positive \( K \)-groups, \( \overline{\Omega}_* = \overline{\Theta}_* \) is multiplication by \( n \).

**Proof.** (1) This is just by direct verification. For example, for any \( N \)-invariant object \( A \), we want to show \( \overline{\Omega}(A) \) is \( N \)-invariant. Choose any \( l \in N \), we compute

\[
(l^* \overline{\Omega}(A))_{(g,x,t)} = l^* (\overline{\Omega}(A)_{(lg,lx,t)}) = l^* \left( \bigoplus_{a \in F} A_{(lga,lx,t)} \right) = \bigoplus_{a \in F} l^*(A_{(lga,lx,t)}) = \bigoplus_{a \in F} A_{(ga,x,t)} = \bigoplus_{a \in F} A_{(ga,x,t)} = \overline{\Omega}(A)_{(g,x,t)}
\]
Similarly if \( \phi \) is \( N \)-invariant, then so is \( \overline{\Omega}(\phi) \). One also needs to check the support conditions which is fairly easy and the functoriality of \( \overline{\Omega} \) is also simple. The verification for \( \overline{\Theta} \) is even more simple and will be omitted. \( \eta^A \) defined in Definition 3.3.4 can also be naturally extended and gives a naturally isomorphism between \( \overline{\Omega} \) and \( \overline{\Theta} \).

(2) For any object \( A \), one defines \( \zeta^A : A^n \to A^n \) by

\[
\zeta^A_{(g',x',t'),(g,x,t)} = \begin{cases} 
0 & \text{if } (g', x', t') \neq (g, x, t) \\
\sum_{a \in F} M(ac_g, a; 1) & \text{if } (g', x', t') = (g, x, t)
\end{cases}
\]

where \( c_g \in F \) is uniquely determined by \( g = l_g c_g \) with \( l_g \in N \). We first check \( \eta^A \) is \( N \)-invariant. For any \( k \in N \), \( (k^* \zeta^A)_{(g',x',t'),(g,x,t)} = k^* (\zeta^A_{(kg',x',t'),(kg,x,t)}) \), this is zero if \( (g', x', t') \neq (g, x, t) \) and when \( (g', x', t') = (g, x, t) \), we have

\[
(k^* \zeta^A)_{(g,x,t), (g,x,t)} = k^* (\zeta^A_{(kg,x,t), (kg,x,t)}) = k^* \left( \sum_{a \in F} M(ac_{kg}, a; 1) \right)
\]

\[
= \sum_{a \in F} k^* (M(ac_{kg}, a; 1)) = \sum_{a \in F} M(ac_{kg}, a; 1) = \sum_{a \in F} M(ac_g, a; 1)
\]

where \( c_{kg} \) is uniquely determined by \( kg = l_{kg} c_{kg} \) with \( l_{kg} \in N \) and \( c_{kg} \in F \). The assignment \( g \mapsto c_g \) can be viewed as the natural homomorphism \( G \to G/N = F \), hence \( c_{g_1 g_2} = c_{g_1} c_{g_2} \) and thus \( c_{kg} = c_g \) in the last equality because \( k \in N \). So \( \zeta^A \) is \( N \)-invariant. This also explains why it is not \( G \)-invariant, because \( c_{kg} \neq c_g \) if \( k \notin N \).

\( \zeta^A \) clearly satisfies the support condition since its support is just the diagonal.

Now define \( (\zeta^A)^{-1} : A^n \to A^n \) by

\[
(\zeta^A)^{-1}_{(g',x',t'),(g,x,t)} = \begin{cases} 
0 & \text{if } (g', x', t') \neq (g, x, t) \\
\sum_{a \in F} M(ac^{-1}_g, a; 1) & \text{if } (g', x', t') = (g, x, t)
\end{cases}
\]

Similarly one checks \( (\zeta^A)^{-1} \) is a morphism in our category. Now one computes

\[
(\zeta^A \circ (\zeta^A)^{-1})_{(g',x',t'),(g,x,t)} = \sum_{(g'',x'',t'')} \zeta^A_{(g',x',t'),(g'',x'',t'')} \circ ((\zeta^A)^{-1})_{(g'',x'',t''),(g,x,t)}
\]

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This is zero when \((g', x', t') \neq (g, x, t)\) and when \((g', x', t') = (g, x, t)\) it becomes

\[
\zeta^A_{(g',x',t'),(g,x,t)} \circ (\zeta^A)^{-1}_{(g,x,t),(g,x,t)}
\]

\[
= \sum_{a \in F} M(ac_g, a;1) \sum_{b \in F} M(bc_g, b;1)
\]

\[
= \sum_{b \in F} M(bc_g, b;1)
\]

\[
= \text{id}
\]

This shows \(\zeta^A \circ (\zeta^A)^{-1} = \text{id}_{A^*}\), similarly one has \((\zeta^A)^{-1} \circ \zeta^A = \text{id}_{A^*}\). Hence \(\eta^A\) is an isomorphism in \(\mathcal{C}_N(G \times X \times [1, \infty), (p \times p)^{-1} \mathcal{E}_G^{X}, q^{-1} \mathcal{F}_G)\)

Finally for any morphism \(\phi : A \to B\), one computes

\[
(\Theta(\phi) \circ \zeta^A)_{(g',x',t'),(g,x,t)}
\]

\[
= \sum_{(g'',x'',t'')} \Theta(\phi)_{(g',x',t'),(g'',x'',t'')} \circ \zeta^A_{(g'',x'',t''),(g,x,t)}
\]

\[
= \Theta(\phi)_{(g',x',t'),(g,x,t)} \circ \zeta^A_{(g,x,t),(g,x,t)}
\]

\[
= \sum_{a \in F} M(ac_g, a;1) \sum_{b \in F} M(bc_g, b;1)
\]

\[
= \sum_{b \in F} M(bc_g, b;1)
\]

\[
= \sum_{a \in F} M(ac_g, a;1)
\]

and

\[
(\zeta^B \circ \Delta^n(\phi))_{(g',x',t'),(g,x,t)}
\]

\[
= \sum_{(g'',x'',t'')} \zeta^B_{(g',x',t'),(g'',x'',t'')} \circ \Delta^n(\phi)_{(g'',x'',t''),(g,x,t)}
\]

\[
= \zeta^B_{(g',x',t'),(g',x',t')} \circ \Delta^n(\phi)_{(g',x',t'),(g,x,t)}
\]

\[
= \sum_{a \in F} M(ac_g, a;1) \sum_{b \in F} M(b, b;1)
\]

\[
= \sum_{a \in F} M(ac_g, a;1)
\]

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This shows $\Theta(\phi) \circ \zeta^A = \zeta^B \circ \Delta^n(\phi)$ for any $\phi : A \to B$, hence $\Theta$ is naturally isomorphic to $\Delta^n$. 

All things together prove Theorem 3.3.1.
Chapter 4: Nonfiniteness of Nils

This Chapter is devoted to the study of Farrell Nil-groups associated to a finite order ring automorphism. Its definition has already been recalled in Section 1.4.3. However, our study has to make use of the categorical description of Farrell Nil-groups. This is recalled in Section 4.1. Most importantly we construct some exact functors which induce the twisted analogue of Verscheibung operators and Frobenius operators on Nil-groups. These operators are the key to the proof of our version of Farrell’s lemma 4.2.2. We then use this key lemma to deduce our various structure theorems of Farrell Nil-groups in the subsequent sections.

4.1 Some Exact Functors

In this section, we define various functors that will be used in our proofs. Let $R$ be an associative ring with unit and $\alpha : R \to R$ be a ring automorphism of finite order, say $|\alpha| = n$. For each integer $i \in \mathbb{Z}$, denote by $R_{\alpha^i}$ the $R$-bimodule which coincides with $R$ as an abelian group, but with bimodule structure given by $r \cdot x := rx$ and $x \cdot r := x\alpha^i(r)$ (where $x \in R_{\alpha^i}$, and $r \in R$). Note that as left (or as right) $R$-modules, $R_{\alpha^i}$ and $R$ are isomorphic, but they are in general not isomorphic as $R$-bimodules. For each right $R$-module $M$ and integer $i$, define a new right $R$-module $M_{\alpha^i}$ as follows: as abelian groups, $M_{\alpha^i}$ is the same as $M$, however the right $R$-module structure on
$M_{\alpha^i}$ is given by $x \cdot r := x\alpha^i(r)$. Clearly $M_{\alpha^n} = M$ and $(M_{\alpha^i})_{\alpha^j} = M_{\alpha^{i+j}}$ as right $R$-modules. We could have defined $M_{\alpha^i} = M \otimes_R R_{\alpha^i}$, however this has the slight disadvantage that the above equalities would not hold – we would only have natural isomorphisms between the corresponding functors.

Let $\mathbf{P}(R)$ denote the category of finitely generated right projective $R$-modules. For each $i \in \mathbb{Z}$, there is an exact functor $S_i : \mathbf{P}(R) \to \mathbf{P}(R)$ given by $S_i(P) := P_{\alpha^i}$, on objects and $S_i(\phi) = \phi$ on morphisms. Note that if we forget about the right $R$-module structures, and just view these as abelian groups and group homomorphisms, then each $S_i$ is just the identity functor. Clearly $S_i \circ S_j = S_j \circ S_i = S_{i+j}$ and $S_n = Id$, so the map $i \mapsto S_i$ defines a functorial $\mathbb{Z}$-action on the category $\mathbf{P}(R)$, which factors through a functorial $\mathbb{Z}_n$-action (recall that $n$ is the order of the ring automorphism $\alpha$).

We are interested in the *Nil-category* $\text{NIL}(R; \alpha)$. Recall that objects of this category are of the form $(P, f)$, where $P$ is an object in $\mathbf{P}(R)$ and $f : P \to P_{\alpha} = S_1(P)$ is a right $R$-module homomorphism which is *nilpotent*, in the sense that a high enough composite map of the following form is the zero map:

$$P \xrightarrow{S_{k-1}(f) \circ S_{k-2}(f) \circ \cdots \circ S_1(f) \circ f} P_{\alpha^k}$$

A morphism $\phi : (P, f) \to (Q, g)$ in $\text{NIL}(R; \alpha)$ is given by a morphism $\phi : P \to Q$ in $\mathbf{P}(R)$ which makes the obvious diagram commutative, i.e. $S_1(\phi) \circ f = g \circ \phi$. We have two exact functors

$$F : \text{NIL}(R; \alpha) \to \mathbf{P}(R), \quad F(P, f) = P$$

$$G : \mathbf{P}(R) \to \text{NIL}(R; \alpha), \quad G(P) = (P, 0)$$

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which give rise to a splitting of the $K$-theory groups $K_i(NIL(R; \alpha)) = K_i(R) \oplus Nil_i(R; \alpha)$, where $Nil_i(R; \alpha) := \ker(\pi_i(NIL(R; \alpha)) \to K_i(R))$, $i \in \mathbb{N}$.

**Remark 4.1.1.** The Farrell Nil-groups $NK_i(R, \alpha)$ mentioned in the introduction coincide, with a dimension shift, with the groups $Nil_i(R; \alpha^{-1})$ defined above. More precisely, one has for every $i \geq 1$ an isomorphism $NK_i(R, \alpha) \cong Nil_{i-1}(R; \alpha^{-1})$ ([41, Theorem 2.1]).

We now introduce two exact functors on the category $NIL(R; \alpha)$ which will play an important role in our proofs. On the level of $K$-theory, one of these yields the twisted analogue of the Verscheibung operators, while the other gives the classical Frobenius operators.

**Definition 4.1.2 (Twisted Verscheibung functors).** For each positive integer $m$, define the twisted Verscheibung functors $V_m : NIL(R; \alpha) \to NIL(R; \alpha)$ as follows. On objects, we set

$$V_m((P, f)) = (P \oplus P_{\alpha-1} \oplus P_{\alpha-2} \oplus \cdots \oplus P_{\alpha-mn}, \overline{f}) = \left( \sum_{i=0}^{mn} P_{\alpha^{-i}}, \overline{f} \right) = \left( \sum_{i=0}^{mn} S_{-i}(P), \overline{f} \right)$$

where the morphism

$$\overline{f} : \sum_{i=0}^{mn} P_{\alpha^{-i}} \to \left( \sum_{j=0}^{mn} P_{\alpha^{-j}} \right)_{\alpha} = \sum_{j=0}^{mn} P_{\alpha^{-j+1}}$$

is defined component-wise by the maps $f_{ij} : P_{\alpha^{-i}} \to P_{\alpha^{-j+1}}$ given via the formula

$$f_{ij} = \begin{cases} \text{id} & \text{if } j = i + 1, 0 \leq i \leq mn - 1 \\ f & \text{if } i = mn, j = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the proof of Lemma 4.1.5 below, we will see that $\overline{f}$ is nilpotent, so that $V_m((P, f))$ does indeed define an object in the category $NIL(R; \alpha)$. If $\phi : (P, f) \to (Q, g)$ is a
morphism in the category $\text{NIL}(R; \alpha)$, we define the morphism

$$V_m(\phi) : \left( \sum_{i=0}^{mn} P_{\alpha^{-i}}, \overline{f} \right) \to \left( \sum_{i=0}^{mn} Q_{\alpha^{-i}}, \overline{g} \right)$$

via the formula $V_m(\phi) = \sum_{i=0}^{mn} S_{-i}(\phi)$. One checks that (i) $\overline{g} \circ V_m(\phi) = S_1(V_m(\phi)) \circ \overline{f}$, (ii) $V_m(id) = id$ and (iii) $V_m(\phi \circ \psi) = V_m(\phi) \circ V_m(\psi)$, so that $V_m$ is indeed a functor. Moreover, $V_m$ is exact because each $S_{-i}$ is exact.

**Definition 4.1.3** (Frobenius functors). For each positive integer $m$, define the Frobenius functors $F_m : \text{NIL}(R; \alpha) \to \text{NIL}(R; \alpha)$ as follows. On objects, we set $F_m((P, f)) = (P, \overline{f})$ where $\overline{f}$ is the morphism defined by the composition

$$P \xrightarrow{S_{mn}(f) \circ S_{mn-1}(f) \circ \cdots \circ S_1(f) \circ f} P_{\alpha^{mn+1}} = P_{\alpha}$$

It is immediate that the map $\overline{f}$ is nilpotent, so that $F_m((P, f))$ is indeed an object in $\text{NIL}(R; \alpha)$. Now if $\phi : (P, f) \to (Q, g)$ is a morphism in the category $\text{NIL}(R; \alpha)$, we define the morphism $F_m(\phi) : (P, \overline{f}) \to (Q, \overline{g})$ to coincide with the morphism $\phi$. It is obvious that $F_m(id) = id$ and $F_m(\phi \circ \psi) = F_m(\phi) \circ F_m(\psi)$, and one can easily check that $\overline{g} \circ \phi = S_1(\phi) \circ \overline{f}$, so that $F_m$ is a genuine functor. Clearly $F_m$ is exact.

**Definition 4.1.4** ($\alpha$-twisting functors). For each $i \in \mathbb{Z}$, we define the exact functor $T_i : \text{NIL}(R; \alpha) \to \text{NIL}(R; \alpha)$ as follows. On objects, we set $T_i((P, f)) = (S_{-i}(P), S_{-i}(f))$, and if $\phi : (P, f) \to (Q, g)$ is a morphism, we set $T_i(\phi)$ to be the morphism $S_{-i}(\phi) : S_{-i}(P) \to S_{-i}(Q)$. Observe that, as with the functors $S_i$ on the category $\mathbf{P}(R)$, the functors $T_i$ define a functorial $\mathbb{Z}$-action on the category $\text{NIL}(R; \alpha)$, which factors through a functorial $\mathbb{Z}_n$-action.

The relationship between these various functors is described in the following Lemma. We will write $G_m$ for the composite exact functor $G_m = F_m \circ V_m$. 88
Lemma 4.1.5. We have the equality $G_m = \sum_{i=0}^{mn} T_i$.

Proof. For any object $(P, f)$ in $NIL(R; \alpha)$, we have $G_m((P, f)) = \left( \sum_{i=0}^{mn} S_{-i}(P), \tilde{f} \right)$, where $\tilde{f} = S_{mn}(f) \circ S_{mn-1}(f) \circ \cdots \circ S_1(f) \circ f$. Note that if we forget the right $R$-module structures, each $S_i$ is the identity functor on abelian groups. So as a morphism of abelian groups, $\tilde{f} = \tilde{f}_{mn+1}$. Now recall that $\tilde{f}$ is a morphism which cyclicly permutes the $mn+1$ direct summands occurring in its source and target. Using this observation, it is then easy to see that $\tilde{f} = \tilde{f}_{mn+1}$ is diagonal and equal to $\sum_{i=0}^{mn} S_{-i}(f)$. So on the level of objects, $G_m$ and $\sum_{i=0}^{mn} T_i$ agree. From this, we also see that $\tilde{f}$ is nilpotent (as was indicated in Definition 4.1.2). It is obvious that they agree on morphisms. \qed

Remark 4.1.6. It is natural to consider the more general case when $\alpha : R \to R$ has finite order in the outer automorphism group of the ring $R$, i.e. there exists $n \in \mathbb{N}$ and a unit $u \in R$ so that $\alpha^n(r) =uru^{-1}, \forall r \in R$. In this situation, we have for any right $R$-module $M$ and integer $m$, an isomorphism $\tau_{m,M} : M_{\alpha^mn} \to M$, $\tau_{m,M}(r) := ru^m$ of right $R$-modules. This gives rise to a natural isomorphism between the functors $S_{mn}$ and $S_0 = Id$. It is then easy to similarly define twisted Verscheibung functors and Frobenius functors, and to verify an analogue of Lemma 4.1.5. However, in this case, we generally do not have that $T_n$ is naturally isomorphic to $T_0$, unless $\alpha$ fixes $u$. This key issue prevents our proof of Farrell’s Lemma 4.2.2(2) below (which is the key to the proof of our main theorems) to go through in this more general setting.

4.2 Non-Finiteness of Farrell Nil-Groups

This section is devoted to proving the first main result of this chapter:

Theorem 4.2.1. Let $R$ be a ring, $\alpha : R \to R$ an automorphism of finite order, and $i \in \mathbb{Z}$. Then $NK_i(R, \alpha)$ is either trivial, or infinitely generated as an abelian group.
We firstly use results from the previous section to establish our analogue of Farrell’s key lemmas from his paper [21].

**Lemma 4.2.2.** The following results hold:

1. \( \forall j \in \mathbb{N}, \) the induced maps \( K_j(V_m), K_j(F_m) : K_j(NIL(R; \alpha)) \to K_j(NIL(R; \alpha)) \) on \( K \)-theory map the summand \( \text{Nil}_j(R; \alpha) \) to itself;

2. \( \forall j, m \in \mathbb{N}, \) the identity \((2 + mn)K_j(G_m) - K_j(G_m)^2 = \mu_{1+mn}\) holds, where the map \( \mu_{1+mn} \) is multiplication by \( 1 + mn; \)

3. \( \forall j \in \mathbb{N} \) and each \( x \in \text{Nil}_j(R; \alpha), \) there exists a positive integer \( r(x), \) such that \( K_j(F_m)(x) = 0 \) for all \( m \geq r(x). \)

**Proof.**

(1) Let \( H_m := \sum_{i=0}^{mn} S_{-i} : \mathbf{P}(R) \to \mathbf{P}(R), \) one then easily checks \( F \circ V_m = H_m \circ F. \) We also have \( F \circ F_m = F. \) Statement (1) follows easily from these.

(2) By the Additivity Theorem for algebraic K-theory, Lemma 4.1.5 immediately gives us that

\[
K_j(G_m) = \sum_{i=0}^{mn} K_j(T_i) = id + m \sum_{i=1}^{n} K_j(T_i)
\]

Now let us evaluate the square of the map \( K_j(G_m): \)

\[
K_j(G_m)^2 = \left( id + m \sum_{i=1}^{n} K_j(T_i) \right) \left( id + m \sum_{i=1}^{n} K_j(T_i) \right)
\]

\[
= id + 2m \sum_{i=1}^{n} K_j(T_i) + m^2 \sum_{i=1}^{n} \sum_{l=1}^{n} K_j(T_{i+l})
\]

\[
= id + 2m \sum_{i=1}^{n} K_j(T_i) + m^2 \sum_{i=1}^{n} \sum_{l=1}^{n} K_j(T_i)
\]

\[
= id + 2m \sum_{i=1}^{n} K_j(T_i) + m^2 n \sum_{l=1}^{n} K_j(T_i)
\]

\[
= id + (2m + m^2 n) \sum_{i=1}^{n} K_j(T_i)
\]
In the third equality above, we used the fact that the $T_i$ functors are $n$-periodic, so that shifting the index on the inner sum by $i$ leaves the sum unchanged. Finally, substituting in the expression we have for $K_j(G_m)$ and the expression we derived for $K_j(G_m)^2$, we see that:

$$(2 + mn)K_j(G_m) - K_j(G_m)^2$$
$$(2 + mn)(id + m \sum_{i=1}^{n} K_j(T_i)) - (id + (2m + m^2n) \sum_{i=1}^{n} K_j(T_i))$$
$$(2 + mn)id - id = \mu_{1+mn}$$

(3) This result is due to Grunewald [43, Proposition 4.6].

\[\square\]

**Proof of Theorem 4.2.1.** The proof now follows easily. Let us focus on the case where $i \geq 1$, as the case $i \leq 1$ has already been established by Grunewald [42] and Ramos [61]. So let us assume that the Farrell Nil-group $NK_i(R, \alpha) \cong Nil_{i-1}(R; \alpha^{-1})$ is non-trivial and finitely generated, where $i \geq 1$. Then one can find arbitrarily large positive integers $m$ with the property that the map $\mu_{1+mn}$ is an injective map from $Nil_{i-1}(R; \alpha^{-1})$ to itself (for example, one can take $m$ to be any multiple of the order of the torsion subgroup of $Nil_{i-1}(R; \alpha^{-1})$). From Lemma 4.2.2(2), we can factor the map $\mu_{1+mn}$ as a composite

$$\mu_{1+mn} = (\mu_{2+mn} - K_j(G_m)) \circ K_j(G_m)$$

and hence there is an arbitrarily large sequence of integers $m$ with the property that the corresponding maps $K_j(G_m)$ are injective. This implies that there are infinitely many integers $m$ for which the map $K_j(F_m)$ is non-zero.

On the other hand, let $x_1, \ldots, x_k$ be a finite set of generators for the abelian group $Nil_{i-1}(R; \alpha^{-1})$. Then from Lemma 4.2.2(3), we have that for any $m \geq \max\{r(x_i)\}$, the map $K_j(F_m)$ is identically zero, a contradiction. This completes the proof.
4.3 Finite Subgroups of Farrell Nil-Groups

In this section, we prove:

**Theorem 4.3.1.** Let $R$ be a ring, $\alpha : R \rightarrow R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $H \leq NK_i(R, \alpha)$ is a finite subgroup, then $\bigoplus_{\infty} H$ also appears as a subgroup of $NK_i(R, \alpha)$. Moreover, if $H$ is a direct summand in $NK_i(R, \alpha)$, then so is $\bigoplus_{\infty} H$.

In order to establish the above theorem, we will need an algebraic lemma for recognizing when two direct summands inside an ambient group jointly form a direct summand.

**Lemma 4.3.2.** Let $G$ be an abelian group and $H < G$, $K < G$ be a pair of subgroups. Suppose that $H \cap K = \{0\}$, and that there are two retractions $\lambda : G \rightarrow H$ and $\rho : G \rightarrow K$ with the property that $\lambda(K) = \{0\}$. Then there exists a subgroup $L < G$, which is isomorphic to $H$, and such that $L \oplus K$ is also a direct summand of $G$.

**Proof.** Consider the homomorphism $Id - \rho : G \rightarrow G$. Let $L = \{h - \rho(h) | h \in H\}$ be the image of $H$ under this homomorphism. We first note that $(Id - \rho)|_H : H \rightarrow L$ is an isomorphism. It is certainly a surjection. Now suppose that $h - \rho(h) = 0$ for some $h \in H$. Then $h = \rho(h) \in K$, which forces $h \in H \cap K = \{0\}$, and hence $h = 0$. This shows that $Id - \rho|_H$ is also an injection. So we now know that $H \cong L$. Next we observe that $L \cap K = \{0\}$. To see this, take any $h - \rho(h) \in L \cap K$. Then since $\rho(h) \in K$, we must also have $h \in K$. But then $h \in H \cap K = \{0\}$, forcing $h = 0$ and hence $h - \rho(h) = 0$.

Now define $\tau : (Id - \rho) \circ \lambda : G \rightarrow L$. For any $h - \rho(h) \in L$, we have

$$\tau(h - \rho(h)) = (Id - \rho)(\lambda(h - \rho(h))) = (Id - \rho)(h) = h - \rho(h)$$
where the second equality holds because \( \lambda(h) = h \) (since \( h \in H \) and \( \lambda \) is a retraction onto \( H \)) and \( \lambda(\rho(h)) = 0 \) (since \( \rho(h) \in K \) and \( \lambda(K) = \{0\} \) by hypothesis). This verifies that the map \( \tau : G \to L \) is a retraction. Clearly \( \tau(K) = \{0\} \) because \( \lambda(K) = 0 \). We finally note that \( \rho(L) = 0 \), because \( \rho(h - \rho(h)) = \rho(h) - \rho(h) = 0 \).

We thus have two orthogonal retractions \( \tau \) and \( \rho \). Now define \( \sigma : G \to L \oplus K \) by \( \sigma(x) = (\tau(x), \rho(x)) \). Since \( L \cap K = \{0\} \) and \( \tau, \rho \) are orthogonal, one easily checks that \( \sigma \) is a retraction. Hence \( L \oplus K \) is a direct summand of \( G \), which proves the lemma. \( \square \)

**Proof of Theorem 4.3.1.** We are now ready to prove Theorem 4.3.1. In order to simplify the notation, we will simply write \( V_m \) for \( K_j(V_m) \), and use a similar convention for \( F_m \) and \( G_m \).

Case \( i \geq 1 \). We first consider the case when \( i \geq 1 \), and recall that \( NK_i(R, \alpha) \cong Nil_{i-1}(R; \alpha^{-1}) \). Let \( H < Nil_{i-1}(R; \alpha^{-1}) \) be a finite subgroup. According to Lemma 4.2.2(3), since \( H \) is finite, there exists an integer \( r(H) = \max_{x \in H} \{ r(x) \} \), so that \( F_m(H) = 0 \) for all \( m > r(H) \). Let \( S \subset \mathbb{N} \) consist of all natural number \( m > r(H) \) such that \( \gcd(1 + mn, |H|) = 1 \). \( S \) contains every multiple of \( |H| \) which is greater than \( r(H) \), so is an infinite set. Consider the morphisms

\[
\begin{align*}
Nil_{i-1}(R; \alpha^{-1}) & \xrightarrow{V_m} Nil_{i-1}(R; \alpha^{-1}) \\
\xrightarrow{F_m} Nil_{i-1}(R; \alpha^{-1})
\end{align*}
\]

so that the composite is the morphism \( G_m \), and define \( H_m \leq Nil_{i-1}(R; \alpha^{-1}) \) to be \( H_m := V_m(H) \). By the defining property of the set \( S \), we have that for \( m \in S \),

\[
(\mu_{2+mn} - G_m) \circ G_m = \mu_{1+mn}
\]

is an isomorphism when restricted to \( H \). Hence \( G_m \) is a monomorphism when restricted to \( H \), forcing \( V_m \) to also be a monomorphism when restricted to \( H \). So for all \( m \in S \), we see that \( H_m \cong H \).
We now claim that there is an \( m \in S \), so that \( H_m \cap H = \{0\} \). Assume not. Then for all \( m \in S \), \( H_m \cap H \neq \{0\} \). Since \( H \) contains only finitely many non-zero elements, and \( S \) is an infinite set, there is a non-zero \( x \in H \) and an infinite subset \( S' \subset S \), such that \( x \in H_m \cap H \) holds for all \( m \in S' \). For each \( m \in S' \), there is \( y_m \in H \) so that \( V_m(y_m) = x \). Again, \( H \) is finite, so we can find a single non-zero \( y \in H \) and an infinite subset \( S'' \subset S \) with the property that for all \( m \in S'' \), we have \( V_m(y) = x \).

Applying \( (\mu_{2+mn} - G_m)F_m \) to this equation, we obtain an infinite set of indices \( m \) with the property that \( ((\mu_{2+mn} - G_m)F_mV_m)(y) = (\mu_{2+mn} - G_m)(F_m(x)) \). Therefore \( (\mu_{2+mn} - G_m)(F_m(x)) = (1 + mn)y \). The right hand side of the equation is non-zero for all \( m \in S'' \), since \( y \in H \) and \( \gcd(1 + mn, |H|) = 1 \). But since \( F_m(H) = 0 \) for all \( m \in S \), the left hand side vanishes, giving us a contradiction. We conclude that there must be an \( m \) so that \( H_m \cap H = \{0\} \) and \( H_m \cong H \). Hence \( H \oplus H < \text{Nil}_{i-1}(R; \alpha^{-1}) \).

Applying the same argument to \( H \oplus H \) and so on, we conclude \( \oplus_{\infty} H < \text{Nil}_{i-1}(R; \alpha^{-1}) \).

Next we claim that, if the original \( H \) was a direct summand in \( \text{Nil}_{i-1}(R; \alpha^{-1}) \), then we can find a copy of \( H \oplus H \) is also a direct summand, and which extends the original direct summand (i.e. the first copy of \( H \) inside the direct summand \( H \oplus H \) coincides with the original \( H \)). Suppose \( H < \text{Nil}_{i-1}(R; \alpha^{-1}) \) is a direct summand, so there exists a retraction \( \rho : \text{Nil}_{i-1}(R; \alpha^{-1}) \to H \). Let \( H_m \) be obtained as above.

We first construct a retraction \( \lambda : \text{Nil}_{i-1}(R; \alpha^{-1}) \to H_m \). Recall that \( \mu_{1+mn} \) is an isomorphism on \( H_m \), so there exists an integer \( l \) so that \( \mu_l \circ \mu_{1+mn} \) is the identity on \( H_m \). We define \( \lambda : \text{Nil}_{i-1}(R; \alpha^{-1}) \to H_m \) to be the composition of the following:

\[
\begin{align*}
\text{Nil}_{i-1}(R; \alpha^{-1}) & \xrightarrow{F_m} \text{Nil}_{i-1}(R; \alpha^{-1}) \xrightarrow{\mu_{2+mn} - G_m} \\
\text{Nil}_{i-1}(R; \alpha^{-1}) & \xrightarrow{\rho} H \xrightarrow{V_m|H} H_m \xrightarrow{\mu_l} H_m
\end{align*}
\]
We claim that $\lambda$ is a retraction onto $H_m$, i.e. $\lambda(x) = x$ for all $x \in H_m$. Note that for $x \in H_m$, we can always find a $y \in H$ so that $x = V_m(y)$. We now evaluate

$$\lambda(x) = (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ F_m)(x)$$

$$= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ F_m)(V_m(y))$$

$$= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ G_m)(y)$$

$$= (\mu_l \circ V_m \circ \rho \circ ((2 + mn)G_m - G_m^2))(y)$$

$$= (\mu_l \circ V_m \circ \rho \circ \mu_{1+mn})(y)$$

$$= (\mu_l \circ \mu_{1+mn})(V_m(y))$$

$$= (\mu_l \circ \mu_{1+mn})(x)$$

$$= x$$

This verifies $\lambda$ is a retraction. Note also that $\lambda(H) = 0$, since $F_m(H) = 0$ follows from the fact that $m \in S$ (recall that integers in $S$ are larger than $r(H)$). Hence we are in the situation of Lemma 4.3.2, and we can conclude that $H \oplus H$ also arises as a direct summand of $\text{Nil}_{i-1}(R; \alpha^{-1})$. Note that, when applying our Lemma 4.3.2, we replaced the second copy $H_m$ of $H$ by some other (isomorphic) subgroup, but kept the first copy of $H$ to be the original $H$. Hence the direct summand $H \oplus H$ does indeed extend the original summand $H$. Iterating the process, we obtain that $\oplus_{\infty} H$ is a direct summand of $\text{Nil}_{i-1}(R; \alpha^{-1})$. This completes the proof of Theorem 4.3.1 in the case where $i \geq 1$.

Case $i \leq 1$. Next, let us consider the case of the Farrell Nil-groups $NK_i(R, \alpha^{-1})$ where $i \leq 1$. For these, the proof of Theorem 4.3.1 proceeds via a (descending) induction on $i$, with the case $i = 1$ having been established above.
We remind the reader of the standard technique for extending results known for $K_1$ to lower $K$-groups. Take the ring $\Lambda \mathbb{Z}$ consisting of all $\mathbb{N} \times \mathbb{N}$ matrices with entries in $\mathbb{Z}$ which contain only finitely many non-zero entries in each row and each column, and quotient out by the ideal $I \triangleleft \Lambda \mathbb{Z}$ consisting of all matrices which vanish outside of a finite block. This gives the ring $\Sigma \mathbb{Z} = \Lambda \mathbb{Z} / I$, and we can now define the suspension functor on the category of rings by tensoring with the ring $\Sigma \mathbb{Z}$, i.e. sending a ring $R$ to the ring $\Sigma(R) := \Sigma \mathbb{Z} \otimes R$, and a morphism $f : R \to S$ to the morphism $Id \otimes f : \Sigma(R) \to \Sigma(S)$. The functor $\Sigma$ has the property that there are natural isomorphisms $K_i(R) \cong K_{i+1}(\Sigma(R))$ (for all $i \in \mathbb{Z}$). Moreover, there is a natural isomorphism $\Sigma(R_\alpha[t]) \cong (\Sigma R)_{Id \otimes \alpha}[t]$, which induces a commutative square

$$
\begin{array}{ccc}
K_i(R_\alpha[t]) & \longrightarrow & K_i(R) \\
\downarrow_{\cong} & & \downarrow_{\cong} \\
K_{i+1}\left((\Sigma \mathbb{Z} \otimes R)_{Id \otimes \alpha}[t]\right) & \longrightarrow & K_{i+1}(\Sigma \mathbb{Z} \otimes R)
\end{array}
$$

By induction, for each $m \in \mathbb{N}$, we can identify $NK_{1-m}(R, \alpha)$ with $NK_1(\Sigma^m R, Id^m \otimes \alpha)$, where $\Sigma^m$ denotes the $m$-fold application of the functor $\Sigma$. Obviously, if the automorphism $\alpha$ has finite order in $\text{Aut}(R)$, the induced automorphism $Id^m \otimes \alpha$ will have finite order in $\text{Aut}(\Sigma^m \otimes R)$. So for the Farrell Nil-groups $NK_i(R, \alpha)$ with $i \leq 0$, the result immediately follows from the special case of $NK_1$ considered above. This completes the proof of Theorem 4.3.1.

### 4.4 A Structure Theorem

For a general ring $R$, we know by Theorem 4.2.1 that a non-trivial Nil-group is an infinitely generated abelian group. While finitely generated abelian groups have a very nice structural theory, the picture is much more complicated in the infinitely
generated case (the reader can consult [63, Chapter 4] for an overview of the theory). If one restricts to abelian (torsion) groups of finite exponent, then it is an old result of Prüfer [58] that any such group is a direct sum of cyclic groups (see [63, item 4.3.5 on pg. 105] for a proof). This fact together with results from our previous sections enable us to deduce a structure theorem for certain Farrell-Nil groups:

**Theorem 4.4.1.** Let $R$ be a countable ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $NK_i(R, \alpha)$ has finite exponent, then there exists a finite abelian group $H$, so that $NK_i(R, \alpha) \cong \bigoplus_{\infty} H$.

*Proof.* Let $R$ be a countable ring and $\alpha : R \to R$ be an automorphism of finite order. Then by Proposition C.0.15 of the Appendix, we know that $NK_i(R, \alpha)$ is a countable group. If in addition $NK_i(R, \alpha)$ has finite exponent, then by the result of Prüfer mentioned above, it follows that $NK_i(R, \alpha)$ decomposes as a countable direct sum of cyclic groups of prime power order, each of which appears with some multiplicity. In view of our Theorem 4.3.1, any summand which occurs must actually occur infinitely many times. Since the exponent of the Nil-group is finite, there is an upper bound on the prime power orders that can appear, and hence there are only finitely many possible isomorphism types of summands. Let $H$ be the direct sum of a single copy of each cyclic group of prime power order which appear as a summand in $NK_i(R, \alpha)$. It follows immediately that $\bigoplus_{\infty} H \cong NK_i(R, \alpha)$. This completes the proof. $\square$

**Corollary 4.4.2.** Let $G$ be a finite group, $\alpha \in \text{Aut}(G)$. Then there exists a finite group $H$, whose exponent divides some power of $|G|$, so that $NK_0(\mathbb{Z}G, \alpha) \cong \bigoplus_{\infty} H$.

*Proof.* Connolly and Prassidis [17] proved $NK_0(\mathbb{Z}G, \alpha)$ has finite exponent when $G$ is finite. KuKu and Tang [49] showed $NK_i(\mathbb{Z}G, \alpha)$ is $|G|$-primary torsion for $i \geq 0$. $\square$
Remark 4.4.3. It is a natural question whether the above Corollary holds in dimensions other than zero. In negative dimensions \( i < 0 \), Farrell and Jones showed in [37] that \( NK_i(\mathbb{Z}G, \alpha) \) always vanishes when \( G \) is finite. In positive dimensions \( i > 0 \), there are partial results. As mentioned in the proof above, Kuku and Tang [49, Theorem 2.2] showed that \( NK_i(\mathbb{Z}G, \alpha) \) is \( |G| \)-primary torsion. Grunewald [43, Theorem 5.9] then generalized their result to polycyclic-by-finite groups in all dimensions. He showed that, for all \( i \in \mathbb{Z} \), \( NK_i(\mathbb{Z}G, \alpha) \) is \( mn \)-primary torsion for every polycyclic-by-finite group \( G \) and every group automorphism \( \alpha : G \to G \) of finite order, where \( n = |\alpha| \) and \( m \) is the index of some poly-infinite cyclic subgroup of \( G \) (such a subgroup always exists). However, although we have these nice results on the possible orders of torsion elements, it seems there are no known results on the exponent of these Nil-groups. This is clearly a topic for future research.

Remark 4.4.4. As an example in dimension greater than zero, Weibel [\textsuperscript{?we}] showed that \( NK_1(\mathbb{Z}D_4) \neq 0 \), where \( D_4 \) denotes the dihedral group of order 8. He also constructs a surjection \( \bigoplus_{\infty}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \to NK_1(\mathbb{Z}D_4) \), showing that this group has exponent 2 or 4. It follows from our Corollary that the group \( NK_1(\mathbb{Z}D_4) \) is isomorphic to one of the three groups \( \bigoplus_{\infty}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \), \( \bigoplus_{\infty}\mathbb{Z}_4 \), or \( \bigoplus_{\infty}\mathbb{Z}_2 \).

4.5 Cokernels of the \( K \)-Theoretic Relative Assembly Map

In applications to geometric topology, the rings of interest are typically integral group rings \( \mathbb{Z}G \). For computations of the \( K \)-theory of such groups, the key tool is provided by the Farrell-Jones Conjecture 1.2.1. The conjecture roughly predicts the \( K \)-theory of \( \mathbb{Z}G \) is determined by the \( K \)-theory of the integral group rings of the virtually cyclic subgroups of \( G \), assembled together in some homological fashion.
In view of this conjecture, one can view the $K$-theory of virtually cyclic groups as the “basic building blocks” for the $K$-theory of general groups. Focusing on such a virtually cyclic group $V$, one can consider the portion of the $K$-theory that comes from the finite subgroups of $V$. This would be the image of the assembly map:

$$H_n^V(E_{FIN}V; K_Z) \to K_n(Z[V])$$

where $E_{FIN}V$ is a model for the classifying space for proper $V$-actions. While this map is always split injective (see [3]), it is not surjective in general. Thus to understand the $K$-theory of a virtually cyclic group, we need to understand the $K$-theory of finite groups, and to understand the cokernels of the above assembly map. According to the Transitivity Principle 1.2.4, the cokernels of the above assembly map can also be interpreted as the obstruction to reduce the family of virtually cyclic groups used in the Farrell-Jones isomorphism conjecture to the family of finite groups.

Our Theorem 4.5.1 gives some structure for the cokernel of the assembly map.

**Theorem 4.5.1.** For any virtually cyclic group $V$, there exists a finite abelian group $H$ with the property that there is an isomorphism:

$$\bigoplus_{\infty} H \cong \text{CoKer} \left( H_0^V(E_{FIN}V; K_Z) \to K_0(Z[V]) \right)$$

The same result holds in dimension $n$ whenever $\text{CoKer} \left( H_n^V(E_{FIN}V; K_Z) \to K_n(Z[V]) \right)$ has finite exponent.

**Proof.** Let $V$ be a virtually cyclic group. Then one has that $V$ is either of the form (i) $V = G \rtimes_{\alpha} \mathbb{Z}$, where $G$ is a finite group and $\alpha \in \text{Aut}(G)$, or is of the form (ii) $V = G_1 *_{H} G_2$, where $G_i$, $H$ are finite groups and $H$ is of index two in both $G_i$.

Let us first consider case (i). In this case, the integral group ring $Z[V]$ is isomorphic to the ring $R_{\hat{\alpha}}[t, t^{-1}]$, the $\hat{\alpha}$-twisted ring of Laurent polynomials over the coefficient
ring $R = \mathbb{Z}[G]$, where $\hat{\alpha} \in \text{Aut}(\mathbb{Z}[G])$ is the ring automorphism canonically induced by the group automorphism $\alpha$. Then it is known (see [20, Lemma 3.1]) that the cokernel we are interested in consists of the direct sum of the Farrell Nil-group $NK_n(\mathbb{Z}G, \hat{\alpha})$ and the Farrell Nil-group $NK_n(\mathbb{Z}G, \hat{\alpha}^{-1})$. Applying Theorem 4.4.1 and Corollary 4.4.2 to these two Farrell Nil-groups, we are done.

In case (ii), we note that $V$ has a canonical surjection onto the infinite dihedral group $D_\infty = \mathbb{Z}_2 \ast \mathbb{Z}_2$, obtained by surjecting each $G_i$ onto $G_i/H \cong \mathbb{Z}_2$. Taking the preimage of the canonical index two subgroup $\mathbb{Z} \leq D_\infty$, we obtain a canonical index two subgroup $W \leq V$. The subgroup $W$ is a virtually cyclic group of type (i), and is of the form $H \rtimes_\alpha \mathbb{Z}$, where $\alpha \in \text{Aut}(H)$. Hence it has associated Farrell Nil-groups $NK_n(\mathbb{Z}H, \hat{\alpha})$.

The cokernel of the relative assembly map for the group $V$ is a Waldhausen Nil-group associated to the splitting of $V$ (see [20, Lemma 3.1]). It was recently shown that this Waldhausen Nil-group is always isomorphic to a single copy of the Farrell Nil-group $NK_n(\mathbb{Z}H, \hat{\alpha})$ associated to the canonical index two subgroup $W \leq V$ (see for example [18], [20], or for an earlier result in a similar vein [50]). Again, combining this with our Theorem 4.4.1 and Corollary 4.4.2, we are done, completing the proof.

4.6 Applications and Concluding Remarks

We conclude this short note with some further applications and remarks.

4.6.1 Waldhausen’s A-theory

Recall that Waldhausen [66] introduced a notion of algebraic $K$-theory $A(X)$ of a topological space $X$. Once the $K$-theoretic contribution has been split off, one is left with the finitely dominated version of the algebraic $K$-theory $A^{fd}(X)$. This finitely
dominated version satisfies the “fundamental theorem of algebraic $K$-theory”, in that one has a homotopy splitting:

\[
A^{fd}(X \times S^1) \simeq A^{fd}(X) \times BA^{fd}(X) \times NA^{fd}_{\pm}(X) \times NA^{fd}_{-}(X)
\]

(4.6.1)

see [46] (the reader should compare this with the corresponding fundamental theorem of algebraic $K$-theory for rings, see [40]). The Nil-terms appearing in this splitting have been studied by Grunewald, Klein, and Macko [44], who defined Frobenius and Verschiebung operations, $F_n, V_n$, on the homotopy groups $\pi_*(NA^{fd}_{\pm}(X))$. In particular, they show that the composite $V_n \circ F_n$ is multiplication by $n$ [44, Proposition 5.1], and that for any element $x \in \pi_i(NA^{fd}_{\pm}(X))$ of finite order, one has $F_n(x) = 0$ for all sufficiently large $n$ (see the discussion in [44, pg. 334, Proof of Theorem 1.1]).

Since these two properties are the only ones used in our proofs, an argument identical to the proof of Theorem 4.3.1 gives the:

**Proposition 4.6.1.** Let $X$ be an arbitrary space, and let $NA^{fd}_{\pm}(X)$ be the associated Nil-spaces arising in the fundamental theorem of algebraic $K$-theory of spaces. Then if $H \leq \pi_*(NA^{fd}_{\pm}(X))$ is any finite subgroup, then

\[
\bigoplus_{\infty} H \leq \pi_*(NA^{fd}_{\pm}(X)).
\]

Moreover, if $H$ is a direct summand in $\pi_*(NA^{fd}_{\pm}(X))$, then so is $\bigoplus_{\infty} H$.

**Remark 4.6.2.** An interesting question is whether there exists a “twisted” version of the splitting in equation (4.6.1), which applies to bundles $X \to W \to S^1$ over the circle (or more generally, to approximate fibrations over the circle), and provides a homotopy splitting of the corresponding $A^{fd}(W)$ in terms of spaces attached to $X$ and the holonomy map.
4.6.2 Cokernels of Assembly Maps

For a general group $G$, one would expect from the Farrell-Jones isomorphism Conjectures that the cokernel of the relative assembly map for $G$ should be “built up”, in a homological manner, from the cokernels of the relative assembly maps of the various virtually cyclic subgroups of $G$ (see for example [51] for an instance of this phenomenon). In view of our Theorem 4.5.1, the following question seems relevant:

**Question:** Can one find a group $G$, an index $i \in \mathbb{Z}$, and a finite subgroup $H$, with the property that $H$ embeds in $\text{CoKer} \left( h_i^G(\mathbb{E}G) \to K_i(\mathbb{Z}[G]) \right)$, but $\bigoplus_{\infty} H$ does not?

In other words, we are asking whether contributions from the various Nil-groups of the virtually cyclic subgroups of $G$ could partially cancel out in a cofinite manner. Note the following special case of this question: is there an example for which this cokernel is a non-trivial finite group?

4.6.3 Exotic Farrell Nil-groups

Our Theorem 4.4.1 establish that, for a countable tame ring, meaning the associated Farrell Nil-groups has finite exponent, the associated Farrell Nil-groups, while infinitely generated, still remain reasonably well behaved, i.e. are countable direct sums of a fixed finite group. In contrast, for a general ring $R$ (or even, a general integral group ring $\mathbb{Z}G$), all we know about the non-trivial Farrell Nil-groups is that they are infinitely generated abelian groups. Of course, the possibility of having infinite exponent *a priori* allows for many strange possibilities, e.g. the rationals $\mathbb{Q}$, or the Prüfer $p$-group $\mathbb{Z}(p^\infty)$ consisting of all complex $p^i$-roots of unity ($i \geq 0$). We can ask:

**Question:** Can one find a ring $R$, automorphism $\alpha \in \text{Aut}(R)$, and $i \in \mathbb{Z}$, so that $NK_i(R, \alpha) \cong \mathbb{Q}$? How about $NK_i(R, \alpha) \cong \mathbb{Z}(p^\infty)$?
Remark 4.6.3. Grunewald [43, Theorem 5.10] proved that for every group $G$ and every group automorphism $\alpha$ of finite order, $NK_i(\mathbb{Q}G,\alpha)$ is a vector space over the rationals after killing torsion elements for all $i \in \mathbb{Z}$. However this still leaves the possibility that they may vanish.

Or rather, in view of our results, the following question also seems natural:

**Question:** What conditions on the ring $R$, automorphism $\alpha \in \text{Aut}(R)$, and $i \in \mathbb{Z}$, are sufficient to ensure $NK_i(R,\alpha)$ is a torsion group of finite exponent? Does $NK_i(\mathbb{Z}G;\alpha)$ have finite exponent for all polycyclic-by-finite groups when $\alpha$ is of finite order?

Finally, while this paper completes our understanding of the finiteness properties of Farrell Nil-groups associated with *finite order* ring automorphisms, nothing seems to be known about the Nil-groups associated with *infinite order* ring automorphisms. This seems like an obvious direction for further research.
Appendix A: Spectrum and Homology Theory

**Definition A.0.4.** Let $G$ be a group. A $G$-homology theory $H^G_*$ is a collection of covariant functors $H^G_n, n \in \mathbb{Z}$ from the category of $G$-CW pairs to the category of abelian groups, together with natural transformations

$$\partial_n(X, A) : H^G_n(X, A) \to H^G_{n-1}(A) := H^G_{n-1}(A, \emptyset)$$

for $n \in \mathbb{Z}$ so that the following axioms are satisfied:

(i) $G$-homotopy invariance: if $f_0, f_1 : (X, A) \to (Y, B)$ are $G$-homotopic maps of $G$-CW pairs, then $H^G_n(f_0) = H^G_n(f_1)$ for $n \in \mathbb{Z}$;

(ii) Long exact sequence of a pair: given a $G$-CW pair $(X, A)$, there is a long exact sequence

$$\cdots \to H^G_{n+1}(X, A) \xrightarrow{\partial_n} H^G_n(A) \xrightarrow{H^G_n(i)} H^G_n(X) \xrightarrow{H^G_n(j)} H^G_n(X, A) \xrightarrow{} \cdots$$

where $i : A \to X$ and $j : X \to (X, A)$ are inclusions;

(iii) Excision: for a $G$-CW pair $(X, A)$ and a $G$-map $f : A \to B$. The canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$H^G_n(F, f) : H^G_n(X, A) \to H^G_n(X \cup_f B, B), \forall n \in \mathbb{Z}$$
(vi) Disjoint union axiom: for a family of $G$-CW complexes $\{X_i : i \in I\}$, the map

$$
\bigoplus_{i \in I} H_n^G(j_i) : \bigoplus_{i \in I} H_n^G(X_i) \to H_n^G(\coprod_{i \in I} X_i)
$$

induced by the inclusions $j_i : X_i \to \coprod_{i \in I} X_i$ is an isomorphism, for all $n \in \mathbb{Z}$.

When $G$ is the trivial group, we get the usual generalized homology theory satisfying the disjoint union axiom. It is well-known that a generalized homology theory is uniquely determined by its values at a point $H_n(pt), n \in \mathbb{Z}$ in an appropriate sense. Likewise, a $G$-homology theory is uniquely determined by its values at homogeneous $G$-spaces $H_n^G(G/H), n \in \mathbb{Z}$ in an appropriate sense.

A general way to construct generalized homology theories is the usage of the notion of spectrum in algebraic topology.

**Definition A.0.5.** A spectrum $E$ is a sequence of pointed CW-complexes $\{E_i : i \in \mathbb{N}\}$ together with a sequence of structure maps $\sigma_i : \Sigma E_i \to E_{i+1}, i \in \mathbb{N}$, where $\Sigma E_i$ is the reduced suspension of the pointed space $E_i$. A map $f$ between spectra $E$ and $F$ is a sequence of maps $f_i : E_i \to F_i$ which makes the obvious diagram commutative.

**Example A.0.6.** For any pointed CW-complex $X$, there is a corresponding spectrum $\{\Sigma^i X : i \in \mathbb{N}\}$, where $\Sigma^i X$ is the $i$-th iterated suspension of $X$. The structure maps are given by identities $id : \Sigma(\Sigma^i X) \to \Sigma^{i+1}X$.

**Definition A.0.7.** Let $E = \{E_i \mid \sigma_i : \Sigma E_i \to E_{i+1}\}$ be a spectrum. For each $k \in \mathbb{Z}$, the $k$-th homotopy groups of $E$ are defined to be the colimit

$$
\pi_k(E) := \text{colim}_{i \to \infty} \pi_{k+i}(E_i)
$$

where the direct system is given by the composition of $\pi_{k+i}(E_i) \to \pi_{k+i+1}(\Sigma E_i) \to \pi_{k+i+1}(E_{i+1})$, with the first map the suspension homomorphism and the second map induced by the structure map $\sigma_i$. 

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Note that negative homotopy groups of a spectrum completely make sense, and can well be non-zero.

Almost all constructions for topological spaces carry over to spectra. For example, one has the notions of mapping cylinder, mapping cone, smash product, etc, for spectra. Spaces and spectra can also interact with each other. In particular, one can form the smash product of a pointed space $X$ and a spectrum $E$, by forming smash product degree-wise. This will result in a spectrum and is usually denoted by $X \wedge E$.

The main usage of spectrum is the construction of generalized homology theories:

**Theorem A.0.8.** Let $E = \{ E_i | \sigma_i : \Sigma E_i \to E_{i+1} \}$ be a spectrum. Then the construction $H_n(X; E) := \pi_n(X_+ \wedge E), n \in \mathbb{Z}$, gives a generalized homology theory, where $X_+$ is disjoint union with a (base) point.

For a proof, see [1]. We have, at a point, $H_n(pt; E) = \pi_n(E)$. The spectrum $E$ or its homotopy groups is usually referred to as the coefficient of the homology theory. The singular homology theory is given by the spectrum of Eilenberg-Maclane spaces.

This construction of generalized homology theories using spectra has been generalized by Davis and Lück [19] to construct $G$-homology theories, using the notion of or$G$-spaces and or$G$-spectra. The following materials are taken from their paper.

**Definition A.0.9.** Let $G$ be a group. The orbit category or$G$ of $G$ is the category with objects homogeneous $G$-spaces $G/H$ and morphisms $G$-equivariant maps.

**Definition A.0.10.** Let $G$ be a group. A contravariant or$G$-space is a contravariant functor $X$ from the orbit category or$G$ to the category of compactly generated Hausdorff spaces. A covariant or$G$-spectrum is a covariant functor $E$ from the orbit category or$G$ to the category of spectra.
Example A.0.11. Let $X$ be a $G$-CW complex. Then $X$ determines a contravariant or $G$-space:

$$G/H \mapsto \text{map}_G(G/H, X) = X^H$$

where $X^H$ is the fixed point set of the subgroup $H$ acting on $X$. This contravariant or $G$-space is usually denoted by $\text{map}_G(?, X)$ or $X^?$.

One can form smash product of a contravariant or $G$-space and a covariant or $G$-spectrum to get an ordinary spectrum.

Definition A.0.12. Let $X$ be a pointed contravariant or $G$-space and $E$ be a covariant or $G$-spectrum. Their smash product $X \wedge_G E$ is an ordinary spectrum with

$$(X \wedge_G E)_n := \left( \bigvee_{H \leq G} \left( X(G/H) \wedge E_n(G/H) \right) \right) / \sim$$

where the equivalence relation $\sim$ is generated by $X(f)(x) \wedge y \sim x \wedge E_n(f)(y)$ for any $G$-map $f : G/H \to G/K$, $x \in X(G/K)$ and $y \in E_n(G/H)$.

Theorem A.0.13. Let $E$ be a covariant or $G$-spectrum. Then the construction

$$H_n^G(X; E) := \pi_n(X^+_\wedge_G E), n \in \mathbb{N}$$

gives a $G$-homology theory.
Appendix B: $K$- and $L$-theory Spectra

Algebraic $K$- and $L$-groups can be defined as homotopy groups of certain spectra. Their constructions are quite technical and we are not going to present them here. Interested readers can refer to [56][62][16]. We will however record their formal properties here. For the proofs of the Farrell-Jones conjecture, only these formal properties are used.

**Theorem B.0.14.** [8, Theorem 5.1] There exists a functor $\mathbb{K}^{-\infty}: \mathcal{A} \mathcal{D} \to \mathbf{Sp}$ from the category of small additive categories to the category of spectra with the following properties:

1. $\pi_i(\mathbb{K}^{-\infty}(R_{\oplus})) = K_i(R)$ for all $i \in \mathbb{Z}$, where $R$ is a ring and $R_{\oplus}$ is the additive category of finitely generated free $R$-modules;

2. If $\phi: \mathcal{A} \to \mathcal{B}$ is an equivalence of additive categories, then $\mathbb{K}^{-\infty}(\phi)$ is a weak equivalence of spectra;

3. If $\mathcal{A}$ is a flasque additive category, then $\mathbb{K}^{-\infty}(\mathcal{A})$ is weakly contractible;

4. If $\mathcal{A} \subseteq \mathcal{U}$ is a Karoubi filtration of additive categories, then

$$\mathbb{K}^{-\infty}(\mathcal{A}) \to \mathbb{K}^{-\infty}(\mathcal{U}) \to \mathbb{K}^{-\infty}(\mathcal{U}/\mathcal{A})$$

is a homotopy fibration sequence of spectra;
5. If $\mathcal{A} = \lim_{i} \mathcal{A}_i$ is a colimit of additive categories over a directed system, then the natural map $\lim_{i} \mathbb{K}^{-\infty}(\mathcal{A}_i) \to \mathbb{K}^{-\infty}(\mathcal{A})$ is a weak equivalence.

Similar results hold for the $L$-theory spectra functor $\mathbb{L}^{<\infty} : \mathcal{A} \mathcal{D}^* \to \mathbf{Sp}$ from the category of small additive categories with involutions to the category of spectra.

An additive category $\mathcal{A}$ is called flasque if there exists an additive functor $\Sigma : \mathcal{A} \to \mathcal{A}$ so that $id_{\mathcal{A}} \oplus \Sigma$ is naturally equivalent to $\Sigma$. The notion of Karoubi filtration can be found in [15].

For a ring $R$, we denote $\mathbb{K}^{-\infty}(R_{\oplus})$ by $\mathbf{K}_R$. $\mathbf{K}_R$ is called the non-connective $K$-theory spectrum of the ring $R$. Non-connective means its negative homotopy groups are non-vanishing in general, and equal the negative $K$-groups of $R$. There also exists a connective $K$-theory spectrum, meaning the negative homotopy groups vanish.
Appendix C: Countable Nil-groups

In this appendix, we give a short discussion on the cardinality of Nil-groups. The following proposition is needed in the proof of our Theorem 4.4.1 – while presumably well-known to experts, we were unable to find it in the literature.

**Proposition C.0.15.** Let $R$ be a countable ring and $\alpha : R \to R$ be a ring automorphism. Then the groups $K_i(R)$ and $NK_i(R; \alpha)$ are countable for all $i \in \mathbb{Z}$.

**Proof.** Since $NK_i(R; \alpha)$ is a subgroup of $K_i(R_\alpha[t])$ and $R_\alpha[t]$ is countable when $R$ is countable, it is enough to show $K_i(R)$ is countable when $R$ is countable. So let us focus on $K_i(R)$.

We first use Quillen’s $+\text{-construction}$ to treat the case where $i \geq 1$. Consider the infinite general linear group $GL(R)$. Being the countable union of countable groups $GL_n(R)$ ($n \in \mathbb{N}$), we see that $GL(R)$ is countable. Hence the simplicial complex spanned by the group elements of $GL(R)$, which is a model for the universal space for free $GL(R)$-actions, is also countable. Then the quotient $BGL(R)$ is of course a countable CW-complex. Performing Quillen’s $+\text{-construction}$ to $BGL(R)$, we obtain the algebraic $K$-theory space $BGL(R)^+$ with $K_i(R) := \pi_i(BGL(R)^+)$, for $i \geq 1$. Note that $BGL(R)^+$ is obtained from $BGL(R)$ by attaching 2-cells and 3-cells indexed by some generating set of the commutator subgroup of $GL(R)$, hence $BGL(R)^+$ is a
countable CW-complex. More details of Quillen’s +-construction can be found, for example, in [64, Theorem 5.2.2].

We now show the homotopy groups of a countable CW-complex is countable. By filtering a countable CW-complex by its countably many finite subcomplexes, it suffices to show homotopy groups of finite CW-complexes are countable. So let us assume $X$ is a finite CW-complex. Since every finite CW-complex has the homotopy type of a finite simplicial complex, we may assume $X$ is a finite simplicial complex. Fix a triangulation $\Delta_i$ of $S^i$. The set of all iterated barycentric subdivisions of $\Delta_i$ is countable. Fix a vertex in $\Delta_i$ and a vertex in $X$ as base points. By simplicial approximation, any element in $\pi_i(X)$ can be represented by a simplicial map from some iterated barycentric subdivision of $\Delta_i$ to $X$. But the set of such simplicial maps is clearly countable, hence $\pi_i(X)$ is countable. Thus $K_i(R)$ is countable when $i \geq 1$.

Now let us consider the case when $i < 1$. First, we consider $i = 0$. Let $\text{Idem}(R)$ be the set of idempotent matrices in $M(R)$, where $M(R)$ is the union of all $n \times n$ matrices over $R$, ($n \in \mathbb{N}$). $GL(R)$ acts on $\text{Idem}R$ by conjugation, denote the quotient by $\text{Idem}R/GL(R)$. This is a semigroup and $K_0(R)$ can be identified with the Grothendieck group associated to this semigroup (see [64, Theorem 1.2.3]). Therefore $K_0(R)$ is countable since $\text{Idem}R$ is countable. Now when $i < 0$, the negative $K$-groups $K_i(R)$ can be inductively defined to be the cokernel of the natural map (see [64, Definition 3.3.1])

$$K_{i+1}(R[t]) \oplus K_{i+1}(R[t^{-1}]) \to K_{i+1}R[t, t^{-1}]$$

Note when $R$ is countable, $R[t], R[t^{-1}]$ and $R[t, t^{-1}]$ are all countable. Hence their $K_0$-groups are all countable. Thus we inductively have $K_i(R)$ are countable for all $i < 0$. This completes the proof. \qed

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Bibliography


