Using Modern Stellar Observables to Constrain Stellar Parameters and the Physics of the Stellar Interior

DISSERTATION

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By

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Abstract

The current state and future evolution of a star is, in principle, specified by only a few physical quantities: the mass, age, hydrogen, helium, and metal abundance. These same fundamental quantities are crucial for reconstructing the history of stellar systems ranging in scale from planetary systems to galaxies. However, the fundamental parameters are rarely directly observable, and we are forced to use proxies that are not always sensitive or unique functions of the stellar parameters we wish to determine. Imprecise or inaccurate determinations of the fundamental parameters often limit our ability to draw inferences about a given system. As new technologies, instruments, and observing techniques become available, the list of viable stellar observables increases, and we can explore new links between the observables and fundamental quantities in an effort to better characterize stellar systems. In the era of missions such as *Kepler*, time-domain observables such as the stellar rotation period and stellar oscillations are now available for an unprecedented number of stars, and future missions promise to further expand the sample.

Furthermore, despite the successes of stellar evolution models, the processes and detailed structure of the deep stellar interior remains uncertain. Even in the
case of well-measured, well understood stellar observables, the link to the underlying parameters contains uncertainties due to our imperfect understanding of stellar interiors. Model uncertainties arise from sources such as the treatment of turbulent convection, transport of angular momentum and mixing, and assumptions about the physical conditions of stellar matter. By carefully examining the sensitivity of stellar observables to physical processes operating within the star and model assumptions, we can design observational tests for the theory of stellar interiors.

I propose a series of tools based on new or revisited stellar observables that can be used both to constrain stellar parameters and the physics of the interior. I examine how the acoustic signature of the location of the base of stellar convective envelopes can be used as an absolute abundance indicator, and describe a novel $^3\text{He}$-burning instability in low mass stars along with the observational signatures of such a process. Finally, I examine the manner in which stellar rotation, observed in a population of objects, can be used as a means to distinguish between different evolutionary states, masses, and ages. I emphasize that rotation periods can be used as age indicators (as often discussed in the literature), but that the interpretation of rotation periods must be made within the context of the full stellar population to arrive at accurate results.
Dedication

To my parents and my husband, for being there, always.
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Publications

Research Publications


Fields of Study

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Chapter 1: Introduction

1.1. Fundamental Stellar Parameters and Stellar Observables

Most of the luminous material in the universe is either composed of stars or illuminated by them. Stars make the cosmic overdensities that are galaxies glow, and provide the warmth and light that sustains life on Earth. The distribution of ages, compositions, and masses of stars that compose a galaxy are our windows into the formation of the system. Stars trace and contribute to the chemical evolution of the universe. They harbor planets, some of which may potentially host life.

In order to understand the many systems comprised of stars, we need to be able to accurately determine the properties of the stars themselves. Once the fundamental parameters of mass, age, and composition are known, we can predict the luminosities, temperatures, evolutionary trajectories, and chemical yields of any given object. These values are critical for the interpretation of stellar systems over all scales. Estimates for the ages of different components of our own Milky Way depend on the ages of stars; estimates of the radii, ages, and compositions of small rocky bodies orbiting other stars depend sensitively on our ability to determine the stellar radius, age and composition of the host star.

While the state of a given star is set by a handful of fundamental stellar parameters (the mass, age, hydrogen, helium, and metal content), these quantities are generally not directly observable. The historic stellar observables are secondary,
surface properties such as the magnitude, color, stellar effective temperature, luminosity, surface gravity, and metallicity ([Fe/H]). We link these observables, through stellar models, to the underlying fundamental stellar parameters. This linkage is fraught with degeneracies when parameters are desired to the few percent level, and many of the surface observables evolve slowly as a function of time. For a middle-aged main sequence star like the Sun, with typical uncertainties on the temperature, luminosity, and metallicity, the inferred mass uncertainties can encompass stars with very different internal structures, and the age uncertainties can be of order the age of the star itself.

We rely on these “standard” observables because they are relatively simple to measure, and have relatively well-understood relationships to the underlying fundamental parameters. They are not the only windows into stellar interiors, but they have traditionally been the most accessible. Recent space-based missions operating in the time domain have expanded the list of quantities that we might consider “standard” observables to stellar oscillations, driven by pressure waves in the stellar envelope, and stellar rotation. These phenomena have been historically difficult to measure or difficult to interpret, and it is only with the substantial increase in the number of successful measurements made to date that we can begin to use these observables as general tools. These novel observables have two primary uses: as tools for inferring the fundamental stellar parameters, and as constraints on our models of stellar interiors.

Stellar evolution codes, despite their simplifying assumptions, are able to reproduce the observed properties of stars remarkably well. However, certain fine details of the internal structure of stars remain poorly constrained: the location and structure of convection zones, degree of overshooting at the boundary of convection zones, degree of internal mixing, transport of angular momentum, stellar dynamos
and the generation of magnetic fields, element diffusion, and details of stellar matter at the pressures and temperatures relevant to the stellar interior. We are ever in search of systems and observables that will allow us to produce more accurate models. The problem, then, exists in both directions. We seek observables that allow us to constrain stellar parameters through models, but also observables that allow us to test the physics of the models themselves, and to refine our picture of the interior workings of stars.

1.2. Stellar Evolution Models

Our investigation is largely theoretical in nature, and utilizes the Yale Rotating Evolution Code (YREC) as our primary tool in the study of how various stellar properties vary as a function of fundamental parameters. The equations of stellar structure for a spherically symmetric star, with mass as the independent variable, are given by (following Kippenhahn & Weigert 1990):

\[
\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho},
\]

\[
\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^2} - \frac{1}{4\pi r^2} \frac{\partial r}{\partial t}^2,
\]

\[
\frac{\partial l}{\partial m} = \epsilon_n - \epsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta P}{\rho \partial t},
\]

\[
\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P} \nabla,
\]

\[
\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \left( \sum_j r_{ij} - \sum_k r_{ik} \right), i = 1, ..., I,
\]

where \(m, r, \rho, P,\) and \(t\) are the mass, radius, density, pressure, and time coordinates, respectively. \(\epsilon_n\) represents the energy generated per unit mass per
second by nuclear reactions, $\epsilon_\nu$ the energy lost due to neutrinos, $c_P$ the specific heat at constant pressure, $\delta$ a thermodynamic derivative defined as $\delta \equiv -\left(\partial \ln \rho / \partial \ln T\right)_P$, and $\nabla \equiv d \ln T / d \ln P$. $X_i$ represents the mass fraction of element $i$, and $r_{ij}$ the reaction rate of the process that generates element $j$ from element $i$.

The code solves the coupled, differential equations of stellar structure, subject to boundary conditions at the surface and the core. YREC, in its non-rotating configuration, is a one-dimensional Henyey-style code, in which the star is broken into a series of mass shells, and the differential equations replaced with difference equations for each shell. The code iteratively derives corrections to trial solutions for the entire run of structure values (pressure, temperature, luminosity, etc.) until the desired precision is reached.

The assumption of hydrostatic equilibrium is enforced; in the Sun, for example, the dynamical timescale is roughly half an hour, and we can safely assume equilibrium. The stellar model is permitted to be out of thermal or chemical equilibrium. The code treats three-dimensional turbulent convective transport through use of the mixing-length theory, in which convection is parameterized by specifying typical scale for convective transport. In practice the stellar model is broken into “interior” and “envelope” solutions, which are joined at a fitting point in mass. The full set of equations is solved in the interior portion of the model, while luminosity is held fixed in the envelope solution. Optimal fitting points are both deep enough in the stellar model that the convergence process is accelerated and stable due to the simplified set of stellar equations, but shallow enough that the assumption of fixed luminosity (and no nuclear burning) at the fitting point is valid. Additional physics, such as opacities, the equation of state, and atmospheric boundary conditions are supplied via input tables. The code can be run in a “1.5-d” rotating configuration, in which the corrections to the structure due to rotational
deformation are included in the solutions, as well as various mixing, hydrodynamical instabilities, and angular momentum transport mechanisms.

The simplifying assumptions and general procedures that YREC utilizes are fairly standard across the subset of one-dimensional evolution codes in use today. Often the most substantial differences between codes are in the choices of microphysics, or whether particular physical processes (such as convective core overshooting) are treated or ignored.

1.3. Modern Observables: Asteroseismology

Over the past decade, asteroseismology has become possible for stars other than the Sun, and opened windows into their interiors that were inaccessible in the past. Asteroseismology makes use of the fact that stars are not static; they support standing waves in their interiors whose frequencies and amplitudes depend on the stellar structure. We identify waves of two different characters: p-modes, where the restoring forces is pressure, and g-modes, where the restoring force is gravity. These oscillations are thought to be stochastically excited by turbulent convection. Waves excited at the star’s resonant frequencies are comparatively long-lived, while waves excited at other frequencies are quickly damped. The p-modes are the primary observable modes that propagate in the envelopes of solar-like stars, and give rise to displacements at the stellar surface which yield small, periodic changes in brightness that can be detected via photometric methods, or radial velocity shifts in spectral lines that are detected via spectroscopic monitoring. Observations of these oscillations are challenging: the typical velocity amplitude for a solar oscillation mode is 15 cm s\(^{-1}\), and the typical photometric variation is of order a few parts per million.
Oscillation modes can be described with spherical harmonics. Modes of low degree, $l$, propagate more deeply into the stellar interior, while modes of high degree sample smaller scales closer to the stellar surface. In this manner, each mode samples the conditions at a different depth in the star, and with enough modes, we can invert the problem and determine the stellar sound speed profile (which in turn depends on pressure, density, and composition). This inversion process has motivated sophisticated modeling of the solar interior, and offers tight constraints on the convective envelope transition (Christensen-Dalsgaard et al. 1991; Bahcall & Pinsonneault 1992), degree of convective overshooting (Basu 1997), rotation profile (Thompson et al. 2003), and helium abundance (Basu & Antia 1995).

Seismic inferences of the solar sound speed are also responsible for tension in the determination of solar oxygen abundance, which will be relevant for later chapters in this dissertation. One dimensional atmosphere models that assume local thermodynamic equilibrium (LTE) (Grevesse & Sauval 1998) yield oxygen abundances higher than the more recent determination by Asplund et al. (2009) using sophisticated 3-d non-LTE atmosphere models. Because oxygen is one of the primary sources of opacity at the location of the convective boundary in the Sun, changes to the oxygen abundance will result in shifts in the location of the base of the convection zone (BCZ). The seismic measurements of the BCZ are consistent with the earlier, higher oxygen abundances, and notably inconsistent with the more recent oxygen abundance determinations.

Seismology of the Sun has a long history, but asteroseismology of other stars is a fairly recent achievement. The first signs of oscillations in the Sun were detected as early as Leighton (1960), and observations continue through the present day with ongoing projects such as GONG (Harvey et al. 1996). Detections of solar-like oscillations (where the primary observed modes are stochastically excited,
damped pressure modes) in stars other than to Sun has lagged substantially behind helioseismology. Brown et al. (1991) detected the first solar-like oscillations in Procyon, later confirmed in Martić et al. (1999). Subsequent discoveries of solar-like oscillations were made in η Boo (Kjeldsen et al. 1995), β Hydri (Bedding et al. 2001), and α Cen (Bouchy & Carrier 2001). In 2003 the MOST satellite was launched (Walker et al. 2003) with the goal of performing asteroseismic measurements from space. CoRoT (Michel et al. 2008), launched in 2006, and especially Kepler (Gilliland et al. 2010), launched in 2009, have vastly increased both the number of stars with seismic detections and the quality of those detections. During the first months of Kepler observations, ∼ 2000 solar-like stars were targeted in search of solar-like oscillations in the spacecraft’s 1 minute cadence mode, and over 500 detections have been reported (Chaplin et al. 2011a, 2014). Hekker et al. (2011) reported nearly 12,000 red giants displaying solar-like oscillation patterns.

Unlike the Sun, only low degree modes are detected in the full disk observations of other stars. In the best cases dozens of modes are measured (Metcalf et al. 2012), in comparison to the Sun, in which over a million modes are detected. Typical solar-like oscillators display an envelope of excess power at the frequencies at which modes are present, with individual modes spaced by regular intervals. The so-called “global seismic parameters” $\nu_{\text{max}}$ (frequency of maximum power of the envelope of excess power) and $\Delta \nu$ (frequency spacing between modes of the same degree and consecutive radial order) in combination with scaling relations provide model-independent determinations of the stellar mass and radius (Kjeldsen & Bedding 1995). With the addition of stellar evolutionary models and careful analysis of individual modes, seismic observations can provide masses, radii, stellar ages, and helium abundances for stars other than the Sun (Mathur et al. 2012; Chaplin et al. 2014).
Asteroseismology has the potential to offer unprecedented views into stellar interiors, and the list of asteroseismic achievements is growing rapidly. Mazumdar et al. (2014) reported the detection of the acoustic signature of the convective boundary in 19 stars in which oscillations were detected with particularly high signal to noise. Deheuvels et al. (2012, 2014) constrained the degree of internal differential rotation in subgiants with mode splittings, and Bedding et al. (2011) demonstrated that asteroseismology can be used to distinguish between core helium and hydrogen shell burning stars. These observations represent our first direct view into the interiors of stars other than our own Sun. The field of asteroseismology is only about two decades old, and it is only in the last few years that seismic data has been available in the quantity and quality necessary to make significant advancements. It is under these conditions that we search for other, unrecognized uses of seismic signatures in the effort to constrain stellar parameters and interiors.

1.4. Modern Observables: Stellar Rotation

All stars rotate, and their rotation rates are linked to fundamental parameters such as age and mass. Despite this connection, stellar rotation has largely been neglected as a viable diagnostic. The reason for this neglect is twofold: observations of all but the most rapidly rotating, active stars are difficult from the ground, and the interpretation of a given rotation period, even when it is successfully measured, can be ambiguous.

Theoretical modeling of rotation has proceeded slowly, because it involves the complex and uncertain physics. Stellar rotation is determined by the combination of initial angular momentum, structural changes that result in evolution of the moment of inertia, mechanism (or lack of) for the loss of angular momentum, and the distribution and transport of angular momentum in the stellar interior. Of
these, only the changes to the moment of inertia are well determined by modern stellar models. Observations of young stars can help to specify the rotation velocities shortly after formation, but without an assumption regarding the angular momentum profile, the total initial angular momentum is unknown. A solid body rotator with a given surface rotation rate, for example, has a lower total angular momentum than a differentially rotating star (with the interior layers being the more rapidly rotating) with the same surface rotation rate. Some evolutionary processes tend to drive a system towards differential rotation both through differential contraction and expansion, and angular momentum loss from the stellar surface. We expect that stellar rotation profiles should fall somewhere between the solid body rotation and rotation with local conservation of angular momentum, but have few observational tests to determine the degree of coupling in the interior. Helio- and asteroseismology have provided information in a handful of systems (Deheuvels et al. 2012, 2014; Tayar & Pinsonneault 2013; Kurtz et al. 2014) that suggest that these limiting cases are valid. The Sun’s radiative zone supports only weak differential rotation to a depth of roughly 0.2R⊙ (Thompson et al. 2003), and requires that physical processes within the star, whether they be hydrodynamical, magnetic, or wave-driven, maintain coupling between the envelope and interior.

We see evidence that stars less massive than ∼1.5M⊙ undergo angular momentum loss, based on the fact that they are typically slow rotators despite observations of uniformly rapid rotation in young clusters. In systems in which we have additional constraints on the stellar ages, we find that the surface rotation rates of these objects drop as a function of time (Skumanich 1972). This is generally attributed to coupling between stellar mass loss and the magnetic field of the star (Weber & Davis 1967): particles streaming away from the surface are essentially forced to co-rotate out to an Alfvén radius (typically many stellar radii) due to the
magnetic field, at which point even a small amount of mass loss can carry away a substantial portion of the angular momentum. Because dynamo generation of the magnetic field is thought to require turbulent convection, this mechanism should operate only in stars with thick convective envelopes. In this picture we expect, and indeed observe, that there is a sharp transition in the observed rotation rates at about 1.5M_{\odot} (Kraft 1967), where stars become cool enough to support a substantial surface convection zones. Stars above this mass appear to rotate with approximately their initial rotation rates; they do not have strong dynamos, deep convective envelopes, or sufficient mass loss to undergo substantial angular momentum loss.

There are two primary means of detecting surface rotation. Rotation broadens spectral lines in the stellar atmosphere, which can be used to measure the rotation velocity modulo an inclination uncertainty of sin i. One can also monitor the photometric brightness of the star. The passage of starspots across the stellar disk will result in periodic dimming, and allow a direct measurement of the surface rotation period. This method is somewhat limited by the fact that starspots generally sample limited stellar latitudes, and we expect some degree of latitudinal differential rotation. In general, spectroscopic methods work well for the more massive, more rapidly rotating stars that have substantial line broadening but few spots, whereas the spot modulation method is suited to the low mass, slowly rotating stars that are generally magnetically active and spotted, but would be difficult to detect spectroscopically because of low rotation velocities of order a few km/s. However, in both cases, stars like the Sun (old and quiet, with periods of several tens of days) are difficult to detect from ground-based observatories. They either require very careful measurements of the line broadening, often at levels beyond the capabilities of modern spectrographs and theory of stellar atmospheres, or long time series (many months) with exceptional photometry (at the millimag level, Basri
et al. 2010) that is challenging or impossible from the ground. For this reason, our picture of stellar rotation as been limited to the young, massive, or active stars. However, in the same manner that Kepler has vastly expanded the extant sample of stars detected in solar-like oscillations, it has also facilitated the detection of tens of thousands of spot-modulation rotation periods (McQuillan et al. 2014; Nielsen et al. 2013). We now have a sample of rotation periods across all spotted spectral types, at a range of evolutionary states and masses, which lends itself to investigations of how we might use these measurements to learn more about the stars themselves and the physics of the interior.

1.5. Scope of the Dissertation

In addition to the aforementioned uses of asteroseismology, we devote Chapter 2 to the discussion to additional diagnostic power of asteroseismic measurements. Motivated by both the Solar oxygen abundance problem and BCZ measurements in stars other than the Sun, we examine the sensitivity of the depth of the CZ to mass, stellar abundances, and input physics, and in particular, the use of a measurement of the acoustic depth to the CZ as an atmosphere-independent, absolute measure of stellar metallicities. We find that for low-mass stars on the main sequence with $0.4M_\odot \leq M \leq 1.6M_\odot$, the acoustic depth to the base of the CZ, normalized by the acoustic depth to the center of the star, $\tau_{cz,n}$, is both a strong function of mass, and varies at the 0.5-1% per 0.1 dex level in $[Z/X]$, and is therefore also a sensitive probe of the composition. We estimate the theoretical uncertainties in the stellar models, and show that combined with reasonable observational uncertainties, we can expect to measure the metallicity to within 0.15 - 0.3 dex for solar-like stars. We discuss the applications of this work to rotational mixing, particularly in the context of
the observed mid F star Li dip, and to distinguishing between different mixtures of heavy elements.

In Chapter 3 we report on the discovery of an instability in low mass stars just above the fully convective threshold (\( \sim 0.35 \, M_\odot \)) that we encountered during the investigation of CZ depth as a function of mass in the previous chapter. In this instability, non-equilibrium \(^3\)He burning creates a convective core, which is separated from a deep convective envelope by a small radiative zone. The steady increase in central \(^3\)He causes the core to grow until it touches the surface convection zone, which triggers fully convective episodes in what we call the “convective kissing instability”. These episodes lower the central abundance and cause the star to return to a state in which is has a separate convective core and envelope. These periodic events eventually cease when the \(^3\)He abundance throughout the star is sufficiently high that the star is fully convective, and remains so for the rest of its main sequence lifetime. The episodes correspond to few percent changes in radius and luminosity, over Myr to Gyr timescales. We discuss the physics of the instability, as well as prospects for detecting its signatures in open clusters and wide binaries. Secondary stars in cataclysmic variables (CVs) will pass through this mass range, and this instability could be related to the observed paucity of such systems for periods between two and three hours. We demonstrate that the instability can be generated for CV secondaries with mass-loss rates of interest for such systems, and discuss potential implications.

Finally, we discuss the use of stellar rotation as a non-standard observable in Chapter 4. Stellar rotation is a strong function of both mass and evolutionary state, and missions such as \textit{Kepler} and CoRoT provide tens of thousands of rotation periods, drawn from stellar populations that contain objects at a range of masses, ages, and evolutionary states. Given a set of reasonable starting conditions and a
prescription for angular momentum loss, we address the expected range of rotation periods for cool field stellar populations ($\sim 0.4 - 2.0M_\odot$). We find that cool stars fall into three distinct regimes in rotation. Rapid rotators with surface periods less than 10 days are either young low-mass main sequence (MS) stars, or higher mass subgiants which leave the MS with high rotation rates. Intermediate rotators (10-40 days) can be either cool MS dwarfs, suitable for gyrochronology, or crossing subgiants at a range of masses. Gyrochronology relations must therefore be applied cautiously, since there is an abundant population of subgiant contaminants. The slowest rotators, at periods greater than 40 days, are lower mass subgiants undergoing envelope expansion. We identify additional diagnostic uses of rotation periods. There exists a period-age relation for subgiants distinct from the MS period-age relations. There is also a period-radius relation that can be used as a constraint on the stellar radius, particularly in the interesting case of planet host stars. The high-mass/low-mass break in the rotation distribution on the MS persists onto the subgiant branch, and has potential as a diagnostic of stellar mass. Finally, this set of theoretical predictions can be compared to extensive datasets to motivate improved modeling.
Chapter 2: The Sensitivity of Convection Zone Depth to Stellar Abundances

2.1. Introduction

A profound transition in our understanding of stars is now underway, and one of the major drivers is the detection of pulsations in large samples of Sun-like stars. Much of the immediate interest and effort has focused on using scaling relationships for pulsations to infer global properties, such as mass, radius, and age. Many new insights into stellar structure will in fact emerge from our new ability to design experiments: for example, using masses outside of binary systems, or ages outside of star clusters. However, our deepest insights are likely to emerge from diagnostics of internal structure uniquely available from seismic data. In this chapter we focus on one such property: the depth of the surface convection zone (CZ). We demonstrate that theory predicts strong mass and composition trends in the depth of the CZ with surprisingly small errors. This raises the prospect of an absolute seismic abundance calibration and rigorous tests of interiors theory. Furthermore, we may be able to test more challenging issues, such as deep mixing in the mid-F star lithium dip, or the mixture of heavy elements.

A rich spectrum of non-radial pulsations is observed in the Sun. The study of solar oscillations, or helioseismology, has yielded insights into solar and stellar structure. In the case of the Sun we have spatially resolved information and can reconstruct the speed of sound as a function of depth. For stars we can only observe
global modes of angular low degree \(^1\), which makes full sound speed reconstructions impractical. The low \(l\) modes have asymptotic frequency separations predicted by theory, however, which can yield valuable information about the global properties of stars. Sharp localized changes in structure will also manifest themselves as deviations from these regular separations; examples include ionization zones and transitions from radiative to convective energy transport. We briefly review the former before introducing the latter, which are the main focus of the current work.

The average relationships between frequencies of mode pairs \((l, n) - (l, n - 1)\) and \((l, n) - (l + 2, n - 1)\) are frequently referred to as the large and small frequency separations, respectively. The former is related to the mean density, while the latter is a measure of the degree of central concentration, and thus helium content, of main sequence stars. The small frequency separation is a potent age diagnostic (Ulrich 1986). A significant advantage of these relationships is that useful information can be extracted from the average differences of many mode pairs, increasing the effective signal to noise. The frequency of maximum power reflects a competition between the spectrum of turbulence generating the sound waves and the acoustic cutoff frequency. The cutoff frequency is empirically observed to follow regular scaling relationships, and the combination of \(\nu_{\text{max}}\) and \(\Delta \nu\) can be used to solve for the mass and radius (Brown et al. 1991; Mosser et al. 2010; Belkacem et al. 2011; Chaplin et al. 2011a).

With better data it is possible to extract entirely new kinds of information. It was recognized early on that the solar oscillations could be used to measure the strength of the helium ionization zone, and thus by extension the solar surface

\(^1\)Modes are specified by the three spherical harmonic quantum numbers: \(n\), the overtone, \(l\), the degree, and \(m\), the azimuthal order. For stars other than the Sun, we can generally detect only modes of low degree \((l \lesssim 3)\).
helium abundance (Gough 1984); Delahaye & Pinsonneault (2006) found a literature average of $0.248 \pm 0.004$. The surface helium is significantly below the initial levels, of order 0.27, required to reproduce the solar luminosity at the solar age. This difference can be attributed to gravitational settling of helium and heavy elements (Aller & Chapman 1960; Noerdlinger 1977; Bahcall & Pinsonneault 1992; Bahcall et al. 1995).

The transition from convective to radiative energy transport induces a discontinuity in the temperature gradient, which in turn produces a characteristic response in the frequencies of the modes which cross this boundary. This phenomenon can be used to infer the depth of the surface convection zone (Christensen-Dalsgaard et al. 1991). The agreement between the theoretically predicted depth and the seismic data was a significant triumph for stellar interiors theory; see for example Bahcall & Pinsonneault (1992).

In the solar context these seismic tools can be used for precision tests of astrophysics; for example, the convection zone depth can be measured to a remarkable precision, of order $5 \times 10^{-4}$ (Basu & Antia 2004). Scalar constraints on convection zone depth and surface helium can be combined to test the absolute solar metal abundance and mixture (Delahaye & Pinsonneault 2006). Mixing sufficient to explain the solar lithium depletion (Weymann & Sears 1965; Pinsonneault et al. 1989) can reduce the effects of settling (Richard et al. 1996; Bahcall et al. 2001) and has a characteristic signature in the sound speed profile. Solar data is even of sufficient quality to detect the signature of metal ionization (Basu & Antia 2008), which can be used to infer the absolute oxygen abundance. In stars it is anticipated that the acoustic radius of the convection zone can be measured to of order 2\% (Ballot et al. 2004). Measurements of the surface helium abundance are more promising in red giants (Miglio et al. 2010), but will be technically more difficult in
main sequence stars; similar comments apply to the extent of convective cores. We therefore focus this study on the surface convection zone depth.

To first order the depth of the surface convection zone is determined by the effective temperature, modified by the surface gravity (see Pinsonneault et al. 2001, for a discussion). Stellar metallicity plays a second order role, in the sense that lower bulk metallicity implies a shallower surface convection zone at fixed effective temperature. Our first priority is therefore to quantify these effects, and determine the robustness of the theoretical predictions. Mixing and element separation processes can also modify the convection zone depth at a detectable level, although it will clearly be more difficult to detect these more subtle shifts in the seismic properties. We therefore also explore our ability to empirically diagnose these phenomena.

We begin with a discussion of our methods in Section 2.2. We discuss our models results, both in terms of the overall trends in the depth of the convection zone, as well as their magnitude in comparison to theoretical and observational uncertainties in Section 2.3. We discuss further potential uses of the depth of the CZ as a diagnostic and conclude in Section 2.4.

2.2. Calculation of the Acoustic Depth and Theoretical Error bars

Our overall approach is similar to that employed in solar model studies. We define a reference model calculation which includes our current best estimates of the input physics. We then calibrate a solar model and run a series of models with different masses and compositions; these define our predicted theoretical trends. We then perform a comprehensive error analysis to estimate the detectability of these signals. This includes both theoretical errors (for example, nuclear reaction
cross-sections or quantum mechanical opacity calculations) and observational errors (for example, the seismic age and mass of the star being tested). Errors are not easy to estimate in some cases, such as for opacities. We therefore use the differences between competing calculations as our measure of these theoretical uncertainties. We also explore some indirect tests of physics not typically included in interiors models, such as rotationally induced mixing. In a full calculation one would properly account for nonlinear effects rather than employing a strict parameter variation study. We briefly discuss some such cases, but our main concern is with laying out the baseline theoretical expectations and their overall reliability. More complex nonlinear calculations would be a logical next step and would be better motivated in the presence of a significant observational database.

2.2.1. Calculation of the Acoustic Depth to the Convection Zone

We choose to focus on the acoustic depth of the convection zone, $\tau_{cz}$, rather than its physical depth, since $\tau_{cz}$ is the asteroseismic observable of the CZ location. In general, the “acoustic depth” is defined (Gough 1990) as,

$$\tau = \int_{R_1}^{R_2} \frac{dR'}{c_s}$$

(2.1)

where $R$ is the radius and $c_s$ is the sound speed. In order to easily compare $\tau_{cz}$ for stars across a wide range of masses, we use the “normalized” acoustic depth, $\tau_{cz,n}$, given by,

$$\tau_{cz,n} = \frac{\tau_{cz}}{\tau_*} = \frac{\int_{R_{cz}} R'}{\int_{R=0}^{R_*} R'}$$

(2.2)

where the sound speed is simply $c_s = \sqrt{\frac{\gamma}{\rho} \frac{dT}{dR}}$. 

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Both $P$ and $\rho$ as a function of radius can be obtained directly from a converged stellar model. In general, the adiabatic exponent, $\Gamma_1$, is a combination of the quantities

$$\Gamma_1 = \left(\frac{d \ln P}{d \ln \rho}\right)_{ad} = \frac{c_p}{c_v} \chi_\rho,$$

(2.3)

where $c_v$ and $c_p$ are the specific heats at constant volume and pressure, respectively, and are related by

$$c_v = c_p - \frac{P}{\rho T} \chi_\rho,$$

(2.4)

where the derivatives $\chi_T$ and $\chi_\rho$ are defined as

$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rho},$$

(2.5)

$$\chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{T}. $$

(2.6)

In the case of our stellar models, the values of $\Gamma_1$ come either from tabulated equations of state, or from a combination of $c_p$ and the derivatives $\chi_T$ and $\chi_\rho$, which can be calculated within the evolution code itself.

2.2.2. Standard Input

We define a standard set of input physics from which we create our fiducial stellar models. Models include both helium and heavy element diffusion, since heavy elements sink with respect to lighter elements in a gravitational potential. We use the procedure of Thoul et al. (1994), which numerically solves the full set of the Burgers (1969) equations for a multicomponent fluid with no restriction on the
number of species considered. For the purposes of computing diffusion coefficients, we treat all heavy elements in the same manner as fully ionized iron (see Bahcall et al. 1995). In reality, the effects of diffusion and settling are modified by at least two other physical processes: radiative levitation and mixing. The first is a small effect for the mass ranges we consider, and our models therefore include no prescription for levitation (see Section 2.4 for discussion). We do, however, account for mixing. Mild envelope mixing is needed to explain the Li and Be abundances of low mass stars (Pinsonneault 1997). Richard et al. (1996) and Bahcall et al. (2001) found that mixing sufficient to explain the observed Li depletion primarily affects the seismology though the reduction of the efficiency of element segregation. We therefore set the diffusion coefficients to 0.8 as in Delahaye & Pinsonneault (2006) to account for the effects of mixing.

The atmosphere and surface boundary conditions are given by the Kurucz (1997) model atmosphere tables at solar composition \(^2\). The convection zone depth is only weakly sensitive to the choice of boundary conditions (Bahcall & Pinsonneault 1992; Delahaye & Pinsonneault 2006). We utilize the recently updated nuclear reaction rates of Adelberger et al. (2011) with weak screening (Salpeter 1954), and employ the mixing length theory of convection (Cox 1968; Vitense 1953). Opacities are from the Opacity Project (OP) (Mendoza et al. 2007) for a Grevesse & Sauval (1998, hereafter GS98) solar mixture, and are interpolated for each composition as needed. These are supplemented with the low temperature opacities of Ferguson et al. (2005), also for the GS98 mixture. The GS98 solar abundances are in good agreement with asteroseismology (see Bahcall et al. 2005; Basu & Antia 2008) in comparison to the more recent solar mixture of Asplund et al. (2009), and are thus our default choice. We discuss the effects of adopting the Asplund et al. (2009)

\(^2\)Models are available at:http://kurucz.harvard.edu/
mixture in Section 2.2.3. The structural effects of rotation and convective overshoot are neglected in the standard models, although we address such processes further in Sections 2.2.3 and 2.4. Semiconvection (see Kippenhahn & Weigert 1990) is nominally included, although is of little importance over the stellar mass range we consider.

We utilize the updated 2006 OPAL equation of state (EOS)\(^3\) (Rogers et al. 1996; Rogers & Nayfonov 2002) and the Saumon et al. (1995) EOS for temperature and density combinations outside of the OPAL tables. To calculate the sound speed throughout each stellar model in a thermodynamically consistent fashion, we use the published values of \(\Gamma_1\) in the OPAL 2006 EOS. \(\Gamma_1\) is determined using the values for \(P\) and \(T\) from the converged model at the end of each timestep, for each shell of the model. Values of \(\Gamma_1\) for the envelope and atmosphere are likewise acquired directly from the OPAL EOS tables, using the values of \(P\) and \(T\) from the envelope integration. For our purposes, we neglect the acoustic thickness of the atmosphere in the calculation of \(\tau_{cz,n}\), because the modes of interest are generally evanescent in this region (but see Section 2.3.5 for further discussion of the atmosphere). The base of the convection zone, \(R_{cz}\), is defined to be the location where the Schwarzschild criterion for convective instability, \(\nabla_{rad} > \nabla_{ad}\), is fullfilled. With the values of \(\Gamma_1\) obtained from the OPAL 2006 EOS, and the structure from the interior and envelope calculations within the code, we perform the integral \(\tau_{cz,n}\) by interpolating the calculated values of \(c_s(R)\) onto an even grid in \(R\), and the integrating the tabulated values using a five-point Newton-Cotes integration formula.

We use a solar calibration to set the value of the mixing-length parameter, \(\alpha\) (the ratio between the convective mixing length and pressure scale height), and the initial composition, \(X\), \(Y\), and \(Z\) such that a \(1M_{\odot}\) model at 4.57 Gyr (see

\(^3\)updated 2006 tables available at http://opalopacity.llnl.gov/opal.html
Bahcall et al. (1995) recovers the solar radius, luminosity, and surface abundance of $R_\odot = 6.9598 \times 10^{10}$ cm, $L_\odot = 3.8418 \times 10^{33}$ ergs s$^{-1}$, and $Z/X = 0.02289$ from GS98, respectively. A calibration using this standard set of physics yields $\alpha = 1.93271$, $X = 0.710040$, and $Z = 0.018338$.

Composition grid

We created a larger grid of models for masses $0.4 \ M_\odot - 1.6 \ M_\odot$ and initial abundances $-1.2 \leq [Z/X] \leq +0.6$ where $[Z/X] = \log_{10} ([Z/X]_{model}/[Z/X]_{\odot,s})$ and $[Z/X]_{\odot,s} = 0.025828$, as opposed to the GS98 surface abundance of 0.02289 (the difference being due to element diffusion). We normalize to this initial solar Z/X throughout the chapter, which amounts to a zeropoint offset of 0.0524 dex between a metallicity scale normalized to initial versus surface solar abundances. We use models with the standard set of physics to investigate the effect of composition on the location of the convective boundary. The mass range is chosen to roughly coincide with the onset of fully convective models on the low-mass end and vanishingly thin convective envelopes on the high-mass end. The choice of metallicities is motivated by the typical distribution we expect to observe in a sample of field stars. Models are evolved until they leave the main sequence, or until 14 Gyr has elapsed, whichever occurs first. For stars with $M \gtrsim 1.3M_\odot$, the convective envelope becomes less massive than the default fitting point ($1.24 \times 10^{-4}M_\odot$) between the interior and envelope solutions. The fitting point is moved to a minimum mass of $1 \times 10^{-7}M_\odot$ to accommodate these models. The grid is composed of models spaced every $0.02 \ M_\odot$ in mass, 0.2 dex in $[Z/X]$, with initial helium mass fractions of 0.24, 0.26, and 0.28 and $Y_{i,s}$ for each combination of mass and metallicity. The result is a grid of some $\sim 2400$ models.
2.2.3. Theoretical Error Bars on the Acoustic Depth

We examine the theoretical error bars on the acoustic depth through comparisons of pairs of model grids. The first grid contains “standard” models in the sense that they represent the results for the set of input physics described in Section 2.2.2. Comparison grids are identical to the standard grid except for a single alteration to the input physics. Both grids are subject to separate solar calibrations. We divide the parameter variations into several distinct classes, based on the nature of variation. Some parameters, such as the diffusion coefficient or nuclear cross sections have well defined and random errors. Changes to other parameters, such as the EOS and opacities, can also shift the location of the convection zone in either direction, but the uncertainties in $\tau_{cz,n}$ incurred from switching between different EOS or opacity tables are systematic in nature. We treat the changes induced in $\tau_{cz,n}$ due to well-motivated variations of these parameters as effective 2σ error bars on $\tau_{cz,n}$ (as in Bahcall & Pinsonneault 1992). There are also uncertainties that arise simply from our inability to measure the mass, radius, composition, and age of real stars with perfect accuracy, which we will describe as “observational” in nature. A final class, which we will term “zeropoint uncertainties” is related to assumptions such as the heavy element mixture or the presence of rotational mixing, and for these cases, the resultant theoretical uncertainties are asymmetric. For example, one can make $\tau_{cz,n}$ smaller by including mixing, but never larger. In these cases, we incur uncertainties in the zeropoint of our relations between various physical parameters and $\tau_{cz}$. These are considered separately from the systematic, random, and observational error sources throughout the chapter, but are discussed here for completeness. We proceed, then, to address the uncertainties from each of these sources in turn. The values of $X, Y, Z,$ and $\alpha$ for each of the physics variations are listed in Table 2.5.
Random uncertainties

Element diffusion in stars allows heavier atoms to sink relative to lighter ones. The effect of diffusion is to situate metals, which are a significant source of opacity, deeper within the star than they would otherwise be, resulting in a deeper convective boundary than in models with no element diffusion. The presence of element diffusion in the Sun produces a 1.7% effect (Bahcall et al. 2001) on the location of the convection zone in Solar models (see also Basu et al. 2000; Bahcall et al. 2004). We construct a calibrated set of models with the helium and heavy metal diffusion coefficients altered by 15% (Thoul et al. 1994), to mimic uncertainty in the strength of diffusion in the interior. Apart from the differing solar calibrations and adjustment of the diffusion coefficients, these models are identical to those run with standard physics.

We address the effect of the nuclear reaction rates on the stellar structure, and adopt error estimates for nuclear reaction cross-sections from Adelberger et al. (2011). The major reactions considered are the primary pp chain reactions $S_{1,1}$ (pp), $S_{3,3}$ ($\text{He}^3 + \text{He}^3$), $S_{3,4}$ ($\text{He}^3 + \text{He}^4$), and CNO cycle $S_{1,14}$ ($\text{P} + N_{14}$), which were each changed by $\pm 4\sigma$.

Systematic uncertainties

We chose a different prescription for the equation of state in an effort to quantify the change in $\tau_{c_{2,n}}$ due to quantum mechanical uncertainties. We use the Saumon et al. (1995) (SCV) EOS instead of the OPAL 2006 EOS (Rogers & Nayfonov 2002) chosen for our standard set of models. We choose this particular variation since the differences between earlier versions of the OPAL EOS are small (see Bahcall et al. 2004), and the relative simplicity of the Yale EOS (Guenther et al. 1992), which treats the interior as fully ionized and solves the Saha equation for the envelope,
is a poor representation of the state of the art to which EOS calculations have progressed. The OPAL and SCV equations of state are both sufficiently modern, and yet have very different approaches to the problem, and so the SCV EOS serves as a useful comparison. Because we draw the value of $\Gamma_1$ in the standard models directly from the EOS tables, we must alter the manner in which we calculate $\Gamma_1$ for the SCV EOS models. Using the relationships between the derivatives $\chi_p$, $\chi_T$ and the specific heat at constant pressure, $c_p$ (Equations 2.3, 2.4, 2.5, 2.6), calculated numerically within YREC, we combine these values to calculate $\Gamma_1$ and the sound speed throughout the star.

There are two primary sources for high temperature opacities in stellar interiors models, the OPAL group (Rogers et al. 1996) and the Opacity Project (OP) (Badnell et al. 2005), each of which approaches the quantum mechanical calculation of the high-temperature opacities in a fundamentally different way. A thorough comparison and discussion of the differences between the two methods is present in Seaton & Badnell (2004). The differences in the Rosseland mean opacity for the conditions found at the base of the solar convection zone are of order 5% (Seaton & Badnell 2004). The opacity plays a significant role in determining the precise location of the base of the convection zone (Bahcall et al. 2001), and so we test the sensitivity of $\tau_{cz,n}$ to our choice of opacity table by running variant models using the OPAL opacities, instead of our default choice of the OP opacities.

We expect that the choice of atmosphere and boundary conditions will be most important for very cool stars. We test the worst-case dependence of $\tau_{cz,n}$ on the choice of atmosphere boundary condition by creating a calibrated grid of models for a grey atmosphere boundary condition. We note that this exercise only quantifies the dependence of $\tau_{cz,n}$ on the boundary condition, since the portion of $\tau_{cz}$ due to the atmosphere, $\tau_{atm}$, is neglected in the calculation of $\tau_{cz,n}$.  


We also consider the importance of convective core overshoot (see Zahn 1991; Maeder 1975) to the determination of $\tau_{cz,n}$. While in principle overshoot in all convective layers is possible, and affects the local composition, we consider convective core overshoot in particular, because the added fuel supplied to the core through overshoot-related mixing could have broader impacts on the physical structure of the star. The addition of core overshoot should primarily affect the high end of our mass range, where models begin to develop convective cores. We choose a core overshoot parameterized by the pressure scale height, with a value of 0.2 pressure scale heights. We do not enforce an adiabatic gradient in the overshoot zone or add envelope “undershooting”; overshooting is treated purely as “overmixing”, and the thermal structure is left unchanged. Observationally, were significant overshooting to be present, observers would see it as an effective change in the location of the convective boundary. Depending on the nature of the overshooting, this could manifest itself as either a zeropoint or mass-dependent shift in the location of the CZ. Here we consider only the manner in which overshoot induced mixing affects the evolution because of increased fuel supply to the core.

Observational Uncertainties

We can expect uncertainties in stellar parameters such as $T_{\text{eff}}$, $M$, $R$, $\bar{\rho}$, $\tau_{cz,n}$, $Y$, and the age simply from the nature of the observations with which they are determined. We likewise consider our assumption of a particular mass-radius relationship as a form of observational uncertainty. In all cases, these are uncertainties dictated by our ability to measure stellar properties, and are therefore of a fundamentally different nature than the uncertainties described above. We also note that these uncertainties are subject to potential rapid improvement, typically on a time scale shorter than those for improvements in opacity tables or equations of state.
Asteroseismic age diagnostics are sensitive to the helium fraction in the stellar core, and thus provide information about how far along its main sequence lifetime a star has progressed (Ulrich 1986). Creevey (2009) suggests that we can obtain the ages of main sequence stars to within 10% of the MS lifetime, and already Metcalfe et al. (2010) present an asteroseismic age for KIC 11026764 accurate to 15% with currently existing Kepler data. Future missions, such as GAIA, which aim to attain precise parallaxes on a large sample of stars, may eventually allow us to better constrain the age based on an absolute luminosity, but we proceed with the assumption that age can be measured to this 10% accuracy, and propagate these uncertainties through our models.

The abundance of helium in stars is notoriously difficult to measure directly and represents another source of observational uncertainty. Since luminosity is also a function of the helium content, constraints on the mass and the luminosity (at fixed \(X, Z\), and age) lead to constraints on the helium, an idea that goes as far back as Schwarzschild (1946). If we take \(L = 4\pi R^2 \sigma T_{eff}^4\) and assume reasonable measurement uncertainties in \(R\) and \(T_{eff}\), the uncertainty in \(L\) is \(\sigma_{L}^2 = \left(\frac{2\sigma_R}{R}\right)^2 + \left(\frac{4\sigma_{T_{eff}}}{T_{eff}}\right)^2\), and the uncertainty in \(Y\) due to that in \(L\) is \(\sigma_{Y}^2 = \sigma_{L}^2 \left(\frac{\partial Y}{\partial L}\right)^2\). Finally, the uncertainty propagated to \(\tau_{cz,n}\) is then \(\sigma_{\tau}^2 = \sigma_{Y}^2 \left(\frac{\partial \tau}{\partial Y}\right)^2\). We calculate these derivatives numerically from our composition grid. We assume that \(R\) can be measured to a fractional uncertainty of 2% and \(T_{eff}\) to within 100K, and assume independent, uncorrelated measurements of each.

Seismic measurements of the surface helium may also be possible in the near future, as in Basu et al. (2004); Houdek & Gough (2007). However, we rely on the method described above for our analysis. Seismic determinations of the helium depend on carefully measuring the amplitude of the glitch signature from the He
ionization zone, rather than just the period of the glitch, which may prove to be substantially more difficult than extracting the depth to the convection zone alone. The relationship between the initial helium and the seismic inference of the helium is also model dependent, since element diffusion will affect the inferred $Y$. Any star in which the He glitch can be successfully extracted may additionally yield a good convection zone depth, and in these cases this alternative method to determine the He abundance may be appropriate. However, because measurements of effective temperature and luminosity are relatively easy and capable of constraining the helium abundance quite well, they may be more practical for the majority of stars at present.

A portion of the observational uncertainties will come directly from the asteroseismic measurements themselves, namely our ability to precisely constrain the large frequency separation and glitch signatures. We assume that the value of $\bar{\rho}$ is attainable to within a relative uncertainty of 1% from asteroseismic measurements (Verner et al. 2011), and that $\tau_{cz,n}$ can be measured to within 2% (Ballot et al. 2004).

We have assumed for our standard grid that a single mixing-length parameter, calibrated for a solar model, is valid over the entire range of masses and compositions we consider. Because we employ only the mixing length theory of convection in these models, we have no ability to test how $\tau_{cz,n}$ changes as a result of different theoretical assumptions about convection, and rather we choose to view variations in $\alpha$ as an uncertainty in the mass-radius relationship (but see Section 2.4 for a further discussion of convection theory). Because the general approach in stellar modeling is to determine the mass and calibrate the model such that the correct radius is recovered, we treat this as an observational error. To test how severely this may impact the inferred depth of the CZ, we construct a grid of many different values of $\alpha$, and choose $\alpha(M)$ such that for all masses considered, the radius of the star in
the “altered physics” grid is about 1% larger than in the single, solar calibrated $\alpha$ case in the standard grid. Observed discrepancies from the theoretical mass-radius relationship are observed to be as large as 10% for cool, low-mass stars in binary systems where they can be well studied (Kraus et al. 2011; Irwin et al. 2009; Bayless & Orosz 2006). A single radius uncertainty encompasses many values of $\alpha$. On both the high and low mass ends of the mass range, the stellar radii become rather insensitive to changes in the mixing-length parameter. For high mass stars, this is because the pressure scale height is small enough that large changes in $\alpha$ itself are physically of little significance. On the low mass end, because stars are nearly fully convective, changes in $\alpha$ tend to shift stars along the main sequence, rather than changing the relation. In both extremes, no change in $\alpha$ is ever sufficient to produce a 10% difference in the radius. Our chosen $\Delta R/R = 0.01$ is achievable over nearly the entire mass range considered with sufficient changes to the value of $\alpha$. We chose $\Delta R/R = 0.01$ because it was achievable over the entire mass range, and allowed us to probe the sensitivity to $\alpha$ across all masses. Had we chosen a somewhat larger value of the radius uncertainty, the uncertainties in $\tau_{cz}$ of models of moderate masses would scale appropriately, and the mass range over which no change in $\alpha$ was ever sufficient to cause the deviation in radius would be somewhat larger. Although scaling these theoretical error bars to larger uncertainties in the stellar mass-radius relation is not unreasonable, it is important to note that for stars with $M \lesssim 0.6M_\odot$ and $M \gtrsim 1.4M_\odot$, the model radius becomes insensitive to $\alpha$ and simple scalings will fail.

Zeropoint uncertainties

Observational and theoretical evidence suggests that some form of mixing operates in both the Sun and other stars, and that this mixing can have effects on the apparent efficiency of diffusion. We know from modelling of the Sun that diffusion alone
does not adequately reproduce solar light-element depletion relative to meteorites (Richard et al. 1996; Bahcall et al. 2001), and that rotationally induced mixing provides a well-motivated physical process by which the observed depletion could be achieved (Pinsonneault et al. 1989). Balachandran (1995) finds that diffusion alone cannot explain the Li abundances in M67, and it is generally believed to be the signature of some form of deep mixing. Deliyannis et al. (1998) likewise finds correlated Li and Be depletion patterns in Hyades F-stars (the strong Li depletion first recognized by Boesgaard & Tripicco 1986) is best supported by a slow mixing of stellar material. The primary consequence of mixing for our purposes is the accompanying decrease in the efficiency of element diffusion, possibly to the point to which it appears that diffusion does not operate at all. Recent observations of NGC 6397 (Korn et al. 2006) and theoretical modelling efforts (Chaboyer et al. 1992; Dotter et al. 2008) found that diffusion must be partially suppressed in order to explain the observed trends in the light elements of metal-poor stars. To mimic the effects of strong mixing, we construct models with no helium or heavy element diffusion.

The mixture of heavy elements, even from the Sun, is another important systematic error source. We consider here both the case of revised Solar heavy element abundances, and the case of $\alpha$-element enhancement in low-metallicity models. In both cases, the relative abundances of important contributors to the opacity are altered with respect to iron, and we seek to investigate the sensitivity of $\tau_{\text{c},\text{n}}$ to changes in the element mixtures.

In the case of the Sun, there exists a well-known tension between the solar CZ depth and sound speed profile inferred from helioseismology versus that implied by the recent solar abundances of Asplund et al. (2004, 2009). These recent abundances are based on non-LTE, 3D radiative transfer calculations of the solar atmosphere
and represent the state-of-the-art in modern atmosphere modeling. However, solar models constructed with this new, low bulk metallicity are in worse agreement with seismic diagnostics such as the surface helium abundance, CZ depth, and solar sound speed profile than models with the older GS98 mixture (Bahcall et al. 2005; Basu & Antia 2008). Although the new solar mixture has a similar iron abundance as the old GS98 mixture, the CNO elements are significantly adjusted, and the oxygen abundance in particular is quite low. Because these elements tend to be completely ionized in the deep interior of stars, they contribute little in the way of opacity in the core, but have significant opacities near the location of the base of the CZ in solar-like stars (Delahaye & Pinsonneault 2006). It is important to note that similar work on the Solar mixture by Caffau et al. (2011), also using a 3D analysis, arrived at a higher oxygen and bulk metallicity than Asplund et al. (2009). Although future work may alleviate the conflict between helioseismic inversions and the Asplund et al. (2009) mixture, we choose to investigate how the lowest published oxygen abundances affect our conclusions regarding the depth of the convection zone. We construct a calibrated set of models for the Asplund et al. (2009) mixture, using OP (Seaton 2005) and Ferguson et al. (2005) low temperature opacity tables adjusted for the change in mixture. The models are calibrated to a surface abundance of $Z/X = 0.0199$. We note that while the pre-main sequence stellar models and opacity tables are adjusted to reflect the difference in abundance pattern, the EOS and atmosphere tables are not. In both cases the correct mass fraction in metals is used, and the errors incurred from the difference in mixture in the atmosphere and EOS should be negligible, since the change of mixture primarily affects the CNO elements and therefore nuclear burning and the highly metal-sensitive opacities.

In the case of metal-poor halo stars, we may also expect that there may be deviations from the Solar mixture due to different chemical enrichment histories.
We consider $\alpha$-enhanced models, with $[\alpha/Fe] = +0.2$ (following Dotter et al. 2008). We compare models with the standard GS98 mixture at $[Z/X], [Fe/H] = -1.0$ to $\alpha$-enhanced models with $[Fe/H] = -1.0$ but $[Z/X] = -0.85$ with $Y_i = 0.247$ in both cases. As with the Solar case, both low and high temperature opacity tables and initial models are adjusted for the change in mixture. We supply EOS and atmospheres with the correct bulk metal abundances, but relative abundances are not adjusted.

Combined Uncertainties

We calculate the magnitude of the uncertainties from each of the error sources above for an assumed reference model at solar composition, with an age of 5 Gyr, with the standard set of physics. In combining the errors from each source, we treat the uncertainties as uncorrelated. We add individual sources of error in quadrature. For example, the random error on $\tau_{cz,n}$, $\sigma_{\tau,rand}$ is composed of $\sigma_{\tau,rand}^2 = \Sigma \sigma_{\text{nuc}}^2 + \sigma_{\text{diff}}^2$, where the individual errors are from the nuclear reaction rates and diffusion coefficients, respectively. Similar combinations of error terms are calculated for each class of uncertainty, random, systematic, and observational. The total uncertainty on $\tau_{cz,n}$ is

$$\sigma_{\tau}^2 = \sigma_{\tau,rand}^2 + \sigma_{\tau,sys}^2 + \sigma_{\tau,obs}^2,$$

where the random, systematic, and observational error bars are added in quadrature. We consider the zeropoint uncertainties separately, since they are of a fundamentally different nature, and have asymmetric effects on the depth of the convection zone.

Although an exhaustive investigation of the crossterms in our theoretical error estimates are beyond the scope of this dissertation, we do comment briefly on a few cases in which we have investigated the role of crossterms with the age uncertainties.
The shapes of the curves in $\tau_{cz,n} - T_{eff}$ and $\tau_{cz,n} - \bar{\rho}$ space depend strongly on age, and so we have singled out this particular error source in which to look for crossterms. We find that for the case in which both the age and helium are simultaneously considered, that the error on $\tau_{cz,n}$ from the combination of age and $Y$ uncertainties is negligible when the interplay between the error sources is considered. Crossterms between age uncertainties and changes in the physics are likewise negligible. The exception is in the case of overshoot models, in which the time dependence of the diffusion and the age uncertainties interact, inflating the error bars by up to 30% for the high mass models when both age and overshoot uncertainties are considered simultaneously. This is not unreasonable, since overshooting is both more important for more massive objects, and affects the amount of fuel in the stellar core. In general, we recommend that one estimate errors, and potential cross-terms, on a source-by-source basis when actual data are available.

2.3. The sensitivity of the convection zone to mass and composition

In this section we present the main predictions of our models. The depth of the convection zone is a strong function of mass and composition, as predicted by interiors theory and recovered in our models. With our theoretical error bars, we find that these trends are a strong, observable effect, even when we account for uncertainties in both the interiors models and observationally derived quantities. The mass and composition dependencies can be viewed in terms of the purely asteroseismic quantities of the mean density ($\bar{\rho}$, here given as $\bar{\rho} = M/R^3$) and normalized acoustic depth $\tau_{cz}/\tau_\ast = \tau_{cz,n}$, and any analysis can in principle rely on solely asteroseismically obtained measurements. Furthermore, the composition dependence of $\tau_{cz,n}$ is more pronounced in $\bar{\rho} - \tau_{cz,n}$ space, as opposed to, for example, $T_{eff} - \tau_{cz,n}$ space: the use of purely asteroseismic variables is not only useful, but
beneficial. Finally, we show that with appropriate observations, various tests of stellar physics, beyond the basic question of the location and presence of a convection zone, are possible.

2.3.1. Mass dependence of the acoustic depth to the CZ

We expect from very simple interiors arguments that the depth of the convection zone must be a strong function of mass. To first order the location of the base of the convection zone is set by the location of the H and He ionization zones, where the adiabatic temperature gradient is suppressed below the radiative temperature gradient and the criterion for convection is satisfied. This is illustrated in Figure 2.1, which shows $R_{cz}/R_*$ versus mass on the left, and $\tau_{cz,n}$ versus $\bar{\rho}$ in the middle and $\tau_{cz,n}$ versus $T_{eff}$ on the right for standard models at solar composition and an age of 1.5, and 10 Gyr. The expected strong mass dependence of the depth of the CZ is clearly present, and the mapping from the $R_{cz}/R_*$ vs. $M$ to $\tau_{cz,n}$ vs. $\bar{\rho}$ planes is a simple one, which preserves the sense of the trends. Models with relatively high masses have vanishingly thin, shallow convection zones, whereas low mass models are nearly fully convective. If our physical understanding of what sets the location of the convection zones is correct, observations of many different stars should display this strong predicted trend (also discussed in Monteiro et al. 2000). In the case in which this strong dependence is not observed, we immediately learn that there is some fundamental physical process that has been neglected or poorly treated in interiors models.

We can also comment here on the basic time dependence of such relationships. At very young ages, the entire mass range we consider (0.4 ≤ $M_\odot$ ≤ 1.4) is on the main sequence. The basic shape of the $\tau_{cz,n}$ vs. $\bar{\rho}$ changes very little between 0.5 and 1.0 Gyr, for example, because the most massive objects we consider have
main sequence lifetimes of at least 1 Gyr. Once we begin to look at later times, however, the low density tail of the curves begins to show significant changes, because the stars that occupy that part of parameter space are progressively less massive (more massive models have evolved off the main sequence and out of our realm of consideration). The mean density of a star decreases over the course of its main sequence lifetime, so it is possible for an older, less massive star to have the same mean density as a young, more massive star. Their convection zones, however, generally not at same relative depth on the main sequence, and so the low density tail of a given $\tau_{cz,n}$ vs. $\bar{\rho}$ curve shifts to deeper convection zones at late times. We incorporate this feature of the relation into our observational error estimates in later sections.

2.3.2. Composition dependence of the acoustic depth to the CZ

While mass should be the primary determinant of the depth of the convection zone, it is clear from similarly simple arguments that the composition should also play some role in the location of the convection zone. The radiative temperature gradient is given by Kippenhahn & Weigert (1990) as $\nabla_{\text{rad}} = \frac{3}{16\pi acG} \frac{P}{mT^4}$ where $a$, $c$, and $G$ are the usual physical constants, $l$ is the luminosity of the shell with mass $m$, $P$ and $T$ the pressure and temperature, and $\kappa$ the opacity. With all else being equal, an increase in the opacity leads to an increase in the radiative temperature gradient, which in turn means that the criterion for convection is satisfied deeper (at higher $T$) within the star. While metals are an almost negligible fraction of the mass, because they contribute significantly to the opacity and are important in determining the precise location of the convection zone. This is indeed what we find, as shown in Figures 2.2, 2.3, 2.4, and 2.5, where $\tau_{cz,n}$ and $\tau_{cz}$ (in seconds) is plotted with respect to $\bar{\rho}$ and $T_{eff}$. We predict that $\tau_{cz,n}$ changes by 0.5-1% per 0.1 dex in $[Z/X]$ over

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most of the mass range we consider. Furthermore, this composition signature is an absolute, rather than relative, measure of stellar abundances. It is most pronounced in the $\tau_{cz} - \bar{\rho}$ plane, rather than in the $\tau_{cz} - T_{eff}$ plane. Therefore, the most preferable space in which to work is also the space in which the composition measurement can be made solely with asteroseismically obtainable variables: spectroscopic and photometric characterization of the stellar parameters is only necessary as an additional constraint on the stellar parameters.

Because our models include gravitational settling, heavy elements tend to sink relative to light ones, and the surface $[Z/X]$ is generally not the same as the initial abundance, and changes as a function of time. Figure 2.6 shows both the difference between the initial and surface abundances. This difference arises because of gravitational settling of heavy elements and would manifest itself as a $T_{eff}$ dependence of the surface $[Z/H]$ in a sample with homogeneous initial composition, such as an open star cluster. If we consider models at fixed $\bar{\rho}/\bar{\rho}_\odot = -0.2$ at 1.0, 5.0, and 10.0 Gyr as in the right panel of Figure 2.6, we find that the difference between the surface and initial abundance is most pronounced for the 10.0 Gyr curve. This is a balance between two competing factors: at earlier times, more massive stars with short settling timescales are still on the main sequence and occupy this density range. At later times, less massive stars occupy this region and have longer settling timescales, but longer MS lifetimes over which settling can occur. This difference between the initial and surface abundances is important for any comparison of asteroseismic and atmospheric abundance measurements: the value of the surface abundance for a given model is a physics and age dependent property.

We also show in Figure 2.7 the fractional difference in $\tau_{cz,n}$ among models of different initial helium abundances at constant, solar $Z/X$ at 1.0 Gyr. We factor the
2.3.3. Uncertainties in the relationship between acoustic depth, density, and effective temperature

The characterization of the uncertainties in the relationships among mass, composition, and \( \tau_{cz,n} \) means that we can not only comment on the existence of an important trend, but also quantify whether it is presently observable. We first discuss the results of the variant models introduced in Section 2.2.3, and then show that these uncertainties are small enough that the depth of the CZ can be used as a precise indicator of composition. We present representative uncertainties in \( \tau_{cz,n} \) due to random, systematic, and observational uncertainties in Figure 2.8 for models with a solar composition and age of 5 Gyr.

The contribution to \( \sigma_\tau \) from sources such as the diffusion coefficients and nuclear reaction rates is below 0.1\%, except on the very low and very high mass ends of the distribution. Contributions from the systematic class of errors are somewhat more significant, with the EOS being the most important source of uncertainty in this particular grouping. As expected, the uncertainties due to overshoot and the choice of boundary condition are small, below 0.05\%.

Uncertainties incurred from errors in our knowledge of the global properties of the star \((M, T_{eff}, R, Y\text{ and age})\) are by far the largest source of error in \( \tau_{cz,n} \). At low mean densities, uncertainty in \( \tau_{cz,n} \) due to uncertainty in the age is the most significant contributor to the observational error, which is unsurprising given the behavior of the curves in Figure 2.1 as a function of age. At low masses the uncertainty in the age itself is large because the objects have very long MS
lifetimes and we can only measure the age asteroseismically to within 10% of that lifetime. However, the change in $\tau_{cz,n}$ with time is comparatively small and so the uncertainties in $\tau_{cz,n}$ induced by age variations are modest for the low-mass models. For higher masses, our ability to measure the age asteroseismically is substantially better, but the shape of the $\tau_{cz,n}$ vs. $\bar{\rho}$ is changing significantly with time because models are evolving off the main sequence and a given density probes very different masses at different times. Therefore, the age-induced uncertainties are largest on the high mass (low density, high $T_{eff}$) parts of the curve.

This does, however, depend on the assumption that the age uncertainties are roughly constant over the mass range we consider, which is unlikely to be the case. For example, phenomena such as so called avoided crossings (Osaki 1975; Aizenman et al. 1977) result in modes with mixed p and g-mode character, which are very sensitive to stellar age, and become accessible as an alternative method of asteroseismic age determination for slightly evolved stars. In reality the means of seismically constraining stellar ages will vary across the HR diagram. However, predicting how these variations will affect our analysis is beyond the scope of this dissertation, and remains an important caveat.

One should note that the uncertainties due to the mass-radius relationship are relatively small, but that we have also chosen a very modest $\Delta R = 1\%$. In principle, these errors can be scaled for larger radius uncertainties in the mid-mass range. On both the large and small mass extremes, however, simple scalings of radius errors using the mixing length $\alpha$ will fail (as mentioned in 2.2.3). On the low mass end, even a very large change in $\alpha$ has only a small effect on $\tau_{cz,n}$. For high masses, however, large changes in $\alpha$ also lead to substantial changes in $\tau_{cz,n}$. Therefore, one must be wary if attempting to scale these particular error bars for larger radius discrepancies for the higher mass stars.
The uncertainties due to unknown helium abundances are also non-negligible contributors to the observational error. On average, given the assumed uncertainties in our ability to measure luminosity, we can hope to constrain the helium mass fraction to within 0.01-0.02, using the technique described in Section 2.2.3. Additional information from parallaxes or asteroseismic determinations of the helium could better constrain this number. It is however, encouraging that we will be able to assign realistic error bars to the helium, as opposed to *ad hoc* estimates.

We combine this suite of systematic, random and observational uncertainties on $\tau_{cz, n}$ and translate this uncertainty into a measure of our ability to measure $[Z/X]$ through $\sigma^2_{[Z/X]} = \sigma^2_{\tau_{cz, n}} \left( \frac{\partial [Z/X]}{\partial \tau_{cz, n}} \right)^2$, where the components of $\sigma_{\tau_{cz, n}}$ are shown in Equation 2.7. Figure 2.9 shows $\sigma_{[Z/X]}$ for each error source, systematic, random, and observational, as well as the combined total error. The result is that $\tau_{cz, n}$ is actually strikingly sensitive to composition, even when we include reasonable theoretical and observational errors. The uncertainty in $[Z/X]$ is in the range of 0.15-0.3 dex over the mass range we consider.

In addition, we alter the compositions and perform the same comparisons for sets of metal-poor and metal-rich models (rather than errors derived for the solar case presented in Figure 2.9). We assume a simple chemical evolution scheme of the form

$$Y = Y_p + \frac{dY}{dZ} Z$$

with $Y_p = 0.246$ and $dY/dZ = 1.0$. We take models with *initial* $Z/X$ ratios at a tenth solar and 2.5 times solar, with the change in $Y$ determined by our chemical evolution assumptions. Examining the fractional differences in $\tau_{cz, n}$ for sets of models at different initial abundances allow us to check whether our ability to
determine \([Z/X]\) depends strongly on the composition. We find that the situation is quite favorable, with \(\sigma_{[Z/X]} \approx 0.2 - 0.3\) for the brightest \([Z/X]_i = -1.0\) models at 10 Gyr (representative of a halo star population) when all assumptions about the observational errors are identical to those in the solar example. In this case of a metal rich object with \([Z/X]_i = +0.4\), \(\sigma_{[Z/X]}\) is similar to that in the solar case. This suggests that \(\tau_{cz,n}\) remains a good indicator of composition across the entire regime of compositions we have considered, provided our assumed observational errors remain representative.

2.3.4. Probing the physics of stellar interiors with acoustic depth diagnostics

Our analysis suggests several interesting tests of the conditions that prevail in stellar interiors using measurements of \(\tau_{cz,n}\), beyond the potential to constrain composition and confirm basic theoretical predictions of interiors models. Because the depth of the convection zone has some sensitivity to the particular physical assumptions of interiors models, we can invert the question confronted above and ask: if we can measure \(\tau_{cz,n}\), and if we can trust our stellar parameters derived by means other than asteroseismology (photospheric metallicities in particular), can we use the patterns we observe in \(\tau_{cz,n}\) to infer something about the physics of the interior? In this section we first describe how one could use measurements of \(\tau_{cz,n}\) as a test of the physics responsible for the observed mid-F star Li depletion. Secondly, we outline the manner in which one could use a large sample of \(\tau_{cz,n}\) measurements to constrain the stellar abundance pattern.
The Li dip

We observe that Li undergoes a severe depletion event in stars of roughly $6200 - 6350K$ (Boesgaard & Tripicco 1986; Balachandran 1995). While the existence of the Li dip is well established observationally, theory has yet to converge upon a mechanism responsible for the effect. Many different mechanisms have been proposed to explain Li depletion in both the Sun and other stars: through mixing by waves, mass loss, diffusion, and rotationally induced mixing (see Pinsonneault 1997, and references therein, for a thorough discussion). We focus here on rotational mixing and imagine the following scenario: if stars undergo an episode of strong rotational mixing at about $\sim 6350K$, and that mixing has the effect of completely erasing the element segregation induced by gravitational settling and diffusion, then we expect a jump in $\tau_{cz,n}$ as the model crosses the Li dip boundary and the underlying physical assumptions change. We find that the no-diffusion models have values of $\tau_{cz,n}$ that are $\sim 6\%$ lower than those in the standard model over the temperature range of 6200-6350 K for models at 1 Gyr (at later times, few models populate this temperature range). We imagine a scenario in which stars are well represented by the standard model curve up until the edge of the Li dip, at which point they undergo a strong mixing event which erases the effects of diffusion, and the star abruptly jumps onto a no-diffusion model curve. If we consider the case in which we observe pairs of stars, one of the Li “peak” at 6200 K, the other in the Li “dip” at 6350 K, the important quantity to consider is the slope, $\Delta \tau_{cz,n}/\Delta T_{eff}$ of the standard models over this temperature range, compared to that between a standard model on the low temperature side of the Li dip, and a no-diffusion model at the high temperature side. We consider the slope $\Delta \tau_{cz,n}/\Delta T_{eff}$ of the standard model and the scatter in that slope present when we introduce our aforementioned changes to the physics, compared to the slope between the standard and no-diffusion models.
over the dip. In the ideal case in which we have perfect measurements of $T_{\text{eff}}$ and $\tau_{cz,n}$, and the only uncertainties are theoretical (not observational), then the jump in the value of $\tau_{cz,n}$ across the Li dip is an $13\sigma$ event. However, when observational error bars due to age, $Y$, and asteroseismic measurement uncertainties are included on the standard-to-no-diffusion model slope, the jump in $\tau_{cz,n}$ is significant at the $1.0\sigma$ level per pair of stars. In $\tau_{cz,n} - \bar{\rho}$ space the significance is slightly decreased, due to the fact that the mapping between $T_{\text{eff}}, \bar{\rho}$ and mass changes slightly between the standard and no diffusion cases, and conspires in this plane to make the jump less visible. We conclude then, that with a sample of $\sim 15$ pairs of stars, if a mixing event is responsible both for removing the signatures of diffusion and providing the means to deplete Li, then a trend in the observed values of $\tau_{cz,n}$ should be visible at the $3\sigma$ level.

There clearly exist some caveats to this prediction, the most important of which is that it is unclear whether rapidly rotating stars of the sort on the hot side of the Li dip will actually display solar-like oscillations and produce reliable measurements of $\tau_{cz,n}$. The detection and interpretation of solar oscillations in rapidly rotating stars is still among the principal challenges in asteroseismology (Reese 2010). Furthermore, our models have neglected the structural effects of rotation, and we have chosen to use no-diffusion models as a simple representation of rotationally mixed stars. Rotationally induced changes to the structure, especially in hot, rapidly rotating Li dip stars, may be important. Nevertheless, we provide a useful test of the mixing in stellar interiors in the temperature range of the Li dip.

**Variant Elemental Mixtures**

We briefly also consider $\alpha$-enhanced models, with $[\alpha/\text{Fe}] = +0.2$ (following Dotter et al. 2008) at fixed $[\text{Fe/H}]$ at 10 Gyr, to mimic an old halo star population. If
one were to consider a pair of stars at the same $\bar{\rho}$, and assume that the theoretical uncertainties on $\alpha$-enhanced stellar models are the same as in the case of a solar mixture, then their measured values of $\tau_{cz,n}$ should be different by of order 1.0-1.3\sigma for $\bar{\rho} < \bar{\rho}_\odot$ with the inclusion of observational uncertainties. In general, the fractional difference between models with solar and $\alpha$-enhanced mixtures is $\Delta \tau_{cz,n} \sim 0.03 - 0.06$ for the stars with $\bar{\rho} < \bar{\rho}_\odot$ at 10 Gyr, which will also be the most likely objects to be detected in missions such as Kepler. Therefore, if one can measure the $[\text{Fe/H}]$ values for several pair of stars at the same age and mean densities, then it may be possible to distinguish between solar and $\alpha$-enhanced mixtures on the basis of the normalized acoustic depth to the CZ. This is another example of the power of pairwise comparison, since the theoretical errors effectively cancel for two stars of the same mean density and age, and it is only observational errors that effect the significance of the difference in $\tau_{cz,n}$.

As we discussed in Section 2.2.3, the recent Asplund et al. (2009) oxygen abundances are contentious in part because the revision implies a solar CZ depth that does not agree with asteroseismic measurements. We investigate here whether we can utilize ensemble measurements of $\tau_{cz,n}$ to learn about the oxygen abundance relative to the total metal abundance of other stars. This question is well-posed in an open cluster situation, in which the stars are of uniform age and composition, and the stellar parameters are somewhat better constrained than in the case of a random field star. The typical difference between standard solar models and Asplund mixture models is $\Delta \tau_{cz,n} \sim 0.005 - 0.01$ for models at solar composition at 1 Gyr. We will focus on a sample of stars, randomly drawn from a uniform distribution in $5500 \leq T_{\text{eff}} \leq 6500$, each with an average combined observational and theoretical uncertainty of $\sigma_{\tau} \sim 0.03$ (including a 0.3 dex $[Z/X]$ uncertainty). The quantity $\tau_{\text{standard}} - \tau_{\text{observed}}$, where $\tau_{\text{standard}}$ is the acoustic depth that would
be measured for standard physics, and $\tau_{\text{observed}}$ is the value measured for stars with an Asplund mixture, quantifies the zeropoint offset induced by the difference in mixture. Given our standard assumptions about the theoretical and observational uncertainties, careful measurements of $\tau_{cz,n}$ for a $\sim 10$ star sample could detect a mixture difference at $3\sigma$. One should note that this is an idealized example: we’ve assumed that all stars are exactly the same age and composition, with exactly the same oxygen abundance, and exactly the same physical processes operating within them. In reality, we could imagine that some effect, such as mixing, might operate differently in stars of different masses, which would dilute the abundance pattern signal, or make it appear anomalously strong, depending on the sense of the mass dependence. Both mixing and low relative oxygen abundances tend to make the CZ more shallow, and so disentangling the zeropoint offset due to a different relative oxygen abundance may in practice be quite challenging. This analysis also relies on a correct theoretical zeropoint calibration: our standard physics models must be anchored correctly, because any theoretical zeropoint offset could be mistaken as a real signal. It is currently unclear how well we can achieve this, as even numerical sources of error become important at the $\Delta \tau_{cz,n} = 0.005$ level. Nevertheless, this is another potentially useful application of careful measurements of $\tau_{cz,n}$. In the solar case, detailed information about the mixture could be obtained through a simultaneous measurement of the surface helium and $R_{cz}$. We anticipate that a similar approach, if practical, will be required in the stellar context.

2.3.5. Caveats

We discuss the caveats to our findings in regards to our treatment of the atmosphere and convection theory, and other physically important elements of stellar interiors, such as rotation, magnetic fields, and radiative levitation.
We have neglected the contribution to the acoustic depth from the atmosphere throughout our discussion: we have only considered the acoustic depth due to the interior and envelope portions of our models. Although all models are run with a Kurucz atmosphere boundary condition ($P$ at $T = T_{\text{eff}}$), we perform the calculation of the “acoustic thickness” of the atmosphere using a grey atmosphere, to allow us to calculate the necessary integrals as a function of radius. While we have already demonstrated that the choice of boundary condition produces small ($\sim 0.5\%$) changes in the normalized acoustic depth, we expect that there are somewhat larger uncertainties associated with the atmosphere itself. We integrate the sound speed in the atmosphere using the assumption that the change in radius is given by $dr = \frac{d\tau}{-\kappa \rho}$ (Cox 1968), where $\tau$ is the optical depth, and $\kappa$ the opacity. For models with a grey atmosphere boundary condition, we find that the acoustic thickness of the atmosphere, $\tau_{\text{atm}}$ is $50 \gtrsim \tau_{\text{atm}} \gtrsim 250$ seconds, with $\tau_{\text{atm}}$ increasing with increasing stellar mass. $\tau_{\text{atm}}$ is typically $5\text{-}7\%$ of $\tau_{\text{cz}}$ for all but the most massive stars with the thinnest convective envelopes, where it is a more significant fraction of $\tau_{\text{cz}}$. $\tau_{\text{atm}}$ is typically $4\%$ of the total acoustic travel time in the interior + envelope regions for all masses. The inclusion of $\tau_{\text{atm}}$ in the normalized acoustic depth can change $\tau_{\text{cz, n}}$ by up to $20\%$ for massive objects with thin convective envelopes, but is typically $5\%$ for stars with $M_* \lesssim 1.0 M_\odot$. A calibrated, standard physics, solar model at $4.57$ Gyr produces a $\tau_{\text{cz}} = 2100$ s, whereas the solar value for $\tau_{\text{cz}}$, which includes surface and atmospheric contributions is $\sim 2200 - 2300$ s (Verner et al. 2004), which suggests our model results are in good agreement with reality. From these arguments, we can reasonably expect that the neglect of the atmosphere may result in a few hundred second offset. One can avoid potential complications in the interpretation of $\tau_{\text{cz, n}}$ due to the neglect of the atmosphere by using differential comparisons of objects in similar locations on the HR diagram.
One can also expect that given the nature of our asteroseismic observables, the derived values should not suffer substantially from uncertain surface term corrections. In both the case of the mean density (derived from the large frequency separation) and the acoustic depth to the CZ (derived from an oscillatory signal in what would otherwise be uniformly spaced frequencies) it is only the relative difference in the surface term among the modes that will bias the measurements. Furthermore, in models where surface convection responsible for the asteroseismic surface terms is treated more carefully than in our MLT approach (Stein & Nordlund 2000), the inclusion of the additional physics significantly improves agreement with high frequency solar modes, but the solar base of the CZ remained essentially unchanged.

We note, however, that we have assumed an unquantified systematic uncertainty in our choice of a single prescription for solar convection. While the near surface convection appears not to be of great importance to our analysis, we have included no test here of the importance of variant convective theories to our results.

As is the case with stellar modeling in general, the relative importance of element separation and mixing is one of our primary uncertainties. Throughout our analysis, we have ignored the effects of rotation, except in the decreased efficiency of diffusion encapsulated in the diffusion coefficients. In terms of determinations of $\tau_{cz}$, we expect that the most important contributions to be in the form of adjustments to the relative efficiencies of mixing and diffusion, which we have shown have an impact of $\tau_{cz}$. For rapidly rotating objects, the rotational splitting of the modes and introduction of new modes of oscillation (Reese 2010) may make mode identification and interpretation challenging.
Magnetic fields may produce changes in the sound speed near the surface of the star, but we expect only a small correction to the sound speed in the deep interior for all but extremely strong internal magnetic fields. Again, since the modes of importance have turning points well below the photosphere, we expect corrections from magnetic fields to be small. Furthermore, Chaplin et al. (2011b) finds that highly active stars are less likely to display detectable solar-like oscillations, which suggests that the primary role of magnetic fields may be in dictating whether we can detect p-modes at all, rather than affecting the acoustic glitch signature itself.

We have also neglected radiative levitation (see Pinsonneault 1997, for discussion), which can selectively levitate some elements relative to others. While our analysis captures the impact of global metal diffusion, it does not account for selective levitation of individual elements. In particular, this can affect elements such as iron, which contributes substantially to the opacity. The effects of radiative levitation are most pronounced in hotter stars with thin surface convection zones. The accuracy of the most massive of the models we consider may therefore suffer from our neglect of radiative levitation.

In general, we advocate pairwise comparisons of measured values of $\tau_{cz,n}$ for stars which one suspects differ significantly in only one way, i.e., testing the mass-$\tau_{cz}$ relation using two stars of very different mass but similar composition and age, or two stars with similar ages and masses but different compositions. Obtaining an accurate zeropoint calibration of this relation is currently challenging. For example, variations in $\tau_{cz,n}$ on the order of 0.005 can be induced due to numerical differences between models with different envelope fitting points. Even in solar models, similar numerical uncertainties due to interpolation can affect the inferred base of the convection zone (Bahcall et al. 2004). Physical effects, such as the presence of envelope undershooting, could also appear as a zeropoint offset in the relation.
Furthermore, as shown in (Bahcall et al. 2004), the interpolation of quantities such as radiative opacity tables is uncertain on the 1-3% level near the base of the CZ in the Sun. The best approach is therefore to compare pairs of interesting stars, in which case zeropoint calibrations will be of less importance.

2.4. Discussion and Summary

We have discussed the factors that affect the location of the base of the convection zone thoroughly, but have neglected the second (and probably more commonly discussed) source of acoustic glitches in asteroseismic spectra: the helium ionization zone. In the Sun, measurements of the helium ionization zone glitch have constrained the surface helium abundance (Basu & Antia 2004), and hopes are high that this will also prove possible in the stellar case. We expect that even if the He ionization zone is sensitive to metallicity, the dynamic range of the effect will be much too small to make precise metal abundance measurements. Furthermore, the ionization zone lies in the outermost layers of the star, and is subject to uncertain surface term corrections, much more so than the deeper base of the convection zone. For these reasons a similar analysis on the effects of mass and metal content on the helium ionization zone is beyond the scope of this dissertation, but we emphasize that the sensitivity is unknown, and it could yet prove to be an interesting diagnostic.

Our entire analysis has focused only on main sequence stars, which are inherently fainter, and their mode amplitudes are smaller than the subgiants, which have recently proven to be a rich source of asteroseismic information (Brandão et al. 2011; Metcalfe et al. 2010; Bedding et al. 2006), and are among the most common stars with detectable solar-like oscillations in the current Kepler ensemble (Chaplin et al. 2011a). A preliminary analysis of models evolved onto the subgiant branch shows similar trends with composition, with composition effects of similar
magnitude. At fixed age, $\tau_{cz,n}$ now increases with decreasing $\bar{\rho}$ and probes models increasingly closer to beginning the ascent up the giant branch with deepening convective envelopes. A full analysis of the sensitivity of $\tau_{cz,n}$ to composition in subgiants is underway. If the sensitivity and theoretical error bars are similar to those on the main sequence, we stand to benefit substantially from extending the analysis to subgiants, which have larger mode amplitudes and higher luminosities, which can help to reduce observational errors.

This unique means of measuring the composition promises a host of interesting applications. We could, for example, test the tendency of planets to be found around hosts of spectroscopically high metallicity (Fischer & Valenti 2005). Differences between the interiors and atmospheres based compositions could help to constrain whether planets are more likely to be found around intrinsically metal rich stars, or whether planets themselves tend to enrich the outer layers of their hosts with heavy elements. A simple comparison of spectroscopically and asteroseismically determined compositions would in itself be an interesting consistency check, and potentially offer insights into the reliability of both methods and physical processes such as element diffusion. The striking sensitivity of the location of the convection zone to composition even at very low metallicities also provides an interesting and relatively rare insight into the interiors of metal-poor stars. We also note that with good constraints on both $Y$ and the absolute metal content of the star, we may be able to constrain the slope of the chemical evolution curve, $dY/dZ$. These are only a handful of the numerous ways in which we can begin to use asteroseismic measurements such as this as novel diagnostics of stellar interiors and stellar populations.

To conclude, we have created a grid of stellar models of different compositions and examined the sensitivity of the acoustic depth to the convection zone as a
function of composition, mass, and our assumptions about the input stellar physics. We make three primary predictions based on the analysis of our models:

1. We predict strong trends in the depth of the convection zone as a function of mass and composition. The asteroseismic CZ depth indicator $\tau_{cz,n}$ can be different by as much as factor of $\sim 2$ between stars of masses 0.4 and 1.4 $M_\odot$. Composition produces changes in $\tau_{cz,n}$ of order 1% per 0.1 dex in $[Z/X]$. $\tau_{cz,n}$ remains sensitive to the composition even at low ($\sim -1.0$) values of $[Z/X]$. These strong scalings provide a simple test of interiors theory, and an absolute abundance measure independent of atmospheric modelling. Furthermore, the problem is well posed in $\tau_{cz,n} - \bar{\rho}$ space, both of which are purely asteroseismic observables.

2. Reasonable estimates of theoretical and observational uncertainties suggest that not only is $\tau_{cz,n}$ sensitive to the composition, but that the uncertainties in the relationship are small. On average, we expect to be able to measure absolute abundances to 0.15-0.3 dex for solar-like stars at 5 Gyr given the assumed observational and theoretical uncertainties.

3. Finally, the measurement of the depth to the convection zone has potential diagnostic power as a means of probing theoretical uncertainties. In particular, we have addressed the manner in which one would use $\tau_{cz,n}$ to test for rotational mixing in Li dip stars, and to test for differences in the relative element abundances in an ensemble of targets.

Measurements of $\tau_{cz,n}$ have the potential to both constrain interiors theory in terms of the balance between diffusion and mixing, element abundance patterns, and the basic prediction of a strongly mass dependent CZ depth. The technique also
and offers a unique, absolute abundance measure, which is inherently useful in the study of the chemical enrichment of the galaxy, and benchmark for comparison to stellar atmosphere derived abundances. This is a powerful tool that can help us to precisely measure stellar parameters and test the physics of stellar interiors.

2.5. Figures and Tables

Fig. 2.1.— Left panel: The physical depth of the convection zone, normalized by the radius of the star, as a function of mass. Middle panel: The acoustic depth to the convection zone, normalized by the acoustic depth from surface to center of the star. Right panel: The normalized acoustic depth as a function of effective temperature. The dotted curve is for solar composition models at 1.0 Gyr, the solid for 5.0 Gyr, and dashed for 10.0 Gyr. All models are on the main sequence, with the central hydrogen fraction $X_c \geq 0.0002$. The top axis in the center and right panels gives the mass for objects at 5.0 Gyr (solid curve). The corresponding discussion can be found in Section 2.3.1.
Fig. 2.2.— The variation of the normalized acoustic depth of the base of the convection zone as a function of composition and $T_{\text{eff}}$. Solid lines represent models of fixed initial $[Z/X]$ (reference to the initial solar abundance) from red/top most (+0.6) to purple/bottom most (−1.2) spaced every 0.2 dex. The composition dependence of the acoustic depth is plotted for three representative ages, and dotted lines are plotted in gray for constant mass, in solar units. All models are stars on the main sequence, with $X_{\text{core}} \geq 0.0002$ and have an initial, solar-calibrated helium of $Y_i = 0.271$. Representative observational error bars on the quantities $\tau_{cz,n}$ and $T_{\text{eff}}$ are shown in the left most panel. See Section 2.3.2 for discussion.
Fig. 2.3.— The variation of the absolute acoustic depth (in seconds) of base of the convection zone as a function of composition and $T_{\text{eff}}$. Solid lines represent models of fixed initial $[Z/X]$ from red/top most (+0.6) to purple/bottom most (−1.2) spaced every 0.2 dex. The composition dependence of the acoustic depth is plotted for three representative ages, and dotted lines are plotted in gray for constant mass, in solar units. All models are stars on the main sequence, with $X_{\text{core}} \geq 0.0002$ and have an initial, solar-calibrated helium of $Y_i = 0.271$. Representative observational error bars on the quantities $\tau_{cz}$ and $T_{\text{eff}}$ are shown in the left most panel. See Section 2.3.2 for discussion.
Fig. 2.4.— The variation of the normalized acoustic depth of the base of the convection zone as a function of composition and $\bar{\rho}$. Solid lines represent models of fixed initial $[Z/X]$ from red/top most (+0.6) to purple/bottom most (−1.2) spaced every 0.2 dex. The composition dependence of the acoustic depth is plotted for three representative ages, and dotted lines are plotted in gray for constant mass, in solar units. All models are stars on the main sequence, with $X_{\text{core}} \geq 0.0002$ and have an initial, solar-calibrated helium of $Y_i = 0.271$. Representative observational error bars on the quantities $\tau_{cz,n}$ and $\bar{\rho}$ are shown in the left most panel. See Section 2.3.2 for discussion.
Fig. 2.5.— The variation of the absolute acoustic depth (in seconds) of base of the convection zone as a function of composition and $\bar{\rho}$. Solid lines represent models of fixed initial $[Z/X]$ from red/top most (+0.6) to purple/bottom most (−1.2) spaced every 0.2 dex. The composition dependence of the acoustic depth is plotted for three representative ages, and dotted lines are plotted in gray for constant mass, in solar units. All models are stars on the main sequence, with $X_{\text{core}} \geq 0.0002$ and have an initial, solar-calibrated helium of $Y_i = 0.271$. Representative observational error bars on the quantities $\tau_{cz}$ and $\bar{\rho}$ are shown in the left most panel. See Section 2.3.2 for discussion.
Fig. 2.6.— Left panel: The fractional difference in $\tau_{cz,n}$ between the surface abundance and initial composition at constant $[Z/X]$. This difference arises because of gravitational settling of heavy elements, and would manifest itself as a $T_{eff}$ dependence of the surface $[Z/H]$ in a mono-composition sample. Center panel: Lines of iso-composition in surface abundance, compared to a reference model at solar surface abundance. Right panel: surface abundance as a function of $\bar{\rho}$ for models of solar composition. The solid line denotes models at 1 Gyr, the dotted at 5 Gyr, and the dashed at 10 Gyr. See Section 2.3.2 for discussion.

Fig. 2.7.— The fractional difference in $\tau_{cz,n}$ between models with an initial solar helium abundance ($Y = 0.271$) and $Y = 0.24$ (solid), $Y = 0.26$ (dotted) and $Y = 0.28$ (dashed) at constant initial $Z/X$ (solar) for models of 1.0 Gyr. $\Delta\tau_{cz,n}/\tau_{cz,n} > 0$ represents models in which the convection zone is deeper than in the standard, solar $Y$ case. See Section 2.3.2 for discussion.
Fig. 2.8.— The fractional uncertainty in the measurement of $\tau_{cz,n}$ for a 5 Gyr old, solar composition star from our each class of uncertainties, from left to right: random, systematic, and observational. See Section 2.2.3 for a discussion of the uncertainty calculations. The sources of uncertainty are coded as follows: left panel: solid line: total, dotted: diffusion coefficients, dashed: nuclear reaction rates. Center panel: solid- total, dotted- EOS, dashed-overshoot, dot dashed-opacity, double-dot dashed-boundary conditions. Right panel: solid-total, dotted-Y, short dashed- age, dot dashed-measurement of $\bar{\rho}$, triple-dot dashed- measurement of $\tau_{cz,n}$, long dashed-mass-radius relation, through changes in $\alpha$. 
Fig. 2.9.— The uncertainty in the measurement of the metallicity of a solar composition, 5 Gyr old star using $\tau_{cz,n}$ as a composition indicator. The dashed (blue) line represents the contribution from asteroseismic uncertainties on the observables, the dashed (purple) from observational uncertainties on the mass-radius relation, Y, and age, the triple-dot-dashed (orange) from systematic uncertainties, long-dashed (red) the zeropoint uncertainties (mixture + mixing), dot-dashed (green) the random uncertainties, and solid (black) the combined systematic, random and observational uncertainties. See Section 2.3.3 for discussion.
Fig. 2.10.— The fractional difference in $\tau_{cz,n}$ (the normalized acoustic depth to the base of the convection zone) between models of different compositions as a function of $T_{\text{eff}}$. Each solid line represents the fractional difference in $\tau_{cz,n}$ between a given $[Z/X]$ ($-1.2, -0.8, -0.2, 0.2, 0.6$, marked for reference in the right panel) and $\tau_{cz,n}$ for a solar composition model. The gray shaded region represents observational and theoretical errors on $\tau_{cz,n}$, both described in detail in Sections 2.2.3.

Fig. 2.11.— The fractional difference in $\tau_{cz,n}$ (the normalized acoustic depth to the base of the convection zone) between models of different compositions as a function of $\bar{\rho}$. Each solid line represents the fractional difference in $\tau_{cz,n}$ between a given $[Z/X]$ ($-1.2, -0.8, -0.2, 0.2, 0.6$, marked for reference in the right panel) and $\tau_{cz,n}$ for a solar composition model. The gray shaded region represents observational and theoretical errors on $\tau_{cz,n}$, both described in detail in Sections 2.2.3.
Fig. 2.12.— Fractional difference in $\tau_{cz,n}$ between a set of models with standard physics, and a set where diffusion has been eliminated entirely to mimic efficient mixing. The solid (blue) line is for models at 1 Gyr, dashed (green) at 5 Gyr, and dot-dashed (red) at 10.0 Gyr. Note that over the temperature range in which the Li dip is observed, the difference is of order 6%, such that over a very narrow temperature range we expect to see an abrupt change in the location of the base of the CZ. For the 5.0 and 10.0 Gyr curves, all stars in the Li dip temperature range have already evolved off of the MS. See Section 2.3.4 for further discussion.
Table 2.1. Theoretical Error Bar Model Grids. A summary of the physics and parameters used to construct each of the grids used to assess the theoretical model uncertainties.
Chapter 3: A $^3$He driven instability near the fully convective boundary

3.1. Introduction

Stars are surprisingly stable for most of their evolution. The exceptions provide windows into important transitions in stellar properties. The helium flash, for example, marks the temporal transition from shell hydrogen burning to core helium burning. Interiors theory also predicts changes in structure as a function of mass. A classic example is the prediction that progressively lower mass models should have progressively deeper surface convection zones, culminating in a transition to a fully convective state at around a third of a solar mass. The global behavior of models near this transition is traditionally thought to be smooth.

We have uncovered a novel instability for models just above the fully convective boundary that has, to the best of our knowledge, never before been noted for these objects. Non-equilibrium $^3$He burning in a marginally stable and small radiative core leads to a steady increase in $^3$He and energy generation, and the development of a small convective core separated from the convective envelope by a radiative buffer zone. Above a critical threshold a transition to a fully convective state occurs. Convective mixing lowers the core $^3$He abundance, quenching the nuclear reactions and causing the core to become radiative again. The sudden drop in energy generation leads to a sudden drop in radius, followed by a gradual recovery as $^3$He production in the core resumes. This phenomenon repeats in a series of pulses, with
an amplitude which is steadily damped as the envelope $^3$He is enriched and the drop in core $^3$He becomes smaller. For non-interacting stars this will apply to a narrow range of masses just above the fully convective boundary. The secondary stars of cataclysmic variables, however, will all pass through this mass range, and the onset of the instability occurs at the upper end of the CV period gap. The astrophysical consequences may therefore be surprisingly broad, and the instability can be triggered for mass-loss rates of interest in the CV context. In this chapter we describe the physics of the instability and present some preliminary assessment of its effects. We describe our stellar models in Section 3.2, discuss the details of the physical processes responsible for the instability in Section 3.3, observational implications of the instability in Section 3.4, and close with a summary in Section 3.5.

### 3.2. Stellar Models

We utilize the Yale Rotating Stellar Evolution Code (YREC) (Pinsonneault et al. 1989; Bahcall & Pinsonneault 1992; Bahcall et al. 1995, 2001) to produce a grid of stellar models with a standard set of input physics. We create a dense grid near the fully convective boundary to investigate the mass range over which this instability operates. Models range in mass from $0.2 - 0.5\,M_\odot$ and are spaced every $0.001\,M_\odot$. Models generally contain $\sim 500$ shells and take $\sim 5000$ timesteps reach to 14 Gyr, at which point the evolution is truncated. All models include helium and heavy element diffusion following the procedure of Thoul et al. (1994), atmosphere and boundary conditions of Allard et al. (2000), nuclear reaction rates of Adelberger et al. (2011) with weak screening (Salpeter 1954), and employ a mixing length theory of convection (Cox 1968; Vitense 1953). Opacities are from the Opacity Project (OP) (Mendoza et al. 2007) for a Grevesse & Sauval (1998) solar mixture, supplemented with the low temperature opacities of Ferguson et al. (2005). We utilize the 2006
OPAL equation of state (Rogers et al. 1996; Rogers & Nayfonov 2002) and the Saumon et al. (1995) EOS for temperature and density combinations outside of the OPAL tables. Convective instability is determined using the Schwarzschild criterion. We impose an initial $^3\text{He}$ mass fraction of $2.95 \times 10^{-5}$, for a Grevesse & Sauval (1998) mixture with the $^3\text{He}/^4\text{He}$ isotopic ratio from Anders & Grevesse (1989) on a series of pre-main sequence starting models. Because $pp$ chain burning is by far the most significant source of $^3\text{He}$, the exact value of this initial abundance is of relatively little importance.

3.3. $^3\text{He}$ Instability

We find that low mass objects at the fully convective boundary undergo non-equilibrium $^3\text{He}$ burning during the early stages of their main-sequence lifetimes which gives rise to periodic fully convective episodes. We discuss the nature of the instability, the conditions under which it operates, and compare our results obtained with YREC to those obtained with the MESA stellar evolution code.

3.3.1. Nature of the Instability

$^3\text{He}$ is produced in low mass stars via proton capture on deuterium; the former can be either primordial or produced from the $pp$ chain. The $pp$ reaction itself has a low cross section but a weak temperature dependence, so $^3\text{He}$ can be generated even in relatively cool environments given sufficient time. $^3\text{He}$ is destroyed by both $^3\text{He} + ^3\text{He}$ ($pp$ I) and $^3\text{He} + ^4\text{He}$ reactions ($pp$ II and $pp$ III); the former is far more important at low temperatures. At high temperatures equilibrium is achieved with a $^3\text{He}$ abundance that decreases as the temperature increases, and net destruction of $^3\text{He}$ can occur. However, at low temperatures the rate of production exceeds the rate of destruction and $^3\text{He}$ is produced (Iben 1967; Rood et al. 1976; see also Boesgaard & Steigman 1985 for a discussion). One therefore expects a $^3\text{He}$ abundance peak.
in the outer layers of higher mass stars (see Figure 1 in Pinsonneault et al. 1989, for an example). For lower mass stars the peak shifts towards the center of the star and the development of a deep convective envelope homogenizes the outer layers. For sufficiently cool central temperatures $^3\text{He}$ is never in equilibrium. Therefore the nuclear energy generation from $^3\text{He} + ^3\text{He}$ burning will depend sensitively on the integrated $^3\text{He}$ production over the lifetime of the star.

If there is a radiative core it is possible to have a local enhancement in the $^3\text{He}$ abundance which is much larger than the average for a fully convective star. This special circumstance can therefore create a small convective core, even in stars far too cool to support the CNO burning responsible for convective cores on the upper main sequence. The growth in central $^3\text{He}$ will cause this core to grow in mass. For a sufficiently small radiative core it is therefore possible to induce a “convective kissing” instability where the central and surface convective regions merge.

To understand the nature of the this instability, we introduce Figure 3.1, which contains plots of the location of the base of the envelope convection zone, core $^3\text{He}$ fraction ($X_{3\text{He}}$), luminosity, and radius of several stellar models of different masses. Fully convective episodes (when $R_{cz} = 0$) are in phase with sudden drops in $X_{3\text{He}}$ in the core. After an episode the fraction of $^3\text{He}$ in the core slowly recovers, along with the luminosity from $^3\text{He}$ burning reactions, whose rates are inflated by the presence of the elevated $^3\text{He}$ abundances. The total luminosity and radius of the star both increase on nuclear timescales in response to the additional burning. The response of the core and envelope convection zones is shown in Figure 3.2, which plots the temperature gradients of a single 0.360 M$_\odot$ model at different stages in the convective cycles. At times early in the cycle ((a), bottom panel of Figure 3.2), the core and envelope convection zones are separated significantly in radius. As time passes and the mean $^3\text{He}$ of the entire star increases with each mixing episode, the
increased \(^3\)He burning inflates the radiative temperature gradient, resulting in a decrease in the separation between the envelope and core convection zones, as seen at times (b) and (c). By time (d) the additional luminosity is sufficient to make the convection zones meet, at which point the star is fully convective, and mixes the \(^3\)He in the core throughout the entire model on a convective overturn timescale. In the case of a 0.36M\(_\odot\) model, the luminosity from the reactions \(^3\)He + \(^3\)He and \(^3\)He + \(^4\)He in the convective core represents between 10 and 35\% of the total model luminosity, depending on the location within the cycle. The dilution of the \(^3\)He in the core results in decreased luminosity and thermal support, and the model radius and luminosity decrease on a thermal timescale. The sudden loss of substantial \(^3\)He burning is somewhat counter-balanced by energy release due to gravitational collapse, which results in modest, few percent changes in the total luminosity over the course of these episodes. The structure returns to a convective core, radiative buffer zone, and convective envelope, while the \(^3\)He again begins to accumulate in the core.

The middle panel of Figure 3.2 shows the temperature gradient at the timestep immediately following the fully convective episodes for several cycles. Each successive cycle begins with higher \(^3\)He abundances throughout the model, since the products of previous cycles are mixed throughout the star. Because the total \(^3\)He abundance is higher at the onset than in previous cycles, the boundaries of the convective core and envelope begin the cycle more closely spaced in radius, and the duration of the cycles decreases. Cycles continue until the \(^3\)He abundance throughout the star is high enough that the model is fully convective for the rest of its MS lifetime.

Figure 3.1 demonstrates the response of the convection zone, central \(^3\)He abundance, radius and luminosity for models of different masses over the course of many convective cycles. More massive models tend to have cycles of longer
duration and greater amplitude, because they are further from the fully convective boundary, and thus require more $^3$He burning to cause them to be convectively unstable. Likewise, lower mass models have shorter cycles with smaller amplitude variations in core $^3$He fraction, luminosity, and radius. In our models (at solar composition $X_i = 0.710342$, $Z_i = 0.018338$, with a solar calibrated mixing length parameter $\alpha = 1.924897$), the instability occurs over the mass range $0.322 \, M_\odot < M < 0.365 \, M_\odot$. For masses $0.266 \, M_\odot < M < 0.322 \, M_\odot$ the star undergoes a brief period in which it has a radiative zone between a convective core and convective envelope without any episodic mixing, but is then fully convective on the remainder of the MS. Models with masses $M \leq 0.266 \, M_\odot$ are fully convective (in that they never have a radiative core) throughout their MS lifetimes. This fully convective boundary is somewhat lower than the typical values quoted in the literature primarily due to our definition: we consider the fully convective boundary to be the division between models that host a radiative core at some point in their lives, but that may be fully convective for the vast majority of their lifetimes, versus models that never have even a transient radiative zone.

### 3.3.2. Checks and Physics

To verify that this instability is visible in other stellar evolution codes apart from YREC, we made use of the publicly available MESA (Paxton et al. 2011, http://mesa.sourceforge.net/) stellar evolution code. We ran solar calibrated models with $Z = 0.016498$, $Y = 0.26275$, mixing length parameter $\alpha = 2.01$ and $Z/X = 0.02289$ (Grevesse & Sauval 1998) at 4.57 Gyr (Bahcall et al. 1995) and the default, out-of-the-box physics and numerical parameters released with MESA version 3372, with the exception of the model timesteps, which we forced to be of order $10^6$ years or smaller. We found the same qualitative behavior with $^3$He at the convective boundary in the MESA models, with some quantitative...
differences. The fully convective boundary in these models occurs for masses $M \leq 0.281 \, M_\odot$, and the mass range over which the convective episodes are present is $0.343 \, M_\odot \leq M \leq 0.375 \, M_\odot$ (versus $0.322 \, M_\odot \leq M \leq 0.365 \, M_\odot$ for the YREC models). This discrepancy may be due to differences in the manner in which the two evolution codes handle the structure changes, burning, and mixing in each timestep (in MESA all three are fully coupled, while in YREC the burning and mixing are performed separately from changes to the structure (Bill Paxton, private communication)). Likewise, small differences in the input physics in the two different codes could also be responsible. Although the timescales differ somewhat, the behavior is qualitatively the same, which is evidence in favor of a physical phenomenon rather than numerical artifact. A MESA model for $M = 0.37 \, M_\odot$ is shown in Figure 3.1 among other YREC models for comparison.

The exact masses at which the star is unstable to these convective episodes is also model dependent within YREC. For example, models run with Kurucz (1997) atmospheres rather than Allard atmospheres become unstable at $M = 0.332 \, M_\odot$ rather than $0.322 \, M_\odot$. However, in both cases the instability occurs over a mass range of $\Delta M = 0.043 \, M_\odot$, consistent with a simple shift in the location of the fully convective boundary. Models run at non-solar metallicities also display unstable behavior over slightly different mass ranges. For example, a $0.34 \, M_\odot$ model with $[Z/X] = -0.4$ (referenced to solar metallicity) displays instabilities with the longer timescales and larger amplitudes of a more massive $\sim 0.36$ solar metallicity object.

Physical processes such as convective overshoot have not been considered here, and will undoubtedly affect the exact location in mass of the instability boundaries. There are two distinct cases in which overshooting may affect the instability; the first is simply by shifting the effective location of the boundaries of the convection zones, resulting in a shift in the masses at which the instability occurs. The second effect is
more subtle, and is related to the interplay and exchange of material between two closely separated convection zones due to overshooting.

The importance of overshooting in a system with closely separated convection zones has been considered in the case of A stars, which are thought to have two thin surface convection zones, one at the hydrogen ionization zone, and another at the deeper second ionization of He. These are formally separated by a radiative buffer, although it is unclear whether they are truly distinct from one another when overshooting is considered, as Freytag et al. (1996); Latour et al. (1981) suggest. In the particular case we consider here, overshooting would serve to connect the core and envelope convection zones either chemically or physically before they formally touch according to the Schwarzschild criterion. YREC treats overshooting as “overmixing” only, and does not adjust the temperature gradients in the overshoot regions, and so is incapable of capturing the full behavior of the system when the CZs are in this close configuration. One could imagine that the dilution of the core $^3$He due to $^3$He poor material mixed in through overshooting would lead to an overall drop in the $^3$He luminosity in the core, a global decrease in the radiative temperature gradient, and therefore a shrinking of the convective core. Likewise, the mixing of $^3$He rich material into the convective envelope might lead to increased burning, and cause the envelope convection zone to deepen slightly. Essentially, if the production timescale for $^3$He in the core is much shorter than the timescale for the mixing of $^3$He poor material into the central CZ, then the effects of overshooting should be minimal. If the mixing timescale is comparable or shorter than the production timescale for $^3$He, then it is possible that the effects of overshooting could prevent these fully convective episodes from occurring at all. However, if the temperature gradients in the overshoot regions are very close to adiabatic, the
boundaries of the two CZs are actually closer in the case of overshooting than they
would otherwise be. Further work on this interaction is therefore of interest.

We offer, however, that the convective velocities in these stars should be
low, and the stellar material strongly stratified. The extent of convective envelope
overshooting in the Sun inferred from helioseismology is less than 0.05 pressure scale
heights (Basu 1997), and both studies of cluster CMDs (Kozhurina-Platais et al.
1997; Sarajedini et al. 1999; VandenBerg & Stetson 2004) and asteroseismology
(Briquet et al. 2011) suggest that the typical extent of core overshooting in
intermediate mass stars is 0.1-0.2 pressure scale heights. There has been very little
observational or theoretical investigation of the amount of overshooting in stars of
a few tenths of a solar mass, but we might argue, given what information we have
about other masses, that the degree of overshooting will be minimal.

Likewise, we have neglected to include the effects of magnetic fields, which
could be significant for very active M-dwarfs, and could potentially stabilize regions
of the model against convection (as in Mullan & MacDonald 2001).

3.4. Observational implications

3.4.1. Single Stars

To investigate whether the luminosity and radius changes associated with the
instability are observable, we focus on open clusters, in which we have a single,
coeval, chemically homogeneous sample of stars. We chose a simple initial mass
function, with $dn/dM = AM^{-\alpha}$, where $A$ is a normalization constant, and $\alpha = 1.3$
(Kroupa 2001). We sample the IMF in 0.001 M$_\odot$ bins, and determine our model
luminosities and effective temperatures at fixed time for each mass. We add noise
to the model HR diagram by assigning luminosities and effective temperatures
randomly sampled from a normal distribution centered on the theoretical values for each stellar mass. We mimic the presence of binaries by randomly selecting half of the stars in the theoretical open cluster, and assigning a companion of mass $bM_{\text{primary}}$ where $b$ is a random number between 0 and 1 drawn from a uniform distribution, motivated by the distribution of binary mass ratios in Raghavan et al. (2010). We assign the binaries the luminosity weighted mean effective temperature of the two components, and add the two luminosities. We do not account for the effects of activity. The presence of the instability produces some weak structure along the single-star MS, which will be challenging to detect in any open cluster survey. However, a few stars lie below the MS, where, with the addition of proper motion or RV membership information, they might be recognized as 'odd' cluster members. Detection of such sources would rely on there being an open cluster with enough members, of an interesting age and metallicity such that this instability is visible.

The presence of this instability could also manifest itself as a scatter in radius as a function of age in M dwarf wide binary systems in which the stellar parameters are well known. Given a sample of stars of the same mass and composition with a variety of ages, one should observe more scatter in the radii of the younger objects in comparison to the older objects. The effect will only be visible if one can disentangle the contributions from stellar activity and observational uncertainty, both of which are current challenges. It is important to note, however, that this is potentially an additional source of scatter in M-dwarf radii which is not related to activity or environment, but rather nuclear burning processes.
3.4.2. Cataclysmic variables

Cataclysmic variables are short period binaries in which a white dwarf accretes mass through Roche lobe overflow from a relatively unevolved donor star. CVs are observed over a wide range of binary periods, ranging from $\sim 8$ hrs to just over 1 hour. The primary means of angular momentum loss are postulated to be magnetic braking associated with the stellar wind of the donor star, and losses through gravitational radiation, which operates most efficiently for the closest orbital separations. One of the most notable features in the CV period distribution is the so-called “period gap” (see Rappaport et al. 1983; Spruit & Ritter 1983) in which we observe a paucity of systems with periods of 2-3 hrs. This gap is usually attributed to an abrupt decrease in the efficiency of magnetic braking at the location of the fully convective boundary, which allows the donor star to relax back into thermal equilibrium and truncate contact. However, observations suggest that the magnetic fields of low-mass, fully-convective stars have simpler geometries, but similar field strengths to their more massive counterparts (Donati & Landstreet 2009). This creates tension in the standard explanation for the period gap.

This instability is interesting in the context of CV systems. Given sufficient time and mass-loss rates, all CV donor stars will pass through the mass range over which these convective episodes occur in single-star models. The response of the donor star radius to mass-loss is a critical element of the CV picture; sudden changes could truncate the mass-loss by bringing the system out of Roche lobe contact, or conversely result in temporarily inflated accretion rates. Although a detailed discussion of the manner in which this instability interacts with the process of regulated mass-loss through Roche-lobe overflow is beyond the scope of this chapter, we do consider the response of single star models to constant mass-loss rates. Under
the right conditions, this instability induces a significant, complicated response in
the stellar radius, at an extremely interesting mass range in CV systems.

We apply constant mass-loss rates between $10^{-8} - 10^{-12}\,M_\odot\,yr^{-1}$ to models of
0.4 $M_\odot$ and a single mass loss rate of $5 \times 10^{-11}\,M_\odot\,yr^{-1}$ to models over a range of
masses (0.4 – 0.7 $M_\odot$), all of which have been evolved to 1 Gyr prior to beginning
mass-loss. This exploration of of mass-loss rates is presented in Figure 3.3. In
general, relatively unevolved models of the lowest masses ($\sim 0.4\,M_\odot$) coupled with
low mass-loss rates ($10^{-10} - 10^{-11}\,M_\odot\,yr^{-1}$) show the largest radius “glitches” due to
convective kissing. The instability that we have identified becomes effective at lower
mass loss rates than those inferred from standard models of CV evolution (see for
example Knigge et al. 2011). This could imply that the physical conditions for the
operation of the mechanism in the CV context do not occur. The standard model
of CV evolution, however, has some significant open problems. Measurements of the
temperatures of white dwarf primaries probe the longer term mass accretion rate
(Townsley & Gansicke 2009), and are not in good agreement with the predictions of
standard models (Knigge et al. 2011). Angular momentum loss prescriptions applied
to CVs differ dramatically in their predicted efficiency. Models developed for the
spin down of single stars predict much lower torques (and implied mass loss rates)
than those traditionally employed in CV studies (Andronov et al. 2003; Ivanova
& Taam 2003). Furthermore, the standard CV theory relies heavily on a strong
decrease in the efficiency of magnetic braking near the fully convective boundary,
generally interpreted as an inefficient magnetic dynamo. Observations of single stars
across the fully convective boundary do not, in fact, show a strong decrease in the
magnetic field strength across this critical mass range, although the field geometry
does appear to change (see Figure 3 of Donati & Landstreet 2009). Disrupted
magnetic braking is also not consistent with the observed spin down of single M
dwarfs (see for example Terndrup et al. 2002). This tension between the necessary inefficiency of magnetic braking for fully convective stars in the standard model and observations of magnetically active fully convective stars with measured spin down rates is significant. In light of these discrepancies, a genuinely different mechanism could provide interesting information and we encourage more quantitative modeling to properly assess its impact.

Whether or not strong radius glitches appear in these models depends sensitively on a combination of the mass-loss rate, initial mass, and evolutionary state of the model. At fixed initial mass, the instability does not manifest itself in models until the mass-loss timescale is greater than the thermal timescale, \( M/\dot{M} \sim M^2/RL \). For high mass loss rates, the mass loss timescale is always smaller than the thermal timescale, the star is out of thermal balance, and no instability is present, even over the mass ranges where we expect it to occur. For lower mass-loss rates, the instability is present and the glitch in radius becomes visible (see Figure 3.3).

The initial mass of the model sets the \(^3\text{He}\) profile and abundance, which in turn determines if the \(^3\text{He}\) abundance in the core is sufficient upon arrival at \( \sim 0.35 \, M_\odot \) to become unstable. Low mass models have \(^3\text{He}\) profiles peaked closer to the central region, and in general the absolute central \(^3\text{He}\) abundance increases with decreasing mass for models of the same initial compositions. Higher mass models, which are more effective at burning \(^3\text{He}\), have \(^3\text{He}\) profiles that peak at larger radii, and generally have more \(^3\text{He}\) depleted cores. Finally, because models of different masses have different nuclear equilibrium timescales for \(^3\text{He}\), in general, the later the mass-loss begins, the less \(^3\text{He}\) is present in the core of the model. These three effects conspire to make the radius glitches most visible in relatively unevolved, low mass models with low mass-loss rates. We note a small radius glitch occurs even in models with low mass-loss rates but insufficient \(^3\text{He}\) to induce the instability,
which occurs as $^3$He is homogenized throughout the star as the model crosses the fully convective boundary; we believe this is the same feature noted in D’Antona & Mazzitelli (1982) and Andronov & Pinsonneault (2004).

Apart from the question of whether this instability affects the degree of mass loss through physically altering the radius of the donor star, one should also note that the composition of any donated material is influenced by the presence of the instability. Both Shen & Bildsten (2009) and Shara (1980) suggest that for $^3$He mass fractions $\gtrsim 10^{-3}$ the ignition of novae may have some sensitivity to the $^3$He abundance, and that the burning of accreted $^3$He on the primary can lead white dwarfs that are more luminous than their average accretion rates suggest. In the case of a 0.4 M$_\odot$ model with a constant mass loss rate of $5 \times 10^{-11}$M$_\odot$yr$^{-1}$, the surface $^3$He abundance increases by a factor of 2 over the course of the first convective episode, from $\sim 5 \times 10^{-4}$ to $10^{-3}$. This increase in the $^3$He abundance of accreted material would appear at masses above the fully convective boundary, where it would otherwise not exist.

We have neglected to include rotational mixing in our models, which could be non-negligible in the case of such close binary systems in which the secondary may be relatively rapidly rotating due to tidal locking, which could help to drive meridional flows (Eddington 1925; Sweet 1950; Zahn 1992) and result in mixing. Furthermore, mixing may also be driven by tidal distortions of the potential due to its close companion (Tassoul & Tassoul 1982; Andronov & Pinsonneault 2004). If the timescale for mixing is shorter than the timescale in between convective kissing episodes, the mixing of material could serve to eliminate episodic mixing events. With the inclusion of tidal deformation, the estimates of the minimum timescale for mixing are of order the timescale between convective episodes for systems at 3 hrs (see Andronov & Pinsonneault 2004, Fig. 9 for timescales). The initial feature in the
curve in Figure 3.3, as the $^3\text{He}$ in the core is first mixed throughout the envelope would remain even in the case of strong mixing, but the subsequent variations in the radius would be damped. Therefore, the precise shape of these curves may be modified in the case in which rotational mixing is important, and our neglect of mixing is an important caveat to this discussion.

3.5. Summary

We have discovered a novel instability in low mass, $M \sim 0.35 \, M_\odot$, stellar models in which a buildup of $^3\text{He}$ in the core results in brief convective episodes within the first few Gyrs on the main sequence. These events are accompanied by changes in the radius and luminosity of the star. We have verified that the same instability appears to occur in the MESA code as well, which suggests that it is not a merely numerical artifact. Stars undergoing these convective episodes may be visible as both single stars or as members of binary systems. We have demonstrated that the radius of the donor star in a CV system could have a sizable response to the onset of the convective kissing instability, given donor stars in which the mass-loss rates, initial stellar mass, and evolutionary state ensures that there is sufficient $^3\text{He}$ available in the core by the time the stellar mass reaches an interesting range. The complicated structure of the glitches, which includes both increases and decreases in radius over the course of the instability, could interact in interesting ways with the process of CV mass-loss, either by severing contact, or by leading to temporarily inflated mass-loss rates, provided they are not suppressed by rotational mixing processes. We suggest that this instability be investigated in the context of full CV evolutionary models, where the interaction between the mass-loss rate and the convective kissing instability could be better quantified.

3.6. Figures
Fig. 3.1.— Behavior of models of several characteristic masses through convective episodes. The panels show radius, luminosity, location of the base of the convective envelope, and core $^3$He fraction from top panel to bottom panel in order, each as a function of time. Curves for YREC models are as follows: solid purple: 0.35 M$_\odot$ model, blue dotted: 0.37 M$_\odot$, green dashed: 0.38 M$_\odot$ (which is too massive to show convective episodes). The red dot-dashed curve shows a MESA 0.37 M$_\odot$ model.
Fig. 3.2.— Temperature gradients and core $^3$He abundance in a $M = 0.360 \, M_\odot$ model. The bottom two panels show the temperature gradients at different times during the evolution of the model, which are labeled on the top panel on a plot of the core $^3$He abundance. In the bottom two panels, the adiabatic temperature gradient is plotted as a thick dashed black line. Any curves that fall below this line represent radiative regions, and any curves that follow it convection zones.
Fig. 3.3.— Zoo of mass vs. radius curves for a sampling of mass-loss rates obtained using a starting model of 0.4 $M_\odot$ evolved to 1 Gyr before mass-loss is initiated. All mass-loss rates are in solar masses per year. The curves correspond to the following mass loss rates: dotted (purple): $5 \times 10^{-10}$ $M_\odot$ yr$^{-1}$, dashed (blue): $1 \times 10^{-10}$, solid (green): $5 \times 10^{-11}$, dot-dashed (orange): $1 \times 10^{-11}$, triple-dot-dashed (red): $5 \times 10^{-12}$. Note that the curve for $5 \times 10^{-12}$ is truncated at 14 Gyr.
Chapter 4: Fast Star, Slow Star; Old Star, Young Star: Subgiant Rotation as a Population and Stellar Physics Diagnostic

4.1. Introduction

Stellar rotation is observed to be both a strong function of mass and evolutionary state. The rotation rates of stars respond to the loss of angular momentum over time and changes to the stellar structure as objects evolve. Furthermore, the fundamental structural differences between hot and cool stars, namely the presence or absence of a convective envelope, produce period distributions with a strong dependence on mass. The fact that rotation is such a sensitive function of stellar mass and evolutionary state means that it can be used as a powerful tool to understand the stars themselves. Rotation is already routinely exploited as a diagnostic of stellar ages in gyrochronology (Barnes 2007; Mamajek & Hillenbrand 2008; Meibom et al. 2009, 2011), in which the strong stellar-spin down due to magnetized winds in cool MS stars is used as a clock. Our focus is different: we aim to show that rotation is not only a useful diagnostic for these cool main sequence stars, but also a powerful tool for studying hot and evolved stars.

On the main sequence there are two rotational regimes, distinguished by effective temperature. Cool stars (< 6200 K) are typically slow rotators, with periods of 10’s of days. These rotation periods are far slower than those one would expect were the star to conserve its angular momentum throughout its pre-main
sequence collapse and contraction, and far slower than observed rotation periods in very young clusters (compare the Mt. Wilson sample of old field stars in Baliunas et al. 1983 to Hartman et al. 2010 for the Pleiades). Furthermore, observations of the rotation period distributions in clusters at a number of different ages have confirmed that these cool stars lose angular momentum and spin down over time (for an extensive compilation, see Irwin & Bouvier 2009). This angular momentum loss proceeds via magnetized stellar winds (Weber & Davis 1967; Schatzman 1962) and is often parameterized using loss laws of the form $dJ/dt \propto \omega^3$ (Kawaler 1988; Krishnamurthi et al. 1997; Sills et al. 2000). The angular momentum loss scales strongly with the rotation velocity, which forces a convergence in the rotation periods for all objects of the same mass at late times: objects born with slow rotation rates lose angular momentum slowly, whereas those born rapidly rotating quickly spin down. Therefore, even a population of cool stars born with a wide range of rotation periods will be slowly rotating with a narrow range of allowed periods at late times (Epstein & Pinsonneault 2014). It is this characteristic of the spin-down that makes gyrochronology possible: it ensures both that rotation will be a strong function of age, and that sensitivity to initial conditions for old objects will be minimal.

In contrast, hot stars (> 6200K) are rapidly rotating. This transition between slow and rapid rotation, known as the Kraft break (Kraft 1967), occurs at roughly $\sim 1.3M_\odot$, where surface convective envelopes become vanishingly thin, and the stars are unable to generate the magnetic winds and drive angular momentum loss. As a result, these objects are rapidly rotating from birth with periods that evolve only mildly over the course of the main sequence. Studies that observed stars at or beyond this boundary have confirmed this general picture (Melo et al. 2001; Wolff & Simon 1997; Zorec & Royer 2012).
While the evolution of the rotation periods on the main sequence is driven by the presence (or lack of) angular momentum loss through winds, structural changes become an important contributor for evolved stars. As stars leave the main sequence their cores contract and envelopes expand, resulting in an increase in their moments of inertia. Even in the most simplistic case in which the star conserves angular momentum and rotates rigidly, this increase in the moment of inertia would lead to a decrease in the surface rotation rate as the star evolves across the subgiant branch. The addition of winds on the subgiant branch allow for further spin-down. Because objects above the Kraft break develop convective envelopes on the subgiant branch, there can be wind-driven angular momentum loss for both main sequence rotational regimes in evolved stages.

In the era of *Kepler* (Borucki et al. 2010) and *CoRoT* (Baglin et al. 2006) we will have access to thousands to tens of thousands of rotation periods (Affer et al. 2012; McQuillan et al. 2013; Nielsen et al. 2013, García et al. 2013, in prep.), all of which will be drawn from a population of stars with many different masses, ages, and evolutionary states. In order to understand the underlying stellar populations and interpret their rotation distributions, we first need a basic set of theoretical predictions for the behavior of rotation across large regions of the HR diagram. Our approach is to choose a prescription for tracking rotational evolution that minimizes our model complexity, allowing us to produce basic predictions for the rotation periods of models across a wide range of masses with minimal assumptions. We will assume that all stars rotate as solid bodies at all times, and that angular momentum is lost through magnetized winds for all objects with sufficiently thick convective envelopes. We draw our initial distribution of rotation periods (as a function of mass) from open clusters where the sensitivity to the somewhat complex pre-main sequence evolution is minimal.
With this set of predictions we show that the strong mass dependence of the surface rotation rate on the main sequence persists onto the subgiant branch, and provides a useful test for the mass scale in asteroseismology and means of identifying more massive stars in the crossing region. By considering evolved objects we show both that there is a strong period-age relationship among subgiant stars that has not been appreciated in the literature, and that the application of the standard gyrochronology relations must be done with some care to avoid mistaking a crossing subgiant for an unevolved dwarf and applying an inappropriate period-age relationship. Additionally, the importance of the envelope expansion in determining the rotation period of stars on the subgiant branch results in a strong correlation between rotation period and radius, which will be particularly useful in the case of transiting planet hosts, in which errors on the stellar radius translate directly into errors on the planet size. Finally, once we are able to compare large datasets of rotation periods to these model predictions, disagreements between our models and the data will motivate the inclusion of missing physics and additional model complexity.

In Section 4.2 we describe the construction of our model grid and treatment of angular momentum evolution, as well as grids to test the effects of variations on the standard physics. In Section 4.3 we show that our simple assumptions preserve the observed rotational properties of both cool and hot stars, and in particular that the Kraft break is naturally recovered. We discuss the manner in which reasonable variations in the input physics affect our results. In Section 4.4 we elucidate the many ways in which we can use stellar rotation as a tool in the context of stellar populations: as a test of the mass scale, age indicator, and constraint on stellar radii. The relationship of these new tools to the existing framework of gyrochronology is
discussed. We discuss uncertainties in our models and future prospects in Section 4.5. Finally, we conclude in Section 4.6.

4.2. Methods

In order to investigate the evolution of the surface rotation rates of a population of stars over time we must specify a reasonable set of starting conditions, here empirically motivated by data from open clusters. We also need a prescription by which we evolve these initial conditions forward in time, which includes processes such as angular momentum (AM) loss, structural evolution, and internal AM transport as a function of mass and composition. In the following section we motivate our initial conditions and prescription for AM evolution. We use observed rotation periods in clusters to calibrate our AM loss law to reproduce both the upper and lower envelopes of the rotation distribution. This calibration sets our starting conditions, and we apply this loss law to a grid of models to examine rotation as a function of age and mass.

4.2.1. Simplifying Assumptions

Treatments of AM evolution can be complex. We opt to construct a set of models based on the simplest set of assumptions that are consistent with the data. There are strong trends in the rotation distribution as a function of effective temperature and gravity that should exist for a variety of reasonable AM evolution models. We choose the simplest case so that we can first investigate the expected range of rotation rates in a base case.

Stars are born with a range of initial rotation periods, and pre-MS AM evolution is complex. At ages less than $\sim 5 - 10$ Myr stars are thought to shed angular momentum through coupling with their protostellar disks. Both accretion-powered
stellar winds (Matt & Pudritz 2005) and disk-locking (Koenigl 1991; Shu et al. 1994) are possible mechanisms for the coupling, but in either case the stellar AM is regulated by interaction with the disk on the timescale of the disk lifetime. The duration of this disk-locking period differs among stars, resulting in a spread in rotation rates, with rapid rotators having been released from their disks at earlier times than slow rotators. There is also evidence that the AM loss law saturates for rapid rotators (Sills et al. 2000; Krishnamurthi et al. 1997), and that differential rotation with depth is necessary to reproduce the young cluster data (Denissenkov et al. 2010).

Rotational evolution on the MS is comparatively simple. Cool stars below the Kraft break rotate slowly, and have therefore forgotten their initial conditions by \( \sim 0.5 \) Gyr (Pinsonneault et al. 1989). Hot stars above the break do not lose AM and the scatter in initial rotation period persists. We have little information about the internal rotation profiles of stars on the MS save that of the Sun. However, the Sun (down to \( 0.2R_\odot \) Thompson et al. 2003) and the rapidly rotating upper envelope of open cluster stars are well approximated as a solid bodies, although uncertainties about the validity of the solid body treat remain for evolved and hot stars (see Zorec & Royer 2012; Deheuvels et al. 2012). In cases in which there are strong internal magnetic fields, solid body rotation is the expectation (Spruit 1987).

We therefore make the following simplifying assumptions. We draw our starting conditions from intermediate age open clusters (\( \sim 0.6 \) Gyr) M37, Praesepe, and the Hyades, supplemented with data from the Pleiades (125 Myr) for rapid rotators. The upper and lower bounds of this period distribution give us a fair indicator of the mean and range of periods as a function of mass, which accounts for the complex pre-MS evolution without requiring that we model it. This distribution can be evolved forward under the simplifying assumption of solid body rotation on the MS
for ages > 0.5 Gyr. We carry the assumption of solid body rotation onto the SGB in our base model.

We also make several simplifying assumptions in the stellar modeling itself. We do not construct fully rotating models. Stellar parameters are tracked with underlying non-rotating models that do not include diffusion, overshoot, or the effects of rotational mixing or deformation. The evolution of the surface rotation period is tracked separately assuming a starting period and appropriate AM loss law. In the following sections we detail our methods for extracting starting conditions from cluster period distributions, the form or our AM loss law on the MS and SGB, and the underlying stellar models that we use to track stellar parameters.

There are several families of published models with more sophisticated treatments of internal angular momentum transport. Hydrodynamic mechanisms for internal angular momentum transport and mixing have been applied in a variety of evolutionary contexts and considering different treatments of the transport mechanisms (Pinsonneault et al. 1989; Heger et al. 2000; Meynet & Maeder 2000). Magnetic angular momentum transport from the Taylor-Spruit dynamo (Spruit 2002) is also included in some codes, including the popular publicly available program MESA (Paxton et al. 2013). The limit of strong magnetic coupling would produce uniform rotation along field lines, and for simple geometries this case is similar to the rigid rotation model employed here. Full evolutionary calculations incorporating wave-driven angular momentum transport are sparser but interesting; see Charbonnel et al. (2013) for recent work and a discussion, and Rogers et al. (2013) for waves in the context of turbulence simulations.
4.2.2. Starting Conditions

In practice, to compile a set of rotation periods across a large mass range (0.4 – 2.0\(M_\odot\)) we must look to several datasets collected in fundamentally different ways. Cool stars are heavily spotted, and therefore have significant spot-modulation in their light curves. Dedicated photometric surveys are capable of determining rotation periods through regular observations of a set of stars. Hot stars, however, generally have only a few spots, if any, and rotation rates for these stars are instead determined using rotational broadening of spectral lines, which are subject to a sin \(i\) ambiguity. We are therefore forced to draw data from multiple sources to cover the entire mass range of interest.

Open clusters represent a rich source of empirical data for our purposes. Many are well-studied, with well-determined ages and rotation rates or periods for hundreds of stars. For our exercise we choose M37 (550 Myr, metallicity of +0.045, Hartman et al. 2009) and the Pleiades (125 Myr, metallicity of +0.003 Hartman et al. 2010). For M37 we utilize the rotation periods from Hartman et al. (2009) obtained from the spot modulation of stellar light curves. These observations consist of the 375 objects in the “clean” sample for which stellar masses are interpolated from the An et al. (2007) isochrones using the cluster parameters from Hartman et al. (2008) and stellar colors from Hartman et al. (2009). Objects fall in the mass range of 0.4 < \(M/M_\odot\) < 1.2. The Pleiades sample is drawn from (Hartman et al. 2010), also from spot-modulation periods. We use the measured rotation periods and inferred masses for all non-binary members in the Hartman et al. sample without adjustment.

A hot star sample is drawn from the Hyades and Praesepe clusters, both of which are similar in age to M37 (An et al. 2007). When we consider the hot stars,
the small age differences between these clusters are not important for rotation. We use the WEDBA\footnote{http://www.univie.ac.at/webda/} open cluster database to draw $v\sin i$’s for Hyades members from Kraft (1965), Abt & Morrell (1995), Reid & Mahoney (2000), and Paulson et al. (2003) and cluster membership from Perryman et al. (1998). Any objects noted as spectroscopic binaries in Perryman et al. (1998) were omitted from the sample. Average values of $V$ and $B - V$ for each star were drawn from the compilation of Mermilliod (1995), and we select only objects with $B - V < 0.8$. In total, 76 objects have $v\sin i$’s, membership information, and photometry from the named sources, and 65 of those objects have $B - V < 0.8$. We draw $v\sin i$’s for Praesepe from Treanor (1960) and McGee et al. (1967) and photometry from Johnson (1952) and Dickens et al. (1968). Object flagged as binaries in the WEBDA database are omitted. A total of 54 objects are selected, 36 of which have $B - V < 0.8$. 

4.2.3. Structural Evolution

We track the evolution of the moment of inertia with stellar models created using the Yale Rotating Evolution Code (YREC, see van Saders & Pinsonneault 2012; Pinsonneault et al. 1989; Bahcall & Pinsonneault 1992; Bahcall et al. 1995, 2001) for $0.4 < M/M_\odot < 2.0$. Models are evolved onto the RGB or to 30 Gyr, whichever occurs first. The models do not include helium or heavy element diffusion. We use the the atmosphere and boundary conditions of Kurucz (1997)\footnote{http://www.univie.ac.at/webda/}, nuclear reaction rates of Adelberger et al. (2011) with weak screening (Salpeter 1954), and employ the mixing length theory of convection (Cox 1968; Vitense 1953) with neither core nor envelope overshoot (primarily important for stars with $M > 1.2M_\odot$). Opacities are from the Opacity Project (OP) (Mendoza et al. 2007) for a Grevesse & Sauval (1998) solar mixture, supplemented with the low temperature opacities of Ferguson
et al. (2005). We utilize the 2006 OPAL equation of state (Rogers et al. 1996; Rogers & Nayfonov 2002).

We use a solar calibration to set the value of the mixing-length parameter, $\alpha$, and the initial composition, $X$, $Y$, and $Z$ such that a $1.0M_\odot$ model at 4.57 Gyr (see Bahcall et al. 1995) recovers the solar radius, luminosity, and surface abundance of $R_\odot = 6.9598 \times 10^{10}$ cm, $L_\odot = 3.8418 \times 10^{33}$ ergs s$^{-1}$, and $Z/X = 0.02289$ from Grevesse & Sauval (1998). A calibration using this standard set of physics yields $\alpha = 1.8098$, $X = 0.71943$, and $Z = 0.01646$. We define the end of the MS as the age at which the core hydrogen fraction drops below $X_c = 0.0002$, and the end of the subgiant branch is defined mathematically as the local minimum on a log $g$ – $T_{\text{eff}}$ diagram in the evolutionary tracks at the base of the giant branch. These stellar models provide the necessary information about radius, luminosity, and moment of inertia as a function of mass and age, but do not include the structural effects of rotational deformation, or account for the effects of rotational mixing.

4.2.4. Angular Momentum Transport and Loss

We must define a set of rules by which we evolve our starting conditions forward in time, which includes both assumptions about the internal transport of AM and the loss of AM via magnetized winds. We will make the assumption of efficient transport of AM for all masses on the MS and SGB; this amounts to treating all models as rigid rotators (in both radiative and convective zones). We assume that there is

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2In models whose effective temperature exceeds the maximum temperature of the Kurucz tables, log $T = 3.95$, the code switches the atmosphere to grey (at $\sim 1.89M_\odot$ for [Fe/H] = 0.0 and $\sim 1.74M_\odot$ for [Fe/H] = −0.2). The differences between the grey and Kurucz atmospheres are not important for the hot stars, but induce shifts in the location of the giant branch on the HR diagram. We therefore run these models are run in two stages: they are evolved from PMS until they cross the log $T = 3.95$ boundary with a Kurucz/grey atmosphere. The models (now with log $T_{\text{eff}} < 3.95$ are then restarted with Kurucz atmospheres and run across the SGB and onto the giant branch. This minimizes, but does not eliminate, the shift in the location of the giant branch induced by the switch between model atmospheres.
angular momentum loss whenever the star supports a sufficiently thick convective envelope. In practice this means that cool stars are subject to magnetic braking throughout their MS lifetimes and onto the SGB. Hot stars born above the Kraft break have very thin convective envelopes on the MS and therefore do not spin down on the MS, but do develop winds on the SGB as their convective envelopes deepen.

There are a number of prescriptions for angular momentum loss through magnetized winds in the literature. Kawaler (1988) assumes that wind losses are proportional to the magnetic flux, and arrives at the functional form \( \frac{dJ}{dt} \propto \omega^3 \). Later modifications to this law, such as those in Krishnamurthi et al. (1997) and Sills et al. (2000), allow for a Rossby scaling and saturation for rotation rates above some \( \omega_{\text{crit}} \), both of which produce better agreement with data from open clusters. In comparison, the Reiners & Mohanty (2012) loss law assumes that the loss is proportional to the magnetic field strength, and therefore derives a form of the loss law that is far more radius-dependent than Kawaler-type variants. However, both formulations fail to adequately reproduce the rapid rotation of stars near the cool side of the Kraft break.

We adopt the Matt & Pudritz (2008) magnetized wind formulation as updated in Matt et al. (2012). We scale the coronal magnetic field strength and mass loss rate relative to the Sun following the prescription of Pinsonneault et al. 2013 (in prep., hereafter PMM). To briefly summarize, the magnetic field is assumed to be linearly proportional to the rotation rate up to a saturation threshold \( \omega_{\text{crit}} \), and proportional to the square root of the atmospheric pressure (due to equipartition considerations). The mass loss rate is assumed proportional to \( L_x \) (Wood et al. 2005), and we adopt \( L_x/L_{\text{bol}} \sim \omega^2 \) (Pizzolato et al. 2003), again up to a saturation threshold. We use a
Rossby-scaled saturation threshold normalized at the Sun. We adopt a $dJ/dt$ of the form:

$$
rac{dJ}{dt} = \begin{cases} 
  f_K K_M \omega \left( \frac{\omega_{\text{crit}}}{\omega_{\odot}} \right)^2 & \omega_{\text{crit}} \leq \omega \frac{\tau_{cz}}{\tau_{cz,\odot}} \\
  f_K K_M \omega \left( \frac{\omega \tau_{cz}}{\omega_{\odot} \tau_{cz,\odot}} \right)^2 & \omega_{\text{crit}} > \omega \frac{\tau_{cz}}{\tau_{cz,\odot}}
\end{cases}
$$

(4.1)

with

$$
\frac{K_M}{K_{M,\odot}} = c(\omega) \left( \frac{R}{R_{\odot}} \right)^{3.1} \left( \frac{M}{M_{\odot}} \right)^{-0.22} \left( \frac{L}{L_{\odot}} \right)^{0.56} \times \left( \frac{P_{\text{phot}}}{P_{\text{phot},\odot}} \right)^{0.44}.
$$

(4.2)

$f_K$ is a constant factor used to scale the loss law to reproduce solar the solar rotation period at solar age, $\omega_{\text{crit}}$ is the saturation threshold, $\tau_{cz}$ is the convective overturn timescale, and $P_{\text{phot}}$ the pressure at the photosphere. The term $c(\omega)$ is drawn from the Matt et al. (2012) prescription and sets the centrifugal correction; it is a small correction for slowly rotating stars, and we set $c(\omega) = \text{const} = 1$.

In practice we track the surface convective envelopes of our models only to a limiting depth, which becomes a problem for stars near the Kraft break. The model fitting point between the envelope and interior cannot be set at arbitrarily small masses (and thus depths) without inducing numerical instability, which in practice means that convective envelopes are tracked only until they pass beyond the fitting point. We have chosen a default fitting point of $\log(M_{\text{interior}}/M_{\text{total}}) = -10^{-8}$ for all of our models, and assume that AM loss ceases when the base of the convection zone moves beyond the fitting point. This value of the fitting point was chosen such that if we assume instead that $\tau_{cz}$ is held constant (as opposed to set to zero) at its last value before the CZ moved beyond the fitting point, that the difference between the constant and zero $\tau_{cz}$ model periods would be less than one day in the periods...
on the MS and SGB. This is an upper limit on the difference in the model periods we would induce were we to choose a more stringent (shallower) fitting point, and assures that our models are insensitive to this purely numerical parameter. Note that this is unimportant for low mass stars, which always have thick convective envelopes, and for high mass stars far above the Kraft break, where convective zones are always negligibly thin on the MS.

Calibration of the Loss Law

We must tune the parameters in our loss law to produce reasonable torques and to reproduce cluster and solar data. We consider two fits to the data: a fit to the rapidly rotating envelope, and a fit to the mostly slowly rotating stars, in order to characterize the range of reasonable rotation periods as a function of mass and time. The fits include 4 free parameters: the saturation threshold $\omega_{\text{crit}}$, the scaling factor $f_K$, used to normalize the loss rate to reproduce the solar value, the initial disk period $P_{\text{disk}}$, and disk-locking timescale $\tau_{\text{disk}}$. We use the Pleiades at 125 Myrs, M37 at 550 Myrs and the Sun with a rotation rate of $P_\odot = 25.4$ days at $t_\odot = 4.57$ Gyr as the reference datasets for the rapid rotators.

We fit the most rapidly rotating sequence allowing all 4 free parameters to vary. We determine the mean rotation rates for the most rapidly rotating 10% of stars in each cluster in bins of 0.1 $M_\odot$ in the mass range of $0.45 \leq M/M_\odot \leq 1.15$, with uncertainties on the mean determined using 1000 bootstrap resamplings (shown as large, red points in Fig. 4.4). We then fit the loss law models to the combined sample of these binned data points in the Pleiades, M37, and the single point for the Sun. The fit amounts to finding a combination of $\omega_{\text{crit}}$, $f_K$, $P_{\text{disk}}$, and $\tau_{\text{disk}}$ that simultaneously reproduces the rapidly rotating sequences for all masses, at all times (125 Myr, 550 Myr, and 4.57 Gyr). The fit the optimization is performed using the
non-linear least squares IDL fitting package MPFIT (Markwardt 2009). We infer the best-fit constants $f_K = 6.575$ and $\omega_{\text{crit}} = 3.394 \times 10^{-5}$ for a disk-locking timescale of $\tau_{\text{disk}} = 0.2810$ Myr and period of $P_{\text{disk}} = 8.134$ days for the rapidly rotating sequence.

For the slowly rotating sequence we allow only $P_{\text{disk}}$ and $\tau_{\text{disk}}$ to vary, exclude the Pleiades from the fit since slow rotators in this young cluster are affected by core-envelope decoupling, which is not accounted for in our models. We also limit the fit to mass above $0.8M_\odot$ since core-envelope decoupling is more important for low masses (Denissenkov et al. 2010). We add additional points to the fit at masses above the Kraft break ($1.5 - 1.9M_\odot$ in steps of $0.1M_\odot$) with rotation velocities of 50 km/s to capture the behavior of the slowly rotating A-star sequence (Abt & Morrell 1995). We infer a disk-locking timescale of $\tau_{\text{disk}} = 5.425$ Myr and period of $P_{\text{disk}} = 13.809$ days for the slow rotators, while $f_K$ and $\omega_{\text{crit}}$ are held at the values derived for the fast rotators.

For comparison, we also fit a more standard modified Kawaler law (Krishnamurthi et al. 1997):

$$\frac{dJ}{dt} = \begin{cases} f_K K w^2 \omega_{\text{crit}}^2 \omega \left( \frac{R}{R_\odot} \right)^{\frac{1}{2}} \left( \frac{M}{M_\odot} \right)^{-\frac{1}{2}} & \omega > \omega_{\text{crit}} \\ f_K K w^3 \left( \frac{R}{R_\odot} \right)^{\frac{1}{2}} \left( \frac{M}{M_\odot} \right)^{-\frac{3}{2}} & \omega \leq \omega_{\text{crit}} \end{cases}$$

using the same procedure, with the same free parameters. For the modified Kawaler law we infer $f_K = 2.656$ and $\omega_{\text{crit}} = 3.126 \times 10^{-5}$, which best fits the rapidly rotating sequence with a disk locking timescale of $\tau_{\text{disk}} = 0.311$ Myr and initial disk rotation period of $P_{\text{disk}} = 7.178$ days.
4.2.5. Parameter Variations

We have made a number of simplifying assumptions in our models, the importance of which should be tested. We present the mechanical details of the models we generate to address these questions in this section; the parameter variations themselves are discussed in detail in Section 4.5.1. There are two subsets of models:

1. We create an alternate set of models that do not lose angular momentum on the SGB. These models are allowed to lose AM on the MS, but conserve AM once they have passed the turnoff, defined here to be when the central hydrogen $X_c < 0.0002$.

2. We construct two grids of interiors models with different compositions: one at solar metallicity, and the other at the mean metallicity of the Kepler stars, $[\text{Fe/H}] = -0.2$, $X = 0.73283$, $Z = 0.01058$. The rotational evolution of both grids is treated in exactly the same manner: the same initial conditions are applied to both populations, and the same prescription for rotational evolution is applied. We do not account for differences in the initial rotation rates of stars as a function of metallicity, which remain observationally ambiguous.

4.3. Results

We present the results of our models of stellar rotation, and show that the chosen form of the wind law naturally reproduces the Kraft break, without fine-tuning. We compare our model predictions to existing observational data both in clusters and the field.
4.3.1. Rotation rate as a function of mass and evolutionary state

Behavior of Representative Models

We begin by considering a small subset of models, and show predictions for the total angular momentum, moments of inertia, and rotation periods as a function of time for a few representative masses in Fig. 4.1 under the “fast launch” conditions. In the second panel (angular momentum as a function of time), we see that more massive stars (bluer/greener curves) lose very little angular momentum on the main sequence, while low mass stars (red curves) undergo substantial loss. High mass stars, which have no winds on the MS, develop deep surface convection zones for the first time on the SGB and begin to lose angular momentum, marked by the steep downturn in the curves of the most massive models after the main sequence turnoff. The second important ingredient in the determination of the surface rotation rate, the moment of inertia, gradually increases over the course of the MS, followed by a steep increase on the SGB as stars undergo envelope expansion. For solid body models, an increase in the moment of inertia results directly in an increase of rotation period. The bottom panel of Figure 4.1 shows the results for these two effects combined: low mass objects spin down and are slowly rotating on the MS, and spin down further on the SGB. High mass objects do not lose angular momentum to stellar winds on the MS, are born rapidly rotating and remain so until they undergo expansion and wind losses on the SGB.

Ensemble Behavior

Our primary results for the full set of models are given in Figure 4.2, where we show period (encoded by color) across many different masses and evolutionary states on a log $g$ – $T_{eff}$ diagram. The diagram is populated by selecting tiles with $\Delta \log g = 0.05$ and $\Delta T_{eff} = 50$ K, calculating the mean rotation period of all models that pass
through each box in log(g)-\(T_{\text{eff}}\), and coloring each box accordingly. The top panel shows the distribution for models evolved with fast launch conditions defined by the loss law fitted to the rapidly rotating cluster sequences. The middle panel shows the same, but for models evolved under the slow launch conditions. The bottom panel shows the simple difference in mean period, tile by tile, between the fast and slow starting conditions.

The location of the Kraft break is clearly visible in the top panel as the abrupt transition between red (rapidly rotating) and green (intermediate rotation) regions. One should note that the break itself is imprinted on the MS starting conditions (denoted by the curve labeled “launch”), but persists well onto the subgiant branch, due to the differences in the angular momentum loss between stars with and without convective envelopes. In the slow launch case the same basic patterns are still present, but the contrast between the rotation rates of objects above and below the Kraft break on the MS is smaller. This contrast is further diminished by the envelope expansion and wind losses on the SGB.

The bottom panel illustrates an important feature: stars born below the Kraft break, regardless of whether they are born rapidly or slowly rotating, have a very narrow range of predicted periods on the majority of the MS and onto the SGB. This is the result of strong \(dJ/dt \propto \omega^3\) dependence in the loss law: rapidly rotating objects are quickly spun down, whereas slowly rotating objects undergo less angular momentum loss. This forces a convergence in the rotation rates at relatively early times (\(\sim 1 \text{ Gyr}\)), and renders the starting conditions unimportant. This insensitivity to initial conditions is one of the reasons that these cool dwarfs are suitable for gyrochronology.
In contrast, the period differences between fast and slow launch conditions for objects born above the Kraft break can be substantial. These objects undergo little or no angular momentum loss, and so MS scatter in the initial rotation periods is preserved well onto the SGB. In practice, this means that we expect to see a range of rotation periods for the hot stars, in comparison to the well-converged cool star sequence.

Figure 4.3 presents the same models in slices through the data at particular evolutionary stages. We show period as a function of the ZAMS $T_{\text{eff}}$ for the starting conditions, at the terminal age MS (TAMS) and end of the subgiant branch (base of the red giant branch, BRGB). In this case, all models are at the same metallicity, and so $T_{\text{eff}}$ can also be viewed as a proxy for mass. Again we see that objects above the Kraft break begin rapidly rotating, with periods of less than a day for the fast launch, compared to the cool stars which have more modest few-to-ten day periods. At the TAMS (dotted curve in Fig. 4.3), the objects above the Kraft break have spun down very little, while the cool stars have periods of $10 < P < 40$ days. All objects undergo significant spin down on the SGB. The rotation period increases by two orders of magnitude for the hot stars by the BRGB (dashed curve) due the combined effects of envelope expansion and stellar winds, while cool stars undergo a similar but more modest response. The difference in the magnitude of the SGB spin down between hot and cool stars is due to two effects. First, the moment of inertia increase is far more substantial for the hot stars (see Fig. 4.1), and second, the hot stars develop convective envelopes while rapidly rotating (in comparison to the cool stars, which always have convective envelopes, and are already rotating very slowly by the SGB), which means that the onset of stellar winds is more important for the massive stars.
4.3.2. Comparison of Model Predictions with Observations

Agreement with Young Clusters

We compare our models to representative young and intermediate-aged open clusters in Figure 4.4. For reference we also indicate the mass dependence that would have been predicted from an extrapolation of a Rossby-scaled Kawaler-style angular momentum loss law.

The upper envelope, given by the red curve in Figure 4.4, is a fit to the $0.5 < M/M_\odot < 1.1$ range, and is, by construction, designed to reproduce the low-mass rapidly rotating sequences in the Pleiades and M37. Objects more massive than $1.2M_\odot$, which here are a mixture of Pleiades, M37, Praesepe, and Hyades stars, were not included in this fit, and yet the model agrees with the data well in this region. The PMM loss law reproduces rotation rates of massive stars far better than the Rossby-scaled Kawaler law, which predicts periods far longer than are actually observed. We naturally reproduce the Kraft break with parameters derived through fits only to low mass stars.

The fit to the lower envelope is denoted by the blue curve in Figure 4.4. The curve is unable to match the data at the lowest masses (recall, however, that the fitting procedure for the slow launch model does not include object below $0.8M_\odot$; see Section 4.2.4). A similar difficulty was noted in Sills et al. (2000): the lowest mass stars in cluster appear to be poorly fit by standard loss laws. More important for our study, however, are stars with masses $\gtrsim 1.0$. The slow and fast launch curves converge for masses $\sim 1.0$, and produce a range of predicted rotation periods for higher masses. The two curves reproduce a reasonable range of rotation rates suggested by the period data from M37 and $v\sin i$ data from the Hyades and Praesepe for the hot stars.
Agreement with Old Clusters

Old open clusters have the potential to be extremely useful for validating our theoretical predictions, since the stellar parameters of cluster members can be well determined. However, there are very few old open clusters, and even fewer with good rotation data. In light of this, we compare our models to data from a single old open cluster. M67 is old enough (3.5-4.0 Gyr; Sarajedini et al. 2009) to have a substantial number of stars near the Kraft break present on the subgiant branch, and the analysis of Canto Martins et al. (2011) provides $v \sin i$ measurements for cluster members. In Figure 4.5 we show our theoretical models overplotted with the cluster $v \sin i$'s from objects classified as subgiants or giants in Canto Martins et al., and corrected by $4/\pi$ to account for mean inclination effects.

We perform a more quantitative test of the agreement between the data and theory for M67. We select a subset of the data that falls on the subgiant branch or early giant branch (roughly log ($g$) $\geq$ 3.5). In the CMD, the subgiant branch is narrow, indicating that the observed log $g$ dispersion is observational scatter. We therefore assign isochrone positions based on $T_{\text{eff}}$ alone.

To test whether our model predictions produce a distribution of rotation rates consistent with those found in M67, we perform the following test. We sample the M67 subgiant distribution with replacement. For each star we determine its location on a 4.0 Gyr rotational isochrone based on its measured effective temperature alone, and then draw from a distribution of inclination angles uniform in $\cos i$ and assign a $\sin i$ value to each source. The model M67 data is then compared to the real $v \sin i$ data using a two sample Kolmogorov-Smirnov test. We repeat this process 1000 times (sampling the distribution, assignment of $i$, and comparison to data) to obtain
a reasonable average $p$ value, and well as the range of $p$ for the slow launch, fast launch, and no SGB wind models.

In the case of M67 we also have additional information from Li studies. The work of Balachandran (1995) suggests that the Li is depleted in the subgiants of M67, and that the depletion patterns require that all stars on the SGB originate from ZAMS temperatures in the range of $6300 - 6600$ K, the so-called “lithium dip”. Our 4.0 Gyr rotational isochrones originate from models with ZAMS effective temperatures in the $6400 - 6600$ K range (also consistent with the modeling in Canto Martins et al. 2011), and we are therefore assured that any differences between the observational and theoretical rotation distributions are due to our treatment of the rotational evolution, not because we are comparing the stars with models that originated from different places on the HR diagram.

We find a median $p$ value of 0.11 for models using a 4.0 Gyr old solar metallicity isochrone under the fast launch conditions, suggesting that the observed and model distributions are consistent with being drawn from the same population. The slow launch case yields a similar result, with $p = 0.31$. In comparison, if we consider our models with no winds on the SGB (discussed in 4.5.1) we find $p = 0.001$, and that the two samples are inconsistent with having been drawn from the same parent distribution (discussed in more detail in Section 4.5.1). If we consider instead a 3.5 Gyr isochrone, we find $p$ values of 0.03 (fast launch), 0.11 (slow launch), and $1.9 \times 10^{-5}$ (no winds). The data and rotational isochrones are shown in Figure 4.5. We also find, however, that the calculated $p$ values vary considerably between realizations of the sample. One must be careful not to overinterpret our results; our analysis suggests only that our rotation models are in good general agreement with the observations. There are likely to be synchronized binary and blue straggler
contaminants even in this sample of subgiants and the sample size is small, which also encourages one not to place undue significance on these statistical tests.

Agreement with Field Data

We can also compare our models to rotation data gathered for field stars. In the case of a field population the stellar parameters are in general more poorly known, and we frequently have no information on the compositions of the objects (which, as we show in Section 4.5.1, has some bearing on the predicted rotation periods). Nevertheless, rotation data, log $g$, and $T_{\text{eff}}$ determinations for field stars exist, and we compare the results of selected studies to our models in Figure 4.6.

The right panel of Figure 4.6 shows field $v \sin i$ data from Lèbre et al. (1999) and Mallik et al. (2003), corrected by $4/\pi$ for the average inclination effect. The Lèbre et al. sample is largely drawn from a collection of subgiants in the Bright Star Catalog, and the Mallik et al. from Hipparcos stars. As in the case of M67, the agreement between the models and the data is fair when one considers that these are lower limits on the rotation velocities. There are points with slower rotation velocities than we predict (which can be the result of $\sin i$ ambiguity), and very few points that are rotating substantially faster than we predict (which cannot be explained by invoking $\sin i$). We do not proceed with more detailed modeling due to the additional uncertainties introduced by star formation history and the range of metallicities found in a field population.

The left-hand panel shows the results of the do Nascimento et al. (2012) analysis of light curves from the CoRoT spacecraft for a second sample of field stars. Periods are determined through spot modulation, and stellar parameters are drawn from Gazzano et al. (2010). Here, unlike in M67, there are clear differences between the data and the models, namely that the highest $T_{\text{eff}}$ objects around 6000
K appear to be rotating more slowly than expected, and objects at 5000-5500 K too rapidly. There are (at least) two possible explanations for this disagreement. There may be deficiencies in our models, although it is unclear why such deficiencies would not be evident in the $v \sin i$ distributions in both the field and M67. There could also be systematic errors in the measurements of the periods and stellar parameters themselves (notably the effective temperatures). The CoRoT light curves extend only to 150 days, while the majority of reported periods are 60 days or longer, meaning that the time series can capture at maximum two full rotation periods. The longest periods that we predict for our cool subgiants are inaccessible with time series of this length. It is unclear whether large samples of rotation periods (rather than $v \sin i$’s) will show similar discrepancies. In either case, this is example of the utility of a set of predictions for rotation periods as a function of mass and evolutionary state: it can either serve as a consistency check, or a test of our AM transport models.

4.4. Stellar Rotation as a Tool

Because rotation is a strong function of mass and evolutionary state, and because sharp features such as a the Kraft break are present and persist in the rotation distribution, it can be used as a powerful diagnostic of the underlying stellar populations. In this section we expand traditional gyrochronology relationships to include subgiants and discuss diagnostic tests of mass and radius. We also quantify the impact of the SGB on field populations.

4.4.1. Stellar Ages

Existing rotational age diagnostics

We are accustomed to thinking of relationships between period and stellar ages in the context of gyrochronology of cool dwarfs (Barnes 2007; Mamajek & Hillenbrand
While these period-age relationships have the potential to be powerful indicators of age in the field, they are applicable only for cool, unevolved stars. The gyrochronology relations rely on AM loss over time to produce a smooth transition from rapidly rotating young stars to slowly rotating old stars. The strong dependence of the loss law on the rotation rate quickly erases the memory of the initial conditions, thus producing a sequence in which all stars of a given mass at a given age rotate at the same rate. Hot stars born above the Kraft break are therefore unsuitable for gyrochronology, since they do not undergo strong wind losses, and have little relationship between their periods and ages on the MS. Likewise, evolved stars that undergo expansion and increase in their moments of inertia are also not suitable for traditional gyrochronology, which assumes that the stellar spin down is due only to AM losses. In clusters it is possible to focus only on stars suitable for gyrochronology; in a field population, these hot and evolved stars will mingle with the cool dwarfs, the effects of which we discuss below.

A more complicated picture: rotational regimes for field populations

Figure 4.7 shows the period as a function of age for both MS stars (top panel) and subgiants (bottom panel). Also shown are slices at fixed effective temperatures (solid black curves). Cool dwarfs ($M < 1.1M_\odot$) in the top panel display the tight period-age relationship that makes gyrochronology possible. Given a period, with some small correction for mass, one can infer the age. The relationship is not as clear for more massive stars ($M > 1.1M_\odot$): here the curves become nearly vertical, indicating that there is little change in the period as a function of age, and that a measurement of the period does not imply a unique stellar age. This is a restatement of the fact that the gyrochronology relations are not applicable to hot stars without strong MS AM loss.
We see in the lower panel that although subgiants may not obey the same period-age relationships as their MS counterparts, these two quantities are in fact related. If we add information about the effective temperature, in particular, we obtain tight sequences that yield unique ages at a given period for subgiants. We can understand this trend as follows: objects leave the MS with a strongly mass and age-dependent rotation rate. Massive stars are born rapidly rotating and do not spin down on the MS, and cooler stars are spun down substantially and have increasingly longer periods the older they are. SGB lifetimes are also strongly mass dependent in the same sense: more massive objects are younger upon arrival on the SGB, and spend less time there. We therefore see that more rapidly rotating stars on the SGB have younger absolute ages, whereas the most slowly rotating stars are also the oldest. This subgiant period-age relationship has not been previously appreciated in the literature. It implies that the diagnostic power of the gyrochronology relationships can be extended to more evolved stars, provided we account for the different mechanisms responsible for their period distributions and spin down.

In Figure 4.8 we combine both the evolved and unevolved stars onto a single diagram for both the fast launch (top panel) and slow launch (bottom panel) conditions, as they would be in a field population. We divide this diagram into three distinct rotational regimes:

1. \( P < 10 \) days: Massive \((M > 1.1M_\odot)\) or young stars. All stars in this period range are either above a solar mass (on the MS or the SGB) or young, rapidly rotating solar/subsolar mass objects.

2. \( 10 < P < 40 \) days: Mixed. Objects in this period range can be low mass MS stars, or subgiants at a range of masses.
3. $P > 40$ days: Subgiants. The vast majority of objects in this period range are low or intermediate mass subgiants.

These three regimes have corresponding designations for traditional gyrochronology: $P < 10$ days denotes young objects, $10 < P < 40$ days objects suitable for gyrochronology, and objects with $P > 40$ days are outside the valid period range for the gyro relations. When we consider a field population we must modify these traditional regimes: rapidly rotating objects can be young or massive. Objects in the $10 < P < 40$ day are good targets for gyrochronology, but we must be very careful about assigning the correct evolutionary state to targets within this range, lest we apply the wrong period-age relationship. Objects with periods longer than 40 days are evolved, and present their own period-age relationships.

The Importance of Subgiants

Not only does the period distribution of subgiants overlap with that of the MS in the critical $10 < P < 40$ day range, but subgiants also represent significant contaminants in surveys of field populations. Subgiant stars will be systematically over-represented in magnitude-limited surveys. We use a TRILEGAL galaxy model with default parameter values (see Girardi et al. 2005, 2012, and the web submission form) for a 5 deg$^2$ field of view (equivalent to the area covered by a single Kepler CCD module) centered on the Kepler field at galactic coordinates $(l, b) = (76.32^\circ, +13.5^\circ)$ to estimate the fraction of subgiants. If we consider objects in a $100K$ slice centered on $5800K$ and define all objects in that slice with log($g$) $\geq$ 4.2 to be subgiants, we find that for a limiting Kepler magnitude of $K_p < 14$, 35% of the stars in the sample are subgiants, 73% of which have masses $M > 1.1M_\odot$. We obtain similar results with a limiting SDSS $g$ magnitude of $g < 14$. We have not included magnitude errors in the analysis, nor tuned the TRILEGAL model to specifically match the Kepler
field, but we expect that a more careful approach will yield a qualitatively similar result: subgiants are an important contaminant in magnitude limited surveys in this temperature range, which requires that any survey seeking to use gyrochronology to determine the ages of stars in such a sample must be able to discriminate between MS and post-MS objects. The case of the two subgiants in Doğan et al. (2013) is an excellent example of this phenomenon. The gyrochronology and asteroseismic ages for these objects do not match, and without the excellent gravity and age information provided by seismology, they would likely have been misclassified as old dwarfs. Failure to identify these frequent subgiant contaminants and treat them accordingly will result in erroneous ages for more than a third of the stars in these surveys.

4.4.2. Period as a Constraint on Stellar Radii

One of the primary science goals of the Kepler mission is to detect transiting extrasolar planets, and statistically characterize the planet population. Because the transit method is sensitive to the ratio of the planetary and stellar radii, one must have precise stellar parameters before accurate measurements of the planet properties are possible. The stellar parameters in the Kepler Input Catalog (KIC Brown et al. 2011), while adequate for their intended purpose of distinguishing giants from dwarfs, are insufficient to determine the stellar radius with the necessary accuracy. Follow-up observations are generally essential for the careful characterization of the star.

We suggest that the stellar rotation period can also offer constraints, at least in a statistical sense, on the radii of the stars at fixed effective temperature. In Figure 4.9 we show theoretical models for the relationship between period and radius for the

\footnote{http://stev.oapd.inaf.it/cgi-bin/trilegal}
fast launch conditions. Radius is not a strong function of period on the MS, because the radii of MS stars change very little over the course of their core-hydrogen burning evolution, but they undergo substantial evolution in their periods due to magnetic braking. On the subgiant branch, however, because both the increase in period and increase in radius are linked to the expansion of the stellar envelope, there exists a correlation at fixed effective temperature between the radius and the rotation period. Therefore, if one has a way of reliably distinguishing between dwarfs and subgiants (via coarse log $g$ measurements or luminosity, for example), a measurement of the period with provide an additional constraint on the stellar radius.

4.4.3. Period as a Test of the Mass Scale

Because period is a strong function of stellar mass, both on the MS and the SGB, measurements of rotation periods could be instrumental in helping to distinguish between massive and low-mass objects, particularly on the subgiant branch. Spectroscopic gravity measurements have notoriously large uncertainties, often of the order of 0.5 dex, which severely hampers classification of stars on the subgiant branch. The addition of a rotation measurement can help to distinguish between masses on the subgiant branch, provided that the determination of log $g$ is sufficient to rule out a MS dwarf.

Using the same TRILEGAL model discussed in Section 4.4.1, we show that one could systematically select massive subgiants for study based on their rotation periods. If we use our “fast launch” model grid to define the regions in a log $g – T_{\text{eff}}$ diagram populated by rapid rotators, and again consider a narrow slice in temperature of 100 K around 5800 K, we find that 24% of subgiants should have $P < 10$ days. Figure 4.7 shows that for the fast launch, all subgiants in this period range are also relatively massive. If we consider the entire long-cadence Kepler
sample with effective temperatures from Pinsonneault et al. (2012), ∼9300 stars fall within this effective temperature slice, which would imply a sample of ∼800 massive, rapidly rotating subgiants available for study.

4.5. Discussion

We have necessarily made simplifying approximations in our models. We have considered rotation across a population, but for that population we have assumed solid body rotation, a single composition, limited stellar demographics, and simplified stellar models. In this section we discuss and quantify the impact of such assumptions. We quantify the importance of both metallicity and uncertain processes such as winds on the SGB for the model predictions of rotation periods in Section 4.5.1. In Section 4.5.2 we discuss the implications of a more realistic stellar population that includes binaries and other subgroups with unusual rotation distributions, as well as the impact of some of our simplified modeling approaches on our results.

Finally, in Section 4.5.3 we also speculate about the use of rotation as a tool in the future. We discuss tests of the physics of angular momentum transport via measurements of the surface rotation rates, as well as using rotation to verify the asteroseismic mass scale. We close in Section 4.5.4 with a discussion of the diagnostic power of rotation in the era of precision parallaxes from missions such as Gaia.

4.5.1. Parameter Variations

We present in this section the results of our parameter variation studies, which provide insight into the importance of physical processes such as continuing AM loss on the SGB through winds, as well as addressing the importance of metallicity to our conclusions.
Winds on the SGB

Hot stars experience minimal AM loss on the MS. However, our diagnostics of stellar activity indicate that subgiants possess active chromospheres and coronae. Studies such as Schrijver & Pols (1993) also argue for the presence of winds based on rotation periods. We therefore expect that massive subgiants will spin down once cool, and have incorporated this into our models. However, the behavior and strength of subgiant AM loss remains uncertain. To quantify the impact of winds SGB on the periods predicted from our models, we compare models run with and without SGB AM loss.

In Figure 4.10 we plot the ratio of the periods of the no-wind models and those from the standard models as a function of time on the SGB. We use models evolved with the fast launch initial conditions where the effects of winds will be most significant. For stars born below the Kraft break (the red curve, for example) objects arrive on the SGB slowly rotating, and have convective envelopes for their entire MS and SGB lifetimes. SGB wind losses are steady, as a result of this slow rotation. The timescale for AM loss $t_{\text{loss}} \propto J_{\text{MSTO}}/ < dJ/dt >$ for the models with winds is much longer than the SGB lifetime, $t_{\text{SGB}}$ (\sim 12 Gyr versus \sim 1.5 Gyr for a 1.0$M_\odot$ model). Therefore, despite the fact that the SGB phase is relatively long and objects undergo AM loss over the entire SGB, the difference between models with and without winds is modest.

Objects born above the Kraft break only develop substantial CZs after arrival on the SGB, which is the reason for the sudden decline in rotation rate for the most massive stars in Figure 4.10. Models with and without wind losses on the SGB are identical until convective envelopes develop and winds begin to operate. Because these stars were rapidly rotating on the MS, the appearance of a surface convection...
zone leads to a strong spin down. However, these stars also have the shortest SGB lifetimes of the objects considered here (0.1 Gyr for a 1.8$M_\odot$ model), limiting the total AM that can be lost.

Stars in the vicinity of the Kraft break (green curves in Fig. 4.10) are the most strongly affected by SGB winds by the time they arrive at the base of the giant branch, due to the combination of relatively rapid rotation and long SGB lifetimes. Here $t_{\text{loss}} \sim t_{\text{SGB}}$, allowing for significant losses. These stars therefore undergo strong braking as their convective envelopes deepen, and have significant SGB crossing times over which to lose AM. These objects are the most strongly affected by the assumptions regarding winds on the SGB.

The effects of wind losses on the SGB are substantial enough that observational datasets should be able to motivate whether or not their inclusion is appropriate. Our preliminary analysis in Section 4.3.2 for the stars in M67 suggests that the inclusion of winds is necessary, although there are additional uncertainties in the stellar physics that could mimic the long periods of a population of subgiants affected by winds.

Metallicity

We have considered mono-abundance model populations, but in reality, objects in a field population will support a range of compositions. Traditional stellar diagnostics have well-known degeneracies between composition, age, and mass, and we therefore investigate the sensitivity of rotation-based stellar diagnostics to composition.

Changes in metallicity affect models in two ways that are important for rotation: lower metallicity models have both shorter total lifetimes, and shallower convective envelopes. Both of these effects tend to decrease the rotation period.
Winds are less effective and therefore drain less AM, and the models have shorter lifetimes over which to lose AM. We therefore expect that at fixed mass and age, the differences between models of different metallicity can be large.

However, if we view the impact of metallicity in terms of observables, namely log \( g \) and \( T_{\text{eff}} \), the differences in the periods induced by composition are somewhat muted. Because rotational behavior is closely linked to effective temperature, we expect that the differences between the two compositions viewed in this plane will be smaller, although not zero, due to age differences. Figure 4.11 shows the difference in rotation period as a function of surface gravity and effective temperature between models at solar composition and those at \( [\text{Fe/H}] = -0.2 \). We construct this diagram in a similar fashion to Figure 4.2, except that each tile is now the difference between the average model period in a given log \( g - T_{\text{eff}} \) box for a solar and \( [\text{Fe/H}] = -0.2 \) grid. As expected, models with higher metallicities tend to rotate more slowly than those of lower metallicities. On average, the low metallicity models at any given location on the diagram are younger, as shown in the bottom panel of the same figure.

The period differences are most severe for objects at the base of the giant branch; at a fixed effective temperature, a more metal rich object has expanded more than a metal poor one due to metallicity dependent shifts in the location of the Hayashi track. The lower MS, while unevolved, contains old objects that have had a long time to accumulate differences in their periods due to age and CZ depth differences. Finally, there is a discrepant patch at \( \sim 6300 \) K due to the fact that the appearance of hooks in the stellar tracks (due to convective cores) is shifted to lower temperatures at higher metallicities, and the morphology of these hooks affects the exact pattern of AM loss, since objects effectively cross from one side of the Kraft break to the other and back over the course of their evolution through the hook. In
this region one is comparing stars that do and do not undergo evolution through a hook, and therefore have different rotation periods.

4.5.2. Other Complications

Astrophysical Backgrounds

We have considered reasonable distributions of rotation periods for single stars on the MS, and quantified a reasonable range in period. We have, not, however, accounted for several classes of astrophysical background sources that will inevitably be present in large datasets of rotation periods:

1. Synchronized binaries: Tidally synchronized binary systems will have rotation periods less than 10 days, and they are relatively common. Duquennoy & Mayor (1991), for example, found binaries with periods less than 10 days to be 4% of the sample in the field, compared with 11% in the Hyades. These objects will contaminate the rapidly rotating sample. RV variability or SED information from photometry should permit us to quantify and distinguish this population from massive single stars.

2. Stellar mergers: The rotation rates of mergers are not radically distinct from those of other massive stars (see for example Mathieu & Geller 2009, for the old open cluster NGC 188). For low mass stars, however, mergers could produce unusually rapid rotation for stars of the same mass and age. Andronov et al. (2006) estimated that ~3% of sub-turnoff stars in old open clusters could be merger products, and such stars would rotate more rapidly on average than single stars. A comparable fraction of halo dwarfs are severely over-depleted in Li (Thorburn 1994), which is consistent with this estimate. Li abundances, for
some mass ranges, should allow us to discriminate between normal and blue straggler stars.

3. Slowly rotating massive stars: There are two mechanisms that can produce slow rotation in massive stars (see Abt & Morrell 1995, for a good discussion). A sub-population of stars have very high magnetic fields (Babcock 1958), which leads to very slow surface rotation. These stars have peculiar spectra caused by element segregation in their outer layers (Michaud 1970). Tidally synchronized binaries will also rotate unusually slowly. Such stars also are subject to unusual surface abundance patterns and are typically referred to as Am stars (Titus & Morgan 1940). Both phenomena appear at surface rotation rates below 90-120 km/s. Stars with an angular momentum history different from that in our models will therefore be reasonably common in massive stars. Abt & Morrell find that 16% of the overall population of A0-A1 stars fall under one category or another. The frequency of chemically peculiar stars rises into the early F domain. The most likely explanation is not a mass dependent trend in frequency, but instead the onset of slower surface rotation, which in turn induces unusual surface abundances.

Each of these classes of objects represents both a background and an opportunity: while they will contaminate any large sample of rotation periods and make interpretation of those periods more challenging, we may also be able to use their unusual rotation rates to identify them as interesting subsets of objects for further study.

Modeling Considerations

We have neglected several physical effects in our models that could be important if we intend to utilize stellar rotation as a precise diagnostic of stellar parameters.
Apart from the uncertainties in the modeling of the AM loss and evolution that we have already discussed, we have also neglected the effects of stellar rotation on the structure itself, and considered models with no diffusion, both of which may be important for stars at and above the Kraft break.

Rotation reduces the effective gravity of the star, and rotating stars have lower effective temperatures and luminosities than non-rotating stars of the same mass and composition. For a 1.6 M_☉ model run in YREC’s fully rotating configuration, with solid body rotation and no wind losses and a ZAMS rotation velocity of ∼ 150 km/s, the corrections to the effective temperature and gravity at the MS turnoff are within ΔT_{eff} = 100 – 300 K throughout the MS and SGB, with Δ log g < 0.03 on the MS, Δ log g < 0.07 on the SGB. Rotation has substantially weaker effects on the structures of low mass stars (see Sills et al. 2000). These effects will need to be taken into account before the mapping between temperature and mass is trustworthy, but the general trends that we focus on throughout this chapter regarding mass, evolutionary state, and period should remain qualitatively unchanged.

Diffusion and gravitational settling are another class of physical effects that we have neglected in our treatment that may induce uncertainty into relationships between T_{eff}, log g, age, and rotation period. Diffusion has the effect of deepening the convection zone and shortening the lifetime of a model with respect to the no diffusion case, both of which, as we have shown for metallicity, affect the predicted rotation period.

4.5.3. Additional Tools

In Section 4.4 we discussed the many ways in which rotation can be used as a tool to understand stellar populations. In this section we discuss additional applications of rotation rates that may represent interesting tools in the near future.
Rotation as a Test of the Asteroseismic Mass Scale

Measurements of rotation period could serve as useful diagnostics in the context of asteroseismology. There are two primary diagnostics of the global oscillation pattern in asteroseismology that can provide information on the mass and radius of the star. The frequency of maximum power, $\nu_{\text{max}}$, is the location of the peak (in power) of the envelope of oscillation modes. The large frequency separation, $\Delta \nu$, is the spacing in frequency between modes of same degree and consecutive radial order. Together, measurements of these quantities in conjunction with the scaling relations (Kjeldsen & Bedding 1995):

$$\frac{\Delta \nu}{\Delta \nu_\odot} = \left( \frac{M}{M_\odot} \right)^{1/2} \left( \frac{R}{R_\odot} \right)^{-3/2}$$

$$\nu_{\text{max}} = \frac{M/M_\odot}{(R/R_\odot)^2 \sqrt{T_{\text{eff}}/T_{\text{eff},\odot}}}$$

provide a measure of the mass and radius. These scaling relations are very general, model independent, and an extremely useful tool for determining stellar parameters. It is, however, important that they be verified. Interferometric measurements of stellar radii have helped to confirm the validity of the scaling in radius (see Huber et al. 2012, for example), but there have been limited tests of the seismic mass scale (but see Miglio et al. 2012). Because of the nature of the sharp transition between slow and rapid rotation as a function of mass, even on the subgiant branch, measurements of rotation period could provide a means to verify stellar masses derived via the scaling relations.

Rotation as a Test of Stellar Physics

We have presented here a set of very simple physical models: we consider only solid body rotation and AM losses from winds, both of which are well motivated by
observations. Our previous discussion of the uses of rotation as a tool relies on the assumption that our models are correct. While our simple case is well motivated, there are more complicated effects, such as differential rotation, that could also be important, and we may find that our theoretical predictions do not agree with the data and motivate additional model complexity.

If we were to consider the addition of differential rotation, for example, we might consider 1) models that have internal differential rotation on the MS, and rapidly rotating cores on the SGB, 2) models that begin as solid bodies on the MS but develop internal differential rotation in radiative zones over the course of the SGB, or 3) models that allow for differentially rotating convective zones as well radiative zones. Each of these combinations would produce a different signature in the surface rotation rates of stars evolving on the SGB, and a pattern of departures from the simple, first-order case we have presented here. Case 1) would result in objects with faster rotation rates than we predict, as the base of the convective envelop dredges up material from deeper, more rapidly rotating regions of the star as the object evolves across the subgiant branch. In contrast, case 3), coupled with wind losses, could produce rotation periods longer than those of the solid body case, because surface layers depleted of AM would not be quickly resupplied with AM from the deeper interior.

Case 2) is among the simplest to address in a more qualitative way. We evolve a solid body model in YREC’s fully rotating configuration of a 1.6 $M_\odot$ star to the MSTO. The model is then allowed to develop differential rotation as it undergoes core contraction and envelope expansion on the SGB. Radiative zones are assumed to conserve specific angular momentum, while convective zones always rotate as solid bodies. We neglect winds both on the MS and SGB for this exercise. Models of this form rotate a maximum of about 1.2 times more slowly (at $\sim 6100$ K) than a solid
body model evolved under YREC’s fully rotating configuration (which accounts for rotational deformation, unlike our standard models). This period difference is the result of a larger effective reservoir of AM available in the solid body case. There AM transport is assumed to be instantaneous, and therefore and AM that would otherwise be sequestered in the core is redistributed throughout the star immediately. In the case of the differentially rotating model, only the angular momentum of the outermost envelope is available, and thus we would observe a slower rotation rate. As the convective envelope deepens in both models the differences in the rotation rates decrease, because limited AM is located in the core of even the differentially rotating case, and both models have large envelopes rotating as solid bodies.

There are many other possibilities suitable for investigation that are beyond the scope of this dissertation. In the absence of large rotation datasets it has been impossible to determine which of these cases is most correct, and it is difficult to motivate the inclusion of additional model complexity. With enough observational data, it will be possible test more complex interiors models by recognizing the pattern of departures between the data and a simple set of models, such as those that we have provided in this chapter. We have begun here with a very simple test case; as large datasets inform our theory, we may find that we are finally well motivated to include more complicated physics in our models.

4.5.4. In the Era of Gaia

Measurements of rotation are highly complementary to the precise stellar parameters that will be obtained by the Gaia mission. Luminosities derived through parallaxes suffer from a mass-composition degeneracy, which means that although the object may be well-placed on a HR-diagram, fundamental parameters such as age and mass remain uncertain. Furthermore, luminosity in sensitive to helium, which unlike
metal content, is not easily measured. As we have shown, rotation is also sensitive to composition, but its dependence is different: luminosity is set by the mean molecular weight, where rotation is set by the effective temperature and age of the star. The addition of rotation information can break the mass-composition degeneracy.

Rotation also remains a means to identify unusual objects, such as stellar mergers and interacting systems, that fall in otherwise typical regions of the HR diagram but have unusual rotation periods (e.g. subturnoff mergers, see Andronov et al. 2006). Gaia will be able to identify likely binaries based on their HR diagram locations with respect to the single star MS, but rotation will be able to determine the fraction of those binaries that have synchronized. Massive binaries will appear as anomalously slowly rotating, whereas low mass binaries will have inflated rotation rates. Our predictions for typical rotation periods, coupled with observations and the precise HR diagram locations provided by Gaia, allow for interesting studies of binary evolution by isolating both pre- and post-merger objects.

Photometric rotation information from Gaia itself will be limited (Distefano et al. 2012), and the periods of solar analogs largely misidentified. However, Distefano et al. (2012) suggests that for rapidly rotating objects with $P < 5$ days, the information from the satellite photometry alone may be adequate to identify the period, meaning that massive stars or synchronized binaries may have measured rotation periods. For broader populations, large-scale spot-modulation or rotation velocity surveys will be necessary to take full advantage of the constraining power of stellar rotation rates.

\footnote{http://sci.esa.int/science-e/www/area/index.cfm?fareaid=26}
4.6. Summary

Massive and precise datasets produced by spacecraft such as *Kepler* and *CoRoT* are transforming stellar and planetary astrophysics. Unlocking the full potential of these rich datasets requires the ability to sort through the complicated mixture of stars observed in field studies. As always, the challenge of sample characterization is to find an observable that varies strongly as a function of the parameters we wish to infer, and to exploit it. Rotation represents such an observable. Historically, it has been impractical to measure rotation in bulk field samples, but modern surveys will yield rotation periods for an unprecedented number of stars, making rotation a viable and useful tool.

Rotation periods are shaped by stellar mass, age, and evolutionary state. A transition from rapid rotation at high mass to slower rotation at lower mass (the Kraft break) is imprinted on the MS and linked to the onset of AM loss from magnetized winds in the cool stars. Such winds progressively spin down cool stars as they age. As stars evolve, their moments of inertia also change, resulting in substantial changes in their surface rotation periods. Mass, age, and evolutionary state are therefore imprinted on the measured surface rotation rates.

We are already accustomed to using rotation as an age diagnostic in gyrochronology, but when we expand our studies beyond the realm of traditional gyrochronology and consider a mixed field population of both evolved and unevolved stars at many different masses, rotation can be used as a tool for a far broader set of investigations. It can provide constraints on stellar mass, evolutionary state, and radius in addition to its use as an age diagnostic, and becomes particularly useful on the subgiant branch, where these quantities can be otherwise difficult to obtain. These qualities make period measurements immediately useful for characterization
of the hosts of transiting planets, where precision stellar radii are essential, and for the verification of the asteroseismic mass scale.

We have shown that stellar rotation in a cool field population of MS and SGB stars falls into three distinct regimes:

1. $P < 10$ days: Young solar-like or massive stars born above the Kraft break

2. $10 < P < 40$ days: MS solar-like stars or crossing massive and intermediate mass subgiants.

3. $P > 40$ days: Primarily solar/ low-mass subgiants

A star’s period, in combination with an effective temperature, therefore immediately makes suggestions regarding its birth mass and current evolutionary state. These regimes also demonstrate that the presence of subgiants will be an important consideration when dealing with large, mixed stellar samples. Subgiants are numerous in magnitude-limited surveys, and represent an important background in the prime period range for gyrochronology (10-40 days). For effective temperatures in the range 5000 – 7000 K, a survey with a magnitude limit of $K_p < 14$ will consist of roughly 35% subgiants. The fraction in bright, seismically interesting samples ( $K_p < 11$) is even higher, at $\sim 40\%$. Any conclusions regarding the age distributions of a field sample determined through gyrochronology that do not account for presence of subgiants will be erroneous. It is essential that the contribution of subgiants to period distributions be recognized: they are paradoxically both an important contaminant and a population of stars for which stellar parameters are highly accessible through measurements of their rotation periods.
We have presented here a set of predictions for the most simplistic physically motivated model of the evolution of the surface rotation rate, based on the existing rotation data. A sample of bright *Kepler* stars with both measured rotation periods and seismic information will be available in the near future, and the exquisite precision of the data will provide powerful tests of these predictions that were not possible in the past. During the preparation of this manuscript, several large samples of rotation period have become available (Affer et al. 2012; Nielsen et al. 2013). Although beyond the scope of this dissertation, our next step is to compare these datasets with our theoretical predictions. The reality of stellar rotation may be far more complex than our simple model: it may be that stars leave the MS with substantial internal differential rotation. This would result in anomalously high observed rotation rates on the SGB as the convective envelope eats into the rapidly rotating interior of such a star. Likewise, it may be the case that the convective envelopes of these evolved stars support substantial differential rotation (as opposed to the theoretical expectation of solid body rotation). While the signature of this differential rotation would be present on the SGB as anomalously slow rotation, it could be confused with other effects, such as unexpectedly efficient winds, which could complicate the interpretation. We can however, examine the surface rotation rates giants after first dredge up; these objects would be sufficiently slowly rotating that winds should not be important, and the presence of differential rotation in the convective envelope could be more clearly identified. In this way, comparison of large datasets to our simple theoretical model will inform us as to whether or not additional complexity is needed to explain the observed distribution of rotation periods across the HR diagram, and we will then be able to motivate the inclusion of additional physics with observations. We stand to learn as much from disagreements between data and models as perfect agreement.
Once we have made these initial comparisons and refinements using small but excellent rotation datasets, rotation-based diagnostics will continue to be useful beyond the era of Kepler and CoRoT, and even in the presence of precise parallaxes (e.g. Gaia). Rotation depends on mass and composition in a fundamentally different manner than luminosity, which will help to break degeneracies between mass and metallicity that will exist even with exceptional parallax measurements. For large time-domain surveys in which parallaxes or extensive spectroscopic follow-up is unavailable, rotation remains a means to differentiate between dwarf and subgiant, evolved and unevolved, low-mass and high-mass stars.

4.7. Figures

Fig. 4.1.— Top panel: Evolutionary tracks for 1.7, 1.5, 1.3, 1.1, and 0.9 solar mass objects, with the bluest objects being the most massive. 2nd panel: Total angular momentum as a function of MS lifetime for the same set of masses. The dotted line marks the location of the main sequence turnoff. 3rd panel: Total moment of inertia for each of the models. Bottom panel: surface rotation period as a function of time. All models are plotted at ages $t > 0.55$ Gyr for $[\text{Fe}/\text{H}] = -0.2$ with the rapidly rotating launch conditions.
Fig. 4.2.— Our theoretical predictions on a $T_{\text{eff}}$ vs. $\log(g)$ diagram. The top panel represent models launched with the rapidly rotating starting conditions, the middle the slow starting conditions, and the bottom the difference (at fixed $\log(g)$ and $T_{\text{eff}}$) between them. Color encodes the rotation period, with bluer colors representing longer rotation periods. Evolutionary tracks for benchmark masses are over plotted in black dotted lines. Solid black lines in the top panel show the launch conditions at 0.55 Gyr (“launch”) and terminal age main sequence (“TAMS”). Models are plotted for ages $0.55 < t < 10.0$ Gyr at $[Fe/H] = -0.2$. 
Fig. 4.3.— Rotation period as a function of ZAMS $T_{\text{eff}}$ at particular evolutionary states: initial rotation periods at 550 Myr (solid line), rotation period on the TAMS (dotted line), and period at the base of the RGB (dashed line) for ages $0.55 < t < 10.0$ Gyr.
Fig. 4.4.— Calibrated wind loss law over plotted on rotation data from the Pleiades (125 Myr) and M37 (550 Myr) (points), and $v \sin i$ data for the Hyades and Praesepe (not included in the fit, and plotted with a $4/\pi$ correction factor to statistically account for a range of inclinations, plotted as arrows.). Solid blue curves show the PMM loss law fit to the slowly rotating sequence at mass intervals denoted by large blue points (10th percentile averages with bootstrapped uncertainties). Red curves represent the loss law fit to the rapidly rotating sequence, and large red points the values to which the model was fit (90th percentile averages with bootstrapped uncertainties). The dotted green curve is for a similar fit using the modified Kawaler law, fit to the rapidly rotating sequence (red points).
Fig. 4.5.— Left panel: Theoretical rotation models on a log $g - T_{\text{eff}}$ diagram, overplotted with the M67 data from Canto Martins et al. (2011). Color encodes rotational velocity, with the reddest colors denoting rapid rotation, for both theoretical contours and observational data. Data $v \sin i$ values are multiplied by a factor of $4/\pi$ to account for average inclination effects. Isochrones for a 4.0 Gyr and 3.5 Gyr solar composition population are shown in solid black and gray curves, respectively. In the lower right is a typical errorbar for the cluster data. Right panel: Comparison of theoretical rotational isochrones (curves) and observational data (points, corrected by a factor of $4/\pi$). Black curves denote 4.0 Gyr isochrones for fast launch (solid), slow launch (dotted), and no SGB wind (dashed).
Fig. 4.6.— Left panel: Rotation velocity is encoded by color on our theoretical log(g)-
$T_{\text{eff}}$ diagram for solar metallicity. Overplotted are $v \sin i$ measurements for field stars
from Lèbre et al. (1999) (diamonds) and Mallik et al. (2003) (triangles), corrected
by $4/\pi$ to account for average inclination effects. Right panel: Rotation period in
days is encoded by color on a $T_{\text{eff}}$ vs. log(g) plot of our theoretical models at solar
composition. Over plotted are the values of $P_{\text{rot}}$ measured by do Nascimento et al.
(2012) with stellar parameters from Gazzano et al. (2010).
Fig. 4.7.— Age versus period for the upper envelope in rotation periods on the MS (top panel) and SGB (lower panel). Green curves represent objects with masses $M \geq 1.5M_\odot$, blue curves those with $1.1 < M/M_\odot < 1.5$, and red those curves with masses $M \leq 1.1M_\odot$. Solid black curves are lines of constant effective temperature in the period-age plane.
Fig. 4.8.— Age versus period for the upper envelope in rotation periods (top panel) and initial periods an order of magnitude slower (bottom panel). Green curves represent objects with masses $M \geq 1.5M_\odot$, blue curves those with $1.1 < M/M_\odot < 1.5$, and red those curves with masses $M \leq 1.1M_\odot$. Solid colored curves are objects on the SGB, dotted curves objects on the MS.
Fig. 4.9.— Radius versus period for the upper envelope in rotation periods on the MS (top panel) and SGB (lower panel). Green curves represent objects with masses $M \geq 1.5M_{\odot}$, blue curves those with $1.1 < M/M_{\odot} < 1.5$, and red those curves with masses $M \leq 1.1M_{\odot}$. Solid black curves are lines of constant effective temperature in the period-radius plane.
Fig. 4.10.— Left panel: $P_{\text{no wind}}/P_{\text{wind}}$ is the ratio of the periods for models without SGB winds to the standard case, which includes SGB wind losses. Mass is given by the color of the curves, with redder color representing lower mass models. Right panel: Period as a function of mass at both the MSTO and BRGB for the standard, fast launch models with SGB wind losses.
Fig. 4.11.— Top panel: The logged absolute value of the difference in period between models at solar metallicity and [Fe/H] = -0.2 at fixed combinations of $T_{\text{eff}}$ and log(g). All models were generated with identical starting conditions and AM evolution schemes. Bottom panel: the difference in age between metal-rich and metal-poor models at fixed log(g) and $T_{\text{eff}}$. 
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