MIXED-INTEGER PROGRAMMING METHODS FOR TRANSPORTATION AND POWER GENERATION PROBLEMS

DISSERTATION

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ABSTRACT

This dissertation conducts theoretical and computational research to solve challenging problems in application areas such as supply chain and power systems. The first part of the dissertation studies a transportation problem with market choice (TPMC) which is a variant of the classical transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. We show that TPMC is strongly $\mathcal{NP}$-complete. We consider a version of the problem with a service level constraint on the maximum number of markets that can be rejected and show that if the original problem is polynomial, its cardinality-constrained version is also polynomial. We propose valid inequalities for mixed-integer cover and knapsack sets with variable upper bound constraints, which appear as substructures of TPMC and use them in a branch-and-cut algorithm to solve this problem. The second part of this dissertation studies a unit commitment (UC) problem in which the goal is to minimize the operational cost of power generators over a time period subject to physical constraints while satisfying demand. We provide several exponential classes of multi-period ramping and multi-period variable upper bound inequalities. We prove the strength of these inequalities and describe polynomial-time separation algorithms. Computational results show the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the UC problem. The last part of this dissertation investigates the effects of uncertain wind power on the UC problem. A two-stage robust model and a three-stage stochastic program are compared.
To my family
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CHAPTER 1
INTRODUCTION

1.1 Background

Every day businesses are faced with complex decisions. In many cases to make these decisions, different types of quantitative tools are necessary and utilized. Some of these problems can be addressed by solving optimization problems. One can find optimization problems in any industry. The general goal is to find a solution that minimizes cost or maximizes some benefit. For example, one of the concerns in the power industry is how to satisfy the electricity demand by using which resource and when while minimizing cost. Another is how to proceed if there is an outage and/or a problem on the electricity transmission network. These problems and many more involve discrete decisions and become especially challenging as the size of the problems grow. Another example is the transportation problem, where the goal is to transport goods from point A to B. Many companies have supply chain systems that involve different versions of this basic problem. However, adding new constraints can make this problem very challenging.

When discrete decisions have to be made, many optimization problems become hard to solve. These discrete decisions can be modeled by binary $x \in \{0,1\}$ or integer $x \in \mathbb{Z}$ variables. In the context of power system problems, one has to decide when or if to start-up a power generator. In a transportation problem one may want to be able to select a set of demand locations instead of sending goods to all
possible locations. **Mixed-integer programming** is an optimization problem class that includes such discrete and continuous decisions. An optimization problem with only continuous variables is called a **linear programming** problem.

Another classification of optimization problems depends on the nature of the data. Problems that have parameters known with certainty are referred to as **deterministic** problems. If we consider the uncertain behavior of some parameters in an optimization problem then we refer to this problem as a **stochastic** problem. The class of optimization models that take into account uncertain behavior of parameters is called **stochastic programming**.

In the following sections, we briefly describe mixed-integer programming and stochastic programming methodologies which are used in this dissertation. We end the chapter with a summary of the problems and results obtained in this dissertation.

### 1.2 Mixed-Integer Programming

A **mixed-integer program** (MIP) can be written in the following format

$$
\min \{ey + hz : Fy + Gz \leq b, y \in \mathbb{Z}_+^n, z \in \mathbb{R}_+^p \},
$$

(1.1)

where $e$ and $h$ are row vectors of size $n$ and $p$, respectively. Matrices $F$ and $G$ have size $m \times n$ and $m \times p$, respectively. Vector $b$ is a column vector of size $m$. All input data is rational. If all variables are integer, then (1.1) is an **integer program** (IP). If all variables are $0-1$, then (1.1) is a **binary integer program** (BIP) or a **0-1 integer program**. Let $x := (y, z)$, $q = n + p$, $A = (F, G)$, $c^1 = (e, h)$, and $S := \{x \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax \leq b\}$.  

An optimization problem $\min \{c^2 x : x \in T\}$, where $c^2 \in \mathbb{R}^q$, $T \in \mathbb{R}^q$, is a **relaxation** of problem (1.1) if $S \subseteq T$ and $c^2 x \leq c^1 x$ for all $x \in S$. One widely used relaxation is the so called **linear programming** (LP) relaxation where the integrality of variables
are relaxed in the MIP problem (1.1), i.e. if we replace $y \in \mathbb{Z}_+^n$ with $y \in \mathbb{R}_+^n$, then we obtain the LP relaxation of (1.1).

Although in some special cases MIP problems can be polynomially solvable, in general they are known to be \textit{NP}-complete. In other words, unless $\mathcal{P} = \mathcal{NP}$ most MIP problems cannot be solved in polynomial time. However, in practice the \textit{branch-and-cut} algorithm is very successful in solving MIP problems. This algorithm combines the advantages of the \textit{cutting plane} and the \textit{branch-and-bound} algorithms. The LP relaxation of a problem is used as a starting point for all three algorithms. Furthermore, all the algorithms repeatedly solve LP problems while introducing new constraints. LP problems are much easier to solve compared to solving MIP problems. In fact LP problems are polynomially solvable [46]. Next, we describe the common methods used in solving MIP problems.

1.2.1 Branch-and-Bound Algorithm

The \textit{branch-and-bound} method was developed by Land and Doig [50] in 1960. To solve a MIP problem with this method, one initially solves the LP relaxation of the given MIP problem. According to the solution of the LP relaxation, the method starts its branching process. For a given solution $\bar{x}$, if $\bar{x} \not\in \mathbb{Z}_+^n \times \mathbb{R}_+^p$ then the method creates two nodes that divide the mixed-integer feasible set with constraints $x_i \leq \lfloor \bar{x}_i \rfloor$ and $x_i \geq \lceil \bar{x}_i \rceil$, $i = 1, \ldots, n$. A node is pruned under three conditions: (i) a feasible solution is found, (ii) problem is infeasible and (iii) the lower bound obtained from the node exceeds the best available upper bound. If a node is not pruned then new nodes are created according to the fractional solution at the current active node and the branching continues. The algorithm ends when all the nodes are explored.
1.2.2 Cutting Plane Algorithm

A cutting plane algorithm is similar to the branch-and-bound algorithm in that it starts with solving the LP relaxation of the given MIP problem and repeatedly solves LP problems until it finds the optimal solution. However, instead of branching and creating nodes at each step the algorithm adds valid inequalities to cut off infeasible solutions. A valid inequality for a given MIP problem (1.1) is an inequality in the form of $\pi x \leq \pi_0$ and is satisfied by all $x \in S$. Before using this algorithm, one needs to find valid inequalities to add to the LP relaxation. Finding a good class of valid inequalities that take into consideration the structure of the specific problem at hand is more of an art than a scientific procedure that has defined steps. Once a class of valid inequalities is available the next step is to find the most violated inequality (if any exists). This is called the separation problem. If a class of valid inequalities is adequate to find the optimal solution of set $S$ for any objective function vector $c \in \mathbb{R}^q$, then this class is enough to describe the convex hull of $S$ (i.e., $\text{conv}(S)$). The valid inequalities that make up the convex hull are called facet-defining inequalities of $\text{conv}(S)$. However, for most MIP problems, it is not possible to find the valid inequalities that define the convex hull.

1.2.3 Branch-and-Cut Algorithm

The branch-and-cut algorithm is a hybrid of the branch-and-bound and the cutting plane algorithms. This hybrid method is widely used in the literature to solve MIP problems. Both the branch-and-bound and the cutting plane algorithms have drawbacks. For example, the branch-and-bound algorithm might create a large branch-and-bound tree and the solution time can suffer due to solving too many LP problems. Similarly, the cutting plane algorithm will most likely add exponentially many valid inequalities and hence slow down the algorithm. In the branch-and-cut algorithm
the cutting plane algorithm is called throughout the branch-and-bound tree. One can decide to call the cutting plane algorithm at every node or for example for the first \(c\) many nodes. In Chapters 2 and 3, we propose branch-and-cut algorithms with exponential classes of valid inequalities defined for the respective problems.

1.3 Stochastic Programming

MIP problems are powerful compared to LP problems because they allow for discrete decisions. However, in the real-world very few problems merely include data known with certainty. To model most problems as realistically as possible often times one needs to consider the stochastic nature of data. In the last part of this dissertation, we take into account the stochastic nature of wind power as a parameter to a power system problem. In this section, we briefly review stochastic programming which is a technique to address uncertainty in optimization problems. For more details we refer the reader to Birge and Louveaux [17] and Kall and Wallace [44].

In a two-stage stochastic program, the stages correspond to the different types of decisions made in the problem. The first-stage decisions usually involve strategic decisions and do not depend on the uncertainty of data. The second-stage decisions depend both on the first-stage decisions and the uncertain data. We typically represent uncertain data with possible scenarios whose probabilities sum to 1. For example in the power industry the decision of when to turn on/off conventional generators for a given time horizon (usually 24 hours) is made a day before the actual electricity dispatch begins. Most conventional generators cannot be started up at a moment’s notice and usually require several hours to start-up thus forcing the turn on/off decisions to be made much earlier than when the demand and/or wind power is realized. Both demand and wind can be categorized as uncertain parameters in a power system setting. In the second stage of the problem the dispatch decisions are
made according to different realizations (also called a \textit{scenario}) of demand and/or wind power.

The classical \textit{two-stage stochastic program} with fixed recourse (originated by Dantzig \cite{28} and Beale \cite{9}) is given below,

\begin{equation}
\min \quad c^\top x + E_\xi[\min \omega q(\omega)^\top y(\omega)] \tag{1.2}
\end{equation}

\begin{equation}
s.t. \quad Ax = b, \tag{1.3}
\end{equation}

\begin{equation}
T(\omega)x + Wy(\omega) = h(\omega), \tag{1.4}
\end{equation}

\begin{equation}
x \geq 0, y(\omega) \geq 0. \tag{1.5}
\end{equation}

The first-stage decisions are represented by $x \in \mathbb{R}^{n_1}_+$. The first-stage parameters are given by vectors and matrices $c, b$ and $A$ with sizes $n_1 \times 1$, $m_1 \times 1$ and $m_1 \times n_1$, respectively. For a given scenario $\omega \in \Omega$ the second-stage decisions are represented by $y(\omega) \in \mathbb{R}^{n_2}_+$. The second-stage parameters are given by $q(\omega), h(\omega)$ and $T(\omega)$ with sizes $n_2 \times 1$, $m_2 \times 1$ and $m_2 \times n_2$, respectively. Note that vector $\xi(\omega)$ represents all the uncertain parameters for a given scenario $\omega$ and $\xi$ includes the data for all possible realizations. For a given scenario $\omega \in \Omega$, let

\begin{equation}
Q(x, \xi(\omega)) = \min_y \{q(w)^\top y | Wy = h(\omega) - T(\omega)x, y \geq 0 \} \tag{1.6}
\end{equation}

be the second stage value function. Then the expected second-stage value function is $Q(x) = E_\xi[Q(x, \xi(\omega))]$. 
We assume a finite and discrete distribution for the uncertain data in the second-stage. Let \( \text{Prob}_\omega \) be the probability of scenario \( \omega \) and \( \sum_{\omega \in \Omega} \text{Prob}_\omega = 1 \). The deterministic equivalent problem (DEP) for (1.2)-(1.5) is given below:

\[
\min \quad c^\top x + \sum_{\omega \in \Omega} \text{Prob}_\omega q(\omega)^\top y(\omega)
\]

\[
\text{s.t.} \quad Ax = b,
\]

\[
T(\omega)x + Wy(\omega) = h(\omega), \quad \omega \in \Omega,
\]

\[
x \geq 0,
\]

\[
y(\omega) \geq 0, \quad \omega \in \Omega.
\]

The DEP is essentially a large-scale LP or MIP problem that considers all possible realizations \( \omega \in \Omega \) of the second-stage problem. The size of the DEP's usually grows extremely fast due to the large number of scenarios that are considered in most problems. Furthermore, the first- and/or second-stage decisions may include integral decisions making the DEP problem even harder to solve.

Next we explain Benders’ decomposition algorithm \([12]\), which is used in Chapter 4 for a two-stage stochastic problem with binary first-stage and continuous second-stage variables. Consider the following relaxation of the first-stage problem.

\[
\min \quad c^\top x + \theta \quad \quad (1.7)
\]

\[
\text{s.t.} \quad Ax = b, \quad \quad (1.8)
\]

\[
x \in \{0, 1\}^n, \theta \geq 0. \quad \quad (1.9)
\]
Consider the dual of (1.6) for any second-stage problem \( \omega \in \Omega \) given below.

\[
\begin{align*}
\max & \quad \pi(\omega)\top (h(\omega) - T(\omega)x) \\
\text{s.t.} & \quad \pi(\omega)\top W \leq q(w).
\end{align*}
\] (1.10) (1.11)

Benders’ decomposition algorithm starts by solving the relaxation of the first-stage problem (1.7)-(1.9). Once the first-stage problem is solved we update all the second-stage problems with the current solution \( \bar{x} \in \{0,1\}^{n_1} \) and solve them. Note that the only updates necessary for the dual problems (1.10)-(1.11) is the objective function. If the dual problem is unbounded then a feasibility cut is added to the first-stage problem (commonly referred to as the master problem). The feasibility cut is

\[
\pi(\omega)\top (h(\omega) - T(\omega)x) \leq 0.
\] (1.12)

If the dual problem has an optimal solution, then one can add an optimality cut for each optimal second-stage problem or add an aggregated cut that considers respective probabilities of these problems. In Chapter 4, we add an aggregated optimality cut to the master problem once all second-stage subproblems have an optimal solution. The aggregated optimality cut is

\[
\sum_{\omega \in \Omega} \text{Prob}_{\omega}(\pi(\omega)\top (h(\omega) - T(\omega)x)) \leq \theta.
\] (1.13)

Note that if one wants to add an optimality cut for each second-stage subproblem, then instead of having a single \( \theta \) in the master problem (1.7)-(1.9) there has to be a \( \theta_\omega \) for each scenario \( \omega \in \Omega \) and the objective function has to include the term

\[
\sum_{\omega \in \Omega} \text{Prob}_{\omega}\theta_\omega.
\]

Thus, in each iteration \( k \), the relaxed master problem is solved and an optimal solution \( \bar{x}^k \) is obtained. Then each second-stage problem is solved given the current
master problem solution \( \bar{x}^k \) and these second-stage problems add feasibility or optimality cuts to the relaxed master problem. This process continues until no more cuts can be added to the relaxed master problem. The pseudo-code for the two-stage Benders’ decomposition algorithm is given next.

**Step 1:** Solve the master problem and obtain the optimal solution \( \bar{x} \).

(The initial master problem is given by (1.7)-(1.9).)

**Step 2:** Initialize \( \text{counter} = 0 \).

**Step 3:** For each second-stage subproblem \( \omega \in \Omega \)

update the dual objective function of (1.10) according to \( \bar{x} \).

**Step 4:** Solve the dual (1.10)-(1.11) of each second-stage problem \( \omega \in \Omega \).

**Step 4.1:** If the dual is unbounded

add a feasibility cut (i.e. (1.12)) to the master problem.

**Step 4.2:** Else if the dual has an optimal solution

increase \( \text{counter} \) by 1.

**Step 4.3:** Else (i.e. the tested problem is infeasible). Go to Step 7.

**Step 5:** If \( \text{counter} = |\Omega| \)

**Step 5.1:** If \( \theta < \sum_{\omega \in \Omega} \text{Prob}(\omega)(\pi(\omega)^\top(h(\omega) - T(\omega)x)) \)

add an optimality cut (i.e. (1.13)) to the master problem.

Go to Step 1.

**Step 5.2:** Else go to Step 7.

**Step 6:** Else (i.e. \( \text{counter} < |\Omega| \)) go to Step 1.

**Step 7:** Terminate.

A *three-stage stochastic program* is very similar to a *two-stage stochastic program*. The interpretation of the first- and the second-stage variables remain the same. The third-stage variables depend on the history up to the second-stage. A general form three-stage stochastic program is
\begin{align*}
\min \ c^1 x^1 + E_{\xi^2} \left[ \min \ c^2(\omega^2)x^2(\omega^2) + E_{\xi^3} \left[ \min \ c^3(\omega^3)x^3(\omega^3) \right] \right] \\
\text{s.t.} \quad W^1 x^1 & = h^1, \\
T^1(\omega^2)x^1 + W^2x^2(\omega^2) & = h^2(\omega^2), \\
T^2(\omega^3)x^2(\omega^2) + W^3x^3(\omega^3) & = h^3(\omega^3), \\
x^1 & \geq 0, x^2(\omega^2), x^3(\omega^3) \geq 0
\end{align*}

where matrix \(W^1\) with size \(m_1 \times n_1\), vectors \(c^1 \in \mathbb{R}^{n_1}\) and \(h^1 \in \mathbb{R}^{m_1}\) are known with certainty. The first-stage decisions are represented by \(x^1\). The second- and third-stage decisions for given scenarios \(\omega^2\) and \(\omega^3\) are represented by \(x^2(\omega^2)\) and \(x^3(\omega^3)\), respectively. Vectors \(\xi^2(\omega^2)\) and \(\xi^3(\omega^3)\) represent the uncertain parameters for given scenarios \(x^2(\omega^2)\) and \(x^3(\omega^3)\), respectively. Furthermore, \(\xi^2\) and \(\xi^3\) include the data for all possible realizations.

One can further generalize a stochastic program with more than two stages and refer to it as a \textit{multi-stage stochastic program}. Benders’ method can be modified to solve multi-stage stochastic programs using a nested structure (we refer the reader to Chapter 6.1 in \cite{Birge1997} by Birge and Louveaux).

\subsection*{1.4 Research Scope and Outline}

The first part of the dissertation studies a transportation problem with market choice (TPMC) which is a variant of the classical transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. While the classical transportation problem is known to be strongly polynomial-time solvable, we show that its market choice counterpart is strongly \(\mathcal{NP}\)-complete. For the special case when all potential demands are no greater than two, the problem reduces in
polynomial time to minimum weight perfect matching in a general graph, and thus can be solved in polynomial time. We also consider a version of the problem with a service level constraint on the maximum number of markets that can be rejected. We show that for the case in which the original problem is polynomial, its cardinality-constrained version is also polynomial. In this case, adding the cardinality constraint to the convex hull of solutions to the original problem does not create any new fractional extreme points. We present valid inequalities for mixed integer cover sets and mixed-integer knapsack sets with variable upper bound constraints, which appear as substructures of TPMC. We use the proposed valid inequalities in a branch-and-cut algorithm to solve this problem.

The second part of this dissertation studies a unit commitment (UC) problem in which the goal is to minimize the operational cost of power generators over a time period subject to physical constraints while satisfying customer demand. In particular, a relaxation of the UC problem with ramping constraints and production limits is considered. Ramping constraints are used to control the change in production level for a generator from one time period to the next. We provide several exponential classes of multi-period ramping and multi-period variable upper bound inequalities. We prove the strength of these inequalities and describe polynomial-time separation algorithms. Our computational results show the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the UC problem with ramping constraints. Furthermore, we present the first complete description of the convex hull of feasible solutions for the two-period ramping relaxation. The two-period inequalities that we describe can be readily used to strengthen ramping formulations without the need for separation. Also we show that optimization of a linear function over the ramping relaxation is polynomial.

One of the challenges of the UC problem is that in practice it is affected by uncertainties. The last part of this dissertation investigates the effects of uncertain
wind power on the UC problem. In addition to the conventional generators that were studied in the second part of this dissertation, wind farms are introduced to the UC problem. In Chapter 4, we compare two different models, namely, a two-stage robust model and a three-stage stochastic program. For both of the proposed models, in the first stage, the unit commitment decisions for a given time horizon for conventional generators are made. These decisions are generally made a day before the actual electricity dispatch begins. Most conventional generators cannot be started up at a moment’s notice and usually require several hours to start-up thus forcing the turn on/off decisions to be made much earlier than when the demand and/or wind power is realized. Due to the nature of wind, wind production levels cannot be scheduled. Unlike conventional generators that use some type of fuel (such as coal) that can be controlled the “fuel” for wind turbines cannot be controlled or easily stored. Thus a good forecast of wind power that ensures the adequate dispatch of energy with a minimum cost is critical.

Wind power forecasting is classified into three categories in terms of time horizon; very-short-term, short-term and medium-term. The longest time horizon is medium-term and can cover up to a week. This time horizon is useful in planning maintenance schedules for wind farms, conventional generators and network lines. The very-short-term and short-term categories can cover up to 9 hours and 72 hours ahead, respectively. Both categories are used in stochastic UC problems that consider uncertain wind. We refer the reader to [58] for more details on wind power forecasting and its relation to power generation problems.

Predicting wind behavior for the not so near future is a challenging task and the reason we propose a model with more than two-stages. For example, a study from Germany suggests that wind power prediction margin error is 9% for 24 hours ahead, 8% for eight hours ahead and 2% for one hour ahead [49]. Suppose that the wind prediction can be made reliably for $n_1$ periods. We divide a given time horizon into
two. Let the length of the time horizon be $n$ and the division be as follows: $[1, n_1]$ and $[n_1 + 1, n]$. At time period 1 wind power predictions for time periods 1 through $n_1$ are used. We call this the second-stage problem. At time period $n_1 + 1$ wind power predictions for time periods $n_1 + 1$ through $n$ are used. We refer to this as the third-stage problem. By this way a model with more than two-stages can use more realistic wind power forecasts. Our stochastic model and method generalizes to any number of stages for the power generation problem, but we focus on a three-stage model.

In a traditional two-stage stochastic program for UC the second stage determines the real-time economic dispatch decisions for the online conventional generators and wind farms for time periods 1 through $n$ \[88, 92\]. Note that earlier two-stage stochastic programs used to solve UC are similar in structure but consider uncertain load in contrast to uncertain energy sources \[18\]. However, because of its high volatility, wind power predictions made at time period 1 for time periods 1 through $n$ are not realistic for large $n$. Therefore, we propose a two-stage robust model in which the decisions for time periods $[1, n_1]$ mimic a traditional two-stage stochastic program and the decisions for time periods $[n_1 + 1, n]$ are made using a robust approach. This is because in the second stage of the problem we have reliable knowledge of wind power scenarios for time periods $[1, n_1]$ but not $[n_1 + 1, n]$.

In the three-stage stochastic program both the second and third stages of the problem determine the real-time economic dispatch decisions for the online conventional generators and wind farms. The difference between the second and the third stages of the problem is the wind power forecast and the time periods that are considered. The second stage of the problem makes dispatch decisions for time periods $[1, n_1]$ and the third stage of the problem makes dispatch decisions for time periods $[n_1 + 1, n]$. In contrast to the two-stage robust model the three-stage stochastic program can make use of the wind power knowledge that is introduced at time period $n_1 + 1$. Thus we
study the benefits of multi-stage formulations on a stochastic UC problem that considers an intermittent energy source. We use two solution methods to solve both the two-stage robust model and the three-stage stochastic program. One solution method is the basic MIP formulation of the models referred to as the deterministic equivalent models and the other solution method is the Benders’ decomposition method.

The remainder of this dissertation is organized as follows. In Section 2, we study a transportation problem with market choice. In Section 3, we study the ramping relaxation of the unit commitment problem. In Section 4, we compare and solve two different models for the unit commitment problem with uncertain wind power. One is a two-stage robust model and the other is a three-stage stochastic program. Finally, in Section 5, we conclude this dissertation and present ideas for future work.
CHAPTER 2
ON THE TRANSPORTATION PROBLEM WITH MARKET CHOICE

2.1 Introduction

We consider a variant of the classical transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. In this problem, if a market is selected its demand must be satisfied fully through shipments from the suppliers. If a market is rejected, then the corresponding potential revenue is lost. The objective is to minimize the total cost of shipping and lost revenues. We refer to this problem as the transportation problem with market choice (TPMC).

More formally, we are given a set of supply and demand nodes that form a bipartite graph $G(V_1 \cup V_2, E)$. The nodes in set $V_1$ represent the supply nodes, where for $i \in V_1$, $s_i \in \mathbb{N}$ represents the capacity of supplier $i$. The nodes in set $V_2$ represent the potential markets, where for $j \in V_2$, $d_j \in \mathbb{N}$ represents the demand of market $j$. The edges between supply and demand nodes have weights that represent shipping costs $w_{ij}$, where $(i, j) \in E$. For each $j \in V_2$, $r_j$ is the revenue lost if the market $j$ is rejected. For a given vector of parameters $\gamma_j$ for $j \in S$ and $S' \subseteq S$, we let $\gamma(S') := \sum_{j \in S'} \gamma_j$, throughout the chapter.

Let $x_{ij}$ be the amount of demand of market $j$ satisfied by supplier $i$ for $(i, j) \in E$, and let $z_j$ be an indicator variable taking a value 1 if market $j$ is rejected and 0
otherwise. A mixed-integer programming (MIP) formulation of the problem is given where the objective is to minimize the transportation costs and the lost revenues due to unchosen markets:

\[
\begin{align*}
\text{min} & \quad \sum_{(i,j)\in E} w_{ij} x_{ij} + \sum_{j\in V_2} r_j z_j \\
\text{s.t.} & \quad \sum_{i:(i,j)\in E} x_{ij} = d_j (1 - z_j) \quad \forall j \in V_2 \tag{2.1b} \\
& \quad \sum_{j:(i,j)\in E} x_{ij} \leq s_i \quad \forall i \in V_1 \tag{2.1c} \\
& \quad z \in \{0,1\}^{|V_2|} \tag{2.1d} \\
& \quad x \in \mathbb{R}_+^{|E|}. \tag{2.1e}
\end{align*}
\]

We refer to problem description (2.1a)-(2.1e) as TPMC. The first set of constraints (2.1b) is the demand constraint. In TPMC either a demand for a market is fully satisfied or rejected altogether, which necessitates the introduction of the additional binary variables. The second set of constraints (2.1c) model the supply restrictions.

TPMC is closely related to the capacitated facility location (CFL) problem. In CFL, given a set of potential facilities \( j \in V_2 \) with capacities \( \bar{d}_j, j \in V_2 \) and customers \( i \in V_1 \) with demands \( \bar{s}_i, i \in V_1 \), we would like to determine which facilities to open so that the demand of all customers can be satisfied from shipments from the open facilities. A MIP formulation of CFL is

\[
\begin{align*}
\sum_{i:(i,j)\in E} \bar{x}_{ij} & \leq \bar{d}_j \bar{z}_j \quad \forall j \in V_2 \tag{2.2a} \\
\sum_{j:(i,j)\in E} \bar{x}_{ij} & = \bar{s}_i \quad \forall i \in V_1 \tag{2.2b} \\
\bar{z} & \in \{0,1\}^{|V_2|} \tag{2.2c} \\
\bar{x} & \in \mathbb{R}_+^{|E|}. \tag{2.2d}
\end{align*}
\]
Therefore one may view the CFL problem as a ‘complement’ of the TPMC problem where the constraints (2.1b) and (2.1c) of TPMC change signs in the constraints (2.2a) and (2.2b) in CFL respectively. Note that there is no straightforward way of ‘complementing’ the variables of TPMC in order to construct an instance of CFL or vice versa. While the CFL problem has been extensively studied with respect to its complexity, polyhedral structure, and approximability ([2] [22] and references therein), TPMC is less understood.

Recently, approximation algorithms and heuristics have been proposed for various supply chain planning and logistics problems with market choice [38, 53]. It is assumed that these problems are uncapacitated or that they have soft capacities. A two-stage approach is utilized in solving these classes of problems that admit a facility location formulation. In the first stage, the problem is to determine a subset of markets and reject the others. In the second stage, the goal is to minimize the production cost and lost revenues due to unselected markets. In particular, for the uncapacitated lot-sizing problem, the facility location formulation is used to model the market choice counterpart. It is shown that the LP relaxation solution can be rounded in a way that guarantees a constant factor approximation algorithm. However, this algorithm relies on scaling continuous variables up, so it does not immediately generalize to our problem with hard capacity constraints (2.1c). Van den Heuvel et al. [90] consider a maximization version of the same problem and show that no constant factor approximation algorithm exists for this version, unless $\mathcal{P} = \mathcal{NP}$. The authors also give several polynomially solvable special cases, and test heuristics for the general case.

The rest of this chapter is organized as follows. In Section 2.2 we explore the complexity of TPMC. We show that while the classical transportation problem admits a strongly polynomial algorithm [48], its market choice counterpart is strongly $\mathcal{NP}$-complete. We also identify a polynomially solvable case when the demands of all potential markets are no more than two. In Section 2.3 we consider a version
of the problem with a service level constraint on the maximum number of markets that can be rejected. We show that for the case in which the original problem is polynomial, its cardinality-constrained version is also polynomial. Furthermore, in this case, we show that adding the cardinality constraint to the convex hull of solutions to the original problem does not create any new fractional extreme points. In Section 2.4 we present methods for constructing valid inequalities for mixed integer cover sets and mixed-integer knapsack sets with variable upper bound constraints, which appear as substructures of TPMC. We show that these methods are useful for generating valid inequalities for TPMC. We also study the strength of the proposed valid inequalities. Our preliminary computations, summarized in Section 2.5, show that there is a reduction in the root gap when our valid inequalities are incorporated to the branch-and-cut algorithm. The work in this chapter can also be found in [25].

2.2 Complexity

We first show that TPMC is strongly \( \mathcal{NP} \)-hard in general.

**Proposition 1.** The decision version of TPMC is \( \mathcal{NP} \)-complete even when:

1. \( s_i = 1 \) for all \( i \in V_1 \), \( d_j = d \geq 3 \) for all \( j \in V_2 \), \( w_{ij} = 0 \) for all \( (i,j) \in E \) and \( r_j = 1 \) for all \( j \in V_2 \).

2. \( |V_1| = 1 \) and \( w_{ij} = 0 \) for all \( (i,j) \in E \).

The proof for Proposition 1 Part 1 is similar to the proof of a related result presented in [73]. For completeness, we provide its proof and the proof of Part 2 in the Appendix. Because the reduction of Part 1 is from the Exact 3-Cover problem, which is strongly \( \mathcal{NP} \)-complete [36], we conclude that TPMC is strongly \( \mathcal{NP} \)-hard even for the case where all demands are equal to three. In contrast, Proposition
Proposition 2. Suppose that $d_j \leq 2$ for all $j \in V_2$. Then there exists a polynomial-time algorithm to solve TPMC.

This result is proven by a polynomial time reduction to a minimum weight perfect matching problem on a general graph (provided in the Appendix). The key ideas of the reduction are based on those presented in [4]. This result can also be proven by a polynomial time reduction to the $b$-matching problem [29], see also Theorem 36.1 in [80].

A matrix $A$ is said to have the Edmonds-Johnson property if the sum of the absolute values of the entries in any column of $A$ is less than or equal to 2. Edmonds and Johnson [29] show that the convex hull of integer solutions to a system $Ax \leq b$, where $A$ has this property is given by the so-called blossom inequalities. Note that the constraint matrix defined by inequalities (2.1b), (2.1c), (2.1e), and $z \in \mathbb{R}^{\lfloor |V_2|}+$ have the Edmonds-Johnson property when $d_j \leq 2$ for all $j \in V_2$. Hence adding the blossom inequalities to the original formulation is enough to give the convex hull of solutions to TPMC in this case. The blossom inequality for TPMC is

$$\sum_{i \in U_1, j \in U_2; (i,j) \in E} x_{ij} + \sum_{j \in U_2} \lfloor d_j/2 \rfloor z_j \leq \left\lfloor \frac{s(U_1) + d(U_2)}{2} \right\rfloor,$$

(2.3)

where $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ such that the sum of total supply in $U_1$ and total demand in $U_2$, $s(U_1) + d(U_2)$, is odd. The separation of blossom inequalities (2.3) is polynomial [39, 52, 70]. We propose other classes of valid inequalities for the general case in Section 2.4.
2.3 TPMC with a cardinality constraint

An important and natural constraint that one may add to the TPMC problem is that of a service level, i.e., the number of rejected markets is restricted to be at most $k$. This restriction can be modeled using a cardinality constraint, $\sum_{j \in V_2} z_j \leq k$, appended to (2.1a)-(2.1e). We call the resulting problem cardinality constrained TPMC (CCTPMC). If we are able to solve CCTPMC in polynomial-time, then we can solve TPMC in polynomial time by solving CCTPMC for all $k \in \{0, \ldots, |V_2|\}$. Therefore by Proposition 1, we obtain that CCTPMC is $\mathcal{NP}$-hard in general. In this section, we examine the specific case where we know that TPMC admits a polynomial-time algorithm.

In light of the proof of Proposition 2, via the reduction to a minimum weight perfect matching problem on a general (non-bipartite) graph $G' = (V', E')$, it is possible to reduce CCTPMC with $d_j \leq 2$ for all $j \in V_2$ to a minimum weight perfect matching problem with a cardinality constraint on a subset of edges (specifically the cardinality constraint is applied only on the edges $(j, j') \in E'$ for each $j \in V_2$; see proof of Proposition 2 in Appendix). To the best of our knowledge, the complexity status of minimum weight perfect matching problem on a general graph with a cardinality constraint on a subset of edges is open. This can be seen by observing that if one can solve minimum weight perfect matching problem with a cardinality constraint on a subset of edges, then one can solve the exact perfect matching problem; see discussion in the last section in [13]. On the other hand, we will prove in this section that CCTPMC with $d_j \leq 2$ for all $j \in V_2$, which is a special case of a minimum weight perfect matching problem with cardinality constraint on a specific subset of edges, in fact admits a polynomial-time algorithm. Our approach will be following: We will prove that the TPMC polytope (when $d_j \leq 2$ for all $j \in V_2$) along with the constraint $\sum_{j \in V_2} z_j \leq k$ is integral. Therefore by invoking the ellipsoid algorithm, it is
possible to solve CCTPMC in polynomial time. This result also allows for solving CCTPMC (when \(d_j \leq 2\) for all \(j \in V_2\)) by a Lagrangian relaxation approach, where we relax the cardinality constraint.

Before we proceed, we briefly note that the intersection of the perfect matching polytope with a cardinality constraint on a strict subset of edges is not always integral.

Example 1. Consider the bipartite graph \(G(V_1 \cup V_2, E)\) with \(V_1 = \{1, 2, 3\}, V_2 = \{4, 5, 6\}, E = \{(1, 4), (1, 5), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6)\},\) and the cardinality constraint \(x_{14} + x_{25} \leq 1\). It is straightforward to show that \(x_{14} = x_{15} = x_{24} = x_{25} = 0.5, x_{26} = x_{35} = 0, x_{36} = 1\) is a fractional extreme point of the intersection of the perfect matching polytope with the cardinality constraint.

Let \(X \in \mathbb{R}_{+}^{|E|} \times \{0, 1\}^{|V_2|}\) be the set of feasible solutions of TPMC. Our main result of this section is presented next.

Theorem 1. Let \(k \in \mathbb{Z}_+\) and \(k \leq |V_2|\). Let \(X^k := \text{conv}(X \cap \{(x, z) \in \mathbb{R}_+^{|E|} \times \{0, 1\}^{|V_2|} | \sum_{j \in V_2} z_j \leq k\}).\) If \(d_j \leq 2\) for all \(j \in V_2\), then \(X^k = \text{conv}(X) \cap \{(x, z) \in \mathbb{R}_+^{|E|} \times [0, 1]^{|V_2|} | \sum_{j \in V_2} z_j \leq k\}\).

Corollary 3. CCTPMC is polynomially solvable when \(d_j \leq 2\) for all \(j \in V_2\).

Observation 1. Theorem 1 is a generalization of the well-known result, Matching Cardinality Theorem: Let \(G(V, E)\) be a graph with \(n\) vertices and \(m\) edges. Let \(M \subset \mathbb{R}^m\) be the matching polytope and let \(M^k \subset \mathbb{R}^m\) be the convex hull of incidence vectors of matchings with at least \(k\) edges. Then \(M^k = M \cap \{x \in \mathbb{R}^m | \sum_{i=1}^m x_i \geq k\}\). (See \[80\] for a proof.)

We construct a bipartite graph \(\hat{G}(\hat{V}^1 \cup \hat{V}^2, \hat{E})\) as follows: \(\hat{V}^1\) is a set of \(n\) vertices corresponding to the \(n\) vertices in \(G\). \(\hat{V}^2\) corresponds to the set of edges of \(G\), i.e., \(\hat{V}^2\) contains \(m\) vertices. We use \((i, j)\) to refer to the vertex in \(\hat{V}^2\) corresponding to
the edge \((i, j)\) in \(E\). The set of edges in \(\hat{E}\) are of the form \((i, (i, j))\) and \((j, (i, j))\) for every \(i, j \in V\) such that \((i, j) \in E\). Now we can construct (the feasible region of) an instance of TPMC with respect to \(\hat{G}(\hat{V}^1 \cup \hat{V}^2, \hat{E})\) as follows:

\[
Q = \{(x, z) \in \mathbb{R}^{2m} \times \mathbb{R}^m \mid x_{i,(i,j)} + x_{j,(i,j)} + 2z_{(i,j)} = 2 \quad \forall (i, j) \in \hat{V}^2 \quad (2.4) \}
\]

\[
\sum_{j: (i,j) \in E} x_{i,(i,j)} \leq 1 \quad \forall i \in \hat{V}^1 \quad (2.5) \]

\[
z_{(i,j)} \in \{0, 1\} \quad \forall (i, j) \in \hat{V}^2. \quad (2.6) \]

We can construct an instance of CCTPMC by adding the constraint \(\sum_{(i,j) \in E} z_{(i,j)} \leq k\) (call this set \(Q^k\)). It is straightforward to verify that the Matching Cardinality Theorem is equivalent to stating \(\text{conv}(Q^k) = \text{conv}(Q) \cap \{(x, z) \mid \sum_{(i,j) \in E} z_{(i,j)} \leq k\} \). Thus, the Matching Cardinality Theorem follows from Theorem 1 applied to the bipartite graph \(\hat{G}\).

Now note that the graph \(\hat{G}\) has a very special structure. In particular, the degree of every node in the second set of vertices \(\hat{V}^2\) is 2. On the other hand, Theorem 1 holds for a general instance of TPMC with \(d_j \leq 2\) for all \(j \in V_2\), i.e. in particular for instances corresponding to general bipartite graphs where the degree of the vertices can be more than 2 and the value of \(d_j\) can be either 1 or 2. \(\square\)

To prove Theorem 1, one approach could be to appeal to the reduction to minimum weight perfect matching problem and then use the well-known adjacency properties of the vertices of the perfect matching polytope. However, as illustrated in Example 1, the integrality result does not hold for the perfect matching polytope on a general graph with a cardinality constraint on any subset of edges. Therefore a generic approach considering the perfect matching polytope appears to be less fruitful. We use an alternative approach to prove this result. In particular, we apply a technique similar to that used in [3]. Consider the following desirable property:
**Definition 1** (Edge Property). Let $T \subseteq \mathbb{R}_+^p \times \{0, 1\}^n$ be some mixed integer set. We say that $T$ satisfies the edge property if for all $(w, r) \in \mathbb{R}^{p+n}$ such that \( \min\{(w^\top x + r^\top z) | (x, z) \in T\} \) is bounded and has at least two optimal solutions, \((x^1, z^1)\) and \((x^2, z^2)\) where \( \sum_{j=1}^n z_j^1 = k^1, \sum_{j=1}^n z_j^2 = k^2 \) and \( k^1 \leq k^2 - 2 \), then there is an optimal solution \((x^3, z^3)\) such that \( \sum_{j=1}^n z_j^3 = k^3 \) and \( k^1 < k^3 < k^2 \).

**Proposition 4.** Let $T \subseteq \mathbb{R}_+^p \times \{0, 1\}^n$ be a mixed integer set such that \( \text{conv}(T) \) is a pointed polyhedron and let \( T^k := \text{conv}(T \cap \{(x, z) \in \mathbb{R}_+^p \times \{0, 1\}^n | \sum_{j=1}^n z_j \leq k\}) \). If $T$ satisfies the edge property, then \( T^k = \text{conv}(T) \cap \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j \leq k\} \).

**Proof.** Assume by contradiction that

\[
T^k \neq \text{conv}(T) \cap \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j \leq k\},
\]

for some \( k = k' \in \{0, 1, \ldots, n\} \). By definition \( T^k = \text{conv}(T \cap \{(x, z) \in \mathbb{R}_+^p \times \{0, 1\}^n | \sum_{j=1}^n z_j \leq k\}) \) so \( T^k \subseteq \text{conv}(T) \cap \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j \leq k\} \) holds for all \( k \in \{0, 1, \ldots, n\} \). By assumption we obtain \( T^{k'} \subseteq \text{conv}(T) \cap \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j \leq k'\} \). Since \( \text{conv}(T) \) is pointed this implies that there exists a vertex \((x', z')\) of \( \text{conv}(T) \cap \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j \leq k'\} \) such that \((x', z') \notin T^{k'} \).

Therefore \( z' \) is fractional and \( \sum_{j=1}^n z_j' = k' \) (if \( \sum_{j=1}^n z_j' < k' \), then this point is also a vertex of \( \text{conv}(T) \), therefore integral and belonging to \( T^{k'} \) - a contradiction).

Since \((x', z')\) is not a vertex of \( \text{conv}(T) \), there exists \((w, r)\) such that the vertex \((x', z')\) is the intersection of the face defined by \( \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j = k'\} \) and an edge of \( \text{conv}(T) \) defined as:
\{(x, z) \in \text{conv}(T) \mid w^\top x + r^\top z = \delta\}, \quad (2.7)

where \(\delta = \min\{w^\top x + r^\top z \mid (x, z) \in \text{conv}(T)\} = w^\top x' + r^\top z'. \) Let \((x^1, z^1)\) and \((x^2, z^2)\) be two feasible points of \(T\) that belong to the edge \((2.7)\) such that \((x', z')\) is a convex combination of \((x^1, z^1)\) and \((x^2, z^2)\). Note that \(\delta = w^\top x' + r^\top z' = w^\top x^1 + r^\top z^1 = w^\top x^2 + r^\top z^2\). Hence, \((x^1, z^1)\) and \((x^2, z^2)\) are two optimal solutions corresponding to the objective function \((w, r)\). Furthermore, due to our selection of \(\delta\), \(\sum_{j \in V_2} z^1_j < k' < \sum_{j \in V_2} z^2_j\). The edge property ensures that there exists an integral optimal solution \((x^3, z^3)\) with \(k^3 = \sum_{j \in V_2} z^3_j = k'\) such that \(\sum_{j \in V_2} z^1_j < k^3 < \sum_{j \in V_2} z^2_j\). However, this implies that \((x^3, z^3)\) belongs to the edge defined by \((2.7)\). Thus, \((x^3, z^3)\) must be a convex combination of \((x^1, z^1)\) and \((x^2, z^2)\) or equivalently, we must have \((x^3, z^3) = (x', z')\) with \(z'\) integral, a contradiction.

Now, we show how edge property and Proposition 4 can be applied to TPMC with an additional constraint that at most \(k\) markets can be rejected. To prove Theorem 1 we use Proposition 4. Similar to the argument in the proof of Proposition 2, we assume that all data are integral, and that \(s_i = 1\) for all \(i \in V_1\) without loss of generality. It is straightforward to verify that the polyhedron \(X\) corresponding to the original instance with \(s_i > 1\) for some \(i \in V_1\) satisfies the edge property if and only if \(X\) corresponding to the corresponding instance with \(s_i = 1\) for all \(i \in V_1\) satisfies the edge property. We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. By hypothesis \(d_j \leq 2\) for all \(j \in V_2\). From Proposition 4 it is sufficient to prove that the edge property holds.

Suppose that \((x^1, z^1)\) and \((x^2, z^2)\) are optimal solutions to \(\min\{w^\top x + r^\top z \mid (x, z) \in X\}\) and that \(x^1\) is fractional. Then we can solve a simple transportation problem with the set of demand nodes \(j\) such that \(z^1_j = 0\). Since all data is integral, there exists an
optimal solution with integral flows. Therefore, we may assume that $x^1$ (and similarly $x^2$) are integral.

**Claim 1.** Suppose we have two feasible solutions of $X$, namely $(x^3, z^3)$ and $(x^4, z^4)$, such that

1. $\sum_{j \in V_2} z_j^3 = k^1 + 1$ and $\sum_{j \in V_2} z_j^4 = k^2 - 1$ and

2. The objective function value of $(x^3, z^3)$ is $\rho - \delta$ and that of $(x^4, z^4)$ is $\rho + \delta$,

where $\rho$ is the objective function value of the solution $(x^1, z^1)$ and $\delta \in \mathbb{R}$,

then the proof of Theorem 1 is complete.

**Proof.** Since $\rho$ is the optimal objective function value, we obtain that $\delta = 0$ since otherwise the objective function value of either $(x^3, z^3)$ or $(x^4, z^4)$ is better than that of $(x^1, z^1)$. Therefore $(x^3, z^3)$ is an optimal solution with $k^1 < \sum_{j \in V_2} z_j^3 < k^2$. Because edge property is satisfied by Proposition 1, the proof of Theorem 1 is complete. \qed

Given an integral point $(\tilde{x}, \tilde{z})$ of $X$, let $S(\tilde{z}) := \{ j \in V_2 \mid \tilde{z}_j = 0 \}$ be the set of nodes in $V_2$ whose demands are met. For $j \in S(\tilde{z})$, let $I_j(\tilde{x}, \tilde{z}) = \{ i \in V_1 \mid \tilde{x}_{ij} > 0 \} = \{ i \in V_1 \mid \tilde{x}_{ij} = 1 \}$ be the set of suppliers that sends one unit to $j$.

Given the optimal solutions $(x^1, z^1)$ and $(x^2, z^2)$, let $F := (S(z^1) \setminus S(z^2)) \cup (S(z^2) \setminus S(z^1))$, $P := S(z^1) \cap S(z^2)$ and $R := V_2 \setminus (F \cup P)$. For $j \in F$, observe that only the set $I_j(x^1, z^1)$ or the set $I_j(x^2, z^2)$ is defined. So for $j \in F$, we define $I_j$ as:

$$I_j := \begin{cases} I_j(x^1, z^1) & \text{if } j \in S(z^1) \setminus S(z^2) \\ I_j(x^2, z^2) & \text{if } j \in S(z^2) \setminus S(z^1) \end{cases} \quad (2.8)$$
As a first step towards constructing \((x^3, y^3)\) and \((x^4, y^4)\) required in Claim 1, we construct a bipartite (conflict) graph \(G^*(U_1 \cup U_2, \mathcal{E})\). The set of nodes is constructed as follows:

1. If \(j \in S(z^1) \setminus S(z^2)\), then \(j \in U_1\) and \(j\) is called a full node. Let \(W_1 = S(z^1) \setminus S(z^2)\) be the set of full nodes of \(U_1\).

2. Similarly, if \(j \in S(z^2) \setminus S(z^1)\), then \(j \in U_2\) and \(j\) is called a full node. Let \(W_2 = S(z^2) \setminus S(z^1)\) be the set of full nodes of \(U_2\).

3. If \(j \in S(z^1) \cap S(z^2)\) and \(d_j = 2\) then we place two copies of node \(j\) in \(U_1\) (call these \(j_1\) and \(j_2\)) and two copies of \(j\) in \(U_2\) (call these \(j_3\) and \(j_4\)). These nodes are called partial nodes of \(j\). Each partial node of \(j\) is distinct: If \(I_j(x^1, y^1) = \{t_1, t_2\}\), then associate (WLOG) \(t_1\) with \(j_1\) and \(t_2\) with \(j_2\), that is define \(I_{j_1} := \{t_1\}\) and \(I_{j_2} := \{t_2\}\). Similarly if \(I_j(x^2, y^2) = \{t_3, t_4\}\), then associate (WLOG) \(t_3\) with \(j_3\) and \(t_4\) with \(j_4\), that is define \(I_{j_3} := \{t_3\}\) and \(I_{j_4} := \{t_4\}\). If \(j \in S(z^1) \cap S(z^2)\) and \(d_j = 1\), then we place one copy of \(j\) in \(U_1\) (call this \(j_1\)) and one copy of \(j\) in \(U_2\) (call this \(j_3\)). Similar to the \(d_j = 2\) case these nodes are called partial nodes of \(j\). If \(I_j(x^1, y^1) = \{t_1\}\) and \(I_j(x^2, y^2) = \{t_3\}\), then set \(I_{j_1} = \{t_1\}\) and \(I_{j_3} = \{t_3\}\). Let \(P = P^1 \cup P^2\), where \(P^1 = \{j \in P : d_j = 1\}\) and \(P^2 = \{j \in P : d_j = 2\}\).

Thus \(U_1 = W_1 \cup \left( \bigcup_{j \in P^2} \{j_1, j_2\} \right) \cup \left( \bigcup_{j \in P^1} \{j_1\} \right)\) and for each element \(a \in U_1\) the set \(I_a\) is well-defined and non-empty. Similarly, \(U_2 = W_2 \cup \left( \bigcup_{j \in P^2} \{j_3, j_4\} \right) \cup \left( \bigcup_{j \in P^1} \{j_3\} \right)\) and for each element \(b \in U_2\) the set \(I_b\) is well-defined and non-empty. Now we construct the edges \(\mathcal{E}\) as follows: For all \(a \in U_1\) and \(b \in U_2\), there is an edge \((a, b) \in \mathcal{E}\) if and only if \(a\) and \(b\) have at least one common supplier, i.e.,

\[ I_a \cap I_b \neq \emptyset \text{ iff } (a, b) \in \mathcal{E}. \]  

(2.9)
Let $G'(V', E')$ be a subgraph of $G^*(U_1 \cup U_2, \mathcal{E})$. Since the elements in $V' \cap (W_1 \cup W_2)$ correspond to unique elements in $V_2$, whenever required we will (with slight abuse of notation) treat $V' \cap (W_1 \cup W_2) \subseteq V_2$.

**Claim 2.** Let $G'(V', E')$ be a subgraph of $G^*(U_1 \cup U_2, \mathcal{E})$ satisfying the following properties:

1. There are no edges in $G^*$ between the nodes in $V'$ and the nodes in $(U_1 \cup U_2) \setminus V'$.

2. For each $j \in P^1$, $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ and for each $j \in P^2$, $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$.

3. $|W_1 \cap V'| = |W_2 \cap V'| + 1$.

Now construct

$$z^3_j = \begin{cases} 
  z^1_j & \text{if } j \in V_2 \setminus (V' \cap F) \\
  1 & \text{if } j \in V' \cap W_1 \\
  0 & \text{if } j \in V' \cap W_2.
\end{cases} \quad (2.10)$$

$$x^3_{ij} = \begin{cases} 
  1 & \text{if } j \in F, z^3_j = 0, i \in I_j \\
  1 & \text{if } j \in P, j_1 \in (U_1 \cup U_2) \setminus V', i \in I_{j_1} \\
  1 & \text{if } j \in P, j_2 \in (U_1 \cup U_2) \setminus V', i \in I_{j_2} \\
  1 & \text{if } j \in P, j_3 \in V', i \in I_{j_3} \\
  1 & \text{if } j \in P, j_4 \in V', i \in I_{j_4} \\
  0 & \text{otherwise.}
\end{cases} \quad (2.11)$$

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and

\[
    z_j^4 = \begin{cases} 
        z_j^2 & \text{if } j \in V_2 \setminus (V' \cap F) \\
        0 & \text{if } j \in V' \cap W_1 \\
        1 & \text{if } j \in V' \cap W_2, \\
        1 & \text{if } j \in F, z_j^4 = 0, i \in I_j \\
        1 & \text{if } j \in P, j_3 \in (U_1 \cup U_2) \setminus V', i \in I_{j_3} \\
        1 & \text{if } j \in P, j_4 \in (U_1 \cup U_2) \setminus V', i \in I_{j_4} \\
        1 & \text{if } j \in P, j_1 \in V', i \in I_{j_1} \\
        1 & \text{if } j \in P, j_2 \in V', i \in I_{j_2} \\
        0 & \text{otherwise.} 
    \end{cases}
\]

(2.12)

\[
    x_{ij}^4 = \begin{cases} 
        1 & \text{if } j \in F, z_j^4 = 0, i \in I_j \\
        1 & \text{if } j \in P, j_3 \in (U_1 \cup U_2) \setminus V', i \in I_{j_3} \\
        1 & \text{if } j \in P, j_4 \in (U_1 \cup U_2) \setminus V', i \in I_{j_4} \\
        1 & \text{if } j \in P, j_1 \in V', i \in I_{j_1} \\
        1 & \text{if } j \in P, j_2 \in V', i \in I_{j_2} \\
        0 & \text{otherwise.} 
    \end{cases}
\]

(2.13)

Then \((x^3, z^3)\) and \((x^4, z^4)\) are feasible solutions of \(X\) that satisfy the requirements of Claim 1.

**Proof.**

1. We verify that \((x^3, z^3)\) is a valid solution to \(X\). A similar proof can be given for the validity of \((x^4, z^4)\). Clearly \(x^3\) and \(z^3\) satisfy the variable restrictions. We verify that the constraint \(\sum_{i: (i,j) \in E} x_{ij}^3 + d_j z_j = d_j\) is satisfied for all \(j \in V_2\). If \(j \in R\), then \(z_j^3 = z_j^1 = 1\) and \(x_{ij}^3 = 0\) for all \((i, j) \in E\); therefore the constraint is satisfied. If \(j \in F\), then using the first and last entry in (2.11), we have \(\sum_{i: (i,j) \in E} x_{ij}^3 + d_j z_j^3 = d_j\). If \(j \in P\), then \(j \in V_2 \setminus (V' \cap F)\). Therefore \(z_j^3 = z_j^1 = 0\). Now it is straightforward to verify that \(\sum_{i: (i,j) \in E} x_{ij}^3 = 2 = d_j\) for each \(j \in P^2\) since \(|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|\) and by the use of the last five entries in (2.11). For \(j \in P^1\) we have \(\sum_{i: (i,j) \in E} x_{ij}^3 = 1 = d_j\) since \(|V' \cap \{j_1\}| = |V' \cap \{j_3\}|\) and by the use of the second, fourth and sixth entries in (2.11).

Now we verify that the constraint \(\sum_{j: (i,j) \in E} x_{ij} \leq 1\) is satisfied for all \(i \in V_1\). Given \(i \in V_1\), assume for contradiction that \(x_{ig}^3 = x_{ih}^3 = 1\) for some \(g, h \in V_2\) and \(g \neq h\).
By construction of \((x^3, z^3)\), \(x^3_{ij} = 0\) for all \(j \in R\). Thus, \(g, h \notin R\). Moreover, since \(\sum_{i:(i,j) \in E} x^3_{ij} + d_j z_j = d_j\) is satisfied for all \(j \in V_2\), we have \(z^3_g = z^3_h = 0\). Now, there are three cases to consider:

(a) \(g, h \in F\). By construction of \(x^3\) we have \(i \in I_g \cap I_h\). Now if \(g \notin V'\) and \(h \notin V'\), then by construction of \(z^3\) (first entry in (2.10)) we have \(z^1_g = z^3_g = 0 = z^3_h = z^1_h\) and thus \(g, h \in S(z^1)\). Therefore by the validity of \((x^1, z^1)\) we have \(I_g \cap I_h = \emptyset\). This contradicts \(i \in I_g \cap I_h\). Now consider the case where \(g \in V'\) and \(h \in V'\). Since \(i \in I_g \cap I_h\) by (2.9) there is an edge between \(g\) and \(h\) in \(G^*(U_1 \cup U_2, \mathcal{E})\). Thus we may assume without loss of generality that \(g \in V' \cap W_1\) and \(h \in V' \cap W_2\). However, this implies that \(z^3_g = 1\), a contradiction. Now, without loss of generality, assume that \(g \in V'\) and \(h \notin V'\). Since \(i \in I_g \cap I_h\) by (2.9) there is an edge between \(g\) and \(h\) in \(G^*(U_1 \cup U_2, \mathcal{E})\). On the other hand, by assumption there is no edge between nodes in \(V'\) and those not in \(V'\), which is the required contradiction.

(b) \(g \in F\) and \(h \in P\). Without loss of generality we may assume that \(g \in W_1\). If \(g \in V'\), then \(z^3_g = 1\), a contradiction. Therefore, we have \(g \notin V'\). Thus \(z^1_g = z^3_g = 0\). Therefore by validity of \((x^1, z^1)\) we have \(i \notin I_h(x^1, z^1)\) or equivalently \(i \in I_h(x^2, z^2)\). Without loss of generality we may assume that \(i \in I_{h_3}\). Note that \(h_3\) belongs to \(V'\) (by the construction of \(x^3\) and the fact that \(x^3_{ih} = 1\) and \(i \in I_{h_3}\)). Since \(i \in I_g\), there exists an edge between \(g\) and \(h_3\). However, since \(g \notin V'\) and \(h_3 \in V'\), we get a contradiction to the fact that there are no edges between the nodes in \(V'\) and the nodes in \((U_1 \cup U_2) \setminus V'\).

(c) \(g, h \in P\). In this case we may assume without loss of generality that \(i \in I_g(x^1, z^1)\) and \(i \in I_h(x^2, z^2)\). Therefore without loss of generality, we
may assume that $i \in I_{g_1}$ and $i \in I_{h_3}$. Since $x_{i g_1}^2 = x_{i h_3}^3 = 1$, we have $g_1 \notin V'$ and $h_3 \in V'$. By assumption on $G'$, this implies that there is no edge between $g_1$ and $h_3$. On the other hand, since $i \in I_{g_1} \cap I_{h_3}$ by (2.9) we have an edge $(g_1, h_3) \in E$, a contradiction.

2. Next we verify that the objective function value of $(x^3, z^3)$ is $\rho - \delta$ and that of $(x^4, z^4)$ is $\rho + \delta$ where $\rho$ is the objective function value of the solution $(x^1, z^1)$ and $\delta \in \mathbb{R}$. This result is verified by showing that $(x^3, z^3)$ and $(x^4, z^4)$ are obtained by ‘symmetrically’ updating demands from $(x^1, z^1)$ and $(x^2, z^2)$ respectively. In particular, we examine each demand node and examine the cost of either satisfying it or not satisfying it in each solution. We consider the different cases next:

(a) $j \in R$. Then $z_j^4 = z_j^3 = z_j^1 = z_j^2 = 1$.

(b) $j \in V' \cap W_1$. Then $z_j^1 = 0$ and $z_j^3 = 1$. On the other hand $z_j^2 = 1$ and $z_j^4 = 0$. Notice that in each solution where $d_j$ is satisfied, this is done by using the same set of input nodes (and thus using the same arcs). Therefore the difference in objective function value between $(x^1, z^1)$ and $(x^3, z^3)$ due to demand node $j$ is $-\sum_{i \in I_j} w_{ij} + r_j$ and the difference in objective function value between the solutions $(x^2, z^2)$ and $(x^4, z^4)$ due to demand node $j$ is $\sum_{i \in I_j} w_{ij} - r_j$.

(c) $j \in V' \cap W_2$. Similar to the above case the difference in objective function value between $(x^1, z^1)$ and $(x^3, z^3)$ due to demand node $j$ is $\sum_{i \in I_j} w_{ij} - r_j$ and the difference in objective function value between $(x^2, z^2)$ and $(x^4, z^4)$ due to demand node $j$ is $-\sum_{i \in I_j} w_{ij} + r_j$.

(d) $j \in F \setminus V'$, then $z_j^1 = z_j^3$ and $z_j^2 = z_j^4$.

(e) $j \in P^2$ such that $j_1, j_2 \in (U_1 \cup U_2) \setminus V'$ and $j_3, j_4 \in (U_1 \cup U_2) \setminus V'$. Then
the demand $d_j$ is satisfied by the nodes in $I_j(x^1, z^1)$ in $(x^1, z^1)$ and $(x^3, z^3)$. Therefore there is no difference in objective function value between $(x^1, z^1)$ and $(x^3, z^3)$ with respect to demand node $j$. Similarly, the demand $d_j$ is satisfied by the nodes in $I_j(x^2, z^2)$ in $(x^2, z^2)$ and $(x^4, z^4)$ and there is no difference in objective function value between $(x^2, z^2)$ and $(x^4, z^4)$ with respect to demand node $j$. We can make a similar argument for the cases: $j \in P^1$ such that $j_1 \in (U_1 \cup U_2) \setminus V'$ and $j_3 \in (U_1 \cup U_2) \setminus V'$, $j_4 \in V'$ without loss of generality. Then the demand $d_j$ is satisfied by the nodes in $(I_{j_1} \cup I_{j_2})$ in $(x^1, z^1)$ and by nodes $(I_{j_2} \cup I_{j_3})$ in $(x^3, z^3)$. Therefore the difference in objective function value between $(x^1, z^1)$ and $(x^3, z^3)$ with respect to demand node $d_j$ is $\sum_{i \in I_{j_1}} w_{ij} - \sum_{i \in I_{j_4}} w_{ij}$. The demand $d_j$ is satisfied by the nodes in $(I_{j_3} \cup I_{j_4})$ in $(x^2, z^2)$ and by the nodes in $(I_{j_1} \cup I_{j_3})$ in $(x^4, z^4)$. Therefore the difference in objective function value between $(x^2, z^2)$ and $(x^4, z^4)$ with respect to demand node $j$ is $\sum_{i \in I_{j_4}} w_{ij} - \sum_{i \in I_{j_1}} w_{ij}$. We can make a similar argument for the cases: $j_1 \in (U_1 \cup U_2) \setminus V'$, $j_2 \in V'$, $j_3 \in V'$, $j_4 \in (U_1 \cup U_2) \setminus V'$, $j_1 \in (U_1 \cup U_2) \setminus V'$, $j_2 \in (U_1 \cup U_2) \setminus V'$, $j_3 \in V'$, $j_4 \in (U_1 \cup U_2) \setminus V'$ and $j_1 \in (U_1 \cup U_2) \setminus V'$, $j_2 \in V'$, $j_3 \in (U_1 \cup U_2) \setminus V'$, $j_4 \in V'$.

(g) $j \in P^2$ such that $j_1 \in V'$, $j_2 \in V'$, $j_3 \in V'$, $j_4 \in V'$. Then the demand $d_j$ is satisfied by the nodes in $(I_{j_1} \cup I_{j_2})$ in $(x^1, z^1)$ and by the nodes in $(I_{j_3} \cup I_{j_4})$ in $(x^3, z^3)$. Therefore, the difference in the objective function value between $(x^1, z^1)$ and $(x^3, z^3)$ with respect to satisfying demand $d_j$ is $\sum_{i \in (I_{j_1} \cup I_{j_2})} (w_{ij} + w_{ij}) - \sum_{i \in (I_{j_4} \cup I_{j_3})} (w_{ij} + w_{ij})$. The demand $d_j$ is satisfied by the nodes in $(I_{j_3} \cup I_{j_4})$ in $(x^2, z^2)$ and by the nodes in $(I_{j_1} \cup I_{j_2})$ in $(x^4, z^4)$. Therefore, the difference in the objective function value between $(x^2, z^2)$
and \((x^4, z^4)\) with regards to satisfying demand \(d_j\) is 
\[
- \sum_{i \in (I_{j_1} \cup I_{j_2})} (w_{ij} + w_{ij}) + \sum_{i \in (I_{j_3} \cup I_{j_4})} (w_{ij} + w_{ij}).
\]
For \(j \in P^1\), we can similarly consider \(j_1\) and \(j_3\) with \(j_1 \in V', j_3 \in V'\).

Therefore, the objective function value of \((x^3, z^3)\) is \(\rho - \delta\) and that of \((x^4, z^4)\) is \(\rho + \delta\) where \(\rho\) is the objective function value of the solution \((x^1, z^1)\) and \((x^2, z^2)\) and \(\delta \in \mathbb{R}\).

3. Finally we verify that \(\sum_{j \in V_2} z^3_j = k^1 + 1\) and \(\sum_{j \in V_2} z^4_j = k^2 - 1\). We prove this for \((x^3, z^3)\). The proof is similar for the case of \((x^4, z^4)\). Observe that if \(j \in R\), then \(z^3_j = z^3_j = 1\). If \(j \in P\), then \(z^1_j = z^3_j = 0\). If \(j \in F \setminus V'\), then \(z^1_j = z^3_j\). If \(j \in W_1 \cap V'\), then \(z^1_j = 0\) and \(z^3_j = 1\) and if \(j \in W_2 \cap V'\), then \(z^1_j = 1\) and \(z^3_j = 0\). Thus \(\sum_{j \in V_2} z^1_j - \sum_{j \in V_2} z^3_j = |V' \cap W_2| - |V' \cap W_1| = -1\), where the last equality is by assumption (3) of \(G'\). Thus, \(\sum_{j \in V_2} z^3_j = k^1 + 1\).

Now the proof of Theorem 1 is complete by showing that a subgraph \(G'(V', E')\) of \(\hat{G}^*(U_1 \cup U_2, E)\) always exists that satisfies the conditions of Claim 2. In order to prove this, we verify a few results.

**Claim 3.** Connected components of \(\hat{G}^*\) are paths or cycles of even length and all the cycles involve only full nodes.

**Proof.** This is evident from the fact that \(\hat{G}^*\) is bipartite and degree of \(a \in (U_1 \cup U_2)\) is bounded from above by \(|I_a|\).

We associate a value \(v_j\) to each node \(j \in U_1 \cup U_2\). In particular:

1. If \(j \in W_1\), then \(v_j = 1\).

2. If \(j \in U_1\) and \(j\) is a partial node, then \(v_j = \frac{1}{2}\).
3. If $j \in U_2$ and $j$ is a partial node, then $v_j = -\frac{1}{2}$.

4. If $j \in W_2$, then $v_j = -1$.

For a subgraph $\tilde{G}(\tilde{V}, \tilde{E})$ of $G^*$ we call $v(\tilde{V}) = \sum_{j \in \tilde{V}} v_j$ the value of the path.

Claim 4. $v(U_1 \cup U_2) = k^2 - k^1 \geq 2$.

Proof. $\sum_{j \in U_1 \cup U_2} v_j = \sum_{j \in W_1} v_j + \sum_{j \in P_2} (v_{j_1} + v_{j_2}) + \sum_{j \in W_2} v_j + \sum_{j \in P_1} (v_{j_3} + v_{j_4}) + \sum_{j \in P_1} v_{j_1} = \vert S(z^1) \setminus S(z^2) \vert - \vert S(z^2) \setminus S(z^1) \vert = \vert S(z^1) \vert - \vert S(z^2) \vert = k^2 - k^1$. \hfill \Box

Claim 5. If $\tilde{G}(\tilde{V}, \tilde{E})$ is a cyclic subgraph of $G^*(U_1 \cup U_2, \mathcal{E})$, then $v(\tilde{V}) = 0$.

Proof. By Claim 3, a cycle has only full nodes. Moreover, since a cycle is of even length, it contains equal number of nodes from $W_1$ and $W_2$. \hfill \Box

Note that a partial node must be a leaf node in a path. Using this observation and by some simple case analysis the following three claims can be verified.

Claim 6. If $\tilde{G}(\tilde{V}, \tilde{E})$ is a path containing exactly one partial node, then $v(\tilde{V}) \in \{-\frac{1}{2}, \frac{1}{2}\}$.

Claim 7. If $\tilde{G}(\tilde{V}, \tilde{E})$ is a path containing two partial nodes, then $v(\tilde{V}) = 0$.

Claim 8. If $\tilde{G}(\tilde{V}, \tilde{E})$ is a path containing only full nodes, then $v(\tilde{V}) \in \{-1, 0, 1\}$.

For the subgraph $\tilde{G}(\tilde{V}, \tilde{E})$, consider a $k \in \tilde{V} \setminus F$ such that $k = j_t$ where $t \in \{1, 2, 3, 4\}$ and $j \in P^2$. Suppose $k = j_1$ or $j_2$, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a mirror path for $j$, if $\tilde{V}$ contains either $j_3$ or $j_4$. Moreover we call one of $j_3$ or $j_4$ (whichever belongs to $\tilde{V}$ or arbitrarily select one of these if both belong to $\tilde{V}$) as the mirror node. Similarly if $k = j_3$ or $j_4$, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a mirror path for $j$, if $\tilde{V}$ contains either $j_1$ or $j_2$. Mirror node is similarly defined in this case. For $j \in P^1$ we consider $k = j_1$ and $k = j_3$. Suppose $k = j_1$, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a mirror path
for $j$, if $\tilde{V}$ contains $j_3$ and we call $j_3$ the mirror node. Similarly if $k = j_3$, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a mirror path for $j$, if $\tilde{V}$ contains $j_1$ and we call $j_1$ the mirror node.

Algorithm constructs $G'(V', E')$ that satisfies all the properties of Claim 2. We next verify that Algorithm is well-defined, that is all the steps can be carried out. Moreover we show that the algorithm generates a subgraph $G'(V', E')$ that satisfies the conditions of Claim 2.

Claim 9. Algorithm is well-defined.

1. At the beginning of Step (3), the total value of all marked paths is 0.

2. Let $\hat{V} := \bigcup_{\tilde{G}(\tilde{V}, \tilde{E}) \text{ is marked before Step (1)}} \tilde{V}$. Then $|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}|$ for all $j \in P^1$.

3. Step (3) is well-defined, that is as long as the algorithm does not terminate, Step (3) can be carried out.

4. At the end of Step (3), the total value of all marked paths is $\frac{1}{2}$.

5. Step (4) is well-defined, that is as long as the algorithm does not terminate, Step (4) can be carried out.

Proof. We prove Claim 9 by induction on the iteration number ($n$) of the algorithm visiting Step (5). When $n = 0$:

1. At the beginning of Step (3) there are no ‘marked’ paths and therefore the total value of all marked paths is 0.

2. $\hat{V} = \emptyset$.

3. By Step (1), we know that there exists no path containing only full nodes with $v(\tilde{V}) = 1$. Moreover by Claim 4 we have $v(U_1 \cup U_2) \geq 2$. Since by Claim 5 all
Algorithm 1 Construction of $G'(V', E')$

**Input:** $G^*(U_1 \cup U_2, \mathcal{E})$.

**Output:** $G'(V', E')$ that satisfies all conditions of Claim 2.

1. If there exists a path $\tilde{G}(\tilde{V}, \tilde{E})$ in $G^*(U_1 \cup U_2, \mathcal{E})$ containing only full nodes with $v(\tilde{V}) = 1$, then set $G' := \tilde{G}$. STOP.

2. Tag all paths in $G^*(U_1 \cup U_2, \mathcal{E})$ as ‘unmarked.’

3. Select a path $\tilde{G}(\tilde{V}, \tilde{E})$ from the set of ‘unmarked’ paths containing a partial node such that $v(\tilde{V}) = \frac{1}{2}$. Tag this path as ‘marked.’ Note that by Claim 6 and Claim 7, $\tilde{V}$ contains a unique partial node $j^*$.

4. Select a path from the list of ‘unmarked’ paths, such that it is a mirror path for $j^*$. Tag this path as ‘marked.’

5. There are three cases:

   (a) The mirror path tagged as ‘marked’ in (4) contains a unique partial node and its value is $\frac{1}{2}$. GO TO Step 6.

   (b) The mirror path tagged as ‘marked’ in (4) contains a unique partial node and its value is $-\frac{1}{2}$. GO TO Step 3.

   (c) The mirror path tagged as ‘marked’ in (4) contains two partial nodes (then its value is 0): One of the partial nodes corresponds to the mirror node. Set $j^*$ to be the other partial node. GO TO Step 4.

6. Set $G'(V', E')$ to be disjoint union of the paths tagged as ‘marked.’ STOP.
cycles have a value of 0, there must exist at least one path with partial nodes with positive value. Since this is only possible (Claim 6 and Claim 7) if there exists exactly one partial node in the path, we see that Step (3) is well-defined.

4. At Step (3) one path is marked which has a value of half.

5. Since one path is tagged as marked in Step (3), it contains exactly one partial node, \( j^* \in P \). Suppose that \( j^* \in P^2 \) and \( j^* = j^*_i \) for some \( i \in \{1, \ldots, 4\} \). Then there exists paths (at least two) which contain the other three partial nodes corresponding to \( j^* \). If \( j^* \in P^1 \) then there exists one path which contains the other partial node. Therefore this step is well-defined.

Now for any \( n \in \mathbb{Z}_+ \), assuming by the induction hypothesis that the result is true for \( n' = 0, \ldots, n - 1 \):

1. Step (3) is arrived at via Step (5b). Let \( n' < n \) be the last iteration when Step (3) is invoked. By the induction hypothesis the total value of all the marked paths at the end of Step (3) in iteration \( n' \) is \( \frac{1}{2} \). From iterations \( n'+1, \ldots, n-1 \), the algorithm alternates between Step (4) and Step (5c). The total value of all the marked paths here is 0. Finally, the value of the last path tagged as marked in Step (4) is \( -\frac{1}{2} \) (since the algorithm invokes Step (5b)). Hence, the total value of all the marked paths is 0 at the beginning of Step (3) in iteration \( n \).

2. Let \( n' < n \) be the last iteration when Step (3) is invoked. By the induction hypothesis \( |\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}| \) for all \( j \in P^2 \) and \( |\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}| \) for all \( j \in P^1 \) where

\[
\hat{V} := \bigcup_{\tilde{G}(\tilde{V}, \tilde{E}) \text{ marked before Step (3) iteration } n'} \tilde{V}.
\]

From iterations \( n'+1, \ldots, n-1 \), the algorithm alternates between Step (4) and Step (5c). Since in iteration \( n-1 \) at Step (4), we add one path that contains only the mirror node to \( j^* \) (the unique partial node from the previous iteration), we arrive at this result.
3. Proof same as that in the case where \( n = 0 \).

4. The total value of paths at the end of Step (3) = value of marked path + total value of previously marked path = \( \frac{1}{2} + 0 \).

5. Step (4) is invoked after either Step (3) or Step (5c). In case we arrive via Step (3), by the induction hypothesis:
\[
|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}| \quad \text{for all } j \in P^2
\]
\[
|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}| \quad \text{for all } j \in P^1
\]
where \( \hat{V} := \bigcup_{\tilde{G}(\tilde{V}, \tilde{E}) \text{ marked before Step (3) iteration } n'} \tilde{V} \). Moreover, the path marked in step (3) contains exactly a unique partial node \( j^* \) then, there must exist an unmarked path containing a mirror node to \( j^* \). In case of we arrive via Step (5c), again the proof is essentially the same by observing that at the start of Step (4), there is a unique partial node \( j^* \) that is not paired with a mirror partial node.

\[
\square
\]

**Claim 10.** Algorithm 1 terminates in finite time.

*Proof.* This is true since there are a finite number of edges and at each iteration of the algorithm at least one unmarked path is tagged as marked.

\[
\square
\]

**Claim 11.** Algorithm 1 generates a subgraph \( G'(V', E') \) that satisfies the properties of Claim 2.

*Proof.* First observe that since the output \( G'(V', E') \) of the algorithm is a disjoint union of paths, there exists no edge between \( V' \) and \( (U_1 \cup U_2) \setminus V' \) in \( \mathcal{E} \), so property 1 is satisfied.
By Claim 9, we have $|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}|$ for all $j \in P^1$ where

$$\hat{V} := \bigcup_{G(\hat{V}, \hat{E}) \text{ is marked before Step 3}} \hat{V}.$$ 

Therefore, it is easily verified that in the last iteration before termination, a path with a unique partial node, which is a mirror node to $j^*$, is marked in Step 4. This is because before termination we arrive at Step (5) implying that the value of the path marked in Step (4) is $1/2$. Hence Claim 6 and Claim 7 imply that there is a unique partial node in this path. Thus, $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ for all $j \in P^1$, so property 2 is satisfied.

Finally, since $v(V') = 1$ and $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ for all $j \in P^1$ we have

$$\sum_{j \in V' \cap W_1} v_j + \sum_{j \in V' \cap W_2} v_j = 1.$$ 

As a result, $|V' \cap W_1| = |V' \cap W_2| + 1$, so property 3 is satisfied.

We showed that the set of solutions to TPMC satisfies the edge property. Theorem 1 then follows from Proposition 4.

Finally we ask the natural question: Does the edge property hold for TPMC when there exist demands that are greater than 2? The next example illustrates that the edge property can fail to hold even if $d_j > 2$ for only one $j \in V_2$.

**Example 2.** Consider an instance of TPMC where $G(V_1 \cup V_2, E)$ is a bipartite graph with $V_1 = \{1, 2, \ldots, 6\}$, $V_2 = \{1, 2, 3, 4\}$, $E = \{(1, 1), (2, 2), (3, 3), (4, 1), (4, 4), (5, 2), (5, 4), (6, 3), (6, 4)\}$, $s_i = 1$, $i \in V_1$, $d_j = 2$, $j = \{1, 2, 3\}$, $d_4 = 3$. For $k = 2$ we obtain a non-integer extreme point of $\text{conv}(T) \cap \{(x, z) \in \mathbb{R}_+^p \times [0, 1]^n | \sum_{j=1}^n z_j \leq k\}$, given by
\[ x_{11} = x_{22} = x_{33} = x_{41} = x_{44} = x_{52} = x_{54} = x_{63} = x_{64} = z_1 = z_2 = z_3 = z_4 = \frac{1}{2}. \]

Therefore, \( T^k \neq \text{conv}(T) \cap \{(x, z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j \leq k\} \) in this example. Next we show how the conflict graph construction fails for this example. In fact, it can be shown that the edge property is not satisfied in this example by using an alternative characterization defined in [3]. Let \( w_{11} = w_{22} = w_{33} = w_{41} = w_{44} = w_{52} = w_{54} = w_{63} = w_{64} = 1 \) and \( r_j = 3, j = \{1, 2, 3\} \) and \( r_4 = 6 \). For \( k = 0 \) the problem is infeasible. For \( k = 1 \), an optimal solution is \( x_{11} = x_{22} = x_{33} = x_{41} = x_{52} = x_{63} = z_4 = 1 \) and all other variables are zero, with an objective function value 12. For \( k = 3 \), an optimal solution is \( x_{44} = x_{54} = x_{64} = z_1 = z_2 = z_3 = 1 \) and all other variables are zero, with an objective function value 12. We show that Algorithm 1 fails to find a subgraph \( G'(V', E') \) of \( G^*(U_1 \cup U_2, \mathcal{E}) \) that satisfies the properties given in Claim 2 for this example. We use two feasible solutions, namely solution for \( k = 1 \) and \( k = 3 \) to build the bipartite graph given in Figure 2.1. Note that \( I_1 = \{1, 4\}, I_2 = \{2, 5\}, I_3 = \{3, 6\} \) and \( I_4 = \{4, 5, 6\} \). In Step 1 of Algorithm 1 we find a path with \( v(\bar{V}) = 1 \) which is \( 1 - 4 - 2 \) then the algorithm stops. We have \( V' = \{1, 4, 2\} \) and \( (U_1 \cup U_2) \setminus V' = \{3\} \). However, property 2 does not hold since there exists an edge between 3 and 4 but \( 3 \in (U_1 \cup U_2) \setminus V' \) and \( 4 \in V' \). Hence, Algorithm 1 fails.

![Figure 2.1: Bipartite Graph G*(U_1 \cup U_2, \mathcal{E}) for Example 2](image-url)
2.4 Valid Inequalities

In this section we give valid inequalities for TPMC and study their strength. First, observe that the variable upper bound inequalities (VUB) for \((i, j) \in E\)

\[ x_{ij} \leq \min\{s_i, d_j\}(1 - z_j) \]  \hspace{1cm} (2.14)

are valid for \(X\).

**Proposition 5.** Let \(I \subseteq V_1, J \subseteq V_2\) such that \(d(J) \geq s(V_1 \setminus I)\). The inequality

\[
\sum_{i \in I, j \in J: (i, j) \in E} x_{ij} + \sum_{j \in J} (\min\{d(J) - s(V_1 \setminus I), d_j\}) z_j \geq d(J) - s(V_1 \setminus I) \]  \hspace{1cm} (2.15)

is valid for \(X\).

**Proof.** Given a feasible solution \((x, z)\) we consider two cases.

1. If \(z_{j'} = 1\) for some \(j' \in J\) such that \(\min\{d(J) - s(V_1 \setminus I), d_{j'}\} = d(J) - s(V_1 \setminus I)\), then the feasible solution satisfies inequality (2.15) because we have

\[
\sum_{i \in I, j \in J: (i, j) \in E} x_{ij} + \sum_{j \in J} (\min\{d(J) - s(V_1 \setminus I), d_j\}) z_j \newline
= \sum_{i \in I, j \in J \setminus \{j'\}: (i, j) \in E} x_{ij} + \sum_{j \in J \setminus \{j'\}} (\min\{d(J) - s(V_1 \setminus I), d_j\}) z_j \newline
+ d(J) - s(V_1 \setminus I) \newline
\geq d(J) - s(V_1 \setminus I)
\]

where the last inequality holds because \(\min\{d(J) - s(V_1 \setminus I), d_j\} \geq 0\) for all \(j \in J\), and all \(x\) and \(z\) variables are non-negative.

2. If \(z_j = 0\) for all \(j \in J\) satisfying \(\min\{d(J) - s(V_1 \setminus I), d_j\} = d(J) - s(V_1 \setminus I)\), then \(\sum_{j \in J} (\min\{d(J) - s(V_1 \setminus I), d_j\}) z_j = \sum_{j \in J} d_j z_j\). Moreover, observe that
\[
\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + s(V_1 \setminus I) \text{ is at least as large as the total flow sent to the demand nodes in } J \text{ in the solution } (x, z), \text{ i.e., } \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + s(V_1 \setminus I) \geq \sum_{j \in J} d_j (1 - z_j).
\]

Therefore we have

\[
\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{j \in J} (\min \{d(J) - s(V_1 \setminus I), d_j\}) z_j + s(V_1 \setminus I) \\
\geq \sum_{j \in J} d_j z_j + \sum_{j \in J} d_j (1 - z_j) = d(J),
\]

so inequality (2.15) is valid.

Next, we give valid inequalities for general mixed-integer sets that are substructures of TPMC.

### 2.4.1 A Coefficient Update Scheme for Mixed-Integer Covers

Consider the mixed integer cover set \( S_1 \) defined by

\[
t + \sum_{j \in J} \beta_j z_j \geq \beta_0 \tag{2.16}
\]

\[
t \geq 0 \tag{2.17}
\]

\[
z_j \in \{0, 1\} \quad \forall j \in J, \tag{2.18}
\]

for given \( \beta_j \geq 0 \) for all \( j \in J \) and \( \beta_0 \geq 0 \). We assume that \( \beta_j \leq \beta_0 \) for all \( j \in J \) without loss of generality. Let \( T_1 = \text{conv}(S_1) \). We refer to inequalities in the form of (2.16) as type-I base inequalities. Note that inequalities (2.15) for TPMC are in the form of (2.16) since we can replace \( \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} \) by \( t \) and \( t \geq 0 \). Therefore, (2.16)-(2.18) is a relaxation of TPMC.
Proposition 6. Given a type-I base inequality (2.16) valid for a mixed-integer program (MIP) with (2.17)-(2.18), let $\tilde{J} := \{j_1, j_2, \ldots, j_p\} \subseteq J$ be a minimal cover, i.e., $\sum_{j \in \tilde{J}} \beta_j > \beta_0$ and $\sum_{j \in \tilde{J} \setminus \{j_k\}} \beta_j \leq \beta_0$ for all $k \in \{1, \ldots, p\}$. Let $\beta_{j_p} \geq \beta_{j_k}$ for all $k \in \{1, \ldots, p\}$. Let $J^* := \tilde{J} \cup \{j \in J : \beta_j \geq \beta_{j_p}\}$, $\beta = \sum_{j \in \tilde{J}} \beta_j - \beta_0$ and $\beta'_0 := \beta_0 - (p-1)\beta$. Then,

$$t + \sum_{j \in J^*} \min \left\{ (\beta_j - \beta), \beta'_0 \right\} z_j + \sum_{j \in J \setminus J^*} \min \left\{ \beta'_0, \beta_j \right\} z_j \geq \beta'_0,$$

(2.19)
is a valid inequality for $S_1$.

Proof. We first claim that $\beta_j \geq \beta$ for all $j \in J^*$. Suppose, without loss of generality, that $\beta_j_1 \leq \beta_j_2 \leq \cdots \leq \beta_j_p$, and recall that $\beta_j \geq \beta_{j_p}$ for all $j \in J^* \setminus \tilde{J}$. Assume by contradiction that $\beta_j_1 < \beta$ or equivalently $\beta_j_1 - \left( \sum_{k=1}^p \beta_{j_k} - \beta_0 \right) < 0$. This is a contradiction to the minimality of the cover $\tilde{J}$.

Next we claim that $\beta'_0 \geq 0$: By the previous claim, we have $\beta_k \leq \beta_{j_k}$ for $k = 1, \ldots, p$. Therefore, we obtain

$$\beta'_0 = \beta_0 - (p-1)\beta \geq \beta_0 - \sum_{k=1}^{p-1} \beta_{j_k} \geq 0,$$

where the last inequality follows from the fact that $\tilde{J}$ is a minimal cover.

Given a feasible solution $(x, z)$, let $J_1 = \{j \in J : z_j = 1\}$ and $J^*_1 = \{j \in J^* : z_j = 1\}$. Consider the following cases:

1. Suppose that there exists $j' \in J^*_1$ such that $\min \left\{ \beta'_0, \beta_{j'} - \beta \right\} = \beta'_0$. Then,

$$t + \sum_{j \in J^*} \min \left\{ (\beta_j - \beta), \beta'_0 \right\} z_j + \sum_{j \in J \setminus J^*} \min \left\{ \beta'_0, \beta_j \right\} z_j \geq t + \sum_{j \in J^* \setminus \{j'\}} \min \left\{ (\beta_j - \beta), \beta'_0 \right\} z_j + \sum_{j \in J \setminus J^*} \min \left\{ \beta'_0, \beta_j \right\} z_j + \beta'_0 \geq \beta'_0,$$

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where the last inequality follows from the fact that all variables are non-negative, \( \beta_j \geq \beta \) for all \( j \in J^* \) and \( \beta'_0 \geq 0 \). The proof for the case where there exists \( j' \in J_1 \setminus J^*_1 \) such that \( \min \{ \beta'_0, \beta_{j'} \} = \beta'_0 \) follows similarly.

2. Suppose that for all \( j \in J^*_1 \), we have \( \min \{ \beta'_0, \beta_{j} - \beta \} = \beta_{j} - \beta \) and for all \( j \in (J_1 \setminus J^*_1) \) we have \( \min \{ \beta'_0, \beta_{j} \} = \beta_{j} \). There are two cases to consider:

(a) Suppose that \( |J^*_1| \leq p - 1 \). In this case,

\[
\begin{align*}
t + \sum_{j \in J^*} (\beta_j - \beta) z_j + \sum_{j \in J_1 \setminus J^*} \beta_j z_j & = t + \sum_{j \in J^*_1} (\beta_j - \beta) + \sum_{j \in J_1 \setminus J^*_1} \beta_j \\
& = t + \sum_{j \in J^*_1} \beta_j + \sum_{j \in J_1 \setminus J^*_1} \beta_j - |J^*_1| \beta \\
& \geq \beta_0 - |J^*_1| \beta \geq \beta_0 - (p - 1) \beta,
\end{align*}
\]

where the first inequality follows because inequality (2.16) is valid and the second inequality follows because of our assumption \( |J^*_1| \leq p - 1 \).

(b) Suppose that \( |J^*_1| \geq p \). In this case,

\[
\begin{align*}
t + \sum_{j \in J^*} (\beta_j - \beta) z_j + \sum_{j \in J_1 \setminus J^*} \beta_j z_j & = t + \sum_{j \in J^*_1} (\beta_j - \beta) + \sum_{j \in J_1 \setminus J^*_1} \beta_j \\
& \geq \sum_{j \in J^*_1} (\beta_j - \beta) \geq \sum_{k=1}^{p} (\beta_{j_k} - \beta) \\
& = \sum_{k=1}^{p} \beta_{j_k} - p \beta = \beta_0 - (p - 1) \beta.
\end{align*}
\]

The second inequality holds since \( |J^*_1| \geq p \) and since \( \beta \leq \beta_{j_1} \leq \beta_{j_2} \leq \cdots \leq \beta_{j_p} \leq \beta \) for \( j \in J^* \setminus J \).

Given type-I base inequalities (2.16) valid for any MIP with \( t \geq 0 \), and \( z_j \in \{0, 1\} \),
\( j \in J \), we can derive a new class of valid inequalities (2.19). Similarly, inequality (2.19) is in the form of (2.16), so this process can be repeated by letting the valid inequality (2.19) be the type-I base inequality to derive other classes of valid inequalities.

Inequality (2.19) is related to the weight inequalities of Weismantel [94] for the 0/1 knapsack polytope. Note that inequality (2.19) is valid when \( J^* \) is replaced with \( \tilde{J} \). After complementing the \( z \) variables, we can show that inequality (2.19) where \( J^* \) is replaced with \( \tilde{J} \) and the condition \( \beta_j \leq \beta'_0 \) for all \( j \in J \setminus \tilde{J} \) is satisfied is equivalent to the weight inequalities for the 0/1 knapsack polytope (ignoring the continuous term \( t \)). However, if \( J^* \supseteq \tilde{J} \) then inequality (2.19) with \( J^* \) dominates inequality (2.19) with \( \tilde{J} \). Additionally if \( J^* = \tilde{J} \) and there exists \( j \in J \setminus \tilde{J} \) such that \( \beta_j > \beta'_0 \) then inequality (2.19) dominates the corresponding weight inequality.

Weismantel also proposes weight-reduction and extended weight inequalities for the 0/1 knapsack polytope. In Example 3 we show that weight-reduction inequalities and inequalities (2.19) are not equivalent. We also show that the extended weight inequality is dominated by the inequalities found using Proposition 6 for this example.

**Example 3.** Consider the type-I base inequality

\[
3z_1 + 4z_2 + 5z_3 + 6z_4 \geq 6, \tag{2.20}
\]

for \( t = 0 \). Next, we give examples of inequality (2.19) for different choices of \( \tilde{J} \).

1. Let \( \tilde{J} = \{1, 4\} \). Then \( J^* = \tilde{J} \) and \( \beta = (3 + 6) - 6 = 3 \). Then corresponding inequality (2.19) defined by this choice of \( \tilde{J} \) is \( \min\{4, 3\}z_2 + \min\{5, 3\}z_3 + 3z_4 \geq 3 \), or

\[
z_2 + z_3 + z_4 \geq 1. \tag{2.21}
\]

2. Let \( \tilde{J} = \{2, 4\} \). Then \( J^* = \tilde{J} \) and \( \beta = (4 + 6) - 6 = 4 \). Then corresponding
inequality (2.19) defined by this choice of \( \tilde{J} \) is 
\[
\min\{3, 2\} z_1 + \min\{5, 2\} z_3 + 2z_4 \geq 2,
\]
or
\[
z_1 + z_3 + z_4 \geq 1. \tag{2.22}
\]

3. Let \( \tilde{J} = \{3, 4\} \). Then \( J^* = \tilde{J} \) and \( \beta = (5 + 6) - 6 = 5 \). Then corresponding inequality (2.19) defined by this choice of \( \tilde{J} \) is 
\[
\min\{3, 1\} z_1 + \min\{4, 1\} z_2 + z_4 \geq 1,
\]
or
\[
z_1 + z_2 + z_4 \geq 1. \tag{2.23}
\]

Inequalities (2.21)-(2.23) dominate the corresponding weight inequalities since for all the inequalities there exists \( j \in J \setminus \tilde{J} \) such that \( \beta_j > \beta_0 \). Inequality (2.23) cannot be obtained by weight-reduction inequalities in [94]. On the other hand, the weight-reduction inequality
\[
3z_1 + z_3 + 2z_4 \geq 2,
\]
cannot be obtained using Proposition 6. For this example, the only valid extended weight inequality is
\[
z_1 + z_2 + 2z_3 + 2z_4 \geq 2,
\]
which is dominated by the inequalities (2.21) and (2.22).
2.4.2 A Coefficient Update Scheme for Mixed-Integer Knapsacks with Variable Upper Bounds

Next, we consider another substructure of TPMC consisting of a mixed integer knapsack and variable upper bound constraints. We define set $S_2$ as follows:

$$\sum_{j \in J} t_j + \sum_{j \in J} \alpha_j z_j \leq \alpha_0$$  \hspace{1cm} (2.24)

$$t_j \leq d_j (1 - z_j) \ \forall j \in J$$  \hspace{1cm} (2.25)

$$z \in \{0, 1\}^{|J|}, t_j \in \mathbb{R}^{|J|}_+,$$  \hspace{1cm} (2.26)

for given $\alpha_j \geq 0$ for all $j \in J$ and $\alpha_0 \geq 0$.

Let $T_2 = \text{conv}(S_2)$. We refer to inequalities in the form of (2.24) as type-II base inequalities. If we replace $t_j := \sum_{i \in I: (i,j) \in E} x_{ij}$, $I \subseteq V_1$ then the sum of relaxation of the supply constraints (2.1c) over $I$ is in the form of (2.24) (with $\alpha_j = 0$ for all $j \in J$) for TPMC, and (2.25) is a relaxation of the demand constraints (2.1b). In this case, we observe that TPMC contains the fixed-charge network flow substructure. Therefore, the lifted flow cover and pack inequalities [6, 7, 10, 71, 84], and submodular inequalities [2, 95] are all valid for TPMC. Furthermore, these inequalities and the blossom inequalities (2.3) are in the form of (2.24). Next we describe valid inequalities for the set $S_2$.

**Proposition 7.** Given the mixed-integer set $S_2$, let $\bar{J} = \{j_1, j_2, \ldots, j_u\} \subseteq J$ such that $d_{j_1} - \alpha_{j_1} \geq d_{j_2} - \alpha_{j_2} \geq \cdots \geq d_{j_u} - \alpha_{j_u}$ and there exists $m = \max\{l \in \{0, \ldots, u - 1\} : \sum_{k=1}^{l} d_{j_k} + \sum_{k=l+1}^{u} \alpha_{j_k} < \alpha_0 - \sum_{j \in J \setminus \bar{J}} \max\{d_j, \alpha_j\}\}$. Let $M = \{j_1, j_2, \ldots, j_m\}$ ($M = \emptyset$ if $m = 0$) and $\alpha = \alpha_0 - \sum_{j \in J \setminus \bar{J}} \max\{d_j, \alpha_j\} - d(M) - \alpha(\bar{J} \setminus M)$. Then the inequality
given by

\[
\sum_{j \in J} t_j + \sum_{j \in \tilde{J}} (\alpha_j + \alpha)z_j + \sum_{j \in J \setminus \tilde{J}} \alpha_jz_j \leq \alpha_0 + (u - m - 1)\alpha
\]

(2.27)

is valid for \( S_2 \).

**Proof.** Given a feasible solution \((t, z)\) to \( S_2 \), let \( \tilde{J}_1 = \{ j \in \tilde{J} : z_j = 1 \} \) and \( \tilde{J}_0 = \{ j \in \tilde{J} : z_j = 0 \} \). Consider the following cases:

1. Suppose that \( u - m - 1 \geq |\tilde{J}_1| \). In this case,

\[
\sum_{j \in J} t_j + \sum_{j \in \tilde{J}} (\alpha_j + \alpha)z_j + \sum_{j \in J \setminus \tilde{J}} \alpha_jz_j = \sum_{j \in J \setminus \tilde{J}_1} t_j + \sum_{j \in \tilde{J}_1} \alpha_j + \sum_{j \in J \setminus \tilde{J}_1} \alpha_jz_j + |\tilde{J}_1|\alpha \\
\leq \alpha_0 + |\tilde{J}_1|\alpha \\
\leq \alpha_0 + (u - m - 1)\alpha.
\]

2. Suppose that \( u - m \leq |\tilde{J}_1| \), or equivalently \( m \geq u - |\tilde{J}_1| = |\tilde{J}_0| \). Then,

\[
\sum_{j \in J} t_j + \sum_{j \in \tilde{J}} (\alpha_j + \alpha)z_j + \sum_{j \in J \setminus \tilde{J}} \alpha_jz_j \\
= \sum_{j \in J \setminus \tilde{J}_1} t_j + \sum_{j \in \tilde{J}_1} \alpha_j + \sum_{j \in J \setminus \tilde{J}_1} \alpha_jz_j + |\tilde{J}_1|\alpha \\
\leq \sum_{j \in J \setminus \tilde{J}_1} \max\{d_j, \alpha_j\} + d(\tilde{J}_0) + \sum_{j \in \tilde{J}_1} \alpha_j + |\tilde{J}_1|\alpha \\
= \alpha_0 - \alpha - d(M) - \alpha(\tilde{J} \setminus M) + d(\tilde{J}_0) + \alpha(\tilde{J}_1) + |\tilde{J}_1|\alpha \\
= \alpha_0 - \left[(d(M) - \alpha(M)) - (d(\tilde{J}_0) - \alpha(\tilde{J}_0))\right] + (|\tilde{J}_1| - 1)\alpha,
\]
where the first inequality holds since
\[
\sum_{j \in J \setminus \tilde{J}} t_j + \sum_{j \in J \setminus \tilde{J}} \alpha_j z_j = \left( \sum_{j \in J \setminus \tilde{J}} t_j + \sum_{j \in J \setminus \tilde{J}} \alpha_j z_j \right) + \sum_{j \in \tilde{J}} t_j \leq \sum_{j \in \tilde{J}} \max\{d_j, \alpha_j\} + d(\tilde{J}),
\]
and the second equality holds because \(\sum_{j \in J \setminus \tilde{J}} \max\{d_j, \alpha_j\} = \alpha_0 - \alpha - d(M) - \alpha(\tilde{J} \setminus M)\).

Furthermore, due to the choice of index \(m\), \(0 < \alpha \leq d_{jm+1} - \alpha_{jm+1}\). Thus, we have
\[
(m - |\tilde{J}_0|)\alpha \leq (m - |\tilde{J}_0|)(d_{jm+1} - \alpha_{jm+1}) \leq \sum_{k=|\tilde{J}_0|+1}^{m} (d_{jk} - \alpha_{jk}).
\]

Moreover, \[-\left[ (d(M) - \alpha(M)) - (d(\tilde{J}_0) - \alpha(\tilde{J}_0)) \right] \leq - \left[ \sum_{k=|\tilde{J}_0|+1}^{m} (d_{jk} - \alpha_{jk}) \right].\]

Thus we have
\[
\alpha_0 + (|\tilde{J}_1| - 1)\alpha - \left[ (d(M) - \alpha(M)) - (d(\tilde{J}_0) - \alpha(\tilde{J}_0)) \right] \leq \alpha_0 + (|\tilde{J}_1| - 1)\alpha - (m - |\tilde{J}_0|)\alpha = \alpha_0 + (u - m - 1)\alpha,
\]
completing the proof.

As in Proposition 6, Proposition 7 can be applied recursively to obtain new non-trivial valid inequalities for TPMC.

Next we give an example illustrating the valid inequalities introduced in this section.
Example 4. Consider an instance of TPMC with a complete bipartite graph, $V_1 = \{1, 2\}$, $V_2 = \{1, 2, 3, 4\}$, $s = (31, 20)$ and $d = (11, 19, 8, 13)$. A valid inequality for $X$ for this instance is

$$x_{21} + x_{22} + x_{23} + x_{24} + 11z_1 + 19z_2 + 8z_3 + 13z_4 \geq 20,$$

which corresponds to inequality (2.15) with $I = \{2\}$ and $J = \{1, 2, 3, 4\}$. Note that $d(J) - s(V_1 \setminus I) = 20 \geq d_j$ for all $j \in J$.

Using (2.28) as the type-I base inequality, we apply the coefficient update in Proposition 6 and let $\tilde{J} = \{1, 4\}$, $J^* = \{1, 2, 4\}$. Then $\beta_1 + \beta_4 = 11 + 13 = 24$ and $(\beta_1 + \beta_4) - \beta_0 = 24 - 20 = 4 = \beta$, and we obtain the corresponding inequality (2.19)

$$x_{21} + x_{22} + x_{23} + x_{24} + 7z_1 + 15z_2 + 8z_3 + 9z_4 \geq 16,$$

which is valid for $X$.

Using (2.29) as the type-I base inequality, we apply the coefficient update in Proposition 6 and let $\tilde{J} = \{3, 4\}$, $J^* = \{2, 3, 4\}$. Then $\beta_3 + \beta_4 = 8 + 9 = 17$ and $(\beta_3 + \beta_4) - \beta_0 = 17 - 16 = 1 = \beta$ and again we obtain the corresponding inequality (2.19)

$$x_{21} + x_{22} + x_{23} + x_{24} + 7z_1 + 14z_2 + 7z_3 + 8z_4 \geq 15,$$

which is valid for $X$.

Now, consider the supply constraint (2.1c) for supplier 2

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 20.$$

Then using (2.31) as the type-II base inequality with $I = \{2\}$ and $J = \{1, 2, 3, 4\}$, we apply the coefficient update in Proposition 7, where we let $\tilde{J} = \{2, 4\}$. Then $\alpha_0 - \sum_{j \in J} \max \{d_j, \alpha_j\} = \alpha_0 - (d_1 + d_3) = 20 - (11 + 8) = 1$. However, all demand
values in set $\tilde{J}$ are greater than 1 so $m = 0$ and $\alpha = \alpha_0 - (d_1 + d_3) - \alpha_2 - \alpha_4 = 20 - (11 + 8) - 0 - 0 = 1$. Then we obtain the corresponding inequality (2.27)

$$x_{21} + x_{22} + x_{23} + x_{24} + z_2 + z_4 \leq 21,$$

which is valid for $X$.

2.4.3 Strength of the Proposed Inequalities

Next we give several facet conditions for inequalities (2.15). Let $V'_2$ be the set of markets. Observe that if $s(V_1) < d_j$ for some $j \in V'_2$ then the demand of market $j$ can never be met in any feasible solution to TPMC. Therefore, we can set $z_j = 1$ for such markets and let $V_2 = \{j \in V'_2 : s(V_1) \geq d_j\}$. In other words, we remove the markets that can never be satisfied from the given set of markets. Therefore, throughout we make the assumption that

$$s(V_1) \geq \max_{j \in V_2} d_j. \quad (2.33)$$

Let $J^\prec = \{j \in J : d_j < d(J) - s(V_1 \setminus I)\}$.

**Theorem 2.** Inequality (2.15) defines a nontrivial facet of $\text{conv}(X)$ only if the following conditions hold:

1. $d(J) > s(V_1 \setminus I)$.

2. There exists $j \in J$ such that $d_j > d(J) - s(V_1 \setminus I)$.

3. $s(V_1) \geq d(J) - \max_{j \in J} \{d_j\} + \max_{j \in V_2 \setminus J} \{d_j\}$.

4. If $s(V_1) < d(J)$ and $I \neq \emptyset$, then $|J^\prec| \geq 2$ and the sum of the smallest two demands in set $J^\prec$ is not greater than $d(J) - s(V_1 \setminus I)$.

5. $I \neq V_1$. 

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6. If $|J| = 1$, then $|V_1 \setminus I| = 1$.

7. $s(V_1) \geq d(J \setminus J^<) + \max_{j \in J^<} \{d_j\}$.

8. If $s(V_1) = d(J)$ and $d_j \geq d(J) - s(V_1 \setminus I)$ for all $j \in J$ then $|I| \leq 1$.

In addition, if the following conditions hold, then (2.15) is a facet of $\text{conv}(TPMC)$:

9. $s(V_1) > d(J) - \max_{j \in J} \{d_j\} + \max_{j \in V_2 \setminus J} \{d_j\}$.

10. There exists $\hat{J} \subsetneq J^<$ such that $d(J \setminus \hat{J}) > s(V_1 \setminus I)$ and $d(J \setminus \hat{J}') > s(V_1 \setminus I)$ where $\hat{J}' = \hat{J} \cup \{k_1\}$ for all $k_1 \in J^< \setminus \hat{J}$.

11. $s(V_1) > \max_{j \in V_2} d_j$.


1. Assume that $d(J) - s(V_1 \setminus I) \leq 0$.

   From validity of inequality (2.15) we have $d(J) - s(V_1 \setminus I) \geq 0$ and combined with the assumption we get $d(J) - s(V_1 \setminus I) = 0$. The resulting inequality is implied by the nonnegativity of $x_{ij}$ and $z_j$ for $i \in I$, $j \in J$, $(i, j) \in E$.

2. Assume that $d_j \leq d(J) - s(V_1 \setminus I)$ for all $j \in J$. Under this assumption inequality (2.15) reduces to

$$\sum_{i \in I, j \in J \setminus (i, j) \in E} x_{ij} + \sum_{j \in J} d_j z_j \geq d(J) - s(V_1 \setminus I). \tag{2.34}$$

We add all the demand constraints (2.1b) in $J$,

$$\sum_{i \in V_1, j \in J \setminus (i, j) \in E} x_{ij} + \sum_{j \in J} d_j z_j = d(J). \tag{2.35}$$

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When we subtract (2.35) from (2.34) we obtain

\[ \sum_{i \in V_1 \setminus I, j \in J: (i,j) \in E} x_{ij} \leq s(V_1 \setminus I). \]  

(2.36)

If \( J \subseteq V_2 \) then inequality (2.36) is weaker than all the supply inequalities (2.1c) in \( V_1 \setminus I \) combined, because \( x_{ij} \geq 0 \) for all \( i \in I, j \in V_2 \setminus J, (i,j) \in E \). If \( J = V_2 \) then inequality (2.36) is dominated by the supply inequalities \( \sum_{j \in V_2 \setminus J} (x_{ij}) \leq s_i \) for all \( i \in V_1 \setminus I \) unless \( |V_1 \setminus I| = 1 \). However, when \( J = V_2 \), \( |V_1 \setminus I| = 1 \) and \( d_j \leq d(J) - s(V_1 \setminus I) \) for all \( j \in J \) inequality (2.15) reduces to a trivial facet.

3. Assume that \( s(V_1) < d(J) - \max_{j \in J \setminus J^<} \{d_j\} + \max_{j \in V_2 \setminus J} \{d_j\} \). Because we have showed that there exists \( j \in J \) such that \( d_j > d(J) - s(V_1 \setminus I) \) we can conclude that \( s(V_1 \setminus I) > d(J) - d_j \geq d(J) - \max_{j \in J} \{d_j\} \). Note that we have to have \( s(I) < \max_{j \in V_2 \setminus J} \{d_j\} \) for \( s(V_1) < d(J) - \max_{j \in J} \{d_j\} + \max_{j \in V_2 \setminus J} \{d_j\} \) to hold because if \( s(I) \geq \max_{j \in V_2 \setminus J} \{d_j\} \), then \( s(V_1) = s(V_1 \setminus I) + s(I) > d(J) - \max_{j \in J} \{d_j\} + \max_{j \in V_2 \setminus J} \{d_j\} \) which would contradict our assumption. Let \( r^* = \arg \max_{j \in V_2 \setminus J} \{d_j\} \). Because (2.15) is a non-trivial facet, it is different from \( z_{r^*} \leq 1 \) and there exists solutions on the face defined by (2.15) with \( z_{r^*} = 0 \). Note that \( \sum_{j \in J \setminus J^<} z_j \leq 1 \) for any point to be on the face defined by inequality (2.15). We consider the following cases:

(a) \( \sum_{j \in J \setminus J^<} z_j = 1 = z_l \) for some \( l \in J \setminus J^< \).

In this case, left-hand side of inequality (2.15) reduces to

\[ \sum_{i \in I, j \in J: (i,j) \in E} x_{ij} + \sum_{j \in J \setminus \{l\}} \left( \min \{d(J) - s(V_1 \setminus I), d_j\} \right) z_j + d(J) - s(V_1 \setminus I) \]

since \( l \in J \setminus J^< \), \( \min \{d(J) - s(V_1 \setminus I), d_l\} = d(J) - s(V_1 \setminus I) \). Thus to
satisfy inequality (2.15) at equality we must have $\sum_{i \in I, j \in J: (i,j) \in E} x_{ij} = 0$, $z_j = 0$ for all $j \in J \setminus \{l\}$ and

$$\sum_{i \in V \setminus I, j \in J \setminus \{l\}: (i,j) \in E} x_{ij} = d(J \setminus \{l\}) \leq s(V \setminus I) - (d_{r^*} - s(I)) = s(V_1) - d_{r^*}$$

where $d_{r^*} - s(I)$ is the amount of demand of market $r^*$ that cannot be satisfied by the suppliers in set $I$. We obtain a contradiction because (2.37) implies that $s(V_1) \geq d(J) - d_t + d_{r^*} \geq d(J) - \max_{j \in J} \{d_j\} + \max_{j \in V \setminus J} \{d_j\}$, since $d_t \leq \max_{j \in J} \{d_j\}$.

(b) $\sum_{j \in J: j < l} z_j = 0$.

Let $\hat{J} = \{j \in J: z_j = 1\}$. Then a point on the face defined by inequality (2.15) satisfies

$$\sum_{i \in I, j \in J: (i,j) \in E} x_{ij} + \sum_{j \in J} d_j = d(J) - s(V \setminus I).$$

This implies that $\sum_{i \in I, j \in J: (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V \setminus I) \geq 0$ because otherwise we would not have a feasible solution. Furthermore,

$$\sum_{i \in V \setminus I, j \in J \setminus \{l\}: (i,j) \in E} x_{ij} = s(V_1 \setminus I).$$

Combining the results we observe that because $s(I) < d_{r^*}$ we cannot send all the demand of $d_{r^*}$ from $s(I)$ so some of the supply from $s(V_1 \setminus I)$ should be sent to $d_{r^*}$ but all the supply $s(V_1 \setminus I)$ is sent to markets in $J \setminus \hat{J}$. We reach a contradiction, we cannot have $z_{r^*} = 0$.

4. Suppose that $s(V_1) < d(J)$ and $I \neq \emptyset$, then not all demand in set $J$ can be met, hence $\sum_{j \in J} z_j \geq 1$. Consider the following cases:
(a) $J^< = \emptyset$. Then inequality $\sum_{j \in J} (d(J) - s(V_1 \setminus I)) z_j \geq d(J) - s(V_1 \setminus I)$ dominates inequality (2.15) since inequality (2.15) has the additional term $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} \geq 0$.

(b) $|J^<| = 1$. Let $J^< = \{k\}$. We apply the coefficient update in Proposition 6 using inequality (2.15) as the type-I base inequality. Let $\tilde{J} = \{j, k\}$ where $j \in J \setminus \{k\}$. Therefore, $\beta = \beta_j + d_k - \beta_0 = d(J) - s(V_1 \setminus I) + d_k - (d(J) - s(V_1 \setminus I)) = d_k$ and the corresponding inequality (2.19) is

$$\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{j \in J \setminus \{k\}} (d(J) - s(V_1 \setminus I) - d_k) z_j + (d_k - d_j) z_k \geq d(J) - s(V_1 \setminus I) - d_k.$$  

If we add $\sum_{j \in J} d_k z_j \geq d_k$ to inequality (2.38) we obtain (2.15). Hence, (2.15) cannot be a facet.

(c) $|J^<| \geq 2$ and $d_{j_1} + d_{j_2} > d(J) - s(V_1 \setminus I)$ where $d_{j_1}$ and $d_{j_2}$ are the two smallest demands in set $J^<$. We use the coefficient update in Proposition 6 using inequality (2.15) as the type-I base inequality. Let $\tilde{J} = \{j_1, j_2\}$. Therefore, $\beta = d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I))$ and the corresponding inequality (2.19) is

$$\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{j \in J \setminus \{j_1, j_2\}} (2(d(J) - s(V_1 \setminus I)) - d_{j_1} - d_{j_2}) z_j \geq 2(d(J) - s(V_1 \setminus I)) - d_{j_1} - d_{j_2}.$$  

Because $d_{j_1}$ and $d_{j_2}$ are the two smallest demands we have $J^* = J$ in Proposition 6. Note that if we add $\sum_{j \in J} (d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I))) z_j \geq$
\[ d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I)) \] to inequality (2.39) we obtain (2.15). Hence, 
\[(2.15) \] cannot be a facet.

5. Assume that \( I = V_1 \). Then inequality (2.15) reduces to

\[
\sum_{i \in V_1, j \in J: (i, j) \in E} x_{ij} + \sum_{j \in J} d_j z_j \geq d(J), \tag{2.40}
\]

Inequality (2.40) is a relaxation of the demand equalities (2.1b) in TPMC. Therefore, if \( I = V_1 \) then all points in TPMC are on the face defined by inequality (2.15), therefore this inequality does not define a proper face.

6. Suppose that \( J = \{j\} \), but \(|V_1 \setminus I| > 1\). Then inequality (2.15) is

\[
\sum_{i \in I: (i, j) \in E} x_{ij} + (d_j - s(V_1 \setminus I)) z_j \geq d_j - s(V_1 \setminus I), \tag{2.41}
\]

where \( d_j > s(V_1 \setminus I) \) from facet condition 1. Subtracting the original demand equality (2.1b) for \( j \) from inequality (2.41), we get

\[
\sum_{i \in V_1 \setminus I: (i, j) \in E} x_{ij} \leq s(V_1 \setminus I)(1 - z_j),
\]

which is dominated by VUB inequalities (2.14) for \( i \in V_1 \setminus I \).

7. Assume that \( s(V_1) < d(J \setminus J^<) + \max_{j \in J^<} \{d_j\} \). Then not all demand for markets in set \( J \setminus J^< \) and the largest demand in set \( J^< \) can be met at the same time. Hence, \( \sum_{j \in J \setminus J^<} z_j + z_m \geq 1 \) where \( m = \arg \max_{j \in J^<} \{d_j\} \). We use Proposition 6 and inequality (2.15) as the type-I base inequality. Let \( \tilde{J} = \{l, m\} \)
8. Assume that $s_l = d(J) - s(V_1 \setminus I) + d_m - (d(J) - s(V_1 \setminus I)) = d_m$. We obtain

\[
\sum_{i \in I} x_{ij} + \sum_{j \in J \setminus I} (d(J) - s(V_1 \setminus I) - d_m)z_j + \sum_{j \in J \setminus \{m\}} d_j z_j + (d_m - d_m)z_m \geq d(J) - s(V_1 \setminus I) - d_m.
\] (2.42)

If we add $\sum_{j \in J \setminus I} d_m z_j + d_m z_m \geq d_m$ to inequality (2.42) we obtain (2.15). Hence, (2.15) cannot be a facet.

8. Assume that $s(V_1) = d(J)$, $d_j \geq d(J) - s(V_1 \setminus I)$ for all $j \in J$ and for contradiction $|I| \geq 2$. Because of assumption $s(V_1) = d(J)$ we have $d_j \geq d(J) - s(V_1 \setminus I) = s(V_1) - s(V_1 \setminus I) = s(I)$ for all $j \in J$. Under these assumptions inequality (2.15) reduces to

\[
\sum_{i \in I, j \in J: (i,j) \in E} x_{ij} + \sum_{j \in J} s(I)z_j \geq s(I).
\] Let $I' = I \setminus \{i'\}$ and $I'' = \{i'\}$ where $i' \in I$ ($I' \neq \emptyset$ and $I'' \neq \emptyset$ because $|I| \geq 2$ by assumption). Consider the following inequalities in the form of inequality (2.15) with set $I$ replaced with sets $I'$ and $I''$, respectively

\[
\sum_{i \in I \setminus \{i'\}, j \in J: (i,j) \in E} x_{ij} + \sum_{j \in J} (s(I) - s_{i'})z_j \geq s(I) - s_{i'},
\] (2.43)

\[
\sum_{j \in J \setminus \{i'\}, j \in E} x_{i'j} + \sum_{j \in J} s_{i'}z_j \geq s_{i'}.
\] (2.44)

Inequality (2.43) is valid because $d(J) - s(V_1 \setminus I') = d(J) - s(V_1 \setminus I) - s_{i'} = s(I) - s_{i'} > 0$. Furthermore, the coefficient of $z_j$ is $\min\{d_j, s(I) - s_{i'}\} = s(I) - s_{i'}$ because of the assumption $d_j \geq d(J) - s(V_1 \setminus I) = s(I)$ for all $j \in J$. Inequality (2.44) is valid because $d(J) - s(V_1 \setminus I'') = s(V_1) - s(V_1 \setminus I'') = s(I'') = s_{i'} > 0$ and similarly the coefficient of $z_j$ is $\min\{s_{i'}, d_j\} = s_{i'}$, because $d_j \geq s(I) \geq s_{i'}$. 

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for all $j \in J$ by assumption. By adding inequalities (2.43) and (2.44) we obtain inequality (2.15) with set $I$. Hence, (2.15) cannot be a facet.

**Sufficiency.** We use the technique in §I.4.3 Theorem 3.6 [64]. We show that inequality (2.15), plus any linear combination of the demand constraints

$$\sum_{i \in V_1; (i,j) \in E} x_{ij} + d_j z_j = d_j$$

for all $j \in V_2$ is the only inequality that is satisfied at equality by all points $(x, z)$ feasible to TPMC that are tight at (2.15), i.e., we show that if all points of TPMC at which (2.15) is tight satisfy

$$\sum_{(i,j) \in E} \alpha_{ij} x_{ij} + \sum_{j \in V_2} \psi_j z_j = \hat{\alpha} \tag{2.45}$$

then

1. $\alpha_{ij} = u_j$, $j \in V_2 \setminus J$, $i \in V_1$, $(i,j) \in E$,
2. $\alpha_{ij} = u_j$, $j \in J$, $i \in V_1 \setminus I$, $(i,j) \in E$,
3. $\alpha_{ij} = \bar{\alpha} + u_j$, $j \in J$, $i \in I$, $(i,j) \in E$,
4. $\psi_j = u_j d_j$, $j \in V_2 \setminus J$
5. $\psi_j = \bar{\alpha} \left( \min \{d(J) - s(V_1 \setminus I), d_j\} \right) + u_j d_j$, $j \in J$,
6. $\hat{\alpha} = \bar{\alpha} \left( d(J) - s(V_1 \setminus I) \right) + \sum_{j \in V_2} u_j d_j$.

In the proof we consider three different types of points at which (2.15) is tight. These points are solutions to TPMC but are subject to additional systems of constraints. Throughout, let $\epsilon$ be a very small number greater than zero unless noted otherwise.
1. Suppose that \( d_l > d(J) - s(V_1 \setminus I) \) for \( l = \arg \max_{j \in J} \{d_j\} \). Consider a point where only markets \( j \in \{r\} \cup J \setminus \{l\} \) are satisfied for some \( r \in V_2 \setminus J \) and constraints

\[
\sum_{i \in I, j : (i, j) \in E} x_{ij} = 0
\]

\[
\sum_{i \in V_1 \setminus I, j : (i, j) \in E} x_{ij} = d(J) - d_l
\]

\[
\sum_{i \in V_1 : (i, r) \in E} x_{ir} = d_r
\]

\[
x_{ij} = 0, \quad i \in V_1, j \in \{l\} \cup V_2 \setminus (J \cup \{r\})
\]

\[
x_{ij} \geq \epsilon, \quad i \in V_1 \setminus I, j \in J \setminus \{l\}
\]

\[
x_{ir} \geq \epsilon, \quad i \in V_1
\]

\[
\sum_{j : (i, j) \in E} x_{ij} \leq s_i - \epsilon, \quad i \in V_1
\]

\[
z_j = 1, \quad j \in \{l\} \cup V_2 \setminus (J \cup \{r\})
\]

\[
z_j = 0, \quad j \in \{r\} \cup J \setminus \{l\}
\]

in addition to the original constraints are satisfied, which we refer to as System 1. We know that a solution to System 1 exists from facet conditions 9 and 11. For a solution to be feasible to System 1, the demand of markets \( j \in \{r\} \cup J \setminus \{l\} \) have to be met, i.e., \( s(V_1) \geq d(J) - \max_{j \in J} \{d_j\} + \max_{j \in V_2 \setminus J} \{d_j\} \). Additionally, we would like to change a given solution by increasing and decreasing the \( x \) values by \( \epsilon \) hence the need for > relationship in facet condition 9.

2. Suppose that \( d_l > d(J) - s(V_1 \setminus I) \) for some \( l \in J \). Consider a point where only
markets $j \in J \setminus \{l\}$ are satisfied and constraints

$$\sum_{i \in I, j \in J(\cdot,i,j) \in E} x_{ij} = 0$$

$$\sum_{i \in V_1 \setminus I, j \in J(\cdot,i,j) \in E} x_{ij} = d(J) - d_l$$

$$x_{ij} = 0, \quad i \in V_1, j \in \{l\} \cup V_2 \setminus J$$

$$x_{ij} \geq \epsilon, \quad i \in V_1 \setminus I, j \in J \setminus \{l\}$$

$$\sum_{j \in V_2(\cdot,i,j) \in E} x_{ij} \leq s_i - \epsilon, \quad i \in V_1 \setminus I$$

$$z_j = 1, \quad j \in \{l\} \cup V_2 \setminus J$$

$$z_j = 0, \quad j \in J \setminus \{l\}$$

in addition to the original constraints are satisfied, which we refer to as System 2. We know that a solution to System 2 exists from facet condition 2 since there exists at least one $j \in J$ such that $s(V_1) \geq s(V_1 \setminus I) > d(J) - d_j$, and from facet condition 11.

3. We define $\hat{J} \subset J$ such that $d(J \setminus \hat{J}) > s(V_1 \setminus I)$. Due to the choice of $\hat{J}$ we have $d_j < d(J) - s(V_1 \setminus I)$ for all $j \in \hat{J}$ so $\hat{J} \subseteq J^<$ (if $d_{j'} \geq d(J) - s(V_1 \setminus I)$ and $j' \in \hat{J}$ then we cannot have $d(J \setminus \hat{J}) > s(V_1 \setminus I)$). In this point, markets in set $\hat{J} \cup V_2 \setminus J$ are rejected and constraints

$$\sum_{i \in I, j \in J(\cdot,i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I)$$

$$\sum_{i \in V_1 \setminus I, j \in J(\cdot,i,j) \in E} x_{ij} = s(V_1 \setminus I)$$

$$x_{ij} = 0, \quad i \in V_1, j \in \hat{J} \cup V_2 \setminus J$$
\[ \begin{align*}
x_{ij} \geq \epsilon, & \quad i \in V_1, j \in J \setminus \hat{J} \\
\sum_{j \in J \setminus \hat{J}} \sum_{(i,j) \in E} x_{ij} \leq s_i - \epsilon, & \quad i \in I \\
z_j = 1, & \quad j \in \hat{J} \cup V_2 \setminus J \\
z_j = 0, & \quad j \in J \setminus \hat{J}
\end{align*} \]

in addition to the original constraints are satisfied, which we refer to as System 3. We consider a set \( \hat{J} \) such that all demand in set \( J \setminus J^c \) is satisfied and \( \sum_{i \in I, j \in J \setminus (i,j) \in E} x_{ij} > 0 \). This is possible due to facet conditions 7, 11, and non-negativity of \( x \) variables.

In order to establish the values of the coefficients \( \alpha_{ij}, \psi_j \) and \( \hat{\alpha} \), we construct a feasible solution to the given systems 1, 2 and 3. Then a small change in the solution is made. By evaluating (2.45) at both solutions, which are on the face defined by (2.15) and comparing the resulting expressions, the possible values of a set of coefficients are obtained.

We start by showing that

1. \( \alpha_{ij} = u_j, \quad j \in V_2 \setminus J, \quad i \in V_1, \quad (i,j) \in E \).

Consider any solution to system 1 with any market \( r \in V_2 \setminus J \) that is satisfied. Choose arbitrary suppliers \( i, i' \in V_1 \) such that \( (i,r), (i',r) \in E \). Construct a new point by decreasing the flow on edge \( (i,r) \) by \( \epsilon \) and increasing the flow on edge \( (i',r) \) by \( \epsilon \). Note that this point is also on the face defined by inequality (2.15). Thus,

\[ \alpha_{ij} = u_j, \quad j \in V_2 \setminus J, \quad i \in V_1, \quad (i,j) \in E. \] (2.46)

2. \( \alpha_{ij} = u_j, \quad j \in J, \quad i \in V_1 \setminus I, \quad (i,j) \in E \). Note that if \( |V_1 \setminus I| = 1 \), then \( \alpha_{ij} = u_j, \quad j \in J \) trivially holds. We condition on the number of markets in set \( J \).

(a) \( J = \{k\} \). Note that, from facet condition 6 we have \( |V_1 \setminus I| = 1 \), so the result holds.
(b) \(|J| \geq 2\). By assumption, \(|V_1 \setminus I| > 1\). Due to facet condition \(2\) there exists \(k \in J\) such that \(d_k > d(J) - s(V_1 \setminus I)\). We consider a solution to system \(2\) with \(l = k\). Choose any market \(j \in J \setminus \{k\}\), any suppliers \(i, i' \in V_1 \setminus I\) such that \((i, j), (i', j) \in E\). Make an \(\epsilon\)-change of flow between the two suppliers \(i, i'\) and market \(j\). Thus,

\[
\alpha_{ij} = u_j, j \in J \setminus \{k\}, i \in V_1 \setminus I, (i, j) \in E.
\]

(2.47)

Next we show that \(\alpha_{ik} = u_k\) for all \(i \in V_1 \setminus I\). If there exists another \(j^*\) such that \(d_{j^*} > d(J) - s(V_1 \setminus I), j^* \neq k\) then we consider a point satisfying System 2 with \(l = j^*\), and use the same argument as before to show that \(\alpha_{ik} = u_k\) for all \(i \in V_1 \setminus I\). If no such \(j^*\) exists then \(d_j \leq d(J) - s(V_1 \setminus I)\) for all \(j \in J \setminus \{k\}\). In this case \(k\) is the only market in \(J\) with \(d_k > d(J) - s(V_1 \setminus I)\). Then from facet condition \(7\) we know that there exists a solution to a variant of System \(3\) with \(\hat{J} \subseteq J < \{k\}\) for some \(j \in J \setminus \{k\}\) (in which we set \(\epsilon = 0\) in case facet condition \(7\) is satisfied at equality), where along with market \(k\) we can satisfy at least one more market, \(j\). Choose suppliers \(i, i' \in V_1 \setminus I\) such that \((i, k), (i', k), (i, j), (i', j) \in E\). Decrease flow on edges \((i, j), (i', k)\) by \(\epsilon\) and increase flow on edges \((i, k), (i', j)\) by \(\epsilon\). Note that since we are using a solution to a variant of system \(3\) in which we set \(\epsilon = 0\) inequality (2.15) is still tight. Thus,

\[
\alpha_{ik} - \alpha_{ij} - \alpha_{i'k} + \alpha_{i'j} = \alpha_{ik} - u_j - \alpha_{i'k} + u_j = \alpha_{ik} - \alpha_{i'k} = 0. \quad (2.48)
\]

Therefore, \(\alpha_{ik} = u_k\) for all \(i \in V_1 \setminus I\).

3. \(\alpha_{ij} = \bar{\alpha} + u_j, j \in J, i \in I, (i, j) \in E\).

Consider a solution to system \(3\) with \(\hat{J} \subseteq J < \). Choose any market \(j \in J \setminus \hat{J}\), any two suppliers \(i, i' \in I\) such that \((i, j), (i', j) \in E\). Make an \(\epsilon\)-change of flow between the two suppliers \(i, i'\) and market \(j\). Thus,

\[
\alpha_{ij} = \alpha^1_{ij}, j \in J \setminus \hat{J}, i \in I, (i, j) \in E.
\]

(2.49)
Let $\alpha_1^j = \bar{\alpha}_j + u_j$, $j \in J \setminus \hat{J}$. Facet condition 10 and definition of $\hat{J}$ (i.e. $\hat{J} \subseteq J^<$) implies that for any $k_1 \in J^<$ we can redefine $\hat{J}$ to either include $k_1$ or not. More specifically, if $k_1 \in \hat{J}$ then market $k_1$ is rejected. To show that $\alpha_{ik_1} = \alpha_{k_1}^1$ for all $i \in I$, $(i, k_1) \in E$ we choose another $\hat{J}$ such that $k_1 \notin \hat{J}$. Using the same argument as before we obtain $\bar{\alpha}_{ij} = \bar{\alpha}_j$, $j \in J \setminus \hat{J}$. Furthermore, since as before we can rearrange set $\hat{J}$ to include or not include any $k_1 \in J^<$ we get $\bar{\alpha}_j = \bar{\alpha}$, $j \in J \setminus \hat{J}$.

4. $\psi_j = u_j d_j$, $j \in V_2 \setminus J$. We rewrite (2.45) using the information obtained until now and get

$$\bar{\alpha} \sum_{i \in I, j \in J : (i, j) \in E} x_{ij} + \sum_{(i, j) \in E} u_j x_{ij} + \sum_{j \in V_2} \psi_j z_j = \hat{\alpha}. \tag{2.52}$$

Consider any solution to system 1 with any market $r \in V_2 \setminus J$ that is satisfied. Then we construct a new solution based on this solution where we set $z_r = 1$ and $x_{ir} = 0$ for all $i \in V_1$, $(i, r) \in E$ and all other variables remain the same. Note that this solution is also on the face defined by (2.15) since $r \in V_2 \setminus J$ and
the new solution is a solution to system 2. We compare face (2.45) evaluated at these two solutions. Thus,

\[ u_r \sum_{i \in V_1; (i,r) \in E} x_{ir} - \psi_r = 0. \]

Because \( \sum_{i \in V_1; (i,r) \in E} x_{ir} = d_r \) we have \( \psi_r = u_r d_r. \)

5. \( \psi_j = \bar{\alpha} \left( \min \{ d(J) - s(V_1 \setminus I), d_j \} \right) + u_j d_j, \ j \in J. \)

We consider 2 cases.

(a) \( d_{j'} < d(J) - s(V_1 \setminus I) \) for some \( j' \in J. \)

We consider a solution to system 3 with \( \hat{J} \) such that \( d(\hat{J}) + d_{j'} \leq d(J) - s(V_1 \setminus I) \). This is a feasible solution due to facet condition 10 where \( k_1 = j' \).

We evaluate (2.52) at this solution and obtain

\[ \bar{\alpha} (d(J \setminus \hat{J}) - s(V_1 \setminus I)) + \sum_{i \in V_1, j \in J \setminus \hat{J}; (i,j) \in E} u_j x_{ij} + \sum_{j \in \hat{J} \cup V_2 \setminus J} \psi_j = \hat{\alpha}. \]

Then we use the same solution except now we set \( z_{j'} = 1, \ x_{ij'} = 0, \ i \in V_1, (i, j') \in E \) (so we redefine \( \hat{J} \) as \( \hat{J}' = \hat{J} \cup \{ j' \} \)) and

\[ \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) - d_{j'} \]

and evaluate (2.52) again. Note that this solution is also on the face defined by (2.15) because we had \( z_{j'} = 0, \ \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) \) and we changed it with \( z_{j'} = 1, \ \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) - d_{j'} \) and the coefficient of \( z_{j'} \) is \( d_{j'} \) in inequality (2.15).

Thus,

\[ \bar{\alpha} (d(J \setminus \hat{J}) - s(V_1 \setminus I) - d_{j'}) + \sum_{i \in V_1, j \in J' \setminus (i,j) \in E} u_j x_{ij} + \sum_{j \in \hat{J} \cup V_2 \setminus J} \psi_j + \psi_{j'} = \hat{\alpha}. \]
Taking the difference between (2.52) evaluated at these two solutions, we obtain

\[ \psi_j' = \bar{\alpha}d_j' + u_j' \sum_{i \in V_1; (i,j') \in E} x_{ij'} = \bar{\alpha}d_j' + u_j'd_j'. \]

(b) \( d_j' \geq d(J) - s(V_1 \setminus I) \) for some \( j' \in J \).

We consider a solution to system 3 with any feasible \( \hat{J} \) such that the right hand side of inequality

\[ \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) \]

is nonnegative and market \( j' \) is satisfied. In the solution we can set

\[ \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = \sum_{i \in I; j' \in \hat{J}; (i,j') \in E} x_{ij'}. \]

This is a feasible solution since \( d_j' \geq d(J) - s(V_1 \setminus I) \) by assumption and we know that for inequality (2.15) to be tight we cannot have \( \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} > d(J) - s(V_1 \setminus I) \). Hence, \( \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} \leq d(J) - s(V_1 \setminus I) \) and we can choose a solution in which a part (or all) of the demand of market \( j' \) is met by suppliers in set \( I \). We use \( \psi_j = \bar{\alpha}d_j + u_jd_j \) for all \( j \in J' \) and recall that markets in set \( \hat{J} \subseteq J' \) are rejected. We evaluate (2.52) at this solution and obtain

\[ \bar{\alpha}(d(J \setminus \hat{J}) - s(V_1 \setminus I) + d(\hat{J})) + u_j' \sum_{i \in I; (i,j') \in E} x_{ij'} \]

\[ + \sum_{i \in V_1 \setminus I, j \in J \setminus \hat{J}; (i,j) \in E} u_j x_{ij} + \sum_{j' \in \hat{J} \cup V_2} u_jd_j = \bar{\alpha}. \]

Then we use the same solution except now we set \( z_j' = 1, z_q = 0, q \in \hat{J} \) (this is still a feasible solution since \( s(V_1) \geq s(V_1 \setminus I) \geq d(J) - d_j' \) by assumption, i.e., once market \( j' \) is rejected all other markets in set \( J \) can be satisfied) and \( \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = 0 \) (implying that \( \sum_{i \in I; (i,j') \in E} x_{ij'} = 0 \)) and reevaluate (2.52). Note that this solution is also on the face defined by (2.15) because we had \( z_j' = 0, z_q = 1, q \in \hat{J}, \) \( \sum_{i \in I, j \in J; (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) \) and we changed it with \( z_j' = 1, z_q = 0, q \in \hat{J}, \)

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\[ \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} = 0 \] and the coefficient of \( z_j' \) is \( d(J) - s(V_1 \setminus I) \). Thus, \( \bar{\alpha}(0) + 0 + \sum_{i \in V_1 \setminus I, j \in J : (i,j) \in E} u_j x_{ij} + \sum_{j \in V_2 \setminus J} u_j d_j + \psi_j' = \hat{\alpha} \).

Taking the difference between (2.52) evaluated at these two solutions, we get \( \bar{\alpha}(d(J) - s(V_1 \setminus I)) + u_{j'} \sum_{i \in V_1 : (i,j') \in E} x_{ij'} - \sum_{i \in V_1, j \in J} u_j x_{ij} + \sum_{j \in J} u_j d_j - \psi_j' = 0 \).

Because \( \sum_{i \in V_1 : (i,j') \in E} x_{ij} = d_{j'} \) and \( \sum_{i \in V_1, j \in J} u_j x_{ij} = \sum_{j \in J} u_j d_j \) we have \( \psi_j' = \bar{\alpha}(d(J) - s(V_1 \setminus I)) + u_{j'} d_{j'} \).

6. \( \hat{\alpha} = \bar{\alpha}(d(J) - s(V_1 \setminus I)) + \sum_{j \in V_2} u_j d_j \). Rewriting equality (2.45), we get

\[
\hat{\alpha} \left( \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{j \in J} \min \{ d(J) - s(V_1 \setminus I), d_j \} z_j \right) + \sum_{(i,j) \in E : j \in V_2} u_j x_{ij} + \sum_{j \in V_2} u_j d_j z_j = \hat{\alpha}.
\]

Evaluating (2.53) at any point \((x, z)\) feasible to TPMC that is tight at inequality (2.15) gives

\[
\hat{\alpha}(d(J) - s(V_1 \setminus I)) + \sum_{j \in V_2} u_j \left( \sum_{i \in V_1 : (i,j) \in E} x_{ij} + d_j z_j \right) = \hat{\alpha}.
\]

From equality (2.1b) in the definition of TPMC we have \( \sum_{i \in V_1 : (i,j) \in E} x_{ij} + d_j z_j = d_j \) for all \( j \in V_2 \). Thus, \( \hat{\alpha} = \bar{\alpha}(d(J) - s(V_1 \setminus I)) + \sum_{j \in V_2} u_j d_j \).

\[ \square \]

Our next result shows that the coefficient update scheme in Proposition 6 is neither
lifting nor coefficient strengthening. We show that both a type-I base inequality \((2.16)\) and the corresponding inequality \((2.19)\) can be facets of \(T_1\) under certain conditions.

**Proposition 8.** If the following conditions hold, then type-I base inequality \((2.16)\) and the corresponding inequality \((2.19)\) are facets of \(T_1\).

1. If there exists \(j \in J^* \setminus J\) with \(\beta_j < \beta_0\) then \(\beta_j - \beta < \beta_0\) and \(\beta(\tilde{J}\setminus\{j_{p},j_{p-1}\}) + \beta_j \leq \beta_0\) where \(\tilde{J} = \{j_1, j_2, \ldots, j_p\}\) and \(\beta_{j_1} \leq \beta_{j_2} \leq \cdots \leq \beta_{j_p}\).

2. For all \(j \in J \setminus J^*\), \(\beta_j < \beta_0\) and \(\beta(\tilde{J}\setminus\{j_p\}) + \beta_j \leq \beta_0\).

**Proof.** We first show that there exists \(\dim(T_1) = |J| + 1\) many affinely independent points that satisfy inequality \((2.19)\) at equality. Consider the following points:

- Let \(t = 0, z_j = 1\) for all \(j \in \tilde{J}, z_j = 0\) for all \(j \in J\setminus\tilde{J}\). In this case, the left-hand side of inequality \((2.19)\) is \(\beta(\tilde{J}) - p\beta = \beta_0 + \beta - p\beta = \beta_0 - (p - 1)\beta = \beta'_0\).

- For each \(j' \in \tilde{J}, t = \beta_{j'} - \beta, z_{j'} = 0, z_j = 1\) for all \(j \in \tilde{J}\setminus\{j'\}, z_j = 0\) for all \(j \in J\setminus\tilde{J}\). In this case, the left-hand side of inequality \((2.19)\) is \(\beta_{j'} - \beta + \beta(\tilde{J}\setminus\{j'\}) - (p - 1)\beta = \beta(\tilde{J}) - p\beta = \beta_0 + \beta - p\beta = \beta_0 - (p - 1)\beta = \beta'_0\). This point also satisfies type-I base inequality \((2.16)\) at equality.

- For each \(j' \in J^* \setminus \tilde{J}\) we consider two cases:

  1. \(\beta_{j'} = \beta_0\).

     Let \(t = 0, z_{j'} = 1, z_j = 0\) for all \(j \in J\setminus\{j'\}\). The left-hand side of inequality \((2.19)\) is \(\min\{((\beta_{j'} - \beta), \beta'_0\} = \min\{((\beta_0 - \beta), \beta_0 - (p - 1)\beta\} = \beta_0 - (p - 1)\beta\) since \(p\) is the number of elements in set \(\tilde{J}\) and \(p \geq 2\), for \(\tilde{J}\) to be a minimal cover. This point also satisfies type-I base inequality \((2.16)\) at equality.

  2. \(\beta_{j'} < \beta_0\).

     Let \(t = \beta_0 - \beta(\tilde{J}\setminus\{j_p,j_{p-1}\}) - \beta_{j'}\), \(z_j = 1\) for all \(j \in \tilde{J}\setminus\{j_p,j_{p-1}\}\), \(z_{j_p} = 0, z_{j_{p-1}} = 0, z_{j'} = 1, z_j = 0\) for all \(J \setminus (\tilde{J}\cup\{j'\})\). From facet
condition 1 we have $\beta_{j'} - \beta < \beta_0'$ hence the left-hand side of inequality (2.19) is $\beta_0 - \beta(\tilde{J} \setminus \{j_p, j_{p-1}\}) - \beta_{j'} + \beta(\tilde{J} \setminus \{j_p, j_{p-1}\}) - (p-2)\beta + \beta_{j'} - \beta = \beta_0 - (p-1)\beta = \beta_0'$. Note that due to facet condition 1, $t \geq 0$. Furthermore, this point also satisfies type-I base inequality (2.16) at equality.

- For each $j' \in J \setminus J^*$ first observe that $\beta_{j'} < \beta_0$ since by definition of $J^*$, $\beta_{j'} < \beta_{j_p} \leq \beta_0$. Let $t = \beta_0 - \beta(\tilde{J} \setminus \{j_p\}) - \beta_{j'}$, $z_j = 1$ for all $j \in \tilde{J} \setminus \{j_p\}$, $z_{j_p} = 0$, $z_{j'} = 1$, $z_j = 0$ for all $J \setminus (\tilde{J} \cup \{j'\})$. From facet condition 2 we have $\beta_{j'} < \beta_0'$ hence the left-hand side of inequality (2.19) is $\beta_0 - \beta(\tilde{J} \setminus \{j_p\}) - \beta_{j'} + \beta(\tilde{J} \setminus \{j_p\}) - (p-1)\beta + \beta_{j'} = \beta_0 - (p-1)\beta = \beta_0'$. Note that due to facet condition 2, $t \geq 0$. Furthermore, this point also satisfies type-I base inequality (2.16) at equality.

In total we have described $1 + |\tilde{J}| + |J^* \setminus \tilde{J}| + |J \setminus J^*| = |J| + 1$ many points. It is easy to see that these points are affinely independent. Furthermore, except for the first described point ($t = 0$, $z_j = 1$ for all $j \in \tilde{J}$, $z_j = 0$ for all $j \in J \setminus \tilde{J}$) all the other $|J|$ many points also satisfy type-I base inequality (2.16) at equality. If we replace the first point with the point $t = \beta_0$, $z_j = 0$ for all $j \in J$, which satisfies the type-I base inequality at equality, then we still get $|J| + 1$ many affinely independent points. Hence, both the type-I base inequality (2.16) and the corresponding inequality (2.19) are facets of $T_1$ under conditions 1 and 2.

Suppose that inequality $\sum_{j \in J} t_j \leq \alpha_0$ is given as a type-II base inequality in the form of (2.24) for set $S_2$, where $\alpha_j = 0$ for all $j \in J$. Assume that there exists $\tilde{J}$ and $m$ such that $\alpha_0 > d(J \setminus \tilde{J})$ and $\alpha_0 - d(J \setminus \tilde{J}) < \max_{j \in \tilde{J}}\{d_j\}$. These conditions imply that $m = 0$ and $\alpha = \alpha_0 - d(J \setminus \tilde{J})$. Then we obtain the corresponding inequality (2.27)

$$\sum_{j \in J} t_j + \sum_{j \in \tilde{J}} \alpha z_j \leq \alpha_0 + (|\tilde{J}| - 1)\alpha,$$ (2.54)
which is valid for $S_2$, under these assumptions.

**Proposition 9.** Inequality (2.54), valid for $S_2$, defines a facet of $T_2$ only if

1. $\tilde{J} \neq \emptyset$.

In addition, if the following conditions hold then (2.54) is a facet of $T_2$:

2. $\alpha_0 < d(J \setminus \tilde{J}) + \min_{j \in J} \{d_j\}$,

3. $\alpha_0 < d(J \setminus \tilde{J}) + \max_{j \in \tilde{J}} \{d_j\} - \max_{j \in J \setminus \tilde{J}} \{d_j\}$,

4. $|J \setminus \tilde{J}| \geq 2$.

**Proof.** Necessity.

Assume that $\tilde{J} = \emptyset$. Then inequality (2.54) reduces to

$$\sum_{j \in J} t_j \leq \alpha_0 - \alpha.$$  \hfill (2.55)

This case implies that $\alpha = \alpha_0 - d(J \setminus \tilde{J}) = \alpha_0 - d(J)$. Thus, inequality (2.55) becomes

$$\sum_{j \in J} t_j \leq d(J)$$

which is dominated by $t_j + d_jz_j \leq d_j$ for all $j \in J$.

**Sufficiency.** We show that there exists $\dim(T_2) = 2|J|$ many affinely independent points that satisfy inequality (2.54) at equality. Let $\epsilon > 0$ be a very small number and $j^* = \arg \max_{j \in \tilde{J}} \{d_j\}$ ($j^*$ exists due to facet condition 1). Consider the following points:

- For each $j' \in \tilde{J}$, let $z_{j'} = 0$, $t_{j'} = \alpha_0 - d(J \setminus \tilde{J})$, $z_j = 1$, $j \in \tilde{J} \setminus \{j'\}$, $t_j = 0$, $j \in \tilde{J} \setminus \{j'\}$, $z_j = 0$, $j \in J \setminus \tilde{J}$, $t_j = d_j$, $j \in J \setminus \tilde{J}$. Note that this is a feasible solution due to the assumption that $\alpha_0 > d(J \setminus \tilde{J})$ and facet condition 2. Furthermore, for each such point we construct another point by increasing $t_{j'}$ by $\epsilon$ and decreasing any $t_j$, $j \in J \setminus \tilde{J}$ by $\epsilon$ ($j$ exists due to facet condition 4). This gives $2|\tilde{J}|$ many points.
For each \( j'' \in J \setminus \tilde{J} \), let \( z_{j''} = 1, t_{j''} = 0, z_j = 0, j \in (J \setminus (\tilde{J} \cup \{j''\})) \cup \{j^*\} \), 
\( t_j = d_j, j \in J \setminus (\tilde{J} \cup \{j''\}) \), \( t_{j^*} = \alpha_0 - d(J \setminus \tilde{J}) + d_{j''}, z_j = 1, j \in \tilde{J} \setminus \{j^*\}, t_j = 0, j \in \tilde{J} \setminus \{j^*\} \). Note that this is feasible due to facet condition 3. For each such point we construct another point by increasing \( t_{j^*} \) by \( \epsilon \) and decreasing any \( t_j, j \in J \setminus (\tilde{J} \cup \{j''\}) \) by \( \epsilon \) (\( j \) exists due to facet condition 4). This gives \( 2|J \setminus \tilde{J}| \) many points.

It is easy to see that these points are affinely independent.

Now, suppose that we start with a type-II base inequality \( \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} \leq s(I) \) in Proposition 7. Note that inequality \( \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} \leq s(I) \) is a relaxation of the supply constraints (2.1c). Let \( t_j = \sum_{i \in I : (i,j) \in E} x_{ij} \) and \( \alpha_j = 0 \) for all \( j \in J \) in inequality (2.24). Suppose that there exists \( \tilde{J} \) and \( m \) such that \( s(I) > d(J \setminus \tilde{J}) \) and \( s(I) - d(J \setminus \tilde{J}) < \max_{j \in J \setminus \tilde{J}} \{d_j\} \). These conditions imply that \( m = 0 \) and \( \alpha = s(I) - d(J \setminus \tilde{J}) \). Then we obtain the inequality

\[
\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{j \in \tilde{J}} \alpha z_j \leq s(I) + (|\tilde{J}| - 1)\alpha,
\]

which is valid for \( X \).

**Proposition 10.** Inequality (2.56), valid for \( X \), defines a facet of \( \text{conv}(X) \) only if

1. \( \tilde{J} \neq \emptyset \).

In addition, if the following conditions hold then (2.56) is a facet of \( \text{conv}(X) \):

2. \( s(V_1) > d(J \setminus \tilde{J}) + \max_{j \in (V_1 \setminus J) \cup \tilde{J}} \{d_j\} \).

3. \( s(I) < d(J \setminus \tilde{J}) + \min_{j \in J \setminus \tilde{J}} \{d_j\} \).

4. \( s(I) \leq d(J \setminus \tilde{J}) + \max_{j \in J \setminus \tilde{J}} \{d_j\} - \max_{j \in J \setminus \tilde{J}} \{d_j\} \).

1. If we replace \( t_j \) by \( \sum_{i \in I : (i,j) \in E} x_{ij} \) for all \( j \in J \) and \( \alpha_0 \) by \( s(I) \) we can use the same argument as in the necessity of facet condition [1] in Proposition [9].

Sufficiency. For the proof we use §1.4.3 Theorem 3.6 [64]. We show that inequality (2.56), plus any linear combination of the demand constraints \( \sum_{i \in V_1 : (i,j) \in E} x_{ij} + d_j z_j = d_j \)
for all \( j \in V_2 \) is the only inequality that is satisfied at equality by all points \((x, z)\)
feasible to TPMC that are tight at (2.56), i.e., we show that if all points of TPMC at which (2.56) is tight satisfy
\[
\sum_{(i, j) \in E} \lambda_{ij} x_{ij} + \sum_{j \in V_2} \omega_j z_j = \hat{\lambda}, \tag{2.57}
\]
then

1. \( \lambda_{ij} = u_j, \ j \in V_2 \setminus J, \ i \in V_1, \ (i,j) \in E, \)
2. \( \lambda_{ij} = u_j, \ j \in J, \ i \in V_1 \setminus I, \ (i,j) \in E, \)
3. \( \lambda_{ij} = \bar{\lambda} + u_j, \ j \in J, \ i \in I, \ (i,j) \in E, \)
4. \( \omega_j = u_j d_j, \ j \in V_2 \setminus \tilde{J}, \)
5. \( \omega_j = \bar{\lambda} \alpha + u_j d_j, \ j \in \tilde{J}, \)
6. \( \hat{\lambda} = \bar{\lambda} \left( s(I) + (|\tilde{J}| - 1) \alpha \right) + \sum_{j \in V_2} u_j d_j. \)

In the proof we consider four different types of points at which (2.56) is tight that make use of the facet conditions. Throughout, let \( \epsilon \) be a very small number greater than zero unless noted otherwise.
1. Consider a point where only markets \( j \in J \setminus \tilde{J} \cup \{r\} \) are satisfied for some \( r \in V_2 \setminus J \), and constraints

\[
\sum_{i \in I, j \in J \setminus (i, j) \in E} x_{ij} = d(J \setminus \tilde{J})
\]

\[
\sum_{i \in V_1, (i, r) \in E} x_{ir} = d_r
\]

\[
x_{ij} = 0, \quad i \in V_1, j \in \tilde{J} \cup V_2 \setminus (J \cup \{r\})
\]

\[
x_{ij} \geq \epsilon, \quad i \in I, j \in J \setminus \tilde{J}
\]

\[
x_{ir} \geq \epsilon, \quad i \in V_1
\]

\[
\sum_{j \in V_2 \setminus (i, j) \in E} x_{ij} \leq s_i - \epsilon, \quad i \in V_1
\]

\[
z_j = 1, \quad j \in \tilde{J} \cup V_2 \setminus (J \cup \{r\})
\]

\[
z_j = 0, \quad j \in \{r\} \cup J \setminus \tilde{J}
\]

in addition to the original constraints are satisfied, which we refer to as System 1. We know that a solution to System 1 exists from assumption \( s(I) > d(J \setminus \tilde{J}) \) and facet condition 2.

2. Consider a point where only markets \( j \in J \setminus \tilde{J} \) are satisfied, and constraints

\[
\sum_{i \in I, j \in J \setminus (i, j) \in E} x_{ij} = d(J \setminus \tilde{J})
\]

\[
x_{ij} = 0, \quad i \in V_1, j \in \tilde{J} \cup V_2 \setminus J
\]

\[
x_{ij} \geq \epsilon, \quad i \in I, j \in J \setminus \tilde{J}
\]

\[
\sum_{j \in V_2 \setminus (i, j) \in E} x_{ij} \leq s_i - \epsilon, \quad i \in I
\]

\[
z_j = 1, \quad j \in \tilde{J} \cup V_2 \setminus J
\]

\[
z_j = 0, \quad j \in J \setminus \tilde{J}
\]
in addition to the original constraints are satisfied, which we refer to as System 2. We know that a solution to System 2 exists from assumption $s(I) > d(J \setminus \tilde{J})$.

3. Consider a point where only markets $j \in J \setminus \tilde{J} \cup \{l\}$ are satisfied for some $l \in \tilde{J}$, and constraints

$$\sum_{i \in I, j \in J: (i,j) \in E} x_{ij} = s(I)$$

$$\sum_{i \in V_1 \setminus I, j \in J: (i,j) \in E} x_{ij} = d(J \setminus \tilde{J}) + d_l - s(I)$$

$$x_{ij} = 0, \quad i \in V_1, j \in \tilde{J} \setminus \{l\} \cup V_2 \setminus J$$

$$x_{ij} \geq \epsilon, \quad i \in V_1, j \in J \setminus \tilde{J} \cup \{l\}$$

$$\sum_{j \in V_2: (i,j) \in E} x_{ij} \leq s_i - \epsilon, \quad i \in V_1 \setminus I$$

$$z_j = 1, \quad j \in \tilde{J} \setminus \{l\} \cup V_2 \setminus J$$

$$z_j = 0, \quad j \in J \setminus \tilde{J} \cup \{l\}$$

in addition to the original constraints are satisfied, which we refer to as System 3. We know that a solution to System 3 exists from facet conditions 2 and 3.

4. Consider a point where only markets $j \in J \setminus (\tilde{J} \cup \{j'\}) \cup \{l^*\}$ are satisfied for $l^* = \arg \max_{j \in J} \{d_j\}$ and some $j' \in J \setminus \tilde{J}$, and constraints

$$\sum_{i \in I, j \in J: (i,j) \in E} x_{ij} = s(I)$$

$$\sum_{i \in V_1 \setminus I, j \in J: (i,j) \in E} x_{ij} = d(J \setminus \tilde{J}) + d_{l^*} - d_{j'} - s(I)$$
in addition to the original constraints are satisfied, which we refer to as System 4.

We know that a solution to system 4 exists from facet conditions 2 and 4.

1. \( \lambda_{ij} = u_j, \ j \in V_2 \setminus J, \ i \in V_1, \ (i, j) \in E. \)

   Consider any solution to system 1 with any market \( j = r \in V_2 \setminus J \) that is satisfied. Choose arbitrary suppliers \( i, i' \in V_1 \) such that \( (i, j), (i', j) \in E \). Construct a new point by decreasing the flow on edge \( (i, j) \) by \( \epsilon \) and increasing the flow on edge \( (i', j) \) by \( \epsilon \). Note that this point is also on the face defined by inequality (2.56). Thus,

   \[
   \lambda_{ij} = u_j, \ j \in V_2 \setminus J, \ i \in V_1, \ (i, j) \in E. \tag{2.58}
   \]

2. \( \lambda_{ij} = u_j, \ j \in J, \ i \in V_1 \setminus I, \ (i, j) \in E. \)

   Consider any solution to system 3 with market \( j \in J \setminus \tilde{J} \cup \{l\} \) satisfied for some \( l \in \tilde{J} \). Choose arbitrary suppliers \( i, i' \in V_1 \setminus I \) such that \( (i, j), (i', j) \in E \). Construct a new point by decreasing the flow on edge \( (i, j) \) by \( \epsilon \) and increasing the flow on edge \( (i', j) \) by \( \epsilon \). Note that this point is also on the face defined by inequality (2.56) since \( i, i' \in V_1 \setminus I \). Thus,

   \[
   \lambda_{ij} = u_j, \ j \in J \setminus \tilde{J} \cup \{l\}, i \in V_1 \setminus I, (i, j) \in E. \tag{2.59}
   \]

Note that since we can use the above argument for any \( l \in \tilde{J} \), we have \( \lambda_{il} = u_i \) for all \( l \in \tilde{J}, i \in V_1 \setminus I, (i, l) \in E. \)

3. \( \lambda_{ij} = \bar{\lambda} + u_j, \ j \in J, \ i \in I, \ (i, j) \in E. \)

   Consider any solution to system 2 Choose arbitrary suppliers \( i, i' \in I \) such that \( (i, j), (i', j) \in E \) for \( j \in J \setminus \tilde{J} \). Construct a new point by decreasing the
flow on edge \((i, j)\) by \(\epsilon\) and increasing the flow on edge \((i', j)\) by \(\epsilon\). Note that this point is also on the face defined by inequality (2.56). Thus,

\[
\lambda_{ij} = \lambda^1_j, j \in J \setminus \tilde{J}, i \in I, (i, j) \in E.
\] (2.60)

Next we consider a solution to system 3 with \(\epsilon = 0\). Choose arbitrary suppliers \(i, i' \in I\) and market \(j \in J \setminus \tilde{J}\) such that \((i, j), (i', j), (i, l), (i', l) \in E\). Construct a new point by decreasing the flow on edges \((i, j), (i', l)\) by \(\epsilon\) and increasing the flow on edges \((i', j), (i, l)\) by \(\epsilon\). Note that this point is also on the face defined by inequality (2.56). Thus,

\[
-\lambda_{ij} + \lambda_{il} + \lambda_{i'j} - \lambda_{i'l} = -\lambda^1_j + \lambda_{il} + \lambda^1_j - \lambda_{i'l} = \lambda_{il} - \lambda_{i'l} = 0.
\] (2.61)

Because \(l\) is any market in set \(\tilde{J}\), \(\lambda_{ij} = \lambda^1_j, j \in \tilde{J}, i \in I, (i, j) \in E\).

Let \(\lambda^1_j = \bar{\lambda}_j + u_j, j \in J\). Next we show that \(\bar{\lambda}_j = \bar{\lambda}, j \in J\). We consider a solution to system 3 with \(\epsilon = 0\). Choose any markets \(j, j' \in J\), any suppliers \(i \in V_1 \setminus I, i' \in I\) such that \((i, j), (i', j), (i, j'), (i', j') \in E\). Decrease flow on edges \((i, j'), (i', j)\) by \(\epsilon\) and increase flow on edges \((i, j), (i', j')\) by \(\epsilon\). Thus,

\[
\lambda_{ij} - \lambda_{ij'} - \lambda_{i'j} + \lambda_{i'j'} = u_j - u_{j'} - \lambda^1_j + \lambda^1_{j'} = 0.
\] (2.62)

By again using \(\lambda^1_j = \bar{\lambda}_j + u_j\) and \(\lambda^1_{j'} = \bar{\lambda}_{j'} + u_{j'}\), we obtain

\[
\bar{\lambda}_j = \bar{\lambda}_{j'} = \bar{\lambda}.
\] (2.63)

4. \(\omega_j = u_jd_j, j \in V_2 \setminus \tilde{J}\).

We rewrite (2.57) using the information obtained until now, and get

\[
\bar{\lambda} \sum_{i \in I, j \in J: (i, j) \in E} x_{ij} + \sum_{(i, j) \in E} u_j x_{ij} + \sum_{j \in V_2} \omega_j z_j = \hat{\lambda}.
\] (2.64)

Consider any solution to system 1 with market \(r \in V_2 \setminus J\) that is satisfied. Then we construct a new solution based on this solution where we set \(z_r = 1\) and
\( x_{ir} = 0 \) for all \( i \in V_1, (i, r) \in E \) and all other variables remain the same. This is a solution to System 2. Thus this solution is also on the face defined by (2.56). We compare inequality (2.57) evaluated at these two solutions. Thus,

\[
\sum_{i \in V_1 \cap (i, r) \in E} x_{ir} - \omega_r = 0. \tag{2.59}
\]

Because \( \sum_{i \in V_1 \cap (i, r) \in E} x_{ir} = d_r \) we have \( \omega_r = u_r d_r, r \in V_2 \setminus J \).

Next we show that \( \omega_j = u_j d_j, j \in J \setminus \tilde{J} \). First we consider a solution to system 3 where we choose \( l = l^* = \arg \max_{j \in \tilde{J} \setminus {l^*}} \{d_j\} \). This is a feasible choice due to facet condition 2. We evaluate (2.64) at this solution, and get

\[
\bar{\lambda}(s(I)) + \sum_{i \in V_1 \cap (i, j) \in E} u_j x_{ij} + \sum_{j \in V_2 \setminus J \cup \tilde{J} \setminus \{l^*\}} \omega_j = \bar{\lambda}. \tag{2.65}
\]

Next we consider a solution to system 4 where some market \( j' \in J \setminus \tilde{J} \) is rejected. We evaluate (2.64) at this solution, and obtain

\[
\bar{\lambda}(s(I)) + \sum_{i \in V_1 \cap (i, j) \in E} u_j x_{ij} + \sum_{j \in V_2 \setminus J \cup \tilde{J} \setminus \{l^*\}} \omega_j + w_{j'} = \bar{\lambda}. \tag{2.66}
\]

We subtract (2.66) from (2.65) and obtain \( u_{j'} \sum_{i \in V_1 \cap (i, j') \in E} x_{ij'} - \omega_{j'} = 0. \) Because \( \sum_{i \in V_1 \cap (i, j') \in E} x_{ij'} = d_{j'} \) we have \( \omega_{j'} = u_{j'} d_{j'}, j' \in J \setminus \tilde{J} \).

5. \( \omega_j = \bar{\lambda} \alpha + u_j d_j, j \in \tilde{J} \).

Consider any solution to system 3 with any market \( l \in \tilde{J} \) that is satisfied. Then
\[ \tilde{\lambda}(s(I)) + \sum_{i \in V_1, j \in J \setminus \tilde{J} \cup \{l\} \cap E} u_j x_{ij} + \sum_{j \in V_2 \setminus J \cup \tilde{J} \setminus \{l\}} \omega_j = \hat{\lambda}. \quad (2.67) \]

We also consider a solution to system 2 where market \( l \in \tilde{J} \) is rejected. Then \[ (2.57) \] reduces to

\[ \tilde{\lambda}(d(J \setminus \tilde{J})) + \sum_{i \in V_1, j \in J \setminus \tilde{J} \cup \{l\} \cap E} u_j x_{ij} + \sum_{j \in V_2 \setminus J \cup \tilde{J}} \omega_j = \hat{\lambda}. \quad (2.68) \]

We subtract \[ (2.68) \] from \[ (2.67) \] and obtain, \[ \tilde{\lambda}(s(I) - d(J \setminus \tilde{J})) + u_l \sum_{i \in V_1; (i,l) \in E} x_{il} - \omega_l = 0. \] Since \( s(I) - d(J \setminus \tilde{J}) = \alpha \) and \( \sum_{i \in V_1; (i,l) \in E} x_{il} = d_l \) we conclude that \( \omega_l = \tilde{\lambda}\alpha + u_l d_l \) for \( l \in \tilde{J} \).

6. \[ \hat{\lambda} = \tilde{\lambda} \left( s(I) + (|\tilde{J}| - 1)\alpha \right) + \sum_{j \in V_2} u_j d_j. \]

We rewrite \[ (2.57) \], and get

\[ \tilde{\lambda} \left( \sum_{i \in I, j \in J \setminus \tilde{J} \cup \{l\} \cap E} x_{ij} + \sum_{j \in \tilde{J}} \alpha z_j \right) + \sum_{(i,j) \in E} u_j x_{ij} + \sum_{j \in V_2} u_j d_j z_j = \hat{\lambda}. \quad (2.69) \]

Evaluating \[ (2.69) \] at any point \((x, z)\) feasible to \( TPMC \) that is tight at inequality \[ (2.56) \] gives

\[ \tilde{\lambda} \left( s(I) + (|\tilde{J}| - 1)\alpha \right) + \sum_{j \in V_2} u_j \left( \sum_{i \in V_1; (i,j) \in E} x_{ij} + d_j z_j \right) = \hat{\lambda}. \]
From the definition of TPMC we have
\[ \sum_{i \in V_1: (i,j) \in E} x_{ij} + d_j z_j = d_j \text{ for all } j \in V_2. \]

Thus, \( \hat{\lambda} = \bar{\lambda} \left( s(I) + (|J| - 1) \alpha \right) + \sum_{j \in V_2} u_j d_j. \) 

Even though Propositions 6 and 7 are general results for mixed-integer cover and knapsack sets \( S_1 \) and \( S_2 \), we observed that many of the facets for TPMC can be derived from the recursive application of these results.

**Example 4.** (Continued.) Observe that inequalities (2.28), (2.29) and (2.30) satisfy all the conditions given in Proposition 8 and inequality (2.32) satisfies all the conditions given in Proposition 10, and hence they are facets of \( \text{conv}(X) \).

Finally, while the blossom inequalities (2.3) are strong for the case that \( d_j \leq 2 \) for all \( j \in V_2 \), they are not facet-defining for the general case of TPMC based on our experience with PORTA [21].

### 2.5 Computational Results

In this section we present our computational results for the TPMC problem. We conduct the experiments on an Intel Xeon x5650 Processor at 2.67GHz with 4GB RAM. We use IBM ILOG CPLEX 12.4 as the MIP solver. We test the TPMC problem for various settings of \( V_1 \) and \( V_2 \). There are 12 combinations of \( V_1 \) and \( V_2 \) as shown in Tables 2.1 and 2.2, in the first column. For each combination, we create 3 instances and report the averages. We observed that most instances of the TPMC problem are solved under a minute for each setting of \( V_1 \) and \( V_2 \). Therefore, we found “hard” instances by continually generating and solving instances until we were able to find 3 that were solved in at least 15 minutes under default CPLEX settings. Problem parameters are generated using a discrete uniform distribution with supply values \( s_i \in [10, 20] \), demand values \( d_j \in [10, 20] \), weights \( w_{ij} \in [20, 50] \).
and lost revenues $r_j \in [5000, 6000]$. In our computations, we impose a time limit of half an hour, and consider the following four algorithms:

1. BB (Branch and Bound): TPMC formulation, (2.1a)-(2.1e) with no cuts,

2. UC (User Cuts): TPMC formulation, (2.1a)-(2.1e) with only user cuts,

3. CD (CPLEX Default Settings): TPMC formulation, (2.1a)-(2.1e) with default CPLEX cuts,

4. UCD (User Cuts and CPLEX Default Settings): TPMC formulation, (2.1a)-(2.1e) with user cuts and default CPLEX cuts.
Table 2.1: Comparison of Algorithms BB and UC

| $|V_1|, |V_2|$ | RGap  | RCuts | EGap   | ECuts | Time (unslvd) | B&C Nodes |
|---|---|---|---|---|---|---|
|   | BB  | UC | BB  | UC | BB  | UC | BB  | UC | BB  | UC |
| 200,230 | 0.79% | 0.76% | - | u8 | 0.66% | 0.67% | - | u189 | 1800(3) | 1800(3) | 176305.3 | 107412 |
| 200,240 | 0.48% | 0.48% | - | u4.3 | 0.42% | 0.41% | - | u186 | 1800(3) | 1800(3) | 168384.7 | 91004.7 |
| 200,250 | 0.21% | 0.20% | - | u4.7 | 0.15% | 0.14% | - | u103.3 | 1800(3) | 1800(3) | 141624.3 | 89966.7 |
| 300,330 | 0.56% | 0.56% | - | u4 | 0.48% | 0.47% | - | u125.7 | 1800(3) | 1800(3) | 75466.3 | 56085.3 |
| 300,340 | 0.78% | 0.77% | - | u6 | 0.71% | 0.71% | - | u123.3 | 1800(3) | 1800(3) | 60125.7 | 50122.7 |
| 300,350 | 0.38% | 0.38% | - | u4.3 | 0.35% | 0.35% | - | u98.3 | 1800(3) | 1800(3) | 51141.3 | 44166.7 |
| 400,430 | 0.26% | 0.24% | - | u4 | 0.23% | 0.20% | - | u74.7 | 1800(3) | 1800(3) | 29565 | 32466.7 |
| 400,440 | 0.25% | 0.25% | - | u4.7 | 0.23% | 0.23% | - | u72.7 | 1800(3) | 1800(3) | 23177.7 | 26600 |
| 400,450 | 0.16% | 0.16% | - | u4.7 | 0.15% | 0.14% | - | u57.7 | 1800(3) | 1800(3) | 22593 | 23733.3 |
| 500,530 | 0.24% | 0.23% | - | u7 | 0.22% | 0.22% | - | u52 | 1800(3) | 1690.5(2) | 18152.3 | 16380 |
| 500,540 | 0.27% | 0.26% | - | u4 | 0.26% | 0.24% | - | u61.3 | 1800(3) | 1800(3) | 16115 | 17573.3 |
| 500,550 | 0.32% | 0.32% | - | u6.3 | 0.31% | 0.30% | - | u21.7 | 1800(3) | 1800(3) | 14767.7 | 18100 |
| Average | 0.39% | 0.38% | - | u5.2 | 0.35% | 0.34% | - | u96.4 | 1800(3) | 1790.9(2.9) | 66451.5 | 47801 |
Table 2.2: Comparison of Algorithms CD and UCD

| $|V_1|,|V_2|$ | RGap  | RCuts  | EGap  | ECuts  | Time (unslnv) | B&C Nodes |
|---------|--------|--------|-------|--------|---------------|-----------|
|         | CD     | UCD    | CD    | UCD    | CD            | UCD       | CD       | UCD       |
| 200,230 | 0.73%  | 0.73%  | 0.19% | 0.19%  | 568.7         | 569,37.3  | 1342.3(1) | 1249.8(1) |
|         | 0.47%  | 0.47%  | 0.36% | 0.34%  | 307.3         | 219.7,101.7 | 1420.9(2) | 1333.9(2) |
| 200,250 | 0.20%  | 0.20%  | 0.07% | 0.07%  | 573.3         | 412,17.3   | 1265.4(1) | 815.6(1)  |
| 300,330 | 0.53%  | 0.54%  | 0.44% | 0.13%  | 164.7         | 178.3,50.3 | 1800(3)  | 1227.1(2) |
| 300,340 | 0.74%  | 0.73%  | 0.17% | 0.15%  | 334.7         | 239,16.3   | 1678.3(1) | 1067.6(1) |
| 300,350 | 0.37%  | 0.37%  | 0.26% | 0.26%  | 161           | 139.3,53.3 | 1025.9(1) | 901.7(1)  |
| 400,430 | 0.23%  | 0.22%  | 0.18% | 0.18%  | 114.7         | 105.7,41   | 1800(3)  | 1234.9(2) |
| 400,440 | 0.24%  | 0.24%  | 0.17% | 0.12%  | 128.7         | 167.7,25   | 1800(3)  | 1216.6(2) |
| 400,450 | 0.15%  | 0.15%  | 0.13% | 0.12%  | 133           | 173.3,23   | 1800(3)  | 1800(3)   |
| 500,530 | 0.23%  | 0.22%  | 0.20% | 0.18%  | 76.3          | 86.3,55    | 1800(3)  | 1661.7(2) |
| 500,540 | 0.26%  | 0.26%  | 0.25% | 0.24%  | 58.3          | 151.3,32   | 1800(3)  | 1800(3)   |
| 500,550 | 0.31%  | 0.31%  | 0.30% | 0.29%  | 26.3          | 68,34.7    | 1800(3)  | 1800(3)   |
| **Average** | 0.37%  | 0.37%  | 0.23% | 0.19%  | **220.6**     | **209.1,40.6** | **1611.1(2)** | **1342.4(1)** | **42981.2** | **31589.3** |
In Tables 2.1 and 2.2, column RGap reports the average percentage integrality gap at the root node just before branching, which is $100 \times \frac{(zub - zrb)}{zub}$, where $zub$ is the objective function value of the best integer solution obtained within time limit and $zrb$ is the best lower bound obtained at the root node. Column RCuts reports the average number of cuts added at the root node. In column EGap, we report the average percentage end gap at termination output by CPLEX, which is $100 \times \frac{(zub - zbest)}{zub}$, where $zbest$ is the best lower bound available within time limit. Column ECuts reports the average number of cuts added after the problem is solved to optimality within the time limit. Column Time (unslvd) reports the average solution time in seconds and the number of unsolved instances in parentheses in cases where not all three instances are solved to optimality within time limit. We denote the user cuts by $u$ and for the other cuts, i.e., cuts added by CPLEX we do not use a prefix. In column B&C Nodes we report the average number of branch-and-cut tree nodes explored. At the end of Tables 2.1 and 2.2 we give the averages of RGap, RCuts, EGap, ECuts, Time (unslvd) and B&C Nodes, respectively. For each value in the tables we report the numbers rounded to the first decimal place except the values of RGap and EGap which are rounded to two decimal places.

User cuts are generated every 10000 B&C nodes. For the variable upper bound inequalities (2.14) we add a violated inequality if $s_i < d_j$, $i \in V_1$, $j \in V_2$, $(i,j) \in E$ and $\bar{x}_{ij} > s_i(1 - \bar{z}_j)$. Recall that inequalities (2.19) are related to the weight inequalities for 0/1 knapsack problems. The exact separation of weight inequalities involves solving 0/1 knapsack problems. Weismantel, Kaparis and Letchford give exact pseudo-polynomial separation algorithms for weight inequalities [45, 94]. The optimization problems for finding the most violated inequalities (2.15) and (2.27) involve nonlinear objectives and constraints that resemble knapsack constraints. Thus, we use a heuristic separation for inequalities (2.15), (2.19) and (2.27). Let $(\bar{x}, \bar{z})$ be a fractional point. The heuristic for finding a violated inequality (2.15) takes $(\bar{x}, \bar{z})$ and
selects sets $I$ and $J$ simultaneously. Set $J$ includes a market with fractional $\bar{z}$ value, and other markets that receive demand from the same suppliers as the market with fractional $\bar{z}$. All the suppliers that do not send demand to markets in set $J$ are placed in set $I$. More details for this heuristic can be found in Algorithm 2. The heuristic for finding a violated inequality (2.19) uses the type-I base inequalities (2.15), and adds the smallest $p$ coefficients of the $z$ variables that exceed the right-hand side, $\beta_0$ to obtain the cover $\tilde{J}$. For all the instances in Tables 2.1 and 2.2 the violated inequality (2.15) (i.e. type-I base inequality) found by the heuristic separation has the coefficients of all the $z$ variables equal to the right-hand side, $\beta_0$. It is easy to see that if at least two coefficients of $z$ variables are not strictly less than the right-hand side, $\beta_0$ in a given type-I base inequality, the new inequality of type (2.19) cannot be a facet of $\text{conv}(X)$. Therefore, for the given instances no violated inequality of type (2.19) is generated. Note that our separation heuristic for inequality (2.19) is different than that of [41, 45, 94] because our choice of set $J$ also impacts the continuous term $t = \sum_{i \in I, j \in J: (i,j) \in E} x_{ij}$, which is not present in their setting. We have three heuristics for finding a violated inequality (2.27). Two of them uses the supply constraints as a base inequality for a certain choice of $J$ (i.e. $\sum_{j \in J: (i,j) \in E} x_{ij} \leq s_i$ for $i \in V_1$ and $J \subseteq V_2$), one of which finds an inequality with $|\tilde{J}| = 1$ and the other finds an inequality with $|\tilde{J}| = |J| - 1$. The details for these heuristics are given by Algorithms 3 and 4, respectively. The third heuristic uses $\sum_{i \in V_1, j \in V_2: (i,j) \in E} x_{ij} \leq s(V_1)$ as a base inequality and finds a violated inequality with $\tilde{J}$ that includes the rejected markets and markets that have fractional $\bar{z}$ values. More details on this heuristic is given in Algorithm 5.

Table 2.1 compares the performance of algorithms BB and UC, to isolate the reduction in the root gap (0.01%) and end gap (0.01%) using our inequalities. Similarly, Table 2.2 compares the performance of the algorithms CD and UCD, to illustrate the
marginal benefit of incorporating our inequalities into default CPLEX, where we
do not observe a reduction in the root gap but observe a reduction in the end gap
(0.04%). Due to the reduction in the integrality gap the number of branch-and-cut
nodes is almost always lower for UC and UCD compared to BB and CD, respectively.
The solution times and the number of unsolved instances are slightly lower for al-
gorithms that include our proposed inequalities. Although the reduction in the root
and end gap are very small when we add our proposed inequalities, due to the nature
of instances one can observe that in some cases it takes at least half an hour to re-
duce a gap by 0.01% (see instances 500,540 and 500,550 in Table 2.1). In conclusion,
our preliminary computational results show that our proposed inequalities does have
some positive effects, but the separation heuristics need to be significantly improved.
Algorithm 2 Heuristic separation for inequalities (2.15)

Input: \((\bar{x}, \bar{z})\)

Output: Sets \(I\) and \(J\) and the corresponding cut for each fractional \(\bar{z}_j\)

\[
I \leftarrow V_1 \\
\text{\(s(V_1 \setminus I) = 0\)} \\
\text{\(d(J) = 0\)} \\
\text{\(\text{tempSupplies} \leftarrow \emptyset\)} \\
\text{\(\text{tempDemand} \leftarrow \emptyset\)} \\
\text{\(\text{switch} = 0\)}
\]

for all the fractional variables \(\bar{z}_j\) do

\[
\text{\(\text{tempDemand} = \{j\}\)} \\
\text{\(J = \{j\}\)}
\]

while \(|\text{tempDemand}| \geq 1\) or \(|\text{tempSupplies}| \geq 1\) do

if \(\text{switch} = 0\) then

for all the supplies \(i\) that have an edge to all nodes \(j\) in \(\text{tempDemand}\) do

if \(\bar{x}_{ij} > 0\) then

\[
I \leftarrow I \setminus \{i\} \\
\text{\(s(V_1 \setminus I) \leftarrow s(V_1 \setminus I) + s_i\)} \\
\text{\(\text{tempSupplies} \leftarrow \text{tempSupplies} \cup \{i\}\)}
\]

end if

end for

\(\text{switch} = 1\)

\(\text{tempDemand} \leftarrow \emptyset\)

end if

if \(\text{switch} = 1\) then

for all demand \(j\) that have an edge to all nodes \(i\) in \(\text{tempSupplies}\) do

if \(\bar{x}_{ij} > 0\) then

\[
J \leftarrow J \cup \{j\} \\
\text{\(d(J) \leftarrow d(J) + d_j\)} \\
\text{\(\text{tempDemand} \leftarrow \text{tempDemand} \cup \{j\}\)}
\]

end if

end for

\(\text{switch} = 0\)

\(\text{tempSupplies} \leftarrow \emptyset\)

end if

end while

if \(d(J) > s(V_1 \setminus I)\) and \(|J| \geq 2\) and \(\max_{j \in J} \{d_j\} > d(J) - s(V_1 \setminus I)\) then

\[
\sum_{i \in I, j \in J, (i,j) \in E} \bar{x}_{ij} + \sum_{j \in J} (\min\{d(J) - s(V_1 \setminus I), d_j\}) \bar{z}_j < d(J) - s(V_1 \setminus I)
\]

add inequality (2.15) with \(I\) and \(J\)

end if

end if

\(I \leftarrow V_1\)

\(s(V_1 \setminus I) = 0\)

\(d(J) = 0\)

\(\text{switch} = 0\)

end for
Algorithm 3 Heuristic separation for inequalities (2.27) that finds $|	ilde{J}| = 1$

Input: $(\bar{x}, \bar{z})$

Output: Sets $I$, $J$, $\tilde{J}$ and the corresponding cut for each fractional $\bar{z}$

$I, J, \tilde{J} \leftarrow \emptyset$
$d(J \setminus \tilde{J}) = 0$
$\alpha = 0$

for all the fractional variables $\bar{z}_j$ do

$J \leftarrow \{j\}$, $\tilde{J} \leftarrow \{j\}$

for all $i$ such that $\bar{x}_{ij} > 0$ do

$I \leftarrow \{i\}$

for all $j^* \neq j$ do

if $\bar{x}_{ij^*} = d_{j^*}$ then

$J \leftarrow J \cup \{j^*\}$

$d(J \setminus \tilde{J}) = d(J \setminus \tilde{J}) + d_{j^*}$

end if

end for

$\alpha = s_i - d(J \setminus \tilde{J})$

if $|J| \geq 2$ and $\sum_{j \in J, (i,j) \in E} \bar{x}_{ij} + \alpha \bar{z}_j > s_i$ then

add inequality (2.27) with $I$, $J$, $\tilde{J}$ and $\alpha$

end if

$I \leftarrow \emptyset$, $J \leftarrow \{j\}$, $d(J \setminus \tilde{J}) = 0$

end for

end for
Algorithm 4 Heuristic separation for inequalities [2.27] that finds $|\tilde{J}| = |J| - 1$

**Input:** $(\bar{x}, \bar{z})$

**Output:** Sets $I$, $J$, $\tilde{J}$ and the corresponding cut for each fractional $\bar{z}_j$

$J_0 \leftarrow \{ j \in V_2 : \bar{z}_j = 0 \}$

$J_1 \leftarrow \{ j \in V_2 : \bar{z}_j = 1 \}$

$I \leftarrow \emptyset$

$\alpha = 0$

$maxd_j \tilde{J} = \max_{j \in \tilde{J}} \{d_j\}$

for all the fractional variables $\bar{z}_j$ do

$\tilde{J} \leftarrow J_1 \cup \{ j \}$

if $maxd_j \tilde{J} < d_j$ then

$maxd_j \tilde{J} = d_j$

end if

end for

for all $i \in V_1$ do

for all $j' \in J_0$ do

if $\bar{x}_{ij'} > 0$ and $s_i > d_{j'}$ and $s_i - d_{j'} < maxd_j \tilde{J}$ then

$\alpha = s_i - d_{j'}$

$I \leftarrow \{ i \}$, $\tilde{J} \leftarrow \tilde{J} \cup \{ j' \}$

if $\sum_{j \in \tilde{J} \cup (i,j) \in E} \bar{x}_{ij} + \alpha \sum_{j \in \tilde{J}} \bar{z}_j > s_i + (|\tilde{J}| - 1)\alpha$ then

add inequality (2.27) with $I$, $J$, $\tilde{J}$ and $\alpha$

end if

end if

end for

end for

end if
Algorithm 5 Heuristic separation for inequalities (2.27) that finds general $\tilde{J}$

**Input:** $(\bar{x}, \bar{z})$

**Output:** Sets $I$, $J$, $\tilde{J}$ and the corresponding cut $\tilde{J}_f$

- $J_f \leftarrow \{j \in V_2 : 0 < \bar{z}_j < 1\}$
- $J_1 \leftarrow \{j \in V_2 : \bar{z}_j = 1\}$
- $I \leftarrow V_1$
- $J \leftarrow V_2$
- $\tilde{J} \leftarrow J_f \cup J_1$
- $\alpha = 0$
- $maxd_j \tilde{J} = \max_{j \in J_f \cup J_1} \{d_j\}$

**if** $s(V_1) - d(V_2 \setminus \tilde{J}) > 0$ and $s(V_1) - d(V_2 \setminus \tilde{J}) < maxd_j \tilde{J}$ **then**

- $\alpha = s(V_1) - d(V_2 \setminus \tilde{J})$
- **if** $\sum_{i \in V_1, j \in V_2 : (i, j) \in E} \bar{x}_{ij} + \alpha \sum_{j \in \tilde{J}} \bar{z}_j > s(V_1) + (|\tilde{J}| - 1)\alpha$ **then**

- add inequality (2.27) with $I$, $J$, $\tilde{J}$ and $\alpha$

**end if**

**end if**

2.6 Proofs of Section 2.2

In this section, we assume that all data are integral.

**Proposition 1.** The decision version of TPMC is $\mathcal{NP}$-complete even when:

1. $s_i = 1$ for all $i \in V_1$, $d_j = d \geq 3$ for all $j \in V_2$, $w_{ij} = 0$ for all $(i, j) \in E$ and $r_j = 1$ for all $j \in V_2$.

2. $|V_1| = 1$ and $w_{ij} = 0$ for all $(i, j) \in E$.

**Proof.** Since TPMC is a mixed integer linear problem with rational data, it is in $\mathcal{NP}$. We present two reductions to verify the two parts of this result.

1. We reduce every instance of the Exact 3-Cover (E3C) problem to an instance of TPMC. An instance of E3C is given as: Let $B$ be a base set where $|B| = 3q$ for some $q \in \mathbb{N}$. Let $C$ be a collection of subsets of $B$ where each subset is of cardinality 3. Does there exist $D \subseteq C$ such that $|D| = q$ and the union of sets in $D$ covers every element of $B$?

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It is well-known that E3C is strongly \( \mathcal{NP} \)-complete \(^{[36]} \). Given an instance of E3C, we construct an instance of TPMC as follows: For every element in \( B \), we construct a node in \( V_1 \) and for every element in \( C \) we construct a node in \( V_2 \). For \( i \in V_1 \), we use the notation \( B(i) \) to denote the element of \( B \) corresponding to node \( i \). Similarly, for \( j \in V_2 \), we let \( C(j) \) denote the element of \( C \) corresponding to node \( j \). We add an edge between \( i \in V_1 \) and \( j \in V_2 \) if \( B(i) \in C(j) \). Let \( s_i = 1 \) for all \( i \in V_1 \). Let \( d_j = 3 \) for all \( j \in V_2 \). Let \( w_{ij} = 0 \) for all \( (i, j) \in E \). Let \( r_j = 1 \) for all \( j \in V_2 \).

Next, we verify that there exists \( D \subseteq C \) such that \( |D| = q \) and \( D \) covers every element of \( B \) if and only if there exists a feasible solution to TPMC with a cost at most \( |C| - q \). Note that the size of the TPMC instance is polynomially bounded by the size of the E3C instance.

\((\Rightarrow)\) Assume that there exists \( \{D_1, \ldots, D_q\} =: D \subseteq C \) such that \( D \) covers every element of \( B \) if and only if there exists a feasible solution to TPMC with a cost at most \( |C| - q \). Note that the size of the TPMC instance is polynomially bounded by the size of the E3C instance.

\((\Leftarrow)\) Consider a solution \((\hat{x}, \hat{z})\) of TPMC such that

\[
\sum_{(i,j) \in E} w_{ij} \hat{x}_{ij} + \sum_{j \in V_2} r_j \hat{z}_j = |C| - q.
\]

(2.70) Since there are \( 3q \) supply nodes, each with a capacity of 1, the demand of at
most $q$ nodes can be satisfied. Therefore, from (2.70), we conclude that there are exactly $q$ nodes whose demands are satisfied. Let $D = \{C(j) | \sum_{i \in V_1} \hat{z}_j = 0\}$. Clearly, $|D| = q$ and $D$ covers every element of $B$. As a result TPMC is strongly $\mathcal{NP}$-complete.

2. We reduce every instance of the Subset Sum (SS) problem to an instance of TPMC. An instance of SS is given as: Let $A$ be a finite set, $a_n \in \mathbb{Z}^+$ be the size of each element $n \in A$ and $B$ be a positive integer. Does there exist a subset $A' \subseteq A$ such that the sum of the sizes of the elements in $A'$ is exactly $B$?

It is well-known that SS is $\mathcal{NP}$-complete [36]. Given an instance of SS, we construct an instance of TPMC as follows: We construct a single node $V_1 = \{1\}$ and for every element in $A$ we construct a node in $V_2$. We add all the edges between the nodes in $V_1$ and $V_2$. Let the single supply be $s_1 = B$. Let demand of market $j$ be $d_j = a_j$ for all $j \in V_2 = A$. Finally, let the unit shipping costs and lost revenues be $w_{1j} = 0$ and $r_j = d_j$, for $j \in V_2$.

Next, we verify that there exists subset $A' \subseteq A$ such that the sum of the sizes of the elements in $A'$ is exactly $B$ if and only if there exists a feasible solution to TPMC with a cost of at most $\sum_{n \in A} a_n - B$. Note that the size of the TPMC instance is polynomially bounded by the size of the SS instance.

$(\Rightarrow)$ Assume that there exists a subset $A' \subseteq A$ such that the sum of the sizes of the elements in $A'$ is exactly $B$. Now construct the solution

$$
\hat{x}_{1j} = \begin{cases} 
a_j & \text{if } j \in A' \\
0 & \text{otherwise},
\end{cases}
$$
\[ \hat{z}_j = \begin{cases} 1 & \text{if } j \notin A' \\ 0 & \text{otherwise.} \end{cases} \]

It is straightforward to verify that \((\hat{x}, \hat{z})\) satisfies all the constraints of TPMC and
\[ \sum_{(i,j)\in E} w_{ij} \hat{x}_{ij} + \sum_{j\in V_2} r_j \hat{z}_j = \sum_{n\in A} a_n - B. \]

\((\Leftarrow)\) Consider a solution \((\hat{x}, \hat{z})\) of TPMC such that
\[ \sum_{(i,j)\in E} w_{ij} \hat{x}_{ij} + \sum_{j\in V_2} r_j \hat{z}_j = \sum_{n\in A} a_n - B. \] \hfill (2.71)

The total demand satisfied by any feasible solution is at most \(B\) since we cannot satisfy more than the supply. Furthermore, since each edge has a cost per unit flow of 0, we have that \(\sum_{(i,j)\in E} w_{ij} \hat{x}_{ij} = 0\). Therefore, from (2.71), the total demand satisfied must equal \(B\). Let the set of satisfied demand nodes be \(A' = \{j \in A : \hat{z}_j = 0\}\), so we have \(\sum_{n\in A'} a_n = B\).

\[ \square \]

**Proposition 2.** Suppose that \(d_j \leq 2\) for all \(j \in V_2\). Then there exists a polynomial-time algorithm to solve TPMC.

**Proof.** We can convert a given instance of TPMC with \(d_j \leq 2\) for all \(j \in V_2\) and arbitrary supplies into an equivalent instance with all supplies equal to 1. Observe that in any feasible solution since \(d_j \leq 2\) for all \(j \in V_2\), no supply can send more than \(2|V_2|\) units. Therefore, we construct an updated instance by replacing supply node \(i \in V_1\) with \(\min\{s_i, 2|V_2|\}\) supply nodes with a capacity of 1 and unit shipping cost to demand node \(j\) of \(w_{ij}\) for \((i,j)\in E\). Notice that the resulting instance is polynomial in the size of the original problem. Therefore from now on we assume that \(s_i = 1\) for all \(i \in V_1\).
We show that TPMC with \( d_j \leq 2 \) for all \( j \in V_2 \) is equivalent to the problem of finding a minimum weight perfect matching on a suitably constructed general graph \( G' = (V', E') \).

1. For each \( i \in V_1 \), we add a corresponding \( i \in V' \) and similarly for each \( j \in V_2 \) we add \( j \in V' \). (When we use notation \( V_1 \subseteq V' \), \( V_1 \) represents the vertices of \( V' \) corresponding to the vertices \( V_1 \) of \( G \); similarly for \( V_2 \).)

2. Let \( M_1 = \{ j \in V_2 : d_j = 1 \} \) and \( M_2 = \{ j \in V_2 : d_j = 2 \} \).

3. For each demand node \( j \in V_2 \), add a node \( j' \in V' \) (note that this is in addition to \( j \in V' \) for \( j \in V_2 \) as described in 1.). Add an edge \((j, j') \in E'\) with a cost of \( r_j \). We refer to the set of nodes \( j' \in V' \) corresponding to \( j \in M_1 \) as \( M'_1 \). (We define \( M'_2 \) similarly.)

4. For each \( i \in V_1 \) such that \((i, j) \in E\) and \( j \in M_1 \), add the edge \((i, j) \in E'\) with cost of \( w_{ij} \).

5. For each \( i \in V_1 \) such that \((i, j) \in E\) and \( j \in M_2 \), add two nodes, \( ij1, ij2 \in V' \). Add edges \((i, ij1), (ij1, ij2), (ij2, j), (ij2, j') \in E'\) with costs \( \frac{w_{ij}}{2}, 0, \frac{w_{ij}}{2}, \frac{w_{ij}}{2} \) respectively.

6. If \( |V_1| \) is odd, we add an additional artificial node \( \{0\} \) to \( V' \). Let \( V'_1 \subseteq V' \) be defined as \( V'_1 := V_1 \cup M'_1 \) if \( |V_1| \) is even and \( V'_1 := V_1 \cup M'_1 \cup \{0\} \) if \( |V_1| \) is odd.

7. For all \( u, v \in V'_1 \) such that \( u \neq v \), add an edge \((u, v) \in E'\) with a cost of 0.

Therefore, the subgraph induced by the nodes in \( V'_1 \) is a complete graph/clique.

Note that the size of the resulting minimum weight perfect matching problem is polynomial in the size of the TPMC problem. Figure 2.2 illustrates the original graph of a TPMC instance, where the demand of market \( A \) is 2 and that of market
Figure 2.3 illustrates the new graph. (The clique induced by \( V_1 \cup \{ B' \} \cup \{ 0 \} \) is not shown.)

Figure 2.2: A TPMC instance

Figure 2.3: Construction of \( G' \)

We next show that any solution to the TPMC problem corresponds to a perfect matching in \( G' = (V', E') \). Consider a feasible solution \((x, z)\) to the TPMC problem. If \( z_j = 0 \) for \( j \in M_1 \), then there exists exactly one supply node \( i \) such that \( x_{ij} = 1 \). For constructing a matching in \( G' \), we choose edge \((i, j)\), where \( i \in V_1 \) and \( j \in M_1 \), thereby covering nodes \( i \) and \( j \) in \( V' \). If \( z_j = 0 \) for \( j \in M_2 \), then there exists two supply nodes \( i_1 \) and \( i_2 \in V_1 \) such that \( x_{i_1 j} = x_{i_2 j} = 1 \). For constructing a matching in \( G' \), without loss of generality, we choose edges \((i_1, i_1 j_1)\), \((i_1 j_2, j)\), \((i_2, i_2 j_1)\) and \((i_2 j_2, j')\), thereby covering nodes \( i_1, i_2, i_1 j_1, i_1 j_2, i_2 j_1, i_2 j_2, j, j' \). If \( z_j = 1 \) for \( j \in V_2 \), then no supply node \( i \) sends demand to \( j \) and for the matching we choose edge \((j, j')\), hence covering nodes \( j \) and \( j' \) in \( V' \). Moreover if \( j \in M_2 \), we choose edges \((i j_1, i j 2)\) for all \((i, j) \in E, i \in V_1 \) in the matching and therefore the nodes \( i j_1, i j 2, j, j' \) are also covered. Hence whether \( z_j = 1 \) or \( z_j = 0 \), and whether \( j \in M_1 \) or \( j \in M_2 \), the
nodes in $V_2$, $M'_2$, and the nodes $ij1, ij2$ for all $(i, j) \in E, j \in M_2$ are always covered by the edges in the matching we have selected thus far. To complete the proof we show how nodes $i \in V'_1$ are also covered in all cases by extending the matching we have until now.

Let $\tilde{M}_1 = \{ j \in M_1 : z_j = 0 \}$, $\tilde{M}_2 = \{ j \in M_2 : z_j = 0 \}$ and $\tilde{V}_1 = \{ i \in V_1 : x_{ij} = 1 \}$. In other words, set $\tilde{M}_1$ represents the nodes $j \in M_1$ whose unit demands are satisfied, set $\tilde{M}_2$ represents the nodes $j \in M_2$ whose demands, $d_j = 2$, are satisfied, and set $\tilde{V}_1$ represents the set of supply nodes that send demand. Observe that the nodes in $\tilde{V}_1$ are also covered in the matching constructed thus far. However, the nodes $j \in V_1 \setminus \tilde{V}_1$, and $j' \in M'_1$ for $j \in \tilde{M}_1$ and $\{0\}$ (if it exists) are not yet covered. Note that $|\tilde{V}_1| = |\tilde{M}_1| + 2|\tilde{M}_2|$. We consider two cases.

1. $|V_1|$ is even. If $|\tilde{V}_1|$ is even, then $|V_1| - |\tilde{V}_1|$ and $|\tilde{M}_1|$ are even. If $|\tilde{V}_1|$ is odd, then $|V_1| - |\tilde{V}_1|$ and $|\tilde{M}_1|$ are odd. Therefore, $|V_1| - |\tilde{V}_1| + |\tilde{M}_1|$ is always even. Thus, we can cover all nodes $i \in V_1 \setminus \tilde{V}_1$ and $j' \in M'_1$ for $j \in \tilde{M}_1$ using $\frac{|V_1| - |\tilde{V}_1| + |\tilde{M}_1|}{2}$ many disjoint edges that exist between them (recall that the subgraph induced by the nodes $i \in V'_1$ form a complete graph).

2. $|V_1|$ is odd. If $|\tilde{V}_1|$ is even, then $|V_1| - |\tilde{V}_1|$ is odd and $|\tilde{M}_1|$ is even. If $|\tilde{V}_1|$ is odd, then $|V_1| - |\tilde{V}_1|$ is even and $|\tilde{M}_1|$ is odd. Therefore, $|V_1| - |\tilde{V}_1| + |\tilde{M}_1|$ is always odd. Recall that when $|\tilde{V}_1|$ is odd we have an additional dummy node $\{0\}$ that forms a fully connected graph with nodes $i \in V_1$ and $j \in M'_1$. Therefore, we obtain an even number of nodes that need to be covered by choosing $\frac{|V_1| - |\tilde{V}_1| + |\tilde{M}_1| + 1}{2}$ disjoint edges.

So we have verified that given any solution to the TPMC problem we can find a perfect matching in $G' = (V', E')$. Moreover, it is straightforward to check that the cost of this matching is equal to the cost of the given solution to TPMC.
Next we show that any solution to the perfect matching in $G' = (V', E')$ corresponds to a feasible solution of the TPMC problem. Let $P$ be the set of edges that are in the perfect matching. If edge $(j', j) \in P$ for $j' \in M'_1$, $j \in M_1$ (or $j \in M_2$, $j' \in M'_2$), then set $z_j = 1$. Set all remaining $z_j = 0$. If edge $(i, j) \in P$ for $j \in M_1$, then we set $x_{ij} = 1$. If edge $(i, ij1) \in P$, then set $x_{ij} = 1$. Set all remaining $x_{ij} = 0$. Note that due to the construction of graph $G'$, a supply node $i \in V_1$ can send at most 1 unit of demand. Similarly for $j \in M_1$ a single edge that has $j$ as one of its endpoints will be selected. For $j' \in M'_2$, $j \in M_2$ if edge $(j, j') \in P$, then for any $i \in V_1$ edges $(ij2, j), (ij2, j') \notin P$. However, if edge $(j, j') \notin P$ then for a perfect matching there must exist exactly two $i_1, i_2 \in V_1$ such that $(i_1j2, j), (i_2j2, j') \in P$. Therefore, for any $j \in M_2$ either the demand is fully satisfied or it is rejected altogether. Finally, it is easy to see that the cost of the solution to the TPMC problem is equivalent to the cost of the corresponding perfect matching in $G'$, completing the proof.
CHAPTER 3
A POLYHEDRAL STUDY OF RAMPING IN UNIT COMMITMENT

3.1 Introduction

With deregulation of the energy industry and higher penetration of wind and other intermittent power supplies, the problem of scheduling power generators to meet the load (demand for energy) over large geographical regions has become increasingly challenging. At the crux of most power system operations is the so-called unit commitment problem (UC), which seeks to determine a minimum cost production schedule of a set of power generators to meet the load while satisfying a host of operational constraints [81]. Two main sets of decisions are made in a UC problem. The first set of decisions determines which generators to turn on/off at each time period, whereas the second set determines the amount of output of each online generator at each time period so that the load is met. Operational constraints on the generators include spinning reserves, min/max electricity output levels, minimum up/down time and ramping up/down limits, among others.

The combinatorial nature of the operational constraints makes the UC problem particularly difficult to solve to optimality for practical large-scale instances. Even small improvements in the quality of solutions affect the price of electricity over large geographical regions and lead to millions of dollars of savings per day for the consumers. Therefore, independent system operators (ISOs) are keen to find provably
optimal solutions to the UC problem. Over the years, various algorithmic approaches such as dynamic programming [55, 83], branch-and-bound [23], Benders decomposition [8], Lagrangian relaxation [63], unit decommitment [87], genetic algorithms [47, 66], simulated annealing [98] and tabu search [56] have been proposed to find near-optimal feasible solutions to the UC problem. The reader is referred to [42, 78] for a review of solution approaches that consider both deterministic and stochastic loads.

Recent advances in mixed-integer programming (MIP) software have made it possible to solve larger instances of the UC problem to optimality. MIP formulations also offer additional modeling flexibility to handle challenging operational constraints. Garver [37] describes the first MIP formulation for the UC problem, which has been used extensively. Li and Shahidehpour [54] show advantages of solving the MIP formulation of a price-based unit commitment (PUBC) problem for a generating company compared to using the Lagrangian relaxation method. However, there has been limited research on the polyhedral structure of the UC problem to strengthen the MIP formulations in order to leverage the advances in the state-of-the-art optimization software. One of the exceptions is the work of Lee et al. [51], which considers a relaxation of the UC problem with only the minimum up/down time constraints. The authors propose alternating up/down inequalities that are valid for this relaxation and show that the formulation is tight. Subsequently, Rajan and Takriti [76] give a compact extended formulation of this relaxation, which includes the additional start-up and shut-down variables.

In this chapter, we consider a different relaxation of the unit commitment problem with ramping constraints and production limits. Ramping constraints are used to control the change in production level for a generator from one time period to the next. Carrión and Arroyo, Frangioni et al. and Wang et al. [20, 32, 93] consider alternative formulations to represent the ramping constraints. These formulations can
be strengthened with the addition of the start-up and shut-down variables. Ostrowski et al. give polynomial classes of upper bound, and two- and three-period ramp-up and ramp-down inequalities to strengthen their formulation. Our polyhedral study complements that of by providing several exponential classes of multi-period ramping and multi-period variable upper bound inequalities. We also give the first complete description of the convex hull of feasible solutions for the two-period ramping relaxation. The two-period inequalities can be readily used to strengthen ramping formulations without the need for separation.

In this chapter we assume that generators start/end their production levels at the minimum output value. However, in general once a generator is turned on (off) it cannot immediately start (stop) production at its minimum power output value but instead it follows an increasing (a decreasing) power trajectory from 0 (minimum power output value) to the minimum power output value (0). The start-up and shut-down process of a generator can take up to several hours. Therefore, start-up and shut-down ramp constraints that involve an increasing and decreasing power trajectory, respectively can be introduced to the problem. The formulation of these power trajectories are studied by Arroyo and Conejo and Morales-España et al. In the latter study the authors also consider piecewise-linear power-trajectories instead of staircase energy-blocks for operating ramp constraints that exclude the start-up and shut-down periods. In other words, they take into account the case in which a generator cannot ramp-up/down immediately to reach its maximum ramp rate until the very end of the hour it is ramped up/down.

The organization of this chapter is as follows. In Section we give an MIP formulation of the unit commitment problem and describe its ramping relaxations. In Section , we show that optimization of a linear function over the ramping relaxation is polynomial. In Section , we study the ramping relaxation of the UC problem with two time periods and develop new inequalities that give the complete
convex hull description of this relaxation. In Section 3.4, we generalize these inequalities to multiple periods and propose several exponential classes of valid inequalities for ramp-up and ramp-down relaxations. Furthermore, we prove the strength of these inequalities and describe polynomial separation algorithms for them. Finally, in Section 3.5, we provide computational results that show the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the unit commitment problem with ramping constraints. The work in this chapter can also be found in [26].

3.1.1 Problem Description

First, we present an MIP formulation of the UC problem. Let \( m \) be the number of generators, \( n \) be the length of the planning horizon. Throughout the chapter, we let \([a, b] := \{ j \in \mathbb{Z} : a \leq j \leq b \} \) (\([a, b] = \emptyset \) if \( a > b \)). In period \( t \), the fixed cost of starting up generator \( g \) is \( f_{t,g} \), the fixed cost of running generator \( g \) is \( h_{t,g} \), and the unit production cost is \( c_{t,g} \), \( t \in [1,n], g \in [1,m] \). For any period \( t \in [1,n] \) the load is given by \( \tilde{d}_t \) and the spinning reserve constant is given by \( r_t \geq 1 \). In the UC problem the physical constraints of generators include limits on production levels. In particular, the maximum production level when a generator \( g \in [1,m] \) is started up (and before shut-down) is given by \( \overline{u}_g \). In addition, the maximum change in production from one operating period to the next, in absolute value, is limited by \( \delta_g > 0 \), and the minimum and maximum production levels in any period are limited by \( \ell_g \) and \( u_g \), respectively. For simplicity of notation, we define \( \overline{u}_g \) as the maximum production level for both the start-up and shut-down of a generator and similarly, the maximum change for both ramp-up and ramp-down is represented by \( \delta_g \). However, these values can be different for start-up, shut-down, ramp-up and ramp-down and all the results in this chapter will still hold with minor modifications. Finally, it is assumed that when a generator \( g \) is turned on, it needs to remain on for at least \( \tau_g \) periods. This is known as the
minimum up time of a generator. Similarly, if a generator \( g \) is turned off, then it needs to remain off for at least \( \nu_g \) periods. This is referred to as the minimum down time of a generator.

Next, we describe the decision variables. For \( g \in [1, m] \), let \( p_{t,g} \) be the production level in period \( t \) for generator \( g \), and \( x_{t,g} \) be 1 if generator \( g \) is operating in period \( t \) and 0 otherwise, for \( t \in [1, n] \). Let \( s_{t,g} \) be 1 if generator \( g \) is started in period \( t \) and 0 otherwise, and \( z_{t,g} \) be 1 if generator \( g \) is stopped in period \( t \) and 0 otherwise, for \( t \in [2, n], g \in [1, m] \). An MIP formulation of the UC problem where the objective is to minimize the operational cost of generators over \( n \) periods is

\[
\begin{align*}
\text{min} & \quad \sum_{g=1}^{m} \left( \sum_{t=1}^{n} c_{t,g} p_{t,g} + \sum_{t=1}^{n} h_{t,g} x_{t,g} + \sum_{t=2}^{n} f_{t,g} s_{t,g} \right) \\
\text{s.t.} & \quad \sum_{g=1}^{m} p_{t,g} \geq \bar{d}_t, \quad t \in [1, n] \\
& \quad \sum_{g=1}^{m} u_g x_{t,g} \geq r_t \tilde{d}_t, \quad t \in [1, n] \\
& \quad \ell_g x_{t,g} \leq p_{t,g}, \quad t \in [1, n], g \in [1, m] \\
& \quad p_{t,g} \leq u_g x_{t,g}, \quad t \in [1, n], g \in [1, m] \\
& \quad x_{t+1,g} - x_{t,g} \leq s_{t+1,g}, \quad t \in [1, n-1], g \in [1, m] \\
& \quad \sum_{i=t-\nu_g+1}^{t} s_{i,g} \leq 1 - x_{t-\nu_g,g}, \quad t \in [\nu_g + 1, n], g \in [1, m] \\
& \quad \sum_{i=t-\nu_g+1}^{t} s_{i,g} \leq x_{t,g}, \quad t \in [\nu_g + 1, n], g \in [1, m] \\
& \quad s_{t,g} - z_{t,g} = x_{t,g} - x_{t-1,g}, \quad t \in [2, n], g \in [1, m] \\
& \quad p_{t+1,g} - p_{t,g} \leq \bar{u}_g s_{t+1,g} + \delta_g x_{t,g}, \quad t \in [1, n-1], g \in [1, m] \\
& \quad p_{t,g} - p_{t+1,g} \leq \bar{u}_g z_{t+1,g} + \delta_g x_{t+1,g}, \quad t \in [1, n-1], g \in [1, m] \\
& \quad s \in \{0, 1\}^{(n-1)m}, x \in \{0, 1\}^{nm}, z \in \mathbb{R}_+^{(n-1)m}, p \in \mathbb{R}_+^{nm}. 
\end{align*}
\]
The objective (3.1a) minimizes the total operating cost, including the power generation, setup and start-up costs. We refer to the feasible set defined by (3.1b)-(3.1l) as $\mathcal{UC}$. Constraints (3.1b) ensure that the load is met in every period $t \in [1, n]$. Constraints (3.1c) are the so-called spinning reserve constraints, which require that the total maximum capacity of all online generators is enough to satisfy a constant factor of the load in every period $t \in [1, n]$. Constraints (3.1d) and (3.1e) are the minimum and maximum production constraints for any period $t$ and generator $g$, $t \in [1, n]$, $g \in [1, m]$. Constraints (3.1f) describe the relationship between the start-up variables $s$ and generator on/off status variables $x$. The minimum up- and down-time restrictions are modeled by constraints (3.1g) and (3.1h), respectively, for any period $t$ and generator $g$, $t \in [1, n]$, $g \in [1, m]$. They ensure that if a generator $g$ is turned on (off), then it stays on (off) for at least $v_g$ ($v_g$) time periods. Equations (3.1i) expresses the shut-down variable $z_{t,g}$ in terms of variables $s_{t,g}$ and $x_{t,g}$, $t \in [2, n]$, $g \in [1, m]$. Ramp-up and ramp-down are described by constraints (3.1j) and (3.1k), respectively. These constraints ensure that the production level in the first (last) period the generator $g$ is started up (shut down) is at most $\bar{u}_g$, and that the absolute value of the difference in production levels from one period to the next is at most $\delta_g$, for $g \in [1, m]$. Note that constraints (3.1i) and (3.1l) ensure that the turn-off variables $z$ are binary, and hence we do not add this restriction to the model. The unit commitment problem formulation given by inequalities (3.1b)-(3.1l) is also studied by Ostrowski et al. [68]. Note that the authors use a slightly different notation for the right-hand side of the spinning reserve inequality (3.1c). Instead of defining spinning reserve constant $r_t$, for a given time period $t \in [1, n]$ as a percentage of load, they use load plus the spinning reserve value as the right-hand side. Thus, $r_t$ and $\tilde{d}_t$ are both defined in power units. The results presented in this chapter are valid for any unit commitment problem formulation that includes as substructures the relaxations we study.

We consider relaxations with a single generator. Hence, in what follows, we drop
the subscript referring to the generators. We define the ramp-up polytope as the convex hull of ramp-up constraints and the production limits for a single generator given by

\[ \ell x_t \leq p_t, \quad t \in [1, n] \quad (3.2a) \]
\[ p_t \leq u x_t, \quad t \in [1, n] \quad (3.2b) \]
\[ x_{t+1} - x_t \leq s_{t+1}, \quad t \in [1, n-1] \quad (3.2c) \]
\[ s_{t+1} \leq 1 - x_t, \quad t \in [1, n-1] \quad (3.2d) \]
\[ s_t \leq x_t, \quad t \in [2, n] \quad (3.2e) \]
\[ p_{t+1} - p_t \leq \bar{u} s_{t+1} + \delta x_t, \quad t \in [1, n-1] \quad (3.2f) \]
\[ s \in \{0, 1\}^{n-1}, x \in \{0, 1\}^n, p \in \mathbb{R}_+^n. \quad (3.2g) \]

We refer to the feasible set defined by constraints (3.2a)-(3.2g) as \( U \). We also define the ramp-down polytope as the convex hull of ramp-down constraints and the production limits for a single generator given by

\[ \ell x_t \leq p_t, \quad t \in [1, n] \quad (3.3a) \]
\[ p_t \leq u x_t, \quad t \in [1, n] \quad (3.3b) \]
\[ x_t - x_{t+1} \leq z_{t+1}, \quad t \in [1, n-1] \quad (3.3c) \]
\[ z_t \leq 1 - x_t, \quad t \in [2, n] \quad (3.3d) \]
\[ z_t \leq x_{t-1}, \quad t \in [2, n] \quad (3.3e) \]
\[ p_{t+1} - p_t \leq \bar{u} z_{t+1} + \delta x_{t+1}, \quad t \in [1, n-1] \quad (3.3f) \]
\[ z \in \{0, 1\}^{n-1}, x \in \{0, 1\}^n, p \in \mathbb{R}_+^n. \quad (3.3g) \]

We refer to the feasible set defined by constraints (3.3a)-(3.3g) as \( D \). Observe that we replace inequality (3.2f) in formulation \( U \) with inequality (3.3f), and represent the
relationship between the shut-down variables \( z \) and on/off variables \( x \) with inequalities \((3.3c)-(3.3e)\), to obtain the formulation \( D \) in the space of \((p, x, z)\).

Throughout the chapter, we make the following assumptions to ignore uninteresting cases:

(A1) \( u \geq \bar{u} \geq \ell \),

(A2) \( u - \ell \geq \delta > 0 \).

Note that if Assumption (A1) does not hold and the maximum production level when a generator \( g \in [1, m] \) is started up/shut down \( \bar{u} \) is strictly smaller than the minimum production level \( \ell \), then we cannot start up or shut-down this generator (from inequalities \((3.1d)\)). In other words, we can fix variables \( s_i \) and \( z_i \) to 0 for all \( t \in [2, n] \), and eliminate them from the formulation. Additionally, if for any generator \( g \in [1, m] \) the maximum production level when it is started up or shut down \( \bar{u} \) is strictly greater than maximum production level \( u \), then we can strengthen the original ramping inequalities \((3.1j)\) and \((3.1k)\) in formulation \( UC \), by replacing \( \bar{u} \) with \( u \). Similarly, if Assumption (A2) does not hold and the maximum ramping rate \( \delta \) is greater than the difference between maximum production and minimum production levels \( u - \ell \), then we can let \( \delta = u - \ell \), because we cannot ramp more than this difference. Note that Assumption (A2) also implies that \( u > \ell \). It can be easily shown that the ramp-up and ramp-down polytopes \( \text{conv}(U) \) and \( \text{conv}(D) \) are full-dimensional under Assumptions (A1)-(A2).

Note that \( U \) and \( D \) also arise as substructures in lot-sizing problems (cf. [73]) with start-ups, production lower and upper bounds, and production smoothing constraints \(|p_{t+1} - p_t| \leq \delta\).
3.2 Optimization over Ramping Polyhedra

In this section, we address the complexity of optimization over the ramping polytope. Consider the optimization problem: 
\[
\min_{t=1}^{n} \left( c_t p_t + h_t x_t \right) + \sum_{t=2}^{n} f_t s_t : (p, x, s) \in U
\]
for any given objective coefficient vector \((c, h, f) \in \mathbb{R}^{3n-1}\). Note that because the objective function involves fixed and variable costs of production, it is concave with respect to the production variables. Therefore, there exists an optimal solution to the optimization problem over \(U\) that is an extreme point of \(\text{conv}(U)\).

Next we characterize the extreme points of \(\text{conv}(U)\). Let \(\Delta_1 = \max\{j \in [1, n] : \ell + j\delta \leq u\}\) and \(\Delta_2 = \max\{j \in [1, n] : \bar{u} + j\delta \leq u\}\). Also, let \(P = \{0, \{\ell + j\delta\}_{j=0}^{\Delta_1}, \{\bar{u} + j\delta\}_{j=0}^{\Delta_1}\} \mid \{1, n\} = 0\) represent a set of production values.

**Lemma 1.** For any extreme point \((p, x, s)\) of \(\text{conv}(U)\), \(p_t \in P\) for all \(t \in [1, n]\).

**Proof.** We prove this lemma by contradiction. Suppose that there exists an extreme point \((p, x, s)\) of \(\text{conv}(U)\) and an index \(t \in [1, n]\) such that \(p_t \notin P\). Note that we cannot have \(0 < p_t < \ell\) in any feasible solution. If \(t > 1\) and \(p_t - p_{t-1} = \delta\), then let \(\kappa_1 < t\) be the smallest index such that \(p_{\kappa_1+1} - p_{\kappa_1} = p_{\kappa_1+2} - p_{\kappa_1+1} = \cdots = p_t - p_{t-1} = \delta\). Otherwise, let \(\kappa_1 = t\). Similarly, if \(t < n\) and \(p_{t+1} - p_t = \delta\), then let \(\kappa_2 > t\) be the largest index such that \(p_{t+1} - p_t = p_{t+2} - p_{t+1} = \cdots = p_{\kappa_2} - p_{\kappa_2-1} = \delta\). If \(t = n\) or \(p_{t+1} - p_t < \delta\), then let \(\kappa_2 = t\). We consider two solutions \((p^1, x, s)\) and \((p^2, x, s)\) such that \(p_j^1 = p_j + \epsilon\) for \(j \in [\kappa_1, \kappa_2]\), and \(p_j^2 = p_j - \epsilon\) for \(j \in [\kappa_1, \kappa_2]\), and \(p_j^1 = p_j^2 = p_j\) for all \(j \notin [\kappa_1, \kappa_2]\), where \(\epsilon > 0\) is an infinitesimally small number.

First, consider the case that there does not exist a period \(j\) with \(\bar{u} + n\delta < p_j < u - n\delta\). (Note that \(\bar{u} + n\delta < p_j < u - n\delta\) can be feasible only if \(\Delta_2 = n\) and \(\bar{u} + n\delta < u - n\delta\).) In this case, we have \(\ell + j_1\delta < p_t < \ell + (j_1 + 1)\delta\) for some \(j_1 \in [0, \Delta_1 - 1]\), and \(\bar{u} + j_2\delta < p_t < \bar{u} + (j_2 + 1)\delta\) for some \(j_2 \in [0, \Delta_2 - 1]\), and \(u - j_3\delta < p_t < u - (j_3 - 1)\delta\) for some \(j_3 \in [1, \Delta_1]\). Note that \(p_j \notin P\) for all \(j \in [\kappa_1, t]\). If \(p_{\kappa_1-1} = 0\), then \(\ell < p_{\kappa_1} = p_t - (t - \kappa_1)\delta < \bar{u}\), else if \(p_{\kappa_1-1} > 0\), then, by definition,
\( p_{\kappa_1} - p_{\kappa_1-1} < \delta \), so in either case it is feasible to increase or decrease \( p_{\kappa_1} \) by \( \epsilon \). Also observe that if \( \kappa_2 = n \), then \( p_{\kappa_2} = p_t + (\kappa_2 - t)\delta < u \), else \( p_{\kappa_2+1} - p_{\kappa_2} < \delta \) by definition, so in either case it is feasible to increase or decrease \( p_{\kappa_2} \) by \( \epsilon \).

Another case to consider is when \( \Delta_2 = n \) and \( \bar{u} + n\delta < u - n\delta \). In this case, we can also have \( \bar{u} + n\delta < p_t < u - n\delta \). First, suppose that there exists \( p_{j-1} = 0 \) for some \( 1 < j \leq t \) and \( p_i > 0 \) for \( j \leq i \leq t \). In this case, we have \( p_t \leq \bar{u} + (t - j)\delta < \bar{u} + n\delta \), which violates the assumption on the value of \( p_t \). Therefore \( p_j > 0 \) for all \( j \in [1, t] \).

Note that because \( p_{\kappa_1-1} > 0 \) and by definition, \( p_{\kappa_1} - p_{\kappa_1-1} < \delta \), it is feasible to increase or decrease \( p_{\kappa_1} \) by \( \epsilon \). Similarly, it is feasible to increase or decrease \( p_{\kappa_2} \), because \( p_{\kappa_2} < u \).

In all cases, both \((p^1, x, s)\) and \((p^2, x, s)\) are feasible, and \((p, x, s) = \frac{1}{2}(p^1, x, s) + \frac{1}{2}(p^2, x, s)\). Therefore, \((p, x, s)\) is not an extreme point, a contradiction.

\( \square \)

Now we are ready to address the complexity of optimization over \( U \).

**Theorem 3.** There is an \( O(n^3) \) algorithm to optimize a linear function over \( U \).

**Proof.** We use Lemma \[1\] and restrict the possible production values in an optimal solution to \( P \). Note that there are \( O(n) \) elements in \( P \). For ease of exposition, we will describe an algorithm for the more interesting case that \( \ell > 0 \) in detail. We define a shortest path problem over a layered network \( G = (N, A) \) with \( n \) layers, in addition to a dummy source node \( 0 \in N \) and a dummy sink node \( n' \in N \). Each layer \( t \) has nodes labeled \((j, t) \in N \) for all \( j \in P \) representing the possible values of production \( p_t \) in period \( t \). There exist arcs \((0, (j, 1)) \in A \) from the source node \( 0 \) to the nodes \((j, 1), j \in P \), with cost \( h_1 + c_1j \) if \( j > 0 \), and cost zero if \( j = 0 \). In addition, there exist arcs \(((j, n), n') \in A \) from the nodes \((j, n), j \in P \) to the sink node \( n' \) with cost zero. For \( t \in [2, n] \) and \( j \in P \setminus \{0\} \), there exist arcs \(((j, t-1), (k, t)) \in A \), from node \((j, t-1) \) to \((k, t) \) for all \( k \in P \) such that \( k - j \leq \delta \), with cost \( h_t + c_tk \) if \( k > 0 \), and cost
zero if \( k = 0 \). Finally, for \( t \in [2, n] \), there are arcs \( ((0, t - 1), (k, t)) \in A \), from node \( (0, t - 1) \) to \( (k, t) \) for \( k \in \mathcal{P} \) such that either \( \ell \leq k \leq \bar{u} \) with cost \( h_t + c_t k + f_t \), or \( k = 0 \) with cost zero. Note that \( G \) is a directed acyclic graph with \( O(n^2) \) nodes and \( O(n^3) \) arcs. Therefore, this shortest path problem can be solved in \( O(n^3) \). For the case that \( \ell = 0 \), the algorithm is similar, except that the nodes \((0, t)\) need to be split into three copies, the first copy representing \((p_t, x_t, s_t) = (0, 0, 0)\), the second one representing \((p_t, x_t, s_t) = (0, 1, 0)\), and the third one representing \((p_t, x_t, s_t) = (0, 1, 1)\). It is easy to see how the arc set and the arc costs should be adjusted to handle this case. Only two more nodes are added to each layer, so the complexity remains \( O(n^3) \) if \( \ell = 0 \).

Similarly, optimization over \( \mathcal{D} \) is polynomial as well. Note that the unit commitment problem (that includes more than production limit and ramping constraints) with a single generator is solvable in \( O(n^3) \) time \[30, 31\].

In Section 3.4 we present different classes of valid inequalities for ramp-up and ramp-down relaxations (\( \mathcal{U} \) and \( \mathcal{D} \), respectively) of formulation \( \mathcal{UC} \). In addition, we give facet conditions and develop exact separation algorithms for the proposed inequalities. Before we study the general ramping polytope, we consider the two-period version \((n = 2)\) first in Section 3.3. This allows us to understand the structure in a simpler variant, before we extend them to the general case.

### 3.3 Two-Period Ramping Polytope

In this section, we study the two-period ramping polytope in detail. The purpose of focusing on the two-period case is to find the ideal ramping constraints for formulating ramping. The linear number of inequalities defined for this case can be readily used in strengthening UC formulations without the need for separation.

For \( t \in [1, n - 1] \), let the corresponding two-period ramp-up polytope be given by
\( \mathcal{U}_t^2 = \{(p_t, p_{t+1}, x_t, x_{t+1}, s_{t+1}) \in \mathbb{R}_+^2 \times \{0,1\}^3 : x_{t+1} - x_t \leq s_{t+1}; s_{t+1} \leq 1 - x_t; s_{t+1} \leq x_{t+1}; p_{t+1} - p_t \leq \bar{u}s_{t+1} + \delta x_t; \ell x_j \leq p_j \leq u x_j, j = t, t+1 \} \). The two-period ramp-down polytope \( \mathcal{D}_t^2 \), for \( t \in [1, n-1] \) is defined similarly.

### 3.3.1 Inequalities for Two-Period Ramp-Up Polytope

Consider the ramping constraint (3.2f), which was first introduced by [68]. Analyzing the left hand side (LHS) of the ramping constraint, in any integral feasible solution, we can see that \( p_{t+1} - p_t \) can be bounded from above based on the values of \( x_{t+1}, x_t, s_{t+1} \), as illustrated in Table 3.1. Any inequality that has a right hand side (RHS) that is no smaller than this upper bound for each feasible value of \( x_{t+1}, x_t, s_{t+1} \) is valid. We see from Table 3.1 that (3.2f) is valid. The strongest possible inequality would attain the upper bounds on LHS as its RHS for all possible feasible values of \( x_{t+1}, x_t, s_{t+1} \).

First, we present new valid inequalities for the two-period ramp-up polytope (\( \mathcal{U}_t^2 \)) for any \( t \in [1, n-1] \). In Section 3.3.2, we present the convex hull of the two-period ramp-up polytope, which includes the new inequalities. Finally, in Section 3.3.3, we present the analogous inequalities for the two-period ramp-down polytope \( \mathcal{D}_t^2 \).

Table 3.1: Comparison of two-period ramp-up inequalities.

| \( x_t \) | \( x_{t+1} \) | \( s_{t+1} \) | Upper bound on LHS | \( p_{t+1} - p_t \) | \( 3.2f \) | \( 3.4 \) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | \( \bar{u} \) | \( \bar{u} \) | \( \bar{u} \) | \( \bar{u} \) | \( \max \{ \bar{u}, \ell + \delta \} \) |
| 1 | 0 | 0 | \( -\ell \) | \( \delta \) | \( \max \{ -\ell, -\bar{u} - \delta \} \) | \( -\ell \) |
| 1 | 1 | 0 | \( \delta \) | \( \delta \) | \( \max \{ \bar{u} - \ell, \delta \} \) | \( \max \{ \bar{u} - \ell, \delta \} \) |

**Proposition 11.** For \( t \in [1, n-1] \), the two-period ramp-up inequality

\[
p_{t+1} - p_t \leq (\bar{u} - \ell - \delta)s_{t+1} + (\ell + \delta)x_{t+1} - \ell x_t
\]

(3.4)

is valid and defines a facet of \( \text{conv}(\mathcal{U}_t^2) \).
Proof. From the column labeled (3.4) in Table 3.1 we see that this inequality is valid. From the following five affinely independent points \((p_t, p_{t+1}, x_t, x_{t+1}, s_{t+1})\) in \(U^2_t\), we see that (3.4) defines a facet of \(\text{conv}(U^2_t)\): \((0, 0, 0, 0, 0), (0, \bar{u}, 0, 1, 1), (\ell, 0, 1, 0, 0), (\ell, \ell + \delta, 1, 1, 0), (u - \delta, u, 1, 1, 0)\).

From the RHS columns in Table 3.1 we see that (3.4) is the strongest ramping inequality one can derive for the left hand side, \(p_{t+1} - p_t\), and therefore dominates the two-period ramping inequality (3.2f) used in the definition of \(UC\).

We now consider the upper bound constraint (3.2b) for period \(t+1\). This inequality can be strengthened if the generator starts up in period \(t+1\), as follows:

**Proposition 12.** For \(t \in [1, n - 1]\), the two-period variable upper bound (VUB) ramp-up inequality

\[
p_{t+1} \leq ux_{t+1} - (u - \bar{u})s_{t+1}
\]

(3.5)

is valid and defines a facet of \(\text{conv}(U^2_t)\).

Proof. When \(s_{t+1} = 0\), inequality (3.5) is the same as (3.2b), which is valid. When \(s_{t+1} = 1\), the production level in period \(t+1\) can be no greater than \(\bar{u}\), which is the right hand side of (3.5). From the following five affinely independent points \((p_t, p_{t+1}, x_t, x_{t+1}, s_{t+1})\) in \(U^2_t\), we see that (3.5) defines a facet of \(\text{conv}(U^2_t)\): \((0, 0, 0, 0, 0), (0, \bar{u}, 0, 1, 1), (\bar{u}, 0, 1, 0, 0), (u, u, 1, 1, 0), (u - \delta, u, 1, 1, 0)\).

Note that inequalities (3.4) and (3.5) are new ramping and upper bound inequalities that can be used to strengthen the existing formulations of the UC problem with start variables. Ostrowski et al. present a polynomial class of VUB inequalities, but these are the same as our two-period VUB inequality (3.5) when \(\bar{v} = 1\), and are not valid for \(U\) when \(\bar{v} \geq 2\) [68]. However, we show that inequality (3.5) is valid for \(U\) for any value of \(\bar{v}\).
3.3.2 Convex Hull for Two-Period Ramp-Up Polytope

In this section, we present the convex hull of the two-period ramp-up polytope \( \text{conv}(\mathcal{U}_t^2) \), and the convex hull of its projection on to the space without start variables for any \( t \in [1, n - 1] \). For conciseness, we do not give the analogous results on \( \text{conv}(\mathcal{D}_t^2) \). We will see that \( \text{conv}(\mathcal{U}_t^2) \) can be described using trivial inequalities and the new ramp-up inequalities (3.4) and (3.5). (No other inequalities are needed.) By completely describing this simpler two-period polytope, we start from the strongest possible two-period ramping inequalities before generalizing them to the multi-period setting, deriving strong valid inequalities for \( \mathcal{U} \) and \( \mathcal{D} \) in Section 3.4.

First, we give the convex hull of the two-period ramp-up polytope \( \text{conv}(\mathcal{U}_t^2) \) for any given \( t \in [1, n - 1] \). Consider the LP

\[
\begin{align*}
    s_{t+1} &\leq x_{t+1} & (3.6a) \\
    s_{t+1} &\geq x_{t+1} - x_t & (3.6b) \\
    s_{t+1} &\geq 0 & (3.6c) \\
    p_{t+1} &\geq \ell x_{t+1} & (3.6d) \\
    p_t &\geq \ell x_t & (3.6e) \\
    p_{t+1} - p_t &\leq (\ell + \delta)x_{t+1} + (\bar{u} - \ell - \delta)s_{t+1} - \ell x_t & (3.6f) \\
    p_{t+1} &\leq u x_{t+1} - (u - \bar{u})s_{t+1} & (3.6g) \\
    p_t &\leq u x_t & (3.6h) \\
    s_{t+1} &\leq 1 - x_t. & (3.6i)
\end{align*}
\]

Observe that (3.6f) is the same as (3.4), and that (3.6g) is the same as (3.5). Note that we drop the integer restrictions on \( x \) and \( s \). We also drop the bounds on variables \( x_t \) and \( x_{t+1} \) because inequalities (3.6a) and (3.6b) imply that \( x_t \geq 0 \), inequalities (3.6c)
and (3.6i) imply that $x_t \leq 1$, inequalities (3.6a) and (3.6c) imply that $x_{t+1} \geq 0$, and inequalities (3.6b) and (3.6i) imply that $x_{t+1} \leq 1$.

**Theorem 4.** $\text{conv}(U_t^2) = \{(p, x, s) \in \mathbb{R}^5 : (3.6a) - (3.6i)\}$.

**Proof.** Note that $\text{conv}(U_t^2)$ is bounded because all the variables are bounded. We will prove this theorem by showing that every extreme point of the polytope defined by (3.6a)-(3.6i) is integral. To do this, we will consider the intersection of five linearly independent inequalities among the inequalities (3.6a)-(3.6i) ($\binom{9}{5} = 126$ possible points). However, all inequalities except inequality (3.6i) intersect at the origin. Therefore, we will let inequality (3.6i) hold at equality (i.e., $s_{t+1} = 1 - x_t$) and choose four out of the remaining eight inequalities (3.6a)-(3.6h) to be tight to obtain extreme points different than the origin. Hence, $\binom{8}{4} = 70$ points have to be considered.

Throughout the proof we make use of the following observations.

**Observation 2.** If $\delta = u - \ell$, then the ramping inequality (3.6f) becomes $p_{t+1} - p_t \leq ux_{t+1} - (u - \bar{u})s_{t+1} - \ell x_t$, and it is not a facet because it is dominated by inequalities (3.6e) and (3.6g). Because it cannot be a facet, inequalities (3.6a)-(3.6e) and (3.6g)-(3.6i) are enough to give $\text{conv}(U_t^2)$ in this case. In the following proof, we assume that $u > \ell + \delta$ if we are considering inequality (3.6f) as an inequality that holds at equality in an extreme point.

**Observation 3.** If inequalities (3.6e) and (3.6h) are both satisfied at equality in any solution, then we must have $x_t = 0$ because $u > \ell$ from Assumption (A2).

Next we consider the different cases where, in addition to inequality (3.6i), four inequalities from inequalities (3.6a)-(3.6h) are satisfied at equality.

1. Assume that inequality (3.6a) is satisfied at equality.

In this case we obtain $s_{t+1} = 1 - x_t = x_{t+1}$. We need three more inequalities from seven remaining inequalities (3.6b)-(3.6h) to be satisfied at equality, i.e., $\binom{7}{3} = 109$
35 cases to consider. Next we show the simplifications in these inequalities, if there are any.

- Inequality (3.6b) reduces to \( x_t \geq 0 \) and \( x_{t+1} \leq 1 \). Thus, if inequality (3.6b) holds at equality, then \( x_t = 0 \), \( x_{t+1} = s_{t+1} = 1 \), which is integral. Therefore, we do not need to consider this case.

- Inequality (3.6c) reduces to \( x_{t+1} \geq 0 \) and \( x_t \leq 1 \). Thus, if inequality (3.6c) holds at equality, then \( s_{t+1} = x_{t+1} = 0 \) and \( x_t = 1 \), which is integral. Therefore, we do not need to consider this case.

- Inequality (3.6f) reduces to \( p_{t+1} \leq \bar{u}x_{t+1} - \ell x_t \), which is dominated by inequalities (3.6a), (3.6e) and (3.6g). Thus, we do not need to consider this case.

- Inequality (3.6g) reduces to \( p_{t+1} \leq \bar{u}x_{t+1} \).

We are only left with the case where inequalities (3.6a), (3.6i) and three out of inequalities (3.6d), (3.6e), (3.6g) and (3.6h) are satisfied at equality. If inequalities (3.6d) and (3.6g) are both satisfied at equality, then we obtain \( (u - \ell)x_{t+1} = (u - \bar{u})s_{t+1} \). If \( \bar{u} > \ell \), then \( x_{t+1} = 0 = s_{t+1} \) and \( x_t = 1 \) at this extreme point, which is integral. If \( \bar{u} = \ell \), then (3.6a), (3.6i), (3.6d) and (3.6g) are linearly dependent so this case cannot correspond to an extreme point. Inequalities (3.6e) and (3.6h) can both be satisfied at equality only if \( x_t = 0 \) (from Observation 3). In this case, we have \( x_{t+1} = 1 = s_{t+1} \), which is also an integral point.

2. Assume that inequality (3.6b) is satisfied at equality (and inequality (3.6a) is not).

In this case we obtain \( s_{t+1} = x_{t+1} - x_t = 1 - x_t \) and \( s_{t+1} < x_{t+1} \). Thus, \( x_{t+1} = 1 \), \( x_t > 0 \) and \( s_{t+1} < 1 \). We need three more inequalities from six
remaining inequalities (3.6c)-(3.6h) to hold at equality, i.e., $\binom{6}{3} = 20$ cases to consider. Next, we show the simplifications in inequalities, if there are any.

- Inequality (3.6c) reduces to $x_t \leq 1$. Thus, if inequality (3.6c) holds at equality, then $x_t = 1 = x_{t+1}$, $s_{t+1} = 0$, which is integral. Therefore, we do not need to consider this case.
- Inequality (3.6d) reduces to $p_{t+1} \geq \ell$.
- Inequality (3.6f) reduces to $p_{t+1} - p_t \leq \bar{u} - (\bar{u} - \delta)x_t$.
- Inequality (3.6g) reduces to $p_{t+1} \leq \bar{u} + (u - \bar{u})x_t$.

So we have to choose three out of inequalities (3.6d)-(3.6h) to hold at equality. There are $\binom{5}{3} = 10$ cases to consider. Note that we cannot have inequalities (3.6e) and (3.6h) both hold at equality, because we assume that $x_t > 0$ (from Observation 3). Hence, there are only seven cases left. Also, if we let inequalities (3.6d) and (3.6g) hold at equality, then $\ell = \bar{u} + (u - \bar{u})x_t = ux_t + \bar{u}(1 - x_t)$ which cannot hold because $0 < x_t \leq 1$, $\bar{u} \geq \ell$ and $u > \ell$ from Assumption (A2). So there are only four cases left.

- If we choose inequalities (3.6e) and (3.6f) to hold at equality, then we have $p_t = \ell x_t$ and $p_{t+1} = \bar{u} - (\bar{u} - \ell - \delta)x_t$. We need to consider two cases. If inequality (3.6g) is chosen as the third inequality that holds at equality, then $(\ell + \delta - \bar{u})x_t = (u - \bar{u})x_t$, which is infeasible because $x_t > 0$ and, from Observation 2, $u > \ell + \delta$. If inequality (3.6d) is chosen as the third inequality that holds at equality, then $(\bar{u} - \ell - \delta)x_t = \bar{u} - \ell$. Because $1 \geq x_t > 0$ in this case and $\delta > 0$, this extreme point is infeasible.

- If we choose inequalities (3.6f) and (3.6h) to hold at equality, then we have $p_t = ux_t$ and $p_{t+1} = \bar{u} + (u - \bar{u} + \delta)x_t$. If inequality (3.6d) is chosen as the third inequality that holds at equality, then $p_{t+1} = \ell = \bar{u} + (u - \bar{u} + \delta)x_t$. 

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So we obtain \((u - \bar{u} + \delta)x_t = \ell - \bar{u}\). Because \(x_t > 0\), \(\delta > 0\) and \(u > \ell\) (from Assumption (A2)), this point is not feasible. If inequality (3.6g) is chosen as the third inequality that holds at equality, then \(p_{t+1} = \bar{u} + (u - \bar{u})x_t = \bar{u} + (u - \bar{u} + \delta)x_t\). Because \(x_t, \delta > 0\), this solution is not feasible.

3. Assume that inequality (3.6c) is satisfied at equality (and inequalities (3.6a) and (3.6b) are not).

In this case, we obtain \(s_{t+1} = 0 = 1 - x_t\), \(s_{t+1} < x_{t+1}\) and \(s_{t+1} > x_{t+1} - x_t\). Thus, \(x_t = 1\), \(x_{t+1} > 0\) and \(x_{t+1} < 1\). We need three more inequalities from five remaining inequalities (3.6d)-(3.6h) to hold at equality, i.e., \(\binom{5}{3} = 10\) cases to consider. Next, we show the simplifications in inequalities, if there are any.

- Inequality (3.6c) reduces to \(p_t \geq \ell\).
- Inequality (3.6f) reduces to \(p_{t+1} - p_t \leq (\ell + \delta)x_{t+1} - \ell\).
- Inequality (3.6g) reduces to \(p_{t+1} \leq ux_{t+1}\).
- Inequality (3.6h) reduces to \(p_t \leq u\).

We divide the cases as follows:

- If we let inequalities (3.6d) and (3.6g) hold at equality, then \(p_{t+1} = ux_{t+1} = \ell x_{t+1}\). Because \(x_{t+1} > 0\) and \(u > \ell\) (from Assumption (A2)), this point is infeasible (from Observation 3).
- If we let inequalities (3.6c) and (3.6h) hold at inequality, then \(p_t = \ell = u\) which is infeasible because \(u > \ell\) (from Assumption (A2)).
- If we choose inequalities (3.6f) and (3.6h) to hold at equality, then \(p_t = u\). If inequality (3.6g) is chosen as the third inequality to hold at equality, then \((u - \ell - \delta)x_{t+1} = (u - \ell)\). Because \(x_{t+1} < 1\) and \(\delta > 0\), this point is infeasible. If inequality (3.6d) is chosen as the third inequality to hold at
equality, then $\delta x_{t+1} = \ell - u$, which is infeasible because $x_{t+1} > 0$ and from Observation 2 we have $u > \ell + \delta$.

- If we choose inequalities (3.6e) and (3.6f) to hold at inequality, then $p_t = \ell$ and $p_{t+1} = (\ell + \delta)x_{t+1}$. We need to consider two cases. If inequality (3.6g) is chosen as the third inequality that is satisfied at equality, then we have $p_{t+1} = ux_{t+1} = (\ell + \delta)x_{t+1}$. This point is infeasible because $x_{t+1} > 0$ and $u > \ell + \delta$ from Observation 2. If inequality (3.6d) is chosen as the third inequality that holds at equality, then we have $p_{t+1} = \ell x_{t+1} = (\ell + \delta)x_{t+1}$.

Because $x_{t+1}, \delta > 0$, this point is infeasible.

4. Assume that inequality (3.6d) is satisfied at equality (and inequalities (3.6a), (3.6b) and (3.6c) are not).

In this case, $p_{t+1} = \ell x_{t+1}$, $s_{t+1} = 1 - x_t$, $s_{t+1} < x_{t+1}$, $s_{t+1} > x_{t+1} - x_t$, and $s_{t+1} > 0$. Hence, $0 < x_t < 1$ and $0 < x_{t+1} < 1$. We need three more inequalities from four remaining inequalities (3.6e)-(3.6h) to hold at equality, i.e., there are 4 cases to consider. Note, from Observation 3 that we cannot have inequalities (3.6c) and (3.6h) both hold at equality because $x_t > 0$. Therefore, inequalities (3.6f) and (3.6g) must hold at equality and one of inequalities (3.6e) or (3.6h) must hold at equality. In this case, inequality (3.6g) becomes $(u - \ell)x_{t+1} + (u - \bar{u})x_t \geq (u - \bar{u})$. Because inequality (3.6g) holds at equality, $x_{t+1} = \frac{(u - \bar{u})}{(u - \ell)}(1 - x_t) = \frac{(u - \bar{u})}{(u - \ell)}s_{t+1}$. Recall that $\bar{u} \geq \ell$, from (A1), so we obtain $x_{t+1} \leq s_{t+1}$, which contradicts our assumption that $x_{t+1} > s_{t+1}$.

5. Assume that inequality (3.6e) is satisfied at equality (and inequalities (3.6a), (3.6b), (3.6c) and (3.6d) are not).

In this case, $p_t = \ell x_t$, $s_{t+1} = 1 - x_t$, $s_{t+1} < x_{t+1}$, $s_{t+1} > x_{t+1} - x_t$, $s_{t+1} > 0$ and $p_{t+1} > \ell x_{t+1}$. Hence, $0 < x_t < 1$ and $0 < x_{t+1} < 1$. We need all three of the
remaining inequalities (3.6f)-(3.6h) to hold at equality. However, this point is not feasible from Observation 3 and the assumption that \( x_t > 0 \).

We have showed that all intersections of five linearly independent constraints among inequalities (3.6a)-(3.6i), if feasible, give integral extreme points. Hence, the proof is complete.

The unit commitment problem can also be formulated without the start-up variables (see [20] and [32]). The ramp-up polytope without start variables is given by

\[
\begin{align*}
\ell x_t & \leq p_t, \quad t \in [1, n] \quad (3.7a) \\
p_t & \leq u x_t, \quad t \in [1, n] \quad (3.7b) \\
p_{t+1} - p_t & \leq \bar{u} - (\bar{u} - \delta) x_t, \quad t \in [1, n - 1] \quad (3.7c) \\
x & \in \{0, 1\}^n, p \in \mathbb{R}_+^n. \quad (3.7d)
\end{align*}
\]

We refer to the feasible set defined by constraints (3.7a)-(3.7d) as \( \text{UNS} \). By projecting out the start-up variables in the convex hull definition of \( \text{U}_t^2 \), for \( t \in [1, n-1] \), we obtain new inequalities for this more compact formulation as well. For a given \( t \in [1, n-1] \), let the two-period ramp-up polytope without start variables be \( \text{UNS}_t^2 = \{(p, x) \in \mathbb{R}_+^2 \times \{0, 1\}^2 : p_{t+1} - p_t \leq \bar{u} - (\bar{u} - \delta)x_t; \ell x_j \leq p_j \leq u x_j, j = t, t+1\} \). Next, we describe the constraints that we use to describe \( \text{conv}(\text{UNS}_t^2) \):

\[
\begin{align*}
p_t & \leq u x_t \quad (3.8) \\
p_t & \geq \ell x_t \quad (3.9) \\
p_{t+1} & \geq \ell x_{t+1} \quad (3.10)
\end{align*}
\]
\[ p_{t+1} \leq \bar{u} x_{t+1} + (u - \bar{u}) x_t \]  \hspace{1cm} (3.11)

\[ x_{t+1} \leq 1 \]  \hspace{1cm} (3.12)

\[ p_{t+1} \leq u x_{t+1} \]  \hspace{1cm} (3.13)

\[ x_t \leq 1, \]  \hspace{1cm} (3.14)

and if \( \bar{u} \leq \ell + \delta \), then

\[ p_{t+1} - p_t \leq \bar{u} x_{t+1} - (\bar{u} - \delta)x_t \]  \hspace{1cm} (3.15)

\[ p_{t+1} - p_t \leq (\ell + \delta)x_{t+1} - \ell x_t, \]  \hspace{1cm} (3.16)

else if \( \bar{u} > \ell + \delta \), then

\[ p_{t+1} - p_t \leq (\bar{u} - \ell - \delta) + (\ell + \delta)x_{t+1} - (\bar{u} - \delta)x_t \]  \hspace{1cm} (3.17)

\[ (u - \ell - \delta)p_{t+1} - (u - \bar{u})p_t \leq \bar{u}(u - \ell - \delta)x_{t+1} - \ell(u - \bar{u})x_t. \]  \hspace{1cm} (3.18)

**Corollary 13.** For \( \bar{u} \leq \ell + \delta \), \( \text{conv}(\mathcal{UNS}_t^2) = \{ (p, x) \in \mathbb{R}^{2n} : (3.8) - (3.16) \} \), and for \( \bar{u} > \ell + \delta \), \( \text{conv}(\mathcal{UNS}_t^2) = \{ (p, x) \in \mathbb{R}^{2n} : (3.8) - (3.14), (3.17), (3.18) \} \).

**Proof.** See Appendix 3.7.

Note that inequalities (3.11) and (3.15)-(3.18) are upper bound and ramping inequalities that can be used to strengthen the existing formulations of the UC problem without start variables. In particular, inequality (3.18) has an interesting structure, because the coefficients of the production variables \( p_t \) and \( p_{t+1} \) are not necessarily -1 and 1, respectively, as is the case in all known inequalities representing ramping.
3.3.3 Inequalities for Two-Period Ramp-Down Polytope

Using the symmetry between ramping up and ramping down constraints, we can derive the ramp-down analogues of the ramp-up inequality (3.4) and the variable upper bound inequality (3.5).

**Proposition 14.** For \( t \in [1, n-1] \), the two-period ramp-down inequality

\[
p_t - p_{t+1} \leq (\bar{u} - \ell - \delta)z_{t+1} + (\ell + \delta)x_t - \ell x_{t+1}
\]

is valid and defines a facet of \( \text{conv}(\mathcal{D}_t^2) \).

**Proof.** Similar to the proof of Proposition 11 using analogous arguments, and substituting for \( z_{t+1} \).

**Proposition 15.** For \( t \in [1, n-1] \), the two-period VUB ramp-down inequality

\[
p_t \leq u x_t - (u - \bar{u})z_{t+1}
\]

is valid and defines a facet of \( \text{conv}(\mathcal{D}_t^2) \).

**Proof.** Similar to the proof of Proposition 12.

3.4 Multi-period Facets for Ramping Relaxations

3.4.1 Ramp-Up Polytope

In this section, we present multi-period variable upper bound (VUB) and two classes of multi-period ramp-up inequalities for \( \mathcal{U} \) for any \( n \geq 2 \). In Section 3.4.2 we study the strength of these inequalities, and present necessary and sufficient conditions under which they define facets of \( \mathcal{U} \).

**Proposition 16.** If \( \bar{u} \geq \ell + \delta \), then for \( 1 \leq t \leq n \) and \( 1 \leq j \leq \min \left\{ n - t, \frac{\bar{u} - \ell}{\delta} \right\} \), the type-I multi-period ramp-up inequality

\[
p_{t+j} - p_t \leq (\ell + j\delta)x_{t+j} + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\} s_{t+i} - \ell x_t
\]

(3.21)
is valid for $U$.

Proof. There are two cases to consider.

**Case 1.** Suppose that $x_{t+j} = 0 (= p_{t+j})$. Because $\min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\} \geq 0$ for $1 \leq i \leq j$, $u \geq \bar{u}$ from (A1) and $p_t \geq \ell x_t$, inequality (3.21) is clearly valid for this case.

**Case 2.** Suppose that $x_{t+j} = 1$.

i. Suppose that the last period when start-up occurred during periods 1 through $t + j$ is $t + k$, where $1 \leq k \leq j$, i.e., $s_{t+k} = 1$, and $s_{t+i} = 0$ for all $k + 1 \leq i \leq j$. Then, $p_{t+j} \leq \min\{(\bar{u} + (j - k)\delta), u\}$. Also $p_t \geq \ell x_t$.

Therefore,

$$p_{t+j} - p_t \leq \min\{(\bar{u} + (j - k)\delta), u\} - \ell x_t$$

$$= (\ell + j\delta)x_{t+j} + \min\{(\bar{u} - \ell - k\delta), (u - \ell - j\delta)\} s_{t+k} - \ell x_t$$

$$\leq (\ell + j\delta)x_{t+j} + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\} s_{t+i} - \ell x_t.$$  

ii. Suppose that the last period when start-up occurred during periods 1 through $t + j$ is $k$, where $k \leq t$, i.e., $s_{t+i} = 0$ for all $1 \leq i \leq j$ and $x_{t+i} = 1$ for all $0 \leq i \leq j$. Therefore,

$$p_{t+j} - p_t \leq j\delta$$

$$= (\ell + j\delta)x_{t+j} + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\} s_{t+i} - \ell x_t.$$  

Observe that the two-period ramping inequality (3.6f) is a special case of type-I
multi-period ramp-up inequality (3.21) with \( j = 1 \) if \( \bar{u} \geq \ell + \delta \). We illustrate type-I
multi-period ramp-up inequalities in Example 4.

Example 4. Let \( u = 7, \ell = 1, \bar{u} = 4, \delta = 1 \) and \( n \geq 4 \). The inequalities (3.21) for
\( t = 2, j = 2 \) and \( t = 1, j = 3 \), respectively, are

\[
\begin{align*}
    p_4 - p_2 & \leq 3x_4 + 2s_3 + s_4 - x_2, \\
    p_4 - p_1 & \leq 4x_4 + 2s_2 + s_3 - x_1.
\end{align*}
\]

Proposition 16 describes a polynomial class of inequalities. Hence, its separation is polynomial. Next, we present another class of multi-period ramp-up inequalities.

Let \( a^+ = \max\{0, a\} \).

Proposition 17. For \( 1 \leq t \leq n \) and \( 1 \leq j \leq \min\left\{ n - t, \frac{u - \ell}{\delta} \right\} \), let \( S \subseteq [t+1, t+j] \),
\( t + j \in S \), \( q = \min\{k \in S\} \), and \( d_i = \max\{k \in S \cup \{t + 1\} : k < i\} \), \( i \in S \). Then the
type-II multi-period ramp-up inequality

\[
p_{t+j} - p_t \leq \bar{u}x_{t+j} + \delta \sum_{i \in S\setminus\{t+1\}} (i - d_i)(x_i - s_i) + \phi(x_q - s_q) - \ell x_t, \tag{3.22}
\]

where \( \phi = (\ell + \delta - \bar{u})^+ \), is valid for \( U \).

Proof. There are two cases to consider.

Case 1. Suppose that \( x_{t+j} = 0 (= p_{t+j}) \). Inequality (3.22) is clearly valid for this case.

Case 2. Suppose that \( x_{t+j} = 1 \). Note that \( x_i - s_i = 0 \), unless the generator is on in
period \( i \) but started up earlier than \( i \), in which case \( x_i - s_i = 1 \).

i. Suppose that the last period when start-up occurred on or before \( t + j \) is
\[ t + k, \text{ where } 1 \leq k \leq j, \text{ i.e., } s_{t+k} = 1, \text{ and } s_{t+i} = 0 \text{ for all } k + 1 \leq i \leq j. \]

Then, \( p_{t+j} \leq \bar{u} + (j - k)\delta. \) Also \( p_t \geq \ell x_t. \) Therefore,

\[
\begin{align*}
  p_{t+j} - p_t & \leq \bar{u} + (j - k)\delta - \ell x_t \\
  & \leq \bar{u} x_{t+j} + \delta \sum_{i \in (S \cap [t+k+1, t+j]) \setminus \{t+1\}} (i - d_i)(x_i - s_i) - \ell x_t \\
  & \leq \bar{u} x_{t+j} + \delta \sum_{i \in S \setminus \{t+1\}} (i - d_i)(x_i - s_i) + \phi(x_q - s_q) - \ell x_t.
\end{align*}
\]

The second inequality is valid because \( x_{t+j} = 1 \) and \( x_i - s_i = 1 \) for all \( i \in [t + k + 1, t + j] \) so \( (j - k)\delta \leq \delta \sum_{i \in (S \cap [t+k+1, t+j]) \setminus \{t+1\}} (i - d_i)(x_i - s_i). \) The last inequality is clearly valid because only non-negative terms are added.

ii. Suppose that the last period when start-up occurred on or before \( t + j \) is \( k, \)

where \( k \leq t, \) i.e., \( s_{t+i} = 0 \) for all \( 1 \leq i \leq j \) and \( x_{t+i} = 1 \) for all \( 0 \leq i \leq j. \)

Therefore,

\[
\begin{align*}
  p_{t+j} - p_t & \leq j\delta \leq (\bar{u} - \delta - \ell + \phi) \\
  & = \bar{u} + (j - 1)\delta + \phi - \ell \\
  & = \bar{u} x_{t+j} + \delta \sum_{i \in S \setminus \{t+1\}} (i - d_i)(x_i - s_i) + \phi(x_q - s_q) - \ell x_t.
\end{align*}
\]

Note that \( \bar{u} - \delta - \ell + \phi \geq 0 \) by the definition of \( \phi. \)

Observe that the two-period inequality (3.6f) is the special case of the type-II multi-period ramp-up inequality (3.22) with \( j = 1 \) if \( \bar{u} \leq \ell + \delta. \) We illustrate type-II multi-period ramp-up inequalities in Examples 4 and 4.
Example 4. Let $u = 6, \ell = 1, \bar{u} = 2, \delta = 2$ and $n \geq 3$. The inequalities (3.22) for $t = 1$ and $j = 2$

\[
p_3 - p_1 \leq 2x_3 + x_2 - s_2 + 2(x_3 - s_3) - x_1
\]

\[
p_3 - p_1 \leq 2x_3 + 3(x_3 - s_3) - x_1
\]

for $S = \{2, 3\}$ and $S = \{3\}$, respectively, are valid. Note, in this case, that $\bar{u} < \ell + \delta$, so $\phi > 0$.

Example 4. Let $u = 6, \ell = 1, \bar{u} = 2, \delta = 1$ and $n \geq 5$. The inequalities (3.22) for $t = 1$ and $j = 4$

\[
p_5 - p_1 \leq 2x_5 + x_3 - s_3 + x_4 - s_4 + x_5 - s_5 - x_1
\]

\[
p_5 - p_1 \leq 2x_5 + 2(x_4 - s_4) + x_5 - s_5 - x_1
\]

\[
p_5 - p_1 \leq 2x_5 + x_3 - s_3 + 2(x_5 - s_5) - x_1
\]

\[
p_5 - p_1 \leq 2x_5 + 3(x_5 - s_5) - x_1
\]

for $S = \{2, 3, 4, 5\}$, $S = \{2, 4, 5\}$, $S = \{2, 3\}$, and $S = \{2, 5\}$, respectively, are valid. Note, in this case, that $\bar{u} = \ell + \delta$, so $\phi = 0$.

In Proposition 18 we show that although there are exponentially many inequalities (3.22), their separation can be done efficiently.

Proposition 18. Given a point $(\bar{p}, \bar{x}, \bar{s}) \in \mathbb{R}^{3n-1}_+$, there is an $O(n^3)$ algorithm to find the most violated inequality (3.22), if any.

Proof. Given $t$, let $j' = \min \left\{ n - t, \frac{u - \ell}{\delta} \right\}$. Consider the longest path problem on a directed acyclic graph $G = (N, A)$ where the vertex set $N$ is given by the source node 0, the sink node $t'$, and nodes $i \in [t, t + j']$ and the arc set, $A$, is given by the arc $(0, t)$ with length $\ell \bar{x}_t - \bar{p}_t$, arc $(t, t + 1)$ with length $\phi(\bar{s}_{t+1} - \bar{x}_{t+1})$, arcs $(t, q)$ with length
\((\delta(q-t-1) + \phi)(\bar{s}_q - \bar{x}_q)\) for \(t + 1 < q \leq t + j'\), arcs \((i, k)\) with length \(\delta(k-i)(\bar{s}_k - \bar{x}_k)\) for \(t + 1 \leq i < k \leq t + j'\), and arcs \((t+a, t')\) with length \(\bar{p}_{t+a} - \bar{u}\bar{x}_{t+a}\) for \(1 \leq a \leq j'\).

The length of the longest path from 0 to \(t'\) in this graph is equal to the violation of inequality (3.22), if any. The visited nodes in the longest path determine the set \(S\). Note that there are \(O(n^2)\) arcs and \(O(n)\) vertices in this directed acyclic graph, so this longest path problem can be solved in \(O(n^2)\) time for a given \(t\). Solving this problem for all \(t\) we have an \(O(n^3)\) time separation algorithm.

Next, we define a class of multi-period variable upper bound (VUB) inequalities for \(U\).

**Proposition 19.** For \(1 \leq t \leq n\), \(0 \leq j \leq \min\left\{t-2, \frac{u - \bar{u}}{\delta}\right\}\) and any \(M \subseteq [t-j+1, t-1]\) the multi-period VUB ramp-up inequality

\[
p_t \leq \bar{u}x_t + \delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}),
\]

where for \(i \in [t-j+1, t]\) \(e_i = \max\{k \in M \cup \{t-j\} : k < i\}\) and if \(j = 0\), then \(e_t = t\) is valid for \(U\).

**Proof.** There are two cases to consider.

**Case 1.** Suppose that \(x_t = 0(= p_t)\). Because \(\delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}) \geq 0\), inequality (3.23) is clearly valid for this case.

**Case 2.** Suppose that \(x_t = 1\).

i. Suppose that the last period when start-up occurred on or before period \(t\) is \(k \in [t-j, t]\). Then,

\[
p_t \leq \bar{u} + (t-k)\delta \leq \bar{u} + \delta \sum_{i \in (M \cap [k+1, t-1]) \cup \{t\}} (i - e_i)
\]

\[
\leq \bar{u}x_t + \delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}).
\]
ii. Suppose that the last period when start-up occurred on or before period \( t \) is \( k \), where \( k \leq t - j - 1 \) then \( p_t \leq u \). Therefore,

\[
\begin{align*}
p_t & \leq \bar{u} + \delta \sum_{i \in \mathcal{M} \cup \{ t \}} (i - e_i) + (u - \bar{u} - j \delta) \\
& = \bar{u} x_t + \delta \sum_{i \in \mathcal{M} \cup \{ t \}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j \delta)(x_{t-j} - s_{t-j}) \\
& = \bar{u} + j \delta + u - \bar{u} - j \delta = u.
\end{align*}
\]

Observe that the two-period variable upper bound inequality (3.6g) is the special case of the multi-period VUB ramp-up inequality (3.23) with \( j = 0 \).

**Example 4** Let \( u = 7, \ell = 1, \bar{u} = 4, \delta = 1 \) and \( n \geq 4 \). The inequalities (3.23) for \( t = 4, j = 2, M = \{ 3 \} \) and \( t = 3, j = 1, M = \emptyset \), respectively, are,

\[
\begin{align*}
p_4 & \leq 4x_4 + (x_4 - s_4) + (x_3 - s_3) + (x_2 - s_2), \\
p_3 & \leq 4x_3 + (x_3 - s_3) + 2(x_2 - s_2).
\end{align*}
\]

Next we show that although there are exponential many inequalities (3.23), their separation can be done efficiently.

**Proposition 20.** Given a point \((\bar{p}, \bar{x}, \bar{s}) \in \mathbb{R}_+^{3n-1}\), there is an \( O(n^3) \) algorithm to find the most violated multi-period VUB ramp-up inequality (3.23), if any.

**Proof.** Given \( t \), let \( k = \min\{t - 2, \frac{u - \bar{u}}{\delta} \} \). Consider the shortest path problem on a directed acyclic graph \( G' = (N', A') \) where the vertex set \( N' = \{0, t', j_{t-k}, \ldots, j_1\} \), and the arc set \( A' \) is given by the arcs \((j_{i_2}, j_{i_1})\) with length \( \delta(i_2 - i_1)(\bar{x}_{j_{i_2}} - \bar{s}_{j_{i_2}}) \) for \( i_1, i_2 \in [t - k, t], i_1 < i_2 \), arcs \((j_i, t')\) with length \( (u - \bar{u} - (t - i)\delta)\(\bar{x}_{j_i} - \bar{s}_{j_i}\) \) for \( i \in [t - k, t] \) and arc \((0, j_i)\) with length \( \bar{u}\bar{x}_{j_i} - \bar{p}_{j_i} \). The value of the shortest path
from 0 to $t'$ in this graph is equal to the violation of inequality (3.23), if any. The
visited nodes in the shortest path determine the set $M$. Note that there are $O(k^2)$
arcs and $O(k)$ vertices in this directed acyclic graph, so this shortest path problem
can be solved in $O(k^2)$ time for a given $t$, where $k$ is $O(n)$. Solving this problem for
all $t$ we have an $O(n^3)$ time separation algorithm.

Next, we study the strength of the inequalities defined in this section.

3.4.2 Strength of the Proposed Inequalities

First, we consider inequality (3.21).

**Proposition 21.** Type-I multi-period ramp-up inequality (3.21) defines a facet of
$\text{conv}(U)$ if and only if $\ell + j\delta < u$.

**Proof.** Necessity. For contradiction assume that $\ell + j\delta \geq u$. From validity of inequality (3.21) we have $\ell + j\delta \leq u$, so $\ell + j\delta = u$. Then inequality (3.21) can be written as $p_{t+j} - p_t \leq (\ell + j\delta)x_{t+j} - \ell x_t = ux_{t+j} - \ell x_t$, and it is dominated by inequalities (3.2a) and (3.2b).

Sufficiency. We use the technique in Theorem 3.6 of §I.4.3 in [64]. We show that inequality (3.21) is the only inequality that is satisfied at equality by all points $(p, x, s) \in U$ that are tight at (3.21), i.e., we show that if all points of $U$ at which inequality (3.21) is tight satisfy

$$\sum_{k=1}^{n} \alpha_k p_k + \sum_{k=1}^{n} \beta_k x_k + \sum_{k=2}^{n} \gamma_k s_k = \alpha_0, \quad (3.24)$$

then

1. $\alpha_0 = 0$,
2. $\alpha_k = 0$, $k \in [1, n] \setminus \{t, t + j\}$,
3. $\alpha_t = -\bar{\alpha}$, $\alpha_{t+j} = \bar{\alpha}$,
4. $\beta_k = 0, \, k \in [1, n] \setminus \{t, t + j\},$
5. $\beta_t = \bar{\alpha} \ell,$
6. $\beta_{t+j} = -\bar{\alpha}(\ell + j\delta),$
7. $\gamma_k = 0, \, k \in [2, t] \cup [t + j + 1, n],$
8. $\gamma_{t+i'} = -\bar{\alpha} \min\{(\bar{u} - \ell - i'\delta), (u - \ell - j\delta)\}, \, i' \in [1, j].$

In order to establish the values of the coefficients $\alpha_k, \beta_k, \gamma_k$ and $\alpha_0$, we construct a feasible solution to $\mathcal{U}$ on the face defined by (3.21). Then a small change in the solution is made to obtain another feasible solution which is on the face defined by inequality (3.21). Comparing the resulting expressions, the possible values of a set of coefficients are obtained. Throughout, let $\epsilon$ be a very small number greater than zero. Also note that from the validity assumption and (A2), $\bar{u} \geq \ell + \delta > \ell$. We start by describing several points feasible to $\mathcal{U}$ that will be used throughout the facet proofs.

We assume that $k \geq 2$ if we set the value of $s_k$. In the following feasible solutions (except for the zero vector (S1)) if the value of a variable is not given, then its value is equal to zero. Let $k_1, k_2 \in [2, n]$ be two periods such that $k_1 < k_2$. 

$$x_r = p_r = 0, \, r \in [1, n], \, s_r = 0, \, r \in [2, n], \quad (S1)$$

$$x_{k_1} = 1, \, p_{k_1} = \ell, \, s_{k_1} = 1, \quad (S2)$$

$$x_{k_1} = 1, \, p_{k_1} = \ell + \epsilon, \, s_{k_1} = 1, \quad (S3)$$

$$x_1 = 1, \, p_1 = \ell, \quad (S4)$$

$$x_r = 1, \, p_r = \ell, \, r \in [k_1, k_2], \, s_{k_1} = 1, \quad (S5)$$

$$x_r = 1, \, p_r = \ell, \, r \in [k_1, k_2 - 1], \, x_{k_2} = 1, \, p_{k_2} = \ell + \epsilon, \, s_{k_1} = 1, \quad (S6)$$
\( x_r = 1, p_r = \ell + (r-t)\delta, r \in [t, t+j], s_t = 1, \quad (S7) \)

\( x_r = 1, p_r = \ell + \epsilon + (r-t)\delta, r \in [t, t+j], s_t = 1, \quad (S8) \)

\( x_r = 1, p_r = \ell, r \in [1, t], x_r = 1, p_r = \ell + (r-t)\delta, r \in [t+1, t+j], \quad (S9) \)

\( x_{k_1-1} = x_{k_1} = 1, p_{k_1-1} = p_{k_1} = \ell, s_{k_1-1} = 1 \) (assuming \( k_1 - 1 \geq 2 \)), \quad (S10) \)

\( x_r = 1, p_r = \bar{u} + (r - k_1), r \in [k_1, t + j], s_{k_1} = 1 \) (if \( \bar{u} + (t + j - k_1)\delta \leq u \)). \quad (S11) \)

Note that points (S3), (S6), (S7), (S8), and (S9) are feasible because \( \bar{u} > \ell \) and \( \ell + j\delta < u \).

Next we show the values of the coefficients \( \alpha_k, \beta_k, k \in [1, n], \gamma_k, k \in [2, n] \) and \( \alpha_0 \).

1. \( \alpha_0 = 0. \)

Consider solution (S1). Clearly, this point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Hence, \( \alpha_0 = 0. \)

2. \( \alpha_k = 0, k \in [1, n] \setminus \{t, t+j\}. \)

Consider the following two cases:

(a) \( k \in [1, t-1] \cup [t+j+1, n]. \)

Consider solution (S2) with \( k_1 = k. \) Clearly, this point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Now, consider solution (S3) with \( k_1 = k. \) This point also satisfies inequality (3.21) at equality and is a valid point because \( \bar{u} > \ell \) by assumption. Then evaluating (3.24) at both solutions we get \( \alpha_k \ell = \alpha_k (\ell + \epsilon). \) Hence, \( \alpha_k = 0. \)

(b) \( k \in [t+1, t+j-1]. \)

Consider solution (S5) with \( k_1 = t \) and \( k_2 = k. \) This point satisfies inequality (3.21) at equality because both the left- and the right-hand
sides of the inequality are equal to $-\ell$. Now, consider solution (S6) with $k_1 = t$ and $k_2 = k$ (this point also satisfies inequality (3.21) at equality). Then evaluating (3.24) at both solutions we get $\alpha_k \ell = \alpha_k (\ell + \epsilon)$. Hence, $\alpha_k = 0$.

3. $\alpha_t = -\bar{\alpha}, \alpha_{t+j} = \bar{\alpha}$.

Consider solution (S7). This point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to $j\delta$. Now, consider solution (S8). This point also satisfies inequality (3.21) at equality. Because we showed that $\alpha_k = 0, k \in [1, n] \setminus \{t, t+j\}$ in part 2, evaluating (3.24) at both solutions we get $\alpha_t \epsilon = -\alpha_{t+j} \epsilon$. Let $\bar{\alpha} := -\alpha_t = \alpha_{t+j}$.

4. $\beta_k = 0, k \in [1, n] \setminus \{t, t+j\}$.

Consider the following two cases:

(a) $k > t + j$.

Consider a solution to $U$ with $x_r = 1, r \in [t, k], s_t = 1, p_{t+i} = \ell + i\delta, i \in [0, j], p_r = \ell + j\delta, r \in [t+j+1, k]$ and all other variables are equal to zero. This point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to $j\delta$. Now, consider the same solution except we set $x_k = 0 = p_k$ (this solution is on the face defined by inequality (3.21)). Evaluating (3.24) at both solutions we get $\alpha_k (\ell + j\delta) + \beta_k = 0$. Because we showed that $\alpha_k = 0$ in part 2 we get $\beta_k = 0$.

(b) $t < k < t + j$.

Consider solution (S5) with $k_1 = t$ and $k_2 = k$. This point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to $-\ell$. Now, consider solution (S5) with
\( k_1 = t \) and \( k_2 = k - 1 \) if \( t < k - 1 \), and solution \((S2)\) with \( k_1 = t \) if \( t = k - 1 \). Both of the points satisfy inequality \((3.21)\) at equality. Note that if \( t = k - 1 \) we use solution \((S2)\) because both \( k_1 = t \) and \( k_2 = k - 1 = t \) and we define \( k_1 < k_2 \) in solution \((S5)\). Evaluating \((3.24)\) at the described solutions we get \( \alpha_k \ell + \beta_k = 0 \). Because we showed that \( \alpha_k = 0 \) in part 2 we get \( \beta_k = 0 \).

(c) \( k \leq t - 1 \) for \( t \geq 2 \).

Consider the following two cases:

i. \( k = 1 \).

Consider solution \((S4)\). This point is on the face defined by inequality \((3.21)\) because both the left- and the right-hand sides of the inequality are equal to zero. Evaluating \((3.24)\) at this solution we get \( \alpha_k \ell + \beta_k = \alpha_0 = 0 \) and since \( \alpha_k = 0 \) (from part 2) we get \( \beta_k = 0 \).

ii. \( k \geq 2 \).

Consider solution \((S10)\) with \( k_1 = k \). This point satisfies inequality \((3.21)\) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Now, consider solution \((S2)\) with \( k_1 = k - 1 \). Then evaluating \((3.24)\) at both solutions we get \( \alpha_k \ell + \beta_k = 0 \), and because \( \alpha_k = 0 \) (from part 2) we get \( \beta_k = 0 \).

5. \( \beta_t = \bar{\alpha} \ell \).

If \( t = 1 \), then we use solution \((S4)\). Because \( \alpha_t = \alpha_1 = -\bar{\alpha} \) (from part 3) evaluating this solution at equality \((3.24)\) we get \( \alpha_1 \ell + \beta_1 = 0 \) so \( \beta_1 = \bar{\alpha} \ell \).

For \( t \geq 2 \) consider solution \((S5)\) with \( k_1 = 1 \) and \( k_2 = t \). This point satisfies inequality \((3.21)\) at equality because both the left- and the right-hand sides of the inequality are equal to \(-\ell \). Now consider solution \((S5)\) with \( k_1 = 1 \) and \( k_2 = t - 1 \). This point is on the face defined by inequality \((3.21)\) because both
the left- and the right-hand sides of the inequality are equal to zero. Then evaluating (3.24) at both solutions we obtain \( \alpha_t \ell + \beta_t = 0 \). Because \( \alpha_t = -\bar{\alpha} \) (from part 3) we get \( \beta_t = \bar{\alpha} \ell \).

6. \( \beta_{t+j} = -\bar{\alpha}(\ell + j\delta) \).

Consider solution (59). This point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to \( j\delta \). Consider the same solution except now we set \( x_{t+j} = p_{t+j} = 0 \). This point is on the face defined by inequality (3.21) because both the left- and the right-hand sides of the inequality are equal to \( -\ell \). Then evaluating (3.24) at both solutions we obtain \( \alpha_{t+j}(\ell + j\delta) + \beta_{t+j} = 0 \). Because \( \alpha_{t+j} = \bar{\alpha} \) (from part 3) we get \( \beta_{t+j} = -\bar{\alpha}(\ell + j\delta) \).

7. \( \gamma_k = 0, \ k \in [2, t] \cup [t + j + 1, n] \).

Consider solution (S2) with \( k_1 = k \). This point satisfies inequality (3.21) at equality because both the left- and the right-hand sides of the inequality are equal to zero unless \( k = t \). If \( k = t \), then both the left- and the right-hand sides of the inequality are equal to \( -\ell \). Evaluating (3.24) at this solution we obtain \( \alpha_k \ell + \beta_k + \gamma_k = 0 \). If \( k \neq t \), then we have \( \alpha_k = \beta_k = 0 \) (from parts 2 and 4) so we get \( \gamma_k = 0 \). If \( k = t \), then because \( \alpha_t \ell = -\bar{\alpha} \ell \) and \( \beta_t = \bar{\alpha} \ell \) (from parts 3 and 5), we get \( \gamma_t = 0 \).

8. \( \gamma_{t+i'} = -\bar{\alpha} \min\{(\bar{u} - \ell - i'\delta), (u - \ell - j\delta)\}, \ i' \in [1, j] \).

Consider a solution to \( \mathcal{U} \) with \( x_{t+i'} = 1, \ p_{t+i'} = \bar{u}, \ s_{t+i'} = 1, \ x_{t+i} = 1, \ p_{t+i} = \min\{\bar{u} + (i - i')\delta, u\}, \ i \in [i' + 1, j] \) and all other variables are equal to zero. This point satisfies inequality (3.21) at equality because either both the left- and the
right-hand sides of the inequality are equal to \( \bar{u} + (j - i')\delta \) or \( u \) depending on the value of \( p_{t+j} \). Evaluating (3.24) at this solution we obtain,

\[
\alpha_{t+i'}\bar{u} + \beta_{t+i'} + \gamma_{t+i'} + \sum_{i=i'+1}^{j-1} (\alpha_{t+i}p_{t+i} + \beta_{t+i}) + \alpha_{t+j}p_{t+j} + \beta_{t+j} = 0. (3.26)
\]

From parts 1-4 and 6 we obtain \( \gamma_{t+i'} = -\bar{\alpha}(p_{t+j} - \ell - j\delta) \). Furthermore, because \( p_{t+j} \) is either \( \bar{u} + (j - i')\delta \) or \( u \), proof is complete.

Next, we study the strength of inequalities (3.22).

**Proposition 22.** Type-II multi-period ramp-up inequality (3.22) defines a facet of \( \text{conv}(U) \) only if the following conditions hold:

1. If \( \bar{u} = \ell \) and \( j \geq 2 \), then \( |S| = 1 \).
2. If \( \bar{u} > \ell + \delta \), then \( j > 1 \).

In addition, if the following conditions hold, then inequality (3.22) is a facet of \( \text{conv}(U) \)

3. \( \ell + j\delta < u \).
4. \( \bar{u} \leq \ell + \delta \).

**Proof.** Necessity.

1. Assume that \( \bar{u} = \ell, j \geq 2 \) and for contradiction \(|S| \geq 2 \).
Let $S = \{ q = q_1, q_2, \ldots, q_\omega = t + j \}$ and $q_0 = t$. Next consider the following inequalities (3.22) with $t = q_i$, $t + j = q_{i+1}$ and $S = \{ q_{i+1} \}$ for $i \in [0, \omega - 1]$:

\[
\begin{align*}
    p_{q_1} - p_t &\leq \ell x_{q_1} + (q_1 - t)\delta(x_{q_1} - s_{q_1}) - \ell x_t & (3.27) \\
    p_{q_2} - p_{q_1} &\leq \ell x_{q_2} + (q_2 - q_1)\delta(x_{q_2} - s_{q_2}) - \ell x_{q_1} & (3.28) \\
    &\vdots & (3.29) \\
    p_{q_{\omega-1}} - p_{q_{\omega-2}} &\leq \ell x_{q_{\omega-1}} + (q_{\omega-1} - q_{\omega-2})\delta(x_{q_{\omega-1}} - s_{q_{\omega-1}}) - \ell x_{q_{\omega-2}} & (3.30) \\
    p_{t+j} - p_{q_{\omega-1}} &\leq \ell x_{t+j} + (q_\omega - q_{\omega-1})\delta(x_{t+j} - s_{t+j}) - \ell x_{q_{\omega-1}} & (3.31)
\end{align*}
\]

Summing inequalities (3.27)-(3.31), we see that inequality (3.22) with $|S| \geq 2$ is dominated when $\bar{u} = \ell$ and $j \geq 2$.

2. Assume that $\bar{u} > \ell + \delta$ and for contradiction $j = 1$.

In this case, inequality (3.22) reduces to $p_{t+1} - p_t \leq \bar{u}x_{t+1} - \ell x_t$ and inequality (3.21) reduces to

\[
p_{t+1} - p_t \leq (\ell + \delta)x_{t+1} + (\bar{u} - \ell - \delta)s_{t+1} - \ell x_t. & (3.32)
\]

Adding $(\bar{u} - \ell - \delta)s_{t+1} \leq (\bar{u} - \ell - \delta)x_{t+1}$, i.e., scaled inequality (3.2e), to (3.32) we obtain inequality (3.22), hence inequality (3.22) cannot be a facet.

**Sufficiency.** We show that inequality (3.22) is the only inequality that is satisfied at equality by all points $(p, x, s) \in U$ that are tight at (3.22), i.e., we show that if all points of $U$ at which inequality (3.22) is tight satisfy

\[
\sum_{k=1}^{n} \alpha_k p_k + \sum_{k=1}^{n} \beta_k x_k + \sum_{k=2}^{n} \gamma_k s_k = \alpha_0, & (3.33)
\]

then

1. $\alpha_0 = 0$,
2. \( \alpha_k = 0, \ k \in [1, n] \setminus \{t, t + j\} \),

3. \( \beta_k = 0, \ k \in [1, n] \setminus (S \cup \{t\}) \),

4. \( \alpha_t = -\bar{\alpha}, \ \alpha_{t+j} = \bar{\alpha} \),

5. \( \beta_t = \bar{\alpha} \ell \),

6. If \(|S| = 1\), then \( \beta_{t+j} = \beta_q = -\bar{\alpha}(\bar{u} + (j - 1)\delta + \phi) \). Otherwise, \( \beta_{t+j} = -\bar{\alpha} (\bar{u} + (t + j - d_{t+j})\delta) \),

7. \( \gamma_k = 0, \ k \in [2, n] \setminus S \),

8. If \(|S| = 1\), then \( \gamma_{t+j} = \gamma_q = \bar{\alpha}((j - 1)\delta + \phi) \). Otherwise, \( \gamma_{t+j} = \bar{\alpha} (t + j - d_{t+j})\delta \),

9. If \( q > t + 1 \) and \( q \neq t + j \), then \( \beta_q = -\bar{\alpha}((q - d_q)\delta + \phi) \). Otherwise, (i.e., \( q = t + 1 \) and \( q \neq t + j \)) \( \beta_q = -\bar{\alpha}\phi \),

10. If \( q > t + 1 \) and \( q \neq t + j \), then \( \gamma_q = \bar{\alpha}((q - d_q)\delta + \phi) \). Otherwise, (i.e., \( q = t + 1 \) and \( q \neq t + j \)) \( \gamma_q = \bar{\alpha}\phi \),

11. \( \beta_i = -\bar{\alpha}(i - d_i)\delta, \ i \in S \setminus \{t + 1, q, t + j\} \),

12. \( \gamma_i = \bar{\alpha}(i - d_i)\delta, \ i \in S \setminus \{t + 1, q, t + j\} \).

Throughout, let \( \epsilon \) be a very small number greater than zero. From facet condition \( 4 \) we have \( \phi = \ell + \delta - \bar{u} \). We start by showing that,

1. \( \alpha_0 = 0 \).

Consider solution \([S1]\). Clearly, this point satisfies inequality \([3.22]\) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Hence, \( \alpha_0 = 0 \).

2. \( \alpha_k = 0, \ k \in [1, n] \setminus \{t, t + j\} \).

Consider the following cases:
(a) \(\bar{u} > \ell\).

Consider solution \((S_2)\) with \(k_1 = k\). Observe that \(x_k - s_k = 0\) unless \(k = 1\). If \(k = 1\), then because \(k \neq t, t \geq 2\) (so \(s_1\) does not exist). Therefore, for any \(k\), inequality \((3.22)\) is satisfied at equality because both the left- and the right-hand sides of the inequality are equal to zero. Next, consider solution \((S_3)\) with \(k_1 = k\). This point also satisfies inequality \((3.22)\) at equality. Evaluating equality \((3.33)\) at both solutions we get \(\alpha_k \epsilon = 0\). Thus, \(\alpha_k = 0\).

(b) \(\bar{u} \leq \ell\).

Recall that we assume that \(\bar{u} \geq \ell\), hence \(\bar{u} = \ell\). Furthermore, \(\phi = \ell + \delta - \bar{u} = \delta\). Consider the following cases:

i. \(k \in [1, t - 1]\).

If \(k = 1\), then we can use solution \((S_4)\) and construct another solution with \(x_1 = 1, p_1 = \ell + \epsilon\). Both solutions satisfy inequality \((3.22)\) at equality. For \(k \geq 2\) consider solution \((S_5)\) with \(k_1 = 1\) and \(k_2 = k\). Inequality \((3.22)\) is satisfied at equality because both the left- and the right-hand sides of the inequality are equal to zero. Next, consider solution \((S_6)\) with \(k_1 = 1\) and \(k_2 = k\). Evaluating equality \((3.33)\) at both solutions we get \(\alpha_k \epsilon = 0\). Thus, \(\alpha_k = 0\).

ii. \(k \in [t + 1, t + j - 1]\).

We use the same points used in part \(2(b)\) for \(k \geq 2\). The only difference is that inequality \((3.22)\) is satisfied at equality but both the left- and the right-hand sides of the inequality are equal to \(-\ell\) due to the choice of \(k\) (\(t + j \in S\) because from facet condition \(1|S| = 1\) and by definition of \(S\)).

iii. \(k \in [t + j + 1, n]\).
Consider a solution to $\mathcal{U}$ with $x_r = 1$, $r \in [t+j,k]$, $s_{t+j} = 1$, (so $x_{t+j} - s_{t+j} = 0$), $p_r = \bar{u}$, $r \in [t+j,k]$ and all other variables are equal to zero. Inequality (3.22) is satisfied at equality because both the left- and the right-hand sides of the inequality are equal to $\bar{u}$. Next, take this solution and only change $p_k = \bar{u} + \epsilon$. This is a feasible solution because the generator is turned on before period $k$ ($s_{t+j} = 1$ and $k > t+j$), and because in this case $\bar{u} = \ell < u$ from (A2). Inequality (3.22) is still satisfied at equality. Evaluating equality (3.33) at both solutions we get $\alpha_k \epsilon = 0$. Thus, $\alpha_k = 0$.

3. $\beta_k = 0$, $k \in [1,n] \setminus (S \cup \{t\})$.

We consider the following cases:

(a) $k \in [1,t-1]$.

If $k = 1$, then we can use solution (S4). We get $\alpha_k + \beta_k = 0$ and because $\alpha_k = 0$ from part 2, $\beta_k = 0$. For $k \geq 2$ consider solution (S5) with $k_1 = 1$ and $k_2 = k$ and $k_1 = 1$ and $k_2 = k - 1$. Evaluating equality (3.33) at both solutions we get $\alpha_k \ell + \beta_k = 0$ and because $\alpha_k = 0$ (from part 2) we obtain $\beta_k = 0$.

(b) $k \in [t+1,t+j-1]$, $k \not\in S$.

Consider solution (S10) with $k_1 = k$. Inequality (3.22) is satisfied at equality in all of the following cases:

i. $k - 1 = t$.

Both the left- and the right-hand sides of the inequality are equal to $-\ell$.

ii. $k - 1 \geq t + 1$.

Note that either $k - 1 \in (S \setminus \{t+1\}) \cup \{q\}$ or $k - 1 \not\in S$. In both
cases, because $t \geq 1$ variable $s_{k-1}$ exists. Therefore, $x_{k-1} - s_{k-1} = 0$ and both the left- and the right-hand sides of the inequality are equal to zero.

Next, we consider solution $(S2)$ with $k_1 = k - 1$. Evaluating equality (3.33) at both solutions we get $\alpha_k \ell + \beta_k = 0$ and because $\alpha_k = 0$ (from part 2) we obtain $\beta_k = 0$.

(c) $k \in [t + j + 1, n]$

Consider the first point given in part 2(b)iii. Then we take this solution and only change $x_k = p_k = 0$. Evaluating equality (3.33) at both solutions we get $\alpha_k \ell + \beta_k = 0$ and because $\alpha_k = 0$ (from part 2) we obtain $\beta_k = 0$.

4. $\alpha_t = -\bar{\alpha}, \alpha_{t+j} = \bar{\alpha}$.

From facet condition 4 we have $\phi = \ell + \delta - \bar{u}$. Consider solution $(S9)$, which is feasible because $\ell + j\delta \leq u$ by the definition of $j$. Observe that no start-up for the generator occurs in the given solution. Inequality (3.22) is satisfied at equality because both the left- and the right-hand sides of the inequality are equal to $j\delta$. (Observe that $x_r - s_r = 1$, $r \in [2, t + j]$.) Next, take this solution and let $p_t = \ell + \epsilon$ and $p_{t+i} = \ell + \epsilon + i\delta$, $i \in [1, j]$ while all other variable values remain the same (this point also satisfies inequality (3.22) at equality and it is feasible because of facet condition 3). Evaluating equality (3.33) at both solutions we get $\epsilon(\alpha_t + \alpha_{t+1} + \cdots + \alpha_{t+j}) = 0$. From part 2 we have $\alpha_r = 0$, $r \in [t + 1, t + j - 1]$ so $\alpha_t = -\alpha_{t+j} = -\bar{\alpha}$.

5. $\beta_t = \bar{\alpha}\ell$.

We can use the same argument as part 5 in the sufficiency proof of Proposition 21.
6. If $|S| = 1$, then $\beta_{t+j} = \beta_q = -\bar{\alpha}(\bar{u} + (j-1)\delta + \phi)$. Otherwise, $\beta_{t+j} = -\bar{\alpha}(\bar{u} + (t+j-d t_{t+j})\delta)$.

Consider the following cases:

(a) $|S| = 1$.

From facet condition 4 we have $\phi = \ell + \delta - \bar{u}$. Consider solution (S9). Next, let $x_{t+j} = p_{t+j} = 0$ while all other variable values remain the same as (S9). Inequality (3.22) is also satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to $-\ell$.

Evaluating equality (3.33) at both solutions we get $\alpha_{t+j}(\ell + j\delta) + \beta_{t+j} = 0$, and because $\alpha_{t+j} = \bar{\alpha}$ from part 4 we obtain $\beta_{t+j} = \beta_q = -\bar{\alpha}(\ell + j\delta) = -\bar{\alpha}(\bar{u} + (j-1)\delta + \phi)$.

(b) $|S| > 1$.

Consider a solution to $U$ with $x_r = 1$, $r \in [d_{t+j}, t+j]$; $p_{d_{t+j}+i} = \bar{u} + i\delta$, $i \in [1, (j-d_{t+j})]$; $s_{d_{t+j}} = 1$ and all other variables are equal to zero. Inequality (3.22) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to $\bar{u} + (t+j-d t_{t+j})\delta$.

Next, consider the same solution except we set $x_{t+j} = p_{t+j} = 0$ which is also on the face defined by (3.22). Therefore, $\alpha_{t+j}(\bar{u} + (t+j-d t_{t+j})\delta) + \beta_{t+j} = 0$.

Because $\alpha_{t+j} = \bar{\alpha}$ from part 4 we get $\beta_{t+j} = -\bar{\alpha}(\bar{u} + (t+j-d t_{t+j})\delta)$.

We use the following observation in the remainder of the proof.

**Observation 4.** For $k \in [2, n] \setminus \{t, t+j\}$, $\beta_k = -\gamma_k$. This can be shown by using solution (S2) with $k_1 = k$. Inequality (3.22) is satisfied at equality since both the left- and the right-hand sides of the inequality are equal to zero whether or not $k \in S$ because $x_k - s_k = 0$. Evaluating equality (3.33) at this solution we get $\alpha_k\ell + \beta_k + \gamma_k = 0$ and since $\alpha_k = 0$ from part 2 we obtain $\beta_k = -\gamma_k$. 

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7. \( \gamma_k = 0 \), \( k \in [2, n] \setminus S \).

If \( k \neq t \), we can easily see this result by using Observation 4 and part 3. If \( k = t \) consider solution \( (S_2) \) with \( k_1 = t \). Then both the left- and the right-hand sides of inequality (3.22) are equal to \( -\ell \). Evaluating equality (3.33) at this solution we get \( \alpha_k \ell + \beta_k + \gamma_k = 0 \). Because \( \alpha_k \ell + \beta_k = 0 \) from parts 4 and 5 we get \( \gamma_k = 0 \).

8. If \( |S| = 1 \), then \( \gamma_{t+j} = \gamma_q = \bar{\alpha}((j-1)\delta + \phi) \). Otherwise, \( \gamma_{t+j} = \bar{\alpha}(t+j-d_{t+j})\delta \).

Consider a solution with \( x_{t+j} = 1 \), \( p_{t+j} = \bar{u} \) and \( s_{t+j} = 1 \). Inequality (3.22) is satisfied at equality at this point since both the left- and the right-hand sides of the inequality are equal to \( \bar{u} \). Evaluating equality (3.33) at this solution we get \( \alpha_t \ell + \beta_t + \gamma_{t+j} = 0 \). From part 4 we have \( \alpha_{t+j} = \bar{\alpha} \). From part 6 if \( |S| = 1 \), then \( \beta_{t+j} = -\bar{\alpha}(\bar{u} + (j-1)\delta + \phi) \), so we obtain \( \gamma_{t+j} = \gamma_q = \bar{\alpha}((j-1)\delta + \phi) \).

Else if \( |S| > 1 \), then \( \beta_{t+j} = -\bar{\alpha}(\bar{u} + (t+j-d_{t+j})\delta) \), so \( \gamma_{t+j} = \bar{\alpha}(t+j-d_{t+j})\delta \).

9. If \( q > t+1 \) and \( q \neq t+j \), then \( \beta_q = -\bar{\alpha}((q-d_q)\delta + \phi) \). If \( q = t+1 \) and \( q \neq t+j \), then \( \beta_q = -\bar{\alpha}\phi \).

Observe that because \( q = \min\{k \in S\} \) if \( q > t+1 \), then we have \( d_q = t+1 \). Consider solution \( (S_7) \). Inequality (3.22) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to \( j \delta \).

Evaluating equality (3.33) at this solution we get \( \alpha_t \ell + \beta_t + \gamma_t + \sum_{k=t+1}^{t+j-1} (\alpha_k (\ell + (k-t)\delta) + \beta_k) + \alpha_{t+j} (\ell + j\delta) + \beta_{t+j} = 0 \), which reduces to

\[
\sum_{i \in S} \beta_i = -\bar{\alpha}(\ell - \bar{u} + (d_{t+j} - t)\delta). \tag{3.34}
\]

Observe that \( \alpha_t \ell + \beta_t + \gamma_t = 0 \) from parts 4, 5 and 7. Next, we consider a solution to \( U \) where \( x_r = 1 \), \( r \in [q, t+j] \), \( p_{q+i} = \bar{u} + i\delta \), \( i \in [0, t+j-q] \), \( s_q = 1 \) and all other variables are equal to zero. Inequality (3.22) is satisfied
at equality at this point because both the left- and the right-hand sides of the inequality are equal to $\bar{u} + (t + j - q)\delta$. (Note that we showed that $\beta_q + \gamma_q = 0$ in Observation 4.) Evaluating equality (3.33) at this solution we get

$$\alpha_q \bar{u} + \beta_q + \gamma_q + \sum_{k=q+1}^{t+j-1} (\alpha_k (\bar{u} + (k - q)\delta) + \beta_k) + \alpha_{t+j} (\bar{u} + (t + j - q)\delta) + \beta_{t+j} = 0$$

which reduces to

$$\sum_{i \in S \setminus \{q\}} \beta_i = -\bar{\alpha}(d_{t+j} - q)\delta.$$  \hspace{1cm} (3.35)

From equations (3.34) and (3.35) we get $\beta_q = -\bar{\alpha}(\ell - \bar{u} + (q - t)\delta) = -\bar{\alpha}(q - d_q)\delta + \phi)$. If $q = t + 1$, then we obtain $\beta_q = -\bar{\alpha}(\ell - \bar{u} + \delta) = -\bar{\alpha}\phi$.

10. If $q > t + 1$ and $q \neq t + j$, then $\gamma_q = \bar{\alpha}((q - d_q)\delta + \phi)$. Otherwise, (i.e., $q = t + 1$ and $q \neq t + j$) $\gamma_q = \bar{\alpha}\phi$.

We can easily see this result by using Observation 4 and part 9.

11. $\beta_i = -\bar{\alpha}(i - d_i)\delta$, $i \in S \setminus \{t + 1, q, t + j\}$.

Let $S = \{q = q_1, q_2, \ldots, q_\omega = t + j\}$. We make use of equation (3.35) and consider solution (S11) with $k_1 = q_2$. Note that for $t + 1 < k_1 < t + j$ solution (S11) is feasible because $\bar{u} + (t + j - k_1)\delta \leq \ell + (t + j - k_1 + 1)\delta < u$, where the last inequality follows from facet condition 4. Inequality (3.22) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to $\bar{u} + (t + j - q_2)\delta$. Evaluating equality (3.33) at this solution we get

$$\sum_{i \in S \setminus \{q, q_2\}} \beta_i = -\bar{\alpha}(d_{t+j} - q_2)\delta.$$  \hspace{1cm} (3.36)

From equations (3.35) and (3.36) we get $\beta_{q_2} = -\bar{\alpha}(q_2 - q)\delta$, where $d_{q_2} = q$. To find the value of $\beta_{q_3}$ we consider solution (S11) with $k_1 = q_3$. Inequality (3.22) is satisfied at equality at this point because both the left- and the right-hand
sides of the inequality are equal to \( \bar{u} + (t+j-q_3)\delta \). Evaluating equality (3.33) at this solution we get

\[
\sum_{i \in S \setminus \{q,q_2,q_3\}} \beta_i = -\bar{\alpha}(d_{t+j} - q_3)\delta. \tag{3.37}
\]

From equations (3.36) and (3.37) we get \( \beta_{q_3} = -\bar{\alpha}(q_3 - q_2)\delta \), where \( d_{q_3} = q_2 \).

We repeat this argument so that the last point we consider is solution \((S11)\) with \( k_1 = q_\omega \) to find the value of \( \beta_{d_{t+j}} \).

12. \( \gamma_i = \bar{\alpha}(i - d_i)\delta \), \( i \in S \setminus \{t + 1, q, t + j\} \).

We can easily see this result by using Observation 4 and part 11.

In Proposition 23 we study the strength of inequalities (3.23).

**Proposition 23.** The multi-period VUB ramp-up inequality (3.23) defines a facet of \( \text{conv}(U) \).

**Proof.** We show that inequality (3.23) is the only inequality that is satisfied at equality by all points \((p, x, s) \in U\) that are tight at (3.23) i.e., we show that if all points of \( U \) at which inequality (3.23) is tight satisfy

\[
\sum_{k=1}^{n} \alpha_k p_k + \sum_{k=1}^{n} \beta_k x_k + \sum_{k=2}^{n} \gamma_k s_k = \alpha_0, \tag{3.38}
\]

then

1. \( \alpha_0 = 0 \),
2. \( \alpha_k = 0 \), \( k \in [1,n] \setminus \{t\} \),
3. \( \beta_k = 0 \), \( k \not\in (M \cup \{t-j,t\}) \),
4. \( \gamma_k = 0 \), \( k \not\in (M \cup \{t-j,t\}) \),
5. \( \beta_t = -\bar{\alpha}(\bar{u} + (t - e_t)\delta) \), where \( \bar{\alpha} = \alpha_t \),

6. \( \gamma_t = \bar{\alpha}(t - e_t)\delta \),

7. \( \beta_k = -\bar{\alpha}(k - e_k)\delta, \quad k \in M \),

8. \( \gamma_k = \bar{\alpha}(k - e_k)\delta, \quad k \in M \),

9. \( \beta_{t-j} = -\bar{\alpha}(u - \bar{u} - j\delta) \),

10. \( \gamma_{t-j} = \bar{\alpha}(u - \bar{u} - j\delta) \).

Throughout, let \( \epsilon \) be a very small number greater than zero. We start by showing that,

1. \( \alpha_0 = 0 \).

Consider solution \((S1)\). Clearly, inequality \((3.23)\) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to zero. Hence, \( \alpha_0 = 0 \).

2. \( \alpha_k = 0, \quad k \in [1, n] \setminus \{t\} \).

We consider the following two cases:

(a) \( k \in [1, t - j - 1] \cup [t + 1, n] \).

If \( k = 1 \), we consider solution \((S4)\) and construct another solution with \( x_1 = 1, \ p_1 = \ell + \epsilon \) (and all other variables are equal to zero), which is feasible because \( u > \ell \) from (A2). Inequality \((3.23)\) is satisfied at equality for both solutions and we get \( \alpha_1 = 0 \) by evaluating \((3.38)\) at both solutions. For \( k \geq 2 \) consider a variant of solution \((S5)\) with \( k_1 = 1, k_2 = k \) and \( p_r = u, \quad r \in [1, k] \). Inequality \((3.23)\) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to zero if \( k \leq t - j - 1 \) and \( u \) if \( k \geq t + 1 \). Then consider another solution
where we only change the value of \( p_k = u - \epsilon \). Evaluating (3.38) at both solutions we obtain \(-\alpha_k \epsilon = 0\) so \( \alpha_k = 0 \).

(b) \( k \in [t - j, t - 1] \).

Consider a variant of solution \( S_5 \) with \( k_1 = 1, k_2 = t \) and \( p_r = u, r \in [1, t] \). Inequality (3.23) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to \( u \). Then consider another solution where we only change the value of \( p_k = u - \epsilon \).

Evaluating (3.38) at both solutions we obtain \(-\alpha_k \epsilon = 0\) so \( \alpha_k = 0 \).

3. \( \beta_k = 0, k \notin (M \cup \{ t - j, t \}) \).

If \( k = 1 \), then consider solution \( S_4 \). Clearly, inequality (3.23) is satisfied at equality at this point. Evaluating (3.38) at this solution we obtain \( \alpha_1 \ell + \beta_1 = 0 \). Because \( \alpha_1 = 0 \) from part 2 we get \( \beta_1 = 0 \). If \( k \geq 2 \) then we consider a variant of solution \( S_{10} \) with \( k_1 = k \) and \( p_{k-1} = p_k = \bar{u} \). Consider the following cases for \( k - 1 \).

(a) \( k - 1 = t \).

Inequality (3.23) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to \( \bar{u} \).

(b) \( k - 1 \in (M \cup \{ t - j \}) \) or \( k - 1 \notin (M \cup \{ t - j, t \}) \).

If \( k - 1 \in (M \cup \{ t - j \}) \), then because \( x_{k-1} - s_{k-1} = 0 \), both the left- and the right-hand sides of inequality (3.23) are equal to zero, and the inequality is tight at this point. If \( k - 1 \notin (M \cup \{ t - j, t \}) \), then clearly inequality (3.23) is satisfied at equality at this point.

Next, we consider a variant of solution \( S_2 \) with \( k_1 = k - 1 \) and \( p_{k-1} = \bar{u} \).

Evaluating (3.38) at both solutions we obtain \( \alpha_k \bar{u} + \beta_k = 0 \). Because \( \alpha_k = 0 \) from part 2 we get \( \beta_k = 0 \).
We use the following observation in the remainder of the proof.

**Observation 5.** For \( k \in [2, n] \setminus \{ t \} \), \( \beta_k = -\gamma_k \). This can be shown by using the point \((S_2)\) with \( k_1 = k \). Inequality (3.23) is satisfied at equality at this point because \( x_k - s_k = 0 \), so both the left- and the right-hand sides of the inequality are equal to zero. Evaluating this solution at equality (3.38) we get \( \alpha_k \ell + \beta_k + \gamma_k = 0 \). Because \( \alpha_k = 0 \) from part 2, we obtain \( \beta_k = -\gamma_k \).

4. \( \gamma_k = 0, \ k \notin (M \cup \{ t - j, t \}) \).

Using Observation 5 and part 3 we can see that this result holds.

5. \( \beta_t = -\bar{\alpha}(\bar{u} + (t - e_t)\delta), \) where \( \bar{\alpha} = \alpha_t \).

Let \( \alpha_t = \bar{\alpha} \). Consider a solution to \( U \) with \( x_r = 1, \ r \in [e_t, t] \), \( p_{e_t+i} = \bar{u} + i\delta \), \( i \in [0, t - e_t] \), \( s_{e_t} = 1 \) and all other variables are equal to zero. Inequality (3.23) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to \( \bar{u} + (t - e_t)\delta \). Observe that \( \bar{u} + j\delta \leq u \) due to the choice of \( j \) in Proposition 19. This observation will be used in the remaining part of the proof. We take this solution and only change the value of \( x_t = p_t = 0 \). Inequality (3.23) is again satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to zero \( (x_{e_t} - s_{e_t} = 0) \). Evaluating (3.38) at both solutions we obtain \( (\bar{u} + (t - e_t)\delta)\alpha_t + \beta_t = 0 \). Because \( \alpha_t = \bar{\alpha} \) we get \( \beta_t = -\bar{\alpha}(\bar{u} + (t - e_t)\delta) \).

6. \( \gamma_t = \bar{\alpha}(t - e_t)\delta \).

Consider a variant of solution \((S_2)\) with \( k_1 = t \) and \( p_t = \bar{u} \). Inequality (3.23) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to \( \bar{u} \). Evaluating (3.38) at this solution we obtain \( \alpha_t\bar{u} + \beta_t + \gamma_t = 0 \). Because \( \beta_t = -\bar{\alpha}(\bar{u} + (t - e_t)\delta) \) from part 5 we get \( \gamma_t = \bar{\alpha}(t - e_t)\delta \).
7. $\beta_k = -\bar{\alpha}(k - e_k)\delta$, $k \in M$.

Let $M = \{m_1, m_2, \ldots, m_b\}$, $m_b = e_t$ and $m_{i-1} = e_{m_i}$, $i \in [2, b]$. Consider a solution to $U$ with $x_r = 1$, $r \in [m_{b-1}, t]$, $p_{m_{b-1}+i} = \bar{u} + i\delta$, $i \in [0, t - m_{b-1}]$, $s_{m_{b-1}} = 1$ and all other variables are equal to zero. Inequality (3.23) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to $\bar{u} + (t - m_{b-1})\delta$. Evaluating (3.38) at this solution we obtain,

$$\alpha_{m_{b-1}}\bar{u} + \beta_{m_{b-1}} + \gamma_{m_{b-1}} + \sum_{i=m_{b-1}+1}^{t} (\alpha_i(\bar{u} + (i - m_{b-1})\delta) + \beta_i) = 0. \quad (3.39)$$

We have $\alpha_{m_{b-1}}\bar{u} + \beta_{m_{b-1}} + \gamma_{m_{b-1}} = 0$ from Observation 5 and part 2. Furthermore, $
\sum_{i=m_{b-1}+1}^{m_{b}} (\alpha_i(\bar{u} + (i - m_{b-1})\delta) + \beta_i) = 0$ and $\sum_{i=m_{b}+1}^{t-1} (\alpha_i(\bar{u} + (i - m_{b-1})\delta) + \beta_i) = 0$ from parts 2 and 3. Therefore, equality (3.39) reduces to $\alpha_{m_b}(\bar{u} + (m_b - m_{b-1})\delta) + \beta_{m_b} + \alpha_t(\bar{u} + (t - m_{b-1})\delta) + \beta_t = 0$. Using parts 2 and 5 we get $\beta_{m_b} = -\bar{\alpha}(m_b - m_{b-1})\delta$ where $m_{b-1} = e_{m_b}$. Observe that we can find $\beta_{m_{b-1}} = -\bar{\alpha}(m_{b-1} - m_{b-2})\delta$ by considering a solution with $x_r = 1$, $r \in [m_{b-2}, t]$, $p_{m_{b-2}+i} = \bar{u} + i\delta$, $i \in [0, t - m_{b-2}]$, $s_{m_{b-2}} = 1$ and all other variables are equal to zero. We repeat this definition of points until we find $\beta_{m_1} = -\bar{\alpha}(m_1 - (t - j))\delta$.

Hence, we can show that $\beta_k = -\bar{\alpha}(k - e_k)\delta$, for $k \in M$.

8. $\gamma_k = \bar{\alpha}(k - e_k)\delta$, $k \in M$.

Using Observation 5 and part 7 we can see that this result holds.

9. $\beta_{t-j} = -\bar{\alpha}(u - \bar{u} - j\delta)$.

Consider a variant of solution (S5) with $k_1 = 1$, $k_2 = t$ and $p_r = u$, $r \in [1, t]$. Inequality (3.23) is satisfied at equality at this point because both the left- and the right-hand sides of the inequality are equal to $u$. Evaluating (3.38) at this solution we obtain $\sum_{i=1}^{t-1} (\alpha_i u + \beta_i) + \alpha_t u + \beta_t = 0$. We have $\alpha_i u = 0$
for all $i \in [1, t - 1]$ from part 2, 
\[
\sum_{i=1}^{t-1} \beta_i = \beta_{t-j} + \sum_{i \in M} \beta_i 
\] from part 3, and 
\[
\sum_{i \in M} \beta_i + \beta_t = -\bar{\alpha}(\bar{u} + j\delta) 
\] from parts 5 and 7. Furthermore, because $\alpha_t = \bar{\alpha}$ we get 
\[
\beta_{t-j} = -\bar{\alpha}(u - \bar{u} - j\delta). 
\]
10. $\gamma_{t-j} = \bar{\alpha}(u - \bar{u} - j\delta)$.

Using Observation 5 and part 9 we can see that this result holds.

Even though we give large classes of valid inequalities for $U$, next example shows that they are not sufficient to completely describe $\text{conv}(U)$ for $n > 2$.

**Example 4**  Consider $U$, where $n = 4$, $u = 7$, $\ell = 1$, $\bar{u} = 4$, and $\delta = 1$. The following inequalities are facets of $\text{conv}(U)$:

\[
p_3 + 2p_4 - p_1 \leq 10x_4 - x_1 + 7x_3 - 3s_3 - 2s_4,
9p_4 - 2p_1 - 3p_2 \leq 45x_4 - 2x_1 - 3x_2 - 9s_4.
\]

Note that these inequalities cannot be expressed as one of the inequalities (3.21), (3.22), (3.23), because some of the production variables have integer coefficients greater than one in absolute value.

### 3.4.3 Ramp-Down Polytope

In this section, we present the ramp-down analogs of the results given in Section 3.4.1. We use the symmetry between ramping up and ramping down constraints. Specifically, reversing time $(n, \ldots, 1)$ is sufficient to obtain this symmetry. Nevertheless, it is important to note that there is a difference in the information start-up and shut-down variables provide. For example in any given solution if $s_t = 1$, then $x_{t-1} = 0$, $x_t = 1$ however, if $z_t = 1$, then $x_{t-1} = 1$ and $x_t = 0$. Figure 3.1 illustrates this difference.
We omit the proofs of our results for the ramp-down polytope, because they follow from their ramp-up counterparts with minor adjustments.

Next, we describe the ramp-down analog of type-I multi-period ramp-up inequality (3.21).

**Proposition 24.** If \( \bar{u} \geq \ell + \delta \), then for \( 1 \leq t \leq n \) and \( 1 \leq j \leq \min \left\{ n - t, \frac{\bar{u} - \ell}{\delta} \right\} \), the type-I multi-period ramp-down inequality

\[
    p_t - p_{t+j} \leq (\ell + j\delta)x_t + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - (j-i+1)\delta), (u - \ell - j\delta)\}z_{t+i} - \ell x_{t+j} \tag{3.40}
\]

is valid for \( \mathcal{D} \). It defines a facet of \( \text{conv}(\mathcal{D}) \) if and only if \( \ell + j\delta < u \).

In the next proposition we give the ramp-down analog of type-II multi-period ramp-up inequality (3.22). This result immediately follows from Proposition 17 by reversing the time.

---

Figure 3.1: Effect of start-up and shut-down variables on production.
Proposition 25. For $1 \leq t \leq n$ and $1 \leq j \leq \min \left\{ n - t, \frac{u - \ell}{\delta} \right\}$, let $S' \subseteq [t+1, t+j]$, $t + 1 \in S'$, $q' = \max\{k \in S'\}$, and $d'_i = \min\{k \in S' \cup \{t + j\} : k > i\}$, $i \in S'$. Then the type-II multi-period ramp-down inequality

$$p_t - p_{t+j} \leq \bar{u}x_t + \delta \sum_{i \in S' \setminus \{t+j\}} (d'_i - i)(x_{i-1} - z_i) + \phi(x_{q'-1} - z_{q'}) - \ell x_{t+j}, \quad (3.41)$$

where $\phi = (\ell + \delta - \bar{u})^+$ is valid for $D$.

As is the case for its ramp-up counterpart, the separation of inequalities (3.41) can be done efficiently.

Proposition 26. Given a point $(\bar{p}, \bar{x}, \bar{z}) \in \mathbb{R}^{3n-1}_+$, there is an $O(n^3)$ algorithm to find the most violated inequality (3.41), if any.

Next, we give conditions under which inequality (3.41) defines a facet.

Proposition 27. Type-II multi-period ramp-down inequality (3.41) defines a facet of $\text{conv}(D)$ only if the following conditions hold:

1. If $\bar{u} = \ell$ and $j \geq 2$ then $|S'| = 1$.

2. If $\bar{u} \geq \ell + \delta$ then $j > 1$.

In addition, if the following conditions hold, then inequality (3.41) is a facet of $\text{conv}(D)$

3. $\ell + j\delta < u$.

4. $\bar{u} \leq \ell + \delta$.

Observe that when $j = 1$ and $\bar{u} \leq \ell + \delta$ inequality (3.41) reduces to $p_t - p_{t+1} \leq (\ell + \delta)x_t - (\ell + \delta - \bar{u})z_{t+1} - \ell x_{t+1}$. Inequality (3.41) can be rewritten as $p_t - p_{t+1} \leq \bar{u}z_{t+1} + \delta x_{t+1} - (\ell + \delta)s_{t+1}$, and because $(\ell + \delta)s_{t+1} \geq 0$ it dominates inequality (3.3f).
Similarly, when $j = 1$ and $\bar{u} > \ell + \delta$ inequality (3.40) dominates inequality (3.3f) because, in this case, inequality (3.40) reduces to $p_t - p_{t+1} \leq \bar{u}z_{t+1} + \delta x_{t+1} - (\ell + \delta)s_{t+1}$.

Finally, we give the ramp-down analog of the multi-period VUB ramp-up inequality (3.23).

**Proposition 28.** For $2 \leq t \leq n$, $0 \leq j \leq \min\left\{t - 2, \frac{u - \bar{u}}{\delta}\right\}$ and any $M \subseteq [t - j + 1, t - 1]$ the multi-period VUB ramp-down inequality

$$p_{t-j-1} \leq \bar{u}x_{t-j-1} + \delta \sum_{i \in M \cup \{t-j\}} (e'_i - i)(x_{i-1} - z_i) + (u - \bar{u} - j\delta)(x_{t-1} - z_t),(3.42)$$

where for $i \in [t - j, t - 1]$ $e'_i = \min\{k \in M \cup \{t\} : k > i\}$, and if $j = 0$, then $e'_i = t$ is valid for $D$ and it defines a facet of conv($D$).

As is the case for its ramp-up counterpart, the separation of inequalities (3.42) can be done efficiently.

**Proposition 29.** Given a point $(\bar{p}, \bar{x}, \bar{z}) \in \mathbb{R}^{3n-1}_+$, there is an $O(n^3)$ algorithm to find the most violated inequality (3.42), if any.

### 3.5 Computational Results

In this section, we report our computational experiments with valid inequalities (3.21), (3.22), (3.23), (3.40), (3.41) and (3.42) in a branch-and-cut algorithm. We test the strength of these inequalities using formulation $\mathcal{UC}$. For test purposes we create eight combinations of numbers of generators ($m$) and time periods ($n$). For each combination we generate 3 instances and report the averages. We add a user cut if it is violated by 0.01 units and we do not limit the number of cuts that are added at each branch and bound node. Problem parameters for each generator are created using a uniform distribution with minimum production value $\ell \in [5000, 10000]$, maximum production value $u \in [10000, 20000]$, minimum up/down time to be between
1 and \( \frac{n}{2} \), maximum production level in the first (and) last operating period value \( \bar{u} \in [\ell, \ell + (0.05)(u - \ell)] \), maximum change in production from one operating period to the next value \( \delta \in [0, (0.05)(u - \ell)] \), production cost \( c_{t} \in [0, 0.1] \), fixed cost of running the generator \( h_{t} \in [200, 900] \) and the fixed cost of starting up the generator \( f_{t} \in [1000, 2000] \), \( t \in [1, n] \). Similarly, for each time period \( t \in [1, n] \) a uniform distribution is utilized for parameters demand \( \tilde{d}_{t} \in [2500m, 7500m] \) and spinning reserve \( r_{t} \in [1, 1.1] \). We conduct the experiments on an Intel Xeon x5650 Processor at 2.67GHz with 4GB RAM. We use IBM ILOG CPLEX 12.4 as the MIP solver. We turn the dynamic search option off, use a single thread, permit only linear reductions in the presolve phase of CPLEX, and impose a time limit of one hour in all our experiments.

Our study complements the polyhedral study of [68], which also takes minimum up and minimum down time into consideration. [68] present two polynomial classes of three-period ramping inequalities, but both of them are valid only when \( \bar{v} \geq 2 \), and are therefore not valid for \( \mathcal{U} \). The authors also present a polynomial class of VUB inequalities, but these are the same as our two-period VUB inequality (3.5) when \( \bar{v} = 1 \), and are not valid for \( \mathcal{U} \) when \( \bar{v} \geq 2 \). As a result, our inequalities cannot be directly compared to the inequalities in [68] analytically. We note that [68] also assume that \( \bar{u} - \delta < \ell \) in all but one class of their inequalities. In contrast, among the six classes of inequalities we propose, four of them are exponential and are valid without any restriction on the data. Only the type-I multi-period ramping inequalities (3.21) and (3.40) are polynomial, and they are valid under the assumption \( \bar{u} \geq \delta + \ell \).

In Table 3.2 we test the strength of our inequalities empirically at the root node, by comparing the performance of three algorithms:

**UCE-N (User Cuts with Exact Separation):** \( \mathcal{UC} \) formulation with only user cuts. Note that user cuts refers to inequalities (3.21), (3.22), (3.23), (3.40),
We use variants of the exact separation algorithms we described in Section 3.4 for inequalities (3.22), (3.23), (3.41) and (3.42). For example, for the separation of inequalities (3.22) given in Proposition 18 instead of generating the most violated cut for each $t$, we generate a violated cut for the smallest $j$ for each $t$, if there are any.

**OC-N ([68] cuts):** $\mathcal{UC}$ formulation with ramping and VUB inequalities of [68] added to the user cuts table for CPLEX (no separation algorithm is implemented), and all CPLEX cuts are turned off.

**CD (CPLEX Default Cut Settings):** $\mathcal{UC}$ formulation with default CPLEX cut settings.

In Table 3.2, we report LPGap, the initial gap percentage at the root node, which is $100 \times \frac{(zub - zlp)}{zub}$, where $zub$ is the objective function value of the best integer solution obtained within time limit (among all the compared algorithms if the optimal solution is unknown) and $zlp$ is the objective function value of the LP relaxation to formulation $\mathcal{UC}$. Column RGap gives the integrality gap percentage at the root node just before branching which is $100 \times \frac{(zub - zrb)}{zub}$, where $zrb$ is the best lower bound obtained at the root node. We report the number of cuts added at the root node by column RCuts. We denote the user cuts added by prefix u, whereas we do not use a prefix for the cuts added by CPLEX. For the gap values in the tables we report the numbers rounded to the second decimal place. For all the tables we report the overall averages in the last row Avg. The time for an unsolved instance is included in the average calculations as 3600 seconds.

Because CPLEX cuts are turned off for algorithms UCE-N and OC-N we can observe the benefits of adding the inequalities defined in this chapter compared to inequalities proposed by [68] without the interference of CPLEX cuts. In all the rows except for the first, we observe that the smallest root gap is found by algorithm
Table 3.2: Comparison of Algorithms UCE-N, OC-N, and CD at the root node.

<table>
<thead>
<tr>
<th>m, n</th>
<th>LPGap</th>
<th>RGap</th>
<th>RCuts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>UCE-N</td>
<td>OC-N</td>
<td>CD</td>
</tr>
<tr>
<td>12, 24</td>
<td>12.83%</td>
<td>3.16%</td>
<td>6.15%</td>
</tr>
<tr>
<td>15, 21</td>
<td>14.94%</td>
<td>4.32%</td>
<td>12.28%</td>
</tr>
<tr>
<td>15, 32</td>
<td>9.89%</td>
<td>3.93%</td>
<td>7.43%</td>
</tr>
<tr>
<td>18, 20</td>
<td>14.15%</td>
<td>2.91%</td>
<td>9.81%</td>
</tr>
<tr>
<td>20, 22</td>
<td>14.95%</td>
<td>2.80%</td>
<td>11.97%</td>
</tr>
<tr>
<td>28, 24</td>
<td>11.15%</td>
<td>1.74%</td>
<td>7.17%</td>
</tr>
<tr>
<td>35, 24</td>
<td>9.90%</td>
<td>1.62%</td>
<td>7.34%</td>
</tr>
<tr>
<td>66, 24</td>
<td>10.65%</td>
<td>0.92%</td>
<td>6.62%</td>
</tr>
<tr>
<td>Avg</td>
<td>12.31%</td>
<td>2.68%</td>
<td>8.60%</td>
</tr>
</tbody>
</table>

UCE-N. Algorithm UCE-N adds the largest number of cuts which also supports this conclusion. On average default CPLEX cut settings (CD) performs better in terms of root gap compared to algorithm OC-N. This result is likely due to the small number of inequalities defined by [68] compared to our inequalities. Solution times are not provided in Table 3.2 because on average the longest solution time takes less than 60 seconds. The run time for algorithm UCE-N is longest because of the large number of cuts added.

Next, in Table 3.3, we compare the following algorithms within a branch-and-cut framework on the same set of instances as those in Table 3.2:

**UCH (User Cuts with a Heuristic Separation):** UC formulation with the cuts defined in this chapter and default CPLEX cuts. For the heuristic separation of inequalities (3.21), (3.22), (3.40) and (3.41) we use the algorithm described in UCE-N, except that for multi-period VUB inequalities (3.23) and (3.42) we set the largest possible value of $j$ to $\lceil n/4 \rceil$. User cuts are generated and added for the first fifty branch-and-cut nodes.
Table 3.3: Comparison of Algorithms UCH, OC and CD.

<table>
<thead>
<tr>
<th>m, n</th>
<th>Alg</th>
<th>RGap</th>
<th>EGap</th>
<th>ECuts</th>
<th>Time(uslvd)</th>
<th>B&amp;C Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>12, 24</td>
<td>UCH</td>
<td>1.30%</td>
<td>0%</td>
<td>114, u625</td>
<td>26</td>
<td>1760</td>
</tr>
<tr>
<td></td>
<td>OC</td>
<td>1.75%</td>
<td>0%</td>
<td>212, u45</td>
<td>60</td>
<td>9054</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>1.89%</td>
<td>0%</td>
<td>206</td>
<td>72</td>
<td>14980</td>
</tr>
</tbody>
</table>

| 15, 21 | UCH  | 2.15% | 0%    | 174, u958 | 187         | 23228     |
|        | OC   | 4.80% | 0%    | 353, u82  | 379         | 86693     |
|        | CD   | 5.43% | 0%    | 436       | 578         | 146615    |

| 15, 32 | UCH  | 2.16% | 0%    | 324, u1766| 388         | 13399     |
|        | OC   | 3.54% | 0%    | 614, u147 | 671         | 42282     |
|        | CD   | 4.62% | 0.25% | 722       | 1407(1)     | 77844     |

| 18, 20 | UCH  | 1.98% | 0%    | 252, u983 | 340         | 23952     |
|        | OC   | 4.29% | 0%    | 508, u112 | 1119        | 124036    |
|        | CD   | 5.36% | 0.90% | 599       | 1828(1)     | 216649    |

| 20, 22 | UCH  | 2.23% | 0.16% | 303, u910 | 340         | 191419    |
|        | OC   | 6.05% | 0.59% | 564, u117 | 1119        | 403366    |
|        | CD   | 6.43% | 0.70% | 599       | 1828(1)     | 425681    |

| 28, 24 | UCH  | 1.27% | 0.21% | 456, u1745| 2752(2)     | 100796    |
|        | OC   | 2.69% | 0.30% | 702, u191 | 2773(2)     | 171457    |
|        | CD   | 3.52% | 0.42% | 785       | 3500(2)     | 199660    |

| 35, 24 | UCH  | 1.20% | 0.21% | 539, u1501| 3052(2)     | 115003    |
|        | OC   | 3.06% | 0.67% | 707, u148 | 3600(3)     | 205150    |
|        | CD   | 3.57% | 1.10% | 812       | 3600(3)     | 201817    |

| 66, 24 | UCH  | 0.59% | 0.12% | 751, u2519| 3104(2)     | 80112     |
|        | OC   | 2.73% | 1.49% | 918, u251 | 3600(3)     | 74095     |
|        | CD   | 3.85% | 2.21% | 947       | 3600(3)     | 105506    |

| Avg   | UCH  | 1.61% | 0.09% | 364, u1376| 1506(7)     | 68709     |
|       | OC   | 3.61% | 0.38% | 572, u137 | 1975(11)    | 139517    |
|       | CD   | 4.33% | 0.70% | 519       | 2273(13)    | 173599    |
**OC** ([68] Cuts): Same as algorithm OC-N, but CPLEX default cuts are also enabled.

**CD (CPLEX Default Cut Settings):** UC formulation with default CPLEX cut settings, as before.

In our experiments with UCH, we restricted the multi-period VUB inequalities (3.23) and (3.42) to those with $j \leq \lceil n/4 \rceil$, because this provided a slight improvement over UCE-N. A similar restriction for the ramping inequalities did not provide any advantage over the exact separation. In Table 3.3 column **EGap** reports the end gap percentage at termination output by CPLEX, which is $100 \times (z_{ub} - z_{best})/z_{ub}$, where $z_{best}$ is the best lower bound available within time limit. Column **ECuts** reports the number of cuts added after the problem is solved to optimality within the time limit. Column **Time** reports the solution time in seconds and the total number of unsolved instances. In column **B&C Nodes** we report the number of branch-and-cut tree nodes explored.

The root gap trend obtained in Table 3.2 changes for algorithm OC and CD. Now that CPLEX cuts are turned on for all the algorithms, the worst performance in terms of root gap, end gap and time is given by algorithm CD. For all the instances, the smallest root gap, end gap, and time values are found by UCH. In addition, the average number of branch-and-cut nodes explored is significantly smaller for UCH than for OC and CD. Overall, algorithm UCH solves more instances to optimality compared to OC and CD within the time limit and solves them several times faster. As a result, algorithm UCH outperforms both OC and CD for all instances.

### 3.6 Conclusion

In this chapter we study the ramp-up and ramp-down relaxations of the unit commitment problem. We show that optimization over the ramping polytope is polynomial.
For the two-period problem, we give a complete linear description of the convex hull of feasible solutions. For the multi-period case, we give large classes of facet-defining inequalities and efficient separation algorithms. However, the convex hull of the feasible set for the case of more than two periods is an open problem that merits further research. Our computational results show the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the UC problem with ramping constraints.

3.7 Proof of Convex Hull of Two-Period Ramping Polytope without Start Variables

Corollary 30. For $\bar{u} \leq \ell + \delta$, $\text{conv}(\mathcal{UNS}_2^2) = \{ (p, x) \in \mathbb{R}^{2n} : (3.8) - (3.16) \}$, and for $\bar{u} > \ell + \delta$, $\text{conv}(\mathcal{UNS}_2^2) = \{ (p, x) \in \mathbb{R}^{2n} : (3.8) - (3.14), (3.17), (3.18) \}$.

Proof. We use Fourier-Motzkin elimination (see [79] and [64]) of variable $s_{t+1}$ from the convex hull of the feasible solutions to the formulation with start-up variables given by inequalities (3.6a)-(3.6i). Inequalities (3.6d), (3.6e) and (3.6h) continue to be facets (given by inequalities (3.8)-(3.10)) because these inequalities do not include variable $s_{t+1}$ in their description. Similarly, if $\bar{u} = \ell + \delta$, then inequality (3.6f) is also a facet, because the coefficient of $s_{t+1}$ is equal to 0. In this case, inequality (3.6f) is equivalent to inequalities (3.15) and (3.16). If $u = \bar{u}$, then inequality (3.6g) is also a facet, because the coefficient of $s_{t+1}$ is equal to 0. In this case, inequality (3.6g) reduces to (3.13).

We need to consider all possible cross products of inequalities (3.6b), (3.6c), and (3.6f) (if $\bar{u} > \ell + \delta$) that provide a lower bound on $s_{t+1}$ with inequalities (3.6a), (3.6g) (if $u > \bar{u}$), (3.6i), and (3.6f) (if $\bar{u} < \ell + \delta$) that provide an upper bound on $s_{t+1}$.

We first consider the cross product of lower-bounding inequalities (3.6b) and (3.6c) with upper-bounding inequalities defined by (3.6a), (3.6g) (if $u > \bar{u}$) and (3.6i).
• The pair of inequalities (3.6b) and (3.6a) gives \( x_t \geq 0 \) which is dominated by inequalities (3.6c) and (3.6b).

• The pair of inequalities (3.6b) and (3.6g) gives inequality (3.11) (if \( u > \bar{u} \)).

• The pair of inequalities (3.6b) and (3.6i) gives inequality (3.12).

• The pair of inequalities (3.6c) and (3.6a) gives \( x_{t+1} \geq 0 \) which is dominated by inequalities (3.6d) and (3.13).

• The pair of inequalities (3.6c) and (3.6g) gives inequality (3.13) (if \( u > \bar{u} \)).

• The pair of inequalities (3.6c) and (3.6i) gives inequality (3.14).

Note that depending on whether the coefficient of \( s_{t+1} \) in inequality (3.6f) is positive or negative, we get a lower-bounding or upper-bounding inequality for \( s_{t+1} \), respectively. Therefore, we consider the following two cases:

**Case 1.** If \( \bar{u} < \ell + \delta \), then inequality (3.6f) is an upper-bounding inequality given by

\[
\begin{align*}
    s_{t+1} &\leq \frac{(\ell + \delta)x_{t+1} - \ell x_t - p_{t+1} + p_t}{(\ell + \delta - \bar{u})}.
\end{align*}
\]

(3.43)

• The pair of inequalities (3.6b) and (3.43) gives inequality (3.15).

• The pair of inequalities (3.6c) and (3.43) gives inequality (3.16).

**Case 2.** If \( \bar{u} > \ell + \delta \), then inequality (3.6f) is a lower-bounding inequality given by

\[
\begin{align*}
    s_{t+1} &\geq \frac{-(\ell + \delta)x_{t+1} + \ell x_t + p_{t+1} - p_t}{(\bar{u} - \ell - \delta)}.
\end{align*}
\]

(3.44)

• The pair of inequalities (3.6a) and (3.44) gives

\[
    p_{t+1} - p_t \leq \bar{u}x_{t+1} - \ell x_t,
\]

(3.45)
which is dominated by inequalities (3.18) and (3.9). To see this, note that multiplying inequality (3.9) by \(-(\bar{u} - \ell - \delta)\) and adding to inequality (3.18) gives inequality (3.45) multiplied by \((u - \ell - \delta)\).

- The pair of inequalities (3.6g) (when \(u > \bar{u}\)) and (3.44) gives

\[
\frac{ux_{t+1} - p_{t+1}}{(u - \bar{u})} \geq \frac{-(\ell + \delta)x_{t+1} + \ell x_t + p_{t+1} - p_t}{(\bar{u} - \ell - \delta)}.
\]

Rearranging terms we get inequality (3.18).

- The pair of inequalities (3.6i) and (3.44) gives inequality (3.17).

\[\Box\]
CHAPTER 4
TWO-STAGE ROBUST AND THREE-STAGE
STOCHASTIC MODELS FOR THE UNIT COMMITMENT
PROBLEM WITH UNCERTAIN WIND POWER

4.1 Introduction

One of the challenges of the UC problem is that in practice it is affected by uncertainties that need to be addressed to find solutions to the problem that are optimal or near-optimal. In general uncertainties stem from load, intermittent energy sources, unexpected transmission and generation outages in the network.

In recent years renewable energy sources such as wind, solar, hydro-electric, tidal power, geothermal and biomass have become increasingly popular. There are many benefits to using renewable energy sources. For example wind and solar power are considered clean sources of energy that are much more environmentally friendly compared to conventional energy technologies. Furthermore, once a renewable energy facility is built the future investments are spent on maintaining it and not on purchasing (most likely importing) energy sources. The EU and U.S. recognize the benefits of renewable energy sources. While the U.S. aims to get 20% of the total energy from renewables by 2030 [75] the EU is more ambitious and targets the same percentage for 2020 [1].

However, some energy sources such as wind are highly volatile. In some cases
wind is unavailable for extended periods of time. This results in the need to operate the wind farm whenever the resource is available and store/curtail the unused energy. At times, it can be beneficial to curtail wind power to obtain a better system operation cost [91]. The variability in wind makes it difficult to integrate it with conventional energy sources such as coal plants. In fact, benefits of wind power such as being a clean power supply can be undermined if conventional generators have to be frequently started-up/shut-down to meet demand. The frequent start-up/shut-down of conventional generators is in general more costly and toxic to the environment [57, 86]. Therefore, good wind power forecasts are crucial in integrating wind farms and conventional generators. Other ways to address this problem include different dispatch modes for wind power.

Due to their nature most renewable energy sources add uncertainty to the UC problem. There are different approaches to managing uncertainty in power systems such as reserve requirements and/or stochastic optimization techniques. To understand the impact of uncertainty of power supplies on the UC problem Tuohy et al. [88] compare existing deterministic and stochastic optimization models that solve this problem. The authors conclude that stochastic optimization results in less costly, of the order of 0.25%, and better performing schedules than deterministic optimization.

Early research on stochastic unit commitment (SUC) problems focuses on uncertain demand. Decomposition algorithms have been developed to solve this large-scale problem. A Lagrangian relaxation type technique called progressive hedging that decomposes the SUC problem to single scenario subproblems is used by Takriti et al. [85]. Carpentier et al. [19] propose an augmented Lagrangian relaxation algorithm while Nowak and Römisch [65] develop a method based on stochastic Lagrangian relaxation of coupling constraints to solve a multi-stage SUC problem. Öztürk et al. [69] use Lagrangian relaxation to solve the UC problem with chance constraints that require the load be met with a specified high probability over the entire time
horizon. In contrast to earlier work Öztürk et al. assume that the random variable for demand is continuous and not discrete. Furthermore, their method accounts for the correlation that exists between the demand of two consecutive hours. Ruiz et al. [77] consider a two-stage stochastic program, where in the first stage of the problem the schedule for the slow generators is planned. In the second stage, given the fixed day-ahead schedule of slow generators, the economic dispatch decisions of the entire system is made. Papavasiliou et al. [72] follow the work of Ruiz et al. [77] but they use a limited number of scenarios and continuous variables for the commitment of fast generators (while Ruiz et al. solve the problem exhaustively as a MIP). More specifically, the authors develop a two-stage SUC model to determine reserve requirements in the presence of wind power. Shiina and Birge [82] propose a column generation algorithm that generates new schedules for power generators to solve the SUC problem. Zheng et al. [97] consider two types of power generators, i.e., quick-start and non-quick-start generators and solve the SUC problem with uncertain demand by a method based on Benders’ decomposition that involves discrete decision variables in both first and second stages. According to our knowledge, this is the first paper that introduces discrete variables in the second stage for a SUC problem with uncertain demand.

Recently, researchers have focused their attention on the uncertainty of the power supply due to the interest in renewable energy sources. Ummels et al. [89] develop a simulation method to assess wind power impacts on unit commitment and dispatch of thermal generation units in the Dutch system for increasing wind power penetrations. The authors show that wind power production reduces total system operating costs and emissions exhaust. Therefore, an energy system that is a mixture of both thermal and intermittent renewable energy sources is a promising alternate to merely using one kind of energy source. Wang et al. [92] present a security-constrained unit commitment (SCUC) algorithm that uses the Benders’ decomposition technique
to account for the volatility of wind power generation. The authors propose a two-stage problem. The first stage consists of unit commitment decisions and the second stage subproblems check the network security constraints according to wind scenarios. Thus, if any network violations occur in a network security check subproblem a Benders cut is generated and added to the master problem. Wang et al. [93] develop a sample average approximation algorithm to solve the UC problem with uncertain wind power output. The authors formulate the problem as a chance-constrained two-stage stochastic program which ensures that, with high probability, a large portion of the wind power output at each operating hour is utilized.

Lately, robust optimization has gained popularity as an alternative to solving problems with uncertainty. For details on robust optimization we refer the reader to [10], [11], [15] and [16]. Unlike stochastic programming where a set of scenarios with a probability distribution is required for the uncertain parameter, robust optimization requires much less information which comes in the form of a deterministic uncertainty set. Both approaches have advantages and disadvantages. For example while finding an exact probability distribution can be difficult, the use of a deterministic uncertainty set (that relies on worst case analysis) can make the problem too conservative. Bertsimas et al. [14] propose a two-stage adaptive robust unit commitment model for the SCUC problem in the presence of nodal net injection uncertainty. The authors consider both power supply and demand uncertainty in their problem formulation. The model is solved by a Benders’ decomposition type algorithm. Jiang et al. [43] and Zhao and Zeng [96] study similar robust UC models. However, papers [43] and [96] do not consider reserve constraints. Jiang et al. propose two solution methods based on a Benders’ decomposition framework to solve their two-stage robust model. One method provides an exact solution and the other a near-optimal solution. The authors combine demand and wind uncertainty by considering wind power as negative demand and make use of polyhedral and cardinal uncertainty sets (defined by
Bertsimas and Sim [15]). Zhao and Zeng first utilize their two-stage robust model that considers a wind uncertainty set to find day-ahead generator schedules. Then they extend their model to include a demand response strategy such that both price levels and generator schedules are determined.

The high penetration of wind power has also led researchers to find reliable and cost effective reserve values. Morales et al. [59] propose a two-stage stochastic program that takes into account uncertain wind power to determine required spinning and non spinning reserve levels and their costs. Ortega-Vazquez and Kirschen [67] present a stochastic optimization method to determine spinning reserve values that minimize the total operating cost and the cost of load shedding. The authors consider uncertain wind power and load forecasts. For more details on problems in the electricity markets that consider uncertainty (particularly stemming from non-dispatchable sources such as wind power) and their solution methods we refer the reader to [24].

Next, we describe the models studied in this chapter.

4.1.1 Our Work

We focus on the problem of integration of a renewable energy source namely, wind power with conventional energy technologies. Thus in addition to the conventional generators that were studied in Chapter 3, wind farms are introduced to the UC problem. We compare two different models: a two-stage robust model and a three-stage stochastic program that use the same parameter information.

Predicting wind behavior is a challenging task. It is especially hard to obtain good wind behavior forecasts for the not so near future. For example, a study from Germany suggests that wind power prediction margin error is 9% for 24 hours ahead, 8% for eight hours ahead and 2% for one hour ahead [49]. Therefore, instead of using wind power forecasts that are predicted at the start of the time horizon for the entire time horizon we split the time horizon into two in terms of wind power
forecast. We use two different wind power forecasts: a wind power forecast at the beginning of the time horizon for the first $n_1$ time periods of the horizon and another at time $n_1 + 1$ for the remaining part of the horizon. By this way we can obtain more accurate wind power forecasts compared to wind power forecasts that are obtained at the start of the time horizon for the entire length of the horizon. This split of the time horizon according to wind power forecast implies that we need to have different sets of scenarios. Let $\Omega_1$ be the set of wind scenarios predicted at the beginning of the time horizon. Let $\Omega_2(\epsilon_1)$ (for a given scenario $\epsilon_1 \in \Omega_1$) be the set of wind scenarios available at time period $n_1 + 1$ for the remaining part of the time horizon.

Due to the nature of wind, wind production levels cannot be scheduled. Thus, for both of the models, in the first stage, the unit commitment decisions for a given time horizon for conventional generators are made. These decisions are generally made a day before the actual electricity dispatch begins. Most conventional generators require several hours to start-up and/or shut-down so these decisions cannot be made in real-time.

In a traditional two-stage stochastic program the second stage determines the real-time economic dispatch decisions for the online conventional generators and wind farms for time periods 1 through $n$ assuming that the uncertainty is fully resolved for those time periods. However, because wind is highly volatile wind power predictions made at time period 1 for time periods 1 through $n$ are not realistic. Therefore we propose a two-stage robust model in which the decisions for time periods $[1, n_1]$ mimic those in the second stage of a traditional two-stage stochastic program and the decisions for time periods $[n_1 + 1, n]$ are made using a robust approach. This is because in the second stage of the problem we have reliable knowledge of wind power scenarios for time periods $[1, n_1]$ but not $[n_1 + 1, n]$. We consider the worst case wind scenarios from set $\Omega_2(\epsilon_1)$ (for a given scenario $\epsilon_1 \in \Omega_1$) so that any solution of the
problem will be feasible to all of the wind power scenarios introduced at time period $n_1 + 1$.

In the three-stage stochastic program, both the second and the third stages of the problem determine the real-time economic dispatch decisions for the online conventional generators and wind farms. The difference between the second and the third stages of the problem is the wind power forecast and the time periods that are considered. The second stage determines the dispatch decisions for time periods $[1, n_1]$ after a reliable forecast of wind in periods $[1, n_1]$ is revealed. Unlike the two-stage robust model which considers the worst case outcomes in periods $[n_1 + 1, n]$, the three-stage stochastic program is a wait-and-see model, which allows the decisions in periods $[n_1 + 1, n]$ to be made in period $n_1 + 1$, after a more accurate forecast of these periods become available. This model can be used in electricity markets that consider intra-day decisions. For example, in Spain, there are six intra-day market sessions [33, 35].

In Figure 4.1 we show the difference between the two-stage robust model and the three-stage stochastic program using a small example. For both of the models let node 0 represent the first stage decisions (i.e., unit commitment decisions) and nodes 1 and 2 represent the dispatch decisions for time periods $[1, n_1]$ for two wind power scenarios. At time period $n_1 + 1$ we observe two more wind power scenarios for both nodes 1 and 2. While the three-stage stochastic program can make a decision in time periods $[n_1 + 1, n]$ according to the different wind power scenarios the two-stage robust model has to consider the worst possible scenarios of nodes 1,1-1,2 and 2,1-2,2. To demonstrate the worst case analysis we combine nodes 1,1 and 1,2 shown in the three-stage stochastic program separately into a single node 1,1-1,2 in the two-stage robust model. The same procedure is done for nodes 2,1 and 2,2. The worst case analysis for the two-stage robust model for time periods $[n_1 + 1, n]$ implies that among
all possible choices of wind power realizations we have to choose the smallest one in order for the solution to be feasible for all wind power realizations.

To solve these two models we use the MIP formulation of the problems referred to as the deterministic equivalent models and the Benders’ decomposition algorithm.

The rest of the chapter is organized as follows. We describe the two models in Sections 4.2 and 4.3 respectively. In Section 4.4 we compare the results of the two models by going over a small example. The Benders’ decomposition algorithms
for both models are explained in Section 4.5. Finally, in Section 4.6 we present our computational results with both the two-stage robust model and the three-stage stochastic program solved by the deterministic equivalent models and the Benders’ decomposition algorithms. The work in this chapter can also be found in [27].

4.2 Deterministic Equivalent of the Three-Stage Stochastic Program

In this section we present the deterministic equivalent problem for the three-stage stochastic program for the UC problem that includes conventional generators and wind farms. In the first-stage of the problem the unit commitment decisions for a given time horizon for conventional generators are made. The second and third stages of the problem determine the real-time economic dispatch decisions for the online conventional generators and wind farms for time periods \([1, n_1]\) and \([n_1 + 1, n]\), respectively.

The first stage decision variables are the unit commitment decisions for the conventional generators for the entire time horizon \(n\). The objective for the first stage problem is to minimize the operational cost of conventional generators and wind farms over \(n\) periods. In the second stage of the problem we assume that wind power forecast is revealed for time periods 1 to \(n_1\) and we determine the dispatch values of both the conventional generators and wind farms. The objective of the second stage problem is to minimize the total cost of power generation via conventional generators and wind farms for time periods 1 to \(n_1\) plus the expected cost of operation for time periods \(n_1 + 1\) to \(n\). In the third stage of the problem we assume that wind power is known for time periods \(n_1 + 1\) to \(n\). The third stage of the problem is almost identical to the second stage but it is solved for different time periods and depends on the value of production variables of conventional generators of the second stage.
problem. The objective of the third stage problem is to minimize the total cost of power generation via conventional generators and wind farms for time periods \(n_1 + 1\) to \(n\).

Whenever possible we utilize the same parameter and variable notation presented in Section 3.1.1. However, we also introduce new notation. From now on a conventional generator belongs to a bus, but we do use the same generator specific parameter names with an additional index for the bus notation. The bus set is divided into two: buses that contain conventional generators and wind farms.

Let \(B\) be the set of all buses. Let \(BG\) be the set of buses with conventional generators, \(BW\) be the set of buses with wind farms and \(\kappa_b\) be the set of generators at bus \(b \in BG\). Let \(n\) be the length of the planning horizon. Let \(\Omega_1\) be the number of scenario tree nodes in stage two. For any stage two node \(\epsilon_1 \in \Omega_1\), let \(\Omega_2(\epsilon_1)\) be the number of scenario tree nodes in stage three. Note that \(\sum_{\epsilon_1 \in \Omega_1} |\Omega_2(\epsilon_1)|\) gives the total number of scenarios. For the wind farms we do not make a decision about unit commitment because wind farms are operated whenever the resource is available. In period \(t \in [1, n_1]\) at stage two node \(\epsilon_1 \in \Omega_1\), lost revenue due to wind curtailment (i.e. wind power that is available but not dispatched or stored) is \(o_{\epsilon_1tb}^1\) and the cost for producing energy from wind that satisfies a fraction of the electricity demand is \(a_{\epsilon_1tb}^1\), \(b \in BW\). In period \(t \in [n_1 + 1, n]\) at stage three node \(\epsilon_2 \in \Omega_1(\epsilon_1)\), given stage two node \(\epsilon_1 \in \Omega_1\) lost revenue due to wind curtailment is \(o_{\epsilon_1\epsilon_2tb}^2\) and the cost for producing energy from wind that satisfies a fraction of the electricity demand is \(a_{\epsilon_1\epsilon_2tb}^2\), \(b \in BW\). For the SUC problem for any period \(t \in [1, n]\) and bus \(b \in B\) the load is given by \(d_{tb}\).

We define random variable \(w_{\epsilon_1tb}^1\) as the production level in period \(t \in [1, n_1]\) for wind farm at bus \(b \in BW\) at its realization \(\epsilon_1 \in \Omega_1\). Similarly, we define random variable \(w_{\epsilon_1\epsilon_2tb}^2\) as the production level in period \(t \in [n_1 + 1, n]\) for wind farm at bus \(b \in BW\) at its realization \(\epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1)\). \(Prob_{\epsilon_1}\) is the probability of stage
two node $\epsilon_1 \in \Omega_1$ and $Prob_{\epsilon_1 \epsilon_2}$ is the conditional probability of stage three node $\epsilon_2 \in \Omega_2(\epsilon_1)$ given the probability of stage two node $\epsilon_1 \in \Omega_1$.

The first stage decision variables are the unit commitment decisions for the conventional generators for the entire time horizon $n$. The same decision variable names (as in Section 3.1.1) with an additional index for the bus notation is used. In the second stage of the problem we assume that wind power forecast is known for time periods $1$ to $n_1$ and we find the dispatch values of both the conventional generators and wind farms. Let $p^1_{\epsilon_1 tbg}$ be the production level at stage two node $\epsilon_1 \in \Omega_1$, in period $t \in [1, n_1]$ at bus $b \in BG$ for generator $g \in \kappa_b$. For wind farm at bus $b \in BW$, let $v^1_{\epsilon_1 tb}$ be the amount of wind power utilized at stage two node $\epsilon_1$ for satisfying demand in period $t$ and $q^1_{\epsilon_1 tb}$ be the amount of wind that is curtailed in period $t$, $t \in [1, n_1]$ at stage two node $\epsilon_1 \in \Omega_1$. In the third stage of the problem we assume that wind power is known for time periods $n_1 + 1$ to $n$. Let $p^2_{\epsilon_1 \epsilon_2 tbg}$ be the production level at stage three node $\epsilon_2 \in \Omega_2(\epsilon_1)$ for a given stage two node $\epsilon_1 \in \Omega_1$ in period $t \in [n_1 + 1, n]$ at bus $b \in BG$ for generator $g \in \kappa_b$. For wind farm at bus $b \in BW$, let $v^2_{\epsilon_1 \epsilon_2 tb}$ be the amount of wind power utilized for satisfying demand at stage three node $\epsilon_2$ for a given stage two node $\epsilon_1$ in period $t$ and $q^2_{\epsilon_1 \epsilon_2 tb}$ be the amount of wind that is curtailed at stage three node $\epsilon_2 \in \Omega_2(\epsilon_1)$ for a given stage two node $\epsilon_1 \in \Omega_1$ in period $t \in [n_1 + 1, n]$.

Objective (4.1) minimizes the total cost of operating conventional generators including the setup and start-up costs and the expected cost of power generation via
conventional generators and wind farms.

\[
\min \sum_{b \in BG} \sum_{g \in \kappa_b} \left( \sum_{t=1}^{n} h_{tbg} x_{tbg} + \sum_{t=2}^{n} f_{tbg} s_{tbg} \right) \quad (4.1)
\]

\[
+ \sum_{\epsilon_1 \in \Omega_1} \sum_{t=1}^{n_1} \left( \sum_{b \in BG} \sum_{g \in \kappa_b} c_{tbg} p_{\epsilon_1 tbg}^1 + \sum_{b \in BW} \left( a_{\epsilon_1 tb}^1 q_{\epsilon_1 tb}^1 + a_{\epsilon_1 tb}^1 v_{\epsilon_1 tb}^1 \right) \right) \quad (4.2)
\]

\[
+ \sum_{\epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2} \sum_{t=n_1+1}^{n} \left( \sum_{b \in BG} \sum_{g \in \kappa_b} c_{tbg} p_{\epsilon_1 \epsilon_2 tbg}^2 + \sum_{b \in BW} \left( a_{\epsilon_1 \epsilon_2 tb}^2 q_{\epsilon_1 \epsilon_2 tb}^2 + a_{\epsilon_1 \epsilon_2 tb}^2 v_{\epsilon_1 \epsilon_2 tb}^2 \right) \right). \quad (4.3)
\]

The unit commitment decisions are generally made a day before the real-time dispatch decisions. Next, we describe the constraints of the first stage problem. The following constraints

\[
\sum_{b \in BG} \sum_{g \in \kappa_b} u_{tbg} x_{tbg} \geq r_t \sum_{b \in B} d_{tb}, \quad t \in [1, n] \quad (4.2)
\]

are the so-called spinning reserve constraints which require that the total maximum capacity of all online conventional generators is enough to satisfy a constant factor of the total load at all the buses in every period \( t \in [1, n] \). Constraints (4.3) describe the relationship between the start-up variables \( s \) and generator on/off status variables \( x \):

\[
x_{(t+1)bg} - x_{tbg} \leq s_{(t+1)bg}, \quad t \in [1, n - 1], b \in BG, g \in \kappa_b. \quad (4.3)
\]

The minimum down- and up-time restrictions are modeled by constraints (4.4) and (4.5), respectively:

\[
\sum_{i=t-\bar{v}_{bg}+1}^{t} s_{ibg} \leq 1 - x_{(t-\bar{v}_{bg})bg}, \quad t \in [\bar{v}_{bg} + 1, n], b \in BG, g \in \kappa_b, \quad (4.4)
\]

\[
\sum_{i=t-\overline{v}_{bg}+1}^{t} s_{ibg} \leq x_{tbg}, \quad t \in [\overline{v}_{bg} + 1, n], b \in BG, g \in \kappa_b. \quad (4.5)
\]

They ensure that if a generator \( g \) at bus \( b \) is turned on (off), then it stays on (off) for at least \( \overline{v}_{g,b} \) (\( \bar{v}_{g,b} \)) time periods. The following equations

\[
s_{tbg} - z_{tbg} = x_{tbg} - x_{(t-1)bg}, \quad t \in [2, n], b \in BG, g \in \kappa_b \quad (4.6)
\]
express the shut-down variables \( z_{tbg} \) in terms of variables \( s_{tbg} \) and \( x_{tbg} \). Binary restrictions and non-negativity of shut-down variables are as follows

\[
s_{tbg} \in \{0, 1\}, \quad z_{tbg} \geq 0, \quad t \in [2, n], \quad x_{tbg} \in \{0, 1\}, \quad t \in [1, n], \quad b \in BG, \quad g \in \kappa_b. \quad (4.7)
\]

Next, we present the constraints for the second stage problem for all stage two nodes \( \epsilon_1 \in \Omega_1 \). Constraints given by

\[
\sum_{b \in BG} \sum_{g \in \kappa_b} p_{\epsilon_1, tbg}^1 + \sum_{b \in BW} v_{\epsilon_1, tb}^1 = \sum_{b \in B} d_{tb}, \quad t \in [1, n_1], \quad \epsilon_1 \in \Omega_1,
\]

ensure that the total load at all the buses is met in every period \( t \in [1, n_1] \). Constraints (4.9), (4.10) and (4.11) are the minimum and maximum production constraints:

\[
\ell_{bg} x_{tbg} \leq p_{\epsilon_1, tbg}^1, \quad t \in [1, n_1], \quad b \in BG, \quad g \in \kappa_b, \quad \epsilon_1 \in \Omega_1, \quad (4.9)
\]

\[
p_{\epsilon_1, tbg}^1 \leq u_{bg} x_{tbg}, \quad b \in BG, \quad g \in \kappa_b, \quad \epsilon_1 \in \Omega_1, \quad (4.10)
\]

\[
p_{\epsilon_1, tbg}^1 \leq u_{bg} x_{tbg} - (u_{bg} - \bar{u}_{bg}) s_{tbg}, \quad t \in [2, n_1], \quad b \in BG, \quad g \in \kappa_b, \quad \epsilon_1 \in \Omega_1. \quad (4.11)
\]

Ramp-up and ramp-down requirements are described by constraints (4.12) and (4.13), respectively:

\[
p_{\epsilon_1, (t+1), bg}^1 - p_{\epsilon_1, tbg}^1 \leq (\bar{u}_{bg} - \ell_{bg} - \delta_{bg}) s_{(t+1),bg} + (\ell_{bg} + \delta_{bg}) x_{(t+1),bg} - \ell_{bg} x_{tbg},
\]

\[
t \in [1, n_1 - 1], \quad b \in BG, \quad g \in \kappa_b, \quad \epsilon_1 \in \Omega_1, \quad (4.12)
\]

\[
p_{\epsilon_1, tbg}^1 - p_{\epsilon_1, (t+1), bg}^1 \leq (\bar{u}_{bg} - \ell_{bg} - \delta_{bg}) z_{(t+1),bg} + (\ell_{bg} + \delta_{bg}) x_{tbg} - \ell_{bg} x_{(t+1),bg},
\]

\[
t \in [1, n_1 - 1], \quad b \in BG, \quad g \in \kappa_b, \quad \epsilon_1 \in \Omega_1. \quad (4.13)
\]

These constraints ensure that the production level in the first (last) period a generator \( g \) at bus \( b \) is started up (shut down) is at most \( \bar{u}_{g,b} \), and that the absolute value of the difference in production levels from one period to the next is at most \( \delta_{bg} \), for \( b \in BG, \quad g \in \kappa_b \). Constraints (4.14) are the ramp-down version of constraints (4.11):

\[
p_{\epsilon_1, tbg}^1 \leq u_{bg} x_{tbg} - (u_{bg} - \bar{u}_{bg}) z_{(t+1),bg}, \quad t \in [1, n_1], \quad b \in BG, \quad g \in \kappa_b, \quad \epsilon_1 \in \Omega_1. \quad (4.14)
\]
The following constraints
\[ v_{e_1tb}^1 + q_{e_1tb}^1 = w_{e_1tb}^1, \quad t \in [1, n_1], \, b \in BW, \, \epsilon_1 \in \Omega_1, \] (4.15)
are the wind energy balance constraints for each time period \( t \in [1, n_1] \) and wind farm \( b \in BW \). For any realization of wind power output \( w_{e_1tb}^1, \epsilon_1 \in \Omega_1 \) wind energy is used to satisfy a fraction of demand and the excess energy is dumped. Non-negativity of stage two variables are as follows
\[ p_{e_1tb}^1 \geq 0, \, b \in BG, \, g \in \kappa_b, \, v_{e_1tb}^1, \, q_{e_1tb}^1 \geq 0, \, b \in BW, \, t \in [1, n_1], \, \epsilon_1 \in \Omega_1. \] (4.16)

Finally we present the constraints for the third stage problem for all stage three nodes \( \epsilon_2 \in \Omega_2(\epsilon_1) \) (given a stage two node \( \epsilon_1 \in \Omega_1 \)). Constraints
\[ \sum_{b \in BG} \sum_{g \in \kappa_b} p_{e_1\epsilon_2tbg}^2 + \sum_{b \in BW} v_{e_1\epsilon_2tb}^2 = \sum_{b \in B} d_{tb}, \quad t \in [n_1 + 1, n], \, \epsilon_1 \in \Omega_1, \, \epsilon_2 \in \Omega_2(\epsilon_1), \] (4.17)
ensure that the total load at all the buses is met in every period \( t \in [n_1 + 1, n] \).
Constraints (4.18) and (4.19) are the minimum and maximum production constraints:
\[ \ell_{bg} x_{tbg} \leq p_{e_1\epsilon_2tbg}^2, \quad t \in [n_1 + 1, n], \, b \in BG, \, g \in \kappa_b, \, \epsilon_1 \in \Omega_1, \, \epsilon_2 \in \Omega_2(\epsilon_1), \] (4.18)
\[ p_{e_1\epsilon_2tbg}^2 \leq u_{bg} x_{tbg} - (u_{bg} - \bar{u}_{bg}) s_{tbg}, \]
\[ t \in [n_1 + 1, n], \, b \in BG, \, g \in \kappa_b, \, \epsilon_1 \in \Omega_1, \, \epsilon_2 \in \Omega_2(\epsilon_1). \] (4.19)

Ramp-up and ramp-down requirements are divided into two sets of constraints in order to show the dependence of the third stage problem on the second stage problem as follows:
\[ p_{e_1\epsilon_2(n_1+1)bg}^2 - p_{e_1n_1bg}^1 \leq (\bar{u}_{bg} - \ell_{bg} - \delta_{bg}) s_{(n_1+1)bg} + (\ell_{bg} + \delta_{bg}) x_{(n_1+1)bg} - \ell_{bg} x_{n_1bg}, \]
\[ b \in BG, \, g \in \kappa_b, \, \epsilon_1 \in \Omega_1, \, \epsilon_2 \in \Omega_2(\epsilon_1), \] (4.20)
Constraints (4.20) and (4.22) show that the third stage conventional generator production variables, \( p_{t_1}^2 - p_{t_2}^2 \), \( t \in [n_1 + 1, n] \) depend on the value of second stage variables \( p_{t_1 b g}^1, b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1) \). Constraints (4.21) and (4.23) are the ramp-up and ramp-down constraints for time periods \( n_1 + 1 \) to \( n \). Constraints (4.24) are the ramp-down version of constraints (4.19):

\[
p_{t_1}^2 - p_{t_2}^2 \leq (\bar{u}_{bg} - \ell_{bg} - \delta_{bg})s_{(t+1)b g} + (\ell_{bg} + \delta_{bg})x_{(t+1)b g} - \ell_{bg}x_{bg},
\]
\[
t \in [n_1 + 1, n - 1], b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1).
\]

Constraints

\[
v_{t_1}^2 + q_{t_1}^2 = w_{t_1}^2, t \in [n_1 + 1, n], b \in BW, \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1),
\]
\[
t \in [n_1 + 1, n], \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1).
\]

The first stage problem constraints are given by (4.2)-(4.7), the second stage
problem constraints are given by (4.8) - (4.16) and the third stage problem constraints are given by (4.17) - (4.26). Hence, the deterministic equivalent for the three-stage stochastic program is given by model (4.1) - (4.26).

4.3 Deterministic Equivalent of the Two-Stage Robust Model

In this section we present the two-stage robust model for the UC problem that includes conventional generators and wind farms and uses the same information as the three-stage stochastic program described in Section 4.2. The first stage decision variables described for the three-stage stochastic program \((x, s, z)\) are used as the first stage decision variables for the two-stage robust model as well. The second stage decision variables are the conventional generator production variables, \(p^1_{t,b,g}, b \in BG, g \in \kappa_b\), the wind power that satisfies a fraction of the demand, \(q^1_{t,b}\) and the wind curtailment variables, \(v^1_{t,b}, b \in BW, \epsilon_1 \in \Omega_1, t \in [1, n]\). Note that compared to the three-stage stochastic program there will be a difference in the constraints that include wind variables for time periods \(t \in [1, n_1]\) and \(t \in [n_1 + 1, n]\). For time periods \(t \in [n_1 + 1, n]\), the model takes into account the worst case wind scenarios. As mentioned in Section 4.1.1 in the two-stage robust model we observe the wind power scenarios for time periods \([1, n_1]\) but make dispatch decisions for time periods \([1, n]\). Therefore, the dispatch decisions for time periods \([n_1 + 1, n]\) are treated as robust decisions i.e., the worst case wind power scenarios are considered for these time periods.

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The objective function for the two-stage robust model is given by (4.27).

\[
\min \sum_{b \in BG} \sum_{g \in \kappa_b} \left( \sum_{t=1}^{n} h_{tbg} x_{tbg} + \sum_{t=2}^{n} f_{tbg} s_{tbg} \right) + \sum_{\epsilon_1 \in \Omega_1} \text{Prob}_{\epsilon_1} \sum_{t=1}^{n_1} \sum_{b \in BG} \sum_{g \in \kappa_b} c_{tbg} p_{1}^{\epsilon_1 tbg}
\]

\[
+ \sum_{\epsilon_1 \in \Omega_1} \text{Prob}_{\epsilon_1} \sum_{t=1}^{n_1} \sum_{b \in BW} \left( o_{\epsilon_1 tbg}^1 q_{\epsilon_1 tbg}^1 + a_{\epsilon_1 tbg}^1 v_{\epsilon_1 tbg}^1 \right)
\]

\[
+ \sum_{\epsilon_1 \in \Omega_1} \text{Prob}_{\epsilon_1} \sum_{t=n_1+1}^{n} \sum_{b \in BW} \left( \left( \sum_{\epsilon_2 \in \Omega_2(\epsilon_1)} o_{\epsilon_1 \epsilon_2 tbg}^2 q_{\epsilon_1 \epsilon_2 tbg}^1 \right) + \sum_{\epsilon_2 \in \Omega_2(\epsilon_1)} a_{\epsilon_1 \epsilon_2 tbg}^2 v_{\epsilon_1 tbg}^1 \right).
\]

The first stage problem for both the two-stage robust model and the three-stage stochastic program is exactly the same. Therefore, next we describe the constraints of the second stage problem for all stage two nodes \( \epsilon_1 \in \Omega_1 \). The following constraints ensure that the total load at all the buses is met in every period \( t \in [1, n] \). Constraints (4.29), (4.30) and (4.31) are the minimum and maximum production constraints:

\[
\ell_{bg} x_{tbg} \leq p_{1}^{\epsilon_1 tbg}, \quad t \in [1, n], b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1,
\]

(4.29)

\[
p_{1}^{\epsilon_1 tbg} \leq u_{bg} x_{1bg}, \quad b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1,
\]

(4.30)

\[
p_{1}^{\epsilon_1 tbg} \leq u_{bg} x_{tbg} - (u_{bg} - \bar{u}_{bg}) s_{tbg}, \quad t \in [2, n], b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1.
\]

(4.31)

Ramp-up and ramp-down requirements are described by constraints

\[
p_{1}^{\epsilon_1 (t+1)bg} - p_{1}^{\epsilon_1 tbg} \leq (\bar{u}_{bg} - \ell_{bg} - \delta_{bg}) s_{(t+1)bg} + (\ell_{bg} + \delta_{bg}) x_{(t+1)bg} - \ell_{bg} x_{tbg},
\]

\[
t \in [1, n - 1], b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1,
\]

(4.32)
\[ p_{1,tbg}^1 - p_{1,(t+1)bg}^1 \leq (\bar{u}_{bg} - \ell_{bg} - \delta_{bg}) z_{(t+1)bg} + (\ell_{bg} + \delta_{bg}) x_{tbg} - \ell_{bg} x_{(t+1)bg}, \]
\[ t \in [1, n - 1], b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1, \quad (4.33) \]

respectively. Constraints (4.34) are the ramp-down version of constraints (4.31):

\[ p_{1,tbg}^1 \leq u_{bg} x_{tbg} - (u_{bg} - \bar{u}_{bg}) z_{(t+1)bg}, \quad t \in [1, n - 1], b \in BG, g \in \kappa_b, \epsilon_1 \in \Omega_1. \quad (4.34) \]

Constraints

\[ v_{1,tb}^1 + q_{1,tb}^1 = w_{1,tb}^1, \quad t \in [1, n_1], b \in BW, \epsilon_1 \in \Omega_1, \quad (4.35) \]

are the wind energy balance constraints for each time period \( t \in [1, n_1] \) and wind farm \( b \in BW \). For any realization of wind power output \( w_{1,tb}^1, \epsilon_1 \in \Omega_1 \), wind energy is used to satisfy a fraction of demand and the excess energy is dumped. The wind balance equation for time periods \( n_1 + 1 \) to \( n \) is not an equality (like (4.35)). We replace the equality we use for time periods \( n_1 + 1 \) to \( n \) with two sets of constraints given by

\[ v_{1,tb}^1 \leq \min_{\epsilon_2 \in \Omega_2(\epsilon_1)} \{ w_{1,tb}^2 \}, t \in [n_1 + 1, n], b \in BW, \epsilon_1 \in \Omega_1, \quad (4.36a) \]
\[ q_{1,tb}^1 \geq \max_{\epsilon_2 \in \Omega_2(\epsilon_1)} \{ w_{1,tb}^2 \} - v_{1,tb}^1, t \in [n_1 + 1, n], b \in BW, \epsilon_1 \in \Omega_1. \quad (4.36b) \]

Constraints (4.36a) consider the worst case (i.e., minimum) for wind power realization as an upper bound on the amount of wind power that can satisfy demand. Constraints (4.36b) consider the worst case (i.e., maximum) for wind power curtailment. Non-negativity of stage two variables are as follows

\[ p_{1,tbg}^1 \geq 0, b \in BG, g \in \kappa_b v_{1,tb}^1, q_{1,tb}^1 \geq 0, b \in BW, t \in [1, n], \epsilon_1 \in \Omega_1. \quad (4.37) \]

The deterministic equivalent for the two-stage robust model is given by (4.27), (4.2) - (4.7), (4.28) - (4.37).
4.4 Comparison of Two Models

We test an instance of the SUC problem with uncertain wind power with the models described in Sections 4.2 and 4.3. Let $BG = \{2\}$, $\kappa_1 = \{1, 2\}$, $\kappa_2 = \{1, 2\}$, $BW = \{2\}$, $n_1 = 3$, $n = 6$, $\Omega_1 = \{1, 2\}$, $\Omega_2(1) = \{1, 2\}$, $\Omega_2(2) = \{1, 2\}$. We have four buses in total, two buses have two conventional generators and the other two buses have wind farms. There are two stage 2 nodes and for each stage 2 node there are two stage 3 nodes (the same wind power scenario structure given in Figure 4.1). In total there are four scenarios with probabilities $0.26190, 0.26190, 0.23810, 0.23810$, respectively. For the four conventional generators at two different buses generator specific parameters are given in Table 4.1. We use generator specific parameter values that are similar to the ones given by Wang et al. [92] in Table 1(a) to obtain a feasible solution. The vector over time periods (6 hours) for $\sum_{b \in B} d_{tb}$ is: $(744, 658, 700, 841, 876, 842)$. The wind power realizations for each stage and time are presented in Tables 4.2 and 4.3. These are the values of uncertain parameters $w^1_{\epsilon_1tb}$, $\epsilon_1 \in \Omega_1$, $t \in [1, n_1]$, $b \in BW$ and $w^2_{\epsilon_1\epsilon_2tb}$, $\epsilon_1 \in \Omega_1$, $\epsilon_2 \in \Omega_2(\epsilon_1)$, $t \in [n_1 + 1, n]$, $b \in BW$. The objective function includes the following costs: start-up, fixed cost of running a generator and variable production cost of conventional generators. Wind variables do not have a cost.

<table>
<thead>
<tr>
<th>(bus,generator)</th>
<th>$u_{bg}$</th>
<th>$\delta_{bg}$</th>
<th>$\ell_{bg}$</th>
<th>$u_{bg}$</th>
<th>$v_{bg}$</th>
<th>$\bar{v}_{bg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>101</td>
<td>59</td>
<td>13</td>
<td>221</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(1,2)</td>
<td>114</td>
<td>56</td>
<td>11</td>
<td>182</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(2,1)</td>
<td>149</td>
<td>56</td>
<td>16</td>
<td>248</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(2,2)</td>
<td>125</td>
<td>50</td>
<td>19</td>
<td>227</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.1: Generator Specific Parameters for Small Test Instance
Table 4.2: Wind Power Realizations for Stage 2 \( (w_{\epsilon_{1,t_{b}}}, \epsilon_1 \in \Omega_1, t \in [1,n_1], b \in BW) \)

<table>
<thead>
<tr>
<th>Node</th>
<th>Wind Farm 1</th>
<th>Wind Farm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>45</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>95</td>
<td>9</td>
</tr>
<tr>
<td>Node 2</td>
<td>117</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>110</td>
</tr>
<tr>
<td></td>
<td>95</td>
<td>28</td>
</tr>
</tbody>
</table>

Table 4.3: Wind Power Realizations for Stage 3 \( (w_{\epsilon^2,\epsilon_2,t_{b}}, \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1), t \in [n_1+1,n], b \in BW) \)

<table>
<thead>
<tr>
<th>Node</th>
<th>Wind Farm 1</th>
<th>Wind Farm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>360</td>
<td>380</td>
</tr>
<tr>
<td></td>
<td>530</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>176</td>
<td>160</td>
</tr>
<tr>
<td>(1,2)</td>
<td>430</td>
<td>234</td>
</tr>
<tr>
<td></td>
<td>399</td>
<td>320</td>
</tr>
<tr>
<td></td>
<td>445</td>
<td>334</td>
</tr>
<tr>
<td>(2,1)</td>
<td>320</td>
<td>260</td>
</tr>
<tr>
<td></td>
<td>240</td>
<td>345</td>
</tr>
<tr>
<td></td>
<td>510</td>
<td>540</td>
</tr>
<tr>
<td>(2,2)</td>
<td>359</td>
<td>369</td>
</tr>
<tr>
<td></td>
<td>330</td>
<td>429</td>
</tr>
<tr>
<td></td>
<td>443</td>
<td>445</td>
</tr>
</tbody>
</table>

The solutions to the described SUC problem instance using the three-stage stochastic program and the two-stage robust model are given in Figures 4.2 and 4.3 respectively. In the figures, at each scenario tree node we present the solutions with a table near the scenario tree node. The unit commitment decisions are given in the table for scenario tree node 0 for both the three-stage stochastic program and the two-stage robust model. For this table x’s represent the generators that are on. The rows represent the time periods and the columns represent the bus and the generator. It can be observed from Figures 4.2 and 4.3 that both models give the same unit commitment decisions for time periods 1, 2 and 3. However, the two-stage robust model suggests that generator 2 at bus 1 be shut-down after time period 3 and the three-stage model suggests that generator 2 at bus 2 be shut-down for the same time periods.
The scenario tree nodes at stage 2 (nodes 1 and 2) in Figures 4.2 and 4.3 show the conventional generator and wind farm dispatch values for time periods 1, 2 and 3. The solutions show that all the available wind power is dispatched to satisfy the load. There is no wind curtailment at this stage for either model. Note that the production values for the generators are almost all different for the compared models.

In Figure 4.2, the scenario tree nodes at stage 3 (nodes (1,1), (1,2), (2,1) and (2,2)) show the conventional generator and wind farm dispatch values for time periods 4, 5 and 6. The results at this stage show that there is ample amount of wind power. In fact, power coming from wind farm 2 is curtailed in almost all time periods. Due to the structure of the two-stage robust model for time periods 4, 5 and 6, in Figure 4.3, we observe wind curtailment for both wind farms.

Next, we analyze the solution of the two-stage robust model for time periods 4, 5 and 6 in more detail. The wind power realizations at nodes (1,1) and (1,2) for time period 4 can be observed in Table 4.3. For example for wind farm 1, wind power realization at node (1,1) is 360 while it is 430 at node (1,2). According to the worst case analysis no more than \(\min\{360, 430\} = 360\) units of wind power can be used to satisfy demand. Furthermore, the wind curtailment amount is calculated as \(\max\{360, 430\} - 360 = 70\) units. We make use of constraints (4.36a) and (4.36b) to find these values, respectively. A similar analysis can be done for time periods 5 and 6 and for the node pairs (2,1) and (2,2).

The objective function value of the three-stage stochastic program is 8850 while the objective function value of the two-stage robust model is 9405 (rounded to the nearest integer). The percentage difference of the objective function values is 6%. The decisions given by the two-stage robust model lead to more expensive outcomes. This is an expected result due to the conservative approach (discussed in the previous paragraph) of the two-stage robust model for time periods \([n_1 + 1, n]\). Note that the
unit commitment decisions are not exactly the same for the compared models after
time period 3.

4.5 Benders’ Method for the Two and Three-Stage Models

In this section we describe the Benders’ decomposition algorithms for both the two-
stage robust model and the three-stage stochastic problem. Recall that the first stage
problem constraints for both the two- and the three-stage models is given by (4.2)-
(4.7). The first stage objective function is the sum of the operating and start-up costs
for the online generators. Thus the objective is given by:

\[
\min \sum_{b \in \mathcal{BG}} \sum_{g \in \kappa_b} \left( \sum_{t=1}^{n} h_{t,lbg} x_{tbg} + \sum_{t=2}^{n} f_{t,lbg} s_{tbg} \right).
\]

(4.38)

To demonstrate the Benders’ decomposition algorithms clearly we use the initial
first stage problem given by (4.39). The first stage decision variables are referred to
as \( y^1 \) and the objective function coefficients are represented by \( d^1 \). We also introduce
a new variable \( \theta \) to the master problem that will be used in adding optimality cuts.

\[
\min d^1 y^1 + \theta
\]

\( \text{s.t. } (4.2) - (4.7) \)

\( \theta \geq 0. \)

4.5.1 Benders’ Method for the Two-Stage Robust Model

For stage two nodes \( \epsilon_1 \in \Omega_1 \) and first stage decision \( y^1 \) let MIP model (4.40) represent
the constraints given by (4.28) - (4.36b). The second stage decision variables are \( y^2_{\epsilon_1} \),
the objective function coefficients are \( d^2_{\epsilon_1} \) (we assume that the probabilities of stage
two subproblems are included in this objective function coefficient) and the right-hand
side constant is \( e^2_{\epsilon_1} \).
Figure 4.2: Three-stage Stochastic Program Solution for Small Test Instance
Figure 4.3: Two-stage Robust Model Solution for Small Test Instance
\[
\min d_{\epsilon_1}^2 y_{\epsilon_1}^2 
\]
\[
\text{s.t. } A_{\epsilon_1}^2 y_{\epsilon_1}^2 \leq c_{\epsilon_1}^2 - A^1 y^1
\]
\[
y_{\epsilon_1}^2 \geq 0.
\]

Then the dual of model (4.40) is given by (4.41) and the dual variables are \(\pi_{\epsilon_1}^2\).

\[
\max (c_{\epsilon_1}^2 - A^1 y^1) \pi_{\epsilon_1}^2
\]
\[
\text{s.t. } A_{\epsilon_1}^2 \pi_{\epsilon_1}^2 \leq d_{\epsilon_1}^2
\]
\[
\pi_{\epsilon_1}^2 \leq 0.
\]

If the dual problem (4.41) for a given stage two node \(\epsilon_1 \in \Omega_1\) is unbounded we add a feasibility cut to the master problem (4.39). The format of the feasibility cut is

\[
(c_{\epsilon_1}^2 - A^1 y^1) \pi_{\epsilon_1}^2 \leq 0.
\]

We add an optimality cut to the master problem (4.39) only if all the stage two subproblems \(\epsilon_1 \in \Omega_1\) have an optimal solution. The optimality cut is given by

\[
\theta \geq \sum_{\epsilon_1 \in \Omega_1} (c_{\epsilon_1}^2 - A^1 y^1) \pi_{\epsilon_1}^2.
\]

The Benders’ decomposition algorithm for the two-stage robust model follows the steps of the algorithm described in Section 1.3.

### 4.5.2 Nested Benders’ Method for the Three-Stage Stochastic Program

To describe the second stage subproblems we use the same notation given in Section 4.5.1. In other words, we make use of descriptions (4.40), (4.41), (4.42) and (4.43).
However, in this section we let MIP model (4.40) represent the constraints given by (4.8) - (4.15).

For stage three nodes \( \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1) \) and first stage decision \( y^1 \) and second stage decision \( y^2_{\epsilon_1} \) let MIP model (4.44) represent the constraints given by (4.17) - (4.25). The third stage decision variables are \( y^3_{\epsilon_1 \epsilon_2} \), the objective function coefficients are \( d^3_{\epsilon_1 \epsilon_2} \) (we assume that the probabilities of stage three subproblems are included in this objective function coefficient) and the right-hand side constants are \( e^3_{\epsilon_1 \epsilon_2} \) and \( f^3_{\epsilon_1 \epsilon_2} \).

\[
\begin{align*}
\min d^3_{\epsilon_1 \epsilon_2} y^3_{\epsilon_1 \epsilon_2} \\
\text{s.t. } A^3_{\epsilon_1 \epsilon_2} y^3_{\epsilon_1 \epsilon_2} & \leq c^3_{\epsilon_1 \epsilon_2} - B^1 y^1 \\
C^3_{\epsilon_1 \epsilon_2} y^3_{\epsilon_1 \epsilon_2} & \leq f^3_{\epsilon_1 \epsilon_2} - G^1 y^1 - H^2_{\epsilon_1} y^2_{\epsilon_1} \\
y^3_{\epsilon_1 \epsilon_2} & \geq 0.
\end{align*}
\] (4.44)

Then the dual of model (4.44) is given by (4.45) and the dual variables are \( \pi^3_{\epsilon_1 \epsilon_2} \) and \( \pi^4_{\epsilon_1 \epsilon_2} \).

\[
\begin{align*}
\max (c^3_{\epsilon_1 \epsilon_2} - B^1 y^1) \pi^3_{\epsilon_1 \epsilon_2} + (f^3_{\epsilon_1 \epsilon_2} - G^1 y^1 - H^2_{\epsilon_1} y^2_{\epsilon_1}) \pi^4_{\epsilon_1 \epsilon_2} \\
\text{s.t. } A^3_{\epsilon_1 \epsilon_2} \pi^3_{\epsilon_1 \epsilon_2} + C^3_{\epsilon_1 \epsilon_2} \pi^4_{\epsilon_1 \epsilon_2} & \leq d^3_{\epsilon_1 \epsilon_2} \\
\pi^3_{\epsilon_1 \epsilon_2}, \pi^4_{\epsilon_1 \epsilon_2} & \leq 0.
\end{align*}
\] (4.45)

If the dual problem (4.45) for a given stage three node \( \epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1) \) is unbounded we add a feasibility cut to the second stage problem \( \epsilon_1 \in \Omega_1 \) or master problem (4.39) according to the value of \( H^2_{\epsilon_1} y^2_{\epsilon_1} \pi^4_{\epsilon_1 \epsilon_2} \). If all the coefficients of second stage variables \( y^2_{\epsilon_1} \) are zero then the cut is added to the master problem. The format of the feasibility cut is

\[
(c^3_{\epsilon_1 \epsilon_2} - B^1 y^1) \pi^3_{\epsilon_1 \epsilon_2} + (f^3_{\epsilon_1 \epsilon_2} - G^1 y^1 - H^2_{\epsilon_1} y^2_{\epsilon_1}) \pi^4_{\epsilon_1 \epsilon_2} \leq 0.
\] (4.46)
We add an optimality cut to a second stage problem $\epsilon_1 \in \Omega_1$ only if all the stage three subproblems $\epsilon_1 \in \Omega_1$, $\epsilon_2 \in \Omega_2(\epsilon_1)$ have an optimal solution. We introduce a new variable $\theta_2$ to the second stage subproblem $\epsilon_1 \in \Omega_1$ as follows

$$\min d^2_{\epsilon_1} y^2_{\epsilon_1} + \theta_2$$

$$\text{s.t. } A^2_{\epsilon_1} y^2_{\epsilon_1} \leq e^2_{\epsilon_1} - A^1 y^1$$

$$y^2_{\epsilon_1}, \theta_2 \geq 0.$$ 

and the optimality cut is given by

$$\theta_2 \geq \sum_{\epsilon_2 \in \Omega_2(\epsilon_1)} (e^3_{\epsilon_1 \epsilon_2} - B^1 y^1) x^3_{\epsilon_1 \epsilon_2} + (f^3_{\epsilon_1 \epsilon_2} - G^1 y^1 - H^2_2 y^2_{\epsilon_1}) x^4_{\epsilon_1 \epsilon_2},$$

The nested Benders’ decomposition algorithm for the three-stage stochastic program is similar to the algorithm described in Section 1.3. Whenever a second stage subproblem $\epsilon_1 \in \Omega$ becomes infeasible, a feasibility cut (4.42) is added to master problem (4.39) and whenever a third stage subproblem $\epsilon_1 \in \Omega$, $\epsilon_2 \in \Omega_2(\epsilon_1)$ becomes infeasible, a feasibility cut is added to either second stage subproblem (4.47) or master problem (4.39). An optimality cut (4.43) is added to master problem (4.39) if all second stage subproblems have an optimal solution, and an optimality cut (4.48) is added to a second stage subproblem (4.47) if all third stage subproblems have an optimal solution. For a given first stage decision $y^1$ we apply the two-stage Benders’ decomposition algorithm described in Section 1.3 to a second stage subproblem $\epsilon_1 \in \Omega_1$ and third stage subproblems $\epsilon_2 \in \Omega_2(\epsilon_1)$. We treat the given second stage subproblem $\epsilon_1 \in \Omega_1$ as the master problem and the third stage subproblems $\epsilon_2 \in \Omega_2(\epsilon_1)$ as the second stage subproblems. The pseudo-code for the nested Benders’ decomposition algorithm is presented next.

**Step 1:** Solve the master problem and obtain $y^1$ (initial master problem is (4.39)).

**Step 2:** $\text{masterOptimalityCutCheck} = 1$. 

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**Step 3:** For each stage two subproblem $\epsilon_1 \in \Omega_1$
update (4.47) according to $y^1$.

**Step 4:** Solve stage two subproblem $\epsilon_1 \in \Omega_1$ and obtain $y^2_{\epsilon_1}$ and $\pi^2_{\epsilon_1}$.

**Step 4.1:** If the primal is infeasible and dual is unbounded

masterOptimalityCutCheck = 0,

add a feasibility cut (4.42) to the master problem.

**Step 4.2:** Else if the problem has an optimal solution

stage2CutsAddedCheck = 1.

While stage2CutsAddedCheck $\geq$ 1

stage2CutsAddedCheck = 0,

stage2OptimalityCutCheck = 1.

**Step 4.2.1:** For each stage three subproblem $\epsilon_1 \in \Omega_1, \epsilon_2 \in \Omega_2(\epsilon_1)$
update (4.45) according to $y^1$ and $y^2_{\epsilon_1}$.

Solve problem and obtain $\pi^3_{\epsilon_1 \epsilon_2}$ and $\pi^4_{\epsilon_1 \epsilon_2}$.

If the dual is unbounded

stage2OptimalityCutCheck = 0.

If coefficients of $y^2_{\epsilon_1}$ are nonzero in (4.46)
add (4.46) to (4.47),
increase stage2CutsAddedCheck by 1.

Else
add (4.46) to (4.39).

Else if the dual is infeasible go to Step 7.

**Step 4.2.2:** If stage2OptimalityCutCheck = 1

If $\theta_2 < \sum_{\epsilon_2 \in \Omega_2(\epsilon_1)} (e^3_{\epsilon_1 \epsilon_2} - B^1 y^1)\pi^3_{\epsilon_1 \epsilon_2}$
$+ (f^3_{\epsilon_1 \epsilon_2} - G^1 y^1 - H^2_{\epsilon_1} y^2_{\epsilon_1})\pi^4_{\epsilon_1 \epsilon_2}$

add optimality cut (4.48) to (4.47),
increase stage2CutsAddedCheck by 1.
**Step 4.2.3**: If $\text{stage2CutsAddedCheck} \geq 1$

resolve (4.47).

If infeasible

add (4.42) to (4.39),

$\text{stage2CutsAddedCheck} = 0$,

$\text{masterOptimalityCutCheck} = 0$.

**Step 4.3**: Else go to Step 7.

**Step 5**: If $\text{masterOptimalityCutCheck} = 1$

If $\theta < \sum_{\epsilon_1 \in \Omega_1} (\epsilon_1^2 - A^1 y^1) \pi_2^2$

add optimality cut (4.43) to (4.39) and go to Step 1.

Else go to Step 7.

Else go to Step 1.

**Step 7**: Terminate.

### 4.6 Computational Results

In this section we present our computational results for both the two-stage robust model and the three-stage stochastic program. These models are solved by using their respective deterministic equivalent models and Benders’ decomposition algorithms. Our goal is to compare the models as well as the proposed solution methods.

#### 4.6.1 Experimental Setup

The conventional generator settings are exactly the same for all the tested instances. There are four buses that contain conventional generators and the combined number of generators over these buses is 14. The details of generator specific limits are given in Table 4.4. The operating costs $h_{tbg}$, start-up costs $f_{tbg}$ and unit production costs $c_{tbg}$ depend on time $t \in [1, n]$, bus $b \in BG$ and generator $g \in \kappa_b$. The operating
costs change between 50 and 100. The start-up costs change between 400 and 500. The unit production costs change between 2 and 5. The time horizon is 24 hours and is split at hour 12 (i.e., $n_1 = 12$, $n = 24$). We used generator specific parameter values that are similar to the ones given by Wang et al. [92] in Table 1(a) to obtain feasible solutions. Every stage two node has exactly two children. Thus the total number of stage three nodes is exactly two times the number of stage two nodes. The probabilities for stage two nodes are randomly generated while sibling stage three nodes have the same probability. We use the wind simulator proposed by Gangammanavar [34] to generate different wind scenarios. The cost of wind power that satisfies partial demand is zero i.e., $a_{11}^1$ and $a_{11}^2$ is zero. However, the wind curtailment costs $o_{11}^1$ and $o_{11}^2$ are non-zero and change between 10 and 20. The time limit is 3600 seconds. The tests are done using CPLEX 12.4, on an Intel Core i5 Processor at 1.8GHz with 4GB 1600 MHz DDR3 Memory. We turn the dynamic search option of CPLEX off and use a single thread for all the experiments.

<table>
<thead>
<tr>
<th>(bus,generator)</th>
<th>$u_{bg}$</th>
<th>$\delta_{bg}$</th>
<th>$\ell_{bg}$</th>
<th>$u_{bg}$</th>
<th>$v_{bg}$</th>
<th>$\bar{v}_{bg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>160</td>
<td>40</td>
<td>17</td>
<td>251</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(1,2)</td>
<td>144</td>
<td>42</td>
<td>19</td>
<td>230</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>(1,3)</td>
<td>199</td>
<td>40</td>
<td>12</td>
<td>221</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>(2,1)</td>
<td>149</td>
<td>49</td>
<td>12</td>
<td>213</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>(2,2)</td>
<td>144</td>
<td>45</td>
<td>13</td>
<td>228</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(2,3)</td>
<td>175</td>
<td>43</td>
<td>11</td>
<td>250</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>(3,1)</td>
<td>150</td>
<td>55</td>
<td>13</td>
<td>200</td>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>(3,2)</td>
<td>156</td>
<td>59</td>
<td>12</td>
<td>259</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>(3,3)</td>
<td>179</td>
<td>58</td>
<td>11</td>
<td>254</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>(4,1)</td>
<td>180</td>
<td>59</td>
<td>16</td>
<td>292</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>(4,2)</td>
<td>178</td>
<td>41</td>
<td>18</td>
<td>294</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(4,3)</td>
<td>188</td>
<td>42</td>
<td>11</td>
<td>230</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>(4,4)</td>
<td>188</td>
<td>54</td>
<td>16</td>
<td>259</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
The tested solution methods in Section 4.6.2 are described below.

- **2DEQ**: Deterministic equivalent formulation of the two-stage robust model (i.e. model given by (4.27), (4.2)-(4.7), (4.28)-(4.37)).

- **2BEN**: Benders’ decomposition applied to the two-stage robust model.

- **3DEQ**: Deterministic equivalent formulation of the three-stage stochastic program (i.e. model given by (4.1)-(4.26)).

- **3BEN**: Nested Benders’ decomposition applied to the three-stage stochastic program.

### 4.6.2 Experiments

We test 20 instances using solution methods 2DEQ, 2BEN, 3DEQ and 3BEN. The results are given in Tables 4.5, 4.6, 4.7 and 4.8. In Table 4.5 all the instances include two wind farms and in Table 4.6 all the instances have four wind farms. Similarly, the instances in Tables 4.7 and 4.8 have two and four wind farms, respectively. In all the experiments the data for conventional generators is exactly the same. The difference between instances within the tables is the number of wind scenarios that are considered. However, for instances that have the same number of wind scenarios but different number of wind farms the demand values also change. This is because the demand parameter depends on the number of buses and its value changes as the number of wind farms increase or decrease.

Column **inst, sw** denotes the instance number and the number of wind power scenarios, respectively. Column **Time(s)** reports the solution time in seconds. In column **2BCuts** we report the number of user cuts added to master problem in algorithm 2BEN. Column **2SObj** reports the objective function value of the optimal solution found by the two-stage robust model. Column **3BMasterCuts** gives the
number of user cuts added to master problem while column 3BStage2Cuts reports
the total number of cuts added to all second stage subproblems in algorithm 3BEN.
Column 3SObj reports the objective function value of the optimal solution found by
the three-stage stochastic program.

Table 4.5: Comparison of solution methods for the two-stage robust model (for in-
stances with two wind farms)

<table>
<thead>
<tr>
<th>inst, sW</th>
<th>Time(s)</th>
<th>2DEQ</th>
<th>2BEN</th>
<th>2BEN</th>
<th>2SObj</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 140</td>
<td>362</td>
<td>30</td>
<td>982</td>
<td>183633</td>
<td></td>
</tr>
<tr>
<td>2, 160</td>
<td>652</td>
<td>39</td>
<td>1210</td>
<td>182988</td>
<td></td>
</tr>
<tr>
<td>3, 180</td>
<td>710</td>
<td>97</td>
<td>1696</td>
<td>183130</td>
<td></td>
</tr>
<tr>
<td>4, 240</td>
<td>1275</td>
<td>34</td>
<td>1357</td>
<td>183159</td>
<td></td>
</tr>
<tr>
<td>5, 400</td>
<td>&gt;3600</td>
<td>41</td>
<td>1721</td>
<td>183043</td>
<td></td>
</tr>
<tr>
<td>6, 440</td>
<td>&gt;3600</td>
<td>84</td>
<td>1656</td>
<td>183101</td>
<td></td>
</tr>
<tr>
<td>7, 480</td>
<td>&gt;3600</td>
<td>83</td>
<td>1929</td>
<td>182968</td>
<td></td>
</tr>
<tr>
<td>8, 500</td>
<td>&gt;3600</td>
<td>58</td>
<td>2106</td>
<td>183136</td>
<td></td>
</tr>
<tr>
<td>9, 600</td>
<td>&gt;3600</td>
<td>162</td>
<td>3494</td>
<td>183356</td>
<td></td>
</tr>
<tr>
<td>10, 700</td>
<td>&gt;3600</td>
<td>109</td>
<td>3275</td>
<td>182824</td>
<td></td>
</tr>
</tbody>
</table>

From columns 2BCuts and 3BMasterCuts in Tables 4.5, 4.6, 4.7 and 4.8, we observe
that the number of cuts added to the master problem do not necessarily increase with
the number of scenarios. The same observation can be made for the total number of
cuts added to second stage problems (by looking under the column 3BStage2Cuts)
in algorithm 3BEN. For both models and solution methods, instances with four wind
farms with the same number of scenarios as the instances with two wind farms require
much less time to be solved. Furthermore, algorithms 2BEN and 3BEN generate more
Benders’ cuts for instances with two wind farms.

All the instances in Tables 4.5 and 4.6 are solved faster by the two-stage Ben-
ders’ decomposition algorithm 2BEN compared to the deterministic equivalent model
Table 4.6: Comparison of solution methods for the two-stage robust model (for instances with four wind farms)

<table>
<thead>
<tr>
<th>inst, sW</th>
<th>Time(s)</th>
<th>2DEQ</th>
<th>2BEN</th>
<th>2SOBJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 140</td>
<td>73</td>
<td>3</td>
<td>276</td>
<td>250568</td>
</tr>
<tr>
<td>2, 160</td>
<td>121</td>
<td>3</td>
<td>258</td>
<td>250325</td>
</tr>
<tr>
<td>3, 180</td>
<td>149</td>
<td>3</td>
<td>373</td>
<td>251123</td>
</tr>
<tr>
<td>4, 240</td>
<td>308</td>
<td>2</td>
<td>234</td>
<td>250653</td>
</tr>
<tr>
<td>5, 400</td>
<td>1118</td>
<td>5</td>
<td>393</td>
<td>251584</td>
</tr>
<tr>
<td>6, 440</td>
<td>1388</td>
<td>6</td>
<td>629</td>
<td>250295</td>
</tr>
<tr>
<td>7, 480</td>
<td>2436</td>
<td>5</td>
<td>738</td>
<td>250672</td>
</tr>
<tr>
<td>8, 500</td>
<td>1732</td>
<td>5</td>
<td>460</td>
<td>249909</td>
</tr>
<tr>
<td>9, 600</td>
<td>2517</td>
<td>6</td>
<td>401</td>
<td>250442</td>
</tr>
<tr>
<td>10, 700</td>
<td>&gt;3600</td>
<td>8</td>
<td>601</td>
<td>249627</td>
</tr>
</tbody>
</table>

2DEQ. In Table 4.5, 6 instances and in Table 4.6, 1 instance is not solved to optimality by 2DEQ within the time limit (indicated by > 3600). Tables 4.5 and 4.6 clearly demonstrate the advantage of a Benders’ decomposition algorithm. The MIP model 2DEQ becomes computationally intractable as the number of scenarios increase while algorithm 2BEN can solve instances with up to 700 scenarios in a few seconds or minutes.

A similar conclusion can be reached from Tables 4.7 and 4.8. The nested Benders’ decomposition algorithm 3BEN outperforms 3DEQ. As expected, the objective function value of the three-stage stochastic program (3SOBJ) is less than the objective function value of the two-stage robust model (2SOBJ) for all the instances. The percentage difference for the overall average of the objective function values of the two models for instances with two wind farms is 2% and for instances with four wind farms is 3%. However, it is much harder to solve the three-stage stochastic program as indicated by the amount of time the solution methods 3DEQ and 3BEN require compared to 2DEQ and 2BEN.
Table 4.7: Comparison of solution methods for the three-stage stochastic program (for instances with two wind farms)

<table>
<thead>
<tr>
<th>inst, sW</th>
<th>Time(s)</th>
<th>3DEQ</th>
<th>3BEN</th>
<th>3BEN</th>
<th>3BEN</th>
<th>3SObj</th>
</tr>
</thead>
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<tr>
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<td>309</td>
<td>1560</td>
<td>34297</td>
<td>180358</td>
<td></td>
</tr>
<tr>
<td>2, 160</td>
<td>1302</td>
<td>358</td>
<td>2466</td>
<td>41183</td>
<td>179740</td>
<td></td>
</tr>
<tr>
<td>3, 180</td>
<td>2085</td>
<td>948</td>
<td>1369</td>
<td>77844</td>
<td>180093</td>
<td></td>
</tr>
<tr>
<td>4, 240</td>
<td>&gt;3600</td>
<td>1076</td>
<td>2760</td>
<td>73862</td>
<td>180138</td>
<td></td>
</tr>
<tr>
<td>5, 400</td>
<td>&gt;3600</td>
<td>1168</td>
<td>7612</td>
<td>96298</td>
<td>180381</td>
<td></td>
</tr>
<tr>
<td>6, 440</td>
<td>&gt;3600</td>
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<td>679</td>
<td>92937</td>
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</tr>
<tr>
<td>7, 480</td>
<td>&gt;3600</td>
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<td>1664</td>
<td>173434</td>
<td>180455</td>
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</tr>
<tr>
<td>8, 500</td>
<td>&gt;3600</td>
<td>982</td>
<td>1535</td>
<td>122473</td>
<td>180745</td>
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<tr>
<td>9, 600</td>
<td>&gt;3600</td>
<td>1530</td>
<td>2306</td>
<td>198873</td>
<td>180662</td>
<td></td>
</tr>
<tr>
<td>10, 700</td>
<td>&gt;3600</td>
<td>1094</td>
<td>1538</td>
<td>164882</td>
<td>180440</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8: Comparison of solution methods for the three-stage stochastic program (for instances with four wind farms)

<table>
<thead>
<tr>
<th>inst, sW</th>
<th>Time(s)</th>
<th>3DEQ</th>
<th>3BEN</th>
<th>3BEN</th>
<th>3BEN</th>
<th>3SObj</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 140</td>
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<td></td>
</tr>
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<td>9789</td>
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</tr>
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<td>22430</td>
<td>244851</td>
<td></td>
</tr>
<tr>
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<td>84</td>
<td>1177</td>
<td>23025</td>
<td>243254</td>
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</tr>
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<td>20650</td>
<td>243568</td>
<td></td>
</tr>
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<td>243079</td>
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<td>749</td>
<td>21304</td>
<td>244157</td>
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</tr>
<tr>
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<td>1624</td>
<td>26860</td>
<td>244097</td>
<td></td>
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</tbody>
</table>
CHAPTER 5

CONCLUSIONS AND FUTURE WORK

We conclude by summarizing the work presented in Chapters 2, 3, 4 and future research directions. In this dissertation we describe problems found in supply chain and power systems with discrete decisions and uncertain data. Branch-and-cut algorithms and decomposition methods are utilized as solution methods.

In Chapter 2 we study a transportation problem with market choice (TPMC). We show that TPMC is strongly \( \mathcal{NP} \)-complete except for a trivial case (when all the demands are no larger than two). We consider a version of the problem with a service level constraint on the maximum number of markets that can be rejected. We show that for the case in which the original problem is polynomial, its cardinality-constrained version is also polynomial. In this case, adding the cardinality constraint to the convex hull of solutions to the original problem does not create any new fractional extreme points. We present valid inequalities for mixed integer cover sets and mixed-integer knapsack sets with variable upper bound constraints, which appear as substructures of TPMC. One can use these inequalities for solving other problems with similar substructures. The separation problem of the proposed valid inequalities could be investigated further as the heuristics used for generating the valid inequalities need to be significantly improved.

In Chapter 3 we study the ramping relaxations of the unit commitment (UC) problem. We show that optimization over the ramping polytope is polynomial. We
propose families of facet-defining inequalities and efficient separation algorithms. For the two-period problem, we give a complete linear description of the convex hull of feasible solutions. However, the convex hull of the feasible set for the case of more than two periods is an open problem that merits further research. Our computations demonstrate the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the UC problem.

In Chapter 4 we study a stochastic UC problem. The uncertainty comes from a renewable energy source, namely wind power. In addition to conventional generators presented in Chapter 3 we introduce wind farms to the problem. We propose two different models and methods to solve the SUC problem with uncertain wind power. A two-stage robust model and a three-stage stochastic program are compared. The motivation of comparing two models comes from the difficulty of predicting wind power for the not so near future. The three-stage stochastic program divides a given time horizon into two (and forms the second and the thirds stages of the problem) and considers different wind power scenarios according to this split. The two-stage robust model is only able to see the wind power scenarios up until the first part of the time horizon split. Therefore, for the time periods that come after this split point a robust approach that considers the worst case wind power scenarios for the remaining part of the time horizon is modeled. Our computations using deterministic equivalent formulations and Benders’ decomposition algorithms for both of the models suggest that solving these problems using a Benders’ decomposition framework is much more effective. Moreover, the computations also show that the solutions obtained by the two-stage robust model are more costly and conservative compared to the solutions obtained with the three-stage stochastic program. As future research, one can analyze models with more than three stages. Furthermore, both of the models can be used within a rolling horizon algorithm in which the models are solved repeatedly as new information about the uncertain parameter becomes available.
BIBLIOGRAPHY


