Essays on Strategic Interaction via Consumer Rewards
Programs

Dissertation

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By

Ross Arthur Brater, B.A., M.A.

Graduate Program in Economics

The Ohio State University

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Dissertation Committee:

Huanxing Yang, Advisor
Dan Levin
Lixin Ye
Abstract

Consumer rewards programs are seemingly ubiquitous, including airline frequent-flyer and hotel frequent-stay programs, fuel rewards perks, and cash-back credit card offers. Regardless of the setting, a common characteristic of these programs is that eligibility for a firm’s reward imposes a switching cost on the eligible consumer, which tends to dampen competition for reward-eligible consumers. Consequently, consumers rationally exhibit reduced demand elasticity in markets where purchasing a particular product leads to eligibility for a future reward. The following chapters examine the use of rewards programs by firms in different environments.

In the first chapter, I study a differentiated-products oligopoly where firms directly compete only with their nearest rivals, and I examine the practice of firms offering loyalty discounts to not only their own repeat customers but also to past customers of an affiliated firm. If rational, forward-looking consumers are uncertain about which firms they will patronize in the future, these types of partnerships between firms enhance the effect that a particular reward has in initially attracting consumers to the partnered firms. In particular, partnerships between firms are most effective when firms who do not normally compete directly decide to partner with one another. However, if too many firms engage in partnerships, the competition-dampening effect of loyalty rewards is reversed: initial competition is heightened as a result of the increased reward effect, and firm profits fall below even the levels that would prevail
without any form of rewards. Consequently, in any stable network of partnerships, only some firms are partnered, and these firms are not local rivals.

The second chapter focuses on the popular phenomenon of firms promising buyers of one product a discount that may be applied towards the future purchase of a separate product, possibly sold by a competing firm. I study a differentiated products market whose firms may partner with one or more firms in a separate market to facilitate consumer rewards programs across markets. I find that these partnerships tend to relax competition in the market where the rewards are generated by reducing consumer price sensitivity and that this effect increases with the number of partnerships formed. Due to the attractiveness of these partnerships to firms in the reward-generating market, the partnership formation process is quite volatile. If firms are permitted sufficient freedom to jointly alter the current partnership network, no stable networks exist. If the degree to which firms may alter an existing network is more limited, stable networks exist where each firm in the market where rewards are redeemed is partnered with a firm in the reward-generating market.

The final chapter examines the use of loyalty rewards by an incumbent monopolist prior to and after potential entry. Because such loyalty rewards generate switching costs for eligible consumers in ensuing periods, an interesting question is whether the implementation of these programs may be used to adversely affect the prospects of potential entrants. I find that this method of entry deterrence is ineffective. Utilizing a rewards program requires honoring any discount or payout previously promised to repeat customers, which, upon entry, makes attracting enough consumers to injure a rival firm an unsatisfactory option for the incumbent firm. Nonetheless, the reduced
price sensitivity of first-time buyers who will later be eligible for a reward allows the incumbent to increase profits while accommodating entry.
For my parents, whose unconditional love and support has been an enduring source of inspiration.
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Vita

November 5, 1984 ......................... Born - Cincinnati, Ohio

2007 ........................................... B.A. in Economics and Mathematics, Ohio Wesleyan University

2009 ........................................... M.A. in Economics, The Ohio State University

2009-present ................................. Graduate Teaching Associate, The Ohio State University

Fields of Study

Major Field: Economics
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1.1 Introduction

Consumer loyalty discounts and frequent-buyer programs have become increasingly popular in recent years. It is not uncommon to see a grocery store patron sort through numerous tags on her keyring to find the correct card to swipe at checkout; indeed, doing so may result in coupons for future purchases being mailed to her or qualify her for a few cents off the price of gas during her next fill-up. Hotels and airlines often offer free rooms or flights, or discounts, to repeat patrons through various “frequent-flyer” or “frequent-stay” programs. Many credit cards offer cash back or other rewards after a certain dollar amount has been spent using the card. The popularity of programs such as these is such that a simple internet search will turn up a plethora of web pages dedicated to tracking the best and worst consumer loyalty rewards.

The appeal of loyalty programs to firms seems obvious at first glance. By rewarding consumers for repeat purchases, loyalty programs impose a switching cost on consumers once an initial purchase has been made. That is, any reward points or discounts that a consumer has earned may be lost if not redeemed. As in typical
models of switching costs, past customers of another firm can only be attracted if a price reduction compensates them for the cost of switching, in this case, this cost is the forgoing of the reward for which they are eligible. As a result, competition among firms for reward-eligible consumers tends to diminish, as firms set nominal prices less aggressively.

This incentive for a firm to raise prices above the level which would prevail without reward programs becomes stronger as the pool of consumers eligible for that reward grows. As a result, forward-looking consumers who will be eligible for a reward in the future become less sensitive to price competition prior to earning the reward. This is the true value of reward programs to firms: fewer consumers are swayed by the same price cut because a price hike is anticipated later; thus, competition for reward-ineligible consumers is relaxed as well.

The benefits of a loyalty program which provides future discounts to repeat customers can be easily understood by following this analysis. However, many rewards programs exist which do not tie consumers to a single firm. A prominent example exists in the airline industry, where carriers have formed “airline alliances,” which permit frequent-flyer miles earned through allied carriers to be used interchangeably when redeemed. A savvy consumer who is deciding whether to book a flight with United Airlines or Southwest Airlines will realize that miles earned from the former carrier, who is a member of Star Alliance, may be spent towards a future flight with an allied carrier such as US Airways. Meanwhile, rewards earned through Southwest, which is not a member of an alliance, will provide more limited options.
This phenomenon is not confined to airlines. So-called “coalition” loyalty rewards programs exist in which participants earn points by making purchases from participating retailers which can be redeemed for discounts from any coalition member. For instance, the Flybuys loyalty card available in Australia allows for interchangeability among points earned at stores such as Target, K-Mart, and Coles supermarkets, among others. Similar programs in different regions include the Nectar loyalty card in the United Kingdom and Italy, the Payback loyalty card in Germany and India, and Brazil’s Dotz card, all of which have been immensely popular in their respective countries.¹

The purpose of this chapter is to examine the benefits that such partnerships might offer to firms, as well as how these benefits might influence the network formation process itself. To this end, I consider a differentiated-products market, built on the circular city oligopoly model of Salop (1979). The feature of primary interest in the model is that firms are permitted to engage in partnerships with one another prior to competition. The resulting partnership network then induces a two-period game of price competition, in which a past customer of a member of a partnership qualifies for a reward that may be redeemed toward a purchase from any member of that same partnership. Specifically, all firms simultaneously announce nominal prices in both periods, with partnerships and unpartnered firms also announcing reward amounts

¹Estimates by Sports Loyalty International indicate that 68% of households in the United Kingdom earn points at least monthly through use of the Nectar card, and 60% of German and Australian households using their Payback and Flybuys cards, respectively. Among households where Dotz is marketed, 48% actively use the card. Details can be found at www.sli21.com.
in the first period. Importantly, in following the previous literature of differentiated-
product markets featuring switching costs, I assume that consumers are uncertain
about which firm they will find most attractive in future periods.²

Several interesting questions are raised by introducing rewards across firms into
the model. The first of these questions is why forming such a partnership might be
desirable in the first place. A primary benefit of allowing a reward to be redeemed
at one's rival is that the reward itself becomes more valuable to consumers in the
first period, and, critically, the marginal effect of an increase in the reward amount
on firms' sales is larger when firms form a partnership. This is due to the fact that,
as a promised reward is increased, the probability that the reward will be redeemed
by an eligible consumer increases at a faster rate if it is offered by a partnership
than if it is offered by a stand-alone firm. Thus, partnered firms do indeed have an
advantage over unpartnered rivals and can sell to larger market shares while offering
higher prices.

A second question is then which types of partnerships will be most beneficial to
firms engaging in them: should a firm partner with a local rival or a firm with whom
it typically does not compete directly? The answer to this question is straightforward: firms do better when they are partnered with those firms who are not local
rivals. The reasoning for this is simple. The effect of rewards on attracting consumers
who are trying to decide between two local rivals is eliminated when these two firms
are partnered. Buying from either firm provides identical benefits in ensuing periods
because the consumer is eligible for the same reward regardless of her purchasing de-
cision. This erodes the competition-dampening effects provided by rewards. Instead,

²Specifically, consumers’ uncertainty about future tastes is modeled identically as in von
by partnering with firms who are not direct competitors, the partnered firms still reap the benefits of forcing all marginal consumers to decide between loyalty programs.

Finally, I also wish to explore which types of partnership networks might prevail in this market. In particular, should each firm be expected to enter into a partnership, or should some firms remain unpartnered? This question is particularly interesting in light of examples of firms such as Southwest Airlines, JetBlue, and Frontier Airlines, which are not members of any airline alliance despite the aforementioned benefits that partnerships can provide. To answer this question, I restrict my attention to partnership networks which are core stable. In essence, these are networks which do not present an opportunity for a group of firms to improve their standing in the induced game by either forming new partnerships or dissolving existing partnerships immediately prior to the start of competition. I find that the only networks which satisfy this criterion are those in which not all firms engage in loyalty partnerships. Certainly, given the preceding discussion, at least some firms should enter into partnership agreements, and these firms ought to partner with firms who are not local rivals. However, if too many partnerships form, all firms will suffer in the induced game that follows. This is a direct consequence of the increased power of the reward of partnered firms; if the reward offered by every offering firm has a sufficiently heightened ability to attract consumers, competition itself is actually enhanced. Thus, the only core stable partnership networks which exist consist of a portion of firms forming partnerships with firms who are not local rivals.

Despite the ubiquity of consumer loyalty programs, the literature devoted to their study is surprisingly sparse. Note that the loyalty programs discussed above share a common feature: a reward earned through a purchase in one period is redeemed in
a future period. This is fundamentally different than pricing schemes which feature quantity discounts for several purchases made at the same time. The aforementioned programs are more in line with traditional switching costs, as discussed above. The extensive literature on switching costs has explored at length the effect of these costs on the competitiveness of the market. Several of these studies involve a similar setup to the present model, and bear mentioning. In a linear city model where consumers’ tastes vary over time, von Weizäcker (1984) assumes firms commit to charging the same price in each period and finds that switching costs actually increase competition. The reason for this result is that current preferences matter less than a prevailing price difference if consumers find it hard to switch in the future, thus the incentive to cut prices is increased. Relaxing the assumption of fixed prices, Klemperer (1987b) uses a similar model to show that firms are actually incentivized to raise prices once a customer base is installed, which reduces consumer price sensitivity—along with competition—for the reasons previously mentioned.

However, there are qualitative differences from canonical models involving switching costs. First, the switching costs imposed on consumers by rewards are endogeneously chosen by firms rather than being what are typically assumed to be exogenous. Second, the promised rewards must be paid out by the firms if repeat purchases are made, rather than being a cost absorbed solely by a consumer making a switch. Obviously, these differences critically affect the profits of the firms choosing the reward amounts and significantly alter the strategy space of firms from that of canonical switching cost models.

3See, e.g., Tirole (1988) for a discussion of various models of this type of nonlinear pricing.

4For an excellent summary of the literature on switching costs, see Farrell and Klemperer (2007).
I am aware of only a handful of models that address consumer loyalty rewards of this nature. The current model is most similar to that of Caminal and Matutes (1990), who also consider a two-period differentiated-product market where firms promise discounts to repeat purchasers. They find that such a rewards program relaxes competition in both periods via the effects of switching costs outlined above, and the equilibrium of my no-partnerships benchmark is qualitatively similar. Prior to this, Banerjee and Summers (1987) considered loyalty rewards of this form in a duopoly market for an undifferentiated product where firms simultaneously announce rewards and then sequentially announce prices. They find the extreme result that rewards programs result in monopoly pricing in both periods. Importantly, none of the preceding work allows for reward partnerships of any kind.

The formation of partnerships between firms, although novel in this setting, is an example of the more general concept of network formation. The network formation literature examines the properties of networks of connected agents who themselves may choose to form or break links, where the utility of each agent depends on the overall structure of the network. In this case, partnerships between firms form the links of a network, and the benefit derived by each firm under a network is merely their equilibrium profits in the induced competition game. The study of network formation has its roots in the matching literature, beginning with the work of Gale and Shapley (1962) on the marriage problem and its extensions. More recently, Jackson and Wolinsky (1996) and others have studied the stability and efficiency properties of more general social and economic networks. Of note, the discussion

5For a thorough review of the matching literature, see Roth and Sotomayor (1989).
of core stable networks, the notion of stability which I employ, receives significant attention in Jackson and Watts (1998).

The remainder of this chapter is arranged as follows. Section 1.2 models competition under the various partnership networks formally, and Section 1.3 presents a benchmark case of the induced game where no partnerships between firms exist. The next two sections present analysis of the induced game where partnerships are taken as given: Section 1.4 considers the case when partnerships have formed between firms who are not local rivals, while Section 1.5 treats the case where local rivals have partnered. Section 1.6 then examines which of these partnership networks satisfy core stability, given the results of the previous sections.

1.2 The Model of Competition

I wish to study the effect that various loyalty reward partnership networks will have on firm strategies and profits in a competitive setting. To this end, I use the following framework to model the competition that is induced by a specific network. With this information, it will be possible to study which networks are core stable. In particular, I assume that each partnership network assigns a level of profit to each firm, which is given by the symmetric subgame-perfect equilibrium of the induced game.

To model competition, I adopt a variation of the circular city of Salop (1979). A continuum of consumers of mass one is uniformly distributed on the unit circle; these consumers must travel along the circle to purchase from firms, which are located at equal intervals on the circle. There are four firms in the market, so the distance
between adjacent firms is $1/4$. Without loss of generality, firms are indexed 1, 2, 3, and 4, so that firm 1 has neighbors 2 and 4, and is located on the opposite point of the circle from firm 3. All firms have constant marginal cost of production $c$.

Competition takes place over two periods. In the first period, each firm $i$ simultaneously announces a price $p^1_i$ and a reward $r_i$. In the second period, each firm simultaneously announces a price $p^2_i$. The rewards announced in period one are coupon rewards which may be applied to period-two prices; that is, if a consumer purchases from firm $i$ in the first period, she will be eligible for a discount of $r_i$ subtracted from firm $i$’s price in the second period.

A partnership formed between two firms is a binding agreement which specifies a common reward amount that will be offered by both firms. Furthermore, the agreement mandates that previous customers of any partner qualify for this discount when purchasing from any partner in the second period. Prior to competition, any rewards specified by partnership agreements are known only to the partners, though partnerships are public information. Importantly, this is not collusion in the sense of price-fixing; any partnered firms still compete in the market in both periods by setting independent prices. For the purposes of this chapter, I limit the number of firms in any particular partnership to two.

6 The number of firms in the market is taken to be exogenously determined. A market with four firms is chosen, as this is the simplest formulation which permits qualitatively different partnerships. Specifically, partnerships between adjacent firms are critically different from partnerships between nonadjacent firms.

7 Rewards may alternatively be viewed as “cash back” payouts that must be paid to eligible consumers.

8 Equivalently, the agreement grants a single firm the choice of reward amount which must then be honored by both firms. In any event, partnered firms face identical optimization problems in equilibrium, hence they will choose the same reward amount (so a partnered firm loses nothing by letting its partner choose the reward).
Consumers have linear transportation costs, so that a consumer realizes disutility of \( xt \) from purchasing from a firm located at distance \( x \) from that consumer, where \( t \) represents the magnitude of consumers’ transportation costs.\(^9\) Consumers further receive disutility in the amount of the price paid for the chosen product. Otherwise, all consumers value all firms’ products equally. So a consumer who travels a distance \( xt \) and pays a price \( p \) for a product receives utility \( R - xt - p \), where \( R \) is the same regardless of the choice of firm and assumed to be large enough that all consumers wish to purchase from a firm in equilibrium. Consumers have unit demand, so that each consumer will purchase exactly one unit from one firm in each period.

The same consumers are present in both periods, but their locations are not fixed across periods. Specifically, consumers are randomly repositioned independently of their original positions. Consumers have rational expectations in that they correctly anticipate the second-period prices of firms. Consumers make purchasing decisions to maximize their total expected utility over both periods, and firms maximize total expected profits across both periods. For simplicity, I assume no discounting for both firms and consumers. The timing of the induced game is as follows. Any partnerships in place are known before firms make their pricing and reward decisions. First, firms simultaneously announce first-period prices and rewards. Consumers then make purchasing decisions for period one. At the beginning of period two, consumers are repositioned and firms make their second-period pricing decisions. Finally, consumers

\(^9\)As in similar models of horizontal differentiation, consumers’ positions need not represent a physical location. Rather, positions may be interpreted to represent the degree to which consumers prefer one product to another.
make purchasing decisions for period two. The solution concept to be used is subgame-perfect Nash equilibrium, and I focus on symmetric equilibria.\(^{10}\)

In Sections 1.3 through 1.5, which discuss the symmetric subgame-perfect equilibria of these induced games, partnerships between firms (or the lack thereof) are taken as given. Section 1.6 uses the resulting profits that firms realize under different partnership networks to examine which of these partnership networks are core stable. I first examine the situation where no partnerships exist before turning to the other possible partnership networks.

1.3 The Market Without Partnerships

The natural benchmark to establish for this model is the equilibrium of the market where no partnerships between firms exist. This benchmark is the circular-city analog of the linear-city coupon rewards model proposed by Caminal and Matutes (1990). Consumers make first-period purchasing decisions based on not only the first-period utility realized by purchasing from each firm but also their expected period-two utility conditional on being eligible for each firm’s reward. These consumers then comprise four groups in the second period, where each group is eligible for the reward of the firm whose product they previously purchased. Because only subgame-perfect equilibria of the induced games are of interest, this section and subsequent sections analyze the equilibrium of the period-two subgame first. Afterwards, the first period is considered, with firms and consumers anticipating the subgame equilibrium in the following period.

\(^{10}\)Importantly, firms need not be truly “symmetric” at the start of the game. For instance, if firm 1 and firm 3 are partnered while firms 2 and 4 are not, then looking for an equilibrium where all firms follow identical strategies is not appropriate. In such a case, I consider equilibria where firms 1 and 3 employ a common strategy and firms 2 and 4 use a (perhaps different) common strategy.
1.3.1 Period Two

In the second period, the rewards announced by each firm in the previous period are public information. From a given firm’s perspective, the market is segmented into four groups: for each firm $i$, a proportion $e_i$ of consumers is eligible for $i$’s reward $r_i$ and only this reward. As consumers are repositioned each period, each of these groups is uniformly distributed on the unit circle, and the sizes of these groups are also public information.

For any two adjacent firms $i$ and $j$, let $x_{ij}$ denote the distance from firm $i$ to the consumer eligible for $r_i$ who is indifferent between purchasing from firms $i$ and $j$. Similarly, let $y_{ij}$ denote the distance from firm $i$ to the consumer eligible for $r_j$ who is indifferent between $i$ and $j$. Finally, let $z_{ij}$ denote the distance from firm $i$ to the indifferent consumers who are eligible for neither firm’s reward (there will be two groups who contain an indifferent consumer at this distance). Then, for instance, firm 1’s period-two profits $\Pi_1^2$ are given by

\[
(p_1^2 - c) [e_1 (x_{14} + x_{12}) + e_4 (y_{14} + z_{12}) + e_2 (z_{14} + y_{12}) + e_3 (z_{14} + z_{12})] - r_1 [e_1 (x_{14} + x_{12})],
\]

(1.1)

where

\[
x_{ij} = \frac{1}{8} + \frac{p_j^2 - p_i^2 + r_i}{2t}, \quad y_{ij} = \frac{1}{8} + \frac{p_j^2 - r_j - p_i^2}{2t}, \quad z_{ij} = \frac{1}{8} + \frac{p_j^2 - p_i^2}{2t}.
\]

Solving the first-order condition for firm 1 gives firm 1’s reaction function:

\[
p_1^2 = \frac{1}{8} t + \frac{1}{2} c + \frac{1}{4} (p_4^2 + p_2^2) + e_1 r_1 - \frac{1}{4} e_4 r_4 - \frac{1}{4} e_2 r_2.
\]

(1.2)
In the canonical circular-city model without rewards, the reaction function is 
\[ p_1 = \frac{1}{8} t + \frac{1}{2} c + \frac{1}{4} (p_4^2 + p_2^2). \]
The additional adjustment made by firm 1 due to the introduction of rewards, given by the final three terms of (1.2), can be explained as follows. Introducing rewards into the model has two effects. The first effect is due to the "switching cost" imposed on consumers who are eligible for different rewards. Taking prices as given, some of the consumers who are eligible for firm \( i \)'s reward that buy from firm \( i \) would have purchased from \( i \)'s neighbors if rewards were not present. However, the promised discount from the stated price makes \( i \)'s product more attractive under rewards to those who are eligible. The additional attracted consumers due to the reward can be observed by inspecting (1.1), where firm 1’s sales are given within the first set of square brackets. Compared to when \( r_1 = 0 \), a reward set at \( r_1 > 0 \) increases sales by \( \frac{e_1 r_1}{t} \). In the canonical no-rewards model, this change in sales is precisely the same that would result from both firm 4 and firm 2 each raising their prices by \( e_1 r_1 \).\(^{11}\) Thus, under rewards, firm 1 responds to the change in sales due to its reward by adjusting its price upwards by the same amount as it would in the no-rewards model if its neighbors made such a price change.

Likewise, the rewards of firm 1’s neighbors cause some consumers to purchase from those firms when they would otherwise have purchased from firm 1 at the stated prices. The loss in sales due to firm 4’s reward is \( \frac{e_4 r_4}{2t} \), which is the same effect that a price reduction by firm 4 of \( e_4 r_4 \) would have in the canonical model. Similarly, the loss in sales due to firm 2’s reward is the same as would be brought about in the no-rewards model by firm 2 reducing its price by \( e_2 r_2 \). Thus, under rewards, \(^{11}\)To observe this, simply note that in the no-rewards model, firm 1’s sales to the segment between firm 1 and its neighbor \( j \) are given by \( z_{1j} \).
adjustments are made by firm 1 which are equal to the reaction it would have to these price increases in the no-rewards framework.

The second effect due to rewards is due to the actual price reduction guaranteed under the rewards program to those consumers who are eligible. The reward program of firm 1, which has so far been described as a discount, can equivalently be viewed as a cash payment to repeat customers. In this light, it should be clear that offering a reward has an effect similar to an increase in the marginal cost of selling to past consumers. Taking prices as given, note that a reduction in price in the model with rewards attracts the same mass of additional consumers as it would without rewards. Under rewards, however, a proportion $e_1$ of these consumers must be paid a reward of $r_1$. Thus, the marginal cost of attracting new consumers has increased from $c$ to $c + e_1r_1$. Thus, firm 1 adjusts its price upwards to compensate for this effect as well.

The above discussion can be summarized by considering the following annotated version of firm 1’s reaction function, given by (1.2), which highlights the adjustments made by firm 1 due to rewards:

$$p_1^2 = \frac{1}{8} t + \frac{1}{2} (c + e_1 r_1) + \frac{1}{4} (p_4^2 - \underbrace{e_4 r_4}_{\text{virtual decrease in } p_4^2} + \underbrace{e_1 r_1}_{\text{virtual increase in } p_4^2} + \underbrace{p_2^2}_{\text{virtual increase in } p_2^2} - \underbrace{e_2 r_2}_{\text{virtual decrease in } p_2^2} + \underbrace{e_1 r_1}_{\text{virtual increase in } p_2^2}).$$

The expressions for the second-period profits of firms 2, 3, and 4 are analogous to (1.1), and reaction functions similar to (1.2) result from solving those firms’ first-order conditions. Solving the four reaction functions simultaneously yields the subgame equilibrium pricing strategies. Because $e_1, e_2, e_3$, and $e_4$ sum to one, and all firms react similarly to one another, the second-period equilibrium prices turn out to be quite simple. For all firms $i$, firm $i$’s period-two equilibrium price is given by

$$p_i^2 = c + \frac{t}{4} + e_i r_i.$$
Without rewards, firms charge \( c + \frac{t}{4} \) in equilibrium. Thus, in the presence of rewards, each firm’s stated price is shaded up from the no-rewards level, and the magnitude of the adjustment is increasing in the mass of consumers eligible for that firm’s reward as well as the reward amount itself. Thus, as in normal models incorporating switching costs, competition is dampened in terms of prices as a result of the switching costs imposed on consumers by the rewards. However, because these switching costs must be paid by the firms, firms’ second-period profits actually fall from the no-rewards level. This can be seen for firm 1, for instance, by substituting these equilibrium pricing strategies into the period-two profits of firm 1:

\[
\Pi^2_1 = \frac{t}{16} - \frac{e_1 r_1}{2t} (2 (1 - e_1) r_1 + e_4 r_4 + e_2 r_2).
\]

(1.3)

Without rewards, firms have equilibrium profits of \( \frac{t}{16} \). As the last term in the above expression is nonpositive, rewards have the effect of reducing firm 1’s second-period profits, and analogous expressions hold for the profits of the remaining firms. Now, I turn to the first period of competition.

1.3.2 Period One

In the first period, consumers consider not only the first-period utility they will realize by purchasing from one of the firms, but also the utility they expect to realize in the following period conditional on their first-period purchasing decision. This utility is uncertain because consumers are uncertain of where they will be located in the following period, and the usefulness of being eligible for a particular reward clearly depends on consumers’ second period locations.
In order to properly understand the optimization problem of firms in this setting, I first analyze consumer decision-making given prices and rewards. I then turn to the firms’ profit-maximization problems themselves.

**Consumer Decisions**

To compare the differences in expected utility due to being eligible for different firms’ rewards, it will be helpful to calculate the utility that a hypothetical consumer who is eligible for no rewards expects to receive in period two prior to learning her position. This can then be compared to the period-two utilities that consumers eligible for each reward expect to realize, yielding relatively simple expressions for the expected period-two “gross benefit” of being eligible for each firm’s reward compared to being eligible for no rewards at all. Comparing these expected gross benefits for different firms allows for calculation of the period-two net benefit of purchasing from one firm versus another.\(^\text{12}\)

Because consumers correctly anticipate second-period prices, the expected period-two utility of a hypothetical consumer who will not be eligible for any rewards is given by

\[
R - \int_0^{z_{12}} (xt + p_1^2) \, dx - \int_0^{z_{21}} (xt + p_2^2) \, dx - \int_0^{z_{23}} (xt + p_2^2) \, dx - \int_0^{z_{32}} (xt + p_3^2) \, dx
- \int_0^{z_{34}} (xt + p_3^2) \, dx - \int_0^{z_{43}} (xt + p_4^2) \, dx - \int_0^{z_{41}} (xt + p_4^2) \, dx - \int_0^{z_{14}} (xt + p_1^2) \, dx.
\]

\(^\text{12}\)The expected net benefits could be calculated directly, but these “gross benefit” expressions turn out to have a very intuitive interpretation not only in this section, but also under each partnership network.
A consumer eligible for, say, firm 1’s reward, on the other hand, will have expected utility of
\[ R - \int_0^{x_{12}} (xt + p_1^2 - r_1) \, dx - \int_0^{y_{21}} (xt + p_2^2) \, dx - \int_0^{z_{23}} (xt + p_2^2) \, dx - \int_0^{z_{32}} (xt + p_3^2) \, dx - \int_0^{z_{34}} (xt + p_3^2) \, dx - \int_0^{y_{41}} (xt + p_4^2) \, dx - \int_0^{x_{14}} (xt + p_1^2 - r_1) \, dx. \]

Let \( GB_i \) denote the expected period-two gross benefit of being eligible for firm \( i \)'s reward compared to being eligible for no rewards. Then subtracting the first of the two previous expressions from the second and evaluating the difference at the second-period equilibrium prices results in
\[ GB_1 = \frac{r_1}{4} + \frac{r_1}{2t} (e_4 r_4 + e_2 r_2 - 2 e_1 r_1) + \frac{r_1^2}{2t}. \]

This expression has a very intuitive interpretation. Note that this benefit may equivalently be written as
\[ r_1 \left( \frac{1}{4} \frac{p_1^2 + p_2^2 - 2p_1^2}{2t} \right) + r_1 \left( \frac{r_1}{2t} + \frac{r_1}{2t} \right) = r_1 (z_{14} + z_{12}) + \frac{r_1}{2} (x_{14} - z_{14} + x_{12} - z_{12}). \]

(1.4)

The intuition behind this expression is as follows. Being eligible for firm 1’s reward is only beneficial in period two if the consumer is located close enough that she will elect to buy from firm 1 again. With probability \( 1 - x_{14} - x_{12} \), she will be too far away from firm 1 for the reward to benefit her and her realized benefit is zero. If she is close enough to firm 1 to repeat purchase, there are two possibilities. First, she may be close enough that she would have purchased from firm 1 even if she were not eligible for the reward. In this case, her period-two utility simply increases by exactly the reward amount \( r_1 \) over what she would have otherwise realized, and this case occurs with probability \( z_{14} + z_{12} \). The first term in (1.4) captures this expected benefit.
The other possibility is that she is in a position where she would have purchased from one of firm 1’s neighbors if she weren’t eligible for the reward, but the switching cost generated by the reward prevents her from switching. If she is located some distance between \( x_{1j} \) and \( z_{1j} \) from firm 1, she purchases from firm 1 again, but she would have purchased from firm 1’s neighbor \( j \) if not for the reward. If she is located at a distance \( x_{1j} \) from firm 1, the reward makes her just indifferent between firm 1 and firm \( j \), whom she would have purchased from otherwise. Thus, at this distance, the benefit of the reward is zero. At a distance \( z_{1j} \) from firm 1, she would have been indifferent between firms 1 and \( j \) without the reward, so her extra utility due to the reward is the full reward amount \( r_1 \). At any intermediate distance, the reward benefit is between these extremes: as the consumer moves from a distance of \( x_{1j} \) from firm 1 to a distance of \( z_{1j} \) from firm 1, her extra utility due to the reward increases from 0 to \( r_1 \) linearly in the distance traveled. Thus, the expected value of the reward benefit conditional on being located in this interval is \( \frac{r_1}{2} \). As the consumer finds herself in an interval like this with probability \( x_{14} - z_{14} + x_{12} - z_{12} \), the final term in (1.4) captures precisely this expected benefit.

In general, the following expression holds for the expected gross benefit of purchasing from firm \( i \) with neighbors \( j \) and \( k \):

\[
GB_i = \frac{r_i}{4} + \frac{r_i}{2t} (e_j r_j + e_k r_k - 2 e_i r_i) + \frac{r_i^2}{2t}.
\]

Let \( n_{ij} \) denote the distance from firm \( i \) to the consumer indifferent between purchasing from firm \( i \) and its neighbor \( j \) in the first period. Then the mass of consumers who are eligible for each reward in period two, i.e., the sales level of each firm in period
one, are given by

\[ e_1 = n_{14} + n_{12}, \quad e_2 = n_{21} + n_{23}, \quad e_3 = n_{32} + n_{34}, \quad e_4 = n_{43} + n_{41}, \]

and \( n_{ji} = \frac{1}{4} - n_{ij} \). Utilizing the above expected benefits, each \( n_{ij} \) is defined implicitly by

\[ F_{ij} \equiv \frac{t}{4} - 2tn_{ij} - p_i^1 + p_j^1 + GB_i - GB_j, \]

where each \( GB_i \) depends on each of the four different cutoffs \( n_{ij} \). This system of implicit functions describes how consumers allocate first-period purchases given firms’ announced prices and rewards. The implicit function theorem can be used to derive expressions for the equilibrium marginal effects on the locations of the indifferent consumers due to incremental changes in prices and rewards. Those expressions, which are derived in Appendix A, can now be used to solve firms’ first-order conditions yielding equilibrium strategies.

**Equilibrium**

Using (1.3), firm 1’s total profit, \( \Pi_1 = (p_1^1 - c)(n_{14} + n_{12}) + \Pi_2^1 \), can be written

\[ \Pi_1 = (p_1^1 - c) e_1 + \frac{t}{16} - \frac{e_1 r_1}{2t} (2 (1 - e_1) r_1 + e_4 r_4 + e_2 r_2), \]

where

\[ e_1 = n_{12} + n_{14}, \quad e_2 = \frac{1}{2} - n_{12} - n_{32}, \quad e_4 = \frac{1}{2} - n_{14} - n_{34}. \]

Invoking symmetry on the first-order conditions for firm 1’s price and reward amount and solving the resulting system yields the symmetric equilibrium strategy, and computational details may be found in the appendix. In equilibrium, firms each sell to one-fourth of the market, and the following approximations hold in equilibrium for
firms’ reward amounts, first-period prices, and total profits respectively:

\[ r^* \approx 0.223323t, \quad p^* \approx c + 0.311512t, \quad \Pi^* \approx 0.127910t. \]

Again, in the absence of rewards, the equilibrium price is \( c + \frac{t}{4} \), so first-period prices rise as a result of the rewards programs. The intuition behind this result pertains to the fact that consumers have a lower first-period price sensitivity under rewards than in the no-rewards model. For any fixed reward amounts, consider a price increase by firm 1, which causes \( e_1 \) to fall. As a result, consumers anticipate a reduction in firm 1’s price in period two relative to all other prices. This, in turn, increases the \( \textit{ex ante} \) probability that a consumer eligible for firm 1’s period-two reward will indeed purchase from firm 1 in the second period. Thus, the net expected benefit of being eligible for firm 1’s reward increases. This causes fewer consumers to be lost by this price increase than would be lost in the model with no rewards; because this effect applies to all firms, competition is dampened in the first period.

The dampening of competition in the first period leads to higher first-period profits for firms. However, due to the reward payouts, as seen above, period-two profits fall despite the fact that competition is also lessened in that period (resulting in higher nominal prices). Without rewards, firms earn profits of \( \frac{t}{16} \) in each period, so total profits across two periods are \( \frac{t}{8} = 0.125t \). Thus, although second-period profits fall when rewards are adopted, total profits do indeed increase.

The case when there are no partnerships between firms serves as a useful benchmark. Not only is the analysis of the various partnership networks very similar, but much of the intuition carries over as well. In what follows, the discussion focuses on the differences that arise between the different networks and the no-partnership
benchmark. I first consider the two possible scenarios when at least one partnership exists between two nonadjacent firms.

1.4 Partnerships Between Nonadjacent Firms

When there is at least one partnership between nonadjacent firms, there is either a single such partnership, or each firm is partnered with the firm located opposite of its own location. Without loss of generality, I will henceforth assume that firms 1 and 3 have partnered in both cases, and that firms 2 and 4 are partnered only in the latter. In either case, firm 1 is *ex ante* symmetric to its partner while firm 4 is *ex ante* symmetric to firm 2. For consistency with the following section, where firm 1 and firm 4 are without loss of generality never partnered, I focus on the optimization problems of firms 1 and 4.

1.4.1 A Single Partnership Between Nonadjacent Firms

Suppose that firms 1 and 3 have formed a partnership, and that their common neighbors, firms 2 and 4, are not partnered. First, consider the second-period sub-game.

In the second period, there are now three distinct groups of consumers. A mass $e_{13}$ of consumers have purchased from a member of the partnership in the previous period and are thus eligible for the reward offered by the members of the partnership. That is, they qualify for a discount of $r_{13}$ should they elect to purchase from firm 1 or firm 3. Proportion $e_{2}$ of consumers have purchased from firm 2 in the previous period, and qualify for a discount of $r_{2}$ off of firm 2’s stated price. The remaining $e_{4}$ consumers previously purchased from firm 4 and thus are eligible for firm 4’s discount
of $r_4$. As usual, each group of consumers is uniformly distributed around the circular city.

Again, let $x_{ij}$, $y_{ij}$, and $z_{ij}$ respectively denote the distances from firm $i$ to the consumers eligible for $i$’s discount, $j$’s discount, and neither discount who are indifferent between purchasing from $i$ and its neighbor $j$. Then firm 1’s profit in period two, $\Pi_1^2$, is given by

$$(p_1^2 - c) \left[ e_{13} (x_{14} + x_{12}) + e_4 (y_{14} + z_{12}) + e_2 (z_{14} + y_{12}) \right] - r_{12} \left[ e_{13} (x_{14} + x_{12}) \right],$$

where

$$x_{ij} = \frac{1}{8} + \frac{p_j^2 - p_i^2 + r_i}{2t}, \quad y_{ij} = \frac{1}{8} + \frac{p_j^2 - r_j - p_i^2}{2t}, \quad z_{ij} = \frac{1}{8} + \frac{p_j^2 - p_i^2}{2t},$$

and $r_k$ is the discount promised by firm $k$ to eligible consumers. Firm 3’s profit function is analogous, as both firms face the same optimization problem given the prices of the remaining firms. Firm 4’s second period profits $\Pi_4^2$ are given by

$$(p_1^2 - c) \left[ e_4 (x_{43} + x_{41}) + e_{13} (y_{43} + y_{41}) + e_2 (z_{43} + z_{41}) \right] - r_4 \left[ e_4 (x_{43} + x_{41}) \right],$$

and firm 2’s profit function is analogous. As usual, solving firms’ first-order conditions yields the subgame equilibrium prices:

$$p_1^2 = p_3^2 = c + \frac{t}{4} + e_{13}r_{13}, \quad p_4^2 = c + \frac{t}{4} + e_4r_4, \quad p_2^2 = c + \frac{t}{4} + e_2r_2.$$

These subgame equilibrium prices are very similar to those in the case of no partnerships. The only difference is that all consumers who purchased from either firm 1 or firm 3 in the previous period are now eligible for the reward offered by the partnership, and firms merely react appropriately to this eligibility in the same manner that they did under no partnerships. Second-period profits are likewise similar. Each firm
in the partnership realizes the same profits as its partner,

\[
\Pi_1^2 = \Pi_3^2 = \frac{t}{16} - \frac{e_{13}r_{13}}{2t} (2 (1 - e_{13}) r_{13} + e_4 r_4 + e_2 r_2),
\]

while non-partnered firms have analogous profits (though they may realize different profits from one another given different first-period sales):

\[
\Pi_4^2 = \frac{t}{16} - \frac{e_4 r_4}{2t} (2 (1 - e_4) r_4 + 2 e_{13} r_{13}), \quad \Pi_2^2 = \frac{t}{16} - \frac{e_2 r_2}{2t} (2 (1 - e_2) r_2 + 2 e_{13} r_{13}).
\]

In period one, consumers and firms anticipate the second-period subgame equilibrium. As under no partnerships, consumers maximize utility by making a purchasing decision that takes into account the first-period utility realized by purchasing from each firm as well as the expected gross benefit of being eligible for firm rewards in the ensuing period. Now, there are three gross benefits to be calculated, as consumers will be eligible for the rewards of either the partnership, firm 4, or firm 2. Calculation of the expected benefits is similar to that when there are no partnerships, and the expected gross benefit of the partnership reward is given by

\[
GB_{13} = \frac{r_{13}}{2} + \frac{r_{13}}{t} (e_4 r_4 + e_2 r_2 - 2 e_{13} r_{13}) + \frac{r_{13}^2}{t},
\]

while the expected gross benefits of the unpartnered firms’ rewards are

\[
GB_4 = \frac{r_4}{4} + \frac{r_4}{2t} (2 e_{13} r_{13} - 2 e_4 r_4) + \frac{r_4^2}{2t}, \quad GB_2 = \frac{r_2}{4} + \frac{r_2}{2t} (2 e_{13} r_{13} - 2 e_2 r_2) + \frac{r_2^2}{2t}.
\]

These expressions are similar to those under no rewards, and the intuition is identical.\(^{13}\) The only difference is that consumers who will be eligible for the partnership’s reward will benefit in the event that they purchase from firm 1 or from firm 3, rather

\(^{13}\)In particular, expressions analogous to (1.4), relating these gross benefits to the period-two indifferent consumer locations can be easily formulated.
than just one of these firms. This highlights a key point: forming a partnership increases the ability of partnered firms to attract additional consumers by increasing the reward amount. That is, fixing firms’ prices, an increase in the reward amount offered by a partnership will provide additional benefit to eligible consumers who become located near either firm in the following period. A similar reward increase by unpartnered firms will only benefit eligible consumers who end up near the firm from whom they have previously purchased. In this sense, the partnership enhances consumers’ sensitivity to the reward offered by partnered firms.

Letting $n_{ij}$ denote the distance from firm $i$ to the consumer indifferent between purchasing from $i$ and its neighbor $j$ in the first period, firm 1’s sales are $n_{14} + n_{12}$, firm 4’s sales are $\frac{1}{2} - n_{14} - n_{34}$, firm 2’s sales are $\frac{1}{2} - n_{12} - n_{32}$, and the mass of consumers who will be eligible for the partnership’s reward in period two is given by $e_1 = n_{14} + n_{12} + n_{32} + n_{34}$. These cutoffs are defined implicitly by the following functions:

\[
F_{14} \equiv t - 2tn_{14} - p_1^1 + p_4^1 + GB_{13} - GB_4,
\]
\[
F_{12} \equiv \frac{t}{4} - 2tn_{12} - p_1^1 + p_2^1 + GB_{13} - GB_2,
\]
\[
F_{34} \equiv \frac{t}{4} - 2tn_{34} - p_3^1 + p_4^1 + GB_{13} - GB_4,
\]
\[
F_{32} \equiv \frac{t}{4} - 2tn_{32} - p_3^1 + p_2^1 + GB_{13} - GB_2.
\]

Again, these implicit functions can be used to find the equilibrium marginal effects on each cutoff of changes in prices and reward amounts in equilibrium.

As firms anticipate the subgame equilibrium in period 2, the total profit functions of firms 1 and 4 may be written as

\[
\Pi_1 = (p_1^1 - c)(n_{14} + n_{12}) + \frac{t}{16} - \frac{e_{13}r_{13}}{2t} (2 (1 - e_{13}) r_{13} + e_2 r_2 + e_4 r_4),
\]

24
\[ \Pi_4 = \left( p_1^1 - c \right) \left( \frac{1}{2} - n_{14} - n_{34} \right) + \frac{t}{16} - \frac{e_4 r_4}{2t} \right) \left( 2(1 - e_4)r_4 + 2e_{13}r_{13} \right). \]

In the symmetric equilibrium, \( p_1^1 = p_3^1, p_2^1 = p_4^1, \) and \( r_2 = r_4. \) As a result, \( n_{14} = n_{12} = n_{32} = n_{34}. \) Imposing symmetry on the first-order conditions for both firms’ prices and reward amounts and solving the resulting system yields the unique symmetric equilibrium of the game induced by this partnership network. The analysis is similar in spirit to that presented in the appendix for the case of no partnerships, and the details are saved for the reader. In the symmetric equilibrium, the consumer indifferent between buying from a partnered firm and its neighbor is located at a distance of \( n_{SP, O} = 0.133798 \) from the partnered firm. This is greater than \( \frac{1}{8}, \) so the partnered firms realize higher first-period sales in equilibrium, and the following approximations hold for the partnered firms’ first-period strategies and total profits:

\[ r_{SP}^{SP, O} \approx 0.166857t, \quad p_{SP}^{SP, O} \approx c + 0.320310t, \quad \Pi_{SP}^{SP, O} \approx 0.136287t. \]

Approximations for the first-period strategies and total profits of the unpartnered firms are as follows:

\[ r_{NP}^{SP, O} \approx 0.240963t, \quad p_{NP}^{SP, O} \approx c + 0.335046t, \quad \Pi_{NP}^{SP, O} \approx 0.125007t. \]

The following proposition thus holds.

**Proposition 1.1.** When a single partnership exists between nonadjacent firms, partnered firms realize higher profits in equilibrium than the equilibrium profit level under no partnerships, and unpartnered firms realize lower profits in equilibrium than the equilibrium profit level under no partnerships.
To understand the intuition behind this result, it is helpful first to understand why partnered firms do not merely offer half of the reward of and the same price as the unpartnered firms in equilibrium. Superficially, it might seem as if there exists such an equilibrium where sales and profits are identical to the no-partnerships equilibrium. After all, the only difference is that consumers who buy from the partnership in period one benefit if they are located near either partner in period two, rather than only near the firm from whom they’ve purchased previously. It can be observed, however, that if the partnership merely matches the price of the non-partnered firms and offers half of their reward, the partnered firms will not maintain sales equal the level which prevails under no partnerships.

Consider a strategy profile under this partnership network where firms all announce the no-partnerships equilibrium prices $p^*$, the unpartnered firms offer the no-partnerships reward $r^*$, and the partnered firms offer half of this reward, $\frac{r^*}{2}$. Suppose, for the sake of argument, that consumers (erroneously) believe that the gross benefit of being eligible for the partnership is the same as that of being eligible for an unpartnered firm’s reward, so that each firm sells to one-fourth of the market. Given consumers’ behavior, firms will then still have identical nominal prices (equal to the no-partnerships equilibrium prices) in period two. Furthermore, the expected benefit of the partnership’s reward conditional on becoming located within $\frac{1}{8}$ of a partner given these rewards and prices is the same as the expected benefit of a firm’s reward conditional on becoming located within $\frac{1}{8}$ of that firm under no partnerships. In either case, the consumer would have purchased from the preferred firm even if not eligible for the reward, so they receive the full reward amount as extra benefit.
Under the partnership, the reward amount is halved, but there are two such regions the consumer may find herself in.

Examining the benefit to the consumers whose buying decision is influenced by the switching cost reveals the consumers’ folly. Consider the period-two location of the consumer who is eligible for neither the partnership’s reward nor firm 4’s reward and is indifferent between purchasing from firm 1 and firm 4, and the period-two location of the consumer who is eligible for the partnership’s reward and is indifferent between these firms. Given these rewards and prices, the former consumer is located at a distance of $\frac{1}{8}$ from firm 1, and the latter consumer is located $\frac{1}{8} + \frac{r^*}{2t}$ from firm 1. The expected benefit of the reward conditional on being located between these two consumers increases from 0 to $\frac{r^*}{2}$ linearly over the distance between them of $\frac{r^*}{4t}$, so the expected benefit conditional on becoming located in this interval is $\frac{r^*}{16t}$. There are four such intervals where the switching cost influences the buyer’s decision, so the total expected benefit conditional on being swayed by the switching cost imposed by the reward is $\frac{r^*}{4t}$. This is only half of the expected benefit conditional on being swayed by the switching cost under no partnerships, as discussed in the preceding section. Evaluating the final terms in the above expressions for $GB_{13}$ and $GB_{4}$ at the proposed reward amounts illustrates this point. Hence, the consumers have erred in their belief that the firms’ rewards are equally attractive. In fact, if these strategies are used, the reward of the partnered firms is worth less in expectation.

14In the language of that discussion, relative to under no partnerships, the difference between, say, $x_{14}$ and $z_{14}$ has been halved. The reward amount itself has also been halved, though the consumer eligible for the partnership’s reward benefits from the reward in twice as many such intervals. As a result, the expected benefit of the partnership’s reward conditional on having one’s purchasing decision changed by the reward falls.
than the reward of the unpartnered firms. Thus, merely halving the reward amount of the unpartnered firms will result in lower sales for the partnered firms.

In equilibrium, the partnership offers a reward higher than half of that of the other firms in equilibrium. In light of the previous discussion, this should not be surprising. On the other hand, that the partnership does actually benefit the partnered firms can be understood by noticing that the primary benefit of the partnership to the member firms is the increased power of increasing the reward to attract consumers. In period one, additional consumers may be attracted in two ways: a decrease in price, or an increase in the reward amount. As noted in the previous section, introducing rewards decreases the price sensitivity of consumers. Forming a partnership, on the other hand, increases the “reward sensitivity” of consumers: an increase in the reward offered by the partnership increases the benefit realized by consumers who buy from either firm in the following period, while an increase in the reward offered by an unpartnered firm increases the benefit realized by only repeat customers of that firm. Thus, all else equal, a marginal increase in the reward of the partnership is more powerful in attracting consumers than a marginal reward increase by an independent firm. This advantage turns out to be decisive: the partnered firms do not offer merely a high enough reward to maintain sales equal to their opponents; rather, they increase the reward enough to acquire larger market shares than their rivals while at the same time charging higher prices than under no partnerships.

Of note is that rewards possess a degree of strategic complementarity. When both firms offer high rewards, the price sensitivity of consumers decreases. Thus, the non-partnered firms, realizing that the partnered firms will offer a reward high enough to sell to larger market shares, respond by increasing their reward amount above the
no-partnerships level. This allows them to charge a higher price as well, higher than even that of the partnered firms. Although their profits fall below the level under no partnerships due to their disadvantage, this tactic permits them to still do better than when rewards are entirely absent from the model.

So far, it can be seen that if a single partnership is formed, the partnering firms realize higher profits than when no partnerships exist. However, the firms who are not partnered suffer as a result of the partnership. An interesting question is then whether forming a second partnership will benefit firms that are facing a partnership between their common neighbors. I next consider the equilibrium that will prevail if this second partnership is formed.

1.4.2 Two Partnerships Between Nonadjacent Firms

Consider the scenario when two partnerships between nonadjacent firms exist. That is, firms 1 and 3 are partnered and their common neighbors, firms 2 and 4, form another partnership. In the second period, the market is segmented into only two groups: $e_{13}$ of the consumers have previously purchased from either firm 1 or firm 3, while the remaining $1 - e_{13}$ previously purchased from firms 2 or 4. Each of these segments is eligible for the reward offered by the appropriate partnership, and both are distributed uniformly. Importantly, note that any two adjacent firms now locally compete over a pool of consumers where each consumer is eligible for exactly one of their rewards. Adopting similar notation to the previous sections, firm 1’s period-two profits $\Pi_1^2$ can be written as

$$
(p_1^2 - c) \left[ e_{13} (x_{14} + x_{12}) + (1 - e_{13}) (y_{14} + y_{12}) \right] - r_{13} \left[ e_{13} (x_{14} + x_{12}) \right],
$$
where
\[ x_{ij} = \frac{1}{8} + \frac{p_j - p_i + r_i}{2t}, \quad y_{ij} = \frac{1}{8} + \frac{p_j - p_j - p_i}{2t}, \]
and \( r_k \) is the reward offered by the partnership to which firm \( k \) belongs (either \( r_{13} \) or \( r_{24} \)). Solving firms’ first-order conditions simultaneously yields the subgame equilibrium prices:

\[ p_1^2 = p_2^2 = c + \frac{t}{4} + e_{13} r_{13}, \quad p_2^2 = p_4^2 = c + \frac{t}{4} + (1 - e_{13}) r_{24}. \]

These equilibrium prices are identical to those under only a single partnership, except that firms 4 and 2 now also share a common pool of eligible consumers and a common reward. Once more, the subgame equilibrium profits look familiar:

\[ \Pi_1^2 = \Pi_3^2 = \frac{t}{16} - \frac{e_{13} r_{13}}{2t} (2(1 - e_{13}) r_{13} + 2(1 - e_{13}) r_{24}), \]
\[ \Pi_4^2 = \Pi_2^2 = \frac{t}{16} - \frac{(1 - e_{13}) r_{24}}{2t} (2 e_{13} r_{24} + 2 e_{13} r_{13}). \]

As usual, in the first period, consumers and firms anticipate the subgame equilibrium in the second period. The period-one expected gross benefits of being eligible for different rewards in period two are again of utmost importance to consumers in the first period. Under two partnerships, there are two such benefits to be calculated: all consumers will be eligible for one partnership’s reward or the other. The expected benefits are familiar-looking and are now given by

\[ GB_{13} = \frac{r_{13}}{2} + \frac{r_{13}}{t} (2 (1 - e_{13}) r_{24} - 2 e_{13} r_{13}) + \frac{r_{13}^2}{t}, \]
\[ GB_{24} = \frac{r_{24}}{2} + \frac{r_{24}}{t} (2 e_{13} r_{13} - 2 (1 - e_{13}) r_{24}) + \frac{r_{24}^2}{t}. \]

Again, let \( n_{ij} \) denote the distance from \( i \) to the consumer indifferent between purchasing from \( i \) and its neighbor \( j \). Then the mass of consumers eligible for the reward
offered by firm 1’s partnership (the sum of the first-period sales of firms 1 and 3) can be written
\[ e_1 = n_{14} + n_{12} + n_{32} + n_{34}, \]
and these four distances are defined implicitly by
\[
F_{14} \equiv \frac{t}{4} - 2tn_{14} - p_1^1 + p_1^4 + GB_{13} - GB_{24},
\]
\[
F_{12} \equiv \frac{t}{4} - 2tn_{12} - p_1^1 + p_2^1 + GB_{13} - GB_{24},
\]
\[
F_{34} \equiv \frac{t}{4} - 2tn_{34} - p_3^1 + p_1^4 + GB_{13} - GB_{24},
\]
\[
F_{32} \equiv \frac{t}{4} - 2tn_{32} - p_3^1 + p_2^1 + GB_{13} - GB_{24}.
\]

Once more, the implicit function theorem can be used to derive the relevant rates of change of these cutoffs with respect to prices and rewards in equilibrium. These expressions can be used to calculate firms’ equilibrium strategy as follows. Firm 1’s total profits are given by
\[
\Pi_1 = (p_1^1 - c) (n_{14} + n_{12}) + \frac{t}{16} - \frac{e_{13}r_{13}}{2t} (2(1 - e_{13})r_{13} + 2(1 - e_{13})r_{24}).
\]

Under symmetry, \( r_{13} = r_{24} = r, \) \( p_i^1 = p^1 \) for all \( i, \) and each cutoff \( n_{ij} = \frac{1}{8}. \) As usual, imposing this symmetry on firm 1’s first-order conditions and utilizing the expressions for the partial derivatives of the indifferent consumers’ locations with respect to prices and rewards (also evaluated under symmetry) allows for the system to be solved for the unique equilibrium price and reward amount. The following approximations hold in equilibrium for firms’ reward amounts, first-period prices, and total profits under this “dual-partnership” network where nonadjacent firms are partnered:
\[
r^{DP,O} \approx 0.143183t, \quad p^{DP,O} \approx c + 0.285225t, \quad \Pi^{DP,O} \approx 0.123556t.
\]

Interestingly, the formation of a second partnership injures the profits of all firms, whether previously partnered or not, yielding the following proposition:
**Proposition 1.2.** When two partnerships between nonadjacent firms exist, all firms realize lower profits in equilibrium than the equilibrium profit levels of both partnered and unpartnered firms under a single such partnership.

In fact, both firms are now worse off than when partnerships are completely absent, as can be seen by combining this result with Proposition 1.1.

**Corollary 1.3.** When two partnerships between nonadjacent firms exist, all firms realize lower profits in equilibrium than the equilibrium profit level under no partnerships.

The intuition behind this result involves the reward and price sensitivity of consumers under partnerships. Namely, firms are back on even footing when it comes to the attractiveness of their rewards: a reward from either partnership provides a benefit to consumers who become located near either member of that partnership in the second period. Recall that under a single partnership, the partnered firms offered a reward higher than half of the no-partnership equilibrium reward amount. For a similar reason, when a second partnership is formed, the previously unpartnered firms now offer a reward above half of the reward they offered when they were not partnered. The firms which were partnered previously anticipate such a response and, importantly, no longer hold an advantage in consumer reward sensitivity, which causes them to reduce their reward. Appropriately, the equilibrium reward amount lower than that offered by the partnered firms when only one partnership exists.

Because both partnerships are incentivized to offer lower reward amounts than were offered by the single partnership, the price sensitivity of consumers buying from partnerships increases. As a result, prices fall when a second partnership is formed.
The effect of the partnerships in this case actually reverses the usual, competition-dampening effect of rewards. In terms of profits and prices, competition is actually enhanced when two partnerships, rather than just a single partnership, exist.\textsuperscript{15} An important implication of this is that, given an existing partnership between nonadjacent firms, the formation of a second such partnership is undesirable as it hurts the profit level of every firm, including the newly partnering firms.

Thus far, partnerships have been assumed to exist between firms who do not compete directly in prices equilibrium. That is, for the partnerships described so far, given prices and rewards, an incremental change in price will not allow one partner to attract consumers from its partner; rather, only firms outside of the partnership will be affected. I now consider partnerships where adjacent firms who compete locally are engaged in partnerships.

### 1.5 Partnerships Between Adjacent Firms

When there is at least one partnership between adjacent firms, without loss of generality, I will henceforth assume that firms 1 and 2 have partnered. Under a single partnership, the remaining firms (3 and 4) are not partnered. Under two partnerships, firms 3 and 4 are also partnered with one another. In either case, firm 1 is \textit{ex ante} symmetric to its partner while firm 4 is \textit{ex ante} symmetric to firm 3. For consistency with the preceding section, I focus on the optimization problems of firms 1 and 4.

\textsuperscript{15}Note that consumers still experience decreased price sensitivity relative to the no-rewards scenario. This can be seen by observing that prices are higher under two partnerships than under no rewards.
1.5.1 A Single Partnership Between Adjacent Firms

As usual, I derive the subgame equilibrium of the second period before turning to the period-one analysis. Under a single partnership between adjacent firms, the market is comprised of three groups in period two: a proportion $e_{12}$ purchased from firm 1 or firm 2 in the first period and is eligible for a reward $r_{12}$ from either of these firms in period two; likewise, proportion $e_4$ (respectively $e_3$) purchased from firm 4 (3) and are eligible for the reward $r_4$ ($r_3$) should they repeat purchase. Again, I focus on the optimization problems of firms 1 and 4. Using familiar notation, firm 1’s period-two profits are given by

$$\Pi_1 = (p_1^2 - c) \left[ e_{12} (x_{14} + z_{12}) + e_4 (y_{14} + z_{12}) + e_3 (z_{14} + z_{12}) \right] - r_{12} \left[ e_{12} (x_{14} + z_{12}) \right],$$

where

$$x_{ij} = \frac{1}{8} + \frac{p^2_i - p^2_j + r_i}{2t}, \quad y_{ij} = \frac{1}{8} + \frac{p^2_j - r_j + p^2_i}{2t}, \quad z_{ij} = \frac{1}{8} + \frac{p^2_j - p^2_i}{2t},$$

where $r_k$ is the discount granted to eligible customers of firm $k$. Importantly, note that consumers who are both eligible for the reward of the partnership and deciding between firms 1 and 2 are not influenced by the reward amount, as it applies regardless of which firm they choose. However, they must still receive the promised discount. Firm 4’s profit function is

$$\Pi_4 = (p_4^2 - c) \left[ e_4 (x_{43} + x_{41}) + e_3 (y_{43} + z_{41}) + e_{12} (z_{43} + y_{41}) \right] - r_4 \left[ e_4 (x_{43} + x_{41}) \right].$$

Firm 2’s profit function is similar to that of firm 1, while firm 3’s profit function is similar to that of firm 4. Solving for the subgame equilibrium yields the usual prices, in terms of eligible consumers and rewards:

$$p_1^2 = p_2^2 = c + \frac{t}{4} + e_{12}r_{12}, \quad p_3^2 = c + \frac{t}{4} + e_3r_3, \quad p_4^2 = c + \frac{t}{4} + e_4r_4.$$
In particular, firms react exactly the same as under no partnerships, taking into account that partnered firms share a pool of eligible consumers. The subgame equilibrium profits of firms 1 and 4 are given by

\[ \Pi_1^2 = \frac{t}{16} - \frac{e_{12} r_{12}}{2t} \left( 2 (1 - e_{12}) r_{12} - (1 - e_{12}) r_{12} + e_4 r_4 \right), \]
\[ \Pi_4^2 = \frac{t}{16} - \frac{e_4 r_4}{2t} \left( 2 (1 - e_4) r_4 + e_3 r_3 + e_{12} r_{12} \right). \]

The profits of firm 4 are very similar to those under no partnerships. However, firm 1’s profits appear different. Firm 4 is not partnered with an adjacent firm, so the usual effects of rewards on profits apply. However, firm 1 is partnered with firm 2, so a smaller proportion of consumers who are eligible for firm 1’s reward purchase from firm 1 than would otherwise, because they are also eligible for the reward from firm 2. Thus, the profit lost due to the rewards payout falls, which is captured by the change in sign of the middle term within the parentheses.

In period one, consumers once again consider three possible expected gross reward benefits, as they will be eligible for the reward of either the partnership, firm 3, or firm 4. The gross expected benefit of being eligible for the partnership’s reward is

\[ GB_{12} = \frac{r_{12}}{4} + \frac{r_{12}}{4} + \frac{r_{12}}{2t} (e_4 r_4 + e_3 r_3 - 2e_{12} r_{12}) + \frac{r_{12}^2}{2t}, \]

and the gross expected benefit of being eligible for the rewards of the other firms are given by

\[ GB_3 = \frac{r_3}{4} + \frac{r_3}{2t} (e_{12} r_{12} + e_4 r_4 - 2e_3 r_3) + \frac{r_3^2}{2t}, \]
\[ GB_4 = \frac{r_4}{4} + \frac{r_4}{2t} (e_{12} r_{12} + e_3 r_3 - 2e_4 r_4) + \frac{r_4^2}{2t}. \]

The first two terms in \( GB_{12} \) are intentionally not combined to illustrate the intuition behind these expressions. These expressions are similar to those under no partnerships, and equivalent expressions similar to (1.4) could be formulated here as well.
Firm 4’s reward, for instance, is only beneficial if the consumer ends up located between firm 4 and firm 3 or between firms 4 and 1, and the expected benefit conditional on being in one of these intervals is a portion of the reward amount. Consumers eligible for the reward of the partnership realize a similar “partial” benefit conditional on being located between firms 1 and 4 or between firms 2 and 3. The difference in the appearance of the expression for the partnership’s reward and those for the rewards of other firms is due to the fact that consumers eligible for the partnership’s reward who end up located between firms 1 and 2 will benefit in the full amount of the reward no matter where they are located within this interval. This happens with probability $\frac{1}{4}$, and the benefit conditional on this occurrence is $r_{12}$. The first term in $GB_{12}$ captures this additional effect.

Under a single partnership between adjacent firms, the consumers in the first period who are located between the partnered firms are able to rationally ignore the above expected gross benefits when choosing between firms 1 and 2. This is a crucial point, and will be returned to later. Regardless of whom these consumers purchase from, they will be eligible for the same reward in the following period. Thus, the distance from firm 1 to the indifferent consumer in this interval is

$$n_{12} = \frac{1}{8} - \frac{p_{21} - p_{11}}{2t}.$$  

However the other indifferent consumers’ locations depend on the differences in the expected benefits of reward eligibility as well as period-one prices. Adopting the usual notation, firm 1’s first-period sales are given by $n_{14} + n_{12}$ and firm 2’s sales are $n_{21} + n_{23}$, so that $e_{12} = \frac{1}{4} + n_{14} + n_{23}$. Also let firm 4’s sales be denoted by $e_4 = \frac{1}{2} - n_{14} - n_{34}$, and firm 3’s sales by $e_3 = \frac{1}{4} - n_{23} + n_{34}$.
These relevant distances to indifferent consumers are implicitly defined by the following functions:

\[
F_{14} \equiv \frac{t}{4} - 2tn_{14} - p_1^1 + p_2^1 + GB_{12} - GB_4,
\]

\[
F_{23} \equiv \frac{t}{4} - 2tn_{23} - p_2^1 + p_3^1 + GB_{12} - GB_3,
\]

\[
F_{34} \equiv \frac{t}{4} - 2tn_{34} - p_3^1 + p_4^1 + GB_3 - GB_4.
\]

As above, the implicit function theorem is used to derive the equilibrium marginal effects of changes in prices and reward amounts on these distances, which are used in evaluating firms’ first-order conditions. The profit functions of firms 1 and 4 are as follows:

\[
\Pi_1 = (p_1^1 - c) \left( n_{14} + n_{12} \right) + \frac{t}{16} - \frac{e_1 r_{12}}{2t} \left( (1 - e_1) r_{12} + e_4 r_4 \right),
\]

\[
\Pi_4 = (p_4^1 - c) \left( \frac{1}{2} - n_{14} - n_{34} \right) + \frac{t}{16} - \frac{e_4 r_4}{2t} \left( 2 (1 - e_4) r_4 + e_3 r_3 + e_1 r_{12} \right).
\]

The process of solving for the equilibrium of this induced game is identical to that used for the network of a single partnership of nonadjacent firms. Imposing symmetry (in this case, symmetry implies that \(p_1^1 = p_2^1, p_4^1 = p_3^1,\) and \(r_3 = r_4,\) hence \(n_{12} = n_{34} = \frac{1}{8}\) and \(n_{14} = n_{12}\)) on the first-order conditions for prices and rewards and solving the resulting system yields the unique symmetric equilibrium. The following approximations hold for firms’ first-period strategies and total profits:

\[
r_{SP,A}^{SP,A} \approx 0.187798 t, \quad p_{SP,A}^{SP,A} \approx c + 0.273209 t, \quad \Pi_{SP,A}^{SP,A} \approx 0.128894 t,
\]

\[
r_{NP}^{SP,A} \approx 0.226093 t, \quad p_{NP}^{SP,A} \approx c + 0.295832 t, \quad \Pi_{NP}^{SP,A} \approx 0.117826 t.
\]

Additionally, the indifferent consumers located between firms 1 and 2 and between firms 3 and 4 are located midway between these firms, while the indifferent consumer
between partnered and unpartnered firms is located at a distance of $0.143270 > 1/8$ from the partnered firm, so partnered firms do have higher sales in period one. Comparing these equilibrium values to those under no partnerships and the case of a single partnership between nonadjacent firms yields the following stronger version of Proposition 1.1.

**Proposition 1.4.** When any single partnership exists, partnered firms realize higher profits in equilibrium than the equilibrium profit level under no partnerships, and unpartnered firms realize lower profits in equilibrium than the equilibrium profit level under no partnerships.

However, under a single partnership, the equilibrium profit level of the partnered firms is higher when the partnered firms are nonadjacent firms than when they are adjacent firms.

The intuition for this result is relatively straightforward. Recall that when adjacent firms are not partnered, competition between them is dampened by rewards (regardless of whether a partnership exists elsewhere). In the first period, consumers experience decreased price sensitivity due to the different rewards offered; in the second period, some consumers are eligible for the reward of each firm, which generates switching costs. However, as noted above, when adjacent firms are partnered, the consumers located between them are completely unaffected by rewards. This erodes the benefit that partners realize from rewards, and prices fall as a result of the heightened competition for these consumers. Additionally, each partner is located next to an unpartnered firm with whom they must also compete, and firms can only set a single price. As a result, competition is heightened in every quadrant of the city relative to when nonadjacent firms are partnered.
However, the partnered firms still realize an advantage in consumer reward sensitivity in those quadrants where consumers consider the partnership’s reward. Even though consumers located between the partnered firms in period two are not influenced by the reward, they still benefit from the reward payout, and consumers in period one do indeed take this benefit into consideration. This heightened power of the reward benefit is still useful to the partnered firms. Relative to the no-partnership case, the increased appeal of the partnership’s reward still allows the partnership to attract a larger number of consumers with a given reward increase in the regions where consumers consider the partnership’s reward when making purchasing decisions. Indeed, in equilibrium, partnered firms have higher sales and profits than unaffiliated firms. However, the partnership’s reward is now completely ineffective in attracting consumers in two quadrants of the city, which limits the effectiveness of rewards in dampening competition. Hence, the benefit of forming a partnership with an adjacent firm is less than that of a partnership with the nonadjacent firm.

The preceding analysis has demonstrated that forming a partnership with a non-adjacent firm results in higher profits than forming a partnership with a local competitor. However, an additional goal is to examine the partnership networks which are core stable. It is known that if a firm’s neighbors have formed a partnership then that firm does not wish to partner with the remaining unaffiliated firm. The next subsection will show that a this result does not hold when an unaffiliated firm faces a partnership which includes only one of its neighbors.
1.5.2 Two Partnerships Between Adjacent Firms

The final partnership network that remains to be considered consists of two partnerships between adjacent firms. Suppose without loss of generality that firms 1 and 2 comprise one partnership and that firms 3 and 4 constitute the other.

In period two, there are two groups of consumers uniformly distributed around the city: proportion \( e_{12} \) have purchased from firm 1 or firm 2 and are eligible for the reward of that partnership (denoted \( r_{12} \)), while the remainder have purchased from one of the other partnership’s members and are eligible for the reward of that partnership (denoted \( r_{34} \)). Firm 1’s period-two profits are given by

\[
\Pi_1^2 = (p_1^2 - c) \left[ e_{12} (x_{14} + z_{12}) + (1 - e_{12}) (y_{14} + z_{12}) \right] - r_{12} \left[ e_{12} (z_{12} + x_{14}) \right],
\]

where

\[
x_{ij} = \frac{1}{8} + \frac{p_j^2 - p_i^2 + r_i}{2t}, \quad y_{ij} = \frac{1}{8} + \frac{p_j^2 - r_j - p_i^2}{2t}, \quad z_{ij} = \frac{1}{8} + \frac{p_j^2 - p_i^2}{2t},
\]

where \( r_k \) denotes the reward offered to eligible customers of firm \( k \). The profit functions of the remaining firms are similar; again, firms are merely adjusting to the change in the eligibility of consumers for rewards. The subgame equilibrium is given by

\[
p_1^2 = p_2^2 = c + \frac{t}{4} + e_{12} r_{12}, \quad p_3^2 = p_4^2 = c + \frac{t}{4} + (1 - e_{12}) r_{34},
\]

as usual. The period-two profits of firm 1 are then

\[
\Pi_1^2 = \frac{t}{16} - \frac{e_{12} r_{12}}{2t} \left( 2 (1 - e_{12}) r_{12} - (1 - e_{12}) r_{12} + (1 - e_{12}) r_{34} \right).
\]

The profits of firm 2 are identical, and the profits of firms 3 and 4 are similar.

In period one, there are now two expected gross benefits to consider: consumers will be eligible for the reward of one partnership or the other. Adopting the usual
notation, the expected gross benefits provided by the partnerships’ rewards are given by

\[
GB_{12} = \frac{r_{12}}{2} + \frac{r_{12}}{2t} (2 (1 - e_{12}) r_{34} - 2 e_{12} r_{12}) + \frac{r_{12}^2}{2t},
\]

\[
GB_{34} = \frac{r_{34}}{2} + \frac{r_{34}}{2t} (2 e_{12} r_{12} - 2 (1 - e_{12}) r_{34}) + \frac{r_{34}^2}{2t}.
\]

These expressions are analogous to the gross benefit of the single partnership’s reward in the previous subsection, and they have an identical interpretation. Now, once again, consumers trying to decide between purchasing from two partnered firms can rationally disregard these gross benefits. That is, all firms compete with their partner for a group of consumers who care only about prices. Thus, using the usual notation for distances to indifferent consumers,

\[
n_{12} = \frac{1}{8} + \frac{p^1_2 - p^1_1}{2t}, \quad n_{34} = \frac{1}{8} + \frac{p^1_4 - p^1_3}{2t}.
\]

However, the indifferent consumers deciding between firms not partnered with one another do indeed weigh the expected gross benefits when making their purchasing decisions. The normal implicit functions characterize the locations of these consumers:

\[
F_{14} \equiv \frac{t}{4} - 2tn_{14} - p^1_1 + p^1_4 + GB_{12} - GB_{34}, \quad F_{23} \equiv \frac{t}{4} - 2tn_{23} - p^1_2 + p^1_3 + GB_{12} - GB_{34}.
\]

Again, I use the implicit function theorem to solve for the equilibrium marginal effects of changes in prices and reward amounts on the locations of the indifferent consumers, which are used to solve the system of first-order conditions. Firm 1’s total profits are

\[
\Pi_1 = (p^1_1 - c) (n_{14} + n_{12}) + \frac{t}{16} - \frac{r_{12}^2}{2t} e_{12} (1 - e_{12}) (r_{12} + r_{34}).
\]

Solving the system of first-order conditions under symmetry (so that \( p^1_i = p^1 \) for all \( i \), \( r_{12} = r_{34} = r \), and \( n_{ij} = \frac{1}{8} \) for each indifferent consumer) yields the unique symmetric equilibrium. The following approximations hold for the equilibrium first-period
strategies and total profits of firms in the “dual-partnership” case when partnerships are between adjacent firms:

\[ r^{DP,A} \approx 0.155407t, \quad p^{DP,A} \approx c + 0.255630t, \quad \Pi^{DP,A} \approx 0.120370t. \]

Comparing these equilibrium values with those from previous sections provides the following result.

**Proposition 1.5.** When two partnerships between adjacent firms exist, all firms realize higher profits in equilibrium than the equilibrium profits of unpartnered firms under a single such partnership; however, all firms realize lower profits in equilibrium than the equilibrium profits of partnered firms under a single such partnership.

Additionally, under dual partnerships, the common equilibrium profit level of firms is higher when the partnered firms are nonadjacent firms than when they are adjacent firms.

This result also permits Corollary 1.3 to be strengthened, as follows.

**Corollary 1.6.** Under any dual-partnership network, firms realize lower equilibrium profits than when no partnerships exist.

The profits of firms in equilibrium are less than those of the partnered firms under a single partnership, yet higher than those of the unpartnered firms. Thus, when faced with a partnership between two adjacent firms, unaffiliated firms can improve their situation by partnering with one another. This is because partnering still increases the power of firms to attract consumers in the first period by increasing rewards. Even though the effect of rewards on consumers located between previously unaffiliated firms disappears when those firms choose to partner, the increased effectiveness of
the partnership’s reward in attracting consumers in the other regions makes such a partnership beneficial for the same reason that forming the first such partnership increases profits for the partnering firms. In this case, forming a second partnership allows the previously unpartnered firms to better compete with their partnered rivals without increasing competition too much.

To understand why a second partnership is desirable to unpartnered firms in this case, but is not desirable if two nonadjacent firms are partnered, consider the effect of increasing the reward on the sales of, say, firm 1 under either-dual partnership network. When two partnerships exist between nonadjacent firms, an increase in the reward offered by firm 1’s partnership attracts additional consumers to firm 1 who are located between firm 1 and either of its neighbors. If firm 1 and firm 2 are partnered under a dual-partnership network, however, the same reward increase only attracts consumers located between firms 1 and 4. Thus, competition in this sense is lessened in period one under partnerships between adjacent firms relative to nonadjacent firms. This allows for a second partnership to help unpartnered firms in this case, even though the opposite is true when nonadjacent firms are partnered.

Additionally, under dual partnerships, profits are lower when adjacent firms are partnered than when nonadjacent firms partner. The intuition for this result is identical to the reasoning behind why nonadjacent partners fare better than adjacent partners under a single partnership. Specifically, the effects that rewards have in decreasing competition between adjacent firms are completely eliminated in the region between neighboring partnered firms, which puts pressure on all firms to reduce prices. This can be observed by noting that prices are indeed lower under adjacent

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dual partnerships than nonadjacent dual partnerships. The end result is a reduction in profits.

Now that the equilibria of the competition game induced by each partnership network are well-understood, I turn to the partnership formation process itself. The following section introduces core stability for partnership networks and identifies which networks satisfy this criterion.

1.6 Core Stability of Partnership Networks

The analysis to this point has taken firms partnerships as given. However, in a realistic setting, firms will ultimately decide whether or not to partner with one another. It seems reasonable in this instance to expect that a partnership formation process should result in partnership networks which are stable in a specific way. In particular, I will be most interested in which networks are core stable; first, I formalize the notion of partnership networks.

A partnership network \( P \) is a partition of the set of firms \( \{1, 2, 3, 4\} \) where each set in \( P \) contains at most two elements. If \( \{i, j\} \in P \), then \( i \) and \( j \) are partnered with one another under \( P \). If firm \( i \) is contained in a singleton set of \( P \), then \( i \) is unpartnered. Let \( \Pi_i(P) \) represent the symmetric subgame-perfect equilibrium profit level realized by firm \( i \) in the game induced by partnership network \( P \).

**Definition 1.1 (Core Stability).** A partnership network \( P \) is core stable if there does not exist any set of firms \( F \) and partnership network \( P' \) such that:

(i) \( \Pi_i(P') \geq \Pi_i(P) \) for all \( i \in F \) (with at least one strict inequality),

(ii) if \( \{i, j\} \in P' \) but \( \{i, j\} \notin P \), then \( i \in F \) and \( j \in F \), and

(iii) if \( \{i, j\} \notin P' \) but \( \{i, j\} \in P \), then \( i \in F \) or \( j \in F \).
<table>
<thead>
<tr>
<th>Partnership Network</th>
<th>Non-partnered Firms</th>
<th>Partnered Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Partnerships</td>
<td>$\Pi^* \approx 0.127910t$</td>
<td>–</td>
</tr>
<tr>
<td>Nonadjacent</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single-Partnership</td>
<td>$\Pi_{NP}^{SP,O} \approx 0.125007t$</td>
<td>$\Pi_{P}^{SP,O} \approx 0.136287t$</td>
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<td>Dual-Partnership</td>
<td>–</td>
<td>$\Pi_{P}^{DP,O} \approx 0.123556t$</td>
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<tr>
<td>Adjacent</td>
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<td>Single-Partnership</td>
<td>$\Pi_{NP}^{SP,A} \approx 0.117826t$</td>
<td>$\Pi_{P}^{SP,A} \approx 0.128894t$</td>
</tr>
<tr>
<td>Dual-Partnership</td>
<td>–</td>
<td>$\Pi_{P}^{DP,A} \approx 0.120370t$</td>
</tr>
</tbody>
</table>

Table 1.1: Equilibrium Profits Under Within-Market Partnership Networks

If $F$ and $P'$ exist which satisfy (i–iii) above for some $P$, so that $P$ is not core stable, then the firms $F$ are said to block network $P$.

In this context, a partnership network $P$ is core stable if there is no group of firms who each prefer the network $P'$ to $P$ and who can change the network from $P$ to $P'$ without the cooperation of the remaining firms. On the other hand, if such a group of firms does exist, then it blocks $P$.

So that the reader may easily assess the desirability of the various partnership networks to firms, Table 1.1 displays firms’ equilibrium profits of each firm under each possible partnership network. The following proposition, which should not be surprising given the discussion that has been presented thus far, serves to summarize the results of the preceding sections.

**Proposition 1.7.** Any partnership network consisting of a single partnership between nonadjacent firms is core stable; furthermore, such partnership networks are the only core stable partnership networks.
Proof. To show that a partnership network consisting of a single partnership between nonadjacent firms is core stable, first note that the partnered firms under such a network strictly prefer this network to any other network. Hence, any potential blocking group may contain only the unpartnered firms. The only way these two firms can change the existing network is by forming a partnership; by Proposition 1.2, this makes them worse off. Hence, this network is core stable.

To complete the proof, it suffices to show that the networks satisfying the following are blocked:

(a) no partnerships,
(b) a single partnership between adjacent firms,
(c) two partnerships between adjacent firms, and
(d) two partnerships between nonadjacent firms.

First, observe that any two nonadjacent firms block the networks which satisfy (a), (b), or (c). Note that, in any of these cases, dissolving any existing partnerships the blocking firms are a part of and forming a partnership between the blocking firms results in a single partnership between the (nonadjacent) blocking firms. Again, such a network is strictly preferred to any other by those who are partnered under it. This leaves only the networks satisfying (d), which are also blocked by any two nonadjacent firms; by Proposition 1.2, dissolving their partnership makes the newly unpartnered firms better off.

This result reinforces the intuition from previous sections. That partnerships of some sort will prevail prior to competition should be clear. Additionally, because partnerships between neighboring firms are relatively ineffective compared to partnerships between nonadjacent firms, the fact that partnerships between local competitors are
not part of any stable networks is not surprising. However, as has been discussed previously, the effectiveness of the most attractive partnerships, those between firms that are not local competitors, declines to the point that all firms are hurt if too many of these partnerships are created. Thus, the partnership formation process should result in only a partial network of partnerships, and these partnerships should not be between local rivals.
Chapter 2: Consumer Rewards Programs in a Multi-Market Setting

2.1 Introduction

In many markets, firms engage in the practice of offering consumers discounts or free products after repeated purchases have been made. Airline frequent-flyer and hotel frequent-stay programs are prominent examples of these so-called “loyalty reward programs,” where enrolled consumers earn free or upgraded services after a certain number of points have been earned through qualifying purchases. Frequent-flyer programs have received much of the attention in the modest literature that exists on consumer rewards programs.

Many rewards programs, however, do not force consumers to continue to purchase the same product in order to redeem a reward. Indeed, such programs often allow consumers to accrue discounts through the purchase of one product which may be redeemed towards the purchase of a different (sometimes completely unrelated) product. One such example is that of grocery store chains offering discounts on gasoline after a certain number of “fuel points” are accrued through the purchase of groceries. Often, the grocery chains in question offer gasoline sales outside of their stores. In
other cases, partnerships exist where fuel points may be applied toward gasoline purchases from non-grocery branded gas stations. In some instances, consumers have the option of redeeming discounts at stations controlled by the grocery chain or a partner. The best-known such example may be the agreement announced in 2010 between The Kroger Company and Shell Oil to honor discounts at gas stations operated by either firm.16

A myriad of other examples exist. Many airlines have partnerships with hotel chains which allow for frequent-flier miles to be converted to frequent-stay points and vice versa. Consumers who book rental cars often have the opportunity to accrue frequent-flier miles while doing so.17 Credit card reward programs often include an option to spend earned “cash back” on gift cards for partnered retailers which carry a face value higher than the foregone amount of cash back.

This chapter examines the gains that entering into these types of partnerships might provide to firms, and analyzes how such partnership networks might arise. Specifically, I consider a model of price competition where a firms compete over two periods in two separate markets for horizontally differentiated products. Consumers are present in both markets, so that firms may reward their customers in one market, which I deem the “upstream” market, by committing to making them eligible for a discount in a later period on products offered in the second market, which I call

16 The Kroger Company owns not only actual Kroger grocery stores, but also the chains Fry’s, Ralphs, Fred Myers, Scott’s, Hilander, QFC, and others. Spending $100 at these stores entitles a patron to a ten-cent discount off the posted per-gallon price of gasoline at both Kroger-controlled stations as well as Shell-branded stations. Additional details can be found at www.kroger.com and www.shell.us.

17 For instance, the website for the Delta Airlines program, SkyMiles, boasts partnerships with Hilton, Starwood, Marriott, and numerous other hotel chains, while offers to earn miles by booking car rentals through Hertz, Alamo, Avis, Budget, and others are listed. Details are available at www.delta.com.
the “downstream” market. To isolate the effects of these “cross-market” rewards, I assume that consumers are only active in one market each period.\textsuperscript{18} Because I wish to analyze the effects of partnering across these markets, I also examine the equilibria of the model when firms offer upstream consumers discounts which apply to not only the purchase of their own downstream products but also the products offered by partner firms who may not have a presence in the upstream market.

In the specification I adopt, for much of the analysis, it matters not whether the upstream firms are themselves present in the downstream market, or if their only influence on the downstream market is through partnerships. Therefore, throughout, I assume that the upstream firms are, in fact, present in both markets, except where noted otherwise. This is again akin to the gasoline discounts mentioned above, where grocery stores offer gasoline sales, but many gas stations exist with no ties to grocery stores in the absence of partnerships.

An important component of the model is that consumers in the upstream market are unsure of their future preferences over firms’ downstream products. This uncertainty captures the flavor of the gasoline discounts example. Consumers who purchase groceries while earning points toward discounts on gasoline may not be certain which gas stations will be most convenient when they need to fill up. The possibility that consumers might later prefer the downstream product of a firm not present in the upstream market (or, more generally speaking, the product of a firm other than that which they patronize initially) makes partnerships attractive to firms.

\textsuperscript{18}It should be noted that the subgame perfect equilibria of competition under this setup also constitute Markov perfect equilibria of a game lasting an arbitrary number of periods where consumers are always active in both markets, reward eligibility lasts only one period and no within-market rewards are permitted.
While the imposition of rewards programs by firms is not a new idea, allowing partnerships to form across markets invites a number of intriguing novel research questions. Chief among these is an attempt to understand why such partnerships are so popular. In particular, if a firm does have a presence in both markets, are they not encouraging consumers to patronize one of their competitors by forming a partnership? Certainly, allowing a downstream competitor to share in a rewards program creates a stronger competitor in that market. However, the true benefit of rewards programs lies in their ability to decrease consumer price sensitivity in the market where the rewards are generated. Consider the market where rewards are redeemed. Here, firms who are obligated to honor reward discounts are incentivized to raise prices when more consumers are eligible for the promised discount. As such, consumers anticipate such a price increase if a price cut is made by an upstream firm. If a partnership exists, then consumers anticipate a price increase not only by the firm reducing its price but also any partners it might have. Because they may end up purchasing from any firm in the downstream market, the formation of a partnership has the effect of further reducing consumer price sensitivity in the upstream market. This results in higher equilibrium prices and profits in the upstream market and, as it turns out, higher total profits for partnering firms.

Given that multiple different partnership arrangements exist, it is also interesting to ask which of these networks are robust to different partnership formation processes where firms form or dissolve partnerships in an effort to maximize their own profitablility. Due to the structure of the model, upstream firms are able to form partnerships without providing benefits to their upstream competitors. This makes forming such partnerships quite lucrative; as a result, the partnership formation process is volatile.
In particular, if firms are able to simultaneously dissolve existing partnerships and form new partnerships in exchange for compensation, then no network is stable. In other words, given any partnership network, an upstream firm can propose a new network which would make itself and any new partners better off than under the present arrangement. This hinges upon upstream firms luring away the the partners of their competitors until the share of profits allocated to the downstream firms is unreasonably high, at which point the partnered upstream firm is better off severing ties. Of course, this then opens the door for new partnerships. However, if firms are restricted to alter the current arrangement by only leaving a partnership (if they are currently partnered) or joining a partnership (if they are currently not in a partnership), stable networks do exist. In any such network, all downstream firms are partnered with an upstream firm. In a stable network, cutting a partner loose will not benefit an upstream firm, nor will leaving a partnership benefit a downstream firm.

In the absence of partnerships, allowing rewards eligibility to apply across markets is not critically different from imposing rewards that apply to the same market in a future period. Banerjee and Summers (1987) and Caminal and Matutes (1990) respectively consider two-period undifferentiated- and differentiated-product duopoly markets where firms promise discounts to repeat purchasers. Their results are similar to the current model when no partnerships among firms exist. However, these models are limited in that they leave no room for partnerships without destroying the effects of rewards altogether.

These earlier models are extended in the first chapter of this dissertation by examining partnerships which allow for interchangeability of firm rewards that can be
applied to future purchases within the same oligopoly market. The main finding is that such partnership networks can actually increase competition if too many partnerships form, and that firms rationally “opt out” of partnerships to prevent such an occurrence. In the current chapter, the opposite is true: when partnerships allow for rewards to be redeemed with firms who do not compete in the market where they are generated, competition is inhibited further as the number of partners with whom the reward may be redeemed rises. The reason for these conflicting results lies in the fact that when a firm gains a partner who is only present in a market where the reward is redeemed, as in the current chapter, the firm gaining the partner does not also strengthen the position of a competitor in the reward-generating market. This latter effect, while absent here, is present when partnerships form within the same market. Hence, forming partnerships across markets is more useful, provided that a firm’s partners are not present in the upstream market, leading to the current result of stable networks where all firms not present in the reward-generating market engage in partnerships.

Caminal and Claici (2007) extends the duopoly model of Caminal and Matutes (1990) by examining a single market where product differentiation and price competition are modeled identically as in the downstream market in the current chapter, but with a large number of firms who cannot engage in partnerships. They find that firms offering loyalty rewards programs in such a market has the procompetitive effect of lower equilibrium prices and profits; essentially, adding firms to the market can reverse the usual dominant effect of rewards, which is to dampen competition. By contrast, I find that adding firms only to the downstream market and extending partnerships to these firms relaxes competition; furthermore, I find that extending
partnerships to a greater number of firms exaggerates the anticompetitive effects of rewards.\textsuperscript{19} Again, the key difference lies in the fact that when rewards are redeemed in a separate market with partners not present in the reward-generating market, competition in the upstream market cannot be heightened. Thus, the effect of adding firms to the model cannot reverse the competition-dampening effects of rewards if the added firms are only present downstream. The contrast of the results of this chapter with the findings of these related works illustrates that, in some sense, a larger number of firms participating in rewards programs leads to drastically different outcomes depending on the structure of the programs used.

The remainder of the chapter proceeds as follows. Section 2.2 describes the model, and Section 2.3 analyzes the equilibrium of the model when firms do not engage in partnerships. Section 2.4 examines the potential increase in firm profits that results from each upstream duopolist engaging in a partnership with a separate downstream firm. Section 2.5 explores the partnership arrangements where only one upstream duopolist engages in partnerships. Finally, Section 2.6 introduces two notions of network stability and characterizes the set of stable partnership networks under these different solution concepts.

2.2 The Model

I investigate competition in separate markets for two goods, $U$ and $D$. The market for good $U$ is served by two firms, 0 and 1. Firms 0 and 1 are also present in the market for good $D$, but there are two additional firms in this market who do not have a presence in the market for good $U$. Competition occurs over two periods, with

\textsuperscript{19}If there are no firms in the downstream market other than the upstream firms, the model of this chapter is strategically equivalent to the coupon rewards model of Caminal and Matutes (1990).
competition in the market for good $U$ occurring in the first period and competition in
the market for good $D$ in the second. Firms 0 and 1 may choose to offer a reward to
customers who purchase good $U$ from them in the first period, which entitles repeat
patrons a discount on good $D$ in the second period. I will refer to the market for good
$U$ as the “upstream” market, and the market for good $D$ as the “downstream” market,
to capture the idea that consumers earn reward eligibility upstream by purchasing
and then carry it downstream for redemption.\textsuperscript{20}

The products provided by firms are differentiated horizontally as follows. In the
upstream market, firms 0 and 1 are located at opposite ends of a Hotelling unit
interval, with a continuum of consumers uniformly distributed along the interval.
The structure of the downstream market is borrowed from Chen and Riordan (2004).
Specifically, in the downstream market, a Hotelling segment links each possible pair
of firms, so that there are six segments. Each segment in the downstream market
contains a unit mass of consumers, so that the upstream segment contains a mass of
six. Consumers’ positions in the upstream market are independent of their positions
in the downstream market, and when consumers make purchasing decisions in the
upstream market, they are uncertain of their positions in the following period.

Within either market, all firms face a constant marginal cost of production $c_G$ for
good $G \in \{U, D\}$. Consumers have unit demand, so that they will buy a single unit
of a good from a single firm in each period. Furthermore, consumers only have taste
for the products offered by the firms at the endpoints of the segment on which they

\textsuperscript{20}By restricting competition to a single period in each market, I isolate the effects of rewards offered
across markets, rather than similar rewards programs which may offer future discounts towards
purchases made in the same market in which they are earned.
are located. Consumers face linear transportation costs, so that traveling a distance $x$ to buy a unit of good $G$ at price $p$ yields utility of $-xt_G - p$.

Competition occurs in the following manner. At the beginning of the first period, firms $i \in \{0, 1\}$ simultaneously announce prices for good $U$, denoted $p_i^U$, and reward amounts, denoted $r^i$. Consumers then make purchasing decisions in this upstream market. In period two, each firm $j$ simultaneously announces a price for good $D$, denoted $p_j^D$, then consumers make purchasing decisions for this downstream market. A consumer who has purchased good $U$ from firm $i$ in the first period becomes eligible for a discount in the amount of $r^i$ off of the announced price for good $D$ of firm $i$ or any firm partnered with firm $i$. Consumers are assumed to have rational expectations; in the first period, they correctly anticipate the equilibrium prices of the second period. In Section 2.3, I solve the model when no partnerships exist.\(^{21}\)

Beginning with Section 2.4, I allow for partnerships between the firms who are upstream duopolists and the other downstream firms. When partnerships are present, the upstream firms may make a monetary transfer to a downstream firm prior the start of competition in exchange for honoring of the reward discount. Hence, in period two, all firms maximize their own profits, while in period one, a partnering duopolist anticipating the downstream equilibrium of the following period attempts to maximize the joint profits of itself and any partners. Consumers make purchasing decisions which maximize expected utility across both periods. For simplicity, I assume no discounting by firms nor consumers, and I also restrict firms to belonging to at most one partnership.

\(^{21}\)Note that model described here is identical to that of upstream firms only being present in the upstream market and the downstream market being comprised of entirely separate firms. In this interpretation, the no-partnerships case corresponds to the upstream firms each partnering with a single downstream firm.
2.3 The Benchmark

I begin by analyzing the market when firms 0 and 1 do not partner with the remaining firms, so that consumers may only redeem rewards if they buy from the same firm in both markets. Denote the (not partnered) firms not present in the upstream market by $N_0$ and $N_1$. I first analyze the subgame equilibrium of competition in the downstream market, which occurs after a period of competition in the upstream market. I then use backward induction to solve for the equilibrium of competition in the upstream market.

After consumers make purchasing decisions in the upstream market, a proportion $e^0$ of consumers will be eligible for the reward of firm 0 in the downstream market; the rest are eligible for firm 1’s reward. Each of these segments of consumers is uniformly distributed across the six Hotelling segments which exist in the downstream market. Henceforth, I adopt the notation that $x_{i,j}$ represents the distance from firm $i$ to the location of the consumer who is eligible to redeem a reward with firm $i$ yet is indifferent between purchasing from firm $i$ or firm $j$ on the segment joining these two firms. Similarly, $y_{i,j}$ denotes the distance to the indifferent consumer eligible to redeem a reward with neither firm, and $z_{i,j}$ denotes the distance to the indifferent consumer who is eligible to redeem a reward with firm $j$. Thus,

$$x_{i,j} = \frac{1}{2} + \frac{p_{D}^j - p_{D}^i + r^i}{2t_D}, \quad y_{i,j} = \frac{1}{2} + \frac{p_{D}^j - p_{D}^i}{2t_D}, \quad z_{i,j} = \frac{1}{2} + \frac{p_{D}^j - p_{D}^i - r^j}{2t_D}$$

Now, the expression for the profit of firm 0 in the downstream market is

$$\Pi_D^0 = \left(p_{D}^0 - c_D\right) \left[e^0 \left(x_{0,1} + x_{0,N0} + x_{0,N1}\right) + \left(1 - e^0\right) \left(z_{0,1} + y_{0,N0} + y_{0,N1}\right) - r^0 e^0 \left(x_{0,1} + x_{0,N0} + x_{0,N1}\right)\right],$$
while firm 1’s profit is given by

\[ \Pi_1^D = (p_1^D - c_D) \left[ (1 - e^0) (x_{1,0} + x_{1,N0} + x_{1,N1}) \\
+ e^0 (z_{1,0} + y_{1,N0} + y_{1,N1}) \right] - r^1 (1 - e^0) (x_{1,0} + x_{1,N0} + x_{1,N1}). \]

The profits for the remaining firms are

\[ \Pi_{N0}^D = (p_{N0}^D - c_D) \left[ y_{N0,N1} + e^0 (z_{N0,0} + y_{N0,1}) + (1 - e^0) (y_{N0,0} + z_{N0,1}) \right] \]
\[ \Pi_{N1}^D = (p_{N1}^D - c_D) \left[ y_{N1,N0} + e^0 (z_{N1,0} + y_{N1,1}) + (1 - e^0) (y_{N1,0} + z_{N1,1}) \right]. \]

Solving the first-order conditions for firms’ prices simultaneously yields the equilibrium of the downstream-market subgame:

\[ p_0^D = c_D + t_D + e^0 r^0, \quad p_1^D = c_D + t_D + (1 - e^0) r^1, \quad p_{N0}^D = p_{N1}^D = c_D + t_D. \]

Thus, in equilibrium, firms who have promised consumers rewards charge higher prices than when rewards are not used (i.e., \( r^0 = r^1 = 0 \)). Importantly, the amount by which prices are raised is increasing in the proportion of consumers eligible for the promised reward. This happens for two reasons. First, consumers eligible for a discount from a firm are faced with a switching cost if they purchase from a different firm downstream, as they will not qualify for the competitor’s discount. Therefore, firms raise prices to take advantage of these “locked-in” consumers, and the incentive to do so rises when there are more of these consumers. Second, eligible consumers, who are present across each segment of the market, must be given the promised discount, which can equivalently be viewed as a cash payment. In effect, firms face an increase in the marginal cost of serving any portion of the market. Firms respond to this increase in costs by raising prices as they would given an exogenous cost increase. However,
the increase in marginal cost also grows as the size of the group of eligible consumers increases, leading to the equilibrium prices above.

In the upstream market, consumers make purchasing decisions anticipating the equilibrium that will follow in the downstream market. Consumers consider stated prices in the upstream market and also the degree to which announced rewards will be useful to them when making ensuing purchases in the downstream market. Letting $U^i$ denote the utility that a consumer buying from firm $i$ in the upstream market expects to realize in the following period in the downstream market, the location of the indifferent consumer in the upstream market, $e^0$, is defined by

$$-e^0 t_U - p_U^0 + U^0 = -(1 - e^0) t_U - p_U^1 + U^1.$$  

It is easiest to interpret the difference between the expected utilities $U^i$ by comparing them to the utility that a hypothetical consumer eligible for neither reward would expect to realize. If $U$ denotes this hypothetical consumer’s utility, then $U^i = U + GB^i$, where $GB^i$ represents the excess utility that being eligible for firm $i$’s reward provides relative to being eligible for no reward.

Consider the situation facing a consumer who becomes located on the segment between firms 0 and 1 and is eligible for firm 0’s reward. Conditional on being located on this segment, her excess utility due to the reward (relative to not being eligible for any reward) can be calculated as follows. If she is located within a distance of $z_{0,1}$ from firm 0, she would buy from firm 0 regardless of being eligible for the reward, but she spends less due to the reward. Thus, her benefit due to the reward if she is located in this interval is $r^0$. On the other extreme, if she is located further than $x_{0,1}$ from firm 0, she buys from firm 1 despite being eligible for 0’s reward. Hence, in this interval, she receives no excess utility from the reward. Finally, if she is located
more than \( z_{0,1} \) from firm 0 but less than \( x_{0,1} \), she would prefer to buy from firm 1 if not for the reward. The switching cost imposed by the reward is enough to keep her loyal to firm 0, but she receives a benefit less than the full value of the reward. If she is located exactly \( z_{0,1} \) from firm 0, she receives the full excess utility of \( r^0 \), and if she is located exactly \( x_{0,1} \) from firm 0, she receives no excess utility. In this interval, her excess utility falls from \( r^0 \) to 0 linearly as her location changes from a distance of \( z_{0,1} \) from firm 0 to a distance of \( x_{0,1} \); conditional on being in this interval, her expected excess utility is then \( r^0/2 \).

Similar analysis holds if she is located on segments between 0 and any other firm. Obviously, her benefit due to being eligible for firm 0’s reward is zero if she is not located on a segment for which firm 0 is an endpoint. Thus, the “gross benefit” provided by eligibility for firm 0’s reward relative to no reward is

\[
GB^0 = \frac{1}{6} \left[ r^0 y_{0,1} + \frac{r^0}{2} (x_{0,1} - y_{0,1}) \right] + \frac{1}{3} \left[ r^0 y_{0,N0} + \frac{r^0}{2} (x_{0,N0} - y_{0,N0}) \right],
\]
as \( p^N_D = p^N_D \) in equilibrium. A similar expression holds for \( GB^1 \). Realizing that the importance of the rewards to the consumers in the upstream market is entirely captured by the net benefit of \( GB^0 - GB^1 \), the equation implicitly defining \( e^0 \) can be written

\[
t_U - 2t_U e^0 - p^0_U + p^1_U + GB^0 - GB^1 = 0, \tag{2.1}
\]
where

\[
GB^0 = \frac{r^0}{4} + \frac{r^0}{12t_D} \left[ (1 - e^0) r^1 - 3 e^0 r^0 \right] + \frac{(r^0)^2}{8 t_D}, \\
GB^1 = \frac{r^1}{4} + \frac{r^1}{12t_D} \left[ e^0 r^0 - 3(1 - e^0) r^1 \right] + \frac{(r^1)^2}{8 t_D}.
\]
Substituting the equilibrium prices of the downstream market into firm 0’s downstream profit level yields $\Pi^0_D = \frac{3}{2} t_D - \frac{1}{2 t_D} r^0 (3 r^0 + r^1) e^0 (1 - e^0)$, so firm 0’s objective in the upstream market is to maximize

$$\Pi^0 = \Pi^0_U + \Pi^0_D = 6 (p^0_U - c_U) e^0 + \frac{3 t_D}{2} - \frac{1}{2 t_D} r^0 (3 r^0 + r^1) e^0 (1 - e^0),$$

where

$$e^0 = \frac{1}{2} + \frac{3 t_D (r^0 - r^1) + 12 t_D (p^1_U - p^0_U)}{3 (r^0)^2 + 2 r^0 r^1 + 3 (r^1)^2 + 24 t_D t_U}.$$

The latter expression results from solving (2.1) for $e^0$. The equilibrium strategies of each upstream firm and firm profits under no partnerships are given by the following proposition, the proof of which is found in Appendix B.1.

**Proposition 2.1.** When no partnerships exist, the equilibrium upstream price charged and reward offered by each firm are

$$p_{U,NP} = c_U + t_U + \frac{12}{49} t_D, \quad r_{NP} = \frac{6}{7} t_D.$$

In equilibrium, the total profits of each firm present in both markets and the profits of the firms present only downstream are respectively given by

$$\Pi_{NP}^0 = 3 t_U + \frac{183}{98} t_D, \quad \Pi_{D,NP}^0 = \frac{3}{2} t_D.$$

In the equilibrium of the model with no rewards, upstream prices are given by $c_U + t_U$, and a firm’s total profits are $3 t_U + 3 t_D / 2$. Hence, relative to a setting where no rewards are used, equilibrium prices have risen in not only the downstream market, but also in the upstream market, leading to higher profits. The primary reason for this is that consumer price sensitivity in the upstream market decreases when
rewards are introduced. Consider a consumer’s response to a price cut by a firm in the upstream market, which results in some consumers on the margin changing their purchasing decisions and buying from this firm rather than the rival firm. However, in the second period, these consumers anticipate a higher price being offered by the firm whose discount they will be eligible for. Such a consumer is more likely to rationally purchase from the firm whose reward they qualify for in the ensuing period, so the utility gain in the first period is partially offset by a reduction in expected utility downstream. Hence, an upstream price cut when rewards are used attracts fewer consumers when rewards are present in the model. Because price cuts are less effective, competition is not as extreme under rewards, and equilibrium prices rise.

A natural question is to ask whether partnering with downstream firms can improve profits. Note that in the current setup, firms not present in the upstream market do not pay rewards, and are left with no incentive to respond to a change in the proportion of consumers eligible for one discount or another, as they must compete with all firms equally. When saddled with the requirement that they pay rewards to eligible consumers, they too are incentivized to raise prices as the mass of consumers eligible for the reward increases. As shall soon be seen, this tends to reduce price sensitivity and competition in the upstream market further as consumers...

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22Consumer price sensitivity refers to the rate at which market share falls as a result of a marginal price increase. It can be seen from the above expression for firm 0’s market share, $e^0$, that consumer price sensitivity is given by

$$\left| \frac{\partial e^0}{\partial p_U} \right| = \frac{12t_D}{3 (r^0)^2 + 2r^0r^1 + 3 (r^1)^2 + 24t_D t_U} < \frac{1}{2t_U}.$$ 

The right-hand side of the above inequality is the consumer price sensitivity in the no-rewards model.
take into account multiple downstream firms charging higher prices in response to an upstream price reduction.

2.4 Symmetric Partnerships

In this section, I analyze the model taking as given that the upstream firms have each partnered with one of the downstream firms who are not present in the upstream market. Denote the firm partnered with firm 0 as $P_0$, and the firm partnered with firm 1 as $P_1$. As before, when competition in the downstream market begins, a proportion $e^0$ of consumers is eligible for firm 0’s reward, which may now be redeemed from firm 0 or firm $P_0$. The remainder are eligible for firm 1’s reward, which they may redeem from firm 1 or firm $1P$. Firm 0’s profits in the downstream market, $\Pi^0_D$, are given by

$$
(\hat{p}^0_D - c_D) \left[ y_{0,P_0} + e^0 (x_{0,1} + x_{0,P_1}) \right] \\
+ (1 - e^0) (z_{0,1} + z_{0,P_1})] - r^0 e^0 (y_{0,P_0} + x_{0,1} + x_{0,P_1}),
$$

and similar expressions hold for other firms. Note that even though consumers eligible for firm 0’s reward who are deciding between buying from, say, firm 0 or firm $0P$ care only about those firms’ announced prices, they still qualify for the discount. This is, of course, of interest to the firms honoring the reward discount. Solving the first order conditions for prices yields the equilibrium of the downstream subgame:

$$
p^0_D = p^{P_0}_D = c_D + t_D + e^0 r^0, \quad p^1_D = p^{P_1}_D = c_D + t_D + (1 - e^0) r^1.
$$

As with no partnerships, consumers in the upstream market are interested in the net benefit provided by one reward versus another. A similar gross benefit calculation to that above can be made. For example, as any partnered firms each have equal
downstream prices in equilibrium:

\[ GB^0 = \frac{1}{6} r^0 + \frac{2}{3} \left[ r^0 y_{0,1} + \frac{r^0}{2} (x_{0,1} - y_{0,1}) \right]. \]

Here, the first term represents the fact that the consumer receives a benefit of the full reward amount when the consumer is positioned between firm 0 and its partner, which happens with probability 1/6. The second term captures the gross benefit when the consumer becomes located on a segment between 0 or its partner and firm 1 or its partner. Obviously, when the consumer is located between firm 1 and its partner, the benefit provided by the reward is zero. Evaluating this expression and that for firm 1’s reward yields

\[ GB^0 = \frac{r^0}{2} + \frac{r^0}{3t_D} \left[ (1 - e^0)r^1 - e^0 r^0 \right] + \frac{(r^0)^2}{6t_D}, \]
\[ GB^1 = \frac{r^1}{2} + \frac{r^1}{3t_D} \left[ e^0 r^0 - (1 - e^0)r^1 \right] + \frac{(r^1)^2}{6t_D}. \]

I assume that monetary transfers are permitted between firms to induce participation in partnerships, so that joint profits are all that is of interest. Now, evaluating the downstream profits of firm 0 at the equilibrium prices yields \( \Pi^0_D = \frac{3}{2} t_D - \frac{1}{t_D} r^0 (r^0 + r^1) e^0 (1 - e^0) \). Because the downstream profits of firm 0 and its partner are identical in equilibrium, firm 0 wishes to maximize

\[ \Pi^J = \Pi^0_U + 2\Pi^0_D = 6 \left( p^0_U - c_U \right) e^0 + 3t_D - \frac{2}{t_D} r_0^0 \left( r^0 + r^1 \right) e^0 (1 - e^0), \]

where

\[ e^0 = \frac{1}{2} + \frac{3t_D (r^0 - r^1) + 6t_D (p^1_U - p^0_U)}{2 \left( (r^0 + r^1)^2 + 6t_D t_U \right)}. \]

Again, the latter expression is derived by solving (2.1) for \( e^0 \) using the new values of \( GB^i \). The equilibrium strategies of each upstream firm and partnership profits
under no partnerships are given by the next proposition, the proof of which is found in Appendix B.1.

**Proposition 2.2.** When two partnerships exist, the equilibrium upstream price and reward of each firm are

\[ p_{U,DP} = c_U + t_U + \frac{2}{3}t_D, \quad r_{DP} = t_D. \]

In equilibrium, the joint profits of either partnership are given by

\[ \Pi_{DP}^J = 3t_U + 4t_D. \]

From Propositions 2.1 and 2.2, some interesting comparisons can be made. First, note that under two partnerships, the equilibrium reward and upstream prices rise relative to the model without partnerships. Secondly, the sum of the equilibrium profits of an upstream duopolist and those of a downstream firm under no partnerships is exceeded by the equilibrium joint profits of a partnership when two partnerships exist, as

\[ \Pi_{NP}^0 + \Pi_{D,NP}^{N0} = 3t_U + \frac{183}{98}t_D + \frac{3}{2}t_D = 3t_U + \frac{165}{49}t_D < 3t_U + 4t_D = \Pi_{DP}^J. \]

The reasoning for these differences is straightforward. Under the two-partnership arrangement, all firms have a downstream price response to upstream market shares. This reduces consumer price sensitivity upstream even more than in the case of no partnerships, which relaxes competition in the upstream market even further. Hence, equilibrium prices and profits in this market rise.

In the downstream market, eligible consumers in four downstream segments, rather than three, face the switching cost imposed by eligibility for a reward. This positive effect on joint profits is mitigated somewhat by the fact that consumers in the downstream segment between partnered firms can disregard the reward when making
their price decision yet still must be given the discount, as must eligible consumers in any segment. However, the fact that consumers anticipate possibly being positioned in any one of these segments makes the reward itself more valuable in expectation. Overall, the power of using the reward to attract consumers in the upstream market has increased, so firms have an incentive to offer higher rewards.

In either partnership arrangement, consumers in the downstream market who are eligible for a discount from a firm pay a lower price when buying from that firm than in the no-rewards model, while those who are not eligible for a firm’s reward are charged a higher price by that firm. In equilibrium, the loss due to honoring discounts and charging a lower than normal price to eligible consumers is balanced out by consumers not eligible for the reward paying a higher than normal price, as the downstream market profits of all firms are equal to those under no rewards.

As evidenced by the increase in profits relative to no partnerships, in equilibrium, the profit-enhancing effects of rewards in both the upstream and downstream markets dominate the profit mitigating effects of a loss of switching costs in a portion of the downstream market along with honoring the reward and applying discounts to prices for eligible consumers.

2.5 One Partnership

In this section, I consider the partnership networks which consist of a single group of partnered firms which includes exactly one upstream duopolist. I shall use the term “partnership” here to refer to the entire group of firms. This captures the idea that a consumer eligible for a partnership’s discount downstream may use the discount interchangeably when purchasing from any partner firm.
2.5.1 A Single Partnership With One Downstream Member

I first consider the partnership arrangement consisting of a single partnership between an upstream duopolist (say, firm 0) and one of the firms present only in the downstream market (denoted firm $P_0$). The setup and solution of the model under this partnership is similar in principle to those of the preceding sections and is outlined in Appendix B.2. However, solving for the equilibrium relies on a numerical solution of the first-order conditions for firms’ reward amounts. This necessitates specifying values for the transportation costs facing consumers in the upstream and downstream markets.\textsuperscript{23}

Numerical approximations for equilibrium upstream prices, rewards, and profit levels under this partnership can be found in Tables 2.1 and 2.2. In each table, for notation purposes, it is assumed that firm 0 has engaged in the partnership discussed in this section, while firm 1 is not partnered. The downstream firm not partnered is denoted firm $N$. Equilibrium values for this case of a single partnership between two firms are denoted by the subscript $2F$, and as usual, the joint profits of the partnered firms are denoted by the superscript $J$. In Table 2.1, the equilibrium prices and reward amounts under each of the partnership arrangements examined thus far are shown for comparative purposes, with prices shown net of marginal costs for space-saving reasons. Table 2.2 displays the sums of both the partnered and unpartnered upstream duopolists and downstream firms in equilibrium under these partnership arrangements.

\textsuperscript{23}It should be noted that the only time the transportation cost parameter of the upstream market, $t_U$, affects the analysis is where it shows up in the denominator of the expression for the location of the indifferent consumer in the upstream market, $e_0^0$. Here, the relevant value is the product of the two transportation costs $t_D t_U$. Hence, specifying a ratio of these two values and letting $t_D$ vary freely is without loss of generality.
Table 2.1: Equilibrium Prices and Rewards for Various Parameter Values.

<table>
<thead>
<tr>
<th>Travel Costs</th>
<th>No Partnerships</th>
<th>A Single Partnership</th>
<th>Dual Partnerships</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Partnered</td>
<td>Not Partnered</td>
<td></td>
</tr>
<tr>
<td>$t_U$</td>
<td>$t_D$</td>
<td>$p_{U,NP}$</td>
<td>$r_{NP}$</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>1.049</td>
<td>0.171</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>1.098</td>
<td>0.343</td>
</tr>
<tr>
<td>1</td>
<td>0.6</td>
<td>1.147</td>
<td>0.514</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>1.196</td>
<td>0.686</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1.245</td>
<td>0.857</td>
</tr>
<tr>
<td>0.8</td>
<td>1</td>
<td>1.045</td>
<td>0.857</td>
</tr>
<tr>
<td>0.6</td>
<td>1</td>
<td>0.845</td>
<td>0.857</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>0.645</td>
<td>0.857</td>
</tr>
<tr>
<td>0.2</td>
<td>1</td>
<td>0.445</td>
<td>0.857</td>
</tr>
</tbody>
</table>

In general, the following proposition holds.

**Proposition 2.3.** When a single partnership exists and firm 0 has partnered with a single firm not present in the upstream market, the equilibrium strategies of the upstream duopolists satisfy

$p_{U,DP} > p^0_{U,2F} > p^1_{U,2F} > p_{U,NP}$

and

$r^0_{2F} > r_{DP} > r_{NP} > r^1_{2F}$.

Furthermore, the firms’ equilibrium profits satisfy

$
\Pi_{2F}^J > \Pi_{DP}^J > \Pi_{NP}^0 + \Pi_{D,NP}^{N0} > \Pi_{2F}^1 + \Pi_{D,2F}^N.
$
<table>
<thead>
<tr>
<th>Travel Costs</th>
<th>No Partnerships</th>
<th>A Single Partnership</th>
<th>Dual Partnerships</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Pi_{NP}^0 + \Pi_{D,NP}^{N0} )</td>
<td>( \Pi_{2F}^J )</td>
<td>( \Pi_{2F}^1 + \Pi_{D,2F}^N )</td>
</tr>
<tr>
<td>( t_U ) ( t_D )</td>
<td>( (3t_U + \frac{165}{49} t_D) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>3.673</td>
<td>3.878</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>4.347</td>
<td>4.756</td>
</tr>
<tr>
<td>1</td>
<td>0.6</td>
<td>5.020</td>
<td>5.635</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>5.694</td>
<td>6.515</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>6.367</td>
<td>7.396</td>
</tr>
<tr>
<td>0.8</td>
<td>1</td>
<td>5.767</td>
<td>6.798</td>
</tr>
<tr>
<td>0.6</td>
<td>1</td>
<td>5.167</td>
<td>6.203</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>4.567</td>
<td>5.613</td>
</tr>
<tr>
<td>0.2</td>
<td>1</td>
<td>3.967</td>
<td>5.043</td>
</tr>
</tbody>
</table>

Table 2.2: Joint Equilibrium Profits for Various Parameter Values.
Importantly, this proposition implies that the equilibrium profits of two potential partners when not partnered is exceeded by the joint profits of those same firms when they form the sole partnership. Moreover, given that a first partnership has been formed, the sum of the equilibrium profits of the remaining, unpartnered firms is exceeded by the joint profits of those firms when they form a second partnership.

The reasoning behind for this result can be understood by appealing to the above intuition for why two partnerships serve the partnering firms better than no partnerships. When one partnership is created, the newly partnered downstream firm is incentivized to increase prices in response to an increase in the mass of consumers eligible for the partnership’s reward. This has the effect of decreasing consumers’ upstream price sensitivity relative to no partnerships, although not to the degree of two partnerships when all firms have this incentive (as happens under two partnerships). The degree to which price competition in the upstream market is relaxed in each partnership arrangement can be readily observed by comparing firms’ equilibrium prices.

Meanwhile, in the downstream market, consumers anticipate gaining excess utility due to eligibility for the partnership’s reward in five segments, which is more than the three that provide excess utility to those eligible for firm 1’s reward. This difference makes a marginal increase in the reward offered by the partnership more useful in attracting upstream consumers than a similar reward increase by firm 1. In equilibrium, the partnership offers a higher reward so that firm 0 can charge a higher price while still selling to a majority of the market. Here, a reward increase by firm 1 is not beneficial, as relatively few consumers are attracted, while all consumers already purchasing must be granted a larger discount in the downstream market. As a result,
the partnered firms realize higher joint profits than the unpartnered firms, whose joint
profits fall below the no-partnership level.

Interestingly, when a single such partnership exists between two firms, the equi-
librium profit level of the downstream firm who is not engaged in a partnership is
given by $\Pi_{D,2F}^N = 3t_D/2$, as in the no-partnerships and no-rewards cases. Therefore,
in general, a downstream firm who is not part of a partnership is not adversely af-
fected by the existence of rewards programs or partnerships. However, as has been
shown, there are gains to forming such a partnership, as an unpartnered downstream
firm is willing to enter into a partnership and honor such programs in exchange for
compensation, which can also be beneficial to the upstream duopolist as the resulting
joint profits of the partnering firms will rise.

2.5.2 A Single Partnership with Two Downstream Members

The second partnership network to be addressed in this section consists of a part-
nership between an upstream duopolist and both downstream firms not present in
the upstream market. Again, the setup and solution of the model under this partner-
ship is familiar and is outlined in Appendix B.3. As under a single partnership with
one downstream member, solving for the equilibrium of this model relies on numeri-
cal methods. However, the intuition behind the equilibrium strategies and profits of
firms is similar to that above, so for the purposes of brevity, I show here only the
result analogous to Proposition 2.3. Here, the equilibrium values under the single
partnership among three firms are denoted by the subscript $3F$, and the unpartnered
duopolist is assumed to be firm 1.
Proposition 2.4. When a single partnership exists and firm 0 has partnered with a both firms not present in the upstream market, the equilibrium strategies of the upstream duopolists satisfy

\[ p_{U,3F}^0 > p_{U,DP}^0 > p_{U,2F}^0 > p_{U,2F}^1 > p_{U,NP} > p_{U,3F}^1 \] (2.2)

and

\[ r_{3F}^0 > r_{2F}^0 > r_{DP} > r_{NP} > r_{2F}^1 > r_{3F}^1. \] (2.3)

Furthermore, the firms’ equilibrium profits satisfy

\[ \Pi_{3F}^J - \frac{3t_D}{2} > \Pi_{2F}^J > \Pi_{DP}^J > \Pi_{NP}^J + \frac{3t_D}{2} > \Pi_{2F}^1 + \frac{3t_D}{2} > \Pi_{3F}^1 + \frac{3t_D}{2}. \] (2.4)

This proposition can be interpreted as follows. Recall that the profit level of an unpartnered downstream firm in any network is given by \( \frac{3t_D}{2} \); hence the first inequality (2.4) indicates that adding the unpartnered downstream firm to the partnership increases the joint profits of these firms relative to a single two-firm partnership. The last inequality in line (2.4) shows that if the two-firm partnership becomes a three-firm partnership, the unpartnered duopolist sees its profits decline. The reason for this is identical to the intuition for why forming a single two-firm partnership benefits the partnering firms and hurts the unpartnered duopolist, except that the magnitude of the reward effects under the single partnership are enhanced by adding a second downstream partner. This can be observed in line (2.2) by noting the increase in the difference between the partnered and unpartnered duopolists’ prices and in line (2.3) by noting the increase in the difference between the rewards offered. As firm 0 adds additional partners, firm 1 finds it harder to compete in the upstream market, because consumers much prefer the reward that they are more likely to redeem.
2.6 Partnership Formation

Until this point, the existence of partnerships or lack thereof has been taken as given. I now consider a process where firms form or dissolve partnerships prior to competition. In this setting, there may be a transfer from (or to) each upstream duopolist to (or from) any partners, with the amounts of those transfers being specified by the duopolist prior to the formation of the partnership. Such a transfer occurs prior to competition, with upstream competitors later maximizing their own profits.\textsuperscript{24} As a result, under any partnership arrangement, the upstream duopolists may redistribute the equilibrium joint profits of their partnership among the partnership members. All unpartnered firms merely keep their own equilibrium profits.

I will refer to a possible such division of equilibrium profits among firms as a feasible allocation.\textsuperscript{25} I will say that a coalition of firms blocks an allocation if its members can make feasible, through the formation or dissolution of partnerships, an allocation which weakly improves the profits of each coalition member and strictly improves the profits of at least one member. The following notions of stability differ in which actions are able to be performed by a potential blocking coalition.

Definition 2.1 (Core Stability). A partnership network is core stable if it supports an allocation $A$ where no coalition of firms exists for which the simultaneous dissolution of any existing partnerships and/or formation of any possible partnerships among

\textsuperscript{24}I emphasize that partnership formation as described here does not entail collusion in the traditional sense of price fixing to maximize joint profits. Once transfers are made, each firm chooses its strategy to maximize its own profits.

\textsuperscript{25}It should be clear that any allocation which is feasible when upstream firms choose rewards that do not maximize the joint profits of the partners is also feasible when joint profits are maximized. Indeed, the former allocation can be improved upon by a coalition of the same two firms utilizing the latter allocation. Hence, the assumption that upstream firms maximize joint profits is without loss of generality.
any coalition members makes feasible an allocation $A'$ that allocates weakly greater profits to all coalition members and strictly greater profits to one member than under allocation $A$.

**Definition 2.2 (Pairwise Stability).** A partnership network is *pairwise stable* if it supports an allocation $A$ where

(a) there exists no downstream firm which can, by leaving an upstream duopolist’s partnership, receive strictly higher profits than under $A$,

(b) there exists no downstream firm which can, by leaving an upstream duopolist’s partnership, make feasible an allocation $A'$ under which all firms remaining in the duopolist’s partnership are allocated weakly higher profits and one is allocated strictly higher profits than under $A$, and

(c) there exists no firm which can be brought into an upstream duopolist’s partnership thereby making feasible an allocation $A'$ under which all members of the new partnership are allocated weakly higher profits and one is allocated strictly higher profits than under $A$.

Notice that core stability implies pairwise stability, which is a much weaker solution concept. While core stable networks are robust to joint deviations by arbitrary groups, pairwise stable networks need only be robust to a single partner wishing to leave a partnership or the mutual desire of an upstream duopolist’s partnership and a potential downstream partner to add the downstream firm to the partnership.\(^{26}\)

\(^{26}\)The notion of pairwise stability as robustness to the unilateral removal of a single edge from, or the bilateral addition of a single edge to, the graph representing a network is a commonly used solution concept in the literature on network formation. See, for example, Jackson and Wolinsky (1996) for further discussion.
It is trivial that if the act of adding a firm to a partnership (including a “one-firm” partnership) results in strictly higher equilibrium joint profits of the newly partnered firms relative to the total profits of those firms under the old partnership network, then the firms in the new partnership form a coalition which blocks any allocation which is feasible under the old partnership network. Note that both pairwise and core stable networks must be robust to such a change to the network. Hence, from Proposition 2.3, the following lemma holds.

**Lemma 2.5.** *Neither the no-partnerships network nor any network of a single partnership between an upstream duopolist and a single downstream firm is core stable or pairwise stable.*

An important remark regarding the structure of the markets is appropriate here. Note that the upstream firms need not be present in both markets for the analysis of the preceding sections to hold. Suppose that the downstream market instead is comprised of four firms which are unaffiliated with the upstream duopolists in the absence of rewards programs. Indeed, in this model, partnering with a single downstream firm and utilizing a rewards program necessarily raises the joint profits of the partnering firms. Otherwise, an equilibrium would exist in the model of Section 2.3 where both upstream duopolists announce reward amounts of zero, which is not the case. As a result, the no-rewards network and any network featuring a single partnership between an upstream duopolist who is not present in the downstream market and a single downstream firm is neither core nor pairwise stable. Forming a single such partnership blocks the no-rewards network, and forming the second partnership consisting of a single upstream and a single downstream firm, which is equivalent to the benchmark model, blocks the single such partnership. Thus, the
assumption that the upstream duopolists are present in both markets is without loss of generality.

Next, I point out that the upstream duopolists never benefit by partnering with one another. Consider any partnership which includes both upstream duopolists. In the upstream market, by the nature of the partnership, consumers buying from firm 0 or firm 1 will be eligible for any offered discounts in the following period. Thus, consumers optimally behave myopically in the upstream market, which trivially yields no-rewards profits for both firms in this market. Then, in the downstream market, any firm must charge the same price to all consumers, as all consumers are eligible for any reward discounts. In equilibrium, regardless of the announced reward, the effective price paid by consumers must be equal to the no-rewards equilibrium price. Hence, any partnership which includes both upstream duopolists results in no-rewards profits.27 Therefore, because dissolving a partnership between the upstream duopolists always leads to either the no-partnerships case, which Pareto dominates the no-rewards case, or at least one partnership, which always permits a Pareto improvement over the no-rewards case for a group of partnered firms, the following lemma holds.

27Although it may not be immediately obvious, that the upstream duopolists do not wish to partner with one another does not critically depend on the fact that all consumers in the downstream market were active in the upstream market. Consider an overlapping generations model where a proportion \(1 - \alpha\) of the consumers from the upstream market are not later present in the downstream market, while the remaining \(\alpha\) “survive” for the downstream market, and these consumers are uncertain about their own survival. Again, when the upstream duopolists partner, consumers rationally behave myopically in the upstream market, as their future eligibility is unaffected by current purchasing decisions. However, if the consumers who exit the market are replaced by identical new consumers, the downstream market is segmented into eligible and non-eligible consumers, as usual. It is easy to show that downstream profits can only weakly decrease in equilibrium relative to the no-rewards level, so any resulting allocation is again blocked by the no-partnerships allocation. The decrease in downstream profits can be observed by inspection of the objective functions of firms 0 and 1 under any of the partnership arrangements I discuss.
Lemma 2.6. *No network in which the upstream duopolists are members of the same partnership is core stable or pairwise stable.*

As has been discussed above and alluded to by Proposition 2.4, formation of partnerships with downstream firms is highly lucrative for the upstream duopolists. To this point, it is clear that if a downstream firm remains unpartnered, then an upstream duopolist and its partners can form a blocking coalition by partnering with that firm. The only remaining question is which, if any, networks are stable under either solution concept. I now answer this question.

Proposition 2.7. *The set of core stable networks is empty.*

*Proof.* By Lemmas 2.5 and 2.6, the only networks to be considered are those in which all downstream firms are in a partnership. Consider the two-partnerships network. Trivially, one upstream duopolist and its rival’s downstream partner form a blocking coalition by abandoning their current partners and forming a single partnership. This leaves only the possibility of a single partnership that includes both downstream firms. Without loss of generality, suppose firm 0 is partnered with both downstream firms. The unpartnered upstream duopolist, firm 1, is willing to distribute up to \( \Pi_{3F}^J - \Pi_{3F}^1 \) to the downstream firms in exchange for abandoning their current partner and forming a new partnership with firm 1. If the downstream firms are not allocated this much in aggregate, then firm 1 and these firms constitute a blocking coalition. On the other hand, if the downstream firms are allocated at least this much, then firm 0 retains only \( \Pi_{3F}^1 \) for itself in the allocation, which is less than firm 0 would retain under no partnerships, which it can achieve by severing ties with its partners.
Proposition 2.8. Any networks in which all downstream firms are members of a partnership are pairwise stable.

Proof. First, consider the two-partnerships network. Consider the allocation where each downstream firm is allocated $3t_D/2$. Dissolving either partnership would then not affect the downstream firm from the dissolved partnership. Meanwhile, the upstream duopolist is keeping $\Pi_{DP}^f - 3t_D/2$, which exceeds $\Pi_{2F}^f$ by 2.4. The latter amount is exactly what they receive should the partnership be dissolved. Therefore, this network is pairwise stable.

Next, consider the partnership where both downstream firms belong to the same partnership, say with firm 0. Consider the allocation under this network where each downstream firm is allocated $3t_D/2$ by firm 0. Firm 1 is not able to form a blocking coalition, as all downstream firms are currently partnered with the other firm. Meanwhile, as long as both downstream firms are receiving at least $3t_D/2$, neither firm wishes to leave the partnership. This means that the only blocking coalition must consist of firm 0 severing ties with a partner. However, firm 0 is keeping $\Pi_{3F}^f - 3t_D$ and either partner is allocated $3t_D/2$, so the joint profits of the duopolist and a single one of its partner is $\Pi_{3F}^f - 3t_D/2$. Severing ties with a partner would leave the partnership with $\Pi_{2F}^f$ to distribute. By Proposition 2.4, $\Pi_{3F}^f - 3t_D/2 > \Pi_{2F}^f$, so this network cannot be blocked and is pairwise stable.

Due to the high benefits and low costs of forming these partnerships, the partnership formation process is quite volatile. No stable networks exist, so long as firms have sufficient freedom to break or form partnerships. On the other hand, if firms are more limited in which actions may be used to profitably disrupt the current network, stable networks do exist whenever all of the downstream firms are members.
of a partnership. This supports the intuition of the previous sections. In particular, from the perspective of an upstream firm, the formation of a partnership or adding a partner to an existing partnership in this setting does nothing to empower one's upstream rival, yet has the effect of decreasing consumer price sensitivity even further and increasing the power of the reward to attract consumers in that upstream market, which increases equilibrium profits.
Chapter 3: Loyalty Rewards Programs in the Face of Entry

3.1 Introduction

Customer loyalty rewards have long been popular with firms and consumers. Casual observation reveals numerous examples of programs that serve to make lower prices available to repeat customers. Airlines provide frequent-flyer programs, while automobile companies offer lower prices for customers who trade in same-make vehicles. Supermarkets mail coupons to shoppers who have previously purchased groceries and offer discounts on gasoline purchases after a level of purchases has been reached. Big box retail stores award gift certificates after a certain dollar amount has been spent, and credit card companies offer cash-back payments after different increments of charges are made. Even coffee shops and pizzerias use frequent-shopper cards to award a free espresso or slice of pizza after a certain number have been purchased.

One effect of such programs is that consumers are saddled with a switching cost after making an initial purchase. When a consumer has qualified for a reward (or has made purchases which will accrue towards a reward), it becomes costly for them to purchase from a different supplier in the future because they will lose the value of the reward. For instance, a consumer interested in purchasing a new car will not only consider cars’ sticker prices, but also will take into account any loyalty discount
offered by the manufacturer of her current vehicle. In order to lure a consumer away from a rival firm, a firm must offer a price below its rival’s price by a margin large enough to compensate for the discount the consumer would otherwise receive.

Some switching costs are not the result of any action by a firm but rather that exist exogenously: examples range from the cost of establishing a relationship with a new provider to inefficiencies that arise from adopting a new operating system on a personal computer. The effect of such a switching cost on consumers typically provides an incentive for firms to initially set low prices in order to “lock in” consumers, who can then be exploited in the periods that follow due to the switching cost. Additionally, switching costs may allow an existing firm to lock in enough of the market that entry by another firm, which would be profitable in the absence of switching costs, is prevented. However, the switching costs generated by loyalty rewards differ from these types of switching costs in several important respects: they are chosen by the firms that offer the rewards and they are costly to those firms in future periods. That is, when a firm makes a choice of how large a reward or discount to offer, they determine precisely the level of the switching costs that consumers will face rather than taking it as given. Additionally, by implementing such a rewards program, the firm must follow through with a lower price to those consumers who qualify. Equivalently, the reward may be viewed as a cash payout, in which case the firm must pay those who redeem the reward. These differences have important implications. In particular, a firm engaging in a rewards program needs to carefully weigh the benefit it gains from the switching cost against the cost of paying out the reward. Importantly, the reward payout reduces the benefit of lock-in, as locked-in

\[28\text{See Farrell and Klemperer (2007) for an excellent review of the literature on switching costs.}\]
consumers are necessarily charged lower prices or are paid a reward. This may reduce the feasibility of limit pricing in early periods for the purposes of entry deterrence.

I consider rewards programs in the context of a two-period environment where a monopolist faces potential entry by a rival firm. Specifically, an incumbent firm is a monopolist in an initial period armed with the knowledge that an entrant who offers a horizontally differentiated good may enter in the following period. The incumbent might be interpreted, for example, as an airline that offers the only flight on a particular route out of a given airport. The entrant might be a rival airline who is considering serving that route. Following the example of differentiated-product models featuring switching costs such as those of von Weizäcker (1984), Klemperer (1987b), and Caminal and Matutes (1990), I make the assumption that consumers are uncertain about how their preferences might change between the two periods. That is, their preferences change over time such that they are not sure which firm’s product will be more appealing in the second period. Extending the airline example, this can be interpreted as a traveler knowing which airline offers the flight which best fits their schedule at the moment but not knowing which will offer a more convenient flight the next time she will fly. A simple reason for this may be that the she merely has not made travel plans yet even though she knows she will take a vacation within the next year. Likewise, the automobile shopper has full information about which cars are currently in production, but likely cannot predict which manufacturer might be producing the vehicle she will like best ten years down the line.

The purpose of this chapter is to study how an incumbent firm might benefit from implementing a rewards program. In particular, I focus on so-called “coupon rewards,” which specify a discount for repeat consumers from the to-be-determined
publicly available period-two price, but do not guarantee any specific price. Alternatively, these coupons could be viewed as a cash payout received upon a repeat purchase. If a rewards program is adopted, the incumbent imposes switching costs upon its previous customers in the following period. An interesting question is then whether this will allow the incumbent to sell to enough of the market in the second period to forestall the entry and maintain the monopoly. Perhaps surprisingly, the answer is no. If the entrant were to find entry profitable in the absence of rewards, then, regardless of the rewards program implemented, the incumbent is powerless to deter entry. Furthermore, the second-period profits of the incumbent actually fall due to the cost of implementing rewards. A second question then arises: given that the incumbent is made strictly worse off in the second period, can using a rewards program actually be beneficial? The answer to this question is yes. Rewards increase the value of purchasing the incumbent’s good in the first period for all consumers due to those consumers’ uncertainty about their preferences in the following period. Moreover, consumers exhibit decreased price sensitivity in the first period. Because the incumbent is incentivized to raise prices as more consumers become eligible for its reward, consumers anticipate a price increase following an early price cut and are thus less responsive to such a cut. The incumbent thus charges a higher price than it would without rewards while at the same time increasing its first-period sales, and the resulting increase in profit is more than enough to compensate for the period-two loss. In contrast to the normal behavior under standard switching costs, rather than pricing low and then later exploiting those facing the switching cost, the incumbent

This is in contrast to other rewards programs which might feature nonlinear pricing within a single period.
using rewards must increase profits from reward-ineligible consumers via a higher first-period price to make up for an unavoidable loss in the future.

The literature on switching costs is extensive; however, given the inherent differences mentioned above between rewards programs and standard models of switching costs, the majority of this work has little bearing on the topic at hand. Of primary interest are the models that examine entry deterrence. One noteworthy example is Klemperer (1987b), who shows that forestalling entry is possible in a two-period model of Cournot competition even when switching costs are low. By contrast, I demonstrate that an incumbent has no hope of preventing entry, even when imposing a large switching cost via rewards. The difference comes down to the fact that, when switching costs are generated by rewards programs, the incumbent is burdened with a higher marginal cost in the second period due to the reward payout. Subgame perfection then rules out the possibility of the incumbent hurting the profit level of the entrant, regardless of what commitment has been made in the prior period.

It should be clear that loyalty rewards programs are a form of price discrimination, based upon the past behavior of consumers. There are several interesting studies that examine such behavior-based price discrimination. Fudenberg and Tirole (2000) consider a two-period duopoly market for a horizontally differentiated product. In their model, firms are able to price discriminate in the second period based on the purchasing decisions of consumers in the first period. They find that firms will optimally charge higher prices to repeat consumers and lower prices to potential “new” consumers. Villas-Boas (1999) reports similar results for a model where firms are infinitely-lived and face overlapping generations of consumers. These findings seem to contradict the notion of loyalty rewards, which have precisely the opposite effect
on prices. In contrast to these results, I find that charging a lower price to repeat
customers is, in fact, optimal, even if doing the opposite were feasible. The reason for
this discrepancy lies in that, in the current model, consumers' preferences change over
time. When consumers' preferences do not change, higher prices to repeat customers
are optimal because consumers' preferences are partially revealed to firms. Firms
then take advantage of this information by extracting additional surplus from those
consumers with higher willingness to pay. However, when consumers' preferences are
not consistent from period to period, less information about current preferences is
revealed by past purchasing decisions. This results in an environment where loyalty
rewards, hence lower prices, are optimal.

Given the seeming ubiquitousness of rewards programs in our everyday lives, stud-
ies that focus on coupon loyalty rewards programs are surprisingly sparse. Banerjee
and Summers (1987) examine a duopoly market for an undifferentiated product, and
find that firms, when given the option of adopting a coupon rewards program, inde-
dependently elect to implement rewards programs and that doing so serves as a collusive
device for firms. Caminal and Matutes (1990) consider a two-period differentiated
duopoly model. They also find that using coupon rewards increases profits for firms.
However, they show that when firms are given the option to use a different form of
rewards, namely, precommitment in the first period to period-two prices for repeat
consumers, a prisoner's dilemma is created which results in both firms precommitting
and in lower equilibrium profits than when no rewards are used. The latter result is
averted in this chapter; a first-period monopolist would not adopt any strategy that
results in lower profits than the no-rewards equilibrium, hence rewards do survive.
To my knowledge, Cairns and Galbraith (1990) is the only extant paper which examines the use of rewards by a monopolist. The authors find that a monopolist present in several markets for unrelated products is able to forestall entry in a only some of these markets by offering a reward program in which rewards may be redeemed towards products offered only by the incumbent. However, this result hinges on several factors. Firstly, the products offered by both firms in each market are assumed to be undifferentiated. Secondly, firms are able to price discriminate between consumers who pay the full stated price for the products offered and consumers who pay only a fraction of the announced price.\footnote{The latter group of consumers can be viewed as agents who need to purchase plane tickets for work but who receive reimbursement from their employers.} In the current model, the combination of horizontal differentiation of products and firms’ inability to price discriminate (other than via the reward discount itself) causes attracting enough consumers after entry to injure a rival to be an undesirable option for the incumbent. Indeed, the monopolist is better accommodating entry but still employing a rewards program to raise profits.

The remainder of the chapter proceeds as follows. Section 3.2 formally establishes the model, and Section 3.3 establishes the benchmark result when the incumbent firm does not use a rewards program. Section 3.4 solves the model of rewards and presents the main results. Section 3.5 concludes.

\section{3.2 The Model}

The framework I adopt consists of a market which operates for two periods. An incumbent firm, $I$, is active in the market during both periods, while an entrant, $E$, is not present in the first period but may enter the market in the second period.
should it anticipate doing so to be profitable. The entrant faces a fixed cost of entry. Each firm $i$ has marginal cost of production $c^i$, and the entrant’s marginal cost is assumed to be less than the incumbent’s, i.e., $c^E \leq c^I$. These firm characteristics are common knowledge. The incumbent sets a price $p_1^I$ in the first period and a price $p_2^I$ in period two. Should the entrant decide to enter, it sets a price $p_2^E$ in the second period. In the first period, the incumbent firm also sets a reward amount $r > 0$. The reward is a coupon reward; if a consumer purchases from the incumbent in the first period, she receives a discount of $r$ off of the incumbent’s second-period price should she elect to purchase from the incumbent again in the second period. That is, consumers who have previously purchased from the incumbent are eligible for a price $p_2^I - r$ in the second period, while new customers must pay the full price $p_2^I$. That is, first-period buyers are assured that they will receive a discount from the price paid by the incumbent’s new consumers in the following period, and are made aware of the dollar amount of the discount.

The firms’ products are horizontally differentiated. Specifically, consumer preferences are modeled as in the familiar linear city model of Hotelling (1929): there is a continuum of consumers (normalized to have a mass of 1) uniformly distributed over the unit interval; the incumbent is located at 0, and the entrant, when and if it chooses to enter, becomes located at 1.\textsuperscript{32} A consumer at position $\theta$ who purchases

\textsuperscript{31}This assumption guarantees that the benchmark market will be fully covered in the second period for any of the monopoly sales levels I allow for the incumbent in the first period. Additionally, because entry is more profitable for lower-cost firms, it is more likely to occur when the potential entrant is of low cost compared to the entrant than when the opposite is true.

\textsuperscript{32}The purpose of this chapter is not to study the strategic choice of location, or product, by the incumbent or the entrant, which are taken as exogenously determined. Rather, I envision a scenario where firms have characteristics which are known to consumers before the game begins. For instance, Chrysler and Ford will likely have difficulty tailoring their products to a specific geographic market, but a dealer of one may very well consider entry in a market where the other is present, and participation in rewards programs may vary by region. Similarly, Delta and Southwest will not
from firm $i$ in period $\tau \in \{1, 2\}$ realizes utility

$$U(\theta, i) = R - |\theta - i| t - p^i_{\tau},$$

where $p^i_{\tau}$ is the (effective) price she faces offered by firm $i$ in period $\tau$ and $t > 0$ measures consumers’ “transportation costs,” so that costs increase linearly in the distance “travelled.”

Consumers have unit demand; that is, in each period each consumer will purchase at most one unit of one firm’s product, and will purchase if doing so results in nonnegative expected utility. Consumers live for both periods; however, consumers’ locations are not fixed across periods. In the first period, consumers are uncertain about their relative willingness to pay for the two firms’ products in the following period. Specifically, consumers are repositioned in the second period independently of their original positions. Consumers have rational expectations; that is, they correctly anticipate the second-period equilibrium prices of firms.

Firms maximize total expected profits across periods, while consumers maximize total expected utility across periods. For simplicity, the discount factor of both the incumbent and the consumers is assumed to be equal to one. The solution concept to be used is subgame-perfect Nash equilibrium. The timing of the game is as follows. At the beginning of period one, the incumbent announces a first-period price and reward be able to change their product for a specific route, and some routes may be ineligible for free or discounted flights.

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33 As is the case for any such differentiated-product model, the position $\theta$ of a consumer in this model need not correspond to a physical location in a city, nor does $t$ have to represent a literal transportation cost. This aspect of the model merely captures the degree to which consumers prefer one firm’s product to the other.

34 I argue below that the central results of this chapter do not hinge on the fact that consumers’ locations in period two are independent of those in period one; the same goes for the fact that all consumers are active in both periods.

35 This does not affect the qualitative results of the chapter. Period two is completely unaffected by this assumption, while the qualitative results pertaining to period one hold provided that the discount factor is sufficiently high.
amount. Consumers then make their purchasing decisions. First-period profits and utility are realized, then the entrant makes its decision of whether to enter or not. At the beginning of the second period, consumers are repositioned, then any active firms simultaneously announce their second-period prices. Finally, consumers make their purchasing decisions and second-period profits and utilities are realized.

3.3 No Rewards: A Benchmark

I first consider a two-period model where the incumbent firm does not offer a rewards program (i.e., \( r = 0 \)). The purpose of this section is to establish a benchmark against which the optimal rewards program may be compared. Because consumers’ purchasing decisions in the first period have no impact on the second period of the game, consumers maximize utility by acting myopically. Similarly, the incumbent’s pricing decision in period one in no way affects the second period of the game, so the incumbent also acts myopically. Because I wish to examine how the implementation of a rewards program can impact the profits that an entrant would realize in the absence of such a program, I assume that the entrant’s cost of entry is such that it can profitably enter the market when rewards are not used; that is, there will be two firms in the market in the second period.\(^{36}\) I first analyze the incumbent’s optimal first-period strategy before proceeding to the familiar Hotelling setup of the second period.

\(^{36}\)It turns out that the entrant enters in the presence of rewards if and only if it would have entered when rewards were not used, so this assumption is necessary if the entrant is to have any impact on the outcome of the game.
3.3.1 Period One: Monopoly

Facing the monopolist who offers a price of $p_M^I$ and no rewards program, each consumer has an outside option of zero, and their decisions whether or not to purchase will have no impact on the second period. A consumer at position $\theta$, then, will buy from the monopolist if and only if

$$R - \theta t - p_M^I \geq 0 \iff \theta \leq \frac{R - p_M^I}{t}.$$  

The measure of consumers who buy from the monopolist, $n_M^I$, is then given by

$$n_M^I (p_M^I) = \begin{cases} 
1 & \text{if } p_M^I < R - t \\
\frac{R - p_M^I}{t} & \text{if } R - t \leq p_M^I \leq R \\
0 & \text{if } p_M^I > R 
\end{cases}$$

The monopolist’s profits are given by

$$\Pi_M^I (p_M^I) = (p_M^I - c^I) n_M^I (p_M^I).$$

I assume that the monopolist can earn positive profits. To earn positive profits, the monopolist must sell to a mass of consumers greater than zero (i.e. $p_M^I < R$), and the price charged by the monopolist must exceed its marginal cost (i.e. $p_M^I > c^I$). Thus, I make the following assumption.

**Assumption 3.1.** $R - c^I > 0$.

Because I wish to examine how the introduction of a rewards program in the presence of a potential entrant will affect the pricing strategy and sales level of the incumbent firm, I focus on the interesting case where incumbent does not cover the entire market as a monopolist not using a rewards program. The necessary and sufficient condition for this to hold is presented in the following assumption.
Assumption 3.2. $R - c^I < 2t$.

Trivially, charging any price $p^I_M \geq R$ yields zero profits, and charging any price $p^I_M < R - t$ yields strictly lower profits than charging $p^I_M = R - t$, which yields profits of $R - c^I - t$. If $p^I_M \in (R - t, R)$, profits are given by

$$(p^I_M - c^I) \frac{R - p^I_M}{t}.$$ 

This expression is strictly concave in $p^I_M$, and the first-order condition implies an optimal price in this interval of

$$p^I_M = \frac{R + c^I}{2}$$

which yields profits of

$$\Pi^I_M \left(\frac{R + c^I}{2}\right) = \frac{(R - c^I)^2}{4t}.$$ 

Assumption 3.1 implies that $\frac{R + c^I}{2} < R$ and Assumption 3.2 implies that $R - t < R - \frac{R - c^I}{2} = \frac{R + c^I}{2}$; therefore, $\frac{R + c^I}{2} \in (R - t, R)$. Furthermore,

$$\Pi^I_M \left(\frac{R + c^I}{2}\right) = \frac{(R - c^I)^2}{4t} > 0 = \Pi^I_M (R)$$

by Assumption 3.1, and

$$\Pi^I_M \left(\frac{R + c^I}{2}\right) = \frac{(R - c^I)^2}{4t} > R - c^I - t = \Pi^I_M (R - t)$$

by Assumptions 3.1 and 3.2.\(^{37}\) Therefore, the price, sales and profit level of the (monopolist) incumbent firm in the first period with no rewards program are given

\(^{37}\)To see this, note that Assumptions 3.1 and 3.2 hold if and only if there exists some $k \in (0, 1)$ such that $R - c^I = 2kt$. Then $\frac{(R - c^I)^2}{4t} = k^2t$ and $R - c^I - t = 2kt - t$. The desired inequality thus holds if and only if

$$k^2t > 2kt - t,$$

which is equivalent to $(k - 1)^2 > 0.$
by

\[ p_M^I = \frac{R + c^I}{2}, \]
\[ n_M^I = \frac{R - c^I}{2t}, \]
\[ \Pi_M^I = \frac{1}{t} \left( \frac{R - c^I}{2} \right)^2. \]

Henceforth, I will refer to these values as the “monopoly” or “no-rewards” values for period one.

### 3.3.2 Period Two: Price Competition

Period two, in the absence of a rewards program, is simply the standard differentiated product model of competition in which the incumbent sets a price \( p_H^I \) and the entrant sets a price \( p_H^E \). I assume that the market will be fully covered in equilibrium; that is, every consumer purchases in equilibrium and gains strictly positive utility by doing so. If this is not the case, then each firm would set its price as if it were a monopolist, so entry into the market has no impact on the incumbent firm’s price or sales level. The following assumption is the necessary and sufficient condition to ensure market coverage for all \( c^E \leq c^I \).

**Assumption 3.3.** \( R - c^I > t. \)

This assumption implies that the sum of the incumbent firm’s monopoly sales level and the hypothetical monopoly sales level of any entrant with equal or lower marginal cost is greater than one. In other words, the market is fully covered upon entry, and firms must strategically alter prices from monopoly levels. To ensure that equilibrium sales levels for both firms are strictly positive, I make the following assumption.

**Assumption 3.4.** \( t > \frac{1}{3} \left( c^I - c^E \right). \)
Because the market is fully covered, the mass of consumers who buy from firm $i$ given prices $p^i_H$ and $p^j_H$ is given by

$$n^i_H (p^i_H, p^j_H) = \frac{1}{2} + \frac{p^j_H - p^i_H}{2t},$$

so that $n^i_H = 1 - n^E_H$ corresponds to the location of the consumer who is indifferent between the two firms. Firm $i$’s profits are given by

$$\Pi^i_H (p^i_H, p^j_H) = (p^i_H - c^i) n^i_H (p^i_H, p^j_H).$$

This expression is strictly concave in $p^i_H$. The first-order conditions for firms $I$ and $E$ yield, respectively, the following reaction functions:

$$p^I_H = \frac{1}{2} (c^I + t + p^E_H)$$
$$p^E_H = \frac{1}{2} (c^E + t + p^I_H).$$

Solving the reaction functions simultaneously yields firms’ equilibrium pricing strategies:

$$p^I_H = c^I - \frac{1}{3} (c^I - c^E) + t$$
$$p^E_H = c^E + \frac{1}{3} (c^I - c^E) + t.$$

Using these prices, one can easily derive the equilibrium market shares

$$n^I_H = \frac{1}{2} - \frac{1}{6t} (c^I - c^E)$$
$$n^E_H = \frac{1}{2} + \frac{1}{6t} (c^I - c^E)$$

and profit levels

$$\Pi^I_H = \frac{1}{2t} \left( t - \frac{1}{3} (c^I - c^E) \right)^2$$
$$\Pi^E_H = \frac{1}{2t} \left( t + \frac{1}{3} (c^I - c^E) \right)^2.$$
In what follows, I will refer to these equilibrium values as the “Hotelling” or “no-rewards” values for period two. This equilibrium is presented as a benchmark and has been studied extensively; hence, I do not discuss its properties further here. I now turn to the model which is the emphasis of this chapter.

3.4 Rewards Programs

This section considers the two-period model where the incumbent firm, as a first-period monopolist, announces a reward amount at the beginning of the first period, along with a first period price. If the entrant elects to enter in the second period, firms engage in standard duopoly price competition, with one exception: the market is segmented into two groups of consumers, those who qualify for the discounted price (or reward payout) and those who face the higher, undiscounted price of the incumbent. Recall that consumers change their locations between periods, so that these groups are both uniformly distributed over the unit interval.\footnote{Note that whether the groups are actually distributed uniformly or are merely distributed uniformly in expectation does not affect the analysis.}

Because a subgame-perfect equilibrium is desired, I first consider the second period of the game and find that the entrant cannot be deterred from entering. Using backward induction, I then consider the first period where the incumbent firm anticipates entry in the ensuing period.\footnote{Formally, the equilibrium strategy profile must specify a pricing strategy by the incumbent when entry does not occur. However, this outcome is always off of the equilibrium path, so I do not discuss it.}

3.4.1 Period Two

In the second period, the market is segmented into two groups: consumers who did elect to purchase from the incumbent in the previous period and those who did not. A consumer who elected not to purchase in period one faces a price of $p^I_2$ offered by the
incumbent and a price of $p^E_2$ offered by the entrant. For now, assume that the market is fully covered in period two (I derive the necessary and sufficient condition for this to be the case in the following section). In the second period, a consumer located at position $\theta$ who did not purchase in period one purchases from the incumbent in period two if and only if

$$\theta \leq \theta = 1/2 + \frac{p^E_2 - p^I_2}{2t}. \quad (3.1)$$

However, a consumer who did purchase in period one faces a price of $p^I_2 - r$ offered by the incumbent, which is lower than that offered to potential "new" customers. Hence, a consumer located at position $\theta$ in period two who did purchase from the incumbent in period one will purchase again from the incumbent if and only if

$$\theta \leq \theta = \frac{1}{2} + \frac{p^E_2 - p^I_2 + r}{2t}. \quad (3.2)$$

Because the market is fully covered in period two, the consumers mentioned above will purchase from the entrant if the relevant inequality is not satisfied. Switching costs for consumers are generated by the rewards program in the following sense. In the absence of rewards programs, a consumer located at $\theta$ such that $\theta < \theta < \theta$ would strictly prefer to purchase from the entrant, regardless of her prior purchasing decision. However, under the rewards program, a repeat customer of the incumbent who is located in this interval gains the additional benefit of the reward; the opportunity cost of switching, the foregoing of the reward, prevents such switching from taking place.

Letting $n^I_1$ denote the proportion of the market who purchased in period one, the total period-two market share of the incumbent is

$$n^I_2 = n^I_1 \left( \frac{1}{2} + \frac{p^E_2 - p^I_2 + r}{2t} \right) + (1 - n^I_1) \left( \frac{1}{2} + \frac{p^E_2 - p^I_2}{2t} \right), \quad (3.3)$$
while the period-two market share of the entrant is

\[ n_2^E = n_1^i \left( \frac{1}{2} + \frac{p_2^I - p_2^E - r}{2t} \right) + (1 - n_1^i) \left( \frac{1}{2} + \frac{p_2^I - p_2^E}{2t} \right). \]

The incumbent firm’s second-period profit function can be written as

\[ \Pi_2^I = (p_2^I - r - c^I) n_1^I \left( \frac{1}{2} + \frac{p_2^E - p_2^I + r}{2t} \right) + (p_2^I - c^I) (1 - n_1^I) \left( \frac{1}{2} + \frac{p_2^E - p_2^I}{2t} \right). \]

Similarly, the entrant’s profit function is given by

\[ \Pi_2^E = (p_2^E - c^E) \left( n_1^I \left( \frac{1}{2} + \frac{p_2^E - p_2^I}{2t} \right) + (1 - n_1^I) \left( \frac{1}{2} + \frac{p_2^E - p_2^I}{2t} \right) \right). \]

Again, both profit functions are strictly concave in firms’ own prices, and the first-order conditions yield firms’ reaction functions:

\[ p_2^I = \frac{1}{2} \left( c^I + t + p_2^E + 2rn_1^I \right) \]
\[ p_2^E = \frac{1}{2} \left( c^E + t + p_2^I - rn_1^I \right). \]

Firms’ period-two best response functions are very similar to those in the model without rewards. The differences, represented by the final term in each function, can be explained as follows. There are two effects of introducing a rewards program. The first is the switching cost that the reward generates. From the perspective of either firm, this effect can be interpreted as a virtual change in the other firm’s price. Fixing the firms’ prices, introducing a reward of amount \( r \) has the same effect on the incumbent as if the entrant increased its price by \( rn_1^I \). The reward causes the incumbent to further attract proportion \( \frac{r}{2t} \) of the \( n_1^I \) consumers who qualify for it; that is, the incumbent’s second-period market share increases by \( \frac{r}{2t} n_1^I \). This is identical to the effect that would result if the entrant increased its price by \( rn_1^I \) in the absence

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of a rewards program. Thus, with rewards, the incumbent increases its price by the same amount as it would if the entrant made such a price increase and there were no rewards. The entrant makes a similar adjustment downward, as if the entrant had reduced his price by $r n_1^f$.

The second effect of the rewards program can be interpreted as a change in the marginal cost of the incumbent. The rewards program, which has been discussed as a discount for repeat consumers, can alternatively be viewed as a cash payment to repeat consumers. This increases the marginal cost of attracting additional consumers. That is, selling to additional consumers who did purchase in the first period incurs upon the incumbent an additional cost of $r$ per unit sold. Fixing the entrant’s price, a reduction in price by the incumbent in the model with rewards increases sales by the same amount as the model without rewards, with proportion $n_1^f$ of these sales being to repeat customers and the remainder being to new customers. Hence, the incumbent’s marginal cost incurred by attracting new consumers has increased from $c^I$ in the model without rewards to $c^I + r n_1^f$ when rewards are used. Thus, the incumbent increases his price to compensate in the same manner it would if its cost $c^I$ were to rise by $r n_1^f$. The entrant, however, experiences no change in cost and makes no such adjustment. This is a crucial difference between the model of this chapter and models where switching costs exist exogenously rather than being created via rewards, in which case the incumbent firm would not have to make the reward payments. In the former environment, this change in marginal cost would be negated for both firms, whereas only the entrant is unaffected in the latter.

The preceding discussion can be summarized via the following annotated reaction functions which illustrate the differences between the Hotelling reaction functions and
those under a rewards program:

\[ p^I_2 = \frac{1}{2} \left( c^I + \text{change in cost} + t + p^E_2 + \text{virtual change in } p^E_2 \right); \quad p^E_2 = \frac{1}{2} \left( c^E + t + p^I_2 - \text{virtual change in } p^I_2 \right) \]

Solving firms’ reaction functions simultaneously yields the second-period subgame equilibrium prices:

\[ \hat{p}^I_2 = c^I - \frac{1}{3} (c^I - c^E) + t + r^{I_1} \]

\[ \hat{p}^E_2 = c^E + \frac{1}{3} (c^I - c^E) + t. \quad (3.5) \]

\[ \hat{p}^I_2 = c^I - \frac{1}{3} (c^I - c^E) + t + r^{I_1} \]

\[ \hat{p}^E_2 = c^E + \frac{1}{3} (c^I - c^E) + t. \quad (3.6) \]

In equilibrium, the incumbent increases his price from the level without rewards. In light of the previous discussion, this is not surprising. The incumbent has both a higher marginal cost and some of its customers face a switching cost; both of these effects cause the incumbent to increase his price.

Due to the asymmetric effects of the implementation of a rewards program on firms’ reaction functions, such a program does not affect the equilibrium price of the entrant. As can be seen from the reaction functions above, that is not to say that the entrant does not consider the impact of the reward on consumers’ decisions. If the incumbent were to deviate from equilibrium by changing the reward amount but not changing its second period price, the entrant would not be best responding by charging its Hotelling equilibrium price. Rather, in equilibrium, the entrant’s reaction to the two adjustments made by the incumbent exactly counteract the single adjustment the entrant would make on its own. As might be expected, this is different from the environment where the incumbent does not create switching costs by using rewards.
Instead, if switching costs exist exogenously, the equilibrium price of the entrant does not correspond to the Hotelling price.\textsuperscript{40}

While all consumers face the Hotelling price offered by the entrant, the incumbent’s potential new customers are faced with a price $p_I^2$ that lies above the Hotelling price. The incumbent’s potential repeat customers, however, face an effective price $p_I^2 - r$ which is lower than the Hotelling price. That is, $p_I^2 \geq p_I^H \geq p_I^2 - r$. Not surprisingly, then, the proportion of consumers eligible for the reward who buy from the incumbent is greater than the incumbent’s no-rewards market share, while the proportion of consumers not eligible for the reward who buy from the incumbent is less than the incumbent’s no-rewards market share, i.e., $\bar{\theta} \geq n_I^H \geq \theta$.

Substituting the period-two equilibrium prices into (3.3) yields the incumbent’s equilibrium period-two market share:

$$n_I^2 = n_I^1 \left( \frac{1}{2} + \frac{1}{3} (c^E - c^I) - \frac{r n_I^1 + r}{2t} \right) + (1 - n_I^1) \left( \frac{1}{2} + \frac{1}{3} (c^E - c^I) - \frac{r n_I^1}{2t} \right)$$

$$= \frac{1}{2} - \frac{1}{6t} (c^I - c^E) = n^I_H.$$

Noting that $n_E^2 = 1 - n_I^2$ and recalling that the entrant’s equilibrium price is the same whether or not rewards are used, the following results has been proven.

\textsuperscript{40}In fact, in this environment, a switching cost $r$ that can is costless for the incumbent results in equilibrium prices

$$p_I^1 = c^I - \frac{1}{3} (c^I - c^E) + t + \frac{1}{3} r n_I^1$$

$$p_E^2 = c^E + \frac{1}{3} (c^I - c^E) + t - \frac{1}{3} r n_I^1;$$

that is, the incumbent would increase its price over the Hotelling level by less than it does when it must pay the amount of switching cost, and the entrant would actually shade its price down from the Hotelling level by the same amount as the incumbent increases its price. This agrees with the intuition behind the reaction functions discussed above. Also note that the entrant’s price is now decreasing in the incumbent’s first-period sales level. This means that the incumbent’s action in period one will impact the profits of the entrant in period two. As shall shortly be demonstrated, this feature is absent in the model with rewards programs.
Proposition 3.1. In any equilibrium where the incumbent firm uses a rewards program, firms’ market shares in the second period are identical to those in the second period when no rewards are used. Furthermore, the entrant’s equilibrium second-period price and profit level are also unaffected by rewards.

Corollary 3.2. There is no rewards program which allows the incumbent to prevent entry.

The implication of this result is that if the entrant can profitably enter when rewards are not used, then it also enters profitably when a rewards program is in place. In other words, the incumbent is not able to forestall entry by implementing a rewards program prior to entry occurring in an effort to generate switching costs for consumers. This stands in contrast to models with exogenous switching costs, where entry may be deterred even in the presence of very low switching costs. The intuition for this difference lies in the added cost of the reward payout. When switching costs exist exogenously, the incumbent firm is able to prevent the entrant from gaining a critical mass of market share if it has sold to enough consumers in the first period because consumers are reluctant to switch. However, under the rewards program, the incumbent has to pay each repeat customer the amount of the reward. This provides the incumbent with an incentive to concede more of its old consumers to the entrant. This incentive grows stronger when the incumbent’s first-period sales level is larger because the effective marginal cost of the incumbent increases in the first-period sales level (recall that the incumbent must pay the reward amount to a proportion \( n_1 \) of any additional consumers it attracts through a price cut in the second period). As a result, subgame perfection requires that the entrant’s profit level be unaffected by the
incumbent’s first-period sales, unlike the situation where switching costs are costless for the incumbent.

The result of the impossibility of entry deterrence is more robust than might be immediately apparent. Appendix C.1 shows that the relaxation of several assumptions which have been made for simplicity do not alter this finding. In particular, this result does not depend critically upon the assumption that all consumers remain in the market in both periods, or that the distributions of consumers’ locations are independent from one period to the next. Furthermore, the above results under rewards are unaffected by allowing for consumers to have quadratic transportation costs, rather than linear costs.\footnote{Although not explicitly addressed in Appendix C.1, it should be clear that the assumption of no discounting does not affect the analysis of period two, so relaxing this assumption renders the above results unchanged.}

In what follows, it will be helpful to express the incumbent firm’s period-two profits in terms of the parameters of the model, the first-period sales level, and the reward amount. Substituting the subgame equilibrium prices into (3.4) and rearranging yields

$$\Pi^{I}_{2} = \frac{1}{2t} \left( t - \frac{1}{3} (c^{I} - c^{E}) \right)^2 - \frac{r^2}{2t} n^{I}_{1} (1 - n^{I}_{1})$$

$$= \Pi^{I}_{H} - \frac{r^2}{2t} n^{I}_{1} (1 - n^{I}_{1}). \quad (3.7)$$

The following result follows from this expression.

**Proposition 3.3.** In any equilibrium where the incumbent firm uses a rewards program, the incumbent firm’s equilibrium second-period profits are strictly less than the second-period profits when no rewards are used.

\footnote{The qualitative results of the rest of the model will be unaffected by allowing for any of these changes, so long as the picture does not change too drastically.}
The interpretation of the above profit function, and therefore this proposition, is as follows. The implementation of the rewards program has two effects on the second period profit, which are easily understood if the reward is viewed as a cost: a cash payout. First, revenue increases above the Hotelling level due to the higher price the incumbent charges as a result of the switching costs consumers face. The increase in revenue is precisely the price markup multiplied by the period-two market share, or \( rn_1^I \cdot n_2^I \). The second effect is that profits fall due to the cost of the reward payout. The actual amount that is paid out is the reward amount \( r \) multiplied by the mass of repeat purchasers. Of the \( n_1^I \) consumers eligible for the reward, \( \bar{\theta} \) actually buy from the incumbent again, hence, the mass of repeat purchasers is given by \( n_1^I \cdot \bar{\theta} \); the total reward payout is thus \( rn_1^I \cdot \bar{\theta} \). This means that the total change in profit is \( rn_1^I \cdot (n_2^I - \bar{\theta}) \). From (3.2) and (3.3), this is seen to be precisely the difference from the Hotelling profit level given in (3.7). In particular, because \( \bar{\theta} \geq n_H^I = n_2^I \), this total change in profits is always nonpositive. Rewards programs, in effect, actually put the incumbent at a disadvantage in the second period due to the reward payout. Note, however, that this loss is decreasing in the first-period sales level as long as that sales level is greater than the no-rewards monopoly sales level.\(^{42}\) The reason for this is that when more consumers buy in the first period, the second-period price increases, which increases revenue in the second period as the total market share is unaffected. Although more consumers are eligible for the reward, a smaller portion of these actually purchase again and claim the reward due to the higher price; this, in turn, further reduces the loss. That is, this loss provides the incumbent with an incentive to increase sales in the first period. This point will be returned to later.

\(^{42}\)Note that Assumption 3.3 implies that the monopolist sells to at least half the market when rewards are not used.
From the preceding discussion, it is clear that the implementation of a rewards program hurts the incumbent's period-two profits, regardless of the reward amount or period-one sales. This underscores another key difference between this model and other models that feature switching costs: the incumbent cannot exploit locked-in consumers in order to increase its profit in the second period and instead is placed at a disadvantage because it must pay the reward amount to those who are locked in. Any increase in total profit must instead come from increased profits in the first period, where the firm is made more attractive to consumers because of the potential reward in the second period coupled with consumers’ uncertainty about their future preferences. Trivially, setting the reward amount equal to zero results in Hotelling profits in the second period, and normal monopoly profits in the first period. The important question is then whether the incumbent can increase its profits in period one by enough to offset the period-two loss incurred by utilizing a rewards program. This question is addressed in the next section, where the first period of the game is considered.

3.4.2 Period One

In period one, the incumbent announces a first-period price $p_1^I$ and a reward amount $r$ which maximizes its total profits, taking into consideration the period-two equilibrium prices. To properly define this optimization problem, I first derive the first-period sales level of the incumbent in terms of the period-one price and the reward amount.

Because consumers get zero utility from not purchasing, the location of the indifferent consumer in period one, and therefore the incumbent’s sales level in the first
period, is the position where the sum of the utility of buying in period one and the expected utility in period two conditional on having done so is equal to the expected utility in period two conditional on not having purchased. Using (3.2), a consumer calculates her expected period-two utility conditional on having purchased in period one as

\[
\int_{\theta}^{\bar{\theta}} (R - \theta t - p_2^I + r) \, d\theta + \int_{\bar{\theta}}^{1} (R - (1 - \theta) t - p_2^E) \, d\theta
= \bar{\theta} \left( R - \frac{\bar{\theta}}{2} t - p_2^I + r \right) + (1 - \bar{\theta}) \left( R - \frac{(1 - \bar{\theta})}{2} t - p_2^E \right).
\] (3.8)

Likewise, using (3.1), a consumer’s expected period-two utility conditional on having not purchased is

\[
\int_{\theta}^{\bar{\theta}} (R - \theta t - p_2^I) \, d\theta + \int_{\bar{\theta}}^{1} (R - (1 - \theta) t - p_2^E) \, d\theta
= \bar{\theta} \left( R - \frac{\theta}{2} t - p_2^I \right) + (1 - \bar{\theta}) \left( R - \frac{(1 - \bar{\theta})}{2} t - p_2^E \right).
\] (3.9)

From the above expressions, the location of the indifferent consumer, \( n_1^I \), is then implicitly defined by the following equation:

\[
R - n_1^I t - p_1^I + \bar{\theta} \left( R - \frac{\bar{\theta}}{2} t - p_2^I + r \right) + (1 - \bar{\theta}) \left( R - \frac{(1 - \bar{\theta})}{2} t - p_2^E \right)
= \bar{\theta} \left( R - \frac{\theta}{2} t - p_2^I \right) + (1 - \bar{\theta}) \left( R - \frac{(1 - \bar{\theta})}{2} t - p_2^E \right).
\]

Substituting the subgame equilibrium prices, of which \( p_2^I \) depends on \( n_1^I \), into the expressions for \( \theta \) and \( \bar{\theta} \) then substituting for the four expressions in the above equation allows it to be solved for the incumbent’s first-period sales level in terms of the reward amount, the first-period price, and the parameters of the model:

\[
n_1^I \left( p_1^I, r \right) = \frac{4t \left( R - p_1^I \right) + 2r \left( t - \frac{1}{3} \left( c - c^E \right) \right) + r^2}{2 \left( r^2 + 2t^2 \right)}.
\] (3.10)

Inspection of \( n_1^I \left( p_1^I, r \right) \) yields an important result.
Proposition 3.4. Implementation of a rewards program decreases the price sensitivity of consumers in the first period. Moreover, a rewards program permits the incumbent to increase sales in the first period over the monopoly sales level while charging a price higher than the monopoly price.

Proof. The price sensitivity of consumers is the marginal loss in sales that results from an incremental price increase, given by \( \left| \frac{\partial n_1}{\partial p_1} \right| = \frac{2t}{r^2 + 2t^2} \). For any \( r > 0 \), this is strictly less than \( \frac{1}{t} \), the price sensitivity of consumers in the first-period of the no-rewards model. To complete the proof, first consider the first-period sales level that results when the no-rewards first-period price \( p_{1M} = \frac{R + c_I}{2} \) is charged by the incumbent and the reward amount \( \bar{r} = t - \frac{1}{3} (c_I - c_E) \) is used (which is strictly positive by Assumption 3.3):

\[
n_1^I (p_{1M}, \bar{r}) = \frac{4t \left( \frac{R - c_I}{2} \right) + 3\bar{r}^2}{2 (\bar{r}^2 + 2t^2)}.
\]

This is greater than \( n_{1M} = \frac{R - c_I}{2} \) if and only if

\[
4t^2 \left( \frac{R - c_I}{2} \right) + 3t\bar{r}^2 > (2\bar{r}^2 + 4t^2) \left( \frac{R - c_I}{2} \right),
\]

or, equivalently (as \( \bar{r} \) is strictly positive), \( 3t > R - c_I \). This is true by Assumption 3.2. Therefore, \( n_1^I (p_{1M}, \bar{r}) > n_{1M} \). Because \( n_1^I \) is continuous in \( p_1^I \), there exists a price \( p' > p_{1M} \) such that \( n_1^I (p', \bar{r}) > n_{1M} \).

The first statement of this proposition says that if the incumbent increases its price, it loses fewer customers when it uses a rewards program than when it does not use rewards. This might initially seem somewhat counterintuitive, as the increase in price has the same effect on consumers’ first-period utility whether or not rewards are present. The intuition lies in the fact that, as a result of the rewards program, the
incumbent’s period-two price is increasing in the first-period sales level. When the incumbent increases his first-period price, it will result in a lower sales level in the first period. Consumers correctly anticipate a lower period-two price from the incumbent (while the entrant’s price remains constant) and also realize that by purchasing from the incumbent they increase the probability that they will end up buying from the incumbent in the second period thus benefitting from the period-two price reduction. Thus, the decrease in purchasers’ expected utility as a result of a price increase is actually reduced when rewards are used.

The second statement of this proposition says that implementing a rewards program permits the incumbent firm to charge a higher price in the first period while, at the same time, increasing first-period sales. The appeal of the rewards program to consumers is thus: unsure of which firm’s product they will find more attractive in the second period, they are willing to spend more on the incumbent’s good in the first period because there is a positive probability that they will find the reward useful later. It has been established that the incumbent can increase its first-period profits through rewards. Next to be shown is that the increased first-period profit can be enough to counteract the second-period loss that results from utilizing rewards.

Henceforth, the arguments of \( n^I_1 \) are suppressed except where necessary. Now, the objective of the incumbent is to maximize total profits, which are the sum of the first- and second-period profits. The first-period profits are simply \( (p^I_1 - c^I) n^I_1 \), so, using (3.7), the profit function can be written

\[
\Pi^I = (p^I_1 - c^I) n^I_1 + \Pi^I_H - \frac{r^2}{2t} n^I_1 (1 - n^I_1) .
\]  

(3.11)

Here, an “interior solution,” is a solution \( (p^I_1, r) \) which maximizes the firm’s objective function and satisfies \( 0 < n^I_1 < 1 \). It can be shown that the unique candidate for an
interior solution \((p_1^I, r)\) to this optimization problem is given by

\[
\hat{p}_1^I = \frac{R + c^I}{2} + \frac{\hat{r}^2}{2t} \left(1 - \frac{R - c^I}{2t}\right) + \frac{\hat{r}^2}{8t} - \frac{\hat{r}}{16t^3} \tag{3.12}
\]

\[
\hat{r} = t - \frac{1}{3} \left(c^I - c^E\right) \tag{3.13}
\]

The derivation of the candidate solution proceeds as follows. First, the first-order condition for the first-period price is solved for the price. This expression for the price is then substituted into the first-order condition for the reward amount. The resulting condition for an interior solution is cubic in the reward amount; however, two of the roots correspond to a first-period sales of zero, which is clearly not optimal (as setting the reward amount equal to zero obviously yields higher total profits). The remaining root yields the candidate optimal reward amount, which is then substituted into the first-order condition for the first-period price, which in turn yields the candidate optimal price. The analytical details are presented in Appendix C.2.

Note that \(\hat{r}\) is positive if and only if Assumption 3.4 holds.\(^{43}\) Recall that Assumption 3.4 is the necessary and sufficient condition for the incumbent firm to realize positive sales in the second period when rewards are not used. That is, there can only be an equilibrium where rewards benefit the incumbent if the incumbent is not completely shut out of the market in the second period when rewards are not used. In other words, if the incumbent cannot realize positive profits in the second period without rewards, there is no rewards program that will help the incumbent increase its total profits. This is not surprising; by Proposition 3.1, period-two market shares are unaffected by rewards. If all consumers know that they won’t be buying from the incumbent in the second period, participating in a rewards program does not benefit

\(^{43}\)Appendix C.2 shows that the reward amount is the same in the optimal corner solution, so Assumption 3.4 covers all cases.
them. The best the incumbent can do is set its normal monopoly price in the first period, as its selection of reward amount does not matter.44

Because the analysis thus far has not relied upon the fact that the reward amount is positive, it can be seen that even if the incumbent had the option of using a negative reward amount, thereby charging a higher price to repeat consumers, it would not elect to do so. This contrasts with models of observed-behavior price discrimination that support such behavior as an equilibrium strategy. Again, this should not be surprising. In those models, a firm uses this type of price discrimination with the aim of charging a higher price to those consumers who prefer its own good while charging a lower price to customers who otherwise would prefer the product of its opponent in order to “poach” them. In the current model, consumers’ locations in the second period are randomized, so prior purchasing decisions reveal nothing about consumers’ current preferences.

Substituting the values for $\hat{p}_I$ and $\hat{r}$ into (3.10) yields the incumbent’s sales level in the first period for the interior solution:

$$\hat{n}_I = \frac{R - c^I}{2t} + \frac{\hat{r}^2}{8t^2}. \quad (3.14)$$

Noting that this expression is positive, the necessary and sufficient condition for $(\hat{p}_I, \hat{r})$ to be an valid solution (that is, it results in $0 \leq n_I \leq 1$) is given by the following assumption:

**Assumption 3.5.** $\hat{r}^2 \leq 4t \left(2t - (R - c^I)\right)$.\n
The intuition for this assumption is straightforward. It merely specifies that the incumbent not cover too much of the market in the first period when it doesn’t use

44Technically speaking, any reward amount paired with the monopoly price would constitute an equilibrium, but no consumers would be swayed by the reward.
rewards (i.e., $R - c^I$ is not too close to $2t$). As has been discussed, the incumbent can end up benefitting from rewards by increasing its sales, even at a higher price. If this assumption does not hold, then the monopolist optimally covers the entire market in the first period, necessitating a corner solution.\footnote{In Appendix C.2, it is demonstrated that the interior solution and corner solutions coincide when this assumption holds as an equality.}

Importantly, this assumption is not necessary for an equilibrium to exist. In fact, if it fails, the optimal corner solution (where $n^I_1 = 1$) is the unique equilibrium. In this situation, the incumbent will charge a first-period price $\hat{p}^I_1$ which is higher than $\hat{p}^I_1$, and will still use the reward $\hat{r}$.\footnote{Here, $\hat{p}^I_1$ denotes the optimal price under the constraint that the market is covered during period one. However, note that $\hat{p}^I_1 > \hat{p}^I_1$ if and only if Assumption 3.5 fails. It should not be surprising that when Assumption 3.5 holds, $\hat{p}^I_1 \leq \hat{p}^I_1$. When parameters are such that an interior solution is optimal, charging a lower price than the equilibrium price would be necessary to sell to the entire market.} Intuitively, charging the lower price $\hat{p}^I_1$ covers the market, but the consumer located at 1 (as well as every other consumer) still strictly prefers buying to not buying. Thus, the incumbent increases its price to extract this “excess” surplus. This equilibrium is otherwise qualitatively similar to the equilibrium when the solution is in the interior, so the remainder of the analysis will focus on the interior solution. For proofs of the claims made in this paragraph, the reader is referred to the end of Appendix C.2.

If this assumption holds, then $(\hat{p}^I_1, \hat{r})$ is the unique optimal price and reward amount as long as selecting it results in higher profits than choosing $(p^I_M, 0)$, i.e., electing not to use rewards, and higher profits than choosing the optimal corner solution. As the following proposition points out, this is indeed the case.

**Proposition 3.5.** The incumbent firm’s equilibrium first-period price and reward amount are given by $\hat{p}^I_1$ and $\hat{r}$, respectively. Furthermore, the equilibrium first-period

price satisfies
\[
\hat{p}_1' = \frac{R + c^l}{2} + \frac{\hat{r}^2}{2t} \left(1 - \frac{R - c^l}{2t}\right) + \frac{\hat{r}^2}{8t} - \frac{\hat{r}^4}{16t^3} > \frac{R + c^l}{2} = p'_M, \tag{3.15}
\]
the equilibrium first-period sales level satisfies
\[
\hat{n}_1' = \frac{R - c^l}{2t} + \frac{\hat{r}^2}{8t^2} > \frac{R - c^l}{2t} = n'_M, \tag{3.16}
\]
and the incumbent’s equilibrium total profit level satisfies
\[
\hat{\Pi}' = \frac{1}{t} \left(\frac{R - c^l}{2} + \frac{\hat{r}^2}{8t}\right)^2 + \Pi_H^I > \frac{1}{t} \left(\frac{R - c^l}{2}\right)^2 + \Pi_H^I = \Pi_M^I + \Pi_H^I. \tag{3.17}
\]

Proof. The first equality in (3.15) restates (3.12), while the inequality follows by noting that \(\frac{R - c^l}{2t} < 1\) by Assumption 3.2 and \(\hat{r} \leq t\) by Assumption 3.4, hence, \(\frac{\hat{r}^2}{8t} > \frac{\hat{r}^4}{16t^3}\). The first equality in (3.16) restates (3.14), and the inequality follows trivially. The first equality in (3.17) is a result of substituting (3.12) and (3.14) into (3.11) and simplifying; the inequality follows trivially. Inequality (3.17) implies that the candidate solution dominates the no-rewards strategy. It remains to show that the candidate solution dominates all corner solutions and to check that 0 \(\leq \theta \leq 1\) and 0 \(\leq \bar{\theta} \leq 1\) (as this latter point has been ignored in the analysis thus far). The remaining details can be found in Appendix C.2.

As might be expected, in light of Proposition 3.4, the incumbent firm charges a higher price than the no-rewards price in the first period while at the same time increasing first-period sales. Furthermore, this increased first-period profit generated by rewards more than makes up for the loss of second-period profits due to the rewards program, as total profits also rise. Note that the ex ante expected benefit consumers derive from the reward is actually less than the payout the firm incurs due to the reward. Consumers who are eligible for the reward and who would buy from
the incumbent in period two even without the reward do see a period-two increase
in utility of the entire amount of the reward as a result of their eligibility. However,
consumers who would buy from the entrant in period two if they were not eligible for
the reward but who purchase from the incumbent again due to the switching cost see
their utility rise by only a portion of the discount offered, the full amount of which
must be paid by the incumbent. Hence, the incumbent is not merely extracting this
excess utility early on via higher prices, as doing so could not benefit the incumbent
enough to cover the loss in the second period if consumer price sensitivity were un-
affected. Rather, that profits rise when rewards are imposed is heavily dependent
on the fact that consumers’ decreased price sensitivity in the first period causes the
incumbent to lose fewer customers with an increase in price than when rewards aren’t
used.

The incumbent firm increases its price from the no-rewards level in response to
the increased willingness to pay and decreased price sensitivity that is generated by
the rewards program; however, the incumbent does not increase the price by so much
that it merely maintains its no-rewards sales level. The incumbent has an incentive
to sell to more consumers for two reasons, as discussed in the previous section. As the
first-period sales level rises, the price that the incumbent will charge in the second
period increases. While more consumers become eligible for the reward, a smaller
portion of these eligible do claim the reward due to the higher price. At the same
time, the total second period market share is unaffected, per Proposition 3.1, so this
increase in price increases second-period revenues. The combination of these two
effects causes an increase in the first-period sales level above the no-rewards level to
have a positive impact on second-period profits as well as first-period profits.
Until this point, the analysis has merely assumed that the market is fully covered in the second period. Given firms’ prices, this will only be true provided that the indifferent consumers located at $\theta$ and $\bar{\theta}$ realize nonnegative utility upon purchasing from either firm. The following assumption is the necessary and sufficient condition for market coverage, and is derived in Appendix C.3.

**Assumption 3.6.** $R - c^f \geq \frac{3}{2} \hat{r} + \frac{1}{2} \hat{r} \hat{n}^I_1$.

The essence of this assumption is that for the market to be fully covered under the second-period equilibrium prices, the optimal reward amount $\hat{r}$ cannot be too large. The intuition for this lies in that the consumer who did not buy in the first period but is indifferent between firms in the second period, located at $\theta$, is hurt *ex post* as a result of the rewards program. Without rewards, she still would not have purchased from the incumbent in the first period, as she must have been located to the right of $\hat{n}^I_1 > n^I_M$. In the second period, without rewards, she would have purchased from the incumbent, as $\theta < n^I_H$. However, the incumbent’s second period equilibrium price is increased from the no-rewards price, by precisely $\hat{r} \hat{n}^I_1$, so this consumer is hurt by the rewards program and may realize negative utility by purchasing if this price markup is too large. Assumption 3.6 guarantees that this consumer will purchase in the second period, thus ensuring that the period-two market, which is covered under no rewards, is indeed covered under rewards.$^{47}$

It may be of interest to note that Assumption 3.6 can be relaxed if the parameter $R$ is permitted to increase from period one to period two. To see why, suppose that $R$ does change between periods one and two, so that $R_\tau$ is the value of this parameter in

$^{47}$Note that this assumption does not violate any other assumptions. For example, $R - c^f \geq 2\hat{r}$ is sufficient for this assumption to hold for any value of $\hat{n}^I_1$. Substituting in the optimal reward amount, this is true if $t \leq \frac{1}{3} (c^f - c^E)$ by Assumption 3.3.
period $\tau$. The only time $R_2$ enters into the analysis, whether rewards are used or not, is to ensure market coverage in the second period. Assumption 3.3 establishes the necessary condition for the no-rewards case, while Assumption 3.6 handles the case when rewards are used. As these assumptions are never used elsewhere in the analysis except where they cancel in (3.8) and (3.9), all other appearances of $R$ correspond to $R_1$. That is, as long as the market is covered, then in equilibrium neither firms nor consumers are affected by a change in $R_2$. Thus, as long as $R_2$ increases by enough, $R_1$ remains unrestricted by Assumption 3.6.

A change in $R$ may occur, for example, if consumers realize higher utility from consumption in period two due to some external event. In the case of automobile markets, the implementation of tax incentives for owning newer, more efficient cars may increase the reservation values of new cars between purchases. Additionally, newer cars have regularly been equipped with more safety features and the like, as well as better gas mileage, than their older counterparts as technology has improved. These changes may also cause an increase in the inherent desirability of the products in question over time. Similarly, if firms are airlines offering different flights on the same departing route out of an airport, this route may become more desirable to consumers due to some event that happens near the other, receiving, end of the route, such as the opening of a vacation resort or tourist attraction.

In some cases, entry may be spurred in the first place by just such an event. Consider the airline example. An increase in $R$ on the departing route in this case will likely apply to other potential routes into and out of the receiving airport. If this happens, the entrant may find establishing itself as a monopolist on previously unoffered routes out of the receiving airport to be profitable. If its cost of entry on
the departing route is decreasing in the number of routes it offers into and out of the receiving airport, as very well may be the case, then entering the departing route may also become profitable even if it was not before the increase in $R$.

### 3.5 Conclusion

This chapter presented a study of the potential benefits of implementing a coupon loyalty rewards program for a monopolist in the presence of a potential entrant. While the incumbent has no hope of deterring entry, it can increase equilibrium profits by starting a coupon rewards program before entry occurs, even though doing so results in lower profits after entry occurs. In particular, because of the reduced price sensitivity that coupons generate for consumers, the incumbent is able to increase its first-period sales while charging a higher price.

These results contrast sharply with the standard switching costs models, which permit entry deterrence on the part of the monopolist by means of low pricing in the first period in order to lock in consumers in the second period, thereby preventing the entrant from recouping the cost of entry. In the current model, the incumbent firm is actually hurt by a large established customer base in the second period, because it must pay the reward amount (or charge a lower price) to those who are repeat customers. In standard models of switching costs, it is often the case that firms focus on attracting consumers early and exploiting them later. By contrast, in this model, the focus is on creating surplus early through rewards, which is immediately extracted in order to make up for a loss later on.

Although coupon loyalty rewards programs are indeed a form of price discrimination based on past behavior, the results of this chapter are substantially different.
than those found in other behavior-based price discrimination studies, where firms typically offer higher prices to repeat customers than to new customers. Critically, when consumers preferences vary from period to period, the results of those studies may fail, leaving rewards programs as the optimal price discrimination strategy.

In summary, coupon rewards programs, which are widely observed in real-world environments, have rarely been studied. The current study is an attempt to better understand the motivation behind such programs, which have not received significant attention thus far. Further work in this area is more than warranted; it is my hope that this work will make a good starting point.
Appendix A: Appendix to Partnerships and Loyalty
Discounts in a Differentiated Products Market

A.1 Solving for the Symmetric Equilibrium Under No Partnerships

This appendix provides the computational details behind solving the period-one first-order conditions of firms when no partnerships exist. Solving for the equilibria of the games induced by other partnership networks proceeds in a similar fashion, though the details are left to the reader.

Recall that, under no partnerships, in period one the expected benefit of being eligible for firm $i$’s reward in period two is

$$GB_i = \frac{r_i}{4} + \frac{r_i}{2t}(e_jr_j + e_kr_k - 2e_ir_i) + \frac{r_i^2}{2t},$$

where firm $i$ has neighbors $j$ and $k$. In general, the consumer located at distance $n_{ij}$ from firm $i$ between $i$ and $j$ is indifferent between these two firms if

$$\frac{t}{4} - 2tn_{ij} - p_i^1 + p_j^1 + GB_i - GB_j = 0.$$

The left-hand side of this equation defines the implicit function $F_{ij}$ given in the body of the chapter.
Because there are only four indifferent consumers, I restrict my attention to the functions defining \( n_{14}, n_{12}, n_{34}, \) and \( n_{32} \). The following expressions for the first-period market shares of each firm (i.e., the mass of consumers who will be eligible for each reward in period two) are helpful:

\[
e_1 = n_{14} + n_{12}, \quad e_2 = \frac{1}{2} - n_{12} - n_{32}, \quad e_3 = n_{32} + n_{34}, \quad e_4 = \frac{1}{2} - n_{32} - n_{14}.
\]

It is easy to show that under symmetry (i.e., \( p^1_i = p^1 \) and \( r_i = r \) for all \( i \)), \( n_{ij} = \frac{1}{8} \) for each of these indifferent consumers. Hence, under symmetry, \( e_i = \frac{1}{4} \) for all \( i \).

In order to solve the first-order conditions for firm 1’s profit maximization problem under symmetry, it is then necessary to evaluate \( \frac{\partial n_{ij}}{\partial p^1_i} \) and \( \frac{\partial n_{ij}}{\partial r_1} \) for \( i = 1, 3 \) and \( j = 2, 4 \) under symmetry. Differentiating \( F_{14}, F_{12}, F_{34}, \) and \( F_{32} \) with respect to \( p^1_1 \), setting these derivatives equal to zero, imposing \( r_1 = r_2 = r_3 = r_4 = r \), and solving the resulting system yields the following values:

\[
\frac{\partial n_{14}}{\partial p^1_1} = \frac{\partial n_{12}}{\partial p^1_1} = -\frac{5r^2t + 2t^3}{4(r^2 + t^2)(4r^2 + t^2)}, \quad \frac{\partial n_{34}}{\partial p^1_1} = \frac{\partial n_{32}}{\partial p^1_1} = \frac{3r^2t}{4(r^2 + t^2)(4r^2 + t^2)}.
\]

(A.1)

Differentiating \( F_{14}, F_{12}, F_{34}, \) and \( F_{32} \) with respect to \( r_1 \), setting these derivatives equal to zero, imposing \( r_1 = r_2 = r_3 = r_4 = r \) and \( n_{14} = n_{12} = n_{34} = n_{32} = \frac{1}{8} \), and solving the resulting system yields the following values:

\[
\frac{\partial n_{14}}{\partial r_1} = \frac{\partial n_{12}}{\partial r_1} = \frac{14r^3 + 5r^2t + 5rt^2 + 2t^3}{16(r^2 + t^2)(4r^2 + t^2)}, \quad \frac{\partial n_{34}}{\partial r_1} = \frac{\partial n_{32}}{\partial r_1} = -\frac{10r^3 + 3r^2t + rt^2}{16(r^2 + t^2)(4r^2 + t^2)}.
\]

(A.2)

Thus, under symmetry

\[
\frac{\partial e_1}{\partial p^1_1} = 2 \frac{\partial n_{14}}{\partial p^1_1}, \quad \frac{\partial e_2}{\partial p^1_1} = \frac{\partial e_4}{\partial p^1_1} = -\frac{\partial n_{14}}{\partial p^1_1} - \frac{\partial n_{34}}{\partial p^1_1}, \quad \frac{\partial e_3}{\partial p^1_1} = 2 \frac{\partial n_{34}}{\partial p^1_1},
\]

and

\[
\frac{\partial e_1}{\partial r_1} = 2 \frac{\partial n_{14}}{\partial r_1}, \quad \frac{\partial e_2}{\partial r_1} = \frac{\partial e_4}{\partial r_1} = -\frac{\partial n_{14}}{\partial r_1} - \frac{\partial n_{34}}{\partial r_1}, \quad \frac{\partial e_3}{\partial r_1} = 2 \frac{\partial n_{34}}{\partial r_1},
\]
where the partial derivatives in the right-hand sides of these equations are evaluated under symmetry, as given by (A.1) and (A.2). Using these expressions allows firm 1’s first-order conditions to be solved for the symmetric equilibrium prices and reward amounts. Recall that firm 1’s total profits are given by

$$\Pi_1 = (p_1 - c) e_1 + \frac{t}{16} - \frac{e_1 r_1}{2t} (2 (1 - e_1) r_1 + e_4 r_4 + e_2 r_2).$$

Imposing symmetry as described above on the first-order condition for firm 1’s price and rearranging yields

$$p_1^1 - c = \frac{(2r^2 + t^2) (11r^2 + 2t^2)}{4t (5r^2 + 2t^2)}. \quad \text{(A.3)}$$

Similarly imposing symmetry on the first-order condition for firm 1’s reward amount and rearranging yields

$$\left(56r^3 t + 20r^2 t^2 + 20rt^3 + 8t^4\right) (p_1^1 - c) = 96r^5 + 14r^4 t + 83r^3 t^2 + 5r^2 t^3 + 14rt^4.$$

Substituting the first of these equations into the second and simplifying yields

$$43r^3 - 10r^2 t + 18rt^2 - 4t^3 = 0.$$

The left-hand side of this equation has a unique real root, which is the equilibrium reward amount when no firms are partnered. I denote this equilibrium reward amount $r^*$. While $r^*$ has a closed-form representation, it is cumbersome, and the numerical approximation presented in the body of the chapter is much more useful. Substituting $r^*$ into (A.3) and solving for $p_1^1$ yields the equilibrium first-period price $p^*$, and both of these equilibrium values can be used to solve for the equilibrium total profits of each firm, denoted by $\Pi^*$. The numerical approximations for these equilibrium values are given in the body of the chapter.
Appendix B: Appendix to Consumer Rewards Programs in a Multi-Market Setting

Appendix

B.1 Solving for the Equilibria

This appendix provides the analytical details of solving for the equilibria of the competition games presented in Sections 2.3 and 2.4, where no partnerships and two partnerships exist, respectively.

B.1.1 No Partnerships

With no partnerships, firm 0 wishes to maximize

$$\Pi^0 = 6(p_U^0 - c_U)e^0 + \frac{3t_D}{2} - \frac{1}{2t_D} r^0 (3r^0 + r^1) e^0 (1 - e^0),$$

where

$$e^0 = \frac{1}{2} + \frac{3t_D (r^0 - r^1) + 12t_D (p_U^1 - p_U^0)}{2 (3 (r^0)^2 + 2r^0r^1 + 3 (r^1)^2 + 24t_D t_U)}.$$  

First, note that

$$\frac{\partial \Pi^0}{\partial p_U^0} = 6 \left[ (p_U^0 - c_U) \frac{\partial e^0}{\partial p_U^0} + e^0 \right] - \frac{1}{2t_D} r^0 (3r^0 + r^1) \left[ (1 - e^0) \frac{\partial e^0}{\partial p_U^0} - e^0 \frac{\partial e^0}{\partial p_U^0} \right],$$

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and

\[
\frac{\partial e^0}{\partial p_U^0} = -\frac{12t_D}{3 (r^0)^2 + 2r^0r^1 + 3 (r^1)^2 + 24t_D r_U}.
\]

Imposing symmetry, \( p^0_U = p^1_U = p_U \) and \( r^0 = r^1 = r \) implies \( e^0 = 1/2 \). Solving the first-order condition for \( p^0_U \) yields

\[
p_U = c_U + t_U + \frac{r^2}{3t_D}.
\]

(B.1)

Next,

\[
\frac{\partial \Pi^0}{\partial r^0} = 6 \left( p^0_U - c_U \right) \frac{\partial e^0}{\partial r^0} - \frac{1}{2t_D} e^0 (1 - e^0) (6r^0 + r^1) - \frac{1}{2t_D} r^0 (3r^0 + r^1) \left[ (1 - e^0) \frac{\partial e^0}{\partial r^0} - e^0 \frac{\partial e^0}{\partial r^0} \right].
\]

Differentiating the expression for \( e^0 \) with respect to \( r^0 \) and imposing symmetry yields

\[
\frac{\partial e^0}{\partial r^0} = \frac{3t_D}{8 (r^2 + 3t_D t_U)}.
\]

Solving the first-order condition for \( r^0 \) by imposing symmetry and substituting from the above equation and (B.1) yields \( r = 6t_D/7 \). Evaluating (B.1) at this reward then gives the equilibrium upstream market price, \( p_U = c_U + t_U + 12t_D/49 \). Evaluating \( \Pi^0 \) at these values yields the equilibrium profit level. Note that the firms present only downstream realize profits of \( 3t_D/2 \) regardless of the upstream strategies of the other firms.

**B.1.2 Two Partnerships**

Under two partnerships, firm 0 wishes to maximize

\[
\Pi_J = 6 \left( p^0_U - c_U \right) e^0 + 3t_D - \frac{2}{t_D} r^0 (r^0 + r^1) e^0 (1 - e^0)
\]
where
\[
e^0 = \frac{1}{2} + \frac{3t_D (r^0 - r^1) + 6t_D (p_U^1 - p_U^0)}{2 (r^0 + r^1)^2 + 6t_D t_U}.
\]

Again, begin by noting that
\[
\frac{\partial \Pi^0}{\partial p_U^0} = 6 \left[ (p_U^0 - c_U) \frac{\partial e^0}{\partial p_U^0} + e^0 \right] - \frac{2}{t_D} r^0 (r^0 + r^1) \left[ (1 - e^0) \frac{\partial e^0}{\partial p_U^0} - e^0 \frac{\partial e^0}{\partial p_U^0} \right],
\]
and
\[
\frac{\partial e^0}{\partial p_U^0} = -\frac{3t_D}{(r^0 + r^1)^2 + 6t_D t_U}.
\]

Imposing symmetry on the first-order condition for \(p_U^0\) and solving yields
\[
p_U = c_U + t_U + \frac{2r^2}{3t_D}. \tag{B.2}
\]

Next,
\[
\frac{\partial \Pi^0}{\partial r^0} = 6 (p_U^0 - c_U) \frac{\partial e^0}{\partial r^0} - \frac{2}{t_D} e^0 (1 - e^0) (2r^0 + r^1)
- \frac{2}{t_D} r^0 (r^0 + r^1) \left[ (1 - e^0) \frac{\partial e^0}{\partial r^0} - e^0 \frac{\partial e^0}{\partial r^0} \right].
\]

Differentiating the expression for \(e^0\) with respect to \(r^0\) and imposing symmetry yields
\[
\frac{\partial e^0}{\partial r^0} = \frac{3t_D}{4 (2r^2 + 3t_D t_U)}.
\]

Solving the first-order condition for \(r^0\) by imposing symmetry and substituting from the above equation and (B.2) yields \(r = t_D\). Evaluating (B.1) at this reward then gives the equilibrium upstream market price, \(p_U = c_U + t_U + 2t_D / 3\). Evaluating firm 0’s objective function at these values yields the equilibrium joint profits of the partnered firms.
B.2 A Single Partnership with One Downstream Member

Suppose that a single partnership exists between firm 0 and a downstream firm, denoted $P_0$, while firm 1 is not partnered with any other firm. Let the remaining firm be denoted $N$. In the downstream market, firm 0’s profits, $\Pi^D_0$, are given by

$$(p^0_D - c_D) \left[ y_{0,P_0} + e^0 (x_{0,1} + x_{0,N}) + (1 - e^0)(z_{0,1} + y_{0,N}) \right] - r^0 e^0 (y_{0,P_0} + x_{0,1} + x_{0,N}),$$

and the profits of its partner, $\Pi^D_{P_0}$, are similar. In this market firm 1 has no partner, so its profits, $\Pi^D_1$, are given by

$$(p^1_D - c_D) \left[ (1 - e^0) (x_{1,0} + x_{1,P_0} + x_{1,N}) + e^0 (z_{1,0} + z_{1,P_0} + y_{1,N}) \right] - r^1 (1 - e^0) (x_{1,0} + x_{1,P_0} + x_{1,N}).$$

Finally, the profits of the sole firm which does not honor rewards in this market are given by

$$\Pi^N_D = (p^N_D - c_D) \left[ e^0 (z_{N,0} + z_{N,P_0} + y_{N,1}) + (1 - e^0)(y_{N,0} + y_{N,P_0} + z_{N,1}) \right].$$

Solving the first-order conditions for firms’ prices simultaneously yields the equilibrium pricing strategies of the downstream market:

$$p^0_D = p^P_{D0} = c_D + t_D + e^0 r^0, \quad p^1_D = c_D + t_D + (1 - e^0) r^1, \quad p^N_D = c_D + t_D.$$  

Here, note that substituting these prices into firm $N$’s profit function yields $\Pi^N_D = 3t_D/2$. Additionally,

$$\Pi^N_D = \Pi^P_{D0} = \frac{3t_D}{2} - \frac{1}{2t_D} r^0 (2r^0 + r^1) e^0 (1 - e^0),$$
and

$$\Pi^1_D = \frac{3t_D}{2} - \frac{1}{2t_D} r^1 (2r^0 + 3r^1) e^0 (1 - e^0).$$

In the upstream market, consumers anticipate $p^0_D = p^0_D$, so the relative benefits of reward eligibility are as follows:

$$GB^0 = \frac{1}{6} r^0 + \frac{1}{3} \left[ r^0 y_{0,1} + \frac{r^0}{2} (x_{0,1} - y_{0,1}) \right] + \frac{1}{3} \left[ r^0 y_{0,N} + \frac{r^0}{2} (x_{0,N} - y_{0,N}) \right]$$

$$= \frac{r^0}{2} + \frac{r^0}{6t_D} [(1 - e^0)r^1 - 2e^0 r^0] + \frac{(r^0)^2}{6t_D},$$

and

$$GB^1 = \frac{1}{3} \left[ r^1 y_{1,0} + \frac{r^1}{2} (x_{1,0} - y_{1,0}) \right] + \frac{1}{6} \left[ r^1 y_{1,N} + \frac{r^1}{2} (x_{1,N} - y_{1,N}) \right]$$

$$= \frac{r^1}{4} + \frac{r^1}{12t_D} [2e^0 r^0 - 3 (1 - e^0) r^1] + \frac{(r^1)^2}{8t_D}.$$
costs in the two markets, they can be solved simultaneously using numerical methods. These equilibrium reward levels can then be used approximate equilibrium prices and profits. For different values of \( t_U \) and \( t_D \), Table 2.1 displays firms’ equilibrium strategies, and Table 2.2 reports equilibrium profits.

### B.3 A Single Partnership with Two Downstream Members

Suppose that a single partnership exists between firm 0 and both firms not present in the upstream market while firm 1 is not partnered with any other firm. Let firm 0’s partners be denoted by \( P_1 \) and \( P_2 \). In the downstream market, firm 0’s profits, \( \Pi^0_D \), are given by

\[
(p^0_D - c_D) \left[ y_{0,P_1} + y_{0,P_2} + e^0 x_{0,1} + (1 - e^0) z_{0,1} \right] - r^0 e^0 (y_{0,P_1} + y_{0,P_2} + x_{0,1}).
\]

Expressions for the profits of firm 0’s partners, \( \Pi^{P_1}_D \) and \( \Pi^{P_2}_D \) are similar. Firm 1’s profits in this market, \( \Pi^1_D \), are given by

\[
(p^1_D - c_D) \left[ (1 - e^0) \left( x_{1,0} + x_{1,P_1} + x_{1,P_2} \right) + e^0 (z_{1,0} + z_{1,P_1} + z_{1,P_2}) \right] - r^1 \left( 1 - e^0 \right) \left( x_{1,0} + x_{1,P_1} + x_{1,P_2} \right).
\]

Firms’ equilibrium pricing strategies in this market are

\[
p^0_D = p^{P_1}_D = p^{P_2}_D = c_D + t_D + e^0 r^0, \quad p^1_D = c_D + t_D + (1 - e^0) r^1,
\]

thus firms’ downstream equilibrium profits are

\[
\Pi^0_D = \Pi^{P_1}_D = \Pi^{P_2}_D = \frac{3t_D}{2} - \frac{1}{2t_D} r^0 (r^0 + r^1) e^0 (1 - e^0),
\]

\[
\Pi^1_D = \frac{3t_D}{2} - \frac{3}{2t_D} r^1 (r^0 + r^1) e^0 (1 - e^0).
\]
In the upstream market, consumers anticipate $p_D^0 = p_D^{P1} = p_D^{P2}$, so the relative benefits of reward eligibility are as follows:

$$GB^0 = \frac{1}{2} r^0 + \frac{1}{2} \left[ r^0 y_{0,1} + \frac{r^0}{2} (x_{0,1} - y_{0,1}) \right] = \frac{3r^0}{4} + \frac{r^0}{4t_D} \left[ (1 - e^0) r^1 - e^0 r^0 \right] + \frac{(r^0)^2}{8t_D},$$

and

$$GB^1 = \frac{1}{2} \left[ r^1 y_{1,0} + \frac{r^1}{2} (x_{1,0} - y_{1,0}) \right] = \frac{r^1}{4} + \frac{r^1}{4t_D} \left[ e^0 r^0 - (1 - e^0) r^1 \right] + \frac{(r^1)^2}{8t_D}.$$ 

Therefore, in the upstream market, firm 0 maximizes

$$\Pi^J = 6 \left( p_U^0 - c_U \right) e^0 + \frac{9t_D}{2} - \frac{3t_D}{2} r^0 \left( r^0 + r^1 \right) e^0 \left( 1 - e^0 \right),$$

and firm 1 maximizes

$$\Pi^I = 6 \left( p_U^1 - c_U \right) (1 - e^0) + \frac{3t_D}{2} - \frac{3t_D}{2} r^1 \left( r^0 + r^1 \right) e^0 \left( 1 - e^0 \right),$$

where

$$e^0 = \frac{1}{2} + \frac{t_D (3r^0 - r^1) + 4t_D (p_U^1 - p_U^0)}{(r^0 + r^1)^2 + 8t_D t_U}.$$

Again, the equilibrium prices (net of costs) can be solved in terms of the reward amounts from the first-order conditions for firm prices. The first-order conditions of the reward values may then expressed in terms of only $r^0, r^1$ and the parameters of the model (namely, $t_D$ and $t_U$). Numerical solving these first-order conditions, after choosing values for the transportation cost parameters, pins down the equilibrium reward levels for the upstream duopolists, which can in turn be used to calculate equilibrium prices and profits.
Appendix C: Appendix to Loyalty Rewards Programs in the Face of Entry

C.1 On the Impossibility of Entry Deterrence

Proposition 3.1 implies that entry cannot be deterred regardless of the rewards program chosen by the incumbent. That the entrant receives precisely the same profits regardless of the strategy of the incumbent may seem to depend critically on several specific assumptions made. However, this appendix will show that the impossibility of entry deterrence is robust to a number of changes to the model. In particular, Proposition 3.1 holds under a variety of changes. Moreover, in the event that Proposition 3.1 may not hold, it is still the case that entry is unavoidable.

I consider several possible adjustments: using quadratic, rather than linear, transportation costs; allowing a portion of consumers to exit the market between the first and second periods; and relaxing the assumption that consumers are redistributed independently of their earlier positions.

C.1.1 Transportation Costs

I first consider introducing quadratic transportation costs rather than the linear costs of the current model. Consider the modified utility of a consumer located at $\theta$
who purchases from the firm located at \( i \) at a price of \( p^i \):

\[
U(\theta, i) = R - |\theta - i|^2 t - p^i.
\]

Although quadratic costs have long been used in linear city models such as this (see, e.g. D’Aspremont et al. (1979)), it is easily shown that when firms are located at the endpoints of the unit interval and the market is covered (as is the case in period two), then firms sales levels (given prices) are equivalent under either specification. In other words, firm \( i \)’s no-rewards level of sales in period two, \( n^i_H(p^i_H, p^j_H) \), is identical in each case, so the no-rewards period-two subgame equilibrium is unaffected by this change. Similarly, the cutoff values \( \underline{\theta} \) and \( \bar{\theta} \) under rewards are the same whether linear or quadratic costs are used, hence the period-two level of sales for each firm under rewards is unaffected as well. Therefore, all of the analysis involving period two holds for quadratic costs as well as linear costs, and Proposition 3.1 remains valid.

C.1.2 Overlapping Generations

Next, suppose that a proportion \( 0 < \gamma \leq 1 \) of consumers remain in the market at the end of the first period, while the rest are replaced by an equal measure of consumers. All consumers are then distributed randomly in the second period. When \( \gamma = 1 \), this model is identical to that in the body of the chapter, and the no-rewards analysis is unaffected by allowing for \( \gamma < 1 \).

Suppose the incumbent’s sales in the first period are given by \( n^I_1 \). Under rewards, in the second period, there are \( \gamma n^I_1 \) consumers eligible for the reward and \( (1 - \gamma n^I_1) \) consumers who are not, and both segments are uniformly distributed over the unit interval. In the model presented in the body of the chapter, these proportions are \( n^I_1 \).
and \((1 - n_1^I)\), respectively. Note that replacing \(n_1^I\) with \(\gamma n_1^I\) throughout the period-two analysis yields the correct results for the model with \(\gamma < 1\), and the intuition behind the results is identical. Importantly, as neither the period-two equilibrium price for the entrant nor firms’ period-two equilibrium market shares depend on \(n_1^I\), these values are unaffected by letting \(\gamma < 1\). Thus, Proposition 3.1 is unaffected as well.

Importantly, all period-two equilibrium values under this specification approach those given in the original model as \(\gamma\) approaches 1. Thus, so long as \(\gamma\) is sufficiently large, allowing for \(\gamma < 1\) does not qualitatively affect the rest of the results of the chapter.

C.1.3 Correlated Consumer Positions

Until now, the rather strong assumption that consumers’ locations in the second period are independent of their positions in the first period has been made. To relax this assumption, I now allow for the possibility that only a proportion \(0 < \rho \leq 1\) of consumers change locations, and these consumers do not know whether they will change positions.\(^{48}\) This introduces (perhaps strong) positive correlation between consumers’ first- and second-period locations. A consumer located at \(\theta\) in period one has a second-period position drawn from a distribution with a mass point at \(\theta\): with probability \((1 - \rho)\), her period-two location will be \(\theta\); with probability \(\rho\), her position will be drawn from a uniform distribution on \([0, 1]\).

Note that when \(\rho = 1\), this model is identical to that in the body of the chapter. Without rewards, the period-two equilibrium is unaffected by allowing for \(\rho < 1\).

\(^{48}\)This guarantees that there is still a unique cutoff \(n_1^I\) in period one so that consumers indexed by \(\theta\) buy if and only if \(\theta \leq n_1^I\).
Under rewards, suppose that the incumbent’s first-period sales are given by \( n_1^I \). In period two, the cutoff values \( \bar{\theta} \) and \( \overline{\theta} \) given by (3.1) and (3.2) are unaffected; that is, consumers who purchased in the first period still purchase from the incumbent if and only if their position is to the left of \( \overline{\theta} \) and consumers who did not purchase in the first period still purchase from the incumbent if and only if their position is to the left of \( \bar{\theta} \). The market is still assumed to be covered: if consumers don’t purchase from the incumbent, they purchase from the entrant.

It will be helpful to define the profits that each firm derives from the segment of consumers whose location has changed. The \( \rho \) consumers whose positions have changed is distributed uniformly over the unit interval, and proportion \( n_1^I \) of these are eligible for the reward. Thus, the incumbent’s profits from this segment are given by \( \rho \) multiplied by the period-two profits from the original model:

\[
\Pi_I^\rho = (p_2^I - r - c^I) \rho n_1^I \bar{\theta} + (p_2^I - c^I) \rho (1 - n_1^I) \bar{\theta}.
\]

Similarly, the entrant’s profits from this segment are given by

\[
\Pi_E^\rho = (p_2^E - c^E) \rho (n_1^I (1 - \overline{\theta}) + (1 - n_1^I) (1 - \bar{\theta})).
\]

Next, consider the consumers who have not changed location. This segment is of mass \( (1 - \rho) \) and also distributed uniformly over the unit interval, but all consumers located to the left of \( n_1^I \) are eligible for the reward, while those to the right are not. The expressions for firms’ profits from this segment vary, depending on the values of \( n_1^I \), \( \bar{\theta} \) and \( \overline{\theta} \). Noting that \( \bar{\theta} < \overline{\theta} \) for all \( r > 0 \), there are three possible cases in equilibrium:

- Case 1: \( \bar{\theta} < \overline{\theta} < n_1^I \)
In Case 1, the incumbent’s sales as a proportion of this segment will be $\bar{\theta}$. That is, only some of the incumbent’s old customers in this segment will purchase again, and no new customers will be attracted. All of the consumers in this segment who do purchase from the incumbent receive the reward.

- Case 2: $\bar{\theta} \leq n_1^I \leq \tilde{\theta}$

In Case 2, The incumbent’s sales as a proportion of this segment are $n_1^I$. All of the old consumers in this segment are repeat purchasers, and no new consumers are attracted. All of the consumers in this segment who do purchase from the incumbent receive the reward.

- Case 3: $n_1^I < \bar{\theta} < \tilde{\theta}$

In Case 3, the incumbent’s sales as a proportion of this segment will be $\bar{\theta}$. All old consumers will buy again from the incumbent, but some new consumers are now attracted. The incumbent will pay the reward to $n_1^I$ of the consumers in this segment.

I proceed by showing that Proposition 3.1 is unaffected for any equilibrium which satisfies Case 1 or Case 3. I then show that even though it may not hold if Case 2 occurs in equilibrium, the incumbent is still powerless to prevent entry. Hence, Corollary 3.2 is always valid.

**Cases 1 and 3**

Suppose that an equilibrium exists which satisfies Case 1 or Case 3. The period-two analysis for these cases is similar to that of the model with $\rho = 1$. For these cases, Table C.1 displays firms’ second-period profit functions, the reaction functions derived from the first-order conditions, the second period equilibrium prices, and the period-two market shares of the incumbent.
The intuition behind the reaction functions and equilibrium prices is identical to that of the model when $\rho = 1$ in either case. In either of these cases, unlike in Case 2, the purchasing decision of marginal consumers in both segments (those who change positions and those who do not) is affected by an incremental change in price by either firm. Thus, the incumbent’s reaction function is again affected by the rewards program in two ways: rewards have the same effect on sales as an increase in the entrant’s price, and rewards increase the marginal cost of attracting additional consumers. The incumbent raises its price in response to each of these effects. The entrant’s reaction is only affected by rewards in a single way: rewards have the same effect on sales as a decrease in the incumbent’s price. Again, the increase in the incumbent’s reaction function is twice that of the decrease in the entrant’s reaction function, so that the entrant’s equilibrium price is precisely equal to the no-rewards price.

In both Case 1 and Case 3, evaluating the incumbent’s market share at the equilibrium prices, i.e.,

$$\theta = \frac{1}{2} + \frac{\hat{p}_E^E - \hat{p}_I^I}{2t},$$

yields the result that $n_I^I = n_H^H$. Therefore, Proposition 3.1 holds for these cases as well. Because the decisions of marginal consumers in both segments are affected by incremental changes in price, subgame perfection again mandates that the incumbent concede the no-rewards market share to the entrant. This is despite the fact that the incumbent could capture a larger share by charging the no-rewards price. The costliness of the reward payout prevents such an action from being profitable, as in the original model.
### Case 1

#### Profit Functions

- **Case 1**
  \[ \Pi^I_2 : \Pi^I_\rho + (p^I_2 - r^I) (1 - \rho) \bar{y} \quad \Pi^I_\rho + (p^I_2 - c^I) (1 - \rho) \bar{y} - r (1 - \rho) n^I_1 \]

- **Case 3**
  \[ \Pi^E_2 : \Pi^E_\rho + (p^E_2 - c^E) (1 - \rho) (1 - \bar{y}) \quad \Pi^E_\rho + (p^E_2 - c^E) (1 - \rho) (1 - \bar{\theta}) \]

### Reaction Functions

- **Case 2**
  \[ p^I_2 : \frac{1}{2} (c^I + t + p^E_2 + 2 (\rho r n^I_1 + (1 - \rho) r)) \quad \frac{1}{2} (c^I + t + p^E_2 + 2 \rho r n^I_1) \]

- **Case 3**
  \[ p^E_2 : \frac{1}{2} (c^E + t + p^I_2 - (\rho r n^I_1 + (1 - \rho) r)) \quad \frac{1}{2} (c^E + t + p^I_2 - \rho r n^I_1) \]

### Equilibrium Prices

- **Case 2**
  \[ \hat{p}^I_2 : c^I - \frac{1}{3} (c^I - c^E) + t + \rho r n^I_1 + (1 - \rho) r \quad c^I - \frac{1}{3} (c^I - c^E) + t + \rho r n^I_1 \]

- **Case 3**
  \[ \hat{p}^E_2 : c^E + \frac{1}{3} (c^I - c^E) + t \quad c^E + \frac{1}{3} (c^I - c^E) + t \]

### Incumbent’s Market Share

- **Case 2**
  \[ n^I_2 : \bar{\theta} + \frac{1}{2} (\rho r n^I_1 + (1 - \rho) r) \quad \bar{\theta} + \frac{1}{2} \rho r n^I_1 \]

### Table C.1: Firm Reactions and Equilibrium Values Upon Entry

#### Case 2

Case 2 is qualitatively different from Cases 1 or 3 in that, of those consumers who do not change locations, no old consumers are lost by the incumbent (unlike in Case 1) nor are any new consumers attracted (unlike in Case 3). Instead, the incumbent’s sales to this segment are precisely equal in both periods, and the exact same consumers are served in each period. In such an equilibrium, the purchasing decision of the marginal consumers in the segment which has not changed location is unaffected by an incremental price change.
Assume that an equilibrium satisfying Case 2 exists. In this case, firms’ second-period profit functions are

\[
\Pi^I_2 = \Pi^I_\rho + (p^I_2 - r - c^I) (1 - \rho) n^I_1
\]

\[
\Pi^E_2 = \Pi^E_\rho + (p^E_2 - c^E) (1 - \rho) (1 - n^E_1).
\]

Firms’ reaction functions are then

\[
p^I_2 = \frac{1}{2} \left( c^I + t + p^E_2 + 2rn^I_1 + 2t \frac{1 - \rho}{\rho} n^I_1 \right)
\]

\[
p^E_2 = \frac{1}{2} \left( c^E + t + p^I_2 - rn^I_1 + 2t \frac{1 - \rho}{\rho} (1 - n^I_1) \right),
\]

and the second-period equilibrium prices are

\[
\hat{p}^I_2 = c^I - \frac{1}{3} (c^I - c^E) + t + rn^I_1 + \frac{2}{3} \frac{1 - \rho}{\rho} (1 + n^I_1) t
\]

\[
\hat{p}^E_2 = c^E + \frac{1}{3} (c^I - c^E) + t + \frac{2}{3} \frac{1 - \rho}{\rho} (2 - n^I_1) t.
\]

Clearly, the entrant’s price is now affected by the incumbent’s first-period sales level. However, it is now always strictly greater than the no-rewards price. The entrant’s equilibrium market share is given by

\[
n^E_2 = \rho \left( (1 - \theta) - \frac{r}{2t} n^I_1 \right) + (1 - \rho) \left( 1 - n^I_1 \right)
\]

\[
= \rho \left( \frac{1}{2} + \frac{1}{6t} (c^I - c^E) \right) + \frac{1}{3} (1 - \rho) \left( 2 - n^I_1 \right),
\]

and the entrant’s equilibrium profit level is

\[
\Pi^E_2 = \frac{\rho}{2t} \left[ t + \frac{1}{3} (c^I - c^E) + \frac{2}{3} \frac{1 - \rho}{\rho} (2 - n^I_1) t \right]^2.
\]

Note that when \( \rho = 1 \), this is precisely equal to the no-rewards profit level. However, if \( \rho < 1 \), Proposition 3.1 may not hold. I now show that the more important result, Corollary 3.2, cannot fail even if Proposition 3.1 does not hold.
Differentiating the entrant’s equilibrium profits with respect to \( \rho \) yields
\[
\frac{\rho (c' - c^E) + 2t (2 - n_1') + \rho t (2n_1' - 1)}{18 \rho^2 t} \cdot \rho (c' - c^E) - 2t (2 - n_1') + \rho t (2n_1' - 1).
\]
Now,
\[
\begin{align*}
\rho (c' - c^E) + 2t (2 - n_1') + \rho t (2n_1' - 1) &> \rho (c' - c^E) + 2\rho t (2 - n_1') + \rho t (2n_1' - 1) \\
&= \rho (c' - c^E) + 3\rho t \\
&> 0,
\end{align*}
\]
so the entrant’s profits are (at least weakly) decreasing in \( \rho \) if
\[
\rho (c' - c^E) - 2t (2 - n_1') + \rho t (2n_1' - 1) \leq 0. \tag{C.1}
\]
Recall that when \( \rho = 1 \), the entrant’s profits are equal to the no-rewards profit level. Thus, if (C.1) is satisfied, then implementing a rewards program when \( \rho < 1 \) weakly increases the profit level of the entrant, and Corollary 3.2 holds.

Finally, I show that (C.1) cannot fail in an equilibrium that satisfies Case 2. This means that there is no equilibrium where entry is deterred, regardless of the value of \( \rho \). Suppose that (C.1) fails. Then
\[
\rho \left( (c' - c^E) + t (2n_1' - 1) \right) > 2t (2 - n_1'),
\]
which implies
\[
(c' - c^E) + t (2n_1' - 1) > 2t (2 - n_1'),
\]
or, equivalently,
\[
(c' - c^E) > 5t - 4n_1't. \tag{C.2}
\]
This also implies that
\[ n_1^I > \frac{5}{4} - \frac{1}{4t} (c^I - c^E) > \frac{1}{2}, \quad (C.3) \]
where the second inequality follows by Assumption 3.4. Additionally, note that \( \frac{r}{2t} \leq 1 \) in equilibrium, otherwise \( \overline{\theta} = \theta + \frac{r}{2t} > 1 \), which cannot be optimal for the incumbent.

Now, for Case 2 to be satisfied in equilibrium, it must be true that \( \overline{\theta} \geq n_1^I \). Substituting the equilibrium prices into \( \overline{\theta} \) yields
\[
\overline{\theta} = \frac{1}{2} - \frac{1}{6t} (c^I - c^E) + \frac{11 - \rho}{3 \rho} (1 - 2n_1^I) + \frac{r}{2t} (1 - n_1^I) \\
< \frac{1}{2} - \frac{1}{6t} (c^I - c^E) + 1 - n_1^I \quad (C.4) \\
< \frac{1}{2} - \frac{1}{6t} (5 - 4n_1^I) t + 1 - n_1^I \\
= \frac{1}{3} (2n_1^I - 1) + 1 - n_1^I, \quad (C.5)
\]
where (C.4) follows by (C.3) and the fact that \( \frac{r}{2t} \leq 1 \), and (C.5) follows by (C.2).

Finally, note that line (C.6) is less than \( n_1^I \) if and only if
\[
\frac{1}{3} (2n_1^I - 1) < 2n_1^I - 1,
\]
which is clearly true by (C.3). Therefore, Case 2 cannot be satisfied in equilibrium if (C.1) fails. This means that in any equilibrium satisfying Case 2, the entrant’s profits are decreasing in \( \rho \). Because \( \rho = 1 \) corresponds to the original model, the entrant’s profits in such an equilibrium can only increase as a result of a rewards program.

The intuition for this result is as follows. In Case 2, the consumers who have changed locations become locked in for the incumbent, while potential new consumers in this segment are locked out. Recall that these consumers are not sensitive to an incremental price change. This provides the incumbent with an additional incentive to increase its price to “milk” these consumers. Similarly, the entrant has its own
segment of consumers who are insensitive to a price change. Meanwhile, as both firms are incentivized to increase their prices on account of the non-moving segment, competition in the moving segment is dampened. As the non-moving segment grows larger (that is, as $\rho$ becomes smaller), these effects are amplified. Hence, the entrant’s profits increase as $\rho$ falls.

Note that all period-two equilibrium values approach those given in the original model as $\rho$ approaches 1, regardless of which of these three cases arises in equilibrium. Thus, so long as $\rho$ is sufficiently large, allowing for $\rho < 1$ does not qualitatively affect the rest of the results of the chapter, despite significantly complicating the analysis.

C.2 Period One Analysis

This appendix presents the derivation of the optimal first-period price and reward amount for the incumbent, which are announced at the beginning of period one. First, the unique candidate optimal interior solution of the incumbent is derived. Next, the optimal corner solution is identified. Finally, it is shown that the candidate interior solution, if indeed a valid solution, dominates the corner solution and hence is the optimal strategy for the incumbent. The final section outlines the situation where the interior solution fails to be interior; in this case, the corner solution is the unique equilibrium.

C.2.1 The Candidate Interior Solution

First, differentiating the incumbent’s profit function, given by (3.11), with respect to $p'_I$, and setting the resulting partial derivative equal to zero yields the first-order
condition for the optimal price:

\[ \frac{\partial \Pi^I}{\partial p^I_1} = \frac{\partial n^I_1}{\partial p^I_1} (p^I_1 - c^I) + n^I_1 - \frac{\partial n^I_1}{\partial p^I_1} (1 - 2n^I_1) \frac{r^2}{2t} = 0. \]  

(C.7)

From (3.10),

\[ \frac{\partial n^I_1}{\partial p^I_1} = -\frac{2t}{r^2 + 2t^2}. \]

Substituting this and the expression for \( n^I_1 \), given by (3.10), into (C.7) allows the first-order condition to be solved for the price in terms of the reward amount:

\[ p^I_1(r) = \frac{R + c^I}{2} + \frac{(2t^2 - r^2) \left[ 2r \left(3t - (c^I - c^E)\right) - 3r^2\right] - 12r^2t \left(R - c^I\right) + 24r^2t^2}{48t^3} \]  

(C.8)

Differentiating the profit function (3.11) with respect to \( r \), the reward amount, and setting the resulting partial derivative equal to zero yields the first-order condition for the reward amount:

\[ \frac{\partial \Pi^I}{\partial r} = \frac{\partial n^I_1}{\partial r} (p^I_1 - c^I) - \frac{\partial n^I_1}{\partial r} (1 - 2n^I_1) \frac{r^2}{2t} - \frac{2r}{2t} n^I_1 (1 - n^I_1) = 0. \]  

(C.9)

At any critical point, both first-order conditions must hold simultaneously. Next is to show that there can be at most one interior critical point; that is, one critical point \((p^I_1, r)\) such that \(0 < n^I_1 < 1\). To do so, I evaluate each of the terms in (C.9) at \( p^I_1 = p^I_1(r) \) and solve for \( r \). The partial derivative of the first-period sales level (3.10) is

\[ \frac{\partial n^I_1}{\partial r} = \frac{(2t^2 - r^2) \left(3t - (c^I - c^E)\right) - 12rt \left(R - p^I_1\right) + 6rt^2}{3 (r^2 + 2t^2)^2}, \]

and evaluating this derivative at \( p^I_1 = p^I_1(r) \) yields

\[ \frac{\partial n^I_1}{\partial r} \bigg|_{p^I_1=p^I_1(r)} = \frac{2 (2t^2 - r^2) \left(3t - (c^I - c^E)\right) - 12rt \left(R - c^I\right) + 12rt^2 + 3r^3}{12t^2 (r^2 + 2t^2)}. \]

Evaluating the sales level at \( p^I_1 = p^I_1(r) \) yields

\[ n^I_1 \left(p^I_1(r), r\right) = \frac{2r \left(3t - (c^I - c^E)\right) + 12t \left(R - c^I\right) - 3r^2}{24t^2}. \]  

(C.10)
Finally,
\[ p_1'(r) - c' = \frac{(2t^2 - r^2) \left[ 2r \left( 3t - (c' - c^E) \right) + 12t \left( R - c' \right) - 3r^2 \right] + 24r^2 t^2}{48t^3}. \]

Substituting each of the last three expressions into (C.9) and rearranging yields the following condition:
\[ \frac{\left[ 3t - (c' - c^E) - 3r \right] \left[ 2r \left( 3t - (c' - c^E) \right) + 12t \left( R - c' \right) - 3r^2 \right]}{144t^3} = 0 \]

The left-hand side of this equation is cubic in \( r \). However, by looking at (C.10), it can be seen that the roots which are defined by
\[ 2r \left( 3t - (c' - c^E) \right) + 12t \left( R - c' \right) - 3r^2 = 0 \]
correspond to \( n_1' = 0 \). This is not an interior solution, nor can it be optimal. \( n_1' = 0 \) implies a first-period profit of zero, and a second-period profit of \( \Pi_H' \). Clearly, the incumbent does better by setting \( r = 0 \), which results in total profits \( \Pi_M' + \Pi_H' \). The unique candidate for an interior solution is thus defined by
\[ 3t - (c' - c^E) - 3r = 0, \]
or, equivalently,
\[ r = t - \frac{1}{3} (c' - c^E). \]

This is the expression given by (3.13). Substituting this into (C.8) yields (3.12).

C.2.2 The Corner Solution

As shown above, any solution resulting in \( n_1' = 0 \) cannot be optimal. Thus, an optimal corner solution must satisfy \( n_1' = 1 \). From (3.10), such a solution must then satisfy \( 1 \leq n_1'(p_1', r) \). From (3.10), this is equivalent to
\[ 2 \left( r^2 + 2t^2 \right) \leq 4t(R - p_1') + 2r \left( t - \frac{1}{3} (c' - c^E) \right) + r^2, \]

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which is equivalent to
\[ p_I^l \leq R - t + r \left( \frac{1}{2} - \frac{1}{6t} (c_I^l - c_E^l) \right) - \frac{r^2}{4t}. \] (C.11)

With \( n_I^l = 1 \), the profit function (3.11) can be rewritten as
\[ \Pi^I = p_I^l - c_I^l + \Pi^{I_H}. \] (C.12)

Thus, (C.11) must hold as an equality. Substituting into the profit function yields
\[ \Pi^I = R - c_I^l - t + r \left( \frac{1}{2} - \frac{1}{6t} (c_I^l - c_E^l) \right) - \frac{r^2}{4t} + \Pi^{I_H}, \]
which is strictly concave in \( r \). The first-order condition yields the optimal corner solution reward:
\[ \tilde{r} = t - \frac{1}{3} (c_I^l - c_E^l) = \hat{r}. \]

Thus, the optimal reward, regardless of whether the solution to the incumbent’s optimization problem is interior or not, is equal to the reward given by (3.13). Substituting this reward into (C.11) (where the inequality holds as an equality) yields the optimal corner solution price:
\[ \tilde{p}_I^l = R - t + \frac{\tilde{r}^2}{4t}. \]

Substituting this price into (C.12), the incumbent’s profits when the optimal corner solution is chosen are equal to
\[ \tilde{\Pi}^I = R - c_I^l - t + \frac{\tilde{r}^2}{4t} + \Pi^{I_H}. \] (C.13)

Next, I show that if the candidate interior solution is, in fact, an interior point, then it dominates the corner solution. This is one of the statements made by Proposition 3.5, the proof of which is also completed shortly.
C.2.3 Completing the Proof of Proposition 3.5

The trivial portions of the proof of Proposition 3.5 are given in the body of the chapter. Now, I show that \((\hat{p}_1^I, \hat{r})\) dominates \((\tilde{p}_1^I, \hat{r})\) if Assumption 3.5 holds; that is, when the candidate interior solution is, in fact, interior, then it dominates the corner solution. The task at hand is to show that \(\hat{\Pi}^I - \tilde{\Pi}^I \geq 0\). Define the function \(f(\hat{r}^2)\) as this difference, which can be written as

\[
f(\hat{r}^2) = \hat{\Pi}^I - \tilde{\Pi}^I \geq \frac{(R - c')^2}{4t} + \hat{r}^2 \left(\frac{(R - c')}{2t} - 1\right) - (R - c') + t + \frac{(\hat{r}^2)^2}{64t^3}.
\]

It is simple to show that when Assumption 3.5 holds as an equality, i.e, \(\hat{r}^2 = 4t \left(2t - (R - c')\right)\), then \(f(\hat{r}^2) = 0\). If this is the case, then the candidate interior solution is not strictly interior; in fact, it coincides with the corner solution. It remains to show that \(f(\hat{r}^2) \geq 0\) for all \(\hat{r}^2 < 4t \left(2t - (R - c')\right)\). It suffices to show that \(f'(\hat{r}^2) < 0\) for all \(\hat{r}^2 < 4t \left(2t - (R - c')\right)\). Taking the derivative of \(f\) yields

\[
f'(\hat{r}^2) = \frac{1}{4t} \left(\frac{R - c'}{2t} - 1\right) + \frac{1}{4t} \left(\hat{r}^2 \right) .
\]

So, when \(\hat{r}^2 < 4t \left(2t - (R - c')\right)\),

\[
f'(\hat{r}^2) < \frac{1}{4t} \left(\frac{R - c'}{2t} - 1\right) + \frac{1}{4t} \left(\frac{2t - (R - c')}{2t}\right) = 0.
\]

This is the desired result. To complete the proof, it remains only to check that \(0 \leq \underline{\theta} \leq \overline{\theta} \leq 1\). If either of these values lies outside of the unit interval, then the analysis used to derive the equilibrium is invalid. Substituting the second-period equilibrium prices, given by (3.5) and (3.6), into (3.1) and (3.2) yields the following:

\[
\theta = \frac{\hat{r} \left(1 - \hat{n}_1^I\right)}{2t} \quad \text{(C.14)}
\]

\[
\overline{\theta} = \frac{\hat{r} + \hat{r} \left(1 - \hat{n}_1^I\right)}{2t} \quad \text{(C.15)}
\]

Clearly, \(0 < \underline{\theta} < \overline{\theta} < \frac{2t}{2t} \leq 1\), as \(\hat{r} \leq t\). This completes the proof.
C.2.4 When Period-One Market Coverage is Optimal

This appendix contains the proof that the optimal corner solution becomes the equilibrium strategy when Assumption 3.5 fails. If the assumption fails, the only possible equilibrium strategy when a rewards program is used is the optimal corner solution \((\tilde{p}^I_1, \hat{r})\), as there is no interior critical point.

Substituting the reward \(\hat{r}\) into the period-two price for the incumbent, and noting that \(n^I_1 = 1\), shows that all consumers effectively face the no-rewards prices in period two if this strategy is used. Therefore, the market is fully covered in period two (because it is covered when rewards aren’t used), \(0 < \tilde{\theta} = n^I_H < 1\), and, from (C.14), \(\bar{\theta} = 0\). This implies that if \(\hat{\Pi}^I \geq \Pi^I_M + \Pi^I_H\) (that is, if the corner solution dominates the no-rewards strategy) then this strategy does, in fact, constitute the unique equilibrium strategy. I now prove that the required inequality holds.

While the number \(\hat{\Pi}^I\) does not represent a profit level in this case (\(\hat{n}^I_1 > 1\), so the sales level used to derive it no longer corresponds to a sales level), the value of \(\hat{\Pi}^I\) still will be helpful. In particular, it is still the case that \(\hat{\Pi}^I > \Pi^I_M + \Pi^I_H\), as this result did not depend on Assumption 3.5. By an argument identical to that used to show that \(\hat{\Pi}^I - \bar{\Pi}^I \geq 0\) when Assumption 3.5 holds, it can be shown that \(\hat{\Pi}^I - \bar{\Pi}^I < 0\) when the assumption fails. Therefore, when the assumption fails, \(\bar{\Pi}^I > \hat{\Pi}^I > \Pi^I_M + \Pi^I_H\), so the corner solution does indeed dominate the no-rewards strategy. From (C.12) and (3.11), \(\bar{\Pi}^I > \hat{\Pi}^I\) implies that

\[
\tilde{p}^I_1 - c^I > (\hat{p}^I_1 - c^I) \hat{n}^I_1 - \frac{\hat{r}^2}{2t} \hat{n}^I_1 (1 - \hat{n}^I_1) \\
> (\hat{p}^I_1 - c^I) \hat{n}^I_1 \\
> \hat{p}^I_1 - c^I,
\]
as \( \hat{n}_1^I > 1 \) and \( 1 - \hat{n}_1^I < 0 \). This, in turn, implies that \( \hat{p}_1^I > \bar{p}_1^I > p_{M1}^I \), as the result given by (3.15) also did not depend on Assumption 3.5. Trivially, the equilibrium first-period sales level is given by \( \hat{n}_1^I = 1 > n_{M1}^I \).

C.3 Market Coverage

This appendix derives the necessary and sufficient condition for the market to be fully covered in the second period, which is given as Assumption 3.6 in the body of the chapter. The analysis used to derive the equilibrium assumed that consumers buy in the second period from the firm which yields them higher utility. By Assumption 3.3, the market is covered when rewards are not used. However, it is easily seen from (3.5) that the incumbent’s second-period price increases when the incumbent uses a rewards program, so without additional restrictions on the parameter values, it is possible that some consumers would realize negative utility by purchasing in the second period. Therefore, it is necessary to impose that the indifferent consumers located at \( \theta \) and \( \bar{\theta} \) do indeed maximize their utility by purchasing in period two given the equilibrium prices. This is also sufficient, as consumers who are not indifferent necessarily realize higher utility than the indifferent consumers by purchasing from their preferred firms. From (3.5) and (C.14), the utility that the indifferent consumer who did not purchase in the first period realizes by purchasing in period two is given by

\[
R - \theta t - \hat{p}_2^I = R - \frac{\hat{r} (1 - \hat{n}_1^I)}{2} - c^I + \frac{1}{3} (c^I - c^E) - t - \hat{r} \hat{n}_1^I \\
= R - c^I - \frac{3}{2} \hat{r} - \frac{1}{2} \hat{r} \hat{n}_1^I.
\]
Similarly, from (C.15) and (3.5), the second-period utility that the indifferent consumer who did purchase in period one realizes by purchasing is given by

\[
R - \theta t - \hat{p}_t + \hat{r} = R - \hat{r} + \hat{r} \left(1 - \hat{n}_{t}^I\right) - c^I + \frac{1}{3} (c^I - c^E) - t - \hat{n}_{t}^I + \hat{r} \\
= R - c^I - \hat{r} - \frac{1}{2} \hat{n}_{t}^I.
\]

It is easily seen that the indifferent consumer who purchased in the first period is strictly better off than the indifferent consumer who did not. Therefore, it is sufficient that the indifferent consumer who did not purchase realizes nonnegative utility by purchasing in period two, i.e.,

\[
R - c^I - \frac{3}{2} \hat{r} - \frac{1}{2} \hat{n}_{t}^I \geq 0.
\]

This is equivalent to Assumption 3.6.
Bibliography


