ULTRASONIC SCATTERING FROM PLANE AND CYLINDRICAL IMPERFECT INTERFACES WITH APPLICATION TO COMPOSITES

DISSERTATION

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by

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Abstract

This work focuses on modeling ultrasonic wave interaction with plane and cylindrical imperfect interfaces and developing theoretical and experimental techniques to characterize fiber-matrix interphases in composites using ultrasonic waves. Micromechanical models are used to describe complicated multiphasial interphases and spring and asymptotic boundary conditions are developed to study the effects of interface imperfections on wave propagation in anisotropic and inhomogeneous media. For plane interfaces the scattering solutions based on the proposed second order boundary conditions are unique and satisfy energy balance, as well as being highly accurate. They also give zero scattering for a homogeneous substrate/interphase/substrate system. For a thin anisotropic layer between identical isotropic semispaces the scattering matrices are obtained in explicit form by decomposing the elastic field into symmetric and antisymmetric parts. Analytical dispersion equations for Stoneley-type interface waves localized in such a system are also given.

For cylindrical interfaces, a transfer matrix approach is developed for study of wave scattering from a multilayered fiber in a solid matrix. Spring and asymptotic boundary conditions are derived from the transfer matrix to represent a thin fiber-matrix interphase and to study the subsequent effect on low frequency ultra-
sonic wave scattering from the fibers. Such boundary condition models are also obtained from the solution of the governing differential equation for elastic fields in a radially-dependent cylindrical medium. A resonance phenomenon is also studied for compliant fiber-matrix interphases using both spring and interphase layer models.

The results for a single fiber in a matrix are extended to composites where wave scattering, attenuation and velocity are investigated both theoretically and experimentally. Different models for wave dispersion and attenuation in nonhomogeneous media are extended to a multiphase unidirectional composite for waves propagating along and normal to fibers. A critical review of these models are done by numerical simulations with an emphasis on the interphase effect, and by comparison with the experimental data measured for a SiC/titanium composite. A static multiphase micromechanical model is also developed using the transfer matrix approach to determine the composite moduli. Finally ultrasonic measurements have been performed on high-temperature composites under fatigue, showing that ultrasonic waves are very sensitive to fatigue-induced damage on the fiber-matrix interphase and thus can be used for interphase characterization.
To My Parents
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CHAPTER I

INTRODUCTION

1.1 Problem Statement

Interfaces can be found in many material systems on a variety of scales, such as grain boundaries in polycrystalline metals, fiber-matrix and interlaminar interfaces in composites, interfaces in metallurgical solid state diffusion bonds, inertial welds and adhesive joints. They play a key role in the mechanical behavior of these structural materials and joints by transferring mechanical loading from one component to another, thus interface failure usually results in total failure of the structure. Therefore it is extremely important to ensure the quality of interfacial bonding. Conventional electronic and optical methods cannot be used for interface observation in depth. Penetrating methods such as nuclear particle or X-ray techniques are not sensitive to the micromechanical conditions on interfaces which determine the quality of interfacial bonding. Ultrasonic waves, high frequency micromechanical vibrations, have the penetrating power and sensitivity to interrogate solid-solid interfaces deep inside solids, and are thus ideal candidates for
development of nondestructive techniques for interface characterization.

The topic of ultrasonic wave interaction with a solid-solid interface has for a long time been of great interest in nondestructive evaluation (NDE), geophysical acoustics and acoustic solid-state devices. These interactions carry important information on interface properties and relate them to the ultrasonic signature. To model these interactions boundary conditions are postulated to describe the solid-solid interface. One idealized situation has the solids welded together to form a perfect interface so that the displacements and stresses are continuous across the interface. This is usually called a welded or perfectly bonded interface. The opposite situation is no interaction between the solids, in other words total disbonding, which is thus called a debonded interface. Another extreme situation describes a slip interface, corresponding physically to an infinitely thin layer of ideal liquid that ensures free transverse slip on the interface. Here we term these classical boundary conditions, calling any boundary conditions intermediate to these extremes non-classical boundary conditions and interfaces with properties different from designed imperfect interfaces.

In practice, the actual interface between two solids is very complicated and usually cannot be described by classical boundary conditions. Its structure depends on the particular type of solid contact: contact formed during solidification, metallurgical solid state bond, dry mechanical or lubricated contact, adhesive bond et cetera. Even grain boundaries in polycrystalline materials are not perfect because
of misfits of the atomic structures of two neighboring grains. Such imperfect interfaces can be described by boundary conditions intermediate between the classical extremes. In many cases a thin layer of different material exists between two solids, for example, forming interphases in adhesive bonding and high temperature composites. Moreover, the imperfect interface itself can be anisotropic under certain circumstances, a porous/cracked interphase with preferred pore/crack orientation being one example and the fiber-matrix interphase being another as shown in Fig. 1.1. So it is important to introduce relatively complex nonclassical boundary conditions to model complicated imperfect interfaces, and to predict the effect of interface imperfections on wave propagation. The challenge is to use as few free parameters as possible in the boundary conditions to simplify the inverse problem of interface property determination. Note that this work is focused on the scenario that the interfacial layer thickness or the size of interfacial microdefects (porosities, cracks) is much smaller than the wavelength of the interrogating ultrasonic wave.

Recently Rokhlin and Wang have proposed a systematic approach to derive the boundary conditions for modeling complicated nonhomogeneous imperfect plane interfaces [1, 2, 3]. They used a thin multiphase interphasial layer to model the nonhomogeneous imperfect interface [1], and then derived the first order asymptotic boundary conditions by asymptotic expansion of the exact transfer matrix (wave solutions) for the thin interphasial layer [2, 3]. Such a boundary-condition transfer matrix relates the stresses and displacements on each side of the inter-
Figure 1.1: The effect of interface imperfection on wave propagation. (a) Imperfect plane interfaces, (b) imperfect cylindrical fiber-matrix interfaces.
facial layer, and includes both coupling terms and inertial terms. The first-order boundary condition theory greatly simplifies the study of wave scattering from and localization of interface waves on the complicated imperfect interface since there is no need to find the wave solutions inside the interphasial layer. It has been shown by numerical simulation for various imperfect interfaces, such as porous interfaces, cracked interfaces and interfaces with inclusions, that the first order asymptotic boundary condition model describes accurately the dynamic behavior of a thin layer between two solids for small layer thickness-to-wavelength ratio, and gives more insight into the effect of different interface conditions on wave scattering and interface wave localization [2, 3].

The first order boundary condition model was extended, in our previous work [4, 5, 6], from representation of an isotropic interfacial layer or an orthotropic interfacial layer in plane of symmetry to a generally anisotropic interfacial layer between two generally anisotropic solids. This generalization is important since waves can propagate in a different plane other than that formed by the interface normal and the preferred orientation of interface imperfections as shown by Fig 1.2. We have found that retention of the coupling and inertial terms in the boundary conditions is essential for precise calculation of reflection and transmission and especially for the calculation of interface wave velocities [4, 5]. In addition, we have discovered conditions for localization of interfacial modes of vibration which can be used for interfacial sensing [4]. This work formed the basis of my master
thesis [6].

There are several drawbacks to using the first order asymptotic boundary condition model to represent a thin interfacial layer. One is that the energy balance is not satisfied, i.e. the sum of the energies of the scattered waves does not equal that of the incident wave. This model gives good results for the reflected field, but performs unsatisfactorily for the transmitted field. Also for the limiting case where the interfacial layer and the two substrates are of the same material, the first order asymptotic boundary condition model fails to give full transmission. In this dissertation we develop the so-called second order asymptotic B.C. model which has the advantages of being more accurate and free of these imperfections, and which is as simple as the first order boundary condition model.

The other focus of this dissertation is theoretical and experimental study of the fiber-matrix interphase effect in composites on elastic wave propagation by extending the results for plane interphases to cylindrical interphases. It is well known that the fiber-matrix interphase plays an important role in determining composite performance. The interphase not only allows load transfer between fibers and matrix but also provides matching of chemical and thermal compatibility between the constituents. In metal and intermetallic matrix composites, special interfacial reaction barrier coatings and compliant coatings are introduced to improve chemical and thermal compatibility. In ceramic matrix composites, the interphase is designed to provide frictional sliding contact between fiber and matrix to prevent
Figure 1.2: Anisotropic imperfect plane interfaces. $\varphi$ is the rotation angle of the wave incident $(1,3)$ plane with respect to the crack or pore orientation. (a) A fractured interface with a distribution of cracks, (b) an interphase with inclusions or pores.
fiber fracture due to matrix cracking. The interphase microstructure and its reaction with other composite constituents have received increasing attention. Despite great efforts toward the development of special fiber coatings to tailor the interphase, interphase mechanical properties remain difficult to measure and interpret. The complexity of such interphases will become even greater if the properties are altered during manufacture or in service by chemical reaction between the constituents, or if micromechanical defects occur.

In our program two approaches to determine the effective elastic moduli of the interphases in composites have been developed recently [7, 8, 9, 10, 11]. The interphase is considered as a thin layer of bulk coating material between the fiber and matrix with distinct properties. Two ultrasonic techniques for interphase characterization are considered: wave scattering (attenuation) and phase velocity. The basic idea of the proposed methods is illustrated schematically in Fig. 1.3. One method uses the ultrasonic phase velocities in the composites to determine the static composite elastic moduli, and the effective interphase moduli are found from the ultrasonically measured composite moduli via micromechanical models. The other uses the frequency dependence of the ultrasonic scattering (attenuation) in the composite. By modeling the scattering of ultrasonic waves by a single fiber with different interphasial conditions, the attenuation data can be related to the interphase moduli in the composites. This last approach forms the basis of the second part of this dissertation. I will address this problem using a transfer matrix
approach for multiphase fiber composites. The treatment of multilayered cylinders is important for ultrasonic characterization of modern high temperature composites since both the fibers and the fiber-matrix interphases can have multilayered structures due to design and to the manufacturing process. We will analyze different boundary condition models for a thin fiber-matrix interphase, describe wave propagation in composites by accounting for multiple scattering from fibers, and validate our analytical results by experimental measurements of both wave attenuation and velocity as functions of frequency.

1.2 Objectives and Dissertation Organization

The overall objective of this work is to develop ultrasonic methods for solid-solid interface characterization. It is summarized by the following:

(i) Improve understanding of ultrasonic wave scattering from plane and fiber-matrix imperfect interphases with an aim of ultrasonic interphase characterization. To achieve this we plan: to develop asymptotic boundary condition models for elastic wave scattering from anisotropic plane interphases, to derive analytical solutions for wave scattering and localization of interface waves using proposed boundary conditions and to generalize the quasi-static spring model for an anisotropic fractured plane interface to account for 3-D coupling due to interface anisotropy.
Figure 1.3: Schematic diagrams for ultrasonic scattering (attenuation) and phase velocity methods for interphase characterization.
(ii) For cylindrical interphasial layers, to develop a transfer matrix representation which allows a systematic treatment of wave scattering from a \( N \)-phase multilayered cylindrical system. To extend the first and second order asymptotic boundary condition models for plane interphases to cylindrical interphases and to study the interphase effect on fiber scattering. To investigate the spring boundary condition model as an effective replacement of a thin fiber-matrix interphase, and to perform a comparative study of different implementations of the spring model with interphase gap and with the interphase gap filled by matrix or fiber material.

(iii) For interphase characterization in composites, to extend wave propagation models in nonhomogeneous media for studies of fiber-induced wave attenuation and dispersion in a unidirectional multiphase fiber composite, and to analyze the interphase effect on the wave propagation characteristics. To study experimentally wave attenuation and dispersion propagating both normal to and along fibers in composites, and to compare the experimental data with different model predictions.

This dissertation is structured as follows. Part I is devoted to the development of the second order asymptotic boundary conditions for a thin anisotropic plane interfacial layer between two solid media. In Chapter II I give a review of the previous work in boundary condition models for imperfect plane interfaces. These
boundary condition models are interpreted using the transfer matrix formulation put forth in this work. Chapter III introduces second-order asymptotic boundary conditions to describe complicated multiphasial anisotropic plane interphases as shown in Fig. 1.2(b). It will be shown that the scattering solutions based on the second order boundary conditions are unique and satisfy energy balance, and are also highly accurate for both the reflected and transmitted fields. They also give zero scattering for a homogeneous system. For a thin anisotropic layer between identical isotropic semispaces the scattering matrices will be obtained in explicit form in Chapter IV by decomposing the elastic field into symmetric and antisymmetric parts. Analytical dispersion equations of Stoneley-type interface waves localized in such a system will be derived in Chapter V where also the interface wave localization conditions will be discussed. Finally for a fractured anisotropic interface with distributed cracks as in Fig. 1.2(a), the generalized quasi-static spring boundary condition model is described in Chapter VI.

Part II focuses on theoretical and experimental studies of elastic wave propagation in high-temperature composites. The interphase effect is analyzed by extending the results for plane interphases in Part I to cylindrical fiber-matrix interphases. A transfer matrix approach for wave scattering from a multilayered fiber in a solid matrix will be developed in Chapter VII. This is important since in modern composites both the fibers and the fiber-matrix interphases can have multilayered structures. Spring and asymptotic boundary conditions will be derived in Chap-
ter VIII to represent the thin fiber-matrix interphase and to study the subsequent
effect on ultrasonic wave scattering from the fibers. Such boundary condition mod-
els will also be obtained from the solution of the governing differential equation for
the elastic field in a radially-dependent cylindrical medium. Chapter IX focuses on
developing simplified low frequency models. Resonances are found and described
in this chapter for extremely compliant fiber-matrix interphases.

The results for a single fiber in a matrix will be further extended to composites
in Chapters X and XI where wave scattering, attenuation and velocity are investi-
gated both theoretically and experimentally. Different models for wave dispersion
and attenuation in nonhomogeneous media will be developed for a multiphase uni-
directional composite for waves propagating along (Chapter XI) and normal to
(Chapter X) the fibers. These models will be analyzed by numerical examples
and by comparison with the experimental data obtained for a SiC/titanium com-
posite. In the analysis, the fiber-matrix interphase properties are determined from
ultrasonically measured composite moduli via a static multiphase micromechanical
model using the transfer matrix concept. Finally Chapter XII gives an example
of nondestructive monitoring of fatigue-induced damage to the fiber-matrix in-
terphase in high-temperature composites using both ultrasonic wave velocity and
attenuation data.
Part I

Ultrasonic scattering from plane imperfect interfaces
CHAPTER II

BACKGROUND. PREVIOUS WORK ON BOUNDARY CONDITION MODELS FOR IMPERFECT PLANE INTERFACES

Let us consider a plane interface $z = 0$ between two elastic solids as shown in Fig. 2.1. Generally we have three reflected waves in the upper medium and three transmitted waves in the lower medium. To facilitate discussion of different boundary condition models, we use the transfer matrix notation as suggested in [2, 3, 5] to describe the boundary conditions across the interface. Specifically a transfer matrix $\mathbf{B}$, which relates the particle displacements $u_i$ and stresses $\sigma_{ik}$ on the top side of the interface to the displacements $u'_i$ and stresses $\sigma'_{ik}$ on the bottom side of the interface, is defined as:

$$(u_x, u_y, u_z, \sigma_{xk}, \sigma_{yk}, \sigma_{zk})^T = \mathbf{B}(u'_x, u'_y, u'_z, \sigma'_{xk}, \sigma'_{yk}, \sigma'_{zk})^T$$

(2.1)

$\mathbf{B}$ is generally a $6 \times 6$ matrix. The elements of $\mathbf{B}$ depend on the interface properties.

In the case of isotropy, if we choose the $(x, z)$ plane to be the incident plane,
Figure 2.1: A general solid-solid plane interface.
the equation (2.1) decouples to two equations; one is

\[(u_z, u_z, \sigma_{zz}, \sigma_{zz})^T = B(u_z', u_z', \sigma_{zz}', \sigma_{zz}')^T\]  \hspace{1cm} (2.2)

\(B\) reduces to a \(4 \times 4\) matrix. The other is a one dimensional \((y)\) equation for a pure SH wave; \(B\) is a \(2 \times 2\) matrix.

In the case of the welded interface, we have continuity of displacement and stress across the interface \([12, 13]\), so \(B\) is just the unit matrix. The other extreme case, slip boundary conditions, implies continuity of the normal components of displacement and stress across the interface, vanishing of the tangential component of stress, and discontinuity of the tangential component of displacement \([14]\), thus

\[\sigma_{zz} = \sigma_{zz}' = 0, \quad \sigma_{zz} = \sigma_{zz}', \quad u_z = u_z', \quad (2.3)\]

and the matrix \(B\) is singular. For an interface intermediate to these two extremes, the boundary conditions become more complex. Intensive studies have addressed this problem because of its conceptual and practical importance. A brief review is given here to provide an overview of previous results and the physical limitations of the models.

### 2.1 Finite Stiffness, Density and Quasi-Static Models

In 1967, Jones and Whitter \([15]\) introduced one normal and one transverse stiffness coefficient to relate stresses and displacements while studying the propagation of waves along a compliant interface between two semi-infinite isotropic media. They
considered the existence of a thin interfacial layer with properties different from those of the semispaces. If the thickness of the interfacial layer is denoted by \( h \), its Young's modulus by \( E \) and shear modulus by \( \mu \), the boundary conditions can be expressed as [15]

\[
\begin{align*}
\sigma_{zz} &= \sigma'_{zz}, \quad \sigma_{zx} = \sigma'_{zx}; \\
\sigma_{zz} &= \frac{E}{h} (u_z - u'_z), \quad \sigma_{zx} = \frac{G}{h} (u_x - u'_x).
\end{align*}
\] (2.4)

The corresponding transfer matrix \( B \) is

\[
B = \begin{bmatrix}
1 & 0 & \frac{1}{K_t} & 0 \\
0 & 1 & 0 & \frac{i}{K_n} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (2.5)

where \( K_t = \frac{G}{h} \) and \( K_n = \frac{E}{h} \) are the normal and transverse stiffness coefficients respectively. The imperfect interface can be viewed as the sum of one normal spring and one transverse spring to allow the discontinuity of displacements but ensure the continuity of stresses. The effect of the mass of the interfacial layer was not considered in their work. When both stiffness coefficients approach infinity, the interface becomes perfect; when \( K_n \) approaches infinity, but \( K_t \) approaches zero, it becomes transverse slip. The finite stiffness boundary conditions describe the behavior of an interface layer when the thickness of the layer is much smaller than the wavelength. Therefore, this approximate boundary condition can only be applied in the low frequency range. We will term this kind of model the finite
stiffness model.

Later, in 1975, Murty [16] studied a similar problem: a thin viscous layer between two loosely bonded solids. In his work, only the transverse spring was used. Schoenberg [17] in 1980 addressed the elastic behavior across linear slip interfaces using both transverse and normal stiffnesses. The comparison between the exact solution and the approximate solution using the finite stiffness model on an SH wave interaction with such an imperfect interface was discussed analytically in his work. He concluded that the finite stiffness model is only good when the thickness of the layer is much smaller than the normal component of the wavelength, and the impedance of the layer is low compared with those of the substrates. For thin bonded layers, the finite stiffness model has been widely used and extensively analyzed by many researchers [18, 19, 20, 21, 22].

Nayfeh and Nassar [23] in 1978 used the inertial terms instead of stiffnesses in their approximate analysis of the influence of bonding materials in laminated composites. In their model, only the mass effect of the interfacial material was considered, the effect of its stiffness being neglected. We term this kind of approach which considers only the inertial effect the density model. One can also consider a more complicated model where both the effects of the stiffnesses and mass of the interfacial layer are considered. We will call this model the stiffness-mass model.

At the same time, Rokhlin et al. in [24] gave asymptotic dispersion equations for interface waves along an isotropic viscoelastic layer, and an effective shear
modulus was introduced later for a multilayered interface to describe the interface properties [25]. These work led directly to the asymptotic boundary condition formulations [2], which will be discussed in the next section.

The models so far mentioned do not work effectively for interfaces with cracks. A novel approach was taken by Baik and Thompson [26] who described a physically imperfect interface containing cracks. In their work, the spring effect (finite stiffness) and density effect (nonzero inertia) were both considered in modelling a cracked interface. A quasi-static model was proposed in which the interface stiffness was calculated using the known static solutions for an elastic body with cracks under tension forces. The spring stiffness terms were obtained for waves of normal incidence via a known solution for stress intensity factors $K_I$ and $K_{II}$, and the inertial terms were introduced by considering the additional mass per unit area along the interface (for a crack, $m = 0$; for volumetric imperfections such as pores and inclusions, $m$ can be either positive or negative). The work was later extended to oblique incidence [27]. The boundary conditions can be written in the 3-dimensional case as:

\[
\frac{\sigma_{xz} + \sigma'_{xz}}{2} = K_{t1}(u_x - u'_x), \quad \frac{\sigma_{yz} + \sigma'_{yz}}{2} = K_{t2}(u_y - u'_y), \\
\frac{\sigma_{zz} + \sigma'_{zz}}{2} = K_n(u_z - u'_z); \quad \sigma_{xx} - \sigma'_{xx} = -\omega^2 m \frac{u_x + u'_x}{2}, \\
\sigma_{yz} - \sigma'_{yz} = -\omega^2 m \frac{u_y + u'_y}{2}, \sigma_{zz} - \sigma'_{zz} = -\omega^2 m \frac{u_z + u'_z}{2}.
\]
The boundary condition (2.1) is now

\[
\begin{pmatrix}
u_x - u'_x \\
u_y - u'_y \\
u_z - u'_z \\
\sigma_{xz} - \sigma'_{xz} \\
\sigma_{yz} - \sigma'_{yz} \\
\sigma_{zz} - \sigma'_{zz}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & \frac{1}{K_{I1}} & 0 & 0 \\
0 & 1 & 0 & 0 & \frac{1}{K_{I2}} & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{K_n} \\
-\omega^2 m & 0 & 0 & 1 & 0 & 0 \\
0 & -\omega^2 m & 0 & 0 & 1 & 0 \\
0 & 0 & -\omega^2 m & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
u_x + u'_x \\
u_y + u'_y \\
u_z + u'_z \\
\sigma_{xz} + \sigma'_{xz} \\
\sigma_{yz} + \sigma'_{yz} \\
\sigma_{zz} + \sigma'_{zz}
\end{pmatrix}
\]

(2.7)

where \(K_n\) is computed from the quasi-static model (actually \(K_I\) for cracks in a elastic body), \(K_{I1}\) and \(K_{I2}\) are estimated by approximations [27] to the \(K_{II}\) and \(K_{III}\) values, and \(m\) is estimated as the additional interface mass. Only the 2-dimensional problem was discussed in references [26, 27]. Pyrak-Nolte and her coauthors have also studied localization of interface waves in fractured interfaces between crusts [28, 29].

The quasi-static model was verified by Achenbach and coauthors [30, 31] as a low frequency limit for a periodic array of cracks in which the exact solution was given using diffraction theory. Note that the comparisons were done for the case of normal incidence. A parametric study of the comparison between the finite stiffness, density, and quasi-static models was conducted by Xu and Datta [32].

Although the above approximate models have been used extensively and have relatively good agreement with experimental results for very small thickness-to-wavelength ratios, they have some disadvantages when applied to an interfacial layer:
(i) The finite stiffness model cannot be applied to an interfacial layer with high density (or high impedance).

(ii) The density model can only be applied to an interface layer with very low modulus (although it is hard to find high density low modulus material) [32].

(iii) The finite stiffness, density and stiffness-mass models lack terms which couple shear displacement to normal stress or normal displacement to shear stress. This results in worse behavior for oblique incidence than for normal incidence.

(iv) None of the finite stiffness, density and stiffness-mass models represents asymptotically the behavior of a physically infinitely thin interfacial layer \((kh \to 0)\).

(v) The quasi-static model works best for normal incidence, because the normal stiffness \(K_n\) is obtained exactly from the known solution for tensile stress. The accuracy of this model is worse for oblique incidence. In any case, the two transverse stiffnesses \(K_{t1}\) and \(K_{t2}\) cannot be obtained exactly. Another limitation is that the quasi-static solutions are not known for all types of imperfections so far.

In addition, these models cannot be extended straightforwardly to account for the coupling effect due to interface anisotropy. The advantage of these models lies in their simplicity, but there is a need to determine the range of their applicability.
since they do not describe asymptotically the behavior of a physically infinitely thin interfacial layer. A mathematical analysis of different approximate boundary condition models mentioned here, with regard mainly to uniqueness of solutions, has been given by Martin [33].

2.2 First Order Asymptotic Boundary Conditions Deduced from an Interfacial Layer Model

It is advantageous to have a higher order approximate model which will work at higher frequencies. A new boundary condition model called the asymptotic boundary condition model which overcomes the disadvantages of the previous models will be discussed in this section. Recently Rokhlin and Wang [1] proposed to approximate the actual imperfect interface by a thin interfacial layer with effective elastic properties. Both the upper and lower boundaries between this layer and the substrates are assumed to be perfect. The properties of the layer are determined using micromechanical analysis developed for multiphase materials. For some practical cases, the interfacial layer model is exact, as for example an adhesive joint, a diffusion bond or the fiber-matrix interphase coating. In other cases, when the interface includes different microdefects such as porosity, inclusions or cracks, the model gives a reasonable approximation to exact scattering models [1].

To describe wave interaction with such an interfacial layer equivalent boundary conditions can be introduced to represent the effect of the interfacial layer. For the
case when the thickness of the interfacial layer is much smaller than a wavelength, Rokhlin and Wang asymptotically expanded the exact solutions for interfacial layers and derived the equivalent first order asymptotic boundary conditions for an isotropic viscoelastic layer [2] and an orthotropic layer in the plane of symmetry [1, 3]. Also they asymptotically expanded the transfer matrix which relates the displacements and stresses on each side of the interfacial layer. The first order asymptotic boundary conditions were later derived for the off-plane-of-symmetry case [5] to analyze the reflection from an anisotropic imperfect interface, and in [4] to study the localization of an interface wave in an anisotropic interfacial layer. Besides this method of asymptotically expanding the boundary-problem solution, other asymptotic methods have recently been proposed by Boström, Bövik and Olsson [35] using series expansions of the governing differential equations and by Wickham [36] using approximations with boundary integral methods. These models [35, 36] give the same level of approximations but are much more complicated in realization.

Before presenting the results obtained by Rokhlin and his coauthors, I introduce a matrix notation for the multilayered system shown in Fig 2.2(a). This matrix formalism has been used for an isotropic multilayered system by Thomson [37] and Haskell [38] in the 50s. In the problem of propagation of elastic waves in layered media, it is usually assumed that each layer is composed of homogeneous material while successive layers are in welded contact with each other. Considering the nth
plate, we can write the displacement and stress vectors on the $n$th boundary by a propagation matrix $B_n$ and the displacement and stress vectors on the $(n-1)$th boundary as [37]:

$$(u_z^{(n)}, u_z^{(n)}, \sigma_z^{(n)}, \sigma_z^{(n)})^T = B_n(u_z^{(n-1)}, u_z^{(n-1)}, \sigma_z^{(n-1)}, \sigma_z^{(n-1)})^T$$

(2.8)

The elements of $B_n$ were given in [37] for isotropy; they depend on the properties of the $n$th layer and the known propagation wave vector through Snell's law. The continuity of displacement and stress vectors across the interfaces enables one to express the quantities in the $n$th layer in terms of corresponding quantities in the first plate. We can write

$$(u_z^{(n)}, u_z^{(n)}, \sigma_z^{(n)}, \sigma_z^{(n)})^T = \prod_{i=1}^{n} B_i(u_z^{(0)}, u_z^{(0)}, \sigma_z^{(0)}, \sigma_z^{(0)})^T = B(u_z^{(0)}, u_z^{(0)}, \sigma_z^{(0)}, \sigma_z^{(0)})^T$$

(2.9)

where $B = \prod_{i=1}^{n} B_i$

For an interfacial layer between two media as shown by Fig. 2.2(b), there is only one layer, so the propagation matrix $B$ of this layer becomes the exact boundary condition transfer matrix in equation (2.2) if one replaces the imperfect interface in Fig. 2.1 by the interfacial layer. The elements of the boundary condition transfer matrix $B$ depend on the properties of the interfacial layer. If the thickness of the layer is much less than the smallest normal component of the wavelength in this elastic system, one can expand the exact matrix asymptotically, therefore obtaining asymptotic boundary conditions. Let $k_z$ denote the smallest normal
component of the wave vector; we always take the following first order asymptotic approximations in $\mathbf{B}$:

$$\cos(k_z h) \approx 1; \quad \sin(k_z h) \approx k_z h \quad (2.10)$$

where $h$ is the thickness of the layer.

An important step was taken by Rokhlin and Wang in [2] in which the asymptotic boundary conditions for a viscoelastic solid between two isotropic solid media were described as:

$$\mathbf{B} = \begin{bmatrix} 1 & b_{12} & \frac{1}{K_t} & 0 \\ \frac{C_{13}}{C_{33}} b_{12} & 1 & 0 & \frac{1}{K_n} \\ -\omega^2 M_p & 0 & 1 & \frac{C_{13}}{C_{33}} b_{12} \\ 0 & -\omega^2 M_n & b_{12} & 1 \end{bmatrix} \quad (2.11)$$

where

$$K_t = \mu / h = C_{55} / h, \quad K_n = C_{33} / h$$

$$b_{12} = -i kh, \quad M_n = \rho h, \quad M_p = q_1 M_n \quad (2.12)$$

$$q_1 = 1 - \frac{C_{11} C_{33} - C_{13}^2}{C_{33} \rho V^2} = 1 - \left( \frac{V_p}{V} \right)^2.$$

Here $C_{ij}$ are the elastic constants and $\rho$ is the density of the layer; $V_p = \frac{C_{11} C_{33} - C_{13}^2}{C_{33} \rho}$ is the longitudinal velocity in an infinite thin plate; $k$ is the projection of the wave number on the interface plane and $V$ is the velocity along the interface $V = V_i / \sin \theta_i$ ($V_i$ and $\theta_i$ relate to the incident wave). The viscosity of the layer can be accounted for by introducing small imaginary parts in the elastic constants.
Figure 2.2: (a) A multilayered elastic system. (b) A thin interfacial layer as a replacement of an imperfect solid-solid interface. $h$ is the thickness of the layer.
One can rewrite equation (2.11) in the following form:

\[
\begin{align*}
\sigma'_{zz} &= K_i (u_x - u'_x) + ikC_{55}u'_z \\
\sigma'_{zz} &= K_n (u_z - u'_z) + ikC_{13}u'_x \\
(\sigma_{zz} - \sigma'_{zz}) + ikh\frac{C_{13}}{C_{33}}\sigma'_{zz} &= -\omega^2 M_p u'_x \\
(\sigma_{zz} - \sigma'_{zz}) + ikh\sigma'_{xx} &= -\omega^2 M_n u'_z
\end{align*}
\]

The authors [2] pointed out that while these boundary conditions follow directly from approximation (2.10) in the exact solutions for a thin layer, they can also be interpreted physically by applying first principles (Hooke’s law for the first pair of equations in (2.13), Newton’s second law for the second pair) to this thin layer.

The finite stiffness, density and stiffness-mass models are subsets of these asymptotic boundary conditions (2.13). Besides the commonly used stiffness and mass terms, the asymptotic boundary conditions also include coupling terms (those terms which include \( b_{12} \)) which couple the shear displacement to the normal stress or the normal displacement to the shear stress. One may notice that the “plate mass” term (\( M_p \)) is different from the normal mass terms (\( M_n \)) by a factor of \( q \). This mass factor \( q \) reflects the effect of the plate motion in the normal direction on the effective plate inertia. It is less than 1 at oblique incidence. Recall that in the quasi-static model, the three mass terms are identical which is the case for asymptotic boundary conditions at normal incidence.

The advantage of this model is that it is applicable not only for small thickness-
to-wavelength ratios, but also over broader frequency ranges than one might expect. It is shown in [2] that asymptotic boundary conditions give a better approximation than other simpler models (the finite stiffness model, for example) at higher frequencies. The accuracy of this approximation is also less dependent on the layer properties and incidence angles. It is also shown that the boundary conditions with only stiffness and/or inertial terms do not represent asymptotically an infinitely thin interfacial layer. In order to ensure an asymptotic representation the terms coupling normal and shear components of displacement and stress should always be included. Readers are referred to reference [2] for detailed discussions.

In conclusion the asymptotic boundary condition model can overcome most of the disadvantages of previous models. This model works much better than previous models for most cases, especially when physically there is an actual layer (as in adhesive bonding or fiber-matrix interphase) and the layer properties are known or can be accurately modelled. The disadvantage is that when the interface properties are unknown or difficult to model with a thin interfacial layer the asymptotic boundary conditions are hard to apply, as for example with unknown cracks and inclusions. Please note that this difficulty also exists when the finite stiffness model and others are applied.

So far, we have not discussed the effect of introducing anisotropy into the boundary conditions. To address this problem and to better model more complex interfacial conditions, Rokhlin and Wang proposed to approximate an actual im-
perfect interface with its complicated properties by a thin interfacial layer with effective elastic properties and to introduce equivalent boundary conditions to model this layer for small thickness-to-wavelength ratios. For some practical cases, the interfacial layer model is exact, as for example an adhesive joint or diffusion bond. In other cases, when the interface is imperfect and includes different microdefects, it can be considered as a multiphase composite layer with certain effective elastic properties [1, 3]. The asymptotic boundary conditions were obtained for the case of a thin orthotropic material with a plane of symmetry coinciding with the incident plane using the same methodology as for isotropy [3].

Let \((x, z)\) be the incident plane, \((x, y)\) the interface plane. The asymptotic transfer matrix \(B\) is

\[
B = \begin{bmatrix}
1 & b_{12} & \frac{1}{K_t} & 0 \\
\frac{C_{13}}{C_{33}} b_{12} & 1 & 0 & \frac{1}{K_n} \\
-\omega^2 M_p & 0 & 1 & \frac{C_{13}}{C_{33}} b_{12} \\
0 & -\omega^2 M_n & b_{12} & 1
\end{bmatrix}.
\]  (2.14)

In the chosen coordinate system, \(B\) has a similar form as in equation (2.11) with its elements depending on the properties of this orthotropic layer. This is due to the fact that the incident plane coincides with a plane of symmetry for this orthotropic layer. The elements are obtained by asymptotically expanding an exact transfer matrix given in reference [34]. In the original work [34], the liquid-solid-liquid interface was considered; the exact solution has been extended to the solid-solid-solid case [1]. In practice, the interface layer is thin, so modelling of
such an interface as a thin orthotropic layer is reasonable and generally appealing. More detailed discussion can be found in [3], in which the cases of reflection and transmission and localization of interface waves for such a thin orthotropic layer were studied.

As an example, a periodic array of interface voids was studied in [1]. The imperfect interface was first modelled as a two-phase interfacial layer (bulk and pores) and then the asymptotic boundary conditions (2.14) were introduced to replace this thin orthotropic layer. The numerical results for reflection and transmission from the cracks were compared with the numerical solution obtained by Achenbach and Kitahara [39]. They show good agreement for small thickness-to-wavelength ratio.

There are several drawbacks to the first order asymptotic boundary condition model for representation of a thin interfacial layer. One is that energy balance is not satisfied, i.e. the sum of the scattered energies does not equal that of the incident wave. Thus for the limiting case where the interfacial layer and the two substrates are of the same material, although the first order asymptotic boundary condition model predicts no mode conversion, it fails to predict full transmission. In addition, the first order asymptotic boundary condition model gives poorer results for the transmitted field than for the reflected field due to the asymmetric nature of its derivation by asymptotic expansion of the transfer matrix to the interface boundary on the reflection side. To solve this problem, the next chapter
develops the so-called second order asymptotic boundary condition model which is not only more accurate and free of these imperfections, but also preserves the simplicity of the first order boundary condition model.

Another critical issue is that all the previous models can only handle 2-D problems; no coupling between in-plane and out-of-plane components of the elastic field has been considered. It is very important to generalize the previous work to elastic systems of general anisotropy, since wave interaction with imperfect interfaces is the effect of interface imperfection orientation has recently become important. Typical examples for (a) a fractured interface with preferred crack orientation and (b) an interphase with cylindrical-like pore inclusions are shown in Fig. 1.2. One sees that the in-plane and out-of-plane parts of the elastic field are coupled even for isotropic substrates due to the interface anisotropy. The first order asymptotic boundary conditions were extended in this case for an arbitrarily oriented orthotropic interfacial layer [4, 5]. The detailed description of the first order boundary condition model for this generally anisotropic interfacial layer will be given in the next chapter to facilitate discussion of the proposed second order boundary condition model.
CHAPTER III

GENERALIZED BOUNDARY CONDITIONS FOR WAVE SCATTERING FROM ANISOTROPIC PLANE INTERFACES BETWEEN DISSIMILAR ANISOTROPIC MEDIA

We will start in this chapter with the exact transfer matrix solution for a generally anisotropic plane interphase between anisotropic substrates, then we will introduce the first and second order asymptotic boundary conditions (B.C.) for this thin interphase and discuss the physical meanings of each term in the asymptotic B.C. models and in their simplified versions. Finally we will give numerical examples for scattering from a thin porous interphase between two generally anisotropic dissimilar substrates. The solutions calculated by different approximate models will be compared with the exact calculated by a multiple-internal-reflection approach.
3.1 The Exact Transfer Matrix Solution for an Anisotropic Plane Interfacial Layer between Two Anisotropic Solids

Let us consider a plane ultrasonic wave incident on an anisotropic layer between two generally anisotropic solids; reflected and transmitted waves appear on the two sides of the interface layer. To describe the elastic motion of a generally anisotropic interfacial layer between two anisotropic substrates, one uses an exact $6 \times 6$ transfer matrix $B$, which relates the particle displacements and stresses on the top $(u_i, \sigma_{zi})$ and bottom $(u'_i, \sigma'_{zi})$ surfaces of the layer (see Fig. 3.1):

\[
(u_x, u_y, u_z, \sigma_{xx}, \sigma_{xy}, \sigma_{zz})^T = B (u'_x, u'_y, u'_z, \sigma'_{xx}, \sigma'_{xy}, \sigma'_{zz})^T,
\]

(3.1)

where $(\cdot)^T$ is the transpose of a vector or matrix. Equation (3.1) can be considered as equivalent boundary conditions for replacement of the interfacial layer.

Let us define the elastic field vector $\vec{U} = (u_x, u_y, u_z, \sigma_{xx}, \sigma_{xy}, \sigma_{zz})^T$. If the $(x, z)$ plane is the incident plane, for any elastic system varying only in the $z$-direction, one represents the governing equations for the elastic field by a system of differential equations:

\[
\frac{\partial \vec{U}(z)}{\partial z} + ik \vec{A} \vec{U}(z) = 0,
\]

(3.2)

where $k$ is the projection of the wave vector on the interface. The six-dimensional tensor $\vec{A}$ was first studied in [40, 41]; its elements depend on the layer elastic properties, density and wave projection $k$. For general anisotropy, $\vec{A}$ can be written in a form given in [5]. Equation (3.2) may be regarded as a six-dimensional form of
the equations of motion. This representation of the elastic field has also been used in [42] to study wave propagation in anisotropic periodically layered composites.

The differential equation (3.2) has the well-known matrix exponential solution

\[ \tilde{U}(z = z_0 + \Delta z) = \exp(-i k \Delta z A) \tilde{U}(z = z_0). \] (3.3)

and for an interfacial layer of thickness \( h \) equation (3.3) becomes

\[ \tilde{U} = \exp(-i k h A) \tilde{U}'. \] (3.4)

Thus the exact transfer matrix \( B \) in equation (3.1) is

\[ B = \exp(-i k h A). \] (3.5)

Equation (3.4) can be considered to be the exact boundary conditions for the modeling of the interfacial layer.

Here we derive \( A \) for an orthotropic interfacial layer rotated about one of its axes of symmetry with respect to the incident plane. For such an orthotropic layer the material coordinate system \((x^0, y^0, z)\) has a rotation of angle \( \varphi \) in the \((x^0, y^0)\) plane from the global coordinates \((x, y, z)\) as shown in Fig. 3.2. The rotated elastic constant tensor \( C \) is related to the material elastic constant tensor \( C^0 \) by the so-called Bond stress transformation matrix \( P \) as [43]

\[ C = PC^0P^T, \] (3.6)
Figure 3.1: Elastic system with an anisotropic interfacial layer between two generally anisotropic substrates. $I$ stands for a quasilongitudinal mode, and $t_f$ and $t_s$ stand for fast and slow quasitransverse modes respectively.
where

$$P = \begin{bmatrix}
\cos^2 \varphi & \sin^2 \varphi & 0 & 0 & 0 & \sin 2\varphi \\
\sin^2 \varphi & \cos^2 \varphi & 0 & 0 & 0 & -\sin 2\varphi \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \varphi & -\sin \varphi & 0 \\
0 & 0 & 0 & \sin \varphi & \cos \varphi & 0 \\
-\frac{\sin 2\varphi}{2} & \frac{\sin 2\varphi}{2} & 0 & 0 & 0 & \cos 2\varphi
\end{bmatrix}.$$  (3.7)

Thus $C^0$ and $C$ have the following forms,

$$C^0 = \begin{bmatrix}
C_{11}^0 & C_{12}^0 & C_{13}^0 & 0 & 0 & 0 \\
C_{12}^0 & C_{22}^0 & C_{23}^0 & 0 & 0 & 0 \\
C_{13}^0 & C_{23}^0 & C_{33}^0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55}^0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^0
\end{bmatrix}, \quad C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{bmatrix}.  \quad (3.8)$$

Note that $C$ is equivalent in the form to the elastic tensor for a monoclinic material.

A simple version of $A$ has been derived as [5]

$$A = \begin{bmatrix}
0 & 0 & 1 & -\frac{S_{55}}{ik} & -\frac{S_{45}}{ik} & 0 \\
0 & 0 & 0 & -\frac{S_{45}}{ik} & -\frac{S_{44}}{ik} & 0 \\
\frac{C_{13}}{C_{33}} & \frac{C_{36}}{C_{33}} & 0 & 0 & 0 & -\frac{1}{ikC_{33}} \\
\frac{\omega^2 \rho_0}{ik} (1 - \frac{Q_{11}}{\rho_0 V^2}) & ikQ_{16} & 0 & 0 & 0 & \frac{C_{13}}{C_{33}} \\
\frac{\omega^2 \rho_0}{ik} (1 - \frac{Q_{66}}{\rho_0 V^2}) & 0 & 0 & 0 & \frac{C_{36}}{C_{33}} \\
0 & 0 & \frac{\omega^2 \rho_0}{ik} & 1 & 0 & 0
\end{bmatrix}.  \quad (3.9)$$

where $\rho_0$ is the density of the layer material and $V = \omega/k$ is the tracing velocity along the layer ($x$-direction). For an incident wave of angle $\theta_i$, $V = V_i/\sin(\theta_i)$, where $V_i$ is the phase velocity of the incident wave. By using the rotated coordinate
Figure 3.2: Global \((x, y, z)\) and material \((x_0, y_0, z)\) coordinate systems.
system \((x, y, z)\), one can represent the incident field straightforwardly and usually simplify the solution for this general anisotropic boundary problem.

In (3.9) the compliances \(S_{44}, S_{55}\) and \(S_{45}\) can be written in terms of the elastic constants as

\[
S_{44} = \frac{C_{55}}{C_{44} C_{55} - C_{45}^2}, \quad S_{55} = \frac{C_{44}}{C_{44} C_{55} - C_{45}^2}, \quad S_{45} = -\frac{C_{45}}{C_{44} C_{55} - C_{45}^2};
\]

and \(Q_{ij}\) are the stiffnesses defined under the condition of plane stress \(\sigma_{zz} = \sigma_{xx} = \sigma_{yy} = 0\):

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} = 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{pmatrix},
\]

(3.11) is Hooke’s law for a very thin anisotropic free plate.

It can be shown that the eigenvalues of \(A\) for a fixed wave projection number \(k\) are:

\[
\lambda_p = \pm \frac{k_{zj}}{k}, \quad j = 1, 2, 3; \quad p = 1, 2, \ldots, 6
\]

where \(k_{zj}\) is the \(z\) component of the wave vector of type \(j\) with the same wave projection \(k\) \((j = l, t_f, o t_s\) which stands for a quasilongitudinal, fast, or slow quasitransverse mode). Thus the determinant of the exact transfer matrix \(B\) is

\[
|B| = \exp(-ikh \text{ trace}(A)) = \exp(-ikh \sum_{p=1}^{6} \lambda_p) = 1,
\]

(3.13) since the trace of \(A\) \((\sum_{p=1}^{6} \lambda_p) = 0\). This guarantees the energy balance for the wave propagation through the whole elastic system after the interfacial layer is
replaced by the exact B.C. (3.4).

To solve for the scattering and interface localization problem for the single layer problem in Fig. 3.1 we need first find the exact transfer matrix $B$ of (3.5), and then solve the boundary equations (3.1). $B$ can be calculated using two approaches, one is to first diagonalize matrix $A$ using standard eigenvector approach [44] as

$$A = M^{-1} \text{diag}\{\lambda_p; p = 1, 2, \ldots, 6\}M,$$  \hspace{1cm} (3.14)

and then $B$ (3.5) can be found as

$$B = M^{-1} \text{diag}\{\exp(-ikh\lambda_p); p = 1, 2, \ldots, 6\}M.$$  \hspace{1cm} (3.15)

The other involves calculation of $B$ using series expansion in parameter $kh$,

$$B = I - ikhA - \frac{1}{2}(khA)^2 + \frac{1}{6}i(khA)^3 + \cdots.$$  \hspace{1cm} (3.16)

An alternative approach to solve the scattering problem in Fig. 3.1 by considering multiple internal reflections inside the interfacial layer directly will be presented in Section 3.3.

### 3.2 The First and Second Order Asymptotic Boundary Conditions

To obtain the simplified first and second order asymptotic B.C. let us expand the exact transfer matrix $B$ of (3.5) in a series in the small parameter $kh$ (interfacial layer thin compared with the wavelength),

$$B = I - ikhA - \frac{1}{2}(khA)^2 + \frac{1}{6}i(khA)^3 + O(khA)^4.$$  \hspace{1cm} (3.17)
This equation will define the order of accuracy of the boundary condition models. Another form of the exact transfer matrix $B$ has been given by Nayfeh [45] but it cannot easily be asymptotically expanded. This also requires solution of the Christoffel equation for the interfacial layer.

### 3.2.1 The first order asymptotic B.C.

Keeping only the first order term in the asymptotic expansion of $B$ in equation (3.17), we have:

$$B = I - i k h A + O(k h A)^2 = B_I + O(k h A)^2. \quad (3.18)$$

We call $B_I = I - i k h A$ the first order asymptotic transfer matrix. The eigenvalues of $B_I$ can be found as:

$$\lambda_p = 1 \pm k_{zj} h, \quad j = 1, 2, 3; \quad p = 1, 2, \ldots, 6, \quad (3.19)$$

where $k_{zj}$ is the $z$ component of the wave vector of type $j$ in the layer material.

So the determinant of $B_I$ is

$$|B_I| = \prod_{p=1}^{6} \lambda_p = \prod_{j=1}^{3} (1 - k_{zj}^2 h^2) = 1 + O(k_{zj} h)^2. \quad (3.20)$$

Therefore when $k_{zj} h = 2 \pi h / \lambda_{zj}$ is small, i.e. when the layer thickness is much smaller than the normal component of the wavelength in the layer material, the determinant of $B_I$ is close, but not equal, to 1.

Replacing $B$ by $B_I$ in equation (3.1) we obtain the first order asymptotic B.C.
relating the elastic fields on both sides of the layer:

\[(u_x, u_y, u_z, \sigma_{xx}, \sigma_{xy}, \sigma_{zz})^T = B_I (u'_x, u'_y, u'_z, \sigma'_{xx}, \sigma'_{xy}, \sigma'_{zz})^T.\]  

(3.21)

This representation of the B.C. (3.21) is valid for \(|khA| \ll 1\), i.e. for layer thickness much smaller than the smallest wavelength in the elastic system. Substituting \(B_I = I - ikhA\) into equation (3.21) one can write

\[\bar{U} - \bar{U}' = -ikhA\bar{U}'.\]  

(3.22)

Note that (3.22) is a finite difference approximation to the differential equation of \(\bar{U}'\) in (3.2) with \(\Delta \bar{U} = \bar{U} - \bar{U}'\) and \(\Delta z = h\). But unlike the exact B.C. (3.3) the first order B.C. (3.22) has a form in which the elastic field vectors on the top (\(\bar{U}\)) and bottom (\(\bar{U}'\)) surfaces of the layer are not interchangeable. This discrepancy will be important in the later discussions. For detailed discussions of the first order B.C. the reader is referred to [5].

### 3.2.2 The second order asymptotic boundary conditions

To improve the accuracy of the approximation, let us take the second order asymptotic expansion of \(B\) in equation (3.17),

\[B = I - ikhA - \frac{1}{2}(khA)^2 + O(khA)^3 = B' + O(khA)^3.\]  

(3.23)

In such a representation the matrix \(B'\) loses its simple form. Also asymmetry between \(\bar{U}\) and \(\bar{U}'\) remains as in (3.22).
Instead of taking directly the first two terms in the series expansion of $B$, let us take the finite difference approximation to the exact differential equation (3.2) in the form [50]:

$$\frac{\bar{U} - \bar{U}'}{h} + i k A \frac{\bar{U} + \bar{U}'}{2} = 0$$

(3.24)

i.e.,

$$\bar{U} - \bar{U}' = -i k h A \frac{\bar{U} + \bar{U}'}{2}.$$  

(3.25)

Comparing the equation to the first order asymptotic B.C. in the form (3.22), one sees that the only difference is that on the right side of (3.22) the elastic field in the lower substrate $\bar{U}'$ is replaced by the average of the elastic fields on the upper and lower surfaces of the interfacial layer ($\bar{U} + \bar{U}')/2$. The representation (3.25) of the B.C. has simplicity comparable to that of the first order asymptotic B.C. (3.22) but, as we show below, preserves correctly all second order terms of the exact solution. To find the order of this approximation let us rewrite equation (3.25) in a transfer matrix form (3.1):

$$\bar{U} = (I + \frac{i k h A}{2})^{-1}(I - \frac{i k h A}{2})\bar{U}' = B_{II} \bar{U}'$$

(3.26)

where the transfer matrix $B_{II}$ is

$$B_{II} = (I + \frac{i k h A}{2})^{-1}(I - \frac{i k h A}{2}),$$

(3.27)

or can be rewritten in another form as

$$B_{II} = 2(I + \frac{i k h A}{2})^{-1} - I.$$  

(3.28)
Expanding (3.28) in a series we have

\[ B_{II} = I - ikhA - \frac{1}{2}(khA)^2 + \frac{1}{4}i(khA)^3 + O(khA)^4. \] (3.29)

Comparing the expansion (3.29) for \( B_{II} \) with (3.17), we note that the transfer matrix \( B_{II} \) is identical in second order to the exact transfer matrix \( B \) (3.5) (the third order term in (3.29) is different from that of the exact expansion (3.17)). So B.C. (3.25) or (3.26) give a second order approximation to the exact solution for the elastic field. Another important feature of this new representation is that the second order B.C. is symmetric with respect to the elastic field vectors on the top and bottom surfaces of the layer. From equation (3.27) and the eigenvalues for \( A \) (3.12), the determinant of \( B_{II} \) is,

\[ |B_{II}| = \prod_{j=1}^{3} \frac{(1 - k_{2j}^2 h^2/2)}{(1 - k_{2j}^2 h^2/2)} = 1. \] (3.30)

**Note:** The other way to derive the second order B.C. (3.25) is to start with the first order B.C. (3.22) and eliminate the asymmetry of the wave solutions with respect to \( \bar{U} \) and \( \bar{U}' \). The first order B.C. (3.22) approximates the solution for the elastic field on the top surface \( \bar{U} \) of the interfacial layer if the elastic field on the bottom surface \( \bar{U}' \) is known. Now assume that the elastic field on the top surface \( \bar{U} \) is known; using the same first order B.C. approach we find the solution on the bottom surface \( \bar{U}' \) to be

\[ \bar{U} - \bar{U}' = -ikhA\bar{U}. \] (3.31)
Then combining (3.31) and (3.22) we have

\[ 2(\vec{U} - \vec{U}') = -ikhA(\vec{U} + \vec{U}'). \]  

(3.32)

Thus we obtain the same second order B.C. as (3.25).

To calculate a single layer problem, the second order B.C. (3.25) can be used directly; for multilayered problems, the transfer matrix representation (3.26) should be used. To calculate \( \mathbf{B}_{II} \) using (3.28), one must invert \( I + ikhA/2 \), which has unit diagonal terms so the inversion should be stable. The total effective transfer matrix \( \mathbf{B}_{II}^{total} \) which relates the elastic fields on the top and the bottom interfaces of an \( N \)-layered system is a product of the second order asymptotic transfer matrices \( \mathbf{B}_{II}^i \) for individual layers

\[ \mathbf{B}_{II}^{total} = \prod_{i=1}^{N} \mathbf{B}_{II}^i. \]  

(3.33)

### 3.2.3 Physical meanings of terms in the asymptotic B.C.

For a monoclinic interfacial layer of thickness \( h \), using the expression for \( A \) in (3.9) we can write B.C. (3.25) in the form:

\[
\left( \begin{array}{c}
\vec{u} - \vec{u}' \\
\vec{\sigma} - \vec{\sigma}'
\end{array} \right) = \left[ \begin{array}{cc}
\mathbf{N} & \mathbf{S} \\
\mathbf{M} & \mathbf{N}^T
\end{array} \right] \left( \begin{array}{c}
\frac{\vec{u} + \vec{u}'}{2} \\
\frac{\vec{\sigma} + \vec{\sigma}'}{2}
\end{array} \right),
\]

(3.34)

where \( \vec{u} \) and \( \vec{u}' \) are the displacement vectors on the top and bottom surfaces respectively, and \( \vec{\sigma} \) and \( \vec{\sigma}' \) the stress vectors. \( \mathbf{N}, \mathbf{S} \) and \( \mathbf{M} \) are the submatrices (3x3) of \(-ikhA\).
To understand the physical meanings of $\mathbf{N}, \mathbf{S}$ and $\mathbf{M}$ we set in (3.34) two of the three submatrices null. Let us first consider the simplified B.C. when $\mathbf{S}$ and $\mathbf{M}$ are nulled, i.e.

$$\tilde{u} - \tilde{u}' = \frac{\mathbf{N} \tilde{\mathbf{u}} + \tilde{\mathbf{u}}'}{2}, \quad \tilde{\sigma} - \tilde{\sigma}' = \frac{\mathbf{N} \tilde{\mathbf{\sigma}} + \tilde{\mathbf{\sigma}}'}{2}; \quad (3.35)$$

where the elements of $\mathbf{N}$ are

$$\mathbf{N} = -i k h \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
\frac{C_{13}}{C_{33}} & \frac{C_{36}}{C_{33}} & 0 \\
\end{bmatrix} = -i \frac{\omega h \sin(\theta_i)}{V_i} \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
\frac{C_{13}}{C_{33}} & \frac{C_{36}}{C_{33}} & 0 \\
\end{bmatrix}. \quad (3.36)$$

We call $\mathbf{N}$ the dynamic coupling matrix ($\mathbf{N}$ becomes null for $\omega = 0$). Only three off-diagonal elements of $\mathbf{N}$ are nonzero and their physical meanings become clearer after expansion of (3.35)

$$u_x - u'_x = -i k h \frac{u_x + u'_x}{2};$$
$$u_z - u'_z = -i k h \frac{C_{13} u_x + u'_x}{C_{33}} - i k h \frac{C_{36} u_y + u'_y}{2};$$
$$\sigma_{xz} - \sigma'_{xz} = -i k h \frac{C_{13} \sigma_{xz} + \sigma'_{xz}}{C_{33}} , \quad \sigma_{zy} - \sigma'_{zy} = -i k h \frac{C_{36} \sigma_{zy} + \sigma'_{zy}}{2};$$
$$\sigma_{zz} - \sigma'_{zz} = -i k h \frac{\sigma_{zz} + \sigma'_{zz}}{2}. \quad (3.37)$$

Three nonzero terms in $\mathbf{N}$ characterize the degree of coupling between the normal and shear displacements and those in $\mathbf{N}^T$ the coupling between normal and shear stresses. The factors $-i k h$ and $-i k h C_{13}/C_{33}$ couple the normal and in-plane shear components, and $-i k h C_{36}/C_{33}$ couples the normal and out-of-plane components.

Note that $(x, z)$ is the incident plane, which explains the difference in the coupling
factors for components along the $x$ and $y$ directions. For an incident plane coincident with one of the symmetry planes of the orthotropic layer ($\varphi = n\pi/2; n = 0, 1$ in Fig. 3.2), the coupling term between the normal and SH components disappears ($C_{36} = 0$). As one sees from (3.36) the elements of $N$ are proportional to $\sin(\theta_i)$ and become zero at normal incidence.

Next if $N$ and $M$ are nulled in (3.34) we have

$$\vec{u} - \vec{u}' = S \frac{\vec{\sigma} + \vec{\sigma}'}{2}, \quad \vec{\sigma} = \vec{\sigma}'; \tag{3.38}$$

where the elements of $S$ are

$$S = \begin{bmatrix} S_{sv} & S_c & 0 \\ S_c & S_{sh} & 0 \\ 0 & 0 & S_n \end{bmatrix} = h \begin{bmatrix} S_{35} & S_{45} & 0 \\ S_{45} & S_{44} & 0 \\ 0 & 0 & \frac{1}{C_{33}} \end{bmatrix}. \tag{3.39}$$

Note that the elements of $S$ are products of the layer compliance and thickness, and can be considered as specific compliances for a group of springs. Specifically, $S_n = \frac{h}{C_{33}}$ is the specific compliance for a normal spring, $S_{sv} = S_{55}h$ for an in-plane shear spring, $S_{sh} = S_{44}h$ for an out-of-plane shear spring, and $S_c = S_{45}h$ for two coupling springs with the same specific compliance. Therefore we call $S$ the specific compliance matrix. Note that unlike the dynamic coupling matrix $N$, $S$ is frequency and incident angle independent. The coupling springs are introduced due to the anisotropy of the material property of the interfacial layer with respect to the incident plane and they relate the in-plane or out-of-plane shear stresses to out-of-plane or in-plane shear displacements respectively. In the plane of symmetry
\((\varphi = 0 \text{ or } 90^\circ)\), the coupling springs disappear \((C_{45} = 0)\), and \(S_{sv} = h/C_{55}\) and \(S_{sh} = h/C_{44}\).

Finally if we null \(N\) and \(S\) in B.C. (3.34) we have

\[\ddot{u} = \ddot{u}', \quad \ddot{\sigma} - \ddot{\sigma}' = M \ddot{\bar{u}} + \ddot{\bar{u}}' = \frac{Mv^2}{2};\]  
(3.40)

where the elements of \(M\) are

\[
M = -\omega^2 \begin{bmatrix}
M_{sv} & M_c & 0 \\
M_c & M_{sh} & 0 \\
0 & 0 & M_n
\end{bmatrix} = -\omega^2 M_n \mathbf{I} + k^2 h \begin{bmatrix}
Q_{11} & Q_{16} & 0 \\
Q_{16} & Q_{66} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\begin{align*}
&= -\omega^2 M_n \begin{bmatrix}
1 - \frac{Q_{11}}{\rho_0 V^2} & -\frac{Q_{16}}{\rho_0 V^2} & 0 \\
-\frac{Q_{16}}{\rho_0 V^2} & 1 - \frac{Q_{66}}{\rho_0 V^2} & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad M_n = \rho_0 h. \quad (3.41)
\end{align*}

One sees that \(M\) relates displacements to stresses and its effect on the B.C. can be described by dynamic inertial terms. Therefore we call \(M\) the inertial matrix. In general there exist a normal inertia with \(M_n = \rho_0 h\), an in-plane shear inertia with \(M_{sv} = M_n(1 - \frac{Q_{11}}{\rho_0 V^2})\), an out-of-plane shear inertia with \(M_{sh} = M_n(1 - \frac{Q_{66}}{\rho_0 V^2})\) and two coupling inertias with \(M_c = -M_n(\frac{Q_{16}}{\rho_0 V^2})\). The coupling inertias appear due to the interface anisotropy and they relate the in-plane or out-of-plane shear displacements to out-of-plane or in-plane shear stresses. In the plane of symmetry the coupling inertial terms disappear \((Q_{16} = 0)\), and the remaining inertial terms in \(M\) can be considered as plate masses for wave motions of different polarizations along the \(x\)-direction (recall that \(\sqrt{Q_{11}/\rho_0}\) is the longitudinal plate wave velocity and \(\sqrt{Q_{66}/\rho_0}\) that of the SH mode).
Note that in general M depends on the incident field, the density and the stiffness of the interfacial layer. M can be decomposed into two parts: one \((-\omega^2M_n I)\) is angle independent, another is angle dependent (proportional to \(\sin^2(\theta_i)\)) and disappears at normal incidence. Therefore at normal incidence M reduces to \(-\omega^2M_n I\). However, for oblique incidence exact expressions for \(M_{sv}, M_{sh}\) and \(M_c\) should be used and it is not uncommon for them to be negative [3].

3.2.4 Simplified B.C. models

The second order B.C. in the form (3.34) can be expanded to the following boundary equations:

\[
\begin{align*}
    u_x - u_x' & = S_{sv} \frac{\sigma_{xx} + \sigma'_{xx}}{2} + S_c \frac{\sigma_{xy} + \sigma'_{xy}}{2} - i\kappa h \frac{u_x + u_x'}{2} \\
    u_y - u_y' & = S_c \frac{\sigma_{xx} + \sigma'_{xx}}{2} + S_{sh} \frac{\sigma_{xy} + \sigma'_{xy}}{2} \\
    u_z - u_z' & = S_n \frac{\sigma_{zz} + \sigma'_{zz}}{2} - i\kappa h \frac{C_{13} u_x + u_x'}{C_{33}} - i\kappa h \frac{C_{36} u_y + u_y'}{C_{33}} \tag{3.42} \\
    \sigma_{xx} - \sigma'_{xx} & = -\omega^2M_{sv} \frac{u_x + u_x'}{2} - \omega^2M_c \frac{u_y + u_y'}{2} - i\kappa h \frac{C_{13} \sigma_{zz} + \sigma'_{zz}}{C_{33}} \\
    \sigma_{xy} - \sigma'_{xy} & = -\omega^2M_c \frac{u_x + u_x'}{2} - \omega^2M_{sh} \frac{u_y + u_y'}{2} - i\kappa h \frac{C_{36} \sigma_{zz} + \sigma'_{zz}}{C_{33}} \\
    \sigma_{zz} - \sigma'_{zz} & = -\omega^2M_n \frac{u_z + u_z'}{2} - i\kappa h \frac{\sigma_{xx} + \sigma'_{xx}}{2}.
\end{align*}
\]

To solve the reflection-transmission problem we use a global coordinate system which is determined by the incidence and interface planes and for which B.C. (3.42) are formulated. In special cases one may prefer the material coordinate system \((x^0, y^0, z)\). The second order B.C. in the material coordinate system are
given in Appendix A.

The B.C. derived may be applied equally well on anisotropic interfaces between isotropic or anisotropic solids. For isotropic substrates, these boundary conditions couple longitudinal, SV and SH waves due to interface anisotropy (for example six refracted waves may appear for a longitudinal oblique incident wave). When the incident plane coincides with a plane of symmetry of an orthotropic layer (i.e. \( \phi = 0^\circ \) or \( 90^\circ \)), and for the isotropic case, the constants \( C_{13} = C_{36} = Q_{16} = S_{45} = 0 \). In this case B.C. (3.42) decompose into in-plane and out-of-plane parts. The in-plane part is

\[
\begin{pmatrix}
    u_x - u'_x \\
    u_z - u'_z \\
    \sigma_{xx} - \sigma'_{xx} \\
    \sigma_{zz} - \sigma'_{zz}
\end{pmatrix}
= 
\begin{pmatrix}
    0 & -ikh & \frac{h}{C_{55}} & 0 \\
    -ikh & \frac{C_{13}}{C_{33}} & 0 & \frac{h}{C_{33}} \\
    -\omega^2 M_{sv} & 0 & 0 & -ikh \frac{C_{13}}{C_{33}} \\
    0 & -\omega^2 M_{n} & -ikh & 0
\end{pmatrix}
\begin{pmatrix}
    \frac{u_x + u'_x}{2} \\
    u_z + u'_z \\
    \frac{\sigma_{xx} + \sigma'_{xx}}{2} \\
    \frac{\sigma_{zz} + \sigma'_{zz}}{2}
\end{pmatrix}
\]

(3.43)

As one can see the second order asymptotic B.C. (3.43) are a subset of the boundary conditions (3.42) for the general anisotropic problem. The out-of-plane part of the B.C. is

\[
\begin{pmatrix}
    u_y - u'_y \\
    \sigma_{zy} - \sigma'_{zy}
\end{pmatrix}
= 
\begin{pmatrix}
    0 & \frac{h}{C_{44}} \\
    -\omega^2 M_{sh} & 0
\end{pmatrix}
\begin{pmatrix}
    \frac{u_y + u'_y}{2} \\
    \frac{\sigma_{zy} + \sigma'_{zy}}{2}
\end{pmatrix}
\]

(3.44)

where \( M_{sh} \) in this case is simply \( \rho_0 h (1 - \frac{C_{66}}{\rho_0 V^2}) \).

As we will see in the next section the second order B.C (3.34) or (3.42) give an asymptotic approximation homogeneous in incident angle, interfacial layer orien-
tation and layer property (density and elastic constants). It is also interesting to consider simplified B.C. by neglecting those terms which are very small compared to the others. First let us consider the case when N can be neglected in (3.34).

We have:

\[
\begin{align*}
  u_x - u'_x &= S_{sv} \frac{\sigma_{zz} + \sigma'_{zx}}{2} + S_c \frac{\sigma_{zy} + \sigma'_{zy}}{2} \\
  u_y - u'_y &= S_c \frac{\sigma_{zx} + \sigma'_{zx}}{2} + S_{sh} \frac{\sigma_{zy} + \sigma'_{zy}}{2} \\
  u_z - u'_z &= S_n \frac{\sigma_{zz} + \sigma'_{zz}}{2} \\
  \sigma_{zx} - \sigma'_{zx} &= -\omega^2 M_{sv} \frac{u_x + u'_x}{2} - \omega^2 M_c \frac{u_y + u'_y}{2} \\
  \sigma_{zy} - \sigma'_{zy} &= -\omega^2 M_c \frac{u_x + u'_x}{2} - \omega^2 M_{sh} \frac{u_y + u'_y}{2} \\
  \sigma_{zz} - \sigma'_{zz} &= -\omega^2 M_n \frac{u_z + u'_z}{2}.
\end{align*}
\]

This approximation is good at small incident angles \( \theta_i \) since \( kh = (\omega h / V_i) \sin(\theta_i) \ll 1 \). Under those conditions the inertial matrix \( M \) can also be approximated by \(-\omega^2 M_n I\) (see equation (3.41)); therefore the B.C. (3.45) can be further simplified as

\[
\begin{align*}
  u_x - u'_x &= S_{sv} \frac{\sigma_{xx} + \sigma'_{xx}}{2} + S_c \frac{\sigma_{xy} + \sigma'_{xy}}{2} \\
  u_y - u'_y &= S_c \frac{\sigma_{xx} + \sigma'_{xx}}{2} + S_{sh} \frac{\sigma_{xy} + \sigma'_{xy}}{2} \\
  u_z - u'_z &= S_n \frac{\sigma_{zz} + \sigma'_{zz}}{2} \\
  \sigma_{xx} - \sigma'_{xx} &= -\omega^2 M_{sv} \frac{u_x + u'_x}{2} \\
  \sigma_{xy} - \sigma'_{xy} &= -\omega^2 M_c \frac{u_x + u'_x}{2} \\
  \sigma_{zz} - \sigma'_{zz} &= -\omega^2 M_n \frac{u_z + u'_z}{2}.
\end{align*}
\]
\[
\sigma_{zz} - \sigma'_{zz} = -\omega^2 M_n \frac{u_z + u'_z}{2}.
\]

B.C. (3.46) is exactly the second order asymptotic B.C. (3.34) for the case of normal incidence and is referred as the stiffness-mass model or shell model. It is important to point out that the anisotropic coupling compliance \(S_c\) should be kept in the B.C. even for the case of normal incidence. From equations (3.10) and (3.39) one can see that the compliance \(S_c\) is zero in the plane of symmetry of the interfacial layer \((\varphi = 0 \text{ or } 90^\circ, C_{45} = 0 \text{ and } S_{45} = 0)\), and \(S_c\) can be neglected when the anisotropy is weak, i.e. when \(C_{45}\) (or \(S_{45}\)) is much smaller than \(C_{44}\) \(S_{55}\)) and \(C_{55}\) \(S_{44}\)). The B.C. (3.46) in the material coordinate system \((x^0, y^0, z)\) are also given in Appendix A.

One can also consider an approximation where the inertial terms are small and can be neglected but the coupling and compliance terms are kept in (3.42):

\[
\begin{align*}
 u_x - u'_x &= S_{xv} \frac{\sigma_{xx} + \sigma'_{xx}}{2} + S_c \frac{\sigma_{yy} + \sigma'_{yy}}{2} - i k h \frac{u_x + u'_x}{2} \\
 u_y - u'_y &= S_c \frac{\sigma_{xx} + \sigma'_{xx}}{2} + S_{yk} \frac{\sigma_{yy} + \sigma'_{yy}}{2} \\
 u_z - u'_z &= S_n \frac{\sigma_{zz} + \sigma'_{zz}}{2} - i k h \frac{C_{13} u_x + u'_x}{2} - i k h \frac{C_{36} u_y + u'_y}{2} \\
 \sigma_{xx} - \sigma'_{xx} &= -i k h \frac{C_{13} \sigma_{zz} + \sigma'_{zz}}{C_{33}} \\
 \sigma_{yy} - \sigma'_{yy} &= -i k h \frac{C_{36} \sigma_{zz} + \sigma'_{zz}}{C_{33}} \\
 \sigma_{zz} - \sigma'_{zz} &= -i k h \frac{\sigma_{zz} + \sigma'_{zz}}{2}.
\end{align*}
\]

(3.47)

The B.C. (3.47) can be used to replace the second order B.C. (3.42) only when the
density of the interfacial layer is extremely small but the interfacial compliances and layer thickness are not negligible.

For an extremely thin low density interfacial layer \((kh \ll 1)\), only the specific compliance matrix \(S\) is kept and the B.C. equation (3.47) may be simplified further after neglecting all the coupling terms,

\[
\bar{\sigma} = \bar{\sigma}'; \quad u_x - u'_x = S_{sv} \sigma'_x + S_c \sigma'_y, \\
u_y - u'_y = S_c \sigma'_x + S_{sh} \sigma'_y, \quad u_z - u'_z = S_n \sigma'_z.
\]

(3.48)

This is referred as the stiffness model. B.C. (3.48) do not give asymptotic approximations at \(kh \to 0\) [2] but in some conditions may give satisfactory results. These B.C. are applicable when the stresses can be considered continuous across the interface (for example for cracked interfaces).

### 3.2.5 Free orthotropic layer

To further consider the physical meaning of the asymptotic B.C., let us apply them to a free orthotropic plate. Assume in (3.34) \(\bar{\sigma} = \bar{\sigma}' = 0\). Substituting the zero stress condition in either the first (3.22) or the second order B.C. (3.34) we have

\[
\mathbf{M} \frac{\ddot{u} + \ddot{u}'}{2} = 0
\]

(3.49)

For B.C. (3.49) to have nontrivial solutions \(|\mathbf{M}| = 0\), i.e.

\[
(Q_{11} - \rho_0 V^2)(Q_{66} - \rho_0 V^2) - Q_{16}^2 = 0
\]

(3.50)
where $V$ is the plate wave velocity. Equation (3.50) is a dispersion equation for waves in a very thin free orthotropic plate. The equivalent dispersion equation in the material coordinate system $(x^0, y^0, z)$ is obtained by applying the zero stress condition in the second order B.C. in the material coordinate system (A.1), and it can be written as

$$(L_{11} - \rho_0 V^2)(L_{22} - \rho_0 V^2) - L_{12}^2 = 0$$

(3.51)

where

$$L_{11} = Q_{11}^0 \cos^2(\varphi) + C_{66}^0 \sin^2(\varphi),$$

$$L_{22} = Q_{22}^0 \sin^2(\varphi) + C_{66}^0 \cos^2(\varphi),$$

$$L_{12} = (Q_{12}^0 + C_{66}^0) \sin(\varphi) \cos(\varphi).$$

The characteristic equations (3.50) and (3.51) describe coupled plate quasilinear and quasitransverse waves propagating off the axis of symmetry in orthotropic plates (ref. for example to [49]). This demonstrates that the asymptotic B.C. introduced here accurately describe the longitudinal and shear motions of a thin anisotropic interfacial layer.

### 3.2.6 Decomposition of B.C. for symmetric and antisymmetric cases

Let us address the decomposition of the second order B.C. when the elastic field can be decoupled into symmetric and antisymmetric parts [51]. Let us first consider
the antisymmetric case. By definition
\[ u'_x = -u_x, \quad u'_y = -u_y, \quad u'_z = u_z; \]
\[ \sigma'_{zx} = \sigma_{zx}, \quad \sigma'_{zy} = \sigma_{zy}, \quad \sigma'_{zz} = -\sigma_{zz}. \] (3.52)

Substituting (3.52) into the second order asymptotic B.C. (3.25), we have
\[ (u_x, u_y, 0, 0, 0, \sigma_{zz})^T = -\frac{ikh}{2} A(0, 0, u_z, \sigma_{zx}, \sigma_{zy}, 0)^T. \] (3.53)

This reduces to
\[ (\sigma_{zx}, \sigma_{zy}, \sigma_{zz})^T = K_A (u_x, u_y, u_z)^T, \] (3.54)
where for a thin orthotropic layer rotated about one its axes of symmetry \( K_A \) is
\[ K_A = \begin{bmatrix}
\frac{2C_{55}}{h} & \frac{2C_{45}}{h} & ikC_{55} \\
\frac{h}{2}C_{45} & \frac{h}{2}C_{44} & ikC_{45} \\
-ikC_{55} & -ikC_{45} & \frac{k^2h}{2}(C_{55} - \rho_0\nu^2)
\end{bmatrix}. \] (3.55)

As each element in \( K_A \) has the dimensions of spring constant per unit thickness, we call \( K_A \) the stiffness B.C. matrix for the antisymmetric field. As one sees, the second order asymptotic solution for the antisymmetric problem depends on only three of the thirteen elastic constants of the interfacial layer \( (C_{44}, C_{55} \text{ and } C_{45}) \).

For the symmetric case, by definition
\[ u'_x = u_x, \quad u'_y = u_y, \quad u'_z = -u_z; \]
\[ \sigma'_{zx} = -\sigma_{zx}, \quad \sigma'_{zy} = -\sigma_{zy}, \quad \sigma'_{zz} = \sigma_{zz}. \] (3.56)

Substituting the above equalities into the second order B.C. (3.25), we have
\[ (0, 0, u_z, \sigma_{zx}, \sigma_{zy}, \sigma_{zz})^T = -\frac{ikh}{2} A(u_x, u_y, 0, 0, 0, \sigma_{zz})^T. \] (3.57)
This reduces to

\[(\sigma_{zz}, \sigma_{zy}, \sigma_{zz})^T = K_S (u_x, u_y, u_z)^T, \quad (3.58)\]

where for the same orthotropic layer \(K_S\) is

\[
K_S = \begin{bmatrix}
\frac{k^2 h}{2} (C_{11} - \rho_0 V^2) & \frac{k^2 h C_{16}}{2} & -ikC_{13} \\
\frac{k^2 h}{2} C_{16} & \frac{k^2 h}{2} (C_{66} - \rho_0 V^2) & -ikC_{36} \\
\frac{ikC_{13}}{2} & \frac{ikC_{36}}{2} & \frac{2C_{33}}{h}
\end{bmatrix}. \quad (3.59)
\]

We call \(K_S\) the stiffness B.C. matrix for the symmetric field. Here the second order asymptotic solution for the symmetric problem depends on the other set of elastic constants of the interfacial layer \(C_{11}, C_{33}, C_{13}, C_{16}, C_{36}\) and \(C_{66}\). Note that the dispersion equation (3.50) for waves in free orthotropic plates can also be obtained by setting the determinant of \(K_S\) to zero.

It is important that when the second order asymptotic B.C. are applied, decomposition into symmetric and antisymmetric parts halves the rank of the system of equations, greatly simplifying the calculation. Further discussion will be given in the next Chapter. Such a decomposition is not possible for the first order asymptotic B.C. because the asymptotic expansion (3.22) is not symmetric with respect to the elastic fields on the two sides of the interfacial layer. The decomposition is specially useful in studying interface wave phenomena where characteristic equations for symmetric and antisymmetric modes naturally decouple, more details will be given in Chapter V.
3.3 Alternative Approach to Find the Scattering From an Anisotropic Plane Interphase between Two Anisotropic Solids

Before we present the numerical results for comparison of different approximate B.C. models given in Section 3.2.4 we will describe here an alternative approach to solve the scattering problem in Fig. 3.1 exactly by considering multiple internal reflections inside the interfacial layer directly as shown in Figure 3.3. This algorithm will be used to calculate the exact solutions to serve as reference for comparison of solutions calculated using approximate B.C. models. To find the reflection-transmission in this problem, one may generalize it to a 2-boundary problem. But it requires solving a 12th degree system of boundary equations to get the transformation coefficients for all the waves (3 in the upper semispace, 6 inside the layer and 3 in the lower semispace). As is known that both mathematically and numerically, the complexity of solving a system of linear equations grows exponentially as the degrees of freedom increases. The task of writing down all the individual terms involved in the boundary conditions is enormous, and too tedious to handle. Also the physics is obscured by under this approach. So it is necessary to consider other solutions for this case. We will describe an alternative algorithm which considers the multiple reflection inside the layer. It is also very easy to obtain the dispersion equations for guided waves propagating along the interfacial layer under the proposed approach.
Figure 3.3: Generally anisotropic interface layer between two generally anisotropic semispaces. $h$ is the thickness the layer.
In order to illustrate the algorithm we start with the simplest case - normal incidence for isotropic materials. The coordinate system and proposed algorithm are shown in Fig. 3.4. If only one mode (one bulk wave) is considered inside the layer, the reflection and transmission coefficients from the interfacial layer are given by well-known expressions which we can write in the form [13] (note that in reference [13], the case of a liquid layer was originally considered):

\[
R = R_{12} + T_{12} \sum_{n=0}^{\infty} \left( \exp(ik_z h) R_{23} \exp(ik_z h) R_{21} \right)^n \exp(ik_z h) R_{23} \exp(ik_z h) T_{21} \\
= R_{12} + \frac{T_{12} T_{21} R_{23} \exp(2ik_z h)}{1 - R_{21} R_{23} \exp(2ik_z h)} \\
(3.60)
\]

\[
T = T_{12} \sum_{n=0}^{\infty} \left( \exp(ik_z h) R_{23} \exp(ik_z h) R_{21} \right)^n \exp(ik_z h) T_{23} \\
= \frac{T_{12} T_{23} \exp(ik_z h)}{1 - R_{21} R_{23} \exp(2ik_z h)} \\
(3.61)
\]

Here \(R_{ij}\) and \(T_{ij}\) (\(i, j = 1, 2\) or \(2, 3\)) are the coefficients of reflection and transmission at the boundaries of the layer and the semispaces (the first subscript denotes the medium on the incident side of the boundary, the second the medium on the other side), and \(h\) is the thickness of the layer. \(k_z\) is the absolute value of the normal component of the wave vector inside the layer. Note that the values of \(k_z\) are the same for waves travelling up and down through the layer.

Consequently, the dispersion equation for guided interface waves is easily obtained by setting the denominator of the reflection coefficient in equation (3.60) to zero

\[
1 - R_{21} R_{23} \exp(2ik_z h) = 0. \\
(3.62)
\]
Figure 3.4: Coordinate system and notations for the proposed algorithm for an isotropic interface layer between two isotropic semispaces.
The same methodology is generalized here for the case of anisotropy. When the interfacial layer is generally anisotropic, there are usually 3 bulk waves excited inside the layer on one boundary. The generalized solutions for reflection and transmission coefficients from the anisotropic layer and the dispersion equation for interface waves can be obtained using the multiple-reflection approach [47, 48] in which the solutions are given for an infinite number of permissible modes in the region of wave interaction. Here, we follow the approach of paper [47] but with the reduction of the number of permissible modes to three.

First we would like to introduce some important notations. For plane waves incident onto the boundaries, in the most general case the reflection and transmission can be described by 3×3 scattering matrices \( R_{i\beta} \) and \( T_{2\beta} \) (\( \beta = 1 \) or 3; the subscripts do not represent components). These scattering matrices will replace the 1-D reflection and transmission coefficients \( R_{i\beta} \) and \( T_{2\beta} \) in Eq. (3.60). Let the subscripts \( t_f \) and \( t_s \) denote the fast and slow quasitransverse wave and \( l \) the quasilongitudinal wave in the layer. The scattering matrices have the form:

\[
R_{ij} = \begin{pmatrix} R_{il}^{ij} & R_{lt}^{ij} & R_{lt_s}^{ij} \\ R_{tl}^{ij} & R_{tt}^{ij} & R_{tt_s}^{ij} \\ R_{ts}^{ij} & R_{tt_s}^{ij} & R_{ts_s}^{ij} \end{pmatrix} \tag{3.63}
\]

and

\[
T_{ij} = \begin{pmatrix} T_{il}^{ij} & T_{lt}^{ij} & T_{lt_s}^{ij} \\ T_{tl}^{ij} & T_{tt}^{ij} & T_{tt_s}^{ij} \\ T_{ts}^{ij} & T_{tt_s}^{ij} & T_{ts_s}^{ij} \end{pmatrix} \tag{3.64}
\]

where \( R_{mn}^{ij} \) and \( T_{mn}^{ij} \) are the reflection and transmission coefficients for the transformation from wave type \( m \) to wave type \( n \) on the boundary between medium \( i \) and
\( j \) (incident wave \( m \) in medium \( i \)). The nondiagonal components of the matrices represent transformations of one type of wave to another.

So if an incident field is represented by a row vector \((A_l, A_{l_f}, A_{l_s})\) (three different incident waves incident at the same time), then the total reflected field \((R_l, R_{l_f}, R_{l_s})\) and the total transmitted field \((T_l, T_{l_f}, T_{l_s})\) on the boundary between medium \(i\) and \(j\) can be found through the scattering matrices \(R_{ij}\) and \(T_{ij}\):

\[
\begin{align*}
(R_l, R_{l_f}, R_{l_s}) &= (A_l, A_{l_f}, A_{l_s})R_{ij} \\
(T_l, T_{l_f}, T_{l_s}) &= (A_l, A_{l_f}, A_{l_s})T_{ij}.
\end{align*}
\] (3.65)

In addition, we need also to define a matrix which describes the wave propagation through the layer from one boundary to another to replace the corresponding term \(\exp(ik_z h)\) in Eq. (3.60). This propagation matrix can be written in the form

\[
E^\pm = \begin{pmatrix}
\exp(ik_{z\pm} h) & 0 & 0 \\
0 & \exp(ik_{z\pm} h) & 0 \\
0 & 0 & \exp(ik_{z\pm} h)
\end{pmatrix}
\] (3.66)

where \(E^-\) represents the propagation matrix for the passage through the layer from the upper boundary to the lower and \(E^+\) represents that for the passage in the opposite direction. \(k_{z\pm}^j\) is defined in the similar way as for \(k_z\) in Eq. (3.60), i.e. the absolute value the normal component of the wave vector for mode \(j\) in passage \(\pm\). Unlike the isotropic case, in general \(E^+\) and \(E^-\) are not equal because of anisotropy. The reason is that the angles and velocities of the incident and reflected waves of the same type are not necessarily the same for different passages, therefore \(k^+_j \neq k^-_j\) when there is no symmetry of the interfacial layer property about \(z\) axis.
Finally let us introduce reflection $\vec{R}$ and transmission $\vec{T}$ row vectors for the interface layer. These vectors are rows of the generalized scattering matrices for the whole system which describe scattering from the whole layer (total reflection field in medium I and total transmission field in medium III). We emphasize that the reflection and transmission coefficients are taken for a particular incident wave (quasi-longitudinal or quasi-transverse). For a given incident wave, the reflection vector $\vec{R} = (R_l, R_{tf}, R_{ts})$ from the whole system can be found by using the same methodology for one mode. Each element (for example, $R_l$) in the total reflection vector $\vec{R}$ is a summation over all the waves of the same type (quasi-longitudinal in this example) leaving the layer after reflection from the first boundary, and after passing once, twice, etc., through the layer (see Figure 3.5(a)). They are [47]:

$$\vec{R} = \vec{T}_{12}E^+ \sum_{n=0}^{\infty} (R_{23}E^+R_{21}E^-)^n \cdot R_{23}E^+T_{21} + \vec{R}_{12}. \quad (3.67)$$

Similarly, we can compute the total transmission vector $\vec{T} = (T_l, T_{tf}, T_{ts})$ through the whole interface system as

$$\vec{T} = \vec{T}_{12}E^- \sum_{n=0}^{\infty} (R_{23}E^+R_{21}E^-)^n T_{23} \quad (3.68)$$

where $\vec{R}_{12}$ and $\vec{T}_{12}$ are the reflection and transmission row vectors for the incident wave incident on the top layer boundary (boundary 1-2). Suppose the type of incident wave is longitudinal, then they are defined as

$$\vec{R}_{12} = (R_{12}^{ll}, R_{12}^{ltf}, R_{12}^{ltt}), \quad \vec{T}_{12} = (T_{12}^{ll}, T_{12}^{ltf}, T_{12}^{ltt}). \quad (3.69)$$
Figure 3.5: (a) Coordinate system and notations for the proposed algorithm of a generally anisotropic interface layer between two generally anisotropic semispaces. (b) An example of one completely multiple reflection path inside the layer.
In the above equations, one can see that the effect of mode conversions inside the layer is taken care of by scattering matrices, and the effect of layer thickness by propagation matrices $E^+$ and $E^-$. The terms inside the geometric series describe the wave behavior through one complete path or multiple reflection inside the layer (see Fig. 3.5(b)). Let us define a matrix $Y$ by

$$Y \equiv R_{23}E^+R_{21}E^-.$$  \hspace{1cm} (3.70)

The diagonal terms in the matrix $Y$ represent the multiple reflection coefficients of one complete path for waves of similar type inside the layer, while nondiagonal terms describe mode conversions. Then the geometric series can be summed as

$$\sum_{n=0}^{\infty} (R_{23}E^+R_{21}E^-)^n = \sum_{n=0}^{\infty} Y^n = (I - Y)^{-1}$$  \hspace{1cm} (3.71)

where $I$ is the unity matrix. Thus the reflection and transmission vectors $\vec{R}$ and $\vec{T}$ can be written in the form

$$\vec{R} = \vec{T}_{12}E^- (I - Y)^{-1} R_{23}E^+T_{21} + \vec{R}_{12},$$  \hspace{1cm} (3.72)

$$\vec{T} = \vec{T}_{12}E^- (I - Y)^{-1} T_{23}.$$  \hspace{1cm} (3.73)

The dispersion equation for guided waves in the interfacial layer can be found by setting the denominator of the reflection coefficient to zero. It takes the simple form

$$I - Y = 0.$$  \hspace{1cm} (3.74)
In this way we can compute exact solutions for reflection-transmission and interface waves for any layer with arbitrary layer thickness between arbitrary semispaces.

This algorithm is much faster than that for solving a 12th degree system of boundary equations; the only difficulty lies in constructing scattering matrices \( \mathbf{R}_{ij} \) and \( \mathbf{T}_{ij} \) and the propagation matrices \( \mathbf{E}^- \) and \( \mathbf{E}^+ \). The unified approach for reflection and transmission of plane waves in generally anisotropic media [46] is used here to compute the \( 3 \times 3 \) scattering matrices \( \mathbf{R}_{2\beta} \) and \( \mathbf{T}_{2\beta} \) and the row vectors \( \mathbf{\tilde{R}}_{12} \) and \( \mathbf{\tilde{T}}_{12} \) on the boundaries between the interface layer and the semispaces (specifically, 2–3, 2–1 and 1–2 in Figure 3.5).

As stated in the paper [46], from Snell's law all the wave vectors lie in one plane, so it is possible to select the coordinate system in such a way that the interface plane \((x, y)\) plane in Figure 3.5) coincides with the \((x_1, x_2)\) coordinate plane and the incidence plane \((x, z)\) plane in Figure 3.5) coincides with the \((x_1, x_3)\) coordinate plane. Therefore the tensor of elastic constants must be first transformed into this coordinate system by equation (3.6). In the selected principal coordinate system all slowness vectors have only two nonzero components \( m_1^\alpha, m_3^\alpha \), and

\[
    m_2^\alpha = 0, \quad \alpha = 0, 1, \ldots, 6. \tag{3.75}
\]

The second property shows that all \( m_1^\alpha \) slowness vector components are equal to one another; therefore, they are known because \( m_1^\alpha \) is known for the incident wave

\[
    m_1^0 = m_1^1 = \ldots = m_1^6. \tag{3.76}
\]
The unknown $m^a_3$ component of the slowness vector may be found by the Christoffel equation, which gives a sixth-order equation in $m_3$:

\[ | \frac{1}{\rho} C_{ijkl} m_j m_l - \delta_{ik} | = 0 \]  \hspace{1cm} (3.77)

Six solutions of Eq. (3.77) will be found in each medium for a given incident wave direction $\vec{n}$, of which only three correspond to physical real solutions. The six roots of $m^{(a)}_3$ define those slowness vectors that satisfy the condition of having equal components of phase velocity parallel to the interface plane. This procedure must be carried out twice: separately for the upper and lower anisotropic substrates. To select from these six the three physical solutions one must take into account that energy flow (acoustic rays) for reflected waves must be directed into the upper substrates and for transmitted waves into the lower substrates. At the critical angle the appropriate ray must be parallel to the interface [46].

The detailed calculation procedure used here can be summarized as follows:

(i) For a given incident wave, the Christoffel equations are solved for each medium to find the slowness and velocity vectors of the reflected and transmitted waves, and therefore to get the displacement vectors (eigenvectors) of all the waves in each medium. The propagation matrices $\mathbf{E}^\pm$ are thus obtained from the now known $m_3$ values.

(ii) Using the displacement vectors thus found, for each wave incident on each of the boundaries between the interface layer and the substrates (1–2, 2–3 and
2–1), the welded boundary conditions are solved to obtain all the elements in the scattering matrices \( \mathbf{R}_{23}, \mathbf{R}_{21}, \mathbf{T}_{23} \) and \( \mathbf{T}_{21} \) and those in the row vectors \( \mathbf{R}_{12} \) and \( \mathbf{T}_{12} \).

(iii) From these scattering matrices \( \mathbf{R}_{23}, \mathbf{R}_{21} \) and propagation matrices \( \mathbf{E}^\pm \), the matrix \( \mathbf{Y} \) is readily computed from equation (3.70).

(iv) Finally, equation (3.72) and (3.73) together give the total reflection and transmission vectors for the layer system, while (3.74) gives the dispersion equation for interface waves inside the layer.

Please note that it is nontrivial to order the elements in the scattering matrices, row vectors and propagation matrices. The reflected or transmitted waves represented by each element must be ordered carefully to guarantee that products of the scattering matrices or the propagation matrices are meaningful. For example, this is essential when calculating the matrix \( \mathbf{Y} \) for one completely multiple reflection path inside the interfacial layer. When the product of matrices \( \mathbf{R}_{23} \mathbf{E}^+ \) and \( \mathbf{R}_{21} \mathbf{E}^- \) is computed, physics requires that the reflected wave whose coefficient represented by the element in the left side matrix \( \mathbf{R}_{23} \mathbf{E}^+ \) should be used as the incident wave of the corresponding element in the right side matrix \( \mathbf{R}_{21} \mathbf{E}^- \). To better understand this, let us take an example: the product \( \mathbf{W} \) of the element (1,2) in matrix \( \mathbf{R}_{23} \mathbf{E}^+ \) and the corresponding element (2,1) in matrix \( \mathbf{R}_{21} \mathbf{E}^- \) can
be written as
\[ W = R_{23}^{ij} \exp(i k^+ h) R_{21}^{ij} \exp(i k^- h). \] (3.78)

Physically, the above equation states that a quasi-longitudinal wave (with amplitude \( A \)) propagating from medium 2 at \( z = -h \) hits the boundary between medium 2 and 3, and produces a reflected fast quasi-transverse wave. This fast quasi-transverse wave propagates back through the medium 2 and hits the boundary between medium 1 and 2, producing another reflected quasi-longitudinal wave back into medium 2 (with amplitude \( A' \) at \( z = -h \)). Then the product \( W \) represents the total transformation coefficient \( A'/A = W \) for this quasi-longitudinal wave after one completely multiple reflection through the layer. If the order of the scattering matrices \( R_{21} \) and \( R_{23} \) or propagation matrices \( E^+ \) and \( E^- \) are wrong, the product will produce physically meaningless results. This is also important when the operation is addition or matrices are operated with row vectors.

So in a numerical algorithm which works for the general anisotropy, a rule to label different kinds of waves in the three generally anisotropic media should be well defined. For simplicity, we simply label waves by their velocities in the program. The identification of the wave type when the complications mentioned before do happen is dealt case by case by analyzing the slowness surfaces. A compromised approach has been taken to ensure the physical meaning of matrix operation. Specifically, for an example, to guarantee meaningful product of scattering matrices (see equation (3.78)), the characteristics (polarization vectors and wave vectors)
of the wave generated by the element in the left side matrix are saved and later used when calculating the corresponding element in the right side matrix as the incident wave. Similar approach is used when matrices or row vectors are added together. Solutions for all types of waves as a function of incident angle will be drawn simultaneously in one figure when any complication in wave ordering is suspected. By law of continuity, the reflection and transmission coefficients for each of the waves can be found correctly even when the order of wave types changes. The justification lies in that as long as we keep all the wave types consistent when doing actual matrix operations, the physical meaning of such operations should be conserved.

The calculations for such a layered system have been performed to provide a reference for the approximate solutions discussed later. The energy balance is always checked to ensure that the computer round-off error is small and the algorithm is stable. While conceptually this exact approach is simple, it requires even for one interface layer solving the Christoffel equations and the boundary equations three times. The complexity introduced by anisotropy is the selection of the appropriate modes. In comparison when the asymptotic boundary conditions are used to represent this anisotropic interfacial layer when the layer thickness is much smaller than the wavelength, it greatly simplifies the solution to scattering and interface wave localization by reducing the multi-boundary problem to the traditional one-boundary interface with imperfect boundary conditions.
3.4 Comparison of Approximate Boundary Condition Models with the Exact - Numerical Examples

To demonstrate the accuracy of the second order asymptotic B.C., let us consider as an example reflection-transmission phenomena for a nickel-nickel imperfect interface. The geometrical configuration is shown in Fig. 3.6. The upper medium is nickel of cubic symmetry and the lower medium nickel of general anisotropy. The material properties of the substrates are the same as in reference [5]. The imperfect interface is modeled by a parallel row of cylindrical pores as shown in Fig. 3.6. The orientation of the pores has deviation angle $\varphi$ from the incident plane (plane $(x, z)$). The matrix embedding the pores or inclusions is taken as isotropic nickel (an anisotropic matrix can be used as well). The effective elastic moduli of the interfacial layer are calculated from Christensen’s 2-phase model [52]. The energy coefficients of the scattered waves are calculated using various approximate B.C. models, and the approximate solutions will be compared with the exact solution obtained using the algorithm described in Section 3.3.

As an example let us consider an interfacial layer with porosity 30% and pore deviation angle from the incident plane $\varphi = 30^\circ$. The effective interfacial elastic properties in the global coordinate system $(x, y, z)$ are given in Table 3.1. The energy coefficients of the scattered waves for an incident quasilongitudinal wave as a function of the nondimensional thickness parameter $h/\lambda_{t0}$ are shown in Fig. 3.7(a) - (d) at incident angle $\theta_i = 60^\circ$. $\lambda_{t0}$ is the wavelength of the slow transverse
Figure 3.6: Imperfectly bonded nickel-nickel interface. The imperfection is modeled by a thin interfacial layer with cylindrical pores; $\varphi$ is the deviation angle of the pore orientation from the incident plane $(x, z)$ and $h$ is the layer thickness.
Table 3.1: The effective elastic properties (in GPa) of a porous interfacial layer $(C = 0.3)$.

<table>
<thead>
<tr>
<th>$C_{11}$</th>
<th>$C_{22}$</th>
<th>$C_{33}$</th>
<th>$C_{12}$</th>
<th>$C_{13}$</th>
<th>$C_{23}$</th>
<th>$C_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>145</td>
<td>115</td>
<td>103</td>
<td>33.1</td>
<td>32.3</td>
<td>36.5</td>
<td>35.9</td>
</tr>
<tr>
<td>$C_{55}$</td>
<td>$C_{66}$</td>
<td>$C_{45}$</td>
<td>$C_{16}$</td>
<td>$C_{26}$</td>
<td>$C_{36}$</td>
<td></td>
</tr>
<tr>
<td>43.6</td>
<td>50.3</td>
<td>-6.63</td>
<td>-14.6</td>
<td>-11.3</td>
<td>3.61</td>
<td></td>
</tr>
</tbody>
</table>

wave inside the layer propagating normal to the interface. The exact solutions are represented by a solid line, the first order asymptotic solutions (3.22) by closed circles, the second order asymptotic solutions (3.42) by open circles, the solutions using the stiffness-mass model (3.46) by a fine dashed line and the solution using the stiffness model (3.48) by a coarse dashed line. The reflected and transmitted quasilongitudinal waves are plotted in Fig. 3.7(a) and (b), while conversion coefficients to the reflected quasi-SV and SH modes are plotted in Fig. 3.7(c) and (d).

All B.C. models give reasonable approximations for very small $h/\lambda_{t0} (< 0.05)$, but for greater $h/\lambda_{t0}$ only the second order solution gives consistently good approximations up to $h/\lambda_{t0} \approx 0.4$. The first order approximation gives very good results for the reflected waves of the same mode (see $r_{li}$ in Fig. 3.7(a)) and of the quasi-SV mode (see $r_{liw}$ in Fig. 3.7(c)), but it fails for the transmitted wave (see $t_{li}$ in Fig. 3.7(b)) and the pure SH mode in the reflection field (see $r_{lsh}$ in Fig. 3.7(d)). The first order B.C. tend to give good predictions of the reflected field rather than
Figure 3.7: The energy coefficients as functions of $h/\lambda_{t0}$ for different B.C. models. The pore deviation angle $\varphi = 30^\circ$, porosity $C = 0.3$ and the incident angle $\theta_i = 60^\circ$. $\lambda_{t0}$ is the wavelength of the slow transverse wave inside the layer propagating normal to the interface. (a) Energy reflection coefficient $r_{ll}$ for a quasilongitudinal wave; (b) energy transmission coefficient $t_{ll}$ for a quasilongitudinal wave; (c) energy conversion coefficient $r_{lu}$ to the reflected quasitransverse wave; (d) energy conversion coefficient $r_{lish}$ to the reflected SH wave.
Fig. 3.7 (continued)

ENERGY TRANSMISSION COEFFICIENT $t_{ii}$

- --- EXACT SOLUTION
- ○○○○ SECOND ORDER
- •••• FIRST ORDER
- ••••• STIFFNESS-MASS
- - - - STIFFNESS

$\theta_i = 60^\circ$

$\varphi = 30^\circ$

$C = 0.3$

(b)
Fig. 3.7 (continued)

- EXACT SOLUTION
- ○○○○ SECOND ORDER
- ●●●● FIRST ORDER
- ---- STIFFNESS-MASS
- --- STIFFNESS

$\theta_i = 60^\circ$
$\varphi = 30^\circ$
$C = 0.3$

(c)
Fig. 3.7 (continued)

\[ \theta_1 = 60^\circ \]
\[ \varphi = 30^\circ \]
\[ C = 0.3 \]
the transmitted field. Note that the mode conversion to the pure SH mode is due to the anisotropy of the interface and the lower substrate. When the properties of the lower substrate, rather than the interface, significantly affect the reflection of a mode of interest, such as the pure SH mode in Fig. 3.7(c), the first order B.C. may give unsatisfactory results for the reflected field. When the incident angle is high enough simplified models generally do not give satisfactory results. In this case (in Fig. 3.7) the coupling matrix N cannot be neglected and the full inertial matrix $M$ should be used instead of the angle independent part ($-\omega^2 M_n I$).

To further support the above conclusions the energy reflection coefficient $r_{ll}$ for a quasilongitudinal wave as a function of incident angle $\theta_i$ is shown in Fig. 3.8(a) at $h/\lambda_{10} = 0.3$. All parameters and symbols are the same as in Fig. 3.7. The spike at $\theta_i = 76^\circ$ corresponds to a critical angle for the transmitted quasilongitudinal wave. To better judge the quality of approximation by various B.C. models, we plot in Fig. 3.8(b) the relative deviation $\delta = [r_{ll}(model) - r_{ll}(exact)]/r_{ll}(exact)$ as a function of the incident angle $\theta_i$. In addition, in Fig. 3.8(c) we give the results for the relative deviation when the nondimensional layer thickness $h/\lambda_{10} = 0.1$. Note that the peaks around $85^\circ$ in Fig. 3.8(b) and (c) are due to the sharp dip in the exact solution (see Fig. 3.8(a)) which is small.

From these figures one sees that the second order B.C. give good and homogeneous approximations for all incident angles. The first order also give reasonable approximations but do not work as well as the second order B.C. especially at
Figure 3.8: Calculations for the reflected quasilongitudinal wave as a function of incident angle $\theta_i$ for different B.C. models. The pore deviation angle $\varphi = 30^\circ$ and porosity $C = 0.3$. $\lambda_{10}$ is the wavelength of the slow transverse wave inside the layer propagating normal to the interface. (a) The energy reflection coefficients $r_\mu$, $h/\lambda_{10} = 0.3$; (b) the relative deviation $\delta$, $h/\lambda_{10} = 0.3$; (c) the relative deviation $\delta$, $h/\lambda_{10} = 0.1$. 
Fig. 3.8 (continued)

- - - - - SECOND ORDER
○○○○○ FIRST ORDER
--- STIFFNESS-MASS
- - - STIFFNESS

RELATIVE DEVIATION $\delta$ FOR $r_{11}$

\[
\frac{h}{\lambda_{t0}} = 0.3 \\
\phi = 30^\circ \\
C = 0.3 \\
\]

(b)

INCIDENT ANGLE $\theta_1$
Fig. 3.8 (continued)

- ○○○○○ SECOND ORDER
- ●●●●● FIRST ORDER
- ——— STIFFNESS—MASS
- ———— STIFFNESS

RELATIVE DEVIATION $\delta$ FOR $r_{ii}$

$h/\lambda_{t0} = 0.1$
$\varphi = 30^\circ$
$C = 0.3$

INCIDENT ANGLE $\theta_i$
the dip above the critical angle (see Fig. 3.8(a)). When the incident angle is less than 10°, there is little difference between the second order solutions and those obtained using the stiffness-mass model (3.46), for the coupling matrix \( N \) and the angle dependent part of the inertial matrix \( M \) are negligible. Recall that \( N \) is proportional to \( \sin(\theta_i) \), and the angle dependent part of \( M \) is proportional to \( \sin^2(\theta_i) \). For high incident angles, the stiffness-mass model gives an unacceptable approximation. The stiffness model also does not give as good an approximation overall as the first and the second order B.C. models. Finally from Fig. 3.8(b) and (c), one concludes that the simplified B.C. give solutions with large differences in deviations from the exact solution as a function of incident angle, and the first and second order B.C. give smaller and homogeneous deviations for different incident angles.

To study the effect of pore orientation on the wave scattering, we calculate the energy mode conversion coefficient of an incident longitudinal incident wave (at 30°) to a reflected SH wave. To exclude the anisotropic effect of the substrates we select them to be isotropic with properties that of the interfacial layer matrix (nickel). The wave conversion on the interface is purely due to the interface anisotropy. The results obtained using different B.C. models are plotted in Fig. 3.9(a) as a function of the pore orientation angle \( \varphi \) for \( h/\lambda_0 = 0.1 \). In the plane of symmetry (\( \varphi = 0° \) or 90°) there is no mode conversion because there is no coupling between the in-plane and out-of-plane elastic fields. Both the mode
conversion coefficient and the deviation of the approximation from the exact reach their maximum values at $\varphi = 45^\circ$ where the interfacial coupling between normal and out-of-plane shear components is strongest. The energy conversion coefficient is very small even at the maximum (about 0.18%). In Fig. 3.9(b) the mode conversion coefficient is plotted as a function of $h/\lambda_0$ for $\varphi = 45^\circ$. One sees that the first and second order approximations are good if $h/\lambda_0 < 0.2$.

There is little difference between the results obtained using the stiffness-mass model and those using the stiffness model, since the transformation from the longitudinal wave to the SH mode is determined mostly by the anisotropic compliance $S_c$ (see stiffness-mass model (3.46)), and the effect of inertia terms is very small. In addition, the coupling term $-ikhC_{36}/C_{33}$ and coupling inertia $M_c$ kept in the first and second B.C. are also small (for example $C_{36}/C_{33} \approx 0.04$), so the simplified stiffness-mass and stiffness models give better approximations compared to Fig 3.7(d) for the case of anisotropic substrates.

In summary we conclude that only the second order B.C. consistently give good approximations to the exact solution for different interfacial layers and incident modes. For nearly-normal incident waves the second order B.C. reduce to the stiffness-mass model since the coupling and additional terms in the inertial elements are negligible. For general cases the effects of different coupling, inertial and compliance terms should be examined carefully before using simplified B.C. models.
Figure 3.9: The energy conversion coefficients to the reflected SH wave for an incident longitudinal wave at 30°. The interfacial layer porosity $C = 0.3$ and the substrates are identical (isotropic nickel). (a) Versus pore orientation angle $\varphi$ in the interfacial layer. $h/\lambda_{t0} = 0.1$, $\lambda_{t0}$ is the wavelength of the slow transverse wave inside the layer propagating normal to the interface; (b) versus $h/\lambda_{t0}$; $\varphi = 45°$. 
Fig. 3.9 (continued)

- EXACT SOLUTION
- ○○○○ SECOND ORDER
- ●●●● FIRST ORDER
- --- STIFFNESS-MASS
- --- STIFFNESS

ISOTROPIC SUBSTRATE

ENERGY CONVERSION COEFFICIENT $r_{sh}$

$\theta_i = 30^\circ$
$\varphi = 45^\circ$
$C = 0.3$

(b)
3.5 Comparison of Different B.C. Models: Energy Balance, Uniqueness and Scattering

The different B.C. models, besides differing in accuracy, are distinguished in important physical aspects: energy balance and amount of scattering from a homogeneous substrate/layer/substrate system (in a correct model there should be no scattering from a layer with properties equal to those of its substrates).

3.5.1 Energy balance

Energy balance is satisfied when the total energy flux of the scattered waves equals that of the incident wave. In order to prove that the second order B.C. satisfy energy balance let us consider the period-averaged energy flow $P$ in the $z$-direction

$$P = \frac{1}{2} \text{Re}[-\bar{\sigma}^T (i\omega \bar{u})^*]$$  \hspace{1cm} (3.79)

where $(\cdot)^*$ corresponds to the complex conjugate of a vector or matrix, and $\text{Re}(\cdot)$ to the real part of the complex quantity.

Now let us define [42] a $6 \times 6$ orthogonal matrix $J$

$$J = \left( \begin{array}{cc} 0_3 & I_3 \\ -I_3 & 0_3 \end{array} \right)$$  \hspace{1cm} (3.80)

with properties

$$JJ^T = I, \quad J^{-1} = J^T, \quad (J^T)^{-1} = J,$$  \hspace{1cm} (3.81)

where $0_3$ and $I_3$ are the $3 \times 3$ null and identity matrices respectively. We can
express the energy flow \( P \) (3.79) using the elastic field vector \( \vec{U} \) as

\[
P = \frac{i\omega}{4} \vec{U}^H \mathbf{J} \vec{U},
\]

where \((\cdot)^H\) corresponds to the Hermitian transpose. Energy balance is satisfied if

\[
\vec{U}^H \mathbf{J} \vec{U} = (\vec{U'}^')^H \mathbf{J} \vec{U'}. \tag{3.83}
\]

In [42] energy balance in a form equivalent to (3.83) was proven for the exact transfer matrix \( \mathbf{B} \). For the second order B.C.: \( \vec{U} = \mathbf{B}_{II} \vec{U'} \) with \( \mathbf{B}_{II} \) given by (3.27), we can rewrite equation (3.83) as

\[
(\vec{U'}^')^H \mathbf{B}_{II}^H \mathbf{J} \mathbf{B}_{II} \vec{U'} = (\vec{U'}^')^H \mathbf{J} \vec{U'}. \tag{3.84}
\]

Hence energy balance holds if

\[
\mathbf{B}_{II}^H \mathbf{J} \mathbf{B}_{II} = \mathbf{J}. \tag{3.85}
\]

Multiplying both sides of (3.85) by \( \mathbf{J}^T \) on the right, and using equation (3.27), we have

\[
[(I + \frac{i\hbar \mathbf{A}}{2})^{-1} (I - \frac{i\hbar \mathbf{A}}{2})]^H \mathbf{J} (I + \frac{i\hbar \mathbf{A}}{2})^{-1} (I - \frac{i\hbar \mathbf{A}}{2}) \mathbf{J}^T = \mathbf{I}. \tag{3.86}
\]

As shown in Appendix B, (3.86) can be simplified to

\[
[(I + \frac{i\hbar \mathbf{A}}{2}) (I - \frac{i\hbar \mathbf{A}}{2})^{-1} (I + \frac{i\hbar \mathbf{A}}{2})^{-1} (I - \frac{i\hbar \mathbf{A}}{2})]^H = \mathbf{I}. \tag{3.87}
\]

Now it is obvious that if the matrices \((I + \frac{i\hbar \mathbf{A}}{2})\) and \((I - \frac{i\hbar \mathbf{A}}{2})^{-1}\) are commutative, the equality (3.87) holds and energy balance is satisfied. To prove this let us first
note that the matrices \((I + \frac{ikhA}{2})\) and \((I - \frac{ikhA}{2})\) are commutative, which is obvious from a direct computation of their left and right products. Next, if the matrices \(F\) and \(G\) are commutative, then \(G^{-1} F G = F\), so \(G^{-1} F = F G^{-1}\), and \(F\) and \(G^{-1}\) are commutative too. Thus the matrices \((I + \frac{ikhA}{2})\) and \((I - \frac{ikhA}{2})^{-1}\) are commutative and the equalities (3.86, 3.87) hold for any matrix \(A\) with properties (B2). Thus the energy balance for the second order B.C. is proven.

It follows from the proof that satisfaction of energy balance is due inherently to the representation of the transfer matrix \(B_{II}\) in the form (3.27) or B.C. models in the form (3.25). If instead of (3.27) one substitutes in (3.86) the series representation (3.29) of \(B_{II}\), which has the same order of approximation to the exact solution, the equality (3.86) will be satisfied to the fourth order \(O(khA)^4\). If one selects instead of \(B_{II}\) the first order transfer matrix \(B_{I} = I - ikhA\) the energy balance will be satisfied to the second order \(O(khA)^2\). The series representation (3.17) in any order will not satisfy energy balance exactly. For simplified models deduced from the second order B.C. (3.42), energy balance is satisfied regardless of the quality of the approximation of the reflection-transmission coefficients. It can be concluded that B.C. models symmetric about the center plane of the interfacial layer all satisfy energy balance. In other words geometric symmetry in the B.C. is necessary for energy conservation (i.e. symmetric with relation to \(\tilde{U}\) and \(\tilde{U}'\)).

While the first order approximation does not exactly satisfy energy balance, it gives a good approximation to the exact solution (in this case the amount of energy
imbalance is somewhat related to the deviation from the exact solution. It is very
difficult to make a general judgement on the order of accuracy of the simplified B.C.
models, in which the coupling terms are neglected, since their accuracy depends on
the frequency, the angle of incidence, interfacial layer thickness and elastic prop-
erties. Fulfillment of energy balance in this case may give a wrong impression of
the approximation accuracy. The second order asymptotic B.C. guarantee energy
balance and quality of approximation.

3.5.2 Uniqueness

The uniqueness of the solution of a boundary problem is established, if the solution
is zero at zero excitation field. Martin [33], using energy principles, considered the
uniqueness of different B.C. models. Here we apply his argument to the second
order B.C. (3.34).

It was shown in [33] that a sufficient condition for uniqueness is

\[ Im[\int_s (\bar{u}^T \bar{\sigma}^* - \bar{u}^T \bar{\sigma}^*) ds] \geq 0 \]  \hspace{1cm} (3.88)

where \( Im \) denotes the imaginary part; \( s \) is the boundary at \( z = 0 \). Since

\[ \bar{u}^T \bar{\sigma}^* - \bar{u}^T \bar{\sigma}^{**} = \frac{1}{2} [(\bar{u}^T - \bar{u}^T)(\bar{\sigma}^* + \bar{\sigma}^{**}) + (\bar{u}^T + \bar{u}^T)(\bar{\sigma}^* - \bar{\sigma}^{**})], \]  \hspace{1cm} (3.89)

from substituting equation (3.34) into (3.89) we get

\[ \bar{u}^T \bar{\sigma}^* - \bar{u}^T \bar{\sigma}^{**} = < \bar{\sigma}^T > S^T < \bar{\sigma}^* > + < \bar{u}^T > M^* < \bar{u}^* > + < \bar{u}^T > (N + N^*)^T < \bar{\sigma}^* > \]  \hspace{1cm} (3.90)
where brackets $< >$ denote averaged values.

Since $(N + N^*)^T$ is a null matrix, substituting (3.90) into (3.88), we get

$$Im\left[ \int_s (\langle \sigma^T \rangle S^T < \sigma^* > + < \bar{w}^T > M^* < \bar{w}^* > ) ds \right] \geq 0,$$

(3.91)

because $S$ and $M$ are real. Therefore uniqueness is established.

The uniqueness of the first order asymptotic B.C. (3.22) cannot be established by this consideration [33]. The sufficient but not necessary condition (3.88) is derived from energy principles, so it is not surprising that uniqueness cannot be established by this method for B.C. models which do not satisfy energy balance exactly. However one can argue that the first and second order B.C. give close approximations to the same solution so that the solutions predicted by the first order B.C. should be unique.

3.5.3 Scattering from a homogeneous layer/substrate system

Let us take the interfacial layer and both substrates to be of identical materials (homogeneous system) and then replace the interfacial layer by various approximate B.C. models. Approximate B.C. models may give scattering for an interfacial layer in a homogeneous system.

As an example let us consider an in-plane transverse wave incident on an interfacial layer at angle $\theta_i$ (layer properties equal to those of the semispaces). To check the effect of scattering we derived the second order analytical solutions us-
ing decomposed boundary conditions for the symmetric and antisymmetric parts of the field as will be described in the next Chapter. The amplitude reflection $R_{tt}$ and transmission $T_{tt}$ coefficients for the transverse waves are:

$$
R_{tt} = \frac{1}{2}(R^a_{tt} + R^s_{tt}) = 0; \quad T_{tt} = \frac{1}{2}(R^a_{tt} - R^s_{tt}) \quad (3.92)
$$

$$
T_{tt} = \frac{4i - 2\hat{h}(\cos \theta_t - \cos \theta_i) + i\hat{h}^2 \cos \theta_t \cos \theta_i}{4i + 2\hat{h}(\cos \theta_t + \cos \theta_i) - i\hat{h}^2 \cos \theta_t \cos \theta_i}
= \exp[2i \arctan(\frac{\hat{h} \cos \theta_t}{2})] \quad (3.93)
$$

where $\hat{h} = k_i h$ with $k_i$ the shear wave vector and $\theta_i$ is the refraction angle for a longitudinal wave ($\cos \theta_i$ cancels after further simplification in (3.93)). $R^a_{tt}$, $R^s_{tt}$ are the antisymmetric and symmetric reflection coefficients respectively. All other scattering coefficients ($R_{tt}$ and $T_{tt}$) are zero. One sees that the amplitude of the direct transmission coefficient $|T_{tt}|$ is 1. Compared with the exact solution of $T_{tt} = \exp(i\hat{h} \cos \theta_i)$, the second order B.C. give a slightly different phase shift of the directly transmitted wave. The difference in phase is third order in $\hat{h} \cos \theta_i$: $(\hat{h} \cos \theta_i)^3/12$.

The correct prediction of zero scattering in this special case for the second order B.C. can be explained physically by the fact that the full matrix $A$, whose eigenvalues are the exact wave solutions of the interfacial layer is used. When layer properties equal those of the semispaces, the model still predicts zero mode conversion and reflection. It can be shown analytically that the first order approximation (3.22) also predicts no scattering, but due to the energy imbalance,
the model predicts energy absorption by the interface and thus does not give full transmission.

When simplified models are used, only part of A is involved in the calculation, so this approximate A no longer gives the exact wave solutions for the interfacial layer. The transmission coefficient $T_t$ for this case is different from (3.93) and its absolute value is not equal to 1. To satisfy energy balance scattering must occur. As an example the energy coefficients for the scattered waves for a shear wave incident on an aluminum/aluminum/aluminum interface at 20° angle calculated using the stiffness-mass and stiffness models are shown in Fig. 3.10(a) and (b) respectively. The simplified approximations predict significant scattering for a reflected ($r_t$) shear wave and reflected ($r_d$) and transmitted ($t_d$) longitudinal waves, especially when the nondimensional layer thickness $h/\lambda_t$ is not small. This demonstrates the importance of retention of coupling matrices in the approximate B.C. The energy coefficients for the directly transmitted shear wave $t_t$ are shown in Fig. 3.10(c) as functions of $h/\lambda_t$ for different B.C. From this figure one sees that only the second order B.C. give full transmission. It is of interest to note that the first order approximation gives the worst prediction of $t_t$ in this particular case. The first order B.C. do not give scattering but predict transmission loss which is equivalent to predicting energy absorption by the interface. The amount of this loss exactly equals the energy unbalance for these B.C. Note that for normal incidence, there will be no scattering for the stiffness-mass model either. However the
stiffness model predicts scattering even at normal incidence.

3.6 Summary

In summary, the second order asymptotic B.C. give solutions which, in addition to their higher accuracy and consistency, satisfy energy balance and uniqueness, and give zero scattering from an interfacial layer with properties equal to those of the substrates. Compared to the exact solution the asymptotic B.C. are relatively simple, since there is no need to describe the wave behavior inside the interfacial layer. The second order asymptotic B.C. have the same simplicity as the first order B.C. and better approximate the exact solution over a greater frequency range. It has been shown that retention of coupling and inertial terms in the asymptotic models greatly improves the quality of the approximation for oblique wave scattering. When the incident angle is very close to the interface normal, the second order B.C. can be replaced by the stiffness-mass model. In that case, the anisotropic coupling compliance should still be kept if the incident plane is not a plane of symmetry and the interfacial anisotropy is significant.

It has also been shown that if the wave field can be decomposed into symmetric and antisymmetric parts, the rank of the system of equations using the second order B.C. is halved. In the next two Chapters we will derive analytical solutions for wave scattering and localization of interface waves by an anisotropic interfacial layer between two identical isotropic substrates.
Figure 3.10: Scattering for a homogeneous Al/Al/Al system. The energy coefficients of refracted waves for an incident shear wave at 20° are plotted as functions of the nondimensional layer thickness $h/\lambda_t$ (the first and second order B.C. give zero scattering in this case). (a) The stiffness-mass B.C. model. The energy conversion coefficients to the scattered waves $r_{tt}, r_{tl}$ and $t_{tl}$. (b) The stiffness model. The energy conversion coefficients to the scattered waves $r_{tt}, r_{td}$ and $t_{td}$. (c) The energy coefficients $t_{ul}$ for the directly transmitted shear wave calculated by different B.C. models.
Fig. 3.10 (continued)

INCIDENT SHEAR WAVE $\theta_1 = 20^\circ$

AL/AL/AL --- STIFFNESS MODEL

ENERGY COEFFICIENTS OF SCATTERED WAVES

$r_{tl}$
$r_{tt}$
$t_{tl}$

$h/\lambda_t$
Fig. 3.10 (continued)

- ○○○○○ SECOND ORDER
- ••••FIRST ORDER
- --- STIFFNESS–MASS
- - - - STIFFNESS

ENERGY TRANSMISSION COEFFICIENT \( t_{st} \)

AL/AL/AL HOMOGENEOUS SYSTEM
INCIDENT SHEAR WAVE AT \( \theta_i = 20^\circ \)

\[ \frac{h}{\lambda_t} \]

(c)
CHAPTER IV

DECOMPOSED SYMMETRIC AND ANTISYMMETRIC BOUNDARY CONDITIONS. ANALYTICAL SOLUTIONS FOR AN ANISOTROPIC LAYER BETWEEN ISOTROPIC HALFSPACES

An interfacial layer between two identical substrates is of particular interest since most solid-solid bonds are created when joining of the same material. In this case if the interphase plane is a plane of symmetry of the elastic system then the wave field can be decomposed into symmetric and antisymmetric parts as illustrated schematically in Fig. 4.1. In Section 3.2.6. we have already shown that the second order B.C. (3.25) naturally decompose into symmetric and antisymmetric parts. In this Chapter we will discuss the decomposed second order B.C. in details and will also derive analytical solutions to scattering from a thin anisotropic interfacial layer between two isotropic substrates. The results for interface wave localization by this interfacial layer will be presented in Chapter V.
Figure 4.1: Decomposition of antisymmetric and symmetric fields. The semispaces are identical and the interface plane is a plane of symmetry.
4.1 The Antisymmetric Part

Recall from Section 3.2.6 that for the antisymmetric part of the elastic field, the stresses and displacements on the opposite surfaces of the interfacial layer must satisfy the conditions:

\[
\begin{align*}
    u_x &= -u'_x, \
    u_y &= -u'_y, \
    u_z &= u'_z, \
    \sigma_{zz} &= \sigma'_{zz}, \
    \sigma_{zy} &= \sigma'_{zy}, \
    \sigma_{xz} &= -\sigma'_{xz}.
\end{align*}
\] (4.1)

Substituting Eqs (4.1) into the second order asymptotic B.C. (3.25), we have

\[
(u_x, u_y, 0, 0, 0, \sigma_{zz})^T = -\frac{ikh}{2} A \ (0, 0, u_z, \sigma_{xx}, \sigma_{zy}, 0)^T.
\] (4.2)

Using the matrix \( A \) given by Eq (3.9) for an off-axis orthotropic layer one obtains

\[
\begin{pmatrix}
    \sigma_{xx} \\
    \sigma_{zy} \\
    \sigma_{zz}
\end{pmatrix} = K_A \begin{pmatrix}
    u_x \\
    u_y \\
    u_z
\end{pmatrix}, \quad K_A = \begin{bmatrix}
    2C_{55} & 2C_{45} & ikC_{55} \\
    \frac{b}{h} & \frac{b}{h} & ikC_{45} \\
    -ikC_{55} & -ikC_{45} & \frac{k^2h}{2}(C_{55} - \rho_0 V^2)
\end{bmatrix}.
\] (4.3)

One sees from (4.3) that each element in \( K_A \) has the units of stiffness (elastic constant per unit length), so we call \( K_A \) the *antisymmetric stiffness matrix*. It is clear that the second order asymptotic solution for the antisymmetric case depends on only three of the thirteen elastic constants for a monoclinic interfacial layer \((C_{44}, C_{55} \text{ and } C_{45})\). The first two diagonal terms in \( K_A \) are in-plane \((C_{55}/h)\) and out-of-plane \((C_{44}/h)\) shear stiffnesses. (Note that "plane" here means the wave propagation plane which is perpendicular to the interface. \(C_{45}/h\) is a shear stiffness coupling between the in-plane and out-of-plane shear stress and displacement.)
The elements $ikC_{55}$ couples the normal stress or displacement to the in-plane shear displacement or stress; $ikC_{45}$ couples the normal stress or displacement to the out-of-plane shear displacement or stress, so these two are called the coupling terms. The $(3,3)$ diagonal entry in $K_A$: $k^2h(C_{55} - \rho_0V^2)/2 = -\omega^2\rho_0h(1 - V_{SV0}^2/V^2)/2$ includes an inertia part $-\omega^2\rho_0h/2$ which is not zero at normal incidence ($k = 0$). $V_{SV0} = \sqrt{C_{55}/\rho_0}$ is the velocity of the in-plane shear wave propagating along the $x$-direction in the interfacial layer material.

As each row in $K_A$ determines a stress component, to examine the effect of different elements in that row, we write Eq. (4.3) in the form

$$
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{zy} \\
\sigma_{zz}
\end{pmatrix}
= \frac{2C_{55}}{h}
\begin{bmatrix}
1 & \frac{C_{45}}{C_{55}} & \frac{ikh}{2} \\
\frac{C_{45}}{C_{44}} & \frac{ikhC_{45}}{2C_{55}} & (\frac{kh}{2})^2(1 - \frac{\rho_0V^2}{C_{55}}) \\
\frac{ikh}{2} & \frac{ikhC_{45}}{2C_{55}} & \frac{C_{45}}{C_{55}}
\end{bmatrix}
\begin{pmatrix}
u_x \\
u_y \\
u_z
\end{pmatrix}.
$$

Let us first consider the case when $C_{45} \ll C_{55}$. Then $K_A$ can be approximated as

$$
K_A = \frac{2C_{55}}{h}
\begin{bmatrix}
1 & 0 & \frac{ikh}{2} \\
0 & \frac{C_{44}}{C_{55}} & 0 \\
-\frac{ikh}{2} & 0 & (\frac{kh}{2})^2(1 - \frac{\rho_0V^2}{C_{55}})
\end{bmatrix}.
$$

Eq. (4.5) can be used when either the anisotropy of the interfacial layer or the rotation of the incident $(x, z)$ plane from the plane of symmetry is small (i.e. $\varphi$ is small in Fig. 3.2). Equation (4.5) is exact when the incident $(x, z)$ plane coincides with the plane of symmetry of the orthotropic layer ($C_{45} = 0$ at $\varphi = 0$). In this
case the in-plane and out-of-plane (SH) elastic motions are decoupled:

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{zz}
\end{pmatrix} = \frac{2C_{55}^0}{h} \begin{bmatrix}
\frac{1}{ikh} & \frac{ikh}{2} \\
-\frac{kh}{2} (1 - \frac{\rho_0 V^2}{C_{55}^0}) & \frac{1}{ikh}
\end{bmatrix} \begin{pmatrix}
u_x \\
u_z
\end{pmatrix},
\]

(4.6)

\[
\sigma_{x\nu} = \frac{2C_{44}^0}{h} u_{\nu}.
\]

(4.7)

One sees that the second order asymptotic B.C. for the in-plane components (4.6) of the antisymmetric motion depend only on \(C_{55}^0\), while the condition (4.7) for the out-of-plane components depend only on \(C_{44}^0\). For the isotropic case, \(C_{44}^0\) and \(C_{55}^0\) in Eqs. (4.6, 4.7) are simply the shear modulus \((\mu_0)\) of the interfacial layer.

The term \(kh/2\) in Eq. (4.4) can be represented as

\[
\frac{kh}{2} = \frac{\omega}{2V_i} h \sin \theta_i = \frac{\pi h}{\lambda_i} \sin \theta_i = \epsilon_i \sin \theta_i,
\]

(4.8)

where \(\lambda_i\) is the wavelength of the incident wave and \(\epsilon_i \equiv \pi h / \lambda_i\) can be considered as a non-dimensional thickness. One sees that \(kh/2\) is small when either \(\epsilon_i\) or \(\sin \theta_i\) is small, i.e. at low frequency or small incidence angle. Now we can rewrite the matrix \(K_A\) in Eq. (4.4) as

\[
K_A = \frac{2C_{55}^0}{h} \begin{bmatrix}
1 & \frac{C_{45}}{C_{55}} & i\epsilon_i \sin \theta_i \\
\frac{C_{45}}{C_{55}} & \frac{C_{44}}{C_{55}} & i\frac{C_{45}}{C_{55}} \epsilon_i \sin \theta_i \\
-i\epsilon_i \sin \theta_i & -i\frac{C_{45}}{C_{55}} \epsilon_i \sin \theta_i & \epsilon_i^2 \sin^2 \theta_i - \epsilon_{SV0}^2
\end{bmatrix},
\]

(4.9)

where \(\epsilon_{SV0} = \pi h / \lambda_{SV0} = \omega h / (2V_{SV0})\) is the non-dimensional layer thickness parameter normalized by the wavelength \(\lambda_{SV0}\) of the SV wave in the layer material \((V_{SV0} = \sqrt{C_{55}/\rho_0})\).
Eq. (4.9) contains terms of different order in $\epsilon_i \sin \theta_i$: zeroth, first and second. The second order term $\epsilon_i^2 \sin^2 \theta_i$ (the incident-wave dependent part) in the (3,3) element can be neglected if $\epsilon_i \sin \theta_i \ll \epsilon_{SV}: \sin \theta_i (\lambda_{SV}/\lambda_i) \ll 1$; or $\sin \theta_i (V_{SV}/V_i) \ll 1$. This implies that the shear-wave speed $V_{SV0}$ on the interphase is much less than the incident wave velocity $V_i$, or else that the incidence is nearly normal ($\theta_i \approx 0$). For interface guided waves $V = V_i/\sin \theta_i$ is simply the phase velocity of the guided mode.

If one neglects, in addition in $K_A$, all the first order terms in $\epsilon_i \sin \theta_i$ (or $kh/2$), one has
\begin{equation}
K_A = \frac{2C_{55}}{h} \begin{bmatrix}
1 & \frac{C_{45}}{C_{55}} & 0 \\
\frac{C_{45}}{C_{55}} & \frac{C_{44}}{C_{55}} & 0 \\
0 & 0 & -\frac{\omega^2 \rho_0 h^2}{4C_{55}}
\end{bmatrix}, \tag{4.10}
\end{equation}
and the B.C. (4.3) simplify to
\begin{align*}
\sigma_{xx} &= \frac{2C_{55}}{h} u_x + \frac{2C_{45}}{h} u_y, \\
\sigma_{xy} &= \frac{2C_{45}}{h} u_x + \frac{2C_{44}}{h} u_y, \tag{4.11}
\end{align*}
\begin{align*}
\sigma_{zz} &= -\frac{\omega^2 \rho_0 h}{2} u_z = -\frac{\omega^2 \rho_0}{2} u_z.
\end{align*}

The B.C. (4.11) are valid when either $\epsilon_i$ or the incident angle $\theta_i$ is small. They are equivalent to the second order B.C. in the case of normal incidence ($\theta_i = 0$ in Eq. (4.9) or $k = 0$ in Eq. (4.3)). Such a B.C. model is a generalized stiffness-mass model for the antisymmetric case of an anisotropic interface as only the effective mass and stiffnesses of the interphase are considered.
Neglecting further the inertial term on the right-hand side of Eq. (4.11) results in the B.C. for the antisymmetric motion that account for only the shear stiffnesses of the interphase:

\[
\begin{align*}
\sigma_{xx} &= \frac{2C_{55}}{h}u_x + \frac{2C_{45}}{h}u_y, \\
\sigma_{xy} &= \frac{2C_{45}}{h}u_x + \frac{2C_{44}}{h}u_y, \\
\sigma_{zz} &= 0.
\end{align*}
\] (4.12)

The normal stress \(\sigma_{zz}\) on the interface now vanishes. The B.C. (4.12) constitute a generalized stiffness model for the thin interfacial layer. They can only be applied for a very thin interfacial layer with very low density or for a cracked interface under antisymmetric loading.

4.2 The Symmetric Part

The symmetric part of the elastic field on the two surfaces of the interfacial layer must satisfy

\[
\begin{align*}
u_x &= u'_x, \\
u_y &= u'_y, \\
u_z &= -u'_z, \\
\sigma_{xx} &= -\sigma'_{xx}, \\
\sigma_{xy} &= -\sigma'_{xy}, \\
\sigma_{zz} &= \sigma'_{zz}.
\end{align*}
\] (4.13)

Thus the second order B.C. (3.25) are

\[
(0, 0, u_z, \sigma_{xx}, \sigma_{xy}, 0)^T = -\frac{i\hbar}{2}A(u_x, u_y, 0, 0, 0, \sigma_{zz})^T. \] (4.14)
For a thin off-axis orthotropic layer represented by Eq. (3.9) this reduces to

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{xy} \\
\sigma_{zz}
\end{pmatrix} = \mathbf{K}_S 
\begin{pmatrix}
u_x \\
u_y \\
u_z
\end{pmatrix}, \quad \mathbf{K}_S = \begin{bmatrix}
\frac{k^2h}{2}(C_{11} - \rho_0 V^2) & \frac{k^2h}{2} C_{16} & -i k C_{13} \\
k^2h C_{16} & \frac{k^2h}{2} (C_{66} - \rho_0 V^2) & -i k C_{36} \\
 i k C_{13} & i k C_{36} & \frac{2 C_{33}}{h}
\end{bmatrix}.
\]

where \( \mathbf{K}_S \) is the symmetric stiffness matrix, which depends on a different and larger set of elastic constants of the interfacial layer, \((C_{11}, C_{33}, C_{13}, C_{16}, C_{36} \text{ and } C_{66})\), than \( \mathbf{K}_A \). Here the \((3,3)\) entry is the normal stiffness \( C_{33}/h \), the off-diagonal coupling terms are all incident wave dependent and both the \((1,1)\) and \((2,2)\) entries are incident-wave and inertia dependent.

To examine the effect of different elements in \( \mathbf{K}_S \), we rewrite \( \mathbf{K}_S \) as

\[
\mathbf{K}_S = \frac{2 C_{33}}{h} \begin{bmatrix}
\left(\frac{kh}{2}\right)^2 C_{11} - \rho_0 V^2 & \left(\frac{kh}{2}\right) C_{16} & -i k h C_{13} \\
\left(\frac{kh}{2}\right)^2 C_{16} & \left(\frac{kh}{2}\right)^2 (C_{66} - \rho_0 V^2) & -i k h C_{36} \\
\frac{i k h C_{13}}{2 C_{33}} & \frac{i k h C_{36}}{2 C_{33}} & \frac{2 C_{33}}{h}
\end{bmatrix}.
\]

(4.16)

For small anisotropy of the interfacial layer or small rotation angle \( \varphi \) the \( C_{16}/C_{33} \ll 1 \) and \( C_{36}/C_{33} \ll 1 \) and \( \mathbf{K}_S \) becomes

\[
\mathbf{K}_S = \frac{2 C_{33}}{h} \begin{bmatrix}
\left(\frac{kh}{2}\right)^2 C_{11} - \rho_0 V^2 & 0 & -i k h C_{13} \\
0 & \left(\frac{kh}{2}\right)^2 (C_{66} - \rho_0 V^2) & 0 \\
\frac{i k h C_{13}}{2 C_{33}} & 0 & 1
\end{bmatrix}.
\]

(4.17)

Equation (4.16) becomes (4.17) when the incident \((x, z)\) plane coincides with the plane of symmetry of the orthotropic layer, the \((x^0, z)\) plane. In this case the
in-plane and out-of plane (SH) elastic motions decouple:

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{zz}
\end{pmatrix} = \frac{2C_{33}^0}{h} \begin{pmatrix}
\left(\frac{kh}{2}\right)^2 \frac{C_{11}^0}{C_{33}} - \rho_0 V^2 - \frac{ikh}{2} \frac{C_{13}^0}{C_{33}} \\
\frac{ikh}{2} \frac{C_{13}^0}{C_{33}}
\end{pmatrix} \begin{pmatrix}
u_x \\
\nu_z
\end{pmatrix},
\]
and

\[
\sigma_{xy} = \frac{k^2 h}{2} (C_{66}^0 - \rho_0 V^2) \nu_y.
\]

One sees that the second order asymptotic B.C. for the in-plane components (4.18) of the symmetric motion depend on \(C_{11}^0, C_{13}^0\) and \(C_{33}^0\), while the out-of-plane components (4.19) depend only on \(C_{66}^0\). For the isotropic case, \(C_{11}^0, C_{13}^0, C_{33}^0\) and \(C_{66}^0\) in Eq. (4.18, 4.19) are replaced by combinations of the Lamé constants \((\lambda_0, \mu_0)\) of the interphase.

Using (4.8) one can rewrite \(K_S\) in Eq. (4.16) as

\[
K_S = \frac{2C_{33}^0}{h} \begin{pmatrix}
\epsilon_i^2 \sin^2 \theta_i \frac{C_{11}}{C_{33}} - \epsilon_{n0}^2 & \epsilon_i^2 \sin^2 \theta_i \frac{C_{16}}{C_{33}} & -i \epsilon_i \sin \theta_i \frac{C_{13}}{C_{33}} \\
\epsilon_i^2 \sin^2 \theta_i \frac{C_{16}}{C_{33}} & \epsilon_i^2 \sin^2 \theta_i \frac{C_{66}}{C_{33}} - \epsilon_{n0}^2 & -i \epsilon_i \sin \theta_i \frac{C_{36}}{C_{33}} \\
i \epsilon_i \sin \theta_i \frac{C_{13}}{C_{33}} & i \epsilon_i \sin \theta_i \frac{C_{36}}{C_{33}} & 1
\end{pmatrix}.
\]

where \(\epsilon_{n0} = \pi h/\lambda_{n0} = \omega h/(2V_{n0})\), \(V_{n0} = \sqrt{C_{33}/\rho_0}\); \(\epsilon_{n0}\) is the non-dimensional thickness parameter normalized by the longitudinal wave propagating normal to the interface in the interfacial layer material. One sees from (4.20) that, like the antisymmetric stiffness matrix \(K_A\) given by Eq. (4.9), the symmetric stiffness matrix \(K_S\) (4.20) also includes terms of different order in \(\epsilon_i \sin \theta_i\).

The incident-wave dependent part with second order terms \(\epsilon_i^2 \sin^2 \theta_i\) in the (1,1) and (2,2) elements can be neglected if \(\epsilon_i^2 \sin^2 \theta_i/\epsilon_{n0}^2 \ll C_{33}/C_{11}\) and \(\ll C_{33}/C_{66}\).
These conditions correspond to

\[
V^2 = \frac{V_i^2}{\sin^2 \theta_i} \gg \frac{C_{11}}{\rho_0} \equiv V_{i0}^2 \quad \text{and} \quad V^2 \gg \frac{C_{66}}{\rho_0} \equiv V_{SH0}^2,
\]

(4.21)

where \( V_{i0} \) and \( V_{SH0} \) are approximately the velocities of longitudinal and SH waves propagating along the \( x \)-direction in the interphase (exact when \( \phi = 0 \)). If (4.21) are satisfied the (1,1) and (2,2) entries in (4.15) become \(-\frac{\omega^2 \rho_0 h}{2}\), independently of the incident wave.

By neglecting in \( \mathbf{K}_S \) all the second order terms in \( \epsilon_i^2 \sin^2 \theta_i \) we have

\[
\mathbf{K}_S = \frac{2C_{33}}{h} \begin{bmatrix}
-\frac{\omega^2 \rho_0 h^2}{4C_{33}} & 0 & -\frac{ikh C_{13}}{2 C_{33}} \\
0 & -\frac{\omega^2 \rho_0 h^2}{4C_{33}} & -\frac{ikh C_{33}}{2 ikh C_{36}} \\
\frac{ikh C_{13}}{2 C_{33}} & \frac{ikh C_{33}}{2 C_{36}} & \frac{1}{C_{33}}
\end{bmatrix}.
\]

(4.22)

In the B.C. (4.22) the coupling between the \( x \) and \( y \) components is omitted. If one further neglects all the first order terms in \( \epsilon_i \sin \theta_i \) (or \( kh/2 \)), one has

\[
\mathbf{K}_S = \begin{bmatrix}
-\frac{\omega^2 \rho_0 h}{2} & 0 & 0 \\
0 & -\frac{\omega^2 \rho_0 h}{2} & 0 \\
0 & 0 & \frac{2C_{33}}{h}
\end{bmatrix},
\]

(4.23)

implying that

\[
\sigma_{xx} = -\frac{\omega^2 \rho_0 h}{2} u_x,
\]

\[
\sigma_{xy} = -\frac{\omega^2 \rho_0 h}{2} u_y,
\]

\[
\sigma_{zz} = \frac{2C_{33}}{h} u_z.
\]

(4.24)
Eqs. (4.22) and (4.24) are valid when either $\epsilon_i$ or $\theta_i$ is small. The B.C. (4.24), with only the diagonal terms remaining in $K_S$ (4.23), are called the stiffness-mass B.C. for the symmetric case. They are exactly the second order B.C. for normal incidence ($\theta_i = 0$ in Eq. (4.20) or $k = 0, V \to \infty$ in Eq. (4.15)). The $x, y$ and $z$ components all decouple. This contrasts to the stiffness-mass B.C. (4.10) for the antisymmetric case, where the coupling between the $x$ and $y$ shear motions remains, due to the terms in $C_{45}$.

If one further neglects the mass ($\rho_0 = 0$) in (4.24), one obtains the stiffness B.C. for the symmetric field that involve only the normal stiffness:

$$\sigma_{xx} = \sigma_{xy} = 0, \quad \sigma_{zz} = \frac{C_{33}}{h} u_z.$$  \hspace{1cm} (4.25)

The shear stresses $\sigma_{zx}$ and $\sigma_{zy}$ on the interface become zero.

### 4.3 Scattering Coefficients on an Anisotropic Layer between Identical Isotropic Semispaces

Using the stiffness matrix representation (4.3, 4.15) for the antisymmetric and symmetric parts of the second order asymptotic B.C., we will derive analytical solutions for scattering coefficients for a thin anisotropic layer between identical isotropic semispaces. Let us assume that the incident field is a combination of longitudinal and transverse (SV and SH) waves with displacement vectors $\vec{u}_i$, $\vec{u}_V$, and $\vec{u}_H$:

$$\vec{u}_i = 2A_i \vec{P}_i e^{ik_i \vec{r} \cdot \vec{r}}, \quad \vec{u}_V = 2A_V \vec{P}_V e^{ik_V \vec{r} \cdot \vec{r}}, \quad \vec{u}_H = 2A_H \vec{P}_H e^{ik_H \vec{r} \cdot \vec{r}}.$$  \hspace{1cm} (4.26)
where $2A_l$, $2A_V$ and $2A_H$ are the amplitudes and $\vec{P}_l$, $\vec{P}_V$ and $\vec{P}_H$ are the polarization vectors of the incident waves. $l$ stands for longitudinal, $V$ and $H$ for SV and SH waves respectively; $k_l$ and $k_t$ are the wave numbers of the longitudinal and transverse waves and $\vec{n}$ and $\vec{r}$ are the wave normal and position vectors. Owing to the isotropy of the semispace, an incident transverse wave of arbitrary polarization can always be decomposed into two transverse waves, the in-plane part (SV) and out-of-plane part (SH). The time factor $e^{-jwt}$ is omitted throughout. We assume that the three different incident waves satisfy Snell's law $k = k_l \sin(\theta_l) = k_t \sin(\theta_t)$, where $\theta_l$ and $\theta_t$ are the incident angles of the longitudinal and transverse waves.

Let us first decompose the elastic field into antisymmetric and symmetric parts as shown in Fig. 4.1, and solve each part of the problem separately using the corresponding stiffness matrix. There are in general three reflected and 3 transmitted waves in each decomposed field for each of the three types of incident waves, due to the interface anisotropy. Thus a combination of scattered waves for all three incident waves gives the antisymmetric or symmetric part of the total scattering. To describe this, let us introduce scattering matrices - $R^a$ and $R^s$ for the reflected field, $T^a$ and $T^s$ for the transmitted field as:

$$R^{(a,s)} = \begin{bmatrix} R_{ll}^{(a,s)} & R_{lh}^{(a,s)} & R_{lv}^{(a,s)} \\ R_{hl}^{(a,s)} & R_{hh}^{(a,s)} & R_{hv}^{(a,s)} \\ R_{lv}^{(a,s)} & R_{hv}^{(a,s)} & R_{vv}^{(a,s)} \end{bmatrix}, \quad T^{(a,s)} = \begin{bmatrix} T_{ll}^{(a,s)} & T_{lh}^{(a,s)} & T_{lv}^{(a,s)} \\ T_{hl}^{(a,s)} & T_{hh}^{(a,s)} & T_{hv}^{(a,s)} \\ T_{lv}^{(a,s)} & T_{hv}^{(a,s)} & T_{vv}^{(a,s)} \end{bmatrix},$$

(4.27)

where $R_{jk}^a$ and $R_{jk}^s$ are the antisymmetric and symmetric parts of the reflection coefficient, and $T_{jk}^{(a,s)}$ are the transmission coefficients. The first subscript denotes
the type of incident wave and the second subscript the reflected wave. From reciprocity, one obtains (from Fig. 4.1) \( \mathbf{T}^a = -\mathbf{R}^a \) and \( \mathbf{T}^s = \mathbf{R}^s \). Note that to use the decomposition procedure shown in Fig. 4.1, the polarization vectors for waves of the same type in the upper and lower semispaces should be chosen to be symmetric about the interface plane. The convention used in this paper is that for a longitudinal wave \( \vec{P}_t = \vec{n} \), for an SH wave \( \vec{P}_H = (0, 1, 0)^T \) and for an SV wave \( \vec{P}_V = \vec{n} \times \vec{P}_H \) in the upper semispace and \( \vec{P}_V = \vec{P}_H \times \vec{n} \) in the lower semispace.

The scattering matrix \( \mathbf{R}^{(a,s)} \) can be found by satisfying the corresponding antisymmetric and symmetric B.C. (4.3, 4.15). Detailed derivation of the system of boundary equations is given in Appendix C. The scattering coefficient equations imply:

\[
\begin{align*}
R_{iH}^{(a,s)} &= \frac{\gamma}{\beta \xi^2} R_{Hi}^{(a,s)}, \\
R_{iV}^{(a,s)} &= -\frac{\gamma}{\beta \xi^2} R_{Vi}^{(a,s)}, \\
R_{HV}^{(a,s)} &= -R_{VH}^{(a,s)}.
\end{align*}
\]  

(4.28)

which satisfies reciprocity of the energy conversion coefficients between different modes [46]. In (4.28) the non-dimensional parameters are \( \alpha = \sin(\theta_i) = k/k_i; \xi = V_i/V_t \) \( (V_i \) and \( V_t \) being respectively the longitudinal and transverse wave velocities in the semispace); \( \beta = \cos(\theta_i) = \sqrt{1 - \alpha^2} \) and \( \gamma = \xi \cos(\theta_i) = \sqrt{\xi^2 - \alpha^2} \). There are only six independent elements in \( \mathbf{R}^{(a,s)} \); the corresponding elements for \( \mathbf{R}^a \) are given below and in Appendix D:

\[
\begin{align*}
R_{ii}^a &= \frac{\Delta^a(-\beta, -\tilde{h})}{\Delta^a(\beta, \tilde{h})}, \\
R_{HI}^a &= -\frac{4\alpha \beta \xi C_{45}}{\Delta^a} \frac{\mu}{\rho} \tilde{h} [4i\beta + (\frac{\rho_0}{\rho} - W)\tilde{h}],
\end{align*}
\]  

(4.29) (4.30)
where $\rho$ is the density and $\mu$ is the shear modulus of the semispace, and

$$
\Delta^a = -8 \frac{C_{44} C_{55} - C_{45}^2}{\mu^2} \beta + 4i \left[ \frac{C_{44}}{\mu} \Delta_r + \frac{C_{55}}{\mu} \beta^2 + \frac{C_{44} C_{55} - C_{45}^2}{\mu^2} (2\alpha^2 a_2 + \frac{\rho_0}{\rho} a_1) \right] h \\
+ 2[\beta \Delta_r + \frac{C_{44}}{\mu} \frac{\rho_0}{\rho} \rho \gamma + \frac{C_{55}}{\mu} (2\alpha^2 a_2 + \frac{\rho_0}{\rho} a_1) \beta - \frac{C_{44} C_{55} - C_{45}^2}{\mu^2} \alpha^2 \gamma] h^2 \\
+ i(-\frac{\rho_0}{\rho} + \frac{C_{55}}{\mu} \alpha^2) \beta \gamma h^3.
$$

$\Delta^a = 0$ is the second order characteristic equation for antisymmetric interface modes localized in the interphase, discussed in detail in the next section. In the above equations, $h = k_i h$, $\alpha = k/k_i$, $W = 2\alpha^2 - 1$, $a_1 = \alpha^2 + \beta \gamma$, $a_2 = -W - 2\beta \gamma$. $\Delta_r = W^2 + 4\alpha^2 \beta \gamma$ is the characteristic function for Rayleigh waves in the semispace. The analytical solutions (4.29-4.30) are accurate only to second order in $h$, the third order terms in $h$ in $\Delta^a$ arising from coupling of the second order in-plane and out-of-plane components. They are not the complete third-order terms, but they are required for the energy balance as discussed in Chapter III.

We obtain for $R^s$

$$
R_{il}^s = -\frac{\Delta^s(\alpha, -\hat{h})}{\Delta^s(\alpha, h)},
$$

$$
R_{HI}^s = -\frac{4\alpha \beta \xi}{\Delta^s \hat{h} (2\frac{C_{36}}{\mu} W + 2 \frac{C_{13} C_{36} - C_{16} C_{33}}{\mu^2} \alpha^2)} \\
+ i \frac{C_{16} (C_{13} + 2\mu - C_{36} (C_{11} - \rho_0 V^2))}{\mu^2} \alpha^2 \beta \hat{h}.
$$

Here

$$
\Delta^s = 8i \frac{C_{33}}{\mu} \beta \gamma + 4[\Delta_r \beta - 2 \frac{C_{13}}{\mu} \alpha^2 \beta a_2 + \frac{C_{33}}{\mu} \frac{\rho_0}{\rho} (a_1 q_1 \beta + q_2 \gamma)] h + 2i \left\{ \frac{C_{66} - \rho_0 V^2}{\mu} \Delta_r \alpha^2 + \frac{C_{11} - \rho_0 V^2}{\mu} \alpha^2 \beta^2 + 2 \frac{C_{16} C_{36} - C_{13} (C_{66} - \rho_0 V^2)}{\mu^2} a_2 \alpha^4 \right\}.
$$
\[ + \left( \frac{C_{33} \rho_0}{\mu} \left( \frac{C_{11} - \rho_0 V^2}{\mu} q_2 - \frac{C_{13}^2}{C_{33} \mu} \right) + \frac{C_{16} C_{33} - 2C_{13} C_{16} C_{36} + C_{13}^2 C_{66}}{\mu^3} \alpha^2 \right] \alpha^2 a_1 \beta h^2 + \frac{C_{16}^2}{\mu^2} (C_{11} - \rho_0 V^2)(C_{66} - \rho_0 V^2) \alpha^4 \beta h^3, \] (4.34)

where the mass factors are \( q_1 = 1 - \frac{C_{11} C_{33} - C_{13}^2}{C_{33} \rho_0 V^2} \) and \( q_2 = 1 - \frac{C_{33} C_{66} - C_{36}^2}{C_{33} \rho_0 V^2} \).

\( \Delta^* = 0 \) is the second order dispersion equation for symmetric interface modes localized in this thin interphase and also discussed in detail in the next section.

Solutions for other elements of \( R^g \) are also given in Appendix D.

The total scattering matrices for the reflected \( R \) and transmitted \( T \) waves are the sums of the symmetric and antisymmetric scattering matrices:

\[ R = \frac{1}{2} R^a + \frac{1}{2} R^s, \] (4.35)

\[ T = \frac{1}{2} T^a + \frac{1}{2} T^s = -\frac{1}{2} R^a + \frac{1}{2} R^s. \] (4.36)

In general, the rank of the system of equations is halved when the second order asymptotic B.C. are applied if the elastic system is symmetric about the midplane. The stiffness matrix representation of the second order asymptotic B.C. greatly simplifies the derivation and study of the wave solutions. As will be discussed in the next Chapter, this representation also has advantages in studying interface wave phenomena since the symmetric and antisymmetric modes are interface waves of different velocities and are naturally decoupled.
Table 4.1: Elastic constants $C_{ij}^0$ (GPa) and density (g/cc) of a porous interphase (porosity $C = 30\%$) in its material coordinate system ($x^0, y^0, z$). Density $\rho_0$ is in g/cc.

<table>
<thead>
<tr>
<th>$C_{11}^0$</th>
<th>$C_{22}^0$</th>
<th>$C_{33}^0$</th>
<th>$C_{12}^0$</th>
<th>$C_{13}^0$</th>
<th>$C_{23}^0$</th>
<th>$C_{44}^0$</th>
<th>$C_{55}^0$</th>
<th>$C_{66}^0$</th>
<th>$\rho_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>63.5</td>
<td>40.6</td>
<td>40.6</td>
<td>18.3</td>
<td>18.3</td>
<td>18.1</td>
<td>11.2</td>
<td>15.2</td>
<td>15.2</td>
<td>1.89</td>
</tr>
</tbody>
</table>

4.4 Numerical Results

To investigate the accuracy of the second order B.C., we calculate the scattering coefficients on a thin orthotropic interfacial layer between identical isotropic semispaces. The material properties of the semispace are $V_1 = 6.20$ Km/s, $V_t = 3.24$ Km/s and density $\rho = 2.70$ g/cc; the imperfect interphase is modeled by a parallel row of cylindrical pores as shown in Fig. 1.2(b). The effective elastic modulus of the interphase are calculated from Christensen’s two-phase model [52]; the $C_{ij}^0$ for an interphase with 30% porosity are listed in Table 4.1. One sees that the elastic modulus of the interphase along the pore orientation (the $x^0$ direction in Fig. 3.2) is greater than in the other two directions. Further we assume the incident ($x, z$) plane deviates from the pore orientation by a rotation angle $\varphi$.

Calculations have been carried out using both second-order B.C. and simplified models. The terms stiffness-mass and stiffness models refer to the B.C. (4.11, 4.24) and (4.12, 4.25) where the stiffnesses and mass are defined through the interfacial layer parameters. First we consider mode conversion of an incident SH wave into reflected and transmitted SV waves when $\varphi = 45^\circ$. The elastic constants $C_{ij}$ for the
Table 4.2: Elastic constants $C_{ij}$ (GPa) for the same material as in Table 4.1 but in the rotated ($\varphi = 45^\circ$) coordinate system $(x, y, z)$.

<table>
<thead>
<tr>
<th>$C_{11}$</th>
<th>$C_{22}$</th>
<th>$C_{33}$</th>
<th>$C_{12}$</th>
<th>$C_{13}$</th>
<th>$C_{23}$</th>
<th>$C_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50.5</td>
<td>50.5</td>
<td>40.6</td>
<td>19.9</td>
<td>18.2</td>
<td>18.2</td>
<td>13.2</td>
</tr>
<tr>
<td>$C_{55}$</td>
<td>$C_{66}$</td>
<td>$C_{45}$</td>
<td>$C_{16}$</td>
<td>$C_{26}$</td>
<td>$C_{36}$</td>
<td></td>
</tr>
<tr>
<td>13.2</td>
<td>16.9</td>
<td>-2.02</td>
<td>-5.73</td>
<td>-5.73</td>
<td>-0.01</td>
<td></td>
</tr>
</tbody>
</table>

interphase with 30% porosity in its rotated coordinate system $(x, y, z)$ are listed in Table 4.2. The excitation of the SV wave is due purely to the anisotropy of the interface about the incident plane. Recall that the antisymmetric part of the reflection coefficient $R_{HV}^2$ is from (C3) and (4.28)

$$R_{HV}^2 = -R_{VH}^2 = \frac{4\beta C_{45}}{\Delta^a - \mu} [h[-2iV + (\rho_0 - 2\alpha^2)\gamma h]]; \quad (4.37)$$

$$\alpha = \sin(\theta_i), \quad \beta = \cos(\theta_i), \quad \gamma = \xi \cos(\theta_i), \quad W = -\cos(2\theta_i).$$

One sees from Eq. (4.37) that the mode conversion coefficient $R_{HV}^2$ is proportional to the modulus $C_{45}$ of the interfacial layer. Note that the SV and SH waves have the same speed due to the isotropy of the semispace, and physically constitute a single transverse wave with its polarization vector rotated from that of the incident SH wave. However, to demonstrate the effect of coupling between the in-plane and out-of-plane parts of the elastic field due to interface anisotropy, we treat them separately here as two different transverse waves.

The energy conversion coefficients $r_{HV} \equiv R_{HV}^2$ and $t_{HV} \equiv T_{HV}^2$ and the phase shifts of the reflected and transmitted SV waves obtained using different B.C. models are plotted in Fig. 4.2 as a function of $h/\lambda_0$ at incident angle $\theta_i = 50^\circ$.
\[ \lambda_0 = \frac{V_0}{f} \] is the wavelength of the slow bulk transverse wave in the layer propagating normal to the interface, where \( f \) is the frequency and \( V_0 = \sqrt{C_{44}^0/\rho_0} \) is the speed. In Fig. 4.2 the exact solutions are represented by solid lines calculated using the algorithm described in Section 3.3., the solutions calculated using the second order asymptotic B.C. (4.3, 4.15) by open circles, the solutions using the stiffness-mass model (4.7, 4.19) by solid circles and the solutions using the stiffness model (4.12, 4.25) by crosses. One sees from Fig. 4.2 that the second order B.C. give close approximations to the exact solutions for both the energy coefficients and the phase shifts up to \( h/\lambda_0 = 0.2 \) in both the reflected and transmitted fields, while other models give solutions deviating significantly from the exact solutions for \( h/\lambda_0 > 0.05 \) for most cases. This demonstrates the necessity of retaining the coupling and inertial terms in the approximate B.C. models and the advantage of using the second order B.C. approach. One sees also from these figures that the transformation between the in-plane and out-of-plane transverse waves is significant, even when the interface anisotropy is not marked, as in the present example where \( C_{45} = -2.02 \) compared to \( C_{55} = 13.2 \). Note that there is no transformation to the longitudinal modes in this case due to high incidence angle.

To illustrate the effect of interface anisotropy on the mode conversion between SV and SH transverse waves, we calculate the energy conversion coefficients \( r_{HV} \) and \( d_{HV} \) of the reflected and transmitted SV waves as a function of the pore orientation angle \( \varphi \) for the same incident SH wave. The nondimensional layer
Figure 4.2: Mode conversion of an incident SH wave into reflected and transmitted SV waves on a porous interphase. $h/\lambda_0$ is the nondimensional layer thickness. The pore orientation angle $\varphi = 45^\circ$ and the incident angle $\theta_i = 50^\circ$. (a) Energy conversion coefficient $r_{HV}$ for the reflected SV wave. (b) Phase shift in radians for the reflected SV wave. (c) Energy conversion coefficient $t_{HV}$ for the transmitted SV wave. (d) Phase shift in radians for the transmitted SV wave.
Fig. 4.2 (continued)

\[ \varphi = 45^\circ \]
\[ \theta_i = 50^\circ \]
Fig. 4.2 (continued)

ENERGY CONVERSION COEFFICIENT ($t_{HV}$)

- EXACT SOLUTION
- o o o o SECOND ORDER
- •••• STIFFNESS–MASS
- ×××× STIFFNESS

$\varphi = 45^\circ$

$\theta_1 = 50^\circ$

(c)
Fig. 4.2 (continued)

PHASE SHIFT OF TRANS. SV WAVE (radian)

- EXACT SOLUTION
- OOOO SECOND ORDER
- •••• STIFFNESS-MASS
- ×××× STIFFNESS

ϕ = 45°
θ_i = 50°

(d)
thickness parameter $h/\lambda_0$ is fixed at 0.1. The results obtained using different B.C. models are plotted in Fig. 4.3 with all the symbols the same as in Fig. 4.2.

Fig. 4.3 shows that only the second order B.C. give a satisfactory approximation to the exact solution for both the reflected and transmitted SV waves. One sees that in the plane of symmetry ($\varphi = 0^\circ$ or $90^\circ$) there is no mode conversion because there is no coupling between the in-plane and out-of-plane elastic fields ($C_{45} = C_{16} = C_{36} = 0$), while at $\varphi = 45^\circ$, where the interfacial coupling between normal and out-of-plane shear components is the strongest, both the mode conversion coefficient and the deviation of the approximation from the exact solution reach their maximum values.

Next let us consider the reflection of a longitudinal incident wave on the same porous interphase ($\varphi = 45^\circ$). The energy coefficient ($r_{tt} = R_{tt}^2$) and phase shift of the reflected wave of the same mode are shown in Fig. 4.4 as a function of $h/\lambda_0$ at the same incident angle ($\theta_i = 50^\circ$). One sees from Fig. 4.4 that all the B.C. models give reasonable approximations to the exact solutions for small $h/\lambda_0$ ($< 0.1$), but for larger $h/\lambda_0$ the second order solution gives the best approximations (up to $h/\lambda_0 \approx 0.4$).

To better judge the performance of the second order B.C. and other simplified models, we calculate the same reflection as a function of incident angle $\theta_i$ at a relatively large value 0.2 of $h/\lambda_0$. The results are shown in Fig. 4.5 for the same porous interphase ($\varphi = 45^\circ$). One sees from Fig. 4.5 that the second order B.C.
Figure 4.3: Energy conversion coefficients of SH to SV waves as functions of fiber orientation angles $\varphi$ for different B.C. models. The incident SH wave angle $\theta_i$ is $50^\circ$ and $h/\lambda_0 = 0.1$. (a) $r_{HV}$; (b) $t_{HV}$. 
Fig. 4.3 (continued)

\[ h/\lambda_0 = 0.1 \]
\[ \theta_i = 50^\circ \]
Figure 4.4: Reflection of a longitudinal wave on a porous interphase as functions of $h/\lambda_0$ for different B.C. models. The pore orientation angle $\varphi = 45^\circ$ and the incident angle $\theta_i = 50^\circ$. (a) Energy coefficient $r_h$ for the reflected longitudinal wave. (b) Phase shift in radians for the reflected longitudinal wave.
Fig. 4.4 (continued)

\[ \varphi = 45^\circ \]
\[ \theta_i = 50^\circ \]

- \(-2.0 \)
- \(-1.5 \)
- \(-1.0 \)
- \(-0.5 \)
- \(0.0 \)

PHASE SHIFT (radian)

- \(0.0 \)
- \(0.1 \)
- \(0.2 \)
- \(0.3 \)
- \(0.4 \)
- \(0.5 \)
- \(0.6 \)

\(h/\lambda_0\)

- EXACT SOLUTION
- SECOND ORDER
- STIFFNESS-MASS
- STIFFNESS

(b)
give good and homogeneous approximations to both the energy coefficient and the phase shift for all incident angles. When this angle is less than 20°, there is little difference between the second order solutions and those obtained using the stiffness-mass model, since the coupling terms and the angle-dependent part of the inertial terms are negligible. The stiffness model does not give a good approximation at this value of $h/\lambda_0$. Note that at normal incidence $r_{ll} \approx 0.2$ due to the existence of the porous interphase, compared to zero reflection if $h/\lambda_0 = 0$.

4.5 Summary

For a thin anisotropic layer between two identical semispaces the second order B.C. can be decomposed for symmetric and antisymmetric cases into stiffness-type relations described by 3×3 matrices. Analytical solutions for wave scattering from an off-axis orthotropic interfacial layer between identical isotropic semispaces are given. They can be applied to predict the effect of interface anisotropy on wave scattering. The great advantage of these analytical solution is that there is no need to find wave solutions in the anisotropic interfacial layer. The numerical results show that retention of coupling and mass terms in the asymptotic models greatly improves the accuracy of the approximation for oblique wave scattering, and the second order asymptotic B.C. give much better results for the transmitted field than the first order asymptotic B.C.

In the next Chapter, the stiffness-matrix type representation of the decom-
Figure 4.5: Reflection of a longitudinal wave on the same porous interphase as functions of incident angles $\theta_i$ for different B.C. models. The pore orientation angle $\varphi = 45^\circ$ and $h/\lambda_0 = 0.2$. (a) Energy coefficient $r_{il}$ for the reflected longitudinal wave. (b) Phase shift in radians for the reflected longitudinal wave.
Fig. 4.5 (continued)

Phaseshift (radian)

- **Exact Solution**
- **Second Order**
- **Stiffness-Mass**
- **Stiffness**

\[ \frac{h}{\lambda_0} = 0.2 \]
\[ \phi = 45^\circ \]
posed elastic fields will be used to study the localization of interface waves by a thin anisotropic interfacial layer between two identical isotropic substrates. The advantage of such decomposition is more significant since the symmetric and antisymmetric modes are interface waves of different velocities and are naturally decoupled.
CHAPTER V

LOCALIZATION OF INTERFACE WAVES BY A THIN INTERFACIAL LAYER

In this chapter we will address the localization of interface-type waves by a thin interfacial layer. Since Stoneley found theoretically that an interface between two isotropic elastic solids can support a localized interface wave in 1924 [53], there have been many studies of conditions for localization of generalized Stoneley waves on interface with welded and non-welded B.C. (e.g. [16, 29, 32, 54, 55, 56, 57, 58] or thin interfacial layers between solids [4, 17, 20, 24, 51]. It was found that for isotropic cases a thin interfacial layer with a shear modulus less than those of the semispaces will tend to localize the elastic energy and in particular that such a layer between identical solids can always support interface waves [24, 25]. Here we will discuss the conditions for localization of interface guided modes in a thin anisotropic layer between identical isotropic semispaces.
5.1 Dispersion Equations for Interface Symmetric and Antisymmetric Modes

Suppose that an interface wave with displacement \( u_j = A f_j(z) e^{ik(z-Vt)} \) propagates in the \( z \) direction along an anisotropic interface layer, where \( V \) is the interface wave velocity. The boundary equations (3.1) must be satisfied by the interface wave alone, which is composed of nonhomogeneous bulk waves. The waves in the semispaces are all evanescent, three in each semispace due to the coupling effect of the interface anisotropy, while the waves inside the interface layer can be either propagating or evanescent. Let the wavelength of the interface wave be much greater than the thickness of the interfacial layer so that the interface wave problem can be solved using the second order B.C. approach. The characteristic equation for the interface wave can be found either by setting the determinant of the boundary condition matrix to zero [4] or equivalently by setting the denominator in the scattering coefficients to zero. The method of calculating the exact solution for the interface wave velocity and numerical comparison of the exact and asymptotic solutions for the anisotropic case are given by Huang and Rokhlin (1992) where dissimilar anisotropic semispaces have been considered.

Here we focus on the second-order characteristic equations obtained in the previous section for a thin off-axis orthotropic layer between two identical isotropic semispaces. The dispersion equations for the antisymmetric and symmetric interface modes are obtained by setting the characteristic functions (4.31) and (4.34)
to zero:

$$\Delta^a = 0, \quad \Delta^s = 0. \quad (5.1)$$

The elastic motions of the antisymmetric and symmetric modes on the interfacial layer are illustrated in Fig. 5.1. If the wave propagation direction is not an axis of symmetry, i.e. if the $x$-direction does not coincide with either the $x^0$- or $y^0$-direction, the interface wave characteristic equation (5.1) cannot be decomposed into dispersion equations for modes with in-plane and out-of-plane SH type polarizations. The plane here is defined by the interface wave normal and interface plane normal, i.e. the $(x, z)$ plane. In this case, the interface modes are not pure modes but quasi modes. The particle motion for the interface wave is ellipsoidal and frequency dependent with strong coupling between the in-plane and out-of-plane displacement fields.

However when the interface wave propagates along a symmetry axis of the orthotropic interfacial layer, the in-plane and out-of-plane vibrations are decoupled. Thus the characteristic equation for guided waves can be factored into two equations: one for the in-plane polarization and one for the SH-polarization. Let us first consider the antisymmetric case; the in-plane part of the second order dispersion equation for an antisymmetric mode propagating in the plane of symmetry along the $x^0$-axis is:

$$\Delta_0^a = 2 \frac{C_{55}^0}{\mu} (i 2 \beta + 2 \alpha^2 a_2 \bar{h} + \frac{\rho_0 a_1}{\rho} h) + 2 \Delta_r \bar{h} + i(\frac{\rho_0}{\rho} \gamma + \frac{C_{55}^0}{\mu} \alpha^2 \gamma) \bar{h}^2 = 0. \quad (5.2)$$
Figure 5.1: Guided waves along a thin interfacial layer. (a) Antisymmetric mode, (b) symmetric mode.
\[ \Delta_{0H}^a = 0 \] is the dispersion equation for the SH-type antisymmetric mode:

\[ \Delta_{0H}^a = \beta \bar{h} + 2i \frac{C_{44}^0}{\mu} = 0, \quad (5.3) \]

which yields no propagating wave solutions since a propagating mode requires that \( \alpha > 1 \) and \( \beta = i\sqrt{\alpha^2 - 1} \) be positive pure imaginary.

It is interesting to note that the characteristic function \( \Delta^a \), given by Eq. (4.31), for an interface mode propagating along an off-symmetry axis direction (the \( x \)-axis) can be represented as

\[ \Delta^a(C_{ij}) = \Delta_{0s}^a(C_{55}) \Delta_{0H}^a(C_{44}) + \frac{C_{45}^2}{\mu^2} \left[ 8(2\alpha^2 a_2 + \frac{\rho_0}{\rho} a_1) \bar{h} + 2\alpha^2 \gamma \bar{h}^2 \right], \quad (5.4) \]

where \( C_{ij} \) are the elastic constants in the rotated coordinate system \( (x, y, z) \) and the functions \( \Delta_{0s}^a \) and \( \Delta_{0H}^a \) are given by Eqs. (5.2) and (5.3). The term in \( C_{45} \) is due physically to coupling between in-plane and out-of-plane vibrations.

It is plain from Eq. (5.2) that in the plane of symmetry the interface wave velocity \( V = V_i / \alpha \) for the antisymmetric mode depends on only one elastic constant \( C_{55}^0 \) (the in-plane shear modulus) of the interphase, on the interfacial layer density \( \rho_0 \), and on the non-dimensional layer thickness \( k_i \bar{h} \). Physically this is due to the particle motions of an antisymmetric wave on the two sides of the layer being in phase in the \( z \)-direction (see Fig. 5.1), so that the wave velocity is insensitive to the normal stiffness \( C_{33}^0 / \bar{h} \) (Rokhlin et al 1980). Equation (5.2) is of second order with respect to \( \bar{h} \) compared to the first order equations obtained by Rokhlin et al [24, 25] for the isotropic case, which have been applied to measurements of the
shear modulus on thin films between solids. As follows from (5.2) the in-plane shear modulus $C_{ss}^{0}$ of the interfacial layer can be found from the experimentally measured velocity of the antisymmetric interface wave for an anisotropic layer:

$$
\frac{C_{ss}^{0}}{\mu} = \left( \frac{i}{2} \frac{\rho_{0} \gamma \bar{h}^2 - \Delta_{s} \bar{h}}{\rho} + 2 \alpha \alpha^{a} a_{2} \bar{h} + \frac{\rho_{0} a_{1}}{\rho} \bar{h} + \frac{i}{2} \alpha^{2} \gamma \bar{h}^2 \right). \tag{5.5}
$$

Here recall that $\mu$ is the shear modulus of the semispaces. For an isotropic interphase the in-plane shear modulus $C_{ss}^{0}$ is simply the shear modulus $\mu_{0}$ of the interphase.

It is also interesting to discuss the results when simplified B.C. models are used. When the stiffness-mass model is used for this interfacial layer as given by B.C. (4.11) the dispersion equation for the antisymmetric mode becomes

$$
\Delta^{a}(C_{ij}) = \Delta_{0}^{a}(C_{ss}) \Delta_{0H}^{a}(C_{44}) + \frac{C_{45}^{2}}{\mu^2} (8\beta - 4i \frac{\rho_{0} a_{1}}{\rho} \bar{h}) = 0 \tag{5.6}
$$

where $\Delta_{0}^{a}(C_{ss})$ has the same form as the characteristic function for the case of symmetry $\Delta_{0}^{a}(C_{ss}^{0})$:

$$
\Delta_{0}^{a} = 2 \frac{C_{ss}^{0}}{\mu} (i2\beta + \frac{\rho_{0} a_{1}}{\rho} \bar{h}) + 2 \Delta_{s} \bar{h} - i \frac{\rho_{0}}{\rho} \gamma \bar{h}^2 = 0. \tag{5.7}
$$

and $\Delta_{0H}^{a}(C_{44})$ has the same form as $\Delta_{0H}^{a}(C_{44}^{0})$ in (5.3).

If we further neglect the inertia terms, the dispersion equation for a simple stiffness model (4.12) is

$$
\Delta^{a}(C_{ij}) = \Delta_{0}^{a}(C_{ss}) \Delta_{0H}^{a}(C_{44}) + \frac{8C_{45}^{2}}{\mu^2} \beta = 0 \tag{5.8}
$$
where

$$\Delta^0 = 2i\frac{C_{55}}{\mu} \beta + \Delta_r \bar{h} = 0, \quad (5.9)$$

and the form of $\Delta^0_{6H}(C_{44})$ stays the same. One sees that the second order term in $\bar{h}$ vanishes in (5.9).

Similarly for a symmetric interface mode propagating along an off-symmetry axis the characteristic function $\Delta^*(4.34)$ can be represented as

$$\Delta^*(C_{ij}) = \Delta^0(C_{11}, C_{13}, C_{33})\Delta^0_{6H}(C_{66}) + 4 \frac{C_{36}^2}{\mu^2} \alpha^2 \gamma \bar{h} + 2i[2 \frac{C_{16}}{\mu^2} a_2 \alpha^4 \bar{h}^2$$
$$+ \frac{(C_{11} - \rho_0 V^2)C_{36}^2 + C_{33}C_{16}^2 - 2C_{13}C_{36}a_1}{\mu^3} \alpha^4 \bar{h}^2$$
$$+ \frac{C_{16}^2}{\mu^2} \alpha^4 \beta \bar{h}^3]. \quad (5.10)$$

The additional terms on the right side of (5.10) are due to coupling between the in-plane and out-of-plane vibrations. The in-plane part of the characteristic function for a symmetric mode propagating along an axis of symmetry (the $x^0$-axis) is:

$$\Delta^0 = 2 \frac{C_{33}}{\mu} (i2\gamma - 2\alpha^2 a_2 \frac{C_{13}}{C_{33}} \bar{h} + \frac{\rho_0 a_1}{\rho} q_1 \bar{h}) + 2\Delta_r \bar{h} + i\beta \frac{(C_{13})^2}{\mu^3} \alpha^2 - \frac{\rho_0}{\rho} q_1 \bar{h}^2, \quad (5.11)$$

and the characteristic function for the SH-type symmetric mode in this case is:

$$\Delta^*_{6H} = 2\beta + i\alpha^2 \frac{C_{66}^0 - \rho_0 V^2}{\mu} \bar{h}. \quad (5.12)$$

The characteristic equation $\Delta^*_{6H} = 0$ does yield a real solution for a propagating wave, since for a propagating mode $\beta$ is a positive pure imaginary number and the interface wave velocity $V > V^0_{SH}$, where $V^0_{SH} = \sqrt{C_{66}^0/\rho_0}$ is the bulk SH wave velocity of the interphase.
We can also consider cases when simpler B.C. models are used. If the stiffness-mass model with B.C. (4.24) is used the dispersion equation becomes

\[ \Delta^*(C_{33}) = \Delta_0^s(C_{33})\Delta_{0H}^s = 0, \quad (5.13) \]

where

\[ \Delta_0^s = 2\frac{C_{33}^0}{\mu}(i2\gamma + \frac{\rho_0 a_1}{\rho} \tilde{h}) + 2\Delta_v \tilde{h} - i\beta \frac{\rho_0}{\rho} \tilde{h}^2 = 0, \quad (5.14) \]

and

\[ \Delta_{0H}^s = 2\beta - i\frac{\rho_0}{\rho} \tilde{h}. \quad (5.15) \]

One sees from the dispersion equations (5.13-5.15) that \( \Delta^s \) depends only on \( C_{33} \) which is independent of the interface wave propagation direction. The in-plane and out-of-plane parts decouple due to neglect of the coupling terms in (4.24). The in-plane part of the dispersion equation (5.14) depends only on \( C_{33}^0 = C_{33} \), and the out-of-plane part involves on no elastic modulus. In contrast, for the antisymmetric case, due to the presence of a coupling spring term (\( C_{45}/h \)) which is wave number independent and thus of order zero, the decoupling of in-plane and SH-type vibrations is not possible as long as such stiffnesses are retained in the B.C. (4.11).

If we further neglect the mass, the dispersion equation for the simple stiffness model (4.25) is

\[ \Delta^*(C_{33}) = \Delta_0^s(C_{33})\Delta_{0H}^s = 0 \quad (5.16) \]
where

$$\Delta_0^* = i4\gamma \frac{C_{03}}{\mu} + 2\Delta_0 \bar{h} = 0, \quad (5.17)$$

and $\Delta_0^{*H} = 2\beta$ which gives the bulk shear wave in the semispace.

Now we consider conditions for the localization of propagating interface modes by a thin anisotropic interfacial layer. If the velocity of a guided mode $V$ is less than the shear velocity $V_\gamma$ in the semispace, we have a “propagating” mode. If the interface wave velocity is higher than that of at least one bulk wave in the semispace, the guided mode leaks energy into the substrate and is called a “leaky” mode. With increasing frequency, the phase velocity of a given mode decreases, approaching the slowest bulk wave velocity allowed in the interfacial layer: $V_{00}$. Thus for an interfacial layer to support propagating modes in a certain frequency range the interface wave velocity must be slower than the shear wave velocity of the semispace and greater than that of the interphase: $V_{00} < V < V_\gamma$. This implies that the interphase should be a “slower” material than the semispace. A mode which is leaky at low frequency may become a propagating mode as the frequency increases. In this case, the frequency at which the propagating guided mode appears is called the cut-off frequency. To find the cut-off frequency, one can set $V = V_\gamma (\alpha = 1, \beta = 0)$ in the dispersion equations and solve for $\bar{h}_c$.

The dispersion equations derived from the exact layered approach are transcendental functions in $\bar{h}$ which give, in addition to the fundamental modes, an infinite number of higher order modes. The analytical dispersion equations shown above
are second order in $\bar{h}$ and therefore should accurately describe the dispersion of the fundamental modes, especially for predictions of the cut-off frequencies $\bar{h}_c$ which are the lowest frequencies allowed for the propagating modes. Let us first consider the localization of antisymmetric interface waves. Making the substitutions $\alpha = 1, \beta = 0, \gamma = i\sqrt{1 - \xi^2} = i/\sqrt{2(1 - \nu)}$, $a_1 = 1, a_2 = -1$ and $\Delta_r = 1$ into the right hand side of (4.31) and equating to zero, one obtains

$$4\left[\frac{C_{44}}{\mu} + \frac{C_{44}C_{55} - C_{45}^2}{\\mu^2}(\frac{\rho}{\rho} - 2)\right] \bar{h}_c + \sqrt{\frac{2}{1 - \nu}}(\frac{C_{44}}{\mu} \frac{\rho}{\rho} - \frac{C_{44}C_{55} - C_{45}^2}{\\mu^2}) \bar{h}_c^2 = 0, \quad (5.18)$$

where $\nu$ is Poisson's ratio, $\mu$ the shear modulus and $\rho$ the density of the semispace; $C_{ij}$ and $\rho_0$ are the elastic constants and density of the interphase. Equation (5.18) has two solutions: one is $\bar{h}_c = 0$ showing that the fundamental antisymmetric mode does not have a cut-off frequency; the other is always negative and thus non-physical. As one can see from the characteristic equation for the SH mode in the plane of symmetry (5.3) the propagating SH-type antisymmetric mode does not exist.

Next consider the symmetric mode. By substituting $\alpha = 1, \beta = 0$ into the right-hand side of (4.34) and equating to zero, one obtains $P\bar{h}_c - Q\bar{h}_c^2 = 0$, where

$$P = \sqrt{\frac{2}{1 - \nu}}\left(\frac{C_{33}}{\mu} \frac{\rho}{\rho} - \frac{C_{33}C_{66} - C_{36}^2}{\\mu^2}\right)$$

$$Q = \frac{\rho_0}{\rho} - \frac{C_{66}}{\mu} + 2\frac{C_{16}C_{36} - C_{13}C_{66}}{\\mu^2} + 2\frac{C_{13}}{\mu} \frac{\rho_0}{\rho} + \frac{C_{33}}{\mu} \frac{\rho_0}{\rho} - \frac{C_{11}}{\mu}\right)$$

$$+ \frac{C_{33}C_{66} - C_{36}^2}{\\mu^2}\left(\frac{C_{11}}{\mu} - \frac{\rho_0}{\rho}\right) - \frac{C_{16}C_{33} - 2C_{13}C_{16}C_{66} + C_{13}^2C_{66}}{\\mu^3}. \quad (5.19)$$

Since $\bar{h}_c = 0$ is a root the fundamental mode of the symmetric wave also has
no cut-off frequency. As one sees from the characteristic equation (5.12) for the SH mode in the plane of symmetry, the solution with $\bar{h}_c = 0$ corresponds to the propagating SH-type symmetric mode. The second positive solution $\bar{h}_c = P/Q$, depends on the modulus and density ratios of the semispace and interphase. For the plane of symmetry, the solution $\bar{h}_c = P/Q$ simplifies to

$$\bar{h}_c = \frac{\sqrt{\frac{2}{1 - \nu}}}{\frac{\mu}{C_{03}^0} \left( \frac{C_{13}^0}{\mu} + 1 \right)^2 + \frac{\rho_0}{\rho} - \frac{C_{11}^0}{\mu}}. \quad (5.20)$$

For the isotropic case Lamé constants $\lambda_0$ and $\mu_0$ should be used to replace $C_{11}^0$, $C_{13}^0$ and $C_{33}^0$ in Eq. (5.20). For $\bar{h} < \bar{h}_c$ the lowest symmetric mode is a leaky wave with complex wave number. With increase of frequency the velocity decreases so that beyond a cut-off frequency it becomes a propagating mode.

We can also write the expression for the cut-off frequencies $\bar{h}_c$ for the stiffness-mass model and the stiffness models, given by Eqs. (5.14) and (5.17) respectively, as

$$\bar{h}_c = \frac{\sqrt{\frac{2}{1 - \nu}}}{\frac{\mu}{C_{33}^0} + \frac{\rho_0}{\rho}}, \quad (5.21)$$

$$\bar{h}_c = \sqrt{\frac{2}{1 - \nu}} \frac{C_{33}^0}{\mu}. \quad (5.22)$$

### 5.2 Numerical Results

To investigate the accuracy of the second order B.C., we calculate the dispersion curves of interface modes on a thin orthotropic interfacial layer between identical
isotropic semispaces. The material properties of the semispace are $V_t = 6.20$ Km/s, $V_s = 3.24$ Km/s and density $\rho = 2.70$ g/cc; the imperfect interphase is modeled by a parallel row of cylindrical pores as shown in Fig. 1.2(b). The effective elastic modulus of the interphase are calculated from Christensen's two-phase model [52]; the $C^0_{ij}$ for an interphase with 30% porosity are listed in Table 4.1. One sees that the elastic modulus of the interphase along the pore orientation (the $x^0$ direction in Fig. 3.2) is greater than in the other two directions. Further we assume the incident $(x, z)$ plane deviates from the pore orientation by a rotation angle $\varphi$.

Calculations have been carried out using both second-order B.C. and simplified models. The terms stiffness-mass and stiffness models refer to the B.C. (4.11, 4.24) and (4.12, 4.25) where the stiffnesses and mass are defined through the interfacial layer parameters. We calculate the dispersion curves of the interface guided modes in the same interphase to assess the applicability of the second order B.C. First, we consider propagation in the plane of symmetry, assuming that the interface wave travels along the pore direction (i.e. the $x^0$ axis); the wave velocities $V/V_t$, normalized by the shear wave velocity $V_t$ in the semispace, of an antisymmetric mode is given in Fig. 5.2(a) as a functions of $h/\lambda_0$. $\lambda_0 = V_0/\omega$ being the wavelength of the slow shear wave propagating normal to the interface. The exact solutions (solid lines) have been calculated as described in Section 3.3, the solution calculated using the second order dispersion equation (5.2) is represented by open circles, the first order solution (neglecting $\tilde{h}^2$ terms in (5.2)) by a dashed line, the solution
using the stiffness-mass model (5.7) by solid circles and the solution using the stiffness model (5.9) by crosses. One can see that the second order approximation works much better than the first order approximations, while the solutions using the simplified models have large deviations even for small values of $h/\lambda_0$, since, at the incident angle $\theta_i = 90^\circ$, the coupling effect between the normal and shear components cannot be neglected. This once again demonstrates the necessity of retaining the coupling and mass terms in the approximate B.C. models and the advantage of using second order approximate B.C.

In Fig. 5.2(b) $V/V_i$ is given as a function of the pore orientation angle $\varphi$ at fixed $h/\lambda_0 = 0.2$. This time the interface wave normal deviates from the $z^0$-direction by $\varphi$ (the pore orientation angle). The second order solutions shown by open circles, comes from the analytical dispersion equation (4.31) or (5.4). The first order solutions also comes from equation (5.4) except that in the right-hand side the $\bar{h}^2$ terms are dropped including those in $\Delta_5^2(C_{55})$ (see Eq. (5.2)). One sees from Fig. 5.2(b) that the second order approximation closely matches the exact solution while the first order solution gives some discrepancy. The interface wave velocity drops due to a decrease of interfacial stiffness along the wave propagation direction as $\varphi$ increases, i.e. as the interface normal deviates more from the pore direction.

For given material parameters it is found that for $h/\lambda_0 < 0.6$ there are no higher order symmetric or antisymmetric modes for both the exact and second order
Figure 5.2: Normalized wave speed of an antisymmetric interface mode guided by a porous interphase (porosity C = 0.3). (a) Dispersion curves obtained using different B.C. models as functions of $h/\lambda_0$ ($\varphi = 0^\circ$), (b) wave speed as a function of pore orientation angle $\varphi$ ($h/\lambda_0 = 0.2$).
Fig. 5.2 (continued)

\[ \frac{h}{\lambda_0} = 0.2 \]

- Exact Solution
- \( \circ \circ \circ \circ \) Second Order
- \( \ldots \ldots \) First Order

Normalized Interface Wave Speed

Pore Orientation Angle \( \varphi \)

(b)
solutions. The cut-off frequency for the symmetric mode is given by Eq. (5.20) as \( \tilde{h}_s = 4.85 \) \( h/\lambda_0 = 1.23 \). For such a large value of \( \tilde{h}_s \) the approximate equation cannot be used to predict the cut-off frequency. The SH-type interface mode does exist, but its velocity is extremely close to the shear wave velocity of the semispace in the frequency range of the calculation, and as shown by Huang and Rokhlin [4], the asymptotic solution does not predict the dispersion behavior of the SH modes well.

It has been found that a much softer and slower interphase tends to localize symmetric interface modes at much lower frequencies [24, 25]. Thus we take the interphase to be extremely light, soft and isotropic, with \( V_i^0 = 1.5 \)Km/s, \( V_t^0 = 0.5 \) Km/s and \( \rho_0 = 0.3 \) g/cc, and calculate the wave velocities of the fundamental symmetric mode for the same semispaces as previously. The normalized symmetric wave velocity is given in Fig. 5.3 as a function of \( h/\lambda_0 \). The second order solution (open circles) is calculated using the analytical equation (5.11), and the first order solution (dashed line) is calculated in the same way except that the second order terms in \( \tilde{h} \) are neglected. One sees from Fig. 5.3 that both the second and first order solutions closely approximate the exact solution for \( h/\lambda_0 < 0.2 \), with the second order solution slightly better. The cut-off frequencies for both the second and first order approximations are the same and can be calculated directly from equation (5.20). The value \( (h_c = 0.0389 \) or \( h/\lambda_0 = 0.0401 \) is extremely close to the exact value, found numerically, \( h/\lambda_0 = 0.0400 \).
Figure 5.3: Normalized symmetric interface wave speed as a function of the non-dimensional thickness parameter $h/\lambda_0$ for an extremely light and soft interphase between the same semispaces as previously.
5.3 Summary

For a thin anisotropic layer between two identical semispaces the second order B.C. can be decomposed for symmetric and antisymmetric cases into stiffness-type relations described by $3 \times 3$ matrices. Analytical solutions for dispersion equations are derived for interface modes on an off-axis orthotropic interfacial layer. They can be applied to predict the effect of interface anisotropy on interface mode dispersion. The characteristic equations for interface modes relate explicitly to the interphase elastic moduli and can be utilized for moduli determination from interface wave velocity measurements. It has been shown that retention of coupling and mass terms in the asymptotic models greatly improves the accuracy of the approximation, which is especially critical for interface waves. The advantage of the asymptotic B.C. compared to the exact solution is their relative simplicity, since there is no need to describe the wave behavior inside the interfacial layer.
CHAPTER VI

GENERALIZED BOUNDARY CONDITIONS FOR CRACKED Interfaces

In the final chapter of Part I, we focus on a fractured interface as shown in Fig. 1.2(a). In the previous three chapters a thin anisotropic interfacial layer between two solids was considered. The layer approach should be used when the imperfect interface can be modeled by a nonhomogeneous multi-phase interfacial layer with a definable thickness. Otherwise, a different solution may be more appropriate, for example, the quasi-static spring model [26] for a fractured interface with a distribution of cracks. This approach is more natural since the quais-static spring model is derived using known solutions for an elastic body with cracks.

To facilitate discussion we modified Fig. 1.2(a) and redrew it as Fig. 6.1. The interface imperfections have preferred orientations that induce anisotropy on the interface. When the in-incident plane (the \((x, z)\) plane) coincides with the interface symmetry axis (the \(z^0\)-axis, i.e. \(\varphi = 0^\circ\)), the in-incident-plane and out-of-plane elastic motions on the interface are decoupled, and the B.C. have been described
Figure 6.1: Wave scattering from a fractured interface with arbitrarily oriented cracks. $\phi$ is the rotation angle of the incident $(x, z)$ plane with respect to the $x^0$-direction.
by the quasi-static (spring) model [26, 27, 59, 60]. However, when the incident plane (the \((x, z)\) plane) is rotated with respect to the symmetry axis (either the \(x^0\)-axis or the imperfection preferred orientation), the in-incident-plane and out-of-plane elastic motions are coupled due to the interface anisotropy even for isotropic semispaces. Thus new generalized spring B.C. are needed to describe these coupling effects, and to be used for study of the effect of the interface imperfection orientation on elastic scattering and localization of guided waves on this interface. Here we will derive analytical solutions for two types of generalized quais-static spring models, stiffness-mass and stiffness models for cracked interfaces.

6.1 Generalized Spring B.C. for Interface Imperfections with Arbitrary Orientations

Let us first consider cases for waves incident in the plane of symmetry on a fractured interface. The B.C. on the imperfect interface can be directly described by the spring model [26]. Specifically when the incident plane coincides with one of the interface symmetry axes (e.g. \(\varphi = 0^\circ\) in Fig. 6.1) the spring B.C. are given in the Baik-Thompson quasi-static model [26, 27] as

\[
\begin{align*}
\frac{\sigma_{zz^0} + \sigma'_{zz^0}}{2} &= K_V^0 (u_{z^0} - u'_{z^0}), \\
\frac{\sigma_{z\gamma^0} + \sigma'_{z\gamma^0}}{2} &= K_H^0 (u_{\gamma^0} - u'_{\gamma^0}), \\
\frac{\sigma_{zz} + \sigma'_{zz}}{2} &= K_n^0 (u_z - u'_z); \\
\sigma_{zz^0} - \sigma'_{zz^0} &= -\omega^2 M \frac{u_{z^0} + u'_{z^0}}{2},
\end{align*}
\]
\[
\sigma_{yz} - \sigma'_{yz} = -\omega^2 M \frac{u_y + u'_y}{2},
\]
\[
\sigma_{zz} - \sigma'_{zz} = -\omega^2 M \frac{u_z + u'_z}{2},
\]

where \(u_y\) and \(\sigma_{yz}\), \(u'_y\) and \(\sigma'_{yz}\) are the particle displacements and stresses on the top and bottom surfaces of the interface in the material coordinate system \((x^0, y^0, z)\), \(K_V^0\) is the stiffness of an in-plane shear spring, \(K_H^0\) is the stiffness of an out-of-plane shear spring, \(K_n^0\) is the stiffness of a normal spring and \(M\) is the mass of the springs. The stiffnesses \(K_V^0, K_H^0, K_n^0\) are obtained for cracked interface using fracture mechanics. One sees from B.C. (6.1) the elastic motions along \(x^0, y^0\) and \(z\) directions are all decoupled on the interface.

When the incident plane (the \((x, z)\) plane) is tilted with respect to the symmetry axis (either the \(x^0\)-axis or the imperfection preferred orientation), the in-incident-plane and out-of-plane elastic motions are coupled due to the interface anisotropy even for isotropic semispaces. To describe these coupling effect, the spring B.C. (6.1) needs to be generalized. Since it is convenient to analyze scattering and interface wave problems by solving B.C. defined in the rotated coordinate system formed by the incident and interface planes \((x, y, z)\), we will derive the generalized spring B.C. in the rotated coordinate system by applying coordinate transformations to the displacements and stresses in B.C. (6.1).

One sees from Fig. 6.1 that the interface plane is a plane of symmetry but the incident plane \((x, z)\) deviates from the plane of symmetry, the \((x^0, z)\) plane, by a
rotation angle $\varphi$. Therefore the stress/displacement transformation matrix for a coordinate rotation from $(x^0, y^0, z)$ to $(x, y, z)$ $T$ is
\[
T = \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (6.2)
Let us define the spring stiffness matrix $K^0$ in the material coordinate system $(x^0, y^0, z)$ as
\[
K^0 = \begin{pmatrix}
K_V^0 & 0 & 0 \\
0 & K_H^0 & 0 \\
0 & 0 & K_n^0
\end{pmatrix},
\] (6.3)
then from Eq. (6.1) we have
\[
\begin{pmatrix}
\sigma_{xx}^0 + \sigma'_{xx} \\
\sigma_{xy}^0 + \sigma'_{xy} \\
\sigma_{zz}^0 + \sigma'_{zz}
\end{pmatrix} = T \begin{pmatrix}
\sigma_{xx}^0 + \sigma'_{xx} \\
\sigma_{xy}^0 + \sigma'_{xy} \\
\sigma_{zz}^0 + \sigma'_{zz}
\end{pmatrix} = TK^0 T^{-1} \begin{pmatrix}
u_x^0 - u_x' \\
u_y^0 - u_y' \\
u_z - u_z'
\end{pmatrix},
\] (6.4)
where $u_i$ and $\sigma_{zi}$, $u_i'$ and $\sigma'_{zi}$ are the particle displacements and stresses on the top and bottom surfaces of the interface in the rotated coordinate system $(x, y, z)$.

Now let us denote the generalized stiffness matrix as $K$, then
\[
K = TK^0 T^{-1} = \begin{bmatrix}
cos^2 \varphi K_V^0 + \sin^2 \varphi K_H^0 & \sin \varphi \cos \varphi (K_H^0 - K_V^0) & 0 \\
\sin \varphi \cos \varphi (K_H^0 - K_V^0) & \sin^2 \varphi K_V^0 + \cos^2 \varphi K_H^0 & 0 \\
0 & 0 & K_n^0
\end{bmatrix}
\]
\[
= \begin{bmatrix}
K_V & K_c & 0 \\
K_c & K_H & 0 \\
0 & 0 & K_n
\end{bmatrix},
\] (6.5)
where $K_V$ is the in-incident plane shear stiffness, $K_H$ is the out-of-plane shear stiffness, $K_n$ is the normal stiffness and $K_c$ is shear stiffness coupling between in-incident plane and out-of-plane shear stresses and displacements. Since the spring
mass is independent of the selection of the coordinate system, one need only do coordinate transformations to the B.C. involving stiffness terms. Thus Eq. (6.1) becomes

\[
\begin{align*}
\frac{\sigma_{xx} + \sigma'_{xx}}{2} &= K_V (u_x - u'_x) + K_c (u_y - u'_y), \\
\frac{\sigma_{xy} + \sigma'_{xy}}{2} &= K_c (u_x - u'_x) + K_H (u_y - u'_y), \\
\frac{\sigma_{zz} + \sigma'_{zz}}{2} &= K_n (u_z - u'_z); \\
\sigma_{zz} - \sigma'_{zz} &= -\omega^2 M \frac{u_x + u'_x}{2}, \\
\sigma_{xy} - \sigma'_{xy} &= -\omega^2 M \frac{u_y + u'_y}{2}, \\
\sigma_{zz} - \sigma'_{zz} &= -\omega^2 M \frac{u_z + u'_z}{2}.
\end{align*}
\]  

Eq. (6.6) is the new generalized spring B.C. defined the rotated coordinate system \((x, y, z)\). Compare B.C. (6.6) with (6.1), one sees that a new shear coupling spring stiffness \(K_c\) is added in the generalized B.C. (6.6) which gives the coupling of the in-incident plane and out-of-plane motions on the interface, and matrix elements \(K_V\) and \(K_H\) are modified. Note that the magnitude of this coupling stiffness \(K_c\) is proportional to the difference between \(K_V^0\) and \(K_H^0\) and \(\sin(2\varphi)\), it vanishes for plane of symmetry \(\varphi = 0^\circ \) or \(90^\circ \).

It is of interesting to point out that physically when \(K_V^0 = K_H^0\), the interface imperfections have the same stiffness values along its two symmetry axes: the \(x^0\)- and the \(y^0\)-axes. A possible example for this would be an interface with a mesh of cracks with equal crack densities for cracks oriented along both the \(x\) - and \(y\)-axes.
In this case, one sees from Eq. (6.5) that the stiffness matrix $K$ equals $K^0$ for any incident plane deviation angle $\varphi$. Thus the spring B.C. has no coupling shear springs and are independent of the selection of the coordinate system. In other words the interface shows isotropy for the spring B.C.

### 6.2 Scattering and Stoneley Wave Localization by an Interface with Arbitrarily Oriented Imperfections between Identical Isotropic Semispaces

Using the spring B.C. representation (6.6) for an interface with arbitrarily oriented imperfections, we have derived analytical solutions for scattering coefficients on such an interface between identical isotropic semispaces [51]. Let us assume that the incident field is a combination of longitudinal and transverse (SV and SH) waves with displacement vectors $\vec{u}_i$, $\vec{v}_i$, and $\vec{u}_{Hi}$ as defined by Eq. (4.26). There are in general three reflected and three transmitted waves for each of the three types of incident waves, due to the interface anisotropy. Thus a combination of scattered waves for all three incident waves gives the total scattering. To describe this, let us introduce scattering matrices - $R$ for the reflected field, and $T$ for the transmitted field as:

$$
R = \begin{bmatrix}
R_{ii} & R_{Hi} & R_{Vi} \\
R_{Hi} & R_{HH} & R_{VH} \\
R_{Vi} & R_{HV} & R_{VV}
\end{bmatrix}, \quad T = \begin{bmatrix}
T_{ii} & T_{Hi} & T_{Vi} \\
T_{Hi} & T_{HH} & T_{VH} \\
T_{Vi} & T_{HV} & T_{VV}
\end{bmatrix},
$$

(6.7)

where in each element, the first subscript denotes the type of incident wave and the second subscript the reflected or transmitted wave. The conventions used for the
polarization vectors for scattered waves are that for a longitudinal wave \( \vec{P}_l = \vec{n} \), for an SH wave \( \vec{P}_H = (0,1,0)^T \) and for an SV wave \( \vec{P}_V = \vec{n} \times \vec{P}_H \) in the upper semispace and \( \vec{P}_V = \vec{P}_H \times \vec{n} \) in the lower semispace.

The scattering matrices \( R \) and \( T \) can be found by solving the generalized spring B.C. (6.6) after substituting stress-displacement relations for the stresses. The dispersion equations for the Stoneley wave localized by the imperfect interface can be found from the common denominator of the scattering coefficients. We have performed such derivations, for an interface between two identical semispaces, by decomposing the elastic field into symmetric and antisymmetric parts. For details of the derivation the reader is referred to Chapter IV. Here we only list the analytical results. Let us first consider an antisymmetric interface wave propagating along a symmetry axis of interface imperfections (e.g. the \( x^0 \)-axis in Fig. 6.1). The B.C. are described by (6.1), and the dispersion equations are

\[
\Delta_0^a = 2K_0^a(2i\beta\mu k_t + a_1\omega^2 M) + 2\Delta_r \mu^2 k_t^2 - i\gamma \omega^2 M \mu k_t = 0, \tag{6.8}
\]

\[
\Delta_{0H}^a = \mu k_t \beta + 2iK_0^a = 0. \tag{6.9}
\]

Eq. (6.8) is for the antisymmetric mode with elastic motions in the \((x^0, z)\) plane and Eq. (6.9) for that with elastic motions along the \( y^0 \)-direction. In the above equations the non-dimensional parameters are \( \alpha = V_t/V = k/k_t, \xi = V_t/V_i \), where \( V \) is the interface wave velocity and \( V_t \) and \( V_i \) are respectively the longitudinal and transverse wave velocities in the semispace; \( \beta = \sqrt{1 - \alpha^2}, \gamma = \xi = \sqrt{\xi^2 - \alpha^2} \)
and \( \Delta_r = (2\alpha^2 - 1)^2 - 4\alpha^2 \beta \gamma \) is the characteristic equation of Rayleigh wave in the semispace material. Note that Eq. (6.9) does not yield a real solution since \( \beta = i\sqrt{\alpha^2 - 1} \) is a positive imaginary number for a propagating mode (\( \alpha > 1 \)).

When the interface wave propagation direction deviates from the symmetric axis of interface imperfections, the B.C. are described by (6.6) and the dispersion equation for the antisymmetric mode becomes a single coupled equation:

\[
\Delta^a(K_V, K_H, K_c, M) = \Delta_0^a(K_V, M)\Delta_{0H}^a(K_H) + 4K_c^2(2\beta \mu k_t - ia_1\omega^2 M) = 0. \tag{6.10}
\]

Here \( \Delta_0^a(K_V, M) \) and \( \Delta_{0H}^a(K_H) \) have the same forms as the characteristic functions (6.8) and (6.9) except the stiffnesses \( K_V \) and \( K_H \) in the general form of (6.5) are used. The terms involving \( K_c \) in Eq. (6.10) give the effect of the coupling between the \((x, z)\) plane and out-of-plane motions on the localized interface wave. One sees from Eqs. (6.8 - 6.10) that the normal stiffness \( K_n \) is not involved in the localization of antisymmetric modes.

For the symmetric mode the dispersion equation is

\[
\Delta^s(K_n, M) = \Delta_0^s(K_n, M)\Delta_{0H}^s(M) = 0, \tag{6.11}
\]

where

\[
\Delta_0^s = 2K_n(2i\gamma \mu k_t + a_1\omega^2 M) + 2\Delta_r\mu^2 k_t^2 - i\beta \omega^2 M \mu k_t, \tag{6.12}
\]

and

\[
\Delta_{0H}^s = 2\mu k_t \beta - i\omega^2 M. \tag{6.13}
\]
One sees from Eq. (6.11) that none of the shear stiffnesses are involved in the equation. Since \( K_n = K_n^0 \), the effect of the interface anisotropy is not reflected in the symmetric interface mode. In addition the elastic motions in the \((x, z)\) plane and along the \(y\)-direction are decoupled. Unlike the antisymmetric SH mode Eq. (6.13) does yield a real solution for the symmetric SH mode and is only dependent on \( M \).

The scattering coefficients have been obtained in two parts: symmetric and antisymmetric parts. The antisymmetric part is:

\[
R_{ii}^a = -\frac{\Delta^a(-\gamma)}{\Delta^a(\gamma)}
\]

\[
R_{HH}^a = -\frac{\Delta^a\Delta^a_{0H}(-\beta) + 4K_c^2(2\beta\mu k_i - ia_1\omega^2 M)}{\Delta^a_0\Delta^a_{0H}(\beta) + 4K_c^2(2\beta\mu k_i - ia_1\omega^2 M)}
\]

\[
R_{VV}^a = -\frac{\Delta^a(-\beta)\Delta^a_{0H} - 4K_c^2(2\beta\mu k_i + ia_1\omega^2 M)}{\Delta^a_0(\beta)\Delta^a_{0H} + 4K_c^2(2\beta\mu k_i - ia_1\omega^2 M)}
\]

\[
R_{HI}^a = -\frac{4\alpha^2\beta^2}{\Delta^a_0}\mu k_i(4i\mu k_i\beta + \omega^2 M)K_c
\]

\[
R_{VH}^a = \frac{4i\beta}{\Delta^a_0}\mu k_i(2\mu k_i W + i\gamma\omega^2 M)K_c
\]

\[
R_{VI}^a = -\frac{4i\alpha^2\beta^2}{\Delta^a_0}[\omega^2 M(2K_V K_H - 2K_c^2 - i\mu k_i\beta K_V) + 2\mu^2 k_i^2 W(2K_H - i\mu k_i\beta)].
\]

The scattering coefficients for the symmetric part are

\[
R_{ii}^s = -\frac{\Delta^s(-\gamma)}{\Delta^s(\gamma)}; \quad R_{VV}^s = -\frac{\Delta^s_0(-\beta)}{\Delta^s_0(\beta)}
\]

\[
R_{HH}^s = -\frac{\Delta^s_{0H}(-\beta)}{\Delta^s_{0H}(\beta)} = \frac{2\mu k_i\beta + i\omega^2 M}{2\mu k_i\beta - i\omega^2 M}
\]

\[
R_{HI}^s = 0; \quad R_{VH}^s = 0
\]

\[
R_{VI}^s = -\frac{4\alpha\beta^2}{\Delta^s}(\omega^2 M K_n + 2\mu^2 k_i^2 W).
\]
In the above equations the non-dimensional parameters can be written in terms of incident angles as \( \alpha = \sin(\theta_i) \), \( \beta = \cos(\theta_i) \), \( \gamma = \xi \cos(\theta_i) \), \( W = -\cos(2\theta_i) \) and \( \Delta_r = \cos^2(2\theta_i) - 4\xi \sin^2(\theta_i) \cos(\theta_i) \cos(\theta_i) \).

Other scattering coefficient equations that are not listed above can be found from the following equalities:

\[
R_{iH}^{(a,s)} = \frac{\gamma}{\beta \xi^2} R_{Hi}^{(a,s)}, \quad R_{iV}^{(a,s)} = -\frac{\gamma}{\beta \xi^2} R_{Vi}^{(a,s)}, \quad R_{HV}^{(a,s)} = -R_{VH}^{(a,s)}. \tag{6.16}
\]

These equalities result from the reciprocity of the energy conversion coefficients between different modes [46].

The total scattering matrices for the reflected \( \mathbf{R} \) and transmitted \( \mathbf{T} \) waves are the sums of the symmetric and antisymmetric scattering matrices:

\[
\mathbf{R} = \frac{1}{2} \mathbf{R}^s + \frac{1}{2} \mathbf{R}^a, \tag{6.17}
\]
\[
\mathbf{T} = \frac{1}{2} \mathbf{T}^s + \frac{1}{2} \mathbf{T}^a = -\frac{1}{2} \mathbf{R}^a + \frac{1}{2} \mathbf{R}^s. \tag{6.18}
\]

Next consider the spring B.C. model where the mass elements \( M \) are neglected in (6.6). In this case the stresses are continuous across the interface. In general the B.C. can be written as

\[
\sigma_{zx} = \sigma'_{zx} = K_V(u_x - u'_x) + K_c(u_y - u'_y), \]
\[
\sigma_{zy} = \sigma'_{zy} = K_c(u_x - u'_x) + K_H(u_y - u'_y), \tag{6.19}
\]
\[
\sigma_{zz} = \sigma'_{zz} = K_n(u_z - u'_z).
\]

For interfaces with B.C. (6.19) where the effect of mass is neglected, the dispersion
equation for an antisymmetric mode is

$$\Delta^a(K_V, K_H, K_c) = (2i\beta K_V^0 + \mu k_l \Delta_r)(\mu k_l \beta + 2i K_H^0) + 4K_c^2 \beta = 0,$$  \hspace{1cm} (6.20)  

and the dispersion equation for the symmetric mode is

$$\Delta^s = 2i\gamma K_n + \mu k_l \Delta_r.$$  \hspace{1cm} (6.21)  

This depends only on the normal stiffness $K_n$. Since the shear stresses of the symmetric mode vanish, there is no localization of the SH-type mode.

The scattering coefficients from the interfaces with B.C. (6.19) can be found by substituting $M = 0$ in Eqs. (6.14) and (6.15) as

$$R_{ll}^a = -\frac{\Delta^a(-\gamma)}{\Delta^a(\gamma)},$$  

$$R_{HH}^a = -\frac{\Delta^a_{0} \Delta^a_{0H}(-\beta) + 8\beta \mu k_l K_c^2}{\Delta^a_{0} \Delta^a_{0H}(\beta) + 8\beta \mu k_l K_c^2},$$  

$$R_{VV}^a = -\frac{\Delta^a_{0}(-\beta) \Delta^a_{0H} - 8\beta \mu k_l K_c^2}{\Delta^a_{0}(\beta) \Delta^a_{0H} + 8\beta \mu k_l K_c^2},$$  

$$R_{Hl}^a = -\frac{16i\alpha \beta^2 \xi}{\Delta^a} \mu^2 k_l^2 K_c$$  \hspace{1cm} (6.22)  

$$R_{VH}^a = \frac{8i\beta W}{\Delta^a} \mu^2 k_l^2 K_c$$  

$$R_{vl}^a = -\frac{8i\alpha \beta W \xi}{\Delta^a} \mu^2 k_l^2 (2K_H - i\mu k_l \beta)]$$  

for the antisymmetric part and

$$R_{ll}^s = -\frac{\Delta^s(-\gamma)}{\Delta^s(\gamma)}; \quad R_{VV}^s = -\frac{\Delta^s(-\beta)}{\Delta^s(\beta)}$$  

$$R_{HH}^s = -1; \quad R_{Hl}^s = 0; \quad R_{VH}^s = 0$$  \hspace{1cm} (6.23)  

$$R_{vl}^s = -\frac{8i\alpha \beta W \xi}{\Delta^s} \mu^2 k_l^2$$
for the symmetric part.

6.3 Numerical Results for Wave Scattering and Stoneley Wave Localization by an Interface with a Periodic Array of Horizontal Strip Cracks

To demonstrate the effect of the interface imperfection orientation on wave scattering and Stoneley wave localization, we give here a detail analysis for a fracture interface with a periodical array of horizontal strip cracks between two similar isotropic semispaces. The structure of the cracked interface is shown in Fig. 6.2(a) in the \((x^0, z)\) plane and (b) in the interface plane \((x^0, y^0)\). The cracks are assumed to be thin in the \(z\)-direction thus the inertial effect of the cracks is neglected. \(s\) is the center-to-center spacing between adjacent cracks, \(w\) is the width of the uncracked portion and \(A = (s-w)/s\) is the crack area fraction in the \((x^0, y^0)\) interface plane. Margetan et al have obtained the spring stiffness constants by calculating the deformation of the cracked interface under static loading, and given the spring B.C. in the material coordinate system \((x^0, y^0, z)\) as [27]:

\[
\begin{pmatrix}
\sigma_{z0}^0 \\
\sigma_{y0}^0 \\
\sigma_{zz}
\end{pmatrix}
= \begin{pmatrix}
\sigma_{x0}^0 \\
\sigma_{y0}^0 \\
\sigma_{zz}
\end{pmatrix}' = K^0 \begin{pmatrix}
u_z^0 - u_z^0 \\
u_{y0}^0 - u_{y0}^0 \\
u_z - u_z'
\end{pmatrix}
= \begin{bmatrix}
K & 0 & 0 \\
0 & (1-\nu)K & 0 \\
0 & 0 & K
\end{bmatrix}
\begin{pmatrix}
u_z^0 - u_z^0 \\
u_{y0}^0 - u_{y0}^0 \\
u_z - u_z'
\end{pmatrix},
\]

(6.24)

where

\[
K = \frac{\pi}{4s} \frac{E}{1-\nu^2} \left\{ \ln \left[ \sec \left( \frac{\pi A}{2} \right) \right] \right\}^{-1}.
\]

(6.25)
crack area fraction: \( A = (s - w)/s \)

Figure 6.2: An interface with a periodic horizontal strip cracks. The structure (a) in the \((x^0, z)\) plane and (b) in the \((x^0, y^0)\) plane. \( s \) is the center-to-center spacing between adjacent cracks, \( w \) is the width of the uncracked portion and \( A \) is the crack area fraction in the \((x^0, y^0)\) interface plane.
Here, $E$ and $\nu$ are Young's modulus and Poisson's ratio for the semispace material. One sees that the stiffness along the crack direction is smaller than those perpendicular to the crack direction, i.e. $K'_H = (1 - \nu)K$, $K'_V = K'_x = K$.

To calculate the scattering coefficients and dispersion curves of interface modes on an interface with arbitrarily oriented cracks, one needs to use the generalized spring B.C. in the coordinate that consists of the incident plane $(x, z)$ and the interface plane as discussed in previous Section. For an arbitrary crack orientation $\varphi$ as shown in Fig. 6.1 the generalized spring B.C. can be found from Eq. (6.5) as

$$
\begin{pmatrix}
\sigma_{xz} \\
\sigma_{zy} \\
\sigma_{zz}
\end{pmatrix} =
K
\begin{pmatrix}
1 - \nu \sin^2 \varphi & -\nu \sin \varphi \cos \varphi & 0 \\
-\nu \sin \varphi \cos \varphi & 1 - \nu \cos^2 \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
u_x - \nu'_x \\
u_y - \nu'_y \\
u_z - \nu'_z
\end{pmatrix}.
$$

(6.26)

Fig. 6.3 shows dependencies of different shear stiffnesses (normalized by $K$) on the incident plane deviation angle $\varphi$. The material properties of the semispace are $E = 75$ Gpa, $\nu = 0.3$ Km/s and density $\rho = 2.70$ g/cc. One sees from the figure that the normalized in-plane and out-of-plane shear stiffnesses $K_V/K$ and $K_H/K$ vary their values between $1 - \nu$ and 1, while the maximum absolute value for the normalized coupling shear spring $K_c$ is $\nu/2$ at $\varphi = 45^\circ$. Since the greatest difference between $K'_V$ and $K'_H$ for this example is 30%, the interface anisotropy is relatively small. Next we will give numerical results for wave scattering and dispersion of interface modes on this cracked interface.
Figure 6.3: Normalized shear stiffnesses $K_V$, $K_H$ and $K_c$ as a function of incident plane deviation angle $\varphi$. The Poisson's ratio for the semispace material is 0.3.
6.3.1 The scattering problem

First we consider mode conversion of an obliquely incident longitudinal wave into reflected and transmitted SH waves. The crack parameters are crack area ratio $A = 0.5$ and crack spacing $s = 0.2$ mm. In this case $K = 9.3 \times 10^{14}$ N/m$^3$ which corresponds to a relative weak interface. The excitation of the SH wave is due purely to the asymmetry of the interface about the incident plane. Note that the mode converted SV and SH waves have the same speed due to the isotropy of the semispase, and physically constitute a single transverse wave with its polarization vector rotated from that of the incident SH wave. However, to demonstrate the effect of coupling between the in-incident plane and out-of plane parts of the elastic field due to interface anisotropy, we treat them separately here as two different transverse waves. The equations for reflection and transmission coefficients of the SH waves can be found from Section 6.2 as:

$$T_{lH} = -R_{lH} = \frac{2i \sin 2\theta_t \cos \theta_t Z_t \omega K_c}{(2i \cos \theta_t K_V + Z_t \omega \Delta_t)(\cos \theta_t Z_t \omega + 2i K_H) + 4 \cos \theta_t K_c^2}$$

(6.27)

where $Z_t = \rho V_t$ is the shear wave impedance of the semispase material, $\theta_t$ and $\theta_l$ are the scattered angles of shear and longitudinal waves. One sees that the mode conversion coefficients are approximately proportional to the shear coupling spring $K_c$. The equality between the reflection and transmission coefficients for the SH waves is due to the requirement of continuity of the out-of plane shear stress on
the interface.

The amplitudes of the reflection coefficients $R_u$ and $R_{IH}$ are plotted in Fig. 6.4 versus incident plane deviation angle $\varphi$ for an incident longitudinal wave at incident angle $\theta_i = 45^\circ$. Here $f = 15$ MHz is the frequency. One sees from Fig. 6.4 that the reflected longitudinal wave gradually decreases in amplitude as the wave-incident plane moves from that perpendicular to crack direction to parallel to crack direction. In contrast the mode conversion to the SH mode increases sharply initially as $\varphi$ increases then slowly reaches its maximum around $\varphi = 45^\circ$, and decreases to zero again at $\varphi = 90^\circ$. Since the anisotropy of the cracked interface is relatively small, the mode conversion to the SH wave is relatively small. The scattering coefficients $R_u$, $T_u$ and $R_{IH}$ versus incident angle $\theta_i$ at $\varphi = 45^\circ$, where the coupling shear stiffness $K_c$ is at is maximum value ($K_c = -0.15$ $K$), are shown in Fig. 6.5.

The conversion coefficients for an incident SV wave into reflected and transmitted SH waves can be found as:

$$T_{VH} = -R_{VH} = \frac{2i \cos 2\theta_i \cos \theta_i Z_t \omega K_c}{(2i \cos \theta_i K_V + Z_t \omega \Delta_r)(\cos \theta_i Z_t \omega + 2i K_H) + 4 \cos \theta_i K_c^2}. \quad (6.28)$$

The mode conversion to the SH wave is purely due to the coupling of the incident plane shear displacement and stress to the out-of-plane shear stress and displacement. Once again the conversion coefficients are directly related to the shear coupling spring $K_c$. Fig. 6.6 shows the conversion coefficients $R_{VH}$, $R_{VV}$
Figure 6.4: Reflection coefficients $R_H$ and $R_{HH}$ of an incident longitudinal wave on a cracked interface vs incident plane deviation angle $\phi$. $f = 15$ MHz and the incident angle $\theta_i = 45^\circ$. 

- $f = 15$ MHz
- $K = 9.3 \times 10^{14}$ N/m$^3$
- $\theta_i = 45^\circ$
\[ f = 15 \text{ MHz} \]
\[ K = 9.3 \times 10^{14} \text{ N/m}^3 \]
\[ K_c = -0.15 \text{ K} \]

Figure 6.5: Reflection coefficients \( R_{ll} \) and \( R_{lh} \) and transmission coefficient \( T_{ll} \) of an incident longitudinal wave on a cracked interface vs incident angle \( \theta_i \) at \( \varphi = 45^\circ \).
and $T_{VV}$ as a function of incident angle $\theta_i$ at $\varphi = 45^\circ$. One sees from the figure that the reflection amplitude of the SH wave is close to that of the SV wave at a particular angle range (between $20^\circ$ to $30^\circ$ in the figure). One may note that the mode conversion to the SH wave at incident angle $\theta_i = 45^\circ$ is zero. This is because that this angle the excited in-plane shear stress of an incident SV wave is zero, thus there exists no excitation of the SH wave.

To illustrate the effect of interface anisotropy on the mode conversion between SV and SH transverse waves, we calculate the conversion coefficients $R_{VH}$ and $R_{VV}$ for an incident SV wave at normal incidence as a function of $\varphi$. The results are shown in Fig. 6.7. One sees that in plane of symmetry ($\varphi = 0^\circ$ or $90^\circ$) there is no mode conversion because there is no coupling between the in-plane and out-of-plane elastic fields ($K_c = 0$), while at $\varphi = 45^\circ$, where the interfacial coupling between in-plane and out-of-plane shear components is the strongest, the mode conversion coefficient reaches its maximum value. The reflected SV and SH waves have the same speed and can not physically be separated, as a result the reflect a single shear wave with modified polarization is scattered. Compared to the incident SV wave, the polarization of the scattered shear wave has an out-of-plane displacement component. The out-of-plane angle of the polarization for the scattered shear wave is dependent on the amplitude ratio of the reflected SV and SH waves, and can be found as

$$\psi = \arctan\left(\frac{-2i K_c}{2i K_H + Z_I \omega}\right).$$  \hfill (6.29)
\[ f = 15 \text{ MHz} \]
\[ K = 9.3 \times 10^{14} \text{ N/m}^3 \]
\[ K_e = -0.15 \text{ K} \]

Figure 6.6: Reflection coefficients \( R_{VV} \) and \( R_{VH} \) and transmission coefficient \( T_{VV} \) of an incident SV wave on a cracked interface vs incident angle \( \theta_i \) at \( \varphi = 45^\circ \).
Figure 6.7: Reflection coefficients $R_{VV}$ and $R_{VH}$ of a normally incident SV wave on a cracked interface vs incident plane deviation angle $\varphi$. 

- $f = 15$ MHz
- $K_t = 9.3 \times 10^{14}$ N/m$^3$
Fig. 6.8 shows out-of-plane angle component of the polarization for a scattered shear wave as a function of \( \varphi \) for an incident SV wave at normal incidence. In this case the cracked interface works like a polarizer as in optics, if forces a polarization change in the scattered wave due to the interface anisotropy. One sees from the figure that the maximum out-of-plane angle component for the polarization of the scattered shear wave is about 9 degrees which occurs around \( \varphi = 45^\circ \). The same phenomenon can be observed for scattered shear waves when the waves are incident at oblique angles.

### 6.3.2 The interface wave problem

In order for interface mode to be effectively localized on an imperfect interface, the interface has to have low stiffnesses [4]. Here we calculate the dispersion curves of the guided modes localized in the interface with cracked area fraction \( A = 0.8 \) \((K = 9.6 \times 10^{11} \text{ N/m}^3)\). Recall that the dispersion equation for an antisymmetric mode along the \( x \)-direction is:

\[
\Delta^a = (2i\beta K_v + Z_l \omega \Delta_r)(\beta Z_l \omega + 2iK_H) + 4i\beta K_v^2 = 0, \tag{6.30}
\]

where \( \beta = \sqrt{1 - (V_f/V_i)^2} \). The normalized antisymmetric interface velocity \( V/V_i \) is given in Fig. 6.9 as a function of \( \varphi \) at \( f = 15 \text{ MHz} \). The interface wave normal deviates from the \( x^0 \)-direction by angle \( \varphi \). One sees from Fig. 6.9 that the normalized interface wave velocity only changes slightly from 0.956 to 0.948 as the wave changes its propagation from perpendicular to parallel to the crack orientation.
Figure 6.8: The angle change of the polarization of the reflected shear wave for a normally incident SV wave on a cracked interface vs incident plane deviation angle $\varphi$. 

$f = 15$ MHz

REFLECTED SHEAR WAVE
Figure 6.9: Normalized antisymmetric interface wave speed as a function of $\varphi$ ($A = 0.8$, $s = 0.2$ mm, $f = 15$ MHz).
This is due to a small continuous drop from $K$ to $(1 - \nu)K$ in the shear stiffness $K_V$ as $\varphi$ increases. For the plane of symmetry when $K_c = 0$ the wave velocity of the in-plane antisymmetric mode is only dependent on $K_V$, this relation has been used for characterization of the in-plane shear stiffness in adhesives between solids by monitoring the interface wave speed, as demonstrated in [24, 25].

The dispersion equation for a symmetric interface mode propagating along the $x$-direction is:

$$\Delta^s = 2i\gamma K_n + Z_i\omega \Delta_r = 0, \quad (6.31)$$

where $\gamma = \sqrt{(V_i/V_t)^2 - (V_i/V)^2}$. One sees the wave speed is dependent only on $K_n$ thus independent of the wave propagation direction. The cut-off frequency of the symmetric mode can be found by substituting $V = V_t$, $\gamma = \sqrt{(V_i/V_t)^2 - 1}$, $\Delta_r = 1$ into Eq. (6.31) as

$$f_c = \sqrt{1 - \frac{V_t^2}{V_i^2}} \frac{K_n}{\pi Z_t}. \quad (6.32)$$

By measuring the interface wave speed of the symmetric mode, one can obtain the normal stiffness of the cracked interface. More detailed discussion of the localization conditions can be found in Chapter V. Fig. 6.10 shows the normalized wave speed of the symmetric mode as a function of frequency for the same cracked interface as in Fig. 6.9. One sees from the figure that there exists a cut-off frequency at $f = 7$ MHz and the wave speed drops rapidly as frequency increases.
Figure 6.10: Normalized symmetric interface wave speed as a function of frequency ($A = 0.8$, $s = 0.2$ mm).
6.4 Summary

For an anisotropic fractured interface with arbitrarily oriented cracks the generalized spring B.C. are introduced. The spring stiffness elements can be obtained from those defined for plane of symmetry. An additional coupling shear spring appears in the B.C. The generalized shear springs depend on the deviation angle of the incident plane from the plane of symmetry. Analytical solutions for wave scattering and dispersion equations for interface modes on such an interface are derived using the generalized spring B.C. (including mass terms). They can be applied to predict the effect of interface imperfection orientation on wave scattering and interface mode dispersion. Numerical examples have been given for an interface with a periodic array of horizontal strip cracks between two identical semispaces. Mode conversions to SH waves and as a result angle changes in polarization of the scattered shear waves are discussed.
Part II

Ultrasonic scattering from multilayered fiber-matrix interphases. Application to ultrasonic characterization of interphase in composites
The second part of this dissertation focuses on extending to cylindrical interphases the transfer matrix and approximate boundary condition approaches presented in Part I for plane interphases to cylindrical interphases. I start with scattering from a single multilayered fiber and an analysis of different boundary condition models for a thin fiber-matrix interphase. Then wave attenuation and dispersion in composites is studied theoretically and experimentally with an emphasis on the effect of fiber-matrix interphases. These are important since both ultrasonic scattering (attenuation) and wave velocity could be utilized for interphase characterization are shown schematically in Fig. 1.3. The results presented here will help to establish multiple scattering models for the study of wave attenuation in composites, and dispersion models to analyze dynamic effect for determination of the composite elastic moduli from ultrasonic phase velocities.

The organization of part II is as follows. A transfer matrix method for wave scattering from a multilayered fiber in a solid matrix is developed in Chapter VII. This is important since in modern composites both the fibers and the fiber-matrix interphases can have multilayered structures. Spring and asymptotic boundary conditions will be derived in Chapter VIII to represent the thin fiber-matrix interphase and to study the subsequent effect on ultrasonic wave scattering from the fibers. Such boundary condition models will also be obtained alternatively from the solution of the governing differential equation for the elastic field in a radially-dependent cylindrical medium. Chapter IX discusses simplified fiber and
boundary condition models for low frequency ultrasonic wave scattering from a multiphase fiber in solid, and addresses a resonance phenomenon for extremely compliant fiber-matrix interphases.

The results for a single fiber in a matrix will be further extended to composites in Chapters X and XI where wave scattering, attenuation and velocity are investigated both theoretically and experimentally. Different models for wave dispersion and attenuation in nonhomogeneous media will be extended to a multiphase unidirectional composite for waves propagating along (Chapter XI) and normal to (Chapter X) the fibers. These models will be analyzed by numerical examples and by comparison with the experimental data measured for a SiC/titanium composite. Finally Chapter XII gives preliminary results of nondestructive monitoring of fatigue-induced damage on the fiber-matrix interphase in high-temperature composites using both ultrasonic wave velocity and attenuation data.
CHAPTER VII

WAVE SCATTERING FROM A MULTILAYERED FIBER IN SOLID. TRANSFER MATRIX APPROACH

7.1 Introduction

Although much work has been done for elastic wave scattering from a cylindrical inclusion in a solid [61, 62, 63, 64, 65, 66] (for a comprehensive list of early publications, see [66]), almost all previous studies address scattering of an elastic wave on a single phase solid cylinder perfectly bonded to the solid matrix. In our own work [67, 68, 69] scattering from multilayered fibers with focus on the effect of fiber-matrix interphases regions has been reported. I will discuss in this chapter the transfer matrix solution for scattering from a multilayered fiber in matrix, followed by a detailed analysis of different B.C. models for representation of a thin fiber-matrix interphase using the transfer matrix formalism in the next chapter.

The treatment of wave scattering by multilayered cylinders is important for ultrasonic characterization of modern high temperature composites where both
the fibers and the fiber-matrix interphases can have multilayered structures due to design and to the manufacturing process. In a recent paper [68] the wave scattering from a three-phase fiber in a solid matrix was obtained by solving a $12 \times 12$ system of boundary equations, since there are four unknown coefficients in each intermediate cylindrical layer, two propagating inward and the other two propagating outward. If one need consider more layers the analysis and computations become tedious. More recently the analysis was extended to multilayered fibers with an arbitrary number of phases using a transfer matrix approach [69]. The transfer matrix formalism has been widely used for multilayered isotropic and anisotropic plane layered system [2, 3, 5, 13, 37, 38, 70] as discussed in Part I and for multilayered solids with spherical interfaces [71, 72].

Let us consider a time-harmonic, plane longitudinal or transverse wave (polarized perpendicular to the fiber axis) incident normally on a multilayered fiber in a solid matrix as shown in Fig. 7.1. The unit scalar $\phi^{in}$ and vector $\psi_x^{in}$ potentials for the incident longitudinal and transverse waves are expanded in Bessel series

$$
\begin{pmatrix}
\phi^{in} \\
\psi_x^{in}
\end{pmatrix} = \sum_{n=-\infty}^{\infty} (i)^n J_n \left( \frac{k_i^N}{k_N^i} r \right) e^{in\theta},
$$

(7.1)

where $k_i^N$ and $k_N^i$ are the wave numbers of the longitudinal and transverse modes in matrix material (phase $N$) respectively; $J_n(\cdot)$ is the $n$th order Bessel function of the first kind. The time-dependent factor $e^{-jwt}$ is omitted throughout.
Figure 7.1: Scattering from an $N-1$-phase multilayered cylinder in a solid matrix. $k^N$ is the incident wave number in the matrix material (phase $N$)
The scattered waves in the matrix are represented as outgoing waves:

\[
\varphi^s = \sum_{n=-\infty}^{\infty} A^N_n H_n(k^N_l r) \exp(\imath n\theta), \quad \psi^s = \sum_{n=-\infty}^{\infty} B^N_n H_n(k^N_l r) \exp(\imath n\theta),
\]

where \( H_n(x) = J_n(x) + i N_n(x) \) is the \( n \)th order Hankel function of the first kind. Thus the total potential fields in the surrounding matrix are \( \varphi^N = \varphi^s + \varphi^{in} \) and \( \psi^N = \psi^s + \psi^{in} \).

The fields in the fiber core \((r < r_1)\) should be finite on the axis \((r = 0)\) and thus have the following forms:

\[
\varphi^1 = \sum_{n=-\infty}^{\infty} A^1_n J_n(k^1_l r) \exp(\imath n\theta), \quad \psi^1 = \sum_{n=-\infty}^{\infty} B^1_n J_n(k^1_l r) \exp(\imath n\theta),
\]

where \( k^1_l \) and \( k^1_t \) are the wave numbers of the longitudinal and transverse modes in the fiber core material (phase 1). The Neumann functions \( N_n(k r) \) are singular at \( r = 0 \), thus do not appear in equation (7.3).

In the intermediate interphase \( j \) between the fiber core and the matrix both incoming and outgoing scattered waves exist and have the following form:

\[
\varphi^j = \sum_{n=-\infty}^{\infty} \left[ A^j_n J_n(k^j_l r) + C^j_n N_n(k^j_l r) \right] \exp(\imath n\theta), \quad (7.4)
\]

\[
\psi^j = \sum_{n=-\infty}^{\infty} \left[ B^j_n J_n(k^j_l r) + D^j_n N_n(k^j_l r) \right] \exp(\imath n\theta), \quad (7.5)
\]

where \( k^j_l \) and \( k^j_t \) are the wave numbers of the longitudinal and transverse modes in the \( j \)th phase.

Using the relations between the displacements and stresses and the scalar and
vector potentials

\[
\begin{align*}
u_r &= \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \psi_z}{\partial \theta}, \\
u_\theta &= \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi_z}{\partial r}, \\
\sigma_{rr} &= \lambda \Delta^2 \varphi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi_z}{\partial \theta} \right) \right], \\
\sigma_{r\theta} &= 2\mu \left[ \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{1}{r^2} \frac{\partial \varphi}{\partial \theta} \right] + \mu \left[ \frac{1}{r^2} \frac{\partial^2 \psi_z}{\partial \theta^2} - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi_z}{\partial r} \right) \right],
\end{align*}
\]

where \( \lambda \) and \( \mu \) are the Lamé constants, we can write the displacements and stresses in any intermediate cylindrical layer (phase \( j \): \( r_{j-1} \leq r \leq r_j \)) in matrix form as:

\[
(u^i_r, u^i_\theta, \sigma^i_{rr}, \sigma^i_{r\theta})^T = \begin{bmatrix} K^j_n \cdot (A^j, B^j, C^j, D^j)^T \end{bmatrix} e^{in\theta},
\]

where the \( 4 \times 4 \) matrix \( K^j_n \), which relates the displacements and stresses to the coefficients \( (A^j, B^j, C^j, D^j)^T \) of the wave solutions in intermediate phase \( j \) (\( r_{j-1} \leq r \leq r_j \)), is

\[
K^j_n = \begin{pmatrix}
\frac{i}{r} J_n(k^j_l r) & -k^i_l J_n(k^i_l r) \\
k^j_l J'_n(k^j_l r) & -\frac{i}{r} J_n(k^i_l r) \\
2i\mu^j \frac{n}{r^2}[k^j_l r J'_n(k^j_l r) - J_n(k^j_l r)] & -\mu^j(k^j_l)^2[2J''_n(k^j_l r) + J_n(k^j_l r)] \\
(k^j_l)^2[2\mu^j J''_n(k^j_l r) - \lambda^j J_n(k^j_l r)] & 2i\frac{n\mu^j}{r^2}[k^j_l r J'_n(k^j_l r) - J_n(k^j_l r)] \\
\frac{i}{r} N_n(k^j_l r) & -k^i_l N_n(k^i_l r) \\
k^j_l N'_n(k^j_l r) & -\frac{i}{r} N_n(k^i_l r) \\
2i\mu^j \frac{n}{r^2}[k^j_l r N'_n(k^j_l r) - N_n(k^j_l r)] & -\mu^j(k^j_l)^2[2N''_n(k^j_l r) + N_n(k^j_l r)] \\
(k^j_l)^2[2\mu^j N''_n(k^j_l r) - \lambda^j N_n(k^j_l r)] & 2i\frac{n\mu^j}{r^2}[k^j_l r N'_n(k^j_l r) - N_n(k^j_l r)]
\end{pmatrix}
\]

where \( \lambda^j, \mu^j \) are the Lamé constants, and \( k^j_l, k^i_l \) are the wave numbers of the intermediate phase \( j \).
7.2 The Transfer Matrix Representation for a Cylindrical Layer

The displacement-stress fields on the inner \((r = r_{j-1})\) and outer \((r = r_j)\) boundaries of the \(j\)th layer thus can be written as

\[
(u_r^j, u_\theta^j, \sigma_{rr}^j, \sigma_{\theta\theta}^j)_T |_{r=r_{j-1}} = [K_n^j |_{r=r_{j-1}} \cdot (A_i^j, B_i^j, C_i^j, D_i^j)_n] e^{i\alpha}, \quad (7.12)
\]

\[
(u_r^j, u_\theta^j, \sigma_{rr}^j, \sigma_{\theta\theta}^j)_T |_{r=r_j} = [K_n^j |_{r=r_j} \cdot (A_i^j, B_i^j, C_i^j, D_i^j)_n] e^{i\alpha}. \quad (7.13)
\]

From equation (7.12) for the field on the inner boundary of the layer one can obtain the unknown wave coefficients as

\[
(A_i^j, B_i^j, C_i^j, D_i^j)_n = [(K_n^j |_{r=r_j})^{-1} \cdot (u_r^j, u_\theta^j, \sigma_{rr}^j, \sigma_{\theta\theta}^j)_T |_{r=r_{j-1}}] e^{-i\alpha}. \quad (7.14)
\]

Substituting them into equation (7.13) one can relate the displacements and stresses on the outer boundary of the layer to those on the inner boundary by a transfer matrix \(T_n^j\) as:

\[
(u_r^j, u_\theta^j, \sigma_{rr}^j, \sigma_{\theta\theta}^j)_T |_{r=r_j} = K_n^j |_{r=r_j} (K_n^j |_{r=r_{j-1}})^{-1} \cdot (u_r^j, u_\theta^j, \sigma_{rr}^j, \sigma_{\theta\theta}^j)_T |_{r=r_{j-1}}
\]

\[
= T_n^j \cdot (u_r^j, u_\theta^j, \sigma_{rr}^j, \sigma_{\theta\theta}^j)_T |_{r=r_{j-1}}, \quad (7.15)
\]

where

\[
T_n^j = K_n^j |_{r=r_j} (K_n^j |_{r=r_{j-1}})^{-1}. \quad (7.16)
\]

The transfer matrix \(T_n^j\) is a function of the material properties and geometry of the \(j\)th cylindrical layer.
Using the transfer matrix for each intermediate cylindrical layer, the stresses and displacements on the outer boundary of the multilayered cylinder \((r = r_{N-1})\) can be directly related to those in the fiber core \((r = r_1)\) as

\[
(u_r^N, u_\theta^N, \sigma_{rr}^N, \sigma_{r\theta}^N)^T |_{r=r_{N-1}} = (u_r^{N-1}, u_\theta^{N-1}, \sigma_{rr}^{N-1}, \sigma_{r\theta}^{N-1})^T |_{r=r_{N-1}} \\
= T_n^{N-1} T_n^{N-2} \ldots T_n^2 \cdot (u_r^2, u_\theta^2, \sigma_{rr}^2, \sigma_{r\theta}^2)^T |_{r=r_1} = T_n \cdot (u_r^1, u_\theta^1, \sigma_{rr}^1, \sigma_{r\theta}^1)^T |_{r=r_1}
\]

(7.17)

where \(T_n\) is the total transfer matrix

\[
T_n = T_n^{N-1} T_n^{N-2} \ldots T_n^2 = \prod_{j=1}^{N-2} T_n^{N-j}.
\]

(7.18)

### 7.3 The Transfer Matrix Solution in the Exponential Form

Next we give an alternative approach to derive the transfer matrix. For a cylindrical system whose properties have only radial dependence, one can obtain a governing matrix differential equation for elastic field in a radially-dependent cylindrical system by combining Newton's second law, the wave equation and Hooke's law
or

\[ \frac{\partial \mathcal{U}_n(r)}{\partial r} = t_n(r) U_n(r) \]  

(7.20)

where \( \lambda, \mu \) and \( \rho \) are functions of radius \( r \), \( \mathcal{U}_n(r) \) denotes the displacement and stress vector \( (u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta})^T_n \), and \( t_n(r) \) is the characteristic matrix of the elastic field at position \( r \).

Equation (7.20) has a well-known exponential solution:

\[ \mathcal{U}_n(r + \Delta r) = \exp\{ \int_r^{r+\Delta r} t_n(r) dr \} \mathcal{U}_n(r). \]  

(7.21)

The exponential matrix function can be represented in the form of series [44] and the transfer matrix for an intermediate layer \( (r_{j-1} \leq r \leq r_j) \) can be found as

\[ T_n^j = \exp\{ \int_{r_{j-1}}^{r_j} t_n^j(r) dr \} = I + \int_{r_{j-1}}^{r_j} t_n^j(r) dr + \int_{r_{j-1}}^{r_j} t_n^j(r) dr \int_{r_{j-1}}^{\tau} t_n^j(\tau) d\tau + \cdots \]  

(7.22)

Recall that equivalently we have found in (7.16) \( T_n^j = K_n^j|_{r=r_j} = (K_n^j|_{r=r_{j-1}})^{-1} \). The representation of the transfer matrix in the series form (7.22) is more convenient for deriving asymptotic B.C. models which we will address in the next Chapter.

7.4 Solution for the Scattering Field

Now we can write the final B.C. equation for the scattering problem. Using the expressions for the incident and scattered waves (7.1), (7.2) and (7.10) we can write the displacement-stress field on the outer boundary of the cylinder as:

\[ (u_r^N, u_\theta^N, \sigma_{rr}^N, \sigma_{r\theta}^N)^T|_{r=r_{N-1}} = K_n^N|_{r=r_{N-1}} (A_n^N + i_n^N, B_n^N + i_n^N, i A_n^N, i B_n^N) e^{in\theta}. \]  

(7.23)
For the wave field in the fiber core (7.3), we have

\[(u_r^1, u_\theta^1, \sigma_\theta^{1r}, \sigma_r^{1\theta})^T_{r=r_1} = K^1_{n|r=r_1}(A^1_n, B^1_n, 0, 0)^T e^{i\theta}. \quad (7.24)\]

Combining equations (7.17), (7.23) and (7.24), one obtains a system of \(4 \times 4\) boundary equations for the \(N\)-phase cylindrical scattering problem:

\[K^N_{n|r=r_{N-1}} \cdot (A^N_n + i^n, B^N_n + i^n, iA^N_n, iB^N_n)^T = T_n K^1_{n|r=r_1} \cdot (A^1_n, B^1_n, 0, 0)^T. \quad (7.25)\]

One can solve (7.25) as

\[(A^N_n + i^n, B^N_n + i^n, iA^N_n, iB^N_n)^T = S_n \cdot (A^1_n, B^1_n, 0, 0)^T. \quad (7.26)\]

where

\[S_n = (K^N_{n|r=r_{N-1}})^{-1} T_n K^1_{n|r=r_1}. \quad (7.27)\]

The four unknown coefficients \(A^N_n, B^N_n, A^1_n\) and \(B^1_n\) can be found by solving the \(4 \times 4\) system of equations (7.26)

\[(A^N_n, B^N_n, A^1_n, B^1_n)^T = D_n^{-1}(i^n, i^n, 0, 0)^T, \quad (7.28)\]

where matrix \(D_n\) is given in terms of elements of matrix \(S_n\) as

\[D_n = \begin{pmatrix}
1 & 0 & S_{11n} & S_{12n} \\
0 & 1 & S_{21n} & S_{22n} \\
i & 0 & S_{31n} & S_{32n} \\
0 & i & S_{41n} & S_{42n}
\end{pmatrix}, \quad (7.29)\]

where \(S_{ijn}\) is the \((i, j)\)th element of \(S_n\) (7.27).

Equation (7.28) gives the total scattering coefficients when both longitudinal and transverse incidence waves are present. For the case of longitudinal wave
incidence, array \((i^n, i^n, 0, 0)^T\) in equation (7.28) should be replaced by \((i^n, 0, 0, 0)^T\).

Similarly for transverse wave incidence \((0, i^n, 0, 0)^T\) should be used. The advantage of using the transfer matrix approach is to eliminate the need for finding the coefficients of the scattered waves in each intermediate cylindrical layer between the cylindrical core and the surrounding matrix.

Once the scattering coefficients \(A_n^N\) and \(B_n^N\) for the longitudinal and transverse waves are found, the scattered fields can be calculated using (7.2). The far field scattering angle dependence \(\varphi_N^N(\theta)\) and \(\psi_{\varphi}^N(\theta)\) can be obtained using the asymptotic expressions for Hankel functions of large argument

\[
H_n(kr) \approx \sqrt{\frac{2}{\pi kr}} \exp\{i[kr - (n + \frac{1}{2}) \frac{\pi}{2}]\}, \quad \text{for} \quad kr \to \infty \tag{7.30}
\]

as

\[
\varphi_N^N(\theta) = \frac{1 - i}{\sqrt{\pi k^N_i r}} \exp(ikr) \sum_{n=-\infty}^{\infty} i^{-n} A_n^N \exp(in\theta), \tag{7.31}
\]

\[
\psi_{\varphi}^N(\theta) = \frac{1 - i}{\sqrt{\pi k^N_i r}} \sum_{n=-\infty}^{\infty} i^{-n} B_n^N \exp(in\theta). \tag{7.32}
\]

The scattering cross-section, which is defined as the total power scattered per unit length divided by the incident wave intensity (power per unit area), is found to be [63]:

\[
Q_l = \frac{1}{k_{(l,l)i}^m} \sum_{n=-\infty}^{\infty} |A_n^N|^2, \quad Q_t = \frac{1}{k_{(l,t)i}^m} \sum_{n=-\infty}^{\infty} |B_n^N|^2, \tag{7.33}
\]

where \(Q_l\) and \(Q_t\) are the non-dimensional cross-sections for scattered longitudinal and transverse waves respectively, while \(k_{(l,i)}^m = k_{(l,t)}^N\) is the incident wave number.
for the longitudinal or transverse mode in the matrix material. The scattering cross-sections in the equations are normalized by the geometric limit $4r_i(r_i = r_{N-1}$ the fiber radius including the outermost interphase). Note that for transverse isotropy, one need only replace $\lambda$ and $\mu$ in equations (7.6-7.9) by $C_{rr} - 2G_t$ and $G_t$ respectively, where $C_{rr}$ is the radial modulus and $G_t$ is the transverse shear modulus.

Since the system matrix for equation (7.25) has a high condition number, especially for high $kr$ values, double precision is used for calculation of the numerical examples. We use the following inequalities to determine the truncation number $M$ in the series summation of the scattering cross-sections:

$$\frac{|A^N_{M+1}|^2}{\sum_{n=-M}^{M}|A^N_n|^2} < 10^{-8}, \quad \text{and} \quad \frac{|B^N_{M+1}|^2}{\sum_{n=-M}^{M}|B^M_n|^2} < 10^{-8}. \quad (7.34)$$

When (7.34) is satisfied the significance of the newly added $(M + 1)$th term on the scattering cross-sections $Q_t$ and $Q_i$ is negligible. Generally $M$ increases for greater $kr$ values. Typical values for $M$ are: 4 for $kr = 0.1$, 7 for $kr = 1$ and 25 for $kr = 10$.

7.5 Summary

In summary we have demonstrated a transfer matrix representation for a cylindrical layer, which allows a systematic treatment of wave scattering from a $N$-phase multilayered cylindrical system. The same transfer matrix can also be obtained by solving a matrix differential equation for the elastic field in a cylindrical medium.
with radially-dependent properties. The effect of each intermediate cylindrical layer is described by a transfer matrix which relates the stresses and displacements on the outer and inner boundaries of the layer. The total transfer matrix, which relates the stresses and displacements in the surrounding medium and the core, is obtained in the form of products of intermediate transfer matrices. The scattering coefficients are found by solving a system of $4 \times 4$ boundary condition equations on the outer-boundary of the multilayered cylinder. The advantage of the transfer matrix formulation lies in its natural application for multilayered fibers and interphases. Also there is no need to find wave solutions in each of the cylindrical layer between the surrounding medium and the core.
CHAPTER VIII

APPROXIMATE BOUNDARY CONDITION MODELS FOR A THIN FIBER-MATRIX INTERPHASE

In a similar fashion as for a thin plane interphase (Chapter III, Part I), approximate boundary condition models are derived in this chapter for a thin fiber-matrix interphase from its transfer matrix representation given in the last chapter. A detailed discussion on the applicability of different approximate B.C models will be conducted.

8.1 Introduction

Specially designed fiber-matrix interphases are created in modern composites to improve fracture toughness, chemical compatibility and matching of thermal expansion coefficients between composite constituents. They are often designed and manufactured in the form of thin layers. For example in metal and intermetallic matrix composites, special interfacial reaction barrier coatings [73] are introduced to improve chemical and thermal compatibility. In ceramic matrix composites,
interphases in the form of compliant coatings are added to provide frictional sliding contact between fiber and matrix to prevent fiber fracture triggered by matrix cracking [74, 75]. For polymer matrix composites various forms of interphases are also widely applied to improve composite performance, such as in carbon/epoxy composites [76, 77].

Since the interphase does the load transfer between fibers and matrix, the interphase properties and the quality of its bonding with the surrounding fiber and matrix are essential in determining composite mechanical performance. Ultrasonic waves can be a useful nondestructive tool to sense these interphase characteristics since wave scattering from fibers is strongly affected by both the interphase elastic properties and its mechanical contact with the neighboring matrix and fiber. Due to complicated manufacturing process and in service conditions, the interphase properties such as thickness and elastic moduli are in-situ parameters and are often difficult to define. One way to get around this problem is to introduce simplified boundary condition (B.C.) models to describe the relation of elastic fields across the interphase directly when studying wave scattering from fibers. In this chapter we will address this problem with an emphasis on the asymptotic B.C. models and spring models as a replacement of a thin fiber-matrix interphase.

Although much work have been done for the elastic wave scattering from a cylindrical inclusion in a solid [61, 62, 63, 64, 65, 66] (for a comprehensive list of early publications, see [66]), almost all the previous studies address scattering of
an elastic wave on a solid cylinder perfectly bonded to the solid matrix. In our own work [67, 68, 69] scattering from multilayered fibers with focus on the effect of fiber-matrix interphasial regions has been reported. A spring B.C. model for the fiber-matrix interphase was used [67, 68, 69] to simplify the analysis of wave scattering from a multiphase SiC fiber in titanium matrix. Here we will apply the transfer matrix solution presented in the last Chapter to derive different B.C. models for representation of a thin fiber-matrix interphase [78].

The organization of this chapter is as follows. In Section 8.2 the first order asymptotic B.C. model is obtained by expansion of the transfer matrix for a thin fiber-matrix interphasial layer in the nondimensional thickness parameter $h/r_f$ (interphase thickness-to-fiber radius ratio), and the second order B.C. model is derived from approximations to a matrix differential equation of elastic fields in a cylindrical medium with radially-dependent properties. The more simplified spring B.C. model is also discussed. Numerical examples are performed to compare different approximate B.C. models with the exact solution using the scattering cross-sections for longitudinal and shear wave incidence on a multiphase SiC fiber in titanium matrix. In Section 8.3 a detailed discussion is devoted to demonstrate, in applying the spring B.C. model to a thin interphase, the advantage of preserving the interphase gap compared to those prescribing the B.C. directly on an interface with gap filled by matrix or fiber material.
8.2 Representation of a Thin Fiber-Matrix Interphase by Approximate B.C. Models

8.2.1 Transfer matrix approximations

To derive approximate B.C. models we will use here a similar approach to that presented in Chapters III-VI for a thin anisotropic plane interphase [50, 51]. We will develop the first and second order asymptotic B.C. models and the spring B.C. model for representation of a thin cylindrical fiber-matrix interphase.

Let us use the SiC/Ti system as a model system for discussion. The SiC fiber is SCS-6 as shown in Fig. 8.1(a); it consists of a carbon core, a SiC shell and a 3μm carbon-rich interphase coating. As shown in the previous chapter we can apply the transfer matrix $T_n^i$ to represent the effect of the actual fiber-matrix interphase as shown in Fig. 8.1(b), where $T_n^i$ can be represented as

\[
T_n^i = \exp\{\int_{r_f}^{r_i} t_n^i(r)dr\}. \tag{8.1}
\]

Here $r_f$ and $r_i$ are the radii of the interphase inner and outer boundary.

When the fiber-matrix interphase is thin, i.e. $h = (r_i - r_f) \ll r_f$ (e.g. the SCS-6 fiber: $h/r_f = 3/68$) and $k_lth \ll 1$, one can simplify the transfer matrix $T_n^i$ by an asymptotic expansion in $h$. From equation (8.1), one has

\[
T_n^i \approx I + h t_n^i(r_f). \tag{8.2}
\]

Now we write equation (7.15) in the form of the B.C. across the interphase gap
Figure 8.1: (a) Scattering from an SCS-6 fiber in a titanium matrix. (b) Replacement of the fiber-matrix interphase by an interphase gap with transfer matrix $T_n^i$, $r_f$ and $r_i$ are the radius of fiber and interphase respectively.
\((r_f \leq r \leq r_i)\) as:

\[
\begin{pmatrix}
  u_r \\
  u_\theta \\
  \sigma_{rr} \\
  \sigma_{r\theta}
\end{pmatrix}^{r=r_i} = \begin{pmatrix}
  1 - \frac{\lambda^i}{r_f(\lambda^i + 2\mu^i)} & -\frac{i\nu_h\lambda^i}{r_f(\lambda^i + 2\mu^i)} & \frac{h}{\lambda^i + 2\mu^i} & 0 \\
  \frac{\lambda^i}{r_f(\lambda^i + 2\mu^i)} & 1 + \frac{h}{r_f} & \frac{h}{\mu^i} & 0 \\
  \frac{4\pi h \mu^i(\lambda^i + \mu^i)}{r_f^2(\lambda^i + 2\mu^i)} - \Omega^2 \rho^i h & \frac{4\pi h \mu^i(\lambda^i + \mu^i)}{r_f^2(\lambda^i + 2\mu^i)} & 1 - \frac{2h}{r_f} & -\frac{i\nu_h}{r_f} \\
  \frac{4\pi h \mu^i(\lambda^i + \mu^i)}{r_f^2(\lambda^i + 2\mu^i)} - \Omega^2 \rho^i h & \frac{4\pi h \mu^i(\lambda^i + \mu^i)}{r_f^2(\lambda^i + 2\mu^i)} & 1 - \frac{2h}{r_f} & -\frac{i\nu_h}{r_f}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u_r \\
  u_\theta \\
  \sigma_{rr} \\
  \sigma_{r\theta}
\end{pmatrix}^{r=r_f}, \quad (8.3)
\]

where \(\lambda^i, \mu^i\) are the Lamé constants, \(\rho^i\) the density of the interphase. We will call B.C. (8.3) the first order asymptotic B.C. model.

Note that the same first order transfer matrix (8.3) can be derived from the representation of \(T^i_n\) in the form of (7.16), i.e. from

\[
T^i_n = K^i_n|_{r=r_i} \cdot (K^i_n|_{r=r_f})^{-1}. \quad (8.4)
\]

Specifically if one takes the following approximations for each elements in matrix \(K^i_n|_{r=r_i}\):

\[
J_n(kr_i) \approx J_n(kr_f) + (kh)J'_n(kr_f); \quad N_n(kr_i) \approx N_n(kr_f) + (kh)N'_n(kr_f), \quad (8.5)
\]

one can simplify the exact B.C. transfer matrix (8.4) to the first order transfer matrix (8.3) through a rather tedious procedure. From the approximate equation (8.5) one sees that the conditions for a valid approximation (8.3) are \(h \ll r_f\) and \(k_{lt}h \ll 1\).
It is interesting to consider two limiting cases. The first is when the \( r_f \to \infty \), thus physically reducing the cylindrical interphase to a plane interphase. In this case the first order transfer matrix (8.3) reduces to

\[
T^i \approx \begin{pmatrix}
1 & 0 & h \\
0 & 1 + \frac{h}{\mu} & 0 \\
-\omega^2 \rho^i h & 0 & 1 \\
0 & 0 - \omega^2 \rho^i h & 1
\end{pmatrix},
\]

(8.6)

which corresponds to the same first order transfer matrix for a wave normally incident on a plane interphase with thickness \( h \), i.e. corresponds to the transfer matrix \( B \) (2.11) for \( k = 0 \) (\( b_{12} = 0 \) and \( M_p = M_n = \rho h \)). Note that the orders of the rows and columns in matrices \( T^i \) and \( B \) are different due to different orders of normal and shear components of the displacement and stress vectors. Further if one replaces the fraction \( n/r_f \) by \( k \) in (8.3) where \( k \) is the projection of the wave number on the interphase, one obtains the first order transfer matrix \( B \) (2.11) for a wave obliquely incident on a plane interphase with thickness \( h \).

The other limiting case is when the frequency approaches zero, i.e. the static case (\( \omega = 0, n = 0 \)). The first order transfer matrix (8.3) reduces to

\[
T^i \approx \begin{pmatrix}
1 - \frac{h \lambda^i}{r_f(\lambda^i + 2\mu^i)} & 0 & h \\
0 & 1 + \frac{1}{r_f} & 0 \\
\frac{4h \mu^i(\lambda^i + \mu^i)}{r_f^2(\lambda^i + 2\mu^i)} & 0 & 1 - \frac{2h \mu}{r_f(\lambda^i + 2\mu^i)} \\
0 & 0 & 1 - \frac{2h}{r_f}
\end{pmatrix},
\]

(8.7)
which corresponds to the first order transfer matrix derived for a thin interphase from static transverse elastic field solutions [79, 80], where the analysis is focused on the interphase effect on the transverse elastic properties of composites reinforced with multilayered fibers.

One may also define the second order asymptotic B.C. as in case of plane interphase (see Section 3.2) by applying the finite difference approximation to the matrix differential equation (7.20)

\[
\frac{\tilde{U}_n(r = r_i) - \tilde{U}_n(r = r_f)}{h} = t_n(r = \bar{r}) \frac{\tilde{U}_n(r = r_i) + \tilde{U}_n(r = r_f)}{2}, \quad \bar{r} = \frac{r_f + r_i}{2}
\]

(8.8)

where on the left side of the equation a finite difference approximation is used in (7.20) to replace the derivative \( \partial \tilde{U}_n(r) / \partial r \), while on the right side the averaged elastic field is used to replace the radially-dependent function \( \tilde{U}_n(r) \). The second order B.C. can be rewritten in the transfer matrix format as:

\[
\tilde{U}_n(r = r_i) = \left[ I - \frac{h}{2} t_n^i(\bar{r}) \right]^{-1} \left[ I + \frac{h}{2} t_n^i(\bar{r}) \right] \tilde{U}_n(r = r_f).
\]

(8.9)

The advantage of the first and second order B.C. models is that Bessel functions are not involved in the calculation of fields in the thin interphasial layer.

It is reasonable to approximate the B.C. across the interphase further by a simple spring model. This model has been used to describe the static elastic field on the fiber-matrix interface in composites by Aboudi [81] and Hashin [82]. In their approach, an imperfect interface with the spring B.C. is taken where the springs are considered on an infinitely thin fiber-matrix boundary. Here we consider the spring
B.C. in a different form to preserve the system geometry by applying the spring B.C. across the interphase gap instead of filling the gap by fiber or matrix materials. The advantage of using spring B.C. in this way will be discussed in Section 8.3. If one keeps in the first order B.C. model (8.3) only the spring stiffnesses terms, one obtains the spring B.C. as:

\[
\begin{pmatrix}
    u_r
    \\
    u_\theta
    \\
    \sigma_{rr}
    \\
    \sigma_{r\theta}
\end{pmatrix}
_{r=r_i} =
\begin{pmatrix}
    1 & 0 & \frac{h}{\lambda^i + 2\mu^i} & 0 \\
    0 & 1 & 0 & \frac{1}{\mu^i} \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
_{r=r_f}
\begin{pmatrix}
    u_r
    \\
    u_\theta
    \\
    \sigma_{rr}
    \\
    \sigma_{r\theta}
\end{pmatrix}
_{r=r_f},
\]  

i.e.

\[
\sigma_{rr}|_{r=r_i} = \sigma_{rr}|_{r=r_f} = K_n (u_r|_{r=r_i} - u_r|_{r=r_f}),
\]

\[
\sigma_{r\theta}|_{r=r_i} = \sigma_{r\theta}|_{r=r_f} = K_t (u_\theta|_{r=r_i} - u_\theta|_{r=r_f}),
\]

where the normal and shear spring stiffnesses are \( K_n = (\lambda^i + 2\mu^i)/h \) and \( K_t = \mu^i/h \).

Note the same spring B.C. (8.10) can be obtained directly by preserving in \( t^n_i \) (7.19) only two terms related to the spring stiffnesses: \( 1/(\lambda^i + 2\mu^i) \) and \( 1/\mu^i \); then the transfer matrix \( T^n_i \) (7.22) reduces exactly to the same form as (8.10) since the higher order terms vanish (\( t^n_i t^n_i = 0 \)).

### 8.2.2 Numerical examples

Next we give numerical examples for longitudinal and shear wave scattering by an SCS-6 fiber in titanium alloy to demonstrate the applicability of different approximate B.C. models. The properties of each phase in the multiphase SCS-6 fiber
Table 8.1: Properties of each phase of the SCS-6 fiber and titanium alloy matrix.

<table>
<thead>
<tr>
<th>Phase</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$\lambda$ (GPa)</th>
<th>$\mu$ (GPa)</th>
<th>$\rho$ (g/cc)</th>
<th>$r$ ($\mu$m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>core (carbon)</td>
<td>41</td>
<td>0.25</td>
<td>16.4</td>
<td>16.4</td>
<td>1.7</td>
<td>18</td>
</tr>
<tr>
<td>shell (SiC)</td>
<td>415</td>
<td>0.17</td>
<td>91</td>
<td>177</td>
<td>3.2</td>
<td>68</td>
</tr>
<tr>
<td>interphase (carbon)</td>
<td>41</td>
<td>0.25</td>
<td>16.4</td>
<td>16.4</td>
<td>1.7</td>
<td>71</td>
</tr>
<tr>
<td>matrix (Ti)</td>
<td>122</td>
<td>0.348</td>
<td>103</td>
<td>45</td>
<td>5.4</td>
<td></td>
</tr>
</tbody>
</table>

where $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $\lambda$ and $\mu$ are the Lamé constants, $\rho$ is the density and $r$ the radius of the boundary of that phase.

are listed in Table 8.1. The elastic properties of the carbon-rich interphase are equal to those of the carbon core; the interphase thickness is 3 $\mu$m. The scattering cross-section as a function of frequency calculated using different approximate B.C. models are shown in Figure 8.2 for both longitudinal (a) and shear (b) wave incidence. In the figure the top axis is frequency in MHz, and the bottom axis is the nondimensional wave number $k^m r_i$ where $k^m$ is the wave number in the matrix and $r_i = 71 \mu$m is the outer radius of the interphase. One sees from the figure that the second order solutions (coarse dashed lines) are almost indistinguishable from the exact (solid lines). The spring (fine dashed lines) and first order (crosses) B.C. models also give good predictions at this interphase thickness and stiffness for low frequencies, especially for $k^m_i r_i < 4$ or $k^i_{s} h < 0.2$.

The difference between different approximations is more pronounced with the increase of the interphase thickness to fiber radius ratio $h/r_f$. Figure 8.3(a) shows the calculated scattering cross-sections versus $h/r_f$ for shear wave incidence using different approximate B.C. models at frequency = 13 MHz ($k^s_i h = 0.08$). The
Figure 8.2: Total scattering cross-sections versus frequency and nondimensional wave number calculated using different B.C. models. (a) Results for longitudinal wave incidence. (b) Results for shear wave incidence.
Fig. 8.2 (continued)

SHEAR WAVE INCIDENCE

FREQUENCY (MHz)

$h/r_f = 3/68$

SCATTERING CROSS-SECTION $Q_i + Q_t$

--- EXACT
--- SPRING B.C.
--- 2ND ORDER B.C.
+++ 1ST ORDER B.C.

$k_t^m r_i$

(b)
fiber radius $r_f = 68 \mu m$ is kept constant. The legends in this figure are the same as in Figure 8.2. One sees from the figure that all the B.C. models give good results for $h/r_f < 0.05$. For greater interphase thickness, the second order B.C. is satisfactory until $h/r_f = 0.2$, while the spring and first order B.C. models begin to deviate significantly from the exact results. To illustrate this more clearly, we present in Figure 8.3(b) the relative deviation in percentage between the exact and the approximate solutions versus $h/r_f$. One sees that the second order B.C. give the best results, while the spring and first order B.C. models are sufficient for $h/r_f < 0.05$.

To check further the applicability of the spring B.C. model let us compare the spring stiffness values to those calculated directly from the stress-displacement-difference ratios across the interphase gap. Specifically, from (8.11) one can find alternative definitions for $K_n$ and $K_t$ as

$$K_n = \sigma^m_r / (u^m_r - u^m_t), \quad K_t = \sigma^m_\theta / (u^m_\theta - u^m_t) \quad (8.12)$$

or

$$K_n = \sigma^f_r / (u^m_r - u^f_t), \quad K_t = \sigma^f_\theta / (u^m_\theta - u^f_\theta) \quad (8.13)$$

where quantities with superscript $m$ are the fields defined on the matrix side of interphase gap, while quantities with superscript $f$ are the fields defined on the fiber side. We have calculated these stress-displacement-difference ratios as functions of the scattered angle $\theta$ using the exact interphase layer model, and compared them
Figure 8.3: (a) Total scattering cross-sections versus interphase thickness parameter $h/r_f$ calculated using different B.C. models. The results are given for an incident shear wave at frequency $= 13$ MHz. (b) The relative percentage deviations from the exact for the approximate B.C. models shown in (a).
Fig. 8.3 (continued)

SHEAR WAVE INCIDENCE FREQUENCY = 13 MHz

- EXACT
- SPRING B.C.
- 2ND ORDER B.C.
- 1ST ORDER B.C.

$\frac{h}{r_f} = \frac{3}{68}$

NORMAIZED INTERPHASE THICKNESS $\frac{h}{r_f}$

(b)
with $K_n$ and $K_t$ values found from $K_n = (\lambda^i + 2\mu^i)/h$ and $K_t = \mu^i/h$. The results are shown in Figure 8.4 for longitudinal wave incidence at frequency $f = 13$ MHz ($k_f^i h = 0.08, k_t^i h = 0.04$), thus the interphase can be considered very thin. The solid lines in the figure are $K_n = (\lambda^i + 2\mu^i)/h$ and $K_t = \mu^i/h$. The coarse dashed lines are ratios calculated using (8.12) and the fine dashed lines are those calculated using (8.13). One sees from the figure that there exists good agreement between the stress-displacement-difference ratios (dashed lines) and the calculated $K_n$ and $K_t$ values (solid lines). However these ratios depend on the scattered angle $\theta$, thus the assumption of homogeneously distributed springs with constant stiffness constants is only approximately true.

To demonstrate applicability of different B.C. approximations to interphase layers with different stiffnesses we calculated the scattering cross-section versus interphase stiffnesses. In the calculations the interphase Young's modulus is varied while the interphase thickness, density and Poisson's ratio are kept constant. The results are shown in Figure 8.5 for incident longitudinal (a) and shear (b) wave at frequency $f = 13$ MHz and interphase thickness $h = 3\mu m$. In the figure, all legends are the same as in Figure 8.2, the horizontal axis is the interphase Young's modulus in log scale, and $E^m = 122$ GPa and $E^f = 415$ Gpa are the Young's moduli for the matrix and fiber respectively. One sees from the figure that the spring B.C. model give good predictions up to interphase stiffness equaling that of the fiber, while the first and second order asymptotic B.C. continue to give good
LONGITUDINAL WAVE INCIDENCE  \( f = 13 \) MHz

![Graph showing comparison of stiﬀnesses](image)

Figure 8.4: Comparison of stiﬀnesses calculated using \( \sigma/(u^m - u^f) \) with \( K_n = (\lambda^i + 2\mu^i)/h \) and \( K_t = \mu^i/h \). Assume a longitudinal wave incident at frequency = 13 MHz.
Figure 8.5: Total scattering cross-sections versus interphase stiffnesses calculated using different B.C. models for an SCS-6 fiber in titanium matrix. (a) Results for longitudinal wave incidence. (b) Results for shear wave incidence.
Fig. 8.5 (continued)

SHEAR WAVE INCIDENCE (13 MHz)

\[ k_t^m r_i = 2 \]
\[ h/r_f = 3/68 \]

DEBONDED FIBER

RIGID FIBER

WELDED B.C. ACROSS THE GAP

SCATTERING CROSS-SECTION \( q_t + q_c \)

INTERPHASE YOUNG'S MODULUS (GPa)

(b)
results for much stiffer interphases.

It is interesting to discuss the limiting cases presented in Figure 8.5. When the interphase Young's modulus is extremely small, the fiber with the interphase acts like a debonded fiber or a cylindrical cavity with radius that of the outer interphase boundary \( r_i \). All the approximate B.C. models give the correct limit. For example, for the spring B.C. model, since \( K_n \) and \( K_\iota \) approach zero in this case, the spring B.C. (8.11) simplifies to

\[
\sigma_{rr}|_{r = r_i} = \sigma_{r\theta}|_{r = r_i} = 0, \quad (8.14)
\]

which is the free stress condition for a cylindrical cavity with radius \( r_i \).

On the other hand, when the interphase Young's modulus is extremely high, the fiber with the interphase acts like a rigid cylinder with radius \( r_i \), i.e. the B.C. should be

\[
u_r|_{r = r_i} = u_\theta|_{r = r_i} = 0. \quad (8.15)
\]

However, neither of the approximate B.C. models gives the correct limit. For the spring B.C. model, the B.C. reduce to welded B.C. across the interphase gap. Specifically, since \( K_n \) and \( K_\iota \) approach infinity, the spring B.C. (8.11) simplifies to

\[
u_r|_{r = r_i} = \nu_r|_{r = r_f}, \quad u_\theta|_{r = r_i} = u_\theta|_{r = r_f}, \quad \sigma_{rr}|_{r = r_i} = \sigma_{rr}|_{r = r_f}, \quad \sigma_{r\theta}|_{r = r_i} = \sigma_{r\theta}|_{r = r_f}. \quad (8.16)
\]

The discrepancy between the resulting B.C. for the rigid interphase (8.15) and rigid springs (8.16) explains the large deviation of the spring approximation for
very high interphase stiffness as shown in Figure 8.5. In summary the spring B.C. model is sufficient for a thin fiber-matrix interphase \( h/r_f < 0.05, k_{int} \ll 1 \) whose stiffness is less than that of the fiber.

Another interesting phenomenon observable from Figure 8.5 is that there exists a scattering cross-section minimum. The minimum is explained by matching of impedances between matrix and the effective fiber formed by the actual fiber and the interphase. The decrease of the effective fiber impedance to a value lower than that of the matrix results in a sharp increase in the scattering cross-section, reaching in the limit the scattering level from a cylindrical cavity for zero interphase moduli (disbond). Detailed analyses of this phenomenon were given in the next chapter.

### 8.3 Discussion of Spring B.C. Model: Springs on Interphase Gap versus those on Interface Boundary with Gap Filled by either Matrix or Fiber

One may note that so far we have kept the structural geometry of the fiber scattering problem by applying the approximate B.C. model across an interphase gap other than on an interface boundary by filling the interphase gap with either fiber or matrix material. This point is illustrated in Figure 8.6 which shows three different ways of applying the spring B.C. model to replace a thin fiber-matrix interphase. The first is to keep the interphase gap as shown by Figure 8.6(b) and to use spring B.C. across the gap as given by (8.11). Relating specifically to the gap between
Figure 8.6: The exact interphase layer model (a) and different spring B.C. models for a thin fiber-matrix interphase. (b) is for the spring B.C. model with the interphase gap connected by springs, (c) for that with the gap filled by matrix and (d) for that with the gap filled by fiber.
matrix and fiber we rewrite the B.C. equation (8.11) as

$$\sigma_{rr}^m|_{r=r_i} = \sigma_{rr}^f|_{r=r_f} = K_n (u_{rr}^m|_{r=r_i} - u_{rr}^f|_{r=r_f}),$$
$$\sigma_{r\theta}^m|_{r=r_i} = \sigma_{r\theta}^f|_{r=r_f} = K_t (u_{r\theta}^m|_{r=r_i} - u_{r\theta}^f|_{r=r_f}),$$

(8.17)

where quantities with superscript m are the fields defined on the matrix side of the gap, while quantities with superscript f are the fields defined on the fiber side. One sees from equation (8.17) that the displacements jump across the interphase gap from r_f to r_i, while the stresses on both sides of the gap are continuous.

The other two ways involve applying the spring B.C. to an interface with the gap filled by either matrix or fiber materials. In this case the spring B.C. on an interface with radius R are:

$$\sigma_{rr}^m|_{r=R} = \sigma_{rr}^f|_{r=R} = K_n (u_{rr}^m|_{r=R} - u_{rr}^f|_{r=R}),$$
$$\sigma_{r\theta}^m|_{r=R} = \sigma_{r\theta}^f|_{r=R} = K_t (u_{r\theta}^m|_{r=R} - u_{r\theta}^f|_{r=R}).$$

(8.18)

Physically, R = r_f implies that the interphase gap is filled by matrix as shown by Figure 8.6(c), while R = r_i corresponds to the case when the gap is filled by fiber as shown by Figure 8.6(d). One sees from equation (8.18) that the displacements jump on the interface boundary r = R, while the stresses on this interface boundary are continuous.

After composite manufacture, the interphase thickness may not be known exactly. In this case there is an advantage in using the spring B.C. on a boundary between matrix and fiber eliminating the gap and filling it with matrix material
(usually the fiber diameter is known). Thus, to describe the mechanical contact across the interphase, one can use just two parameters $K_n$ and $K_t$. Of course additional error may be introduced in this simplification as we will see below. To compare these different cases we carried out a numerical analysis similar to that in the previous section for the three variations of the spring B.C. Figure 8.7 shows the scattering cross-sections for incident longitudinal (a) and shear (b) waves versus the nondimensional wave number calculated exactly and using different spring B.C. models. The parameters are the same as in Figure 8.2 (interphase Young's modulus $E^i = 41$ GPa, and thickness $h = 3\mu m$). One sees from the figure that the spring B.C model (8.17) with interphase gap (the fine dashed line) gives the best results, while those (8.18) with gap filled by either matrix (crosses) or fiber (coarse dashed lines) material may deviate considerably from the exact even at low frequencies. It seems that the spring B.C. model with gap filled by fiber gives better results than the model with gap filled by matrix.

Figure 8.8 shows the longitudinal and shear wave scattering cross-sections versus interphase thickness to fiber radius ratio $h/r_f$ at frequency $= 13$ MHz. The fiber radius $r_f = 68\mu m$ is kept constant. All other parameters are the same as in Figure 8.3. One sees from the figure that the spring B.C. model with interphase gap consistently gives the best results, while the spring B.C. model with gap filled by fiber material gives better results than the model with gap filled by matrix material. Note that the latter model predicts a constant cross-section as interphase
LONGITUDINAL WAVE INCIDENCE

FREQUENCY (MHz)

![Graph showing scattering cross-sections versus frequency and nondimensional wave number calculated using different spring B.C. models.]{(a)}

Figure 8.7: Total scattering cross-sections versus frequency and nondimensional wave number calculated using different spring B.C. models. (a) Results for longitudinal wave incidence. (b) Results for shear wave incidence.
Fig. 8.7 (continued)

SHEAR WAVE INCIDENCE

FREQUENCY (MHz)

\[ h/r_f = 3/68 \]

SCATTERING CROSS-SECTION \( q_i + q_t \)

\[ k_t^m r_i \]

(b)
thickness increases since the fiber radius is constant.

Finally we calculated the scattering cross-section versus interphase stiffness using different spring B.C. models. In the calculations the interphase Young’s modulus is varied while the interphase thickness, density and Poisson’s ratio are kept constant. The results are shown in Figure 8.9 for both (a) longitudinal and (b) shear wave incidence at frequency = 13 MHz and \( h = 3 \mu m \). One sees from the figure that the spring B.C. model with gap filled by matrix gives totally unacceptable results in the scattering cross-section transition region to the limit of debonded fiber. The spring B.C. model with interphase gap gives good approximation for interphase stiffness below that of the fiber. The spring B.C. model with gap filled by fiber material gives acceptable results for less stiff interphases. Note that due to the difference in matrix and interphase densities and Poisson’s ratios (see Table 8.1) the spring B.C. model with gap filled by matrix does not give satisfactory results even when the interphase Young’s modulus equals that of the matrix.

One can interpret the results in Figure 8.9 by examining the limits for very compliant and stiff interphases. First for an extremely compliant interphase the spring B.C. models with gap and with gap filled by fiber material yields the same free stress B.C. (8.14) (it provides the correct radius of the cylindrical cavity). However the spring B.C. model with gap filled by matrix reduces to a cavity of a smaller radius:

\[
\sigma_{rr}|_{r=r_f} = \sigma_{r\theta}|_{r=r_f} = 0. 
\] (8.19)
LONGITUDINAL WAVE INCIDENCE

FREQUENCY = 13 MHz

--- EXACT
- - INTERPHASE GAP
- - INTERFACE WITH GAP FILLED BY FIBER
+++++ INTERFACE WITH GAP FILLED BY MATRIX

SCATTERING CROSS-SECTION $Q_t + Q_s$

$\frac{h}{r_t} = 3/68$

NORMALIZED INTERPHASE THICKNESS $h/r_t$

(a)

Figure 8.8: Total scattering cross-sections versus interphase thickness calculated using different spring B.C. models at frequency = 13 MHz. (a) Results for longitudinal wave incidence. (b) Results for shear wave incidence.
Fig. 8.8 (continued)

SHEAR WAVE INCIDENCE
FREQUENCY = 13 MHz

SCATTERING CROSS-SECTION $Q_t+Q_r$

$\frac{h}{r_f}=3/68$

NORMALIZED INTERPHASE THICKNESS $h/r_f$

(b)
Figure 8.9: Total scattering cross-sections versus interphase stiffnesses calculated using different spring B.C. models for an SCS-6 fiber in titanium matrix.  (a) Results for longitudinal wave incidence.  (b) Results for shear wave incidence.
Fig. 8.9 (continued)

Shear wave incidence (13 MHz) \( k_t^m r_i = 2 \)

- Exact
- Interphase gap
- Interface with gap filled by fiber
- Interface with gap filled by matrix

Scattering cross-section \( q_t + q_t \)

Interphase Young's modulus (GPa)

\( h/r_f = 3/68 \)
Equation (8.19) is the free stress condition for a cylindrical cavity with radius equal to interphase inner boundary $r_f$. When the interphase Young’s modulus is extremely high, none of the spring B.C. models give the rigid B.C. on the outer boundary of the interphase; instead they give welded B.C. in different forms. The two spring B.C. models without gap give welded B.C. on an interface with the radius of the fiber $R$:

$$
\begin{align*}
  u_r|_{r=R} &= u_r|_{r=R}, & u_\theta|_{r=R} &= u_\theta|_{r=R}, & \sigma_{rr}|_{r=R} &= \sigma_{rr}|_{r=R}, & \sigma_{r\theta}|_{r=R} &= \sigma_{r\theta}|_{r=R},
\end{align*}
$$

(8.20)

where $R = r_f$ is for the spring B.C. model with gap filled by matrix material and $R = r_i$ for that with gap filled by fiber material; while the spring B.C. with gap give the welded B.C. across the interphase gap as given by (8.16).

### 8.4 Summary

In this chapter the first and second order asymptotic B.C models are derived to replace a thin fiber-matrix interphase by asymptotically expanding the transfer matrix for the interphase layer. The spring B.C. model is investigated as an effective replacement of a thin fiber-matrix interphase. Different ways of implementing spring B.C. models with springs connecting the fiber and matrix through the interphase gap or with gap filled by matrix or fiber material are studied. Numerical analysis shows that although the second order asymptotic B.C give the best overall results, the spring approximation with springs connecting the interphase gap is sufficient for a thin ($h/r_f < 0.05$, $k_{i,t}^i h \ll 1$) interphase which is softer than the
fiber. For even thinner and softer interphases the spring B.C. model with gap filled with fiber material may also be used.
CHAPTER IX

LOW FREQUENCY APPROXIMATION FOR ULTRASONIC WAVE SCATTERING BY A MULTIPHASE FIBER IN A SOLID

9.1 Introduction

In the next four chapters we will discuss wave propagation in a SiC/titanium composite. In this chapter we will progressively simplify the model for low frequency ultrasonic scattering from a 3-phase SiC fiber in titanium alloy matrix. The SiC fiber is SCS-6 with diameter approximately 140 μm, widely used in ceramic, metal and intermetallic matrix composites. It consists of a core, a shell and an interphasic coating layer. Sinclair and Addison [66] have analyzed a similar system for perfect fiber-matrix interphases by studying the diffraction spectrum of an incident longitudinal wave by a single SCS-6 fiber embedded in a Ti-6-4 matrix. Matikas and Kapur studied shear wave back reflection [83, 84] from such a fiber-matrix interface with the aim of determining interface properties. Chu and Rokhlin [7, 8, 9, 10] have determined interphasic elastic properties and interphasic
damage for ceramic and intermetallic matrix SiC composites by measuring bulk wave phase velocities along different directions in the composites. The effective interphasial moduli were found from ultrasonically measured composite moduli (velocity data) using micromechanical models (see schematics Fig. 1.3). The dynamic effect-induced error in using velocity data to determine the static composite moduli will be discussed both theoretically and experimentally in Chapters X and XI for waves propagating normal to and along fibers in composites. More recently, fatigue damage on the fiber-matrix interphases in SiC/Ti-15-3 composites has been assessed using both the ultrasonic wave velocity and scattering (attenuation) measurements as functions of the different stages of fatigue life cycle [11], these results will be presented in Chapter XII.

9.2 Simplified Fiber-Matrix Models in the Low Frequency Range

As an example we consider scattering from a SiC (SCS-6) fiber manufactured by Textron, whose structure is shown schematically in Fig. 9.1. The SCS-6 fiber consists of a SiC shell deposited by the CVD process on a carbon core. In addition a 3 μm carbon-rich coating (fiber-matrix interphase) is deposited on the fiber surface which serves as the fiber-matrix interphase to increase composite fracture toughness and to prevent the fiber-matrix chemical reaction. Although we solved the problem of single 3-phase fiber scattering exactly by the approach outlined in
Figure 9.1: Geometry of the elastic wave scattering from an SCS-6 fiber in a matrix titanium. The SEM photo of the SCS-6 fiber is also shown here.
Chapter VIII, the effect of the fiber-matrix interphase on the scattering is hidden in the complicated structure of the solution. To better understand the role of the interphase on scattering and to develop effective inversion schemes for interphase characterization from scattering data, we replace the multiphase fiber by simplified fiber models for wave incident at low frequencies. The low frequency ultrasound propagating in composites with large diameter fibers is important since higher frequency component of the ultrasonic wave can not propagate effective in these composites due to higher attenuation.

Let us first replace the fiber core and shell by an equivalent homogenous fiber based on the available micromechanical model \cite{52}. We define the equivalent fiber moduli so that the composite with embedded equivalent fibers has the same moduli as the composite with actual multiphase fibers (core plus shell) in the static limit \cite{9, 10}. An alternative approach is to replace the SiC shell and carbon core by a homogeneous SiC fiber of the same diameter, i.e. to fill the fiber core with the fiber shell material. The second approach is valid when the radius of the fiber core is only a small fraction of that of the fiber shell so that the scattering is dominated by the fiber shell in the low frequency range.

As an example we calculate scattering cross-sections for a single SiC fiber in a titanium alloy matrix. In Chapters X and XI we will perform theoretical and experimental studies on this SiC/Ti composite (fiber fraction 24\%) for attenuation and dispersion of wave propagating normal to and along fibers. The properties of
the different fiber phases and of the matrix are listed in Table 8.1, except the properties of the carbon-rich interphase coating are taken as experimentally determined for the SiC/Ti composites. The transverse elastic properties for the interphase are found as radial modulus $C_{rr} = 31$ GPa and transverse shear modulus $G_t = 4.6$ GPa, and they are determined from a multiphase generalized self-consistent model [79, 80] using ultrasonically measured composite moduli, the details will be given in the next chapter. The density of the interphase coating is $\rho = 2.1$g/cc. One notes that there exists an unusually high ratio of radial-to-transverse shear modulus ratio for this interphase. This is due to the carbon anisotropy and inclusions of SiC particles as discussed in [9, 10]. It is found that the carbon-rich interphase coating consists of two different microstructure zones. One zone has randomly oriented basic structure units (BSU) as in the fiber core; the other has a structure similar to pyrolytic graphite with a preferred basal plane orientation normal to the radial direction. In addition there exist 25% SiC particles in the interphase zone. These microstructure differences account for the anisotropy in elastic moduli and higher density of the interphase than the isotropic carbon core. The properties of the equivalent fiber replacing the fiber core and shell are given in Table 9.1. One sees from the table that the equivalent fiber is transversely isotropic.

Fig. 9.2 shows the calculated total scattering cross-section $Q_t + Q_\ell$ from such a fiber-matrix interphase. The cross-section is shown versus frequency (bottom axis) and wave number (top axis). In Fig. 9.2(a) the results are shown for a longitudinal
Table 9.1: Properties of the equivalent fiber

<table>
<thead>
<tr>
<th>$E_a$ (GPa)</th>
<th>$C_{rr}$ (GPa)</th>
<th>$\nu_a$</th>
<th>$\nu_t$</th>
<th>$G_a$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>390</td>
<td>384</td>
<td>0.17</td>
<td>0.20</td>
<td>158</td>
</tr>
<tr>
<td>$\kappa$ (GPa)</td>
<td>$G_i$ (GPa)</td>
<td>$\rho$ (g/cc)</td>
<td>$r$ (µm)</td>
<td></td>
</tr>
<tr>
<td>236</td>
<td>148</td>
<td>3.1</td>
<td>68</td>
<td></td>
</tr>
</tbody>
</table>

where $E_a$ is the axial Young’s modulus, $\nu_a$ the axial Poisson's ratio, $G_a$ the axial shear modulus; $C_{rr}$ is the radial modulus, $\nu_t$ the transverse Poisson’s ratio, $G_i$ the transverse shear modulus and $\kappa$ the transverse bulk modulus.

incident wave (versus $k_i^m r_i$) and (b) for a transverse incident wave (versus $k_i^m r_i$), where $k_i^m$ and $k_t^m$ are the wave numbers for the matrix, and $r_i$ is the fiber radius 71 µm (including the 3 µm interphasial layer). The solid lines in the figure represent results when the actual SiC fiber with carbon core and SiC shell is used, the coarse dashed lines are for the results obtained using the equivalent fiber defined above and the fine dashed lines are for the results obtained using the homogeneous SiC fiber defined above. One sees from the figure that the results obtained using both the homogeneous SiC and equivalent fibers agree well with those predicted by the exact 4-phase model at low frequency: $f < 20$ MHz ($k_i^m r_i < 1.5$ or $k_t^m r_i < 3.0$).

In this frequency range the greatest fiber radius-to-wavelength ratio is less than 0.2. The volume fraction of the carbon core is only 7% of the SCS-6 fiber, thus the fiber shell plays a major role in determining both equivalent fiber properties and scattering, which explains the negligible discrepancy between the scattering for the homogeneous SiC and the equivalent fiber models. If the volume fraction of the fiber core is not small the equivalent fiber model should have a significant
advantage over the homogeneous SiC fiber.

Since the interphasial layer between fiber and matrix is usually very thin it is reasonable to approximate the interphase by distributed springs with normal and transverse stiffnesses [7, 9, 51, 26, 82, 84, 86]. As one sees from Table 8.1, the interphasial layer thickness-to-fiber-radius ratio \( h/r_f \) in this composite is 3/68. The spring boundary conditions for the fiber-matrix interphase are

\[
\sigma^m_{rr}|_{r=r_i} = \sigma^f_{rr}|_{r=r_f} = K_n (u^m_r|_{r=r_i} - u^f_r)|_{r=r_f}, \quad \sigma^m_{r\theta}|_{r=r_i} = \sigma^f_{r\theta}|_{r=r_f} = K_t (u^m_{\theta}|_{r=r_i} - u^f_{\theta});
\]

where

\[
K_n = \frac{C_{rr}}{h}, \quad K_t = \frac{G_t}{h}
\]

(9.1)

are the normal and transverse stiffnesses and \( h \) the interphase thickness. Note that phase \( f \) in (9.1) is either a homogenous equivalent fiber or the SiC fiber shell with a diameter of \( r_f \).

Fig. 9.3 shows a comparison of total scattering cross-sections calculated versus frequency using spring and layer models for the same fiber-matrix system. The solid lines in the figure represent results for the actual multiphase fiber (core and shell) with the carbon-rich interphase. The fine dashed lines are for the spring model when the interphasial layer (\( C_{rr} = 31 \text{ GPa}, G_t = 4.6 \text{ GPa} \)) is replaced by distributed springs (\( K_n = 1.0 \times 10^{16} \text{ N/m}^3, K_t = 1.5 \times 10^{15} \text{ N/m}^3 \)). The coarse dashed lines result from the model in which the actual multiphase fiber is further
Figure 9.2: Scattering cross-sections from a 3-phase SiC fiber in titanium with a carbon-rich interphase. The dashed lines represent results when the actual fiber is replaced by either a homogeneous SiC or an equivalent fiber. (a) longitudinal wave incidence, (b) transverse wave incidence.
Fig 9.2 (continued)

\[ k_t^m \ r_f \]

```
TOTAL SCATTERING CROSS-SECTION \( q_t + q_f \)

--- HOMOGENEOUS SiC FIBER
--- EQUIVALENT FIBER
--- SiC FIBER WITH A CARBON CORE

SHEAR WAVE INCIDENCE
```

FREQUENCY \( f \) MHz

(b)
replaced by the equivalent fiber. The solutions obtained using both multiphase and
equivalent fiber models with interphasial springs (dashed lines) agree well with the
exact solutions for the 4-phase composite model in the low frequency range $f < 20$
MHz ($k_i^m r_i < 1.5$ or $k_i^m r_i < 3.0$). In this frequency range, the greatest interphase-
layer-thickness-to-wavelength ratio is about 0.04. We have so far demonstrated
that an equivalent homogeneous fiber with proper spring stiffnesses can be used to
describe scattering in the 4-phase composite in the low frequency range.

To illustrate the effect of the fiber-matrix interphase on ultrasonic wave scattering,
we show in Fig. 9.4 the dependence of scattering cross-section on the interphase
stiffness at a fixed frequency $f = 13$ MHz ($k_i^m r_i = 1.0$ or $k_i^m r_i = 2.0$). In the figure,
the bottom axis is the interphase radial modulus $C_{rr}$ whose value varies from 100
Gpa to 0.1 Gpa, simulating the weakening of the interphase. The top axis is the
corresponding normal $K_n$ or shear $K_t$ spring stiffness. The ratio of the interphase
radial-to-transverse shear modulus and the thickness and density of the interphase
are kept constant. The calculation is done for both the exact 4-phase model with
interphase layer (the solid line) and the equivalent fiber with the spring springs
(the dashed line). One sees from Fig. 9.4 that the interphase stiffness reduction
first leads to a slight scattering cross-section decrease to a minimum, then a sharp
increase with further interphase stiffness reduction. Finally it reaches the limit de-
finied by the scattering cross-section from a cylindrical cavity of the same diameter
(debonded fiber). This can be explained by impedance matching. For example, for
Figure 9.3: Scattering cross-sections from a 3-phase SiC fiber in titanium with a carbon-rich interphase. The dashed lines represent results when the interphase is replaced by distributed springs. (a) longitudinal wave incidence, (b) transverse wave incidence.
Fig 9.3 (continued)

\[ k_t^m \, r_i \]

--- EQUIVALENT FIBER+SPRINGS
--- ACTUAL FIBER+SPRINGS
--- ACTUAL FIBER+INTERPHASE

TOTAL SCATTERING CROSS-SECTION \( q_i + q_t \)

SHEAR WAVE INCIDENCE

FREQUENCY \( f \) MHz

(b)
the longitudinal wave the fiber impedance (37 g/cc \cdot Km/s) is higher than that of the matrix (32 g/cc \cdot Km/s). The initial cross-section reduction with decrease of interphase stiffness is due to reduction of the impedance of effective fiber (including the interphase), which leads to impedance match between the effective fiber and the matrix. The further decrease of the effective fiber impedance to a value lower than that of the matrix results in a sharp increase in the scattering cross-section. One sees from Fig. 9.4 that there is a significant effect of spring stiffnesses on the scattering cross-section suggesting that the scattered wave can be used for interphase characterization. One also notes that the model of equivalent fiber with springs gives a close approximation to that of actual fiber with the interphase layer. The agreement for the shear wave incidence is worse than that for the longitudinal wave incidence due to a greater wave number \(k_i^n r_i = 2.0\) versus \(k_i^n r_i = 1.0\).

The above explanation of the scattering cross-section minimum is rather qualitative. To be able to clarify this phenomena quantitatively we plot in Fig. 9.5(a) the dependence of scattering cross-section on interphase stiffness at a much lower frequency \(f = 1.3\) MHz \((k_i^n r_i = 0.1)\) calculated using the exact 4-phase model. In the calculation, the interphase is assumed isotropy with Poisson’s ratio equaling 0.25. At such low frequencies the scattering is mainly determined by the difference between the radial moduli \(C_{rr}\) of the matrix and the fiber. The radial modulus of the equivalent fiber \((C_{rr}^{eq} = 386\) GPa), calculated from Christensen’s static two-phase model for the fiber core and shell, is about twice that of the matrix
Figure 9.4: Scattering cross-section from the SiC fiber in titanium matrix as a function of variable interphase radial modulus $C_{rr}$. The frequency is 13 MHz. (a) Longitudinal wave incidence. (b) Transverse wave incidence.
Fig 9.4 (continued)

**SHEAR SPRING STIFFNESS** $K_t$ (N/m$^3$)

**SHEAR WAVE INCIDENCE**

$k_t^m r_i = 2.0$, $f = 13$ MHz

**SCATTERING CROSS-SECTION** $Q_t + Q_i$

**INTERPHASE RADIAL MODULUS** $C_{rr}$ (GPa)

(b)
\(C_{rr}^m = 193 \text{ GPa}\). Once again one sees that the interphase stiffness reduction first leads to a slight scattering cross-section decrease to a minimum (a slightly different position than that in Fig. 9.4(a)), then a sharp increase with further interphase stiffness reduction. Finally it reaches the limit defined by the scattering cross-section from a cylindrical cavity of the same diameter (debonded fiber). The limit of scattering for very high interphase stiffness corresponds to that from an absolutely rigid fiber. The initial cross-section reduction with decrease of interphase stiffness is due to reduction of the radial modulus of the effective fiber (including the actual fiber and the interphase), which leads to better stiffness matching between the effective fiber and the matrix. The further decrease of the effective fiber stiffness to a value lower than that of the matrix results in a sharp increase in the scattering cross-section.

To illustrate quantitatively the assertion of fiber-matrix stiffness matching, we calculate the radial modulus of the effective fiber versus the interphase radial modulus using the multiphase generalized self-consistent model for the 3-phase fiber (core, shell and interphase) [79, 80]. Fig. 9.5(b) shows the radial modulus of the effective fiber \(C_{rr}^e/C_{rr}^m\), normalized by that of the matrix, versus the interphase radial modulus. Recall from Fig. 9.5(a) that the scattering cross-section minimum is reached when the interphase radial’s modulus \(C_{rr} \approx 19.0 \text{ GPa}\). One sees from Fig. 9.5(b) that the radial modulus of the effective fiber equals that of the matrix \((C_{rr}^e/C_{rr}^m =1)\) occurring around the interphase radial’s modulus \(C_{rr} \approx 19.5 \text{ GPa}\).
Figure 9.5: (a) Longitudinal wave scattering cross-section versus interphase radial modulus at 1.3 MHz. (b) The normalized effective fiber moduli \( \frac{C_{rr}^{e}}{C_{rr}^{m}} \) versus the interphase Young’s modulus \( E \), where \( C_{rr}^{e} \) and \( C_{rr}^{m} \) are the radial stiffnesses of the effective fiber and matrix respectively.
Fig 9.5 (continued)

NORMAL SPRING STIFFNESS $K_n$ (N/m$^3$)

NORMALIZED RADIAL MODULUS $(C_{rr}/C^m)$

INTERPHASE RADIAL MODULUS $C_{rr}$ (GPa)

$C_{rr} = 19.5$ GPa
This provides an almost perfect match with the interphase moduli at which the minimum scattering occurs.

9.3 Resonance Phenomena for a Compliant Fiber-Matrix Interphase at Low Frequencies

A very interesting resonance phenomenon has been observed for a very compliant fiber-matrix interphase at low frequencies. We have done calculations for the scattering problem shown in Fig. 9.4 with much more compliant fiber-matrix interphase stiffnesses. The results are plotted in Fig. 9.6 showing numerous strong and sharp resonances at the same positions for both longitudinal and shear wave incidence. In the figure the frequency \( f \) is fixed at 13 MHz \((k_{l}^{\infty}r_i = 1.0\) or \(k_{l}^{\infty}r_i = 2.0\)), and the interphase radial modulus \( C_{rr} \) varies from 0.001 Gpa to 1.0 Gpa. The solid line is calculated using the exact 4-phase model (fiber core plus shell plus interphase layer), the coarse dashed line is for the model with equivalent fiber plus the interphase gap connected by springs and the fined dashed line is for the model with equivalent fiber plus interfacial springs where the gap is filled by the fiber materials. One sees from the figure that both spring models (dashed lines) agree well with the exact solution (the solid line) but fail to predict the resonance peaks at this frequency. There is also almost no difference between these two spring models, since the interphase is very compliant (see the discussions in Chapter VIII). The resonances can be seen as the spring-mass resonance phenomenon for a cylinder.
weakly connected to a surrounding solid medium by distributed springs.

The resonance phenomena are markedly different at much lower frequencies. In Fig. 9.7 we plotted the same scattering cross-section at a fixed frequency $f = 1.3$ MHz ($k_l^n r_i = 0.1$ or $k_l^n r_i = 0.2$) for both (a) longitudinal and (b) shear wave incidence. Here the vertical axis is in log scale. One sees that the number of resonances reduces dramatically, to one for longitudinal wave incidence and to two for shear wave incidence. The width of the major resonance peak is also much wider than those in Fig. 9.6, thus this resonance has more practical importance since it exists for a much larger range of interphase stiffness. At this low frequency, both spring models (dashed lines) predict all the resonance peaks. The spring model with the interphase gap (the coarse dashed line) gives results which are almost indistinguishable from the exact solution (the solid line) at this frequency, while the spring model with gap filled by fiber (the fine dashed line) deviates a little from the exact solution around the resonance peaks. Mathematically it is much simpler to study the resonances analytically using the spring model with gap filled by fiber, also asymptotic expansion of Bessel functions can be used at this low frequency ($k_l^n r_i = 0.1 \ll 1$).

To identify which mode contributes to the resonance, we plotted in Fig. 9.8 the cross-section of the scattered (a) longitudinal and (b) shear waves for longitudinal wave incidence. The parameters are the same as in Fig 9.7(a). One sees from the figure that the resonance exists in both scattered longitudinal and shear waves.
Figure 9.6: Resonances in the scattering cross-section from the SiC fiber in titanium matrix for small interphase radial moduli $C_{rr}$. The frequency is 13 MHz. (a) Longitudinal wave incidence. (b) Transverse wave incidence.
Fig. 9.6 (continued)

**SHEAR SPRING STIFFNESS** $K_t \text{ (N/m}^3\text{)}$

**SHEAR WAVE INCIDENCE**

$k_t^m r_t = 2.0$, $f = 13$ MHz

**INTERPHASE RADIAL MODULUS** $C_{rr} \text{ (GPa)}$

(b)
**Figure 9.7:** Resonances in the scattering cross-section from the SiC fiber in titanium matrix for compliant interphase radial moduli $C_{rr}$. The frequency is 1.3 MHz. (a) Longitudinal wave incidence. (b) Transverse wave incidence.
Fig. 9.7 (continued)

- - - FIBER CORE + SHELL + INTERPHASE
--- --- EQUIVALENT FIBER + SPRINGS (GAP)
---------- EQUIVALENT FIBER + SPRINGS
           (GAP FILLED BY FIBER)

SHEAR SPRING STIFFNESS $K_t$ (N/m$^3$)

SHEAR WAVE INCIDENCE
$k_t^m r_i = 0.2$, $f = 1.3$ MHz

INTERPHASE RADIAL MODULUS $C_{rr}$ (GPa)
The cross-section of the scattered longitudinal wave is higher than that of the shear wave for interphase stiffnesses away from the resonance peak, but smaller than that of the shear wave for interphase stiffnesses close to the resonance peak.

We also plotted in Fig. 9.9 the cross-section of the scattered (a) longitudinal and (b) shear waves for shear wave incidence. The parameters are the same as in Fig 9.7(a). One sees from the figure that the major resonance peak exists in both scattered longitudinal and shear waves, while the second sharper resonance peak is due to the mode-converted longitudinal wave. In this case the cross-section of the scattered longitudinal wave is the dominant factor.

Now let us focus on the major resonance peak around $C_{rr} = 0.04$ GPa. We have calculated the scattering cross-sections versus $k_i^m r_i$ and $k_i^m r_i$ at interphase radial modulus $C_{rr} = 0.04$ GPa. The results are shown in Fig 9.10 for both (a) longitudinal and shear wave incidence. One sees from the figure that in addition to the resonance peak at $f = 1.3$ MHz, there is also a resonance peak at a much higher frequency $f = 8.5$ MHz ($k_i^m r_i = 0.7$ or $k_i^m r_i = 1.4$). The spring models agree well with the first resonance peak at lower frequencies and also at higher frequency in general but fail to predict the second resonance peak.

In general for compliant fiber-matrix interphases and low frequency incident waves there exists an interphase stiffness and frequency domain where such resonances appear. The spring model with gap filled by fiber provides a possibility to analyze the conditions for resonances analytically by asymptotically expanding the
Figure 9.8: Scattering from the SiC fiber in titanium matrix versus interphase radial moduli $C_{rr}$ for longitudinal wave incidence. The frequency is 1.3 MHz. The cross-section for (a) the scattered longitudinal wave, and (b) the scattered transverse wave.
Fig. 9.8 (continued)

--- FIBER CORE+SHELL+INTERPHASE
--- EQUIVALENT FIBER+SPRINGS (GAP)
--- EQUIVALENT FIBER+SPRINGS
   (GAP FILLED BY FIBER)

NORMAL SPRING STIFFNESS $K_n$ (N/m$^3$)

SCATTERED SHEAR WAVE $Q_t$

L-WAVE INCIDENCE

$k_i^{\text{in}}r_i=0.1$, $f=1.3$MHz

INTERPHASE RADIAL MODULUS $C_{rr}$ (GPa)

(b)
Figure 9.9: Scattering from the SiC fiber in titanium matrix versus interphase radial moduli $C_{rr}$ for shear wave incidence. The frequency is 1.3 MHz. The cross-section for (a) the scattered longitudinal wave, and (b) the scattered transverse wave.
Fig. 9.9 (continued)

FIBER CORE + SHELL + INTERPHASE
--- --- EQUIVALENT FIBER + SPRINGS (GAP)
--- --- --- --- EQUIVALENT FIBER + SPRINGS (GAP FILLED BY FIBER)

SHEAR SPRING STIFFNESS \( K_t \) (N/m\(^3\))

SCATTERED SHEAR WAVE \( Q_t \)

SHEAR WAVE INCIDENCE
\( k_t^m r_t = 0.2, \ f = 1.3 \text{MHz} \)

INTERPHASE RADIAL MODULUS \( C_{rr} \) (GPa)

(b)
Figure 9.10: Total scattering cross-section from the SiC fiber in titanium matrix versus nondimensional wave number. The interphase radial modulus $C_{rr}$ is 0.04 GPa. (a) Longitudinal wave incidence. (b) Transverse wave incidence.
Fig. 9.10 (continued)

FREQUENCY (MHz)

SCATTERING CROSS-SECTION $Q_i+Q_t$

SHEAR WAVE INCIDENCE
$C_{rr} = 0.04$ GPa

- FIBER CORE+ SHELL+ INTERPHASE
- EQUIVALENT FIBER+SPRINGS (GAP)
- EQUIVALENT FIBER+SPRINGS (GAP FILLED BY FIBER)

$k_t^m r_i$

(b)
Bessel functions in the scattering cross-sections. Note that not all the cylindrical waves of order \( n \) are critical for the resonance. As an example let us consider the longitudinal cross-section for longitudinal wave incidence around the first resonance peak at \( f = 1.3 \text{ MHz} \ (k^m_r = 0.1) \). The results are shown in Fig. 9.11 where the solid line for \( Q_l \) calculated using the exact model, the fine dashed line is for \( Q_l \) calculated using the spring model with gap filled by fiber, the crosses are the zeroth order of \( Q_l(n = 0) \) calculated using the spring model with gap filled by fiber while the coarse dashed line is the first order of \( Q_l(n = 1) \). One sees from the figure that there is a small deviation between the exact solution (solid line) and the predictions by the spring model with gap filled by fiber (dashed lines), but the spring model should be sufficient for study of the resonance condition. One also sees from the figure that the resonance is determined by the first order of \( Q_l(n = 1) \) (the coarse dashed line) which is almost indistinguishable from the infinite summation of \( Q_l \) (the fine dashed line) on \( n \) around resonance peak, while the zeroth order does not contribute to the resonance.

9.4 Summary

It is demonstrated that an equivalent homogeneous fiber with springs between the matrix and the fiber can be used to describe scattering in the low frequency range. Numerical examples show that scattering is significantly affected by the interphase stiffnesses and thus can be used for interphase characterization. The minimum
Figure 9.11: Longitudinal scattering cross-section from the SiC fiber in titanium matrix versus nondimensional wave number for longitudinal wave incidence. The interphase radial modulus $C_{rt}$ is 0.04 GPa.
of the scattering cross-section versus interphase stiffness is shown to be due to matching of stiffness between the effective fiber (actual fiber plus interphase) and the matrix. It is also shown that for compliant fiber-matrix interphases there exist resonance peaks in the scattering cross-sections, and those resonances at very low frequencies can be analyzed analytically by the simplified spring models. \( \alpha \)
CHAPTER X

ATTENUATION AND DISPERSION OF ELASTIC WAVE PROPAGATING NORMAL TO FIBERS IN A COMPOSITE. THEORY AND EXPERIMENT

The results for a single fiber in a matrix will be extended in this and the next chapter to composites where wave attenuation and velocity are investigated both theoretically and experimentally. An important motivation for performing such studies is to provide necessary data to supplement recent efforts in our laboratory for development of ultrasonic methods for interphase characterization in high-temperature composites [7, 8, 9, 10, 11, 79, 87]. The problem of interphase characterization is important since the interphase transfers the load from fiber to matrix and the quality of the bond between the interphase and the surrounding materials (fiber or matrix) determines the overall mechanical performance of the composite.

Chu and Rokhlin have recently determined fiber-matrix interphase effective elastic properties from ultrasonically measured composite moduli [7, 8, 9, 10, 79].
The method is based on measurements of low frequency ultrasonic wave velocities in different directions in the composite and relating them to the static composite elastic moduli. The interphase properties were determined using the static generalized self-consistent micromechanical model for inversion. However the relation between the wave velocity measured by low frequency ultrasound and its static limit, i.e. the frequency dependence of the wave velocity, has not yet been fully studied. Thus error may be introduced when using the velocity data to determine the static composite moduli if dispersion is significant. In this paper we will focus on dispersion and attenuation of waves propagating normal to fibers [89]. The results for wave propagating along fibers will be given in the next chapter.

10.1 Introduction

Due to the increasing importance of composites in high performance light structures, there is interest in their nondestructive characterization. Application of ultrasonic waves is most promising for this purpose since wave propagation is directly affected by the mechanical properties of fibers, matrix and fiber-matrix interphases. In this paper we will discuss frequency dependences of fiber scattering-induced ultrasonic velocity and attenuation at propagation perpendicular to the fiber. Analysis of wave propagation along the fiber direction in a unidirectional composite will be presented in the next chapter.

A significant amount of theoretical work has been done on determining the
wave propagation characteristics in a multi-scatterer medium [92, 93, 94, 95, 96, 97, 90, 98, 99, 100, 101, 102, 103, 104]. Most of the models start with the analysis of a single scatterer. The total scattering effects by all the scatterers are accounted for using different approaches. The first is called the independent scatter model [90], where the scatterers are assumed to be independent and the wave attenuation is found by summing the energy scattered by individual scatterers. In this model the wave velocity can be calculated from attenuation using the Kramers-Kronig relations [102]. Another method is called the effective medium approach. Urick and Ament [93] considered scattering from an imaginary thin slab of unknown properties to represent the actual multi-scatterer media. For particulate composites, the scattering from an imaginary sphere of unknown effective properties is considered [101].

Finally, different averaged wave methods are suggested where the complex wave number in a multi-scatterer medium is found using a statistical averaging procedure [92, 95]. The basic approximation is that the wave field incident at the jth scatterer may be represented by the total wave field that would exist at the scatterer if the scatterer was not there. An incident wave plus the scattering from all other scatterers except the jth is considered as the exciting field, which, in turn, causes a scattered field. A configurational average value of the exciting field is obtained as the total average field. This result is also widely referred as the Waterman-Truell multiple scattering model. For isotropic scatterers a similar solution was obtained
earlier by Foldy [92], which can be considered as the first order low frequency approximation to the Waterman-Truell solution.

Extensive comparative studies of the effective medium and average wave methods have been reported (see, for examples, [105, 106, 107]). It has been shown that these two approaches are equivalent in the low frequency limit. Specifically it was pointed out by McClements et al [107] that these two methods ([92] or [95] and [101]) have implicitly the same first order terms in the wave number. It was also understood [96, 97, 108] that the Waterman-Truell multiple scattering solutions have similar forms to those of Urick and Ament [93] obtained using the effective medium approach with thin slabs. This implies that there is an assumption of supposition of thin slabs in the Waterman-Truell model.

The Waterman-Truell multiple scattering model has received some validation and has been widely accepted for analysis of multiple scattering effects. In this paper we will utilize this model and its variations to study the frequency dependences of wave attenuation and velocity in unidirectional fiber-composites. A comparison of various modifications of the Waterman-Truell model with the experimental results will be conducted. The focus will be on wave attenuation and dispersion at low frequencies where the experimental data in high temperature composites are obtainable. The applicability of the independent scatter model will also be discussed. The model composite system used in this work for analytical and experimental studies is an unidirectional titanium alloy matrix composite reinforced
with 24% 140 µm SiC fibers (SCS-6). Since the SCS-6 fiber has a multilayered structure (carbon core + SiC shell) and a 3 µm carbon-rich interphase coating, the theoretical models developed account for the multilayered cylindrical structure of the composite. We will end our analyses with discussions of the fiber-matrix interphase effects on wave attenuation and dispersion in this composite, and of dynamic corrections for determination of static composite moduli from ultrasonic wave velocities.

10.2 Theory

In this section we will describe the theoretical models used for prediction of wave attenuation and velocity in an unidirectional composite reinforced by randomly distributed identical multilayered fibers as shown in Fig. 10.1. The fiber itself may have a multiphase structure. Since the fibers have different density and elastic moduli than the matrix, a plane wave incident on a fiber will be scattered, resulting in wave attenuation. The interference of scattered waves from different fibers leads to changes in phase velocity compared to that calculated based on composite static elastic moduli. Since the scattering by a single fiber with a finite diameter is frequency-dependent, one would expect that the velocity and attenuation of the wave in composites depend on frequency too. To characterize the wave propagation in a composite, one may use the complex wave number \( \beta = \omega/V^c + i\alpha \), where \( \omega \) is the circular frequency, \( V^c \) is the wave velocity and \( \alpha \) is the attenuation coefficient.
Figure 10.1: Elastic wave propagation normal to fibers in an unidirectional composite. The example shown here is for an SCS-6 fiber composite.

COMPLEX WAVE NUMBER $\beta = \frac{\omega}{v^c} + i\alpha$
10.2.1 Scattering from a single multiphase fiber in matrix

Before we discuss the predictions of wave attenuation and velocity in composites of different models, we briefly describe the results for scattering from a single multiphase fiber in matrix which forms the basis for later discussions. Let us assume a longitudinal or transverse plane wave (polarized perpendicular to the fiber axis) incident normally on the interface between the matrix and the multiphase fiber as shown in Fig. 10.2. The scattered waves in the matrix are represented in the form of outgoing waves:

$$\varphi^s = \sum_{n=-\infty}^{\infty} A_n^m H_n(k_1^m r) \exp(in\theta), \quad \psi^s = \sum_{n=0}^{\infty} B_n^m H_n(k_1^m r) \exp(in\theta),$$

where $H_n(x) = J_n(x) + iN_n(x)$, $N_n(x)$ is a Bessel function of the second kind or Neumann function, and $H_n(x)$ is a Hankel function of the first kind.

In Chapter VII, we have presented a transfer matrix procedure for solving the scattering from a multiphase fiber. Once the scattering coefficients $A_n^m$ and $B_n^m$ are found the total scattering cross-section $\gamma$, which is defined as the total power scattered per unit length divided by the incident wave intensity, is found to be [61]:

$$\gamma = 4r_f(Q_i + Q_t),$$

where $r_f$ is the fiber radius (including the interphase), and

$$Q_i = \frac{1}{k_1^m r_f} \sum_{n=0}^{\infty} |A_n^m|^2, \quad Q_t = \frac{1}{k_1^m r_f} \sum_{n=0}^{\infty} |B_n^m|^2.$$
Figure 10.2: Geometry of elastic wave scattering from a multi-phase fiber embedded in a solid matrix. The example shown here is for a SCS-6 fiber in titanium matrix.
Here \( Q_l \) and \( Q_t \) are the normalized (by \( 4\pi f \)) scattering cross-sections for longitudinal and transverse waves respectively and \( k_i^m \) is the wave number of the incident wave (longitudinal or transverse). Other important parameters are the forward and backward scattering amplitudes which are defined as

\[
\begin{align*}
  f(0) &= \sum_{n=-\infty}^{\infty} (-i)^n (A^n, B^n)_n, \\
  f(\pi) &= \sum_{n=-\infty}^{\infty} (i)^n (A^n, B^n)_n.
\end{align*}
\] (10.4)

In the above equations \( A^n \) is used for the longitudinal incident wave and \( B^n \) is used for the transverse incident wave.

### 10.2.2 Independent scatter and Waterman-Truell multiple scattering models

#### Independent scatter model

The independent scatter model assumes that the individual scatterers (fibers) can be regarded as independent of each other, thus neglecting multiple scattering. The independent scatter model is applicable when the fiber fraction is small or when the impedance difference between the fiber and matrix is small so that the fibers can be considered as weak scatterers rendering multiple scattering effects negligible. Let \( I_0 \) be the initial wave intensity at \( x = 0 \), where \( x \) is the wave propagation distance, so the wave intensity at position \( x \) is \( I = I_0 e^{-2\alpha x} \). If the scattering cross-section \( \gamma \) (10.2) is known then the amount of energy scattered is \( n_s \gamma I \), where \( n_s = c/(\pi r_f^2) \) is number of fibers per unit area, \( c \) is the fiber fraction and \( r_f \) the fiber radius.
Thus [90]

\[ \frac{dI}{dx} = -2\alpha I = -n_s \gamma I, \]  

(10.5)

i.e. the attenuation coefficient \( \alpha \) can be found as

\[ \alpha \approx \frac{1}{2} n_s \gamma. \]  

(10.6)

Since the single fiber scattering cross-section \( \gamma \) depends on frequency, so does the attenuation coefficient \( \alpha \) (10.6).

The independent scattering model does not deal directly with the effect of scattering on wave velocity. However the wave velocity can be found from the Kramers-Kronig relations for the real \( (\alpha) \) and imaginary \( (\omega/V) \) parts of a complex wave number \( \beta \) if the medium is linear, causal and passive [102, 109, 110, 111]. In applying the Kramers-Kronig relations to waves in a composite one can write \( V^c(\omega) \) in terms of \( \alpha(\omega) \) using Hilbert transforms as [102]

\[ V^c(\omega) = V_0^c [1 + \frac{2\omega^2 V_0^c}{\pi} \int_0^\infty \frac{\alpha(\omega')d\omega'}{\omega'^2(\omega'^2 - \omega^2)}]^{-1}, \]  

(10.7)

where \( V_0^c \) is the zero-frequency velocity limit in composite: \( V_0^c \equiv V^c(\omega = 0) \), and the Cauchy principal values of the integral along the real axis of \( \omega \) are taken in (10.7).

There exist several difficulties in generalizing (10.7) to determine the wave dispersion in composites using \( \alpha(\omega) \) calculated from the independent scatter model (10.6). The first is the determination of the zero-frequency velocity \( V_0^c \). One way
to find \( V'_0 \) is to use the composite moduli and density as suggested by Beltzer [102], where the moduli are determined from available static micromechanical models for composites (see for example [52]). However, since the attenuation is calculated by summing the scattered energy from individual fibers, each embedded in the matrix medium, there is no theoretical foundation for coupling together two completely different models. The second is the definition of attenuation at high frequencies. In the high frequency limit, statistical homogenization of the composite and the assumption that the plane wave is only slightly perturbed by the inhomogeneities breaks down. Nevertheless using the Kramers-Kronig relation and the attenuation determined from the independent scatter model for velocity calculation may provide some insight into the wave dispersion behavior in composites at very low frequencies.

**Waterman-Truell multiple scattering models**

In cases where the multiple reflections from fibers become significant, the fibers (scatterers) can not be considered as acting independently (this may occur at a high fiber fraction). Thus multiple scattering models must be considered. Here we focus on the Waterman-Truell approach which finds the wave number of an average wave in a multi-scatterer medium [95]. This procedure consists of averaging a joint probability distribution for a given configuration of scatterers over all configurations of scatterers. The basic approximation is that the wave field incident
at the jth scatterer may be represented by the total wave field that would exist
at the scatterer if the scatterer were not there. An incident wave plus the scat-
tering from all other scatterers except the jth produces the exciting field, which,
in turn, causes a scattered field. A configurational average value of the exciting
field is obtained as the total average field. For a unidirectional composite having N
identical randomly distributed fibers, the multiple scattering field yields a complex
wave propagation constant β in the form

\[ \left( \frac{\beta}{k_m} \right)^2 = \left[ 1 - \frac{2i n_s f(0)}{(k_m)^2} \right]^2 - \left[ \frac{2i n_s f(\pi)}{(k_m)^2} \right]^2, \]  

(10.8)

where \( k_m \) is the wave number in the matrix, and \( f(0) \) and \( f(\pi) \) are the forward and
backward scattering amplitudes (10.4) from a single fiber embedded in the matrix
medium as shown in Fig. 10.3. Since \( f(0) \) and \( f(\pi) \) are frequency dependent, both
\( V^c \) and \( \alpha \) depend on frequency. Eq. (10.8) gives the basic formula for various mod-
ified multiple scattering models discussed below. At very low frequencies \( (kr \ll 1) \)
the fiber acts like an isotropic scatterer, i.e. having amplitudes for \( f(0) \) and \( f(\pi) \)
about equal, so Eq. (10.8) simplifies to

\[ \left( \frac{\beta}{k_m} \right)^2 \approx 1 - \frac{4i n_s f(0)}{(k_m)^2}. \]  

(10.9)

The solution in the form of Eq. (10.9) was given by Foldy [92] using a similar
average wave approach for isotropic scatterers. Note that the second order terms
in Eq. (10.8) do not appear in the equation. Eq. (10.9) is valid for either low
concentration or weak scatterers, which implies very low frequencies.
Figure 10.3: In the Waterman-Truell multiple scattering model: the single fiber scattering problem is considered for a fiber embedded in matrix.
For composites with very high fiber fractions, (10.8) and (10.9) give incorrect velocity limits in the full-packed case where the fiber fraction reaches unity. The reason is that the forward and backward scattering amplitudes are found for a single fiber embedded in matrix material, which is not quite true for the high fiber fraction limit. We next consider two interesting variations of the Waterman-Truell solutions by defining the single fiber scattering problem differently for composites with higher fiber fractions. First, since the velocity of the incident wave is also affected by the fibers, one may consider the fibers embedded in a composite medium as shown in Fig. 10.4(a). The properties of the composite medium can be calculated using the multiphase generalized self-consistent model as described in [79, 80]. One drawback of this approach is that the fiber fraction is artificially increased compared to the true value. The other variation is to take a concentric cylinder which consists of the fiber as the cylinder core and the matrix as the annulus, and embed it in the composite medium as shown in Fig. 10.4(b). In this case the fiber fraction is preserved. Since in both variations the properties of the composite medium are static, implicitly the low frequency approximation is assumed. In the next section we will discuss two other variations where the composite medium is treated as a dynamic effective medium.
Figure 10.4: Modified Waterman-Truell scattering models where single fiber scattering is considered for (a) a fiber embedded in a composite medium, (b) a composite cylinder embedded in a composite medium.
Self-consistent and generalized self-consistent multiple scattering models

Sayers and Smith [98, 99] have defined the effective medium surrounding the scatterers in a self-consistent manner as shown in Fig. 10.5(a). In their theory the density of the effective medium is taken the same as the composite, and two types of scatterers are considered embedded in the effective medium, one being the fiber and the other the matrix (with the same fiber diameter). The unknown wave number of the effective medium $k^e$ is defined as the same as the complex wave number $\beta$ needing to be determined. In other words equation (10.8) is solved in a self consistent manner:

$$\left(\frac{\beta}{k^e}\right)^2 = \left[1 - \frac{2i n_s \bar{f}(0)}{(k^e)^2}\right]^2 - \left[\frac{2i n_s \bar{f}(\pi)}{(k^e)^2}\right]^2$$

(10.10)

where $\bar{f}(0)$ and $\bar{f}(\pi)$ are the forward and backward amplitudes of a single scatterer averaged over the two types of scatterers by their volume fractions, and are functions of the unknown wave number $k^e$ of the embedding medium. The above equation can be solved by iteration where the wave number of the matrix $k^m$ is taken as the initial guess for $k^e$.

However one drawback of the self-consistent model (10.10) is that it does not give the same zero-frequency velocity limit as that determined from the well-known static solution for composite moduli [52, 79, 82]. It is essential for the multiple-scattering model to predict the correct zero-frequency velocity limits since low
frequency velocity measurements have been widely used for measurements of static composite moduli. One solution to this problem is to modify the self-consistent multiple-scattering model such that a concentric cylinder, with fiber as the core and matrix as the annulus, is considered embedded in the unknown effective medium as shown in Fig. 10.5(b) in line with the static generalized self-consistent model [52]. This dynamic model was developed by Mal and Yang [104] for study of SH shear wave propagation, and by Huang and Rokhlin [88] for longitudinal and SV shear wave propagation in composites. The forward and backward scattering amplitude \( f(0) \) and \( f(\pi) \) are found for scattering from the concentric cylinder by waves incident from the effective medium and (10.8) is solved in a self-consistent manner similar to (10.10). In other words

\[
\left( \frac{\beta}{k^e} \right)^2 = \left[ 1 - \frac{2\,i\,n_s\,f(0)}{(k^e)^2} \right]^2 - \frac{\left( 2\,i\,n_s\,f(\pi) \right)^2}{(k^e)^2}
\]

Equation (10.11) can be solved by an iteration procedure where the wave number of the static composite \( k^c \) is taken as the initial guess for \( k^e \). In this case the solution of equation (10.11) after the first step of iteration is the same as that (Fig. 10.4(b)) given by the modified Waterman-Truell model where a fiber with matrix layer is embedded in the static composite medium. Note that in both (10.10) and (10.11) the unknown wave numbers for both longitudinal and shear waves have to be found together during the iteration process, since \( f(0) \) and \( f(\pi) \) depend on both the longitudinal and shear wave numbers of the unknown effective
Figure 10.5: Self-consistent modifications of Waterman-Truell scattering models where the single scatterer problem is found for (a) a scatterer (fiber or matrix) and (b) a composite cylinder embedded in an effective medium in a self-consistent manner.
medium.

10.3 Experimental Measurement of Wave Dispersion and Attenuation in a SiC/Ti Unidirectional Composite

10.3.1 Samples description

The composite samples are titanium alloy matrix reinforced with about 24% SiC fibers (SCS-6). Each fiber consists of a carbon core with a SiC shell along with a 3 μm carbon-rich interfacial layer placed between the fiber and the matrix. The fiber fraction is 24%. The grain scattering has been found negligible due to the large fiber size and homogeneous matrix, thus the attenuation and dispersion of the sound wave is dominated by the effect of fiber-matrix interphases.

The as-received sample is 50/25/15 mm (length/width/height) with fibers in the length direction. Since in the actual measurement the attenuation of the sound wave sets the upper limit of sample thickness, and the bulk wave requirement sets the lower limit of thickness, the as-received sample was subsequently sliced into pieces of desirable thicknesses and orientations as shown in Fig. 10.6. The samples with fiber normal to the through-thickness direction were used in this paper, while others with fibers in the through-thickness direction were used in [112] for study of wave dispersion and attenuation along fibers. The fibers are distributed randomly as shown by the scanning electron micrographs (SEM) in Fig. 10.7. As one can easily see from the figure there is a 3 μm carbon-rich interfasial layer between the fiber and the matrix. The fiber is formed by chemical vapor deposition (CVD)
Figure 10.6: The slicing of an as-received SiC/Ti unidirectional composite into samples with desirable thickness and orientations.
Figure 10.7: SEM photograph of a SiC/Ti unidirectional composite sample. In the center of the fiber one can see the carbon core. The 3 \( \mu \text{m} \) thin interphasial layer of carbon-rich coating is between the SiC shell and the titanium matrix.
Table 10.1: Transverse properties of each phase in a SCS-6/Ti composite.

<table>
<thead>
<tr>
<th>Phase</th>
<th>$C_{rr}$ (GPa)</th>
<th>$G_t$ (GPa)</th>
<th>$\rho$ (g/cc)</th>
<th>$r$ (µm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>core (carbon)</td>
<td>49</td>
<td>16</td>
<td>1.7</td>
<td>18</td>
</tr>
<tr>
<td>shell (SiC)</td>
<td>446</td>
<td>177</td>
<td>3.2</td>
<td>68</td>
</tr>
<tr>
<td>interphase (carbon-rich coating)</td>
<td>31</td>
<td>4.6</td>
<td>2.1</td>
<td>71</td>
</tr>
<tr>
<td>matrix (titanium alloy)</td>
<td>193</td>
<td>45</td>
<td>5.4</td>
<td>-</td>
</tr>
</tbody>
</table>

where $C_{rr}$ is the radial modulus, $G_t$ is the transverse shear modulus, $\rho$ is the density and $r$ the radius of the boundary of that phase.

of two layers of SiC on the carbon core, one with coarse grains, one with fine grains [9, 10]. There is a pyrolytic carbon coating between the carbon core and the inner layer of the SiC shell. The SiC microstructure does not have a noticeable effect on elastic wave scattering and SiC shell is assumed homogeneous in modelling as for the carbon core.

The material properties of the fiber core and shell and of the matrix are listed in Table 10.1. The titanium alloy matrix properties are obtained experimentally on a matrix sample cut from the excess matrix at the edges of the composite sample. The grain scattering in the matrix material has been found very small, thus the attenuation in the composite is attributed to the wave interaction with the fibers. The fiber properties are obtained from published data as discussed in [9, 10]. The interphase elastic properties are in-situ properties, and depends on the chemical reaction and mechanical contact between the pyrolytic carbon coating and the surrounding SiC and titanium alloy. As illustrated in [9, 10] for SCS-6 fibers in ceramic and inter-metallic composites, the interphase properties can vary
significantly and exhibit strong anisotropy.

10.3.2 Determination of interphase moduli

To find the interphase moduli which are in-situ properties, we measured composite moduli using low frequency velocity data in the planes perpendicular and parallel to the fibers. The details of the bulk wave velocity measurement method and the procedure for determination of interphasical properties have been described in [7, 8, 9, 10, 79]. The velocity measurements were done along different directions in each plane for both longitudinal and shear waves using a 5 MHz transducer. The data shown in Fig. 10.8 are obtained from two specimens differently cut from a single piece of composite, one 2.7 mm thick with fibers parallel to the sample surface (open circles in Fig. 10.8), the other is 2.8 mm thick with fibers in the through-thickness sample direction (closed circles in Fig. 10.8). These velocities are used to determine the composite moduli from the Christoffel equations through a least square procedure as discussed in [7]. The composite moduli thus determined are dynamic and not necessarily equal to their static values due to dispersion. As will be shown the greatest dispersion-induced error in thus-determined static composite moduli occurs along the fiber direction which is about 4% [112] and is negligible in the transverse direction (see below), one can conclude that these composite moduli closely reflect their true static values.

The interphase moduli are reconstructed from the ultrasonically determined
Figure 10.8: Measured bulk wave velocity data versus propagation angle in the composite in planes (a) parallel to the fibers and (b) perpendicular to the fibers.
composite moduli using the multiphase generalized self consistent model [79, 80]. The reconstructed interphase transverse elastic moduli are: $C_{rr} = 31$ GPa and $G_t = 4.6$ GPa. The unusually high transverse bulk-to-shear modulus ratio is due to the interphase anisotropy. Specifically, as described in [9, 10], the interphase can be considered as a mixture of the same isotropic carbon as in the fiber core and pyrolytic carbon with a preferred basal plane orientation along the circumferential direction, plus additional reinforcement of 25% fine SiC particles. The interphasial moduli obtained are "effective" in the sense that they include the effect of imperfect contact between the interphase and the surrounding media. The density of the interphase is 2.1 g/cc, calculated by considering 25% SiC particles embedded in the carbon coating.

10.3.3 Techniques for measurements of frequency dependences of wave velocity and attenuation

Immersion ultrasonic wave attenuation and velocity measurements were performed in the through-thickness direction at normal incidence using the through-transmission setup. The samples are mounted on a rotation table which is interfaced with a host computer via a DC motor controller. Alignment is achieved by maximizing the reflection echo received by each transducer in the pulse-echo mode. Excitation and reception of the signals is done using a Panametrics 5052PR pulser-receiver and a LeCroy 9400 digital oscilloscope. The amplitude spectrum of the received signal is obtained from an FFT of the time-domain signal after digitization and averaging.
The water temperature is stabilized at 29.8 ± 0.1° by thermal circulators during measurements.

The through-transmitted signals were measured twice for each experiment, once with the sample present \(T_1(t)\) and once without the sample \(T_0(t)\). Let \(T_0(f)\) and \(T_1(f)\) represent the complex Fourier spectra of these two signals:

\[
T_0(f) = \int_{-\infty}^{\infty} T_0(t)e^{i\omega t} dt, \quad T_1(f) = \int_{-\infty}^{\infty} T_1(t)e^{i\omega t} dt.
\] (10.12)

One can find wave attenuation and dispersion in the composite using the magnitude and phase of the deconvolved spectrum \(T_1(f)/T_0(f)\) respectively [113]. Let us first consider the frequency dependence of the phase velocity [113, 114]. Suppose the reference time delays for signals \(T_0(t)\) and \(T_1(t)\) after digitization are \(t_0\) and \(t_1\), the phase spectrum of the deconvolved function \(T_1(f)/T_0(f)\) is

\[
\phi[T_1(f)/T_0(f)]V^c(f) = \omega h/V^c - \omega h/V_0 - 2\pi f(t_0 - t_1) + 2n\pi,
\] (10.13)

where \(f\) is the frequency, \(h\) the sample thickness, \(\phi(\cdot)\) the phase of a complex number, \(n\) is the correction integer for spurious “2π” errors, \(V_0 = 1.51\) Km/s - longitudinal wave velocity in water. Thus the phase velocity of the longitudinal wave in the sample is found as

\[
V^c(f) = \frac{\omega h}{\omega h/V_0 - 2\pi f(t_0 - t_1) - \phi[T_1(f)/T_0(f)] + 2n\pi}.
\] (10.14)

Once the frequency dependence of the composite velocity is obtained, the at-
Attenuation $\alpha(f)$ in this composite is calculated from [113]

$$
\alpha(f) = \frac{\ln[|\frac{T_0(f)}{T_1(f)}| \cdot t(f) \cdot D]}{h},
$$

(10.15)

where $t(f) = 4\rho_0 V_0 \rho^2 V_c^c(f)/\{[\rho^2 V_c^c(f)]^2 - (\rho_0 V_0)^2\}$ is the energy transmission coefficient for the sample-water interface calculated using the measured velocity $V_c^c(f)$ in the composite and $\rho_0 = 1$ g/cc is density of the water. $D$ is the beam diffraction correction for these two through-transmitted signals ($D$ is only significant at the low frequency limit and is calculated using Bass's diffraction expression [115]).

### 10.4 Experimental Results. Comparison with Theory

The wave velocity and attenuation measurements have been performed on several samples with different thicknesses cut from the same composite panel. As an example the time domain signals of the directly through-transmitted wave measured using a pair of 15-20 MHz transducers are shown in Fig. 10.9(a). Here $T_0(t)$ is the reference signal measured with the sample absent, $T_1(t)$ is the directly transmitted signal through the sample, and $T_1(t)$ is the through-transmitted signal after complete internal reflection inside the sample. The attenuation setting of the pulser/receiver for the reference measurements without the sample is different from that with the sample. One sees from the figure that the second through-transmitted signal $T_2(t)$ is masked by the reverberation noise from multiply-scattered signals inside the composites and has a low signal-to-noise ratio; thus we only use the
directly through-transmitted signals $T_1(t)$ and the reference signal $T_0(t)$ for determinations of both wave velocity and attenuation. The frequency spectra $|T_1(f)|$ and $|T_0(f)|$ are shown in Fig. 10.9(b). The wave velocity and attenuation are determined from the phase and magnitude of the deconvolved spectrum $T_1(f)/T_0(f)$ using equations (10.14) and (10.15).

The typical results for wave attenuation (in Np/cm) and velocity (in Km/s) versus frequency are shown in Fig. 10.10 by open circles for a sample with thickness $h = 2.7$ mm. The experimental results were measured by two pairs of transducers (15-20 MHz and 5-10 MHz) to expand the measurable frequency range. Note that the averaged values were taken where the results obtained using the two transducer pairs overlapped. The predictions by various models are also shown in the figure for comparison. The crosses are calculated using the independent scatter model (10.6, 10.7). The coarse dashed line is the Waterman-Truell solution (10.8), the fine dashed line is the modified Waterman-Truell solution where a fiber surrounded by a matrix layer is embedded in a composite medium (see Fig. 10.4(b)) and the solid line is for the generalized self-consistent Waterman-Truell model where a fiber surrounded by a matrix layer is embedded in a dynamic effective medium (see Fig. 10.5(b) and (10.11)). The properties of each phase used in the calculations are the same as those in Table 10.1. Recall that the interphase moduli were determined independently using bulk wave velocity measurements obtained by a 5 MHz transducer.
Figure 10.9: Ultrasonic results for a SiC/Ti unidirectional composite sample (fibers perpendicular to the surface normal) measured in a transmission mode using 15 and 20 MHz broadband immersion transducers. (a) The time domain signals of the through-transmitted waves; (b) the spectra of the directly through-transmitted waves with ($T_1(f)$) and without ($T_0(f)$) the sample.
Figure 10.10: Experimental data and their comparison with different models. (a) Frequency dependence of wave attenuation. (b) Frequency dependence of wave velocity.
Let us first consider the attenuation data shown in Fig. 10.10(a). One sees from the figure that agreement between the independent scatter model (the crosses) or the Waterman-Truell model (the coarse dashed line) and the experimental data is very good in the frequency range measured. This indicates that the multiple scattering effect is insignificant and this can be attributed to two factors. The first is that the impedance difference between the effective fiber (including the interphase) and the matrix is small (37 g/cc \cdot Km/s versus 32 g/cc \cdot Km/s), thus the fibers can be considered weak scatterers although the fiber fraction is not very small (24%). The impedance of the effective fiber is calculated from its elastic moduli determined by the multiphase generalized self-consistent model [79, 80] using the properties for each fiber phase (fiber core, shell and interphase) listed in Table 10.1. The other is that the greatest fiber-radius-to-wavelength ratio is less than 0.2 in the frequency range measured, thus the scattering occurs in the low frequency region. One also sees from the figure that the other two modified Waterman-Truell models give much lower attenuations than those given by the Waterman-Truell model. This is because these models consider a fiber surrounded by a matrix layer embedded either in a static composite medium (the fine dashed line, see Fig. 10.4(b)) or in a dynamic effective medium (the solid line, see Fig. 10.5(b)) where the impedance difference between the scatterer and its surrounding medium is much smaller than that between the effective fiber and matrix. Therefore the forward and backward scattering amplitudes are much smaller, resulting in much
less attenuation.

The corresponding results for dispersion are shown in Fig. 10(b) (the symbols are the same as in Fig. 10.10(a)). The measured velocity (open circles) has little dispersion, changing from 6.40 Km/s to 6.32 Km/s for frequency varying from 2 to 17 MHz. The velocities calculated using the Kramers-Kronig relations (10.7) and the attenuation data determined by the independent scatter model are shown in the figure by crosses. In the calculation the static wave velocity $V_0^c$ is found from the static composite moduli determined by the multiphase generalized self-consistent model [79, 80]. Surprisingly, while the attenuation predicted by independent scattering model is very close to the experimental data, the wave velocities found by the Kramers-Kronig relations have the greatest deviations from the experimental data for the frequency range measured. This indicates that caution should be used when applying the Kramers-Kronig relations to composite materials, as discussed previously. Both the Waterman-Trueell model (the coarse dashed line) and the modified Waterman-Trueell models, where a fiber surrounded by a matrix layer is embedded either in a static composite medium (the fine dashed line, see Fig. 10.4(b)) or in a dynamic effective medium (the solid line, see Fig. 10.5(b)), are in good agreement with the experimental data for wave velocities.

From the comparison of the experimental data with the model predictions, it appears that wave attenuation and velocity in composites are controlled by different scale factors. Attenuation is controlled by a local effect and can be described
by scattering models where a fiber is considered immersed in unbound matrix. However dispersion is affected by multiple interactions from a large number of fibers and is a more global phenomenon. Thus dispersion is best described by models where a fiber surrounded by a matrix layer is considered immersed in a composite or effective medium. Discussion of different models is continued in the next section.

10.5 Comparison of Different Models, and Analysis of a Dynamic Effect on Interphase Property Determinations

The comparison of wave attenuation and velocity calculated using all the models discussed in Section 10.2 is given in Fig. 10.11. In the figure, the points are calculated using Foldy's isotropic multiple scattering model (10.9), the open squares are for the modified Waterman-Truell model where the fiber is considered embedded in the composite medium (Fig. 10.4(a)) and the open triangles are for the Sayers-Smith self-consistent model ((10.10), see Fig. 10.5(a)).

Let us first consider the attenuation spectra shown in Fig. 10.11(a). One sees from the figure that the predictions of the Foldy isotropic scatterer model (points) are very close to those calculated from the Waterman-Truell model (the coarse dashed line) especially for frequencies not too high, indicating that at low frequencies the fibers act more like isotropic scatterers and the difference between the Foldy solution (10.8) and the Waterman-Truell solution (10.9) is small. It is
Figure 10.11: Calculated longitudinal wave attenuation spectra and dispersion for a unidirectional titanium alloy composite reinforced with 24% SiC fibers using different models. (a) Attenuation versus frequency, (b) velocity versus frequency.
Fig. 10.11 (continued)

+++- INDEPENDENT SCATTER MODEL
••••• FOLDY ISOTROPIC MULTIPLE SCATTERING MODEL
— WATERMAN-TRUELL MULTIPLE SCATTERING MODEL
---• MODIFIED W-T MODEL (SEE FIG. 10.4(b))
——• MODIFIED W-T MODEL (SEE FIG. 10.5(b))
△△△△ SAYERS-SMITH MODEL (SEE FIG. 10.5(a))
○○○○○ EXPERIMENTAL DATA

PHASE VELOCITY (Km/s)

FREQUENCY f (MHz)

(b)
interesting to note that all the modified Waterman-Truell models predict much lower attenuation than the Waterman-Truell model (the coarse dashed line). This is because when the matrix is replaced by either the static composite medium or the dynamic effective medium, and/or the fiber is replaced by a concentric cylinder with fiber as its core and matrix as its annulus, the impedance matching between the scatterer and its surrounding medium is much better than that between the fiber and the matrix. At low frequencies the figure clearly shows the decrease of scattering due to progressive improvement of impedance matching. The modified Waterman-Truell model (the fine dashed line, see Fig. 10.4(b)) where a fiber with matrix layer is considered embedded in the composite medium giving the lowest attenuation. The Sayers-Smith self consistent model (10.10) (open triangles, see Fig. 10.5(a)) predicts attenuation close to that of the Waterman-Truell multiple scattering model at lower frequencies, approaching that of the generalized self-consistent Waterman-Truell model (the solid line, see Fig. 10.5(b)) at higher frequencies.

Results for dispersion are shown in Fig. 10.11(b). The solutions calculated from the modified Waterman-Truell model where a fiber is considered embedded in the composite medium (see Fig. 10.4(a)) do not appear in the figure since the predicted velocities are too high (i.e. 6.7 Km/s) for the scale presented in the figure. The reason for such a significant increase of wave velocity in this model is that in the model (Fig. 10.4(a)) the matrix is replaced by the composite medium which has
much higher moduli. The other models all give similar results among themselves at low frequencies. Three models give the same zero-frequency velocity limits. The first is the independent scatter model (crosses) where its zero-frequency velocity limit is chosen not in the matrix but as that determined by the static composite moduli calculated using the multiphase generalized self consistent model [79, 80]. The other two are the modified Waterman-Truell models, where the single scatterer problem is solved for a fiber surrounded by a matrix layer embedded either in a composite medium (the fine dashed line, see Fig. 10.4(b)) or in a dynamic effective medium (the solid line, see Fig. 10.5(b)). This is because in these models the actual fiber fraction and geometry are preserved as in the composite. Compared with this zero-frequency velocity limit the original Waterman-Truell model (the coarse dashed line, see Fig. 10.3) where the fiber is considered immersed in unbound matrix gives a slightly smaller value while Foldy's model (the fine dashed line) gives a slightly higher value. One also sees that the Sayers-Smith self consistent model (open triangles, see Fig. 10.5(a)) gives a slightly higher zero-frequency velocity. Consideration of the zero-frequency limit of the ultrasonic wave velocity is important since these velocities are often used to determine the static composite moduli.

We apply the generalized self-consistent Waterman Truell multiple scattering model where a fiber with a matrix layer is considered embedded in a dynamic effective medium ((10.11), see Fig. 10.5(b)) to analyze the dynamic effects of dis-
persion in the composite, since this model gives the correct zero-frequency velocity limit and seems to give the best fit to the experimental data (the solid lines and open circles in Figs. 10.10 and 11(b)). To demonstrate the interphase effect on wave velocity we change the interphase radial and transverse shear moduli by factors of 10 in both directions (interphase thickness and density are kept constant). Fig. 10.12 shows the dispersion curves for both (a) longitudinal and (b) shear waves versus frequency. One sees from this figure that the interphase stiffness has a very strong effect on the wave velocity. Thus measurements of both longitudinal and shear wave velocities normal to fibers can be used to obtain the interphase transverse moduli. One also sees from the figure that the dispersion for different interphase stiffnesses is small especially at low frequencies. Thus corrections for the dynamic effect are not necessary when using the velocity measured by low frequency ultrasound to determine the static composite moduli.

To illustrate directly the dynamic effect on accuracy of interphase moduli determination from velocity data, we calculate wave velocities versus interphase stiffnesses using both dynamic and static composite models. The results are plotted in Fig. 10.13 for (a) longitudinal and (b) transverse waves. for (a) longitudinal and (b) transverse waves. In this figure the solid lines are for velocities calculated using the generalized self-consistent Waterman Truell model (10.11) (see Fig. 5(b)) at 5 MHz, and the dashed lines are for those calculated from the static composite moduli. The composite moduli are related to interphase, fiber and matrix moduli
Figure 10.12: The dispersion curves of both (a) longitudinal and (b) shear waves versus frequency for different interphase stiffnesses.
Figure 10.13: Calculated wave velocities for (a) longitudinal and (b) transverse waves versus interphase stiffnesses using dynamic and static composite models. The calculations using the generalized self-consistent Waterman Truell model at 5 MHz are given by the solid lines, and velocities calculated using the static multiphase generalized self-consistent model are given by the dashed lines.
by the static multiphase generalized self-consistent model [79, 80]. The interphase stiffnesses for various SiC fiber ceramic matrix composites (CMC) and metal matrix composites (MMC) [7, 8, 9, 10] are indicated in the figure. One sees from this figure that in the interphase stiffness range for these composites, the predicted ultrasonic velocities at 5 MHz coincide with their zero-frequency limits. Thus both longitudinal and shear velocities measured by low frequency ultrasound can be used directly for determination of both static composite and interphase moduli. If higher accuracy is desired, the dynamic effect can be corrected using the modified Waterman-Truell model where a fiber with a matrix layer is considered immersed either in a composite medium (see Fig. 10.4(b)) or a dynamic effective medium (10.11, see Fig. 10.5(b)).

10.6 Summary

We have performed a theoretical and experimental study of fiber scattering-induced attenuation and dispersion in a unidirectional random fiber-composite for wave propagating perpendicular to fibers. It is found that the attenuation data are close to those predicted by the independent scattering model and the Waterman-Truell solutions at low frequencies. This shows that the multiple scattering effect in this composite is insignificant due to close impedance matching between the effective fiber (including the multilayered fiber and the interphase) and matrix. On the other hand the wave velocity data are very close to those predicted by
the modified Waterman-Truell models where a fiber with a matrix layer is considered embedded either in a composite medium (Fig. 10.4(b)) or in a dynamic effective medium (Fig. 10.5(b)). This indicates that attenuation is determined by a local effect and can be described by scattering models where fiber is considered immersed in unbound matrix, while dispersion is affected more by multiple scattering and should be described by models where a concentric cylinder (fiber with matrix layer) is considered immersed in a composite or effective medium. It is shown numerically that both longitudinal and shear wave velocities are sensitive to the interphase stiffnesses and can be used for determination of interphase transverse moduli. It has also been found from both experimental and simulated data that dispersion normal to fibers is small, so the ultrasonic wave velocities can be used for determination of static composite and interphase moduli without needing correction for dynamic effects. æ
CHAPTER XI

WAVE PROPAGATION ALONG FIBERS IN A COMPOSITE. THEORY AND EXPERIMENT

11.1 Introduction

In the previous chapter we presented theoretical and experimental study of wave dispersion and attenuation in directions perpendicular to the fibers in a unidirectional composite. We found that the dispersion was small so the wave velocity measured normal to fibers could be used to determine the static composite transverse moduli and the transverse interphase moduli. In this chapter we will focus on wave propagation along the fibers and show that the wave dispersion is significant.

Numerous studies of waves propagating along rods have been performed. For a list of earlier publications the reader is referred to an excellent review by Thurston [116]. Recently efforts have been reported on using leaky axisymmetric modes along a single fiber in solid matrix to characterize the interface conditions between the fiber and the embedded matrix [117, 118, 119, 120]. Although these works
demonstrated interesting effects useful for evaluation of a single fiber embedded in a solid matrix, these results cannot be directly transferred to characterization of a composite in the fiber direction since the wave characteristics in composites are markedly different from those for waves in a single fiber.

There have also been theoretical and experimental studies performed in seventies on wave propagation in composite in the fiber direction [121, 122, 123, 124]. These works were performed for composites reinforced by homogeneous fibers with perfect interface between fiber and matrix. In this paper we will develop a model for wave propagation in a composite with multilayered fibers and compare the theoretical predictions with our experimental data for an unidirectional SiC/Ti composite. The ability to treat composites with multilayered fibers is important since modern high temperature composites are designed and manufactured with multilayered fibers and fiber-matrix interphases. For example, the Textron 140 μm diameter SiC fiber (SCS-6) is manufactured by the CVD process with a carbon core, two-layered SiC shell and a 3 μm carbon-rich coating serving as an interphasial layer between the fiber and the matrix. Since the fiber has a large diameter the fiber effect dominates the wave propagation characteristics which is not the case in graphite epoxy composites where the fiber diameter is only about 25 μm.

The organization of the chapter is as follows. In section 11.2 we introduce the multilayered composite cylinder model, with fiber as the core and matrix as the
annulus, to represent a unidirectional multiphase composite. The dispersion equation is found by satisfying the mixed boundary conditions on the outer boundary of the composite cylinder. A transfer matrix formalism is developed to facilitate the derivation of the dispersion equation of the lowest axisymmetric mode for the multilayered composite cylinder. Numerical calculations are given for a unidirectional titanium alloy composite reinforced with 24% SCS-6 fibers. The discussion focuses on the effect of interphase on the wave dispersion and on comparison of the zero-frequency velocity limit determined from the dynamic composite cylinder model with velocities calculated from the static composite axial elastic modulus determined by the static multiphase generalized self consistent model. Section 11.3 describes experimental results for propagation along the fibers in the SiC/Ti composite. The ultrasonic wave velocity and attenuation have been measured in the frequency range of 5-15 MHz. The experimental velocity data are compared with the model predictions. Finally the necessity of the dynamic effect correction for determination of composite and interphase moduli from velocity data is given in Section 11.4.

11.2 Multilayered Cylinder Model for Dispersion of Waves Propagating along Fibers in a Unidirectional Composite

To describe dispersion of waves in the fiber direction in an unidirectional composite, we begin by modeling the composite as a hexagonal array of concentric composite
cylinder cells. Due to symmetry the composite can be represented by one cell with mixed boundary conditions on the outer boundary as shown in Fig. 11.1. The cell consists of a cylindrical core of the multilayered fiber and a cylindrical annulus of matrix material. We define the radius of the multilayered cell \( r_m \) by the requirement that the area fiber fraction in the composite cell equals that in the composite, i.e. \( r_f^2/r_m^2 = c \), where \( c \) is the fiber fraction and \( r_f \) is the fiber radius (including the interphase). If the fibers can be grouped into hexagonal arrays as shown in the figure axisymmetry can be assumed in this composite, therefore mixed boundary conditions, zero shear stresses and radial displacements, can be prescribed on the outer boundary of the composite cylinder cell. The wave dispersion along the fibers in the composite can be found from the axisymmetric guided modes in the composite cylinder cell. The representation of a composite with a unit cell with mixed boundary conditions has been used for analysis of static composite axial properties [125], and for derivation of an approximate dispersion equation in a composite with homogeneous fibers using a continuum mixture theory [123].

**11.2.1 A transfer matrix solution for an axisymmetric mode in a multilayered composite cylinder**

To generalize the discussion to composites with multiphase fibers, we consider here a \( N \)-phase multilayered cylinder as shown in Fig. 11.2. In the figure, phase 1 is the fiber core, phase \( N \) is the matrix annulus and phase \( j \) \( (r_{j-1} \leq r \leq r_j) \) is the
Figure 11.1: Composite cylinder model for waves propagating along the fibers in a unidirectional fiber composite.
Figure 11.2: The cross-section of the $N$-phase multilayered cylinder.
intermediate layer, such as the fiber shell or interphase coating layers. Assume that $\beta$ is the wave number of the axisymmetric mode propagating along the axial direction ($z$-direction), which is constant for partial waves in each phase of the composite cylinder. The solutions for the partial waves of axisymmetric mode in a cylindrical layer $j$ (Fig. 11.2) in the form of scalar $\varphi$ and vector $\psi_\theta$ potentials are

$$\varphi^j = [A_j J_0(k_{rt}^j r) + C_j N_0(k_{rt}^j r)]e^{i\beta z}, \quad \psi_\theta^j = [B_j J_1(k_{rt}^j r) + D_j N_1(k_{rt}^j r)]e^{i\beta z}, \quad (11.1)$$

where $J_0$ and $J_1$ are the zeroth and first order Bessel functions, $N_0$ and $N_1$ are the zeroth and first order Neumann functions, $k_{rt}^j$ and $k_{rt}^j$ are the radial components of the wave numbers for the longitudinal and shear partial waves in phase $j$; and $r$ and $\theta$ are the radius and polar angle of the cylindrical system (see Fig. 11.2).

The axisymmetric potential fields in the fiber core ($r \leq r_1$) are

$$\varphi^1 = A_1 J_0(k_{rt}^1 r)e^{i\beta z}, \quad \psi_\theta^1 = B_1 J_1(k_{rt}^1 r)e^{i\beta z}, \quad (11.2)$$

where $k_{rt}^1$ and $k_{rt}^1$ are the radial components of the longitudinal and shear wave numbers. Terms related to $N_0$ and $N_1$ do not appear in equation (11.2) due to their singularities at $r = 0$. In equations (11.1-11.2) the harmonic term $\exp(-i\omega t)$ is omitted and relations $(\beta)^2 + (k_{rl}^j)^2 = (k_{rt}^j)^2$ and $(\beta)^2 + (k_{rt}^j)^2 = (k_{rt}^j)^2$ are assumed.

The mixed boundary conditions are prescribed on the outer boundary $r = r_N$

$$\sigma_{rz}^N = 0, \quad u_r^N = 0 \quad \text{for} \quad r = r_N. \quad (11.3)$$

The dispersion equation for the wave number $\beta$ of the axisymmetric wave propagation along the fibers can be found by satisfying the boundary condition equation
(11.3).

To facilitate the derivation of the dispersion equation for the $N$-layered composite cylinder, we utilize a transfer-matrix approach to relate the axisymmetric elastic field in the fiber core to those on the outer boundary of the multilayered cylinder. Specifically, using the relations:

$$u_z = \frac{\partial \varphi}{\partial z} + \frac{1}{r} \frac{\partial (\psi \varphi r)}{\partial r},$$

$$u_r = \frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial z},$$

$$\sigma_{rz} = \mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right),$$

$$\sigma_{rr} = \lambda \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2\mu \frac{\partial u_r}{\partial r},$$

(11.4) (11.5)

where $\lambda$ and $\mu$ are the Lamé constants, we can write the displacement and stress field from the expressions for potentials (11.1) as

$$\frac{u_z}{e^{i\beta z}} = i\beta J_0(k_{rt}r)A + \left[ -\frac{1}{r} J_1(k_{rt}r) + k_{rt} J_1'(k_{rt}r) \right] B + i\beta N_0(k_{rt}r)C + \left[ -\frac{1}{r} J_1(k_{rt}r) + k_{rt} J_1'(k_{rt}r) \right] D$$

$$\frac{u_r}{e^{i\beta z}} = k_{rt} J_0'(k_{rt}r)A - i\beta J_1(k_{rt}r)B + k_{rt} N_0'(k_{rt}r)C - i\beta N_1(k_{rt}r)D$$

$$\frac{\sigma_{rz}}{e^{i\beta z}} = \mu \left\{ 2i\beta k_{rt} J_0'(k_{rt}r)A + \left[ (\beta^2 - \frac{1}{r^2}) J_1(k_{rt}r) + \frac{k_{rt}}{r} J_1'(k_{rt}r) + (k_{rt})^2 J_1''(k_{rt}r) \right] B + 2i\beta k_{rt} N_0'(k_{rt}r)C + \left[ (\beta^2 - \frac{1}{r^2}) N_1(k_{rt}r) + \frac{k_{rt}}{r} N_1'(k_{rt}r) + (k_{rt})^2 N_1''(k_{rt}r) \right] D \right\}$$

$$\frac{\sigma_{rr}}{e^{i\beta z}} = [-\lambda \beta^2 J_0(k_{rt}r) + \lambda \frac{k_{rt}}{r} J_0'(k_{rt}r) + (\lambda + 2\mu) k_{rt}^2 J_0''(k_{rt}r)]A - 2i\mu k_{rt} J_1'(k_{rt}r)B + [-\lambda \beta^2 N_0(k_{rt}r) + \lambda \frac{k_{rt}}{r} N_0'(k_{rt}r) + (\lambda + 2\mu) k_{rt}^2 N_0''(k_{rt}r)] C - 2i\mu k_{rt} N_1'(k_{rt}r) D.$$

(11.6)
Equation (11.6) can be written in a matrix form:

$$ (u_z, u_r, \sigma_{rz}, \sigma_{rr})^T = K \cdot (A, B, C, D)^T e^{i\beta z}, \quad (11.7) $$

where the $4 \times 4$ matrix $K$, which relates the axisymmetric displacement-stress vector to the coefficients $(A, B, C, D)$ of the wave solution, is

$$ K = \begin{pmatrix}
  i\beta J_0(k_{rt} r) & \frac{1}{r} J_1(k_{rt} r) + k_{rt} J'_1(k_{rt} r) \\
  k_{rt} J'_0(k_{rt} r) & -i\beta J_1(k_{rt} r) \\
  2i\mu k_{rt} J'_0(k_{rt} r) & K(3, 2) \\
  K(4, 1) & -2i\mu k_{rt} J'_1(k_{rt} r)
\end{pmatrix} $$

Here

$$ K(3, 2) = \mu [(\beta^2 - \frac{1}{r^2}) J_1(k_{rt} r) + \frac{k_{rt}}{r} J'_1(k_{rt} r) + (k_{rt})^2 J''_1(k_{rt} r)], $$

$$ K(4, 1) = -\lambda \beta J_0(k_{rt} r) + \lambda \frac{k_{rt}}{r} J'_0(k_{rt} r) + (\lambda + 2\mu) k_{rt} J''_0(k_{rt} r), $$

$$ K(3, 4) = \mu [(\beta^2 - \frac{1}{r^2}) N_1(k_{rt} r) + \frac{k_{rt}}{r} N'_1(k_{rt} r) + (k_{rt})^2 N''_1(k_{rt} r)] $$

and

$$ K(4, 3) = -\lambda \beta^2 N_0(k_{rt} r) + \lambda \frac{k_{rt}}{r} N'_0(k_{rt} r) + (\lambda + 2\mu) k_{rt}^2 N''_0(k_{rt} r). $$

Thus for any intermediate cylindrical layer $(r_{j-1} \leq r \leq r_j)$, a transfer matrix $T_j$ can be defined to relate the displacements and stresses on the inner boundary of the layer to those on the outer boundary as:

$$ (u_z, u_r, \sigma_{rz}, \sigma_{rr})^T |_{r = r_j} = K_j |_{r = r_j} \cdot (A_j, B_j, C_j, D_j)^T e^{i\beta z} $$

$$ = K_j |_{r = r_j} (K_j |_{r = r_{j-1}})^{-1} (u_z, u_r, \sigma_{rz}, \sigma_{rr})^T |_{r = r_{j-1}} = T_j (u_z, u_r, \sigma_{rz}, \sigma_{rr})^T |_{r = r_{j-1}} \quad (11.8) $$
where

$$T_j = K_j|_{r=r_1}(K_j|_{r=r_{j-1}})^{-1}. \quad (11.9)$$

The transfer matrix $T_j$ depends on the material properties and layer thickness and is uniquely defined for the cylindrical layer $j$.

Using the transfer matrix for each intermediate cylindrical layer, the stresses and displacements on the outer boundary of the composite cylinder ($r = r_N$) can be directly related to those in the fiber core ($r = r_1$) as

$$\begin{align*}
\left(u_z, u_r, \sigma_{rz}, \sigma_{rr}\right)^T|_{r=r_N} &= T(u_z, u_r, \sigma_{rz}, \sigma_{rr})^T|_{r=r_1},
\end{align*} \quad (11.10)$$

where $T$ is the total transfer matrix

$$T = T_NT_{N-1} \cdots T_2 = \prod_{j=0}^{N-2} T_{N-j}. \quad (11.11)$$

For the field in the fiber core (11.2), we have

$$\left(u_z, u_r, \sigma_{rz}, \sigma_{rr}\right)^T|_{r=r_1} = K_1|_{r=r_1} \cdot (A_1, B_1, 0, 0)^T e^{i\beta z}. \quad (11.12)$$

Thus combining equations (11.10) and (11.12), we can write the displacement-stress field on the outer boundary of the composite cylinder as:

$$\left(u_z, u_r, \sigma_{rz}, \sigma_{rr}\right)^T|_{r=r_N} = TK_1|_{r=r_1} \cdot (A_1, B_1, 0, 0)^T e^{i\beta z} = S(A_1, B_1, 0, 0)^T e^{i\beta z}, \quad (11.13)$$

where

$$S = TK_1(r = r_1). \quad (11.14)$$
Finally substituting the mixed boundary conditions (11.3) into (11.13) as:

\[ S_{21}A_1 + S_{22}B_2 = 0 \]  \hspace{1cm} (11.15)
\[ S_{31}A_1 + S_{32}B_2 = 0, \]  \hspace{1cm} (11.16)

where \( S_{ij} \) are the \((i,j)\)th element of matrix \( S \) (11.14), we obtain the dispersion equation as

\[ S_{21}S_{32} - S_{31}S_{22} = 0. \]  \hspace{1cm} (11.17)

The wave solution \( \beta \) of this equation describes the dispersion of the axisymmetric mode propagating along the fibers in a multiphase unidirectional composite. The wave velocity in the composite \( V^c \) is found to be \( V^c = \omega / \beta \). Note that the wave number \( \beta \) thus obtained is real implying zero attenuation for this guided mode.

Equation (11.13) can also be used to obtain the dispersion equation for the axisymmetric mode in a \( N \)-phase multilayered cylinder with free boundary conditions:

\[ \sigma_{rr} = 0, \sigma_{rx} = 0 \text{ for } r = r_N. \]  \hspace{1cm} (11.18)

The corresponding dispersion equation is found to be

\[ S_{31}S_{42} - S_{41}S_{32} = 0. \]  \hspace{1cm} (11.19)

The transfer matrix formalism can also be applied to derive the dispersion equation for a wave propagating along a multilayered fiber \((N-1)\) phase) in an unbound matrix (phase \( N \)), i.e. the radius of the matrix layer is taken as infinite in
the composite unit cell. In this case, the displacements and stresses in an unbound matrix have the following relation:

\[(u_z^N, u_r^N, \sigma_{rz}^N, \sigma_{rr}^N)^T = K^N(A^N, B^N, iA^N, iB^N)e^{i\beta z}; \tag{11.20}\]

and the elastic field vector \((u_z^N, u_r^N, \sigma_{rz}^N, \sigma_{rr}^N)^T|_{r=r_{N-1}}\) on the outer boundary of the fiber can be related to the coefficients in fiber core using the transfer matrix approach similar to equation (11.13) as:

\[(u_z^N, u_r^N, \sigma_{rz}^N, \sigma_{rr}^N)^T|_{r=r_{N-1}} = T_{N-1} T_{N-2} \cdots T_2 K_1|_{r=r_1} \cdot (A_1, B_1, 0, 0)^T e^{i\beta z}. \tag{11.21}\]

Therefore the coefficients in the matrix are related to those in the fiber core by

\[
(A^N, B^N, iA^N, iB^N) = (K^N|_{r=r_{N-1}})^{-1}|_{r=r_{N-1}} (u_z^N, u_r^N, \sigma_{rz}^N, \sigma_{rr}^N)^T|_{r=r_{N-1}} e^{-i\beta z} \\
= (K^N|_{r=r_{N-1}})^{-1}|_{r=r_{N-1}} T_{N-1} T_{N-2} \cdots T_2 K_1|_{r=r_1} (A_1, B_1, 0, 0)^T = S'(A_1, B_1, 0, 0)^T, \tag{11.22}
\]

where

\[
S' = (K^N|_{r=r_{N-1}})^{-1} T_{N-1} T_{N-2} \cdots T_2 K_1|_{r=r_1}. \tag{11.23}\]

Equation (11.22) is a 4x4 linear system of equations for four unknown coefficients \(A_1, B_1\) and \(A^N\) and \(B^N\), and can be rewritten as:

\[
S'_{11} A_1 + S'_{12} B_1 = A^N \tag{11.24}
\]
\[
S'_{21} A_1 + S'_{22} B_1 = B^N \tag{11.25}
\]
\[
S'_{31} A_1 + S'_{32} B_1 = iA^N \tag{11.26}
\]
\[
S'_{41} A_1 + S'_{42} B_1 = iB^N, \tag{11.27}
\]
where $S'_{ij}$ are the $(i,j)$th element of matrix $S'$ (11.23). Finally the dispersion equation for the axisymmetric mode along a $N - 1$ phase cylinder in an unbounded matrix is found by setting the determinant of the system of equations (11.24-11.27) to zero:

$$S'_{11} S'_{22} + i S'_{11} S'_{42} - S'_{21} S'_{12} - i S'_{21} S'_{32} + i S'_{31} S'_{22} - S'_{31} S'_{42} - i S'_{41} S'_{12} + S'_{41} S'_{32} = 0. \quad (11.28)$$

### 11.2.2 Numerical examples

As an example let us consider wave dispersion for a unidirectional titanium alloy composite reinforced with 24% SiC (SCS-6) fibers, which is also used for measurements discussed later. The cross-section of the unit cell for the SiC/Ti composite is shown in Fig. 11.3. The properties of each phase are given in Table 11.1. The SCS-6 fiber consists of a carbon core with a SiC shell plus a 3 μm carbon-rich interphases layer placed between the fiber and the matrix. In the calculation the interphase Young’s modulus will be varied to simulate different interphase stiffnesses, while other interphase properties (thickness = 3μm, Poisson’s ratio = 0.285 and density = 2.1 g/cc) are kept constant.

The interphase moduli are reconstructed from the ultrasonically determined composite moduli (see Section 10.3) using the multiphase generalized self consistent model [79, 80]. The interphase moduli obtained are effective since they depend on the chemical reaction and mechanical contact between the interphases layer (pyrolytic carbon coating) and the surrounding SiC and titanium alloy. The
Figure 11.3: The cross-section of the composite cylinder model for a unidirectional SiC/Ti composite.
Table 11.1: Properties of each phase in a SCS-6/Ti-15-3 composite.

<table>
<thead>
<tr>
<th>Phase</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$\lambda$ (GPa)</th>
<th>$\mu$ (GPa)</th>
<th>$\rho$ (g/cc)</th>
<th>$r$ (µm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>core (carbon)</td>
<td>41</td>
<td>0.25</td>
<td>16.4</td>
<td>16.4</td>
<td>1.7</td>
<td>18</td>
</tr>
<tr>
<td>shell (SiC)</td>
<td>415</td>
<td>0.17</td>
<td>91.4</td>
<td>177</td>
<td>3.2</td>
<td>68</td>
</tr>
<tr>
<td>interphase</td>
<td>28.3</td>
<td>0.285</td>
<td>16.4</td>
<td>16.4</td>
<td>2.1</td>
<td>71</td>
</tr>
<tr>
<td>matrix (Ti)</td>
<td>122</td>
<td>0.35</td>
<td>103</td>
<td>45.1</td>
<td>5.4</td>
<td>146*</td>
</tr>
</tbody>
</table>

where $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $\lambda$ and $\mu$ are the Lamé constants, $\rho$ is the density and $r$ the radius of the boundary of that phase.

* The matrix radius is calculated from fiber fractions $r_m = 71/(0.24)^{0.5}$ µm.

Transverse bulk and shear moduli and axial shear moduli can be reliably reconstructed by this method; they are: $\kappa_t = 26$ GPa, $G_t = 4.6$ GPa and $G_a = 11$ GPa. As described in [9, 10], the interphase can be considered as a mixture of the same isotropic carbon as in the fiber core and pyrolytic carbon with a preferred basal plane orientation along the circumferential direction, plus additional reinforcement with about 25% concentration of fine SiC particles. The effective axial Young’s modulus $E_1$ and Poisson’s ratio $\nu_1$ are not available due to poor sensitivity of the velocity along the fibers to the interphase moduli, as partially discussed in the previous section. Considering the interphase as isotropic we estimated the axial elastic constant as $C_{11} = \kappa_t + G_t = 37$ GPa. Then from $C_{11}$ and $G_a$ we found the interphase axial Young’s modulus $E_1 = 28.3$ GPa and Poisson’s ratio $\nu_1 = 0.285$. The density of the interphase is 2.1 g/cc, calculated by considering 25% SiC particles embedded in the carbon coating.

The low frequency part of the zeroth order axisymmetric mode calculated using
the composite cylinder model with mixed boundary conditions is shown in Fig. 11.4 for different interphase Young's moduli \( E^i = 2.83, 28.3 \) and 283 GPa. The dispersion curves for the composite cylinder with free boundary conditions are also shown in the figure by dashed lines for comparison. In the figure the top axis is the nondimensional wave number \( k_l r_f \), where \( k_l \) is the longitudinal wave number in the matrix and \( r_f \) is the fiber radius including the interphase, and the bottom axis is the frequency. Only low \( k_l r_f \) values are considered since measurements can only be done in the corresponding frequency range for this composite. One sees from Fig. 11.4 that the interphase stiffness has a moderate effect on the dispersion. The velocity in composite (composite cylinder with mixed boundary conditions) is higher than that for a corresponding fiber with matrix layer (cylinder with free boundary conditions).

It is interesting to examine the zero-frequency velocity limits of both groups of dispersion curves in Fig. 11.4. The velocities calculated independently from static composite moduli are shown by solid circles. The static composite moduli used are the axial elastic modulus \( C_{11}^c \) and axial Young's modulus \( E_{11}^c \), \( l \) is the fiber direction. The zero-frequency velocity limits \( V_{f=0}^c \) (dashed lines) for a layered cylinder with free boundary conditions, as for the case of a free single phase cylinder (see [116]), equal those (solid circles) calculated using the composite axial Young's modulus \( E_{11}^c \)

\[
V_{\text{static}}^c = \sqrt{E_1^c / \rho^c}, \tag{11.29}
\]
Figure 11.4: Calculated lowest axisymmetric mode in a unidirectional titanium alloy composite reinforced with 24% SiC fibers using the composite cylinder model with mixed boundary conditions (the solid lines). The dashed lines are for the free composite cylinder. Three different fiber-matrix interphase Young’s moduli are used: 2.83, 28.3, and 283 GPa.
where $\rho^c$ is the composite density. We calculated $E^c_1$ used in equation (11.29) by a recursive substitution method [82, 126, 79, 80] for a multiphase unidirectional composite utilizing the generalized self-consistent model [52]. The details are given in Appendix E. The equality between $V^c(f = 0)$ and $V^c_{\text{static}}$ is expected since the same model with layered fiber surrounded by matrix layer is used in both dynamic and static models.

For waves propagating along fibers in the composite, when the mixed boundary conditions are used the zero-frequency velocity limits $V^c(f = 0)$ (solid lines) (as for the case of an unidirectional composite with single phase fibers) equal those calculated using the composite axial elastic modulus $C^c_{11}$

$$V^c(f = 0) = \sqrt{C^c_{11}/\rho^c}, \quad \text{where} \quad C^c_1 = E^c_1 + 4(\nu^c_1)^2\kappa^c_1. \quad (11.30)$$

Here $\nu^c_1$ the composite axial Poisson's ratio and $\kappa^c_1$ the transverse bulk modulus. $\nu^c_1$ and $\kappa^c_1$ are also calculated by the recursive substitution method for a multiphase unidirectional composite and the details are given in Appendix E.

One may note from Fig. 11.4 that at the zero-frequency velocity limits the interphase stiffness has much stronger effects on velocities of composite (solid lines) than on velocities of the layered cylinder with free boundary conditions (dashed lines). This is because the composite elastic axial modulus $C^c_{11}$ is more sensitive to the interphase stiffness than the composite Young's modulus $E^c_1$. Since $C^c_{11} = E^c_1 + 4(\nu^c_1)^2\kappa^c_1$, where the axial Poisson's ratio $\nu^c$ is almost constant, one can conclude
that the sensitivity of $C_{11}^c$ to the interphase stiffness is mostly determined by that of the composite transverse bulk modulus $\kappa_i^c$.

11.3 Experimental Measurement of a SiC/Ti Unidirectional Composite. Comparison with Theory

In this section we present ultrasonic results for measurements of frequency dependencies of attenuation and dispersion for waves propagating along the fiber direction in the SiC/Ti unidirectional metal matrix composite, and compare the experimental data with the predictions by the composite cylinder model with mixed boundary condition discussed in the previous section. The sample description and experimental technique have been discussed in the previous chapter.

The through-transmitted signals are measured using two pairs of transducers: 15-20 MHz and 5-10 MHz. The results for different samples are similar. Fig. 11.5 shows the typical time domain signals of the directly through-transmitted wave measured using a pair of transducers with center frequencies at 15 and 20 MHz. Here $T_0(t)$ is the reference signal measured with the sample absent, $T_1(t)$ is the directly transmitted signal through the sample, and $T_2(t)$ is the through-transmitted wave after a complete internal reflection inside the sample. The attenuation setting of the pulser/receiver for the reference measurements $T_0(t)$ without the sample is different from that with the sample. The change of the period with time in the through-transmitted signals $T_1(t)$ and $T_2(t)$ indicated the existence of strong wave
Figure 11.5: The time domain signals of the waves transmitted through a SiC/Ti unidirectional composite sample measured using 15 and 20 MHz broadband immersion transducers.
dispersion along the fiber direction. The lower frequency part of the wave spectrum has higher velocity and arrives earlier, resulting in a greater period in the initial part of the through-transmitted signal. The second through-transmitted signal $T_2(t)$ is garbled with noise from the composites and its precursor is hardly identifiable, thus it is not used for final signal analysis. The magnitude of the frequency spectra of the directly through-transmitted waves with $(T_1(f))$ and without $(T_0(f))$ the sample are shown in Fig. 11.6. One sees from the figure that the higher frequency part of the signal after some propagation in the sample is much more strongly attenuated than the lower frequency part. The attenuation and dispersion are determined from the magnitude and phase of the deconvolved spectrum $T_1(f)/T_0(f)$ using equations (10.14) and (10.15).

A typical result for the longitudinal wave velocity (in Km/s) and attenuation (in Np/cm) versus frequency is shown in Fig. 11.7 by open circles for one sample with thickness $h = 2.8$ mm. The experimental data presented in the previous chapter for wave propagating normal to the fibers are also given for comparison by closed circles. The prediction by the lowest axisymmetric mode of the composite cylinder model with mixed boundary conditions is given by the solid line. In the calculation the properties of each phase in the composite system are taken those listed in Table 11.1. Finally the fine dashed line is the prediction by a approximate continuum mixture theory [123], which is a first order approximation to the exact solution. The difference at low frequencies can be explained by the use in the
Figure 11.6: The spectra magnitudes of the directly through-transmitted waves with $|T_1(f)|$ and without $|T_0(f)|$ the sample.
Figure 11.7: (a) Measured frequency dependence of longitudinal wave velocity (the open circles). The solid line is calculated using the composite cylinder theory. (b) Measured longitudinal wave attenuation spectrum (the open circles). The attenuation data for wave propagating perpendicular to the fiber direction are also given here by the closed circles.
Table 11.2: Properties of the effective fiber including the interphase.

<table>
<thead>
<tr>
<th>$E_1$ (GPa)</th>
<th>$\nu_1$</th>
<th>$G_1$ (GPa)</th>
<th>$\kappa_t$ (GPa)</th>
<th>$G_t$ (GPa)</th>
<th>$\rho$ (g/cc)</th>
<th>$r$ (µm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>360</td>
<td>0.18</td>
<td>98</td>
<td>148</td>
<td>65</td>
<td>3.01</td>
<td>71</td>
</tr>
</tbody>
</table>

where $E_1$ is the axial Young's modulus, $\nu_1$ the axial Poisson's ratio, $G_1$ the axial shear modulus; $G_t$ the transverse shear modulus and $\kappa_t$ the transverse bulk modulus. Note that thus obtained effective fiber moduli are obtained from static micro-mechanical models and are only accurate at very low frequencies, and only the axial moduli are used in the calculations.

calculations a 2-phase fiber-matrix system with an effective fiber replacing the multilayered fiber (core plus shell and the interphase). The effective fiber properties are calculated from the multiphase generalized self-consistent model [79, 80], and are given in Table 11.2.

One sees from Fig. 11.7(a) that unlike the case of propagation normal to the fibers (the closed circles), the velocity along the fibers (open circles) has strong dispersion, showing a continuous decrease with frequency. The velocity calculated using the composite cylinder model (the solid line) compared satisfactorily with the measurements (open circles). Some differences may be due to the assumption of an hexagonal arrangement in the composite cylinder model, while in actual composite the fibers are randomly distributed. Finally the agreement between the experimental data and the continuum mixture theory is acceptable only at low frequencies.

In Fig. 11.7(b) the attenuation spectrum measured for waves propagating both perpendicular and parallel to the fibers are given together for comparison. One sees from this figure that at low frequencies ($5$ MHz $< f < 10$ MHz) the atten-
uation for waves propagating along the fiber direction is much smaller than that for wave propagating perpendicular to the fiber direction. At higher frequencies \( f > 10 \text{ MHz} \), the wave attenuation along the fiber direction increases sharply and becomes higher than that for waves propagating perpendicular to the fibers. The effective attenuation of the waves along the fiber direction is caused mostly by the strong wave dispersion, resulting in pulse distortion and spread in time. The attenuation increase at about 7 MHz clearly corresponds to a significant increase of dispersion as can be observed from Fig. 11.7(a). In contrast the scattering effect is the dominant contributor to attenuation in directions normal to the fibers.

11.4 Discussions of Dynamic Effects on the Use of Velocity Data for Determination of Composite and Interphase Moduli

Finally we address dynamic effects on the use of velocity measured in the fiber direction for determination of composite and interphase moduli. Unlike for waves propagating normally to the fibers where the dispersion is insignificant as shown in the previous chapter, the dispersion along the fibers is strong. To illustrate the difference between the velocity calculated from static composite elastic moduli \( V^c_{\text{static}} = (C^c_{ij}/\rho^c) \) and the ultrasound wave velocity \( V^c_f \), we plotted in Fig. 11.8 both \( V^c_{\text{static}} \) (the fine dashed line) and \( V^c(f = 5 \text{ MHz}) \) (the solid line) versus interphase axial Young’s modulus \( E^\text{I} \). The interphase stiffness at 2.83 and 28.3 GPa indicated, by vertical dashed lines, the region where the interphase stiffness
of most high temperature materials falls.

Let us take point A on the solid curve \( E_1^i = 28.3 \text{ GPa}, \ V^c(f = 5 \text{ MHz}) = 6.84 \text{ Km/s} \) as an example. If one uses the velocity measured at 5 MHz as its static limit (point B on the dashed line), the error in velocity due to the dynamic effect is the difference between points A and B \( V^{c}_{\text{static}} = 6.84 \text{ Km/s}, \ V^c(f = 5 \text{ MHz}) = 6.97 \text{ Km/s} \) which is about 2%. Thus use of \( V^c(f = 5 \text{ MHz}) \) to determine the static composite elastic modulus \( C_{11}^c \) would introduce about 4% error. However, if one uses this velocity \( V^c(f = 5 \text{ MHz}) \) to determine the interphase layer stiffness using static micromechanical models for the composite, the mistake will be significant. One has to first shift point A horizontally to point \( C \) \( V^{c}_{\text{static}} = 6.84 \text{ Km/s} \) on the dashed line and take the corresponding interphase modulus on the horizontal axis \( E_1^i = 7.1 \text{ GPa} \). This gives the difference between the true interphase value (28.3 GPa) and the determined one (7.1 GPa). Thus dynamic correction for determination of interphase moduli from velocity measured along fibers is critical.

### 11.5 Summary

In this chapter we have developed a transfer matrix approach for the composite cylinder model to predict wave dispersion along the fibers in a unidirectional multiphase composite. Numerical examples show that this model gives the same zero-frequency velocity limit as that calculated independently from the composite axial
Figure 11.8: Calculated wave velocities versus interphase stiffnesses using dynamic and static composite models. The calculations using the composite cylinder model with mixed boundary conditions at 5 MHz are given by the solid line, and velocities calculated using the static composite axial elastic modulus $C_{11}^c$ are given by the dashed line.
elastic moduli \( \sqrt{C_{11}^e / \rho^e} \) determined by the multiphase generalized self-consistent model. It is found wave dispersion is moderately affected by the interphase stiffness. It is also found that the composite axial Young’s modulus is much less affected by the interphase stiffness than is the composite axial elastic modulus. We have also shown that dynamic corrections are not mandatory when the velocity measured by low frequency ultrasound is used to determine axial composite elastic moduli \( C_{11}^e \), but is necessary if it is used to determine the interphase axial Young’s modulus \( E_i^1 \).

We have also performed velocity and attenuation measurements in a unidirectional SiC/Ti composite (fiber fraction 24%) in the frequency range of 5-15 MHz. Unlike for waves propagating perpendicular to the fibers where the dispersion is found to be insignificant, for waves propagating along the fibers the dispersion is strong. The wave velocity data show a continuous decrease as frequency increases, and a reasonably good comparison with theory. At low frequencies \( f < 10 \text{ MHz} \) wave attenuation in the fiber direction is much smaller than in the perpendicular directions. At higher frequencies the attenuation in the fiber direction increases rapidly accompanied by strong dispersion.
CHAPTER XII

EXAMPLE. EVALUATION OF FATIGUE-INDUCED INTERPHASE DAMAGE IN METAL MATRIX COMPOSITES

In the last chapter of Part II we will discuss an application to demonstrate the sensitivity of the ultrasonic wave attenuation and velocity methods for characterization of interphase damage. The focus is on monitoring of fatigue damage in a crossply SiC/Ti-15V-3Cr-3Al-3Sn metal matrix composite.

12.1 introduction

It is well known that the fiber-matrix interphase plays an important role in determining composite performance. The interphase not only allows load transfer between fibers and matrix but also provides matching of chemical and thermal compatibility between the constituents. In metal and intermetallic matrix composites, special interfacial reaction barrier coatings and compliant coatings are introduced to improve chemical and thermal compatibility. In ceramic matrix composites, the
interphase is designed to provide frictional sliding contact between fiber and matrix to prevent fiber fracture due to matrix cracking. The interphase microstructure and its reaction with other composite constituents have received increasing attention. Despite great efforts in the development of special fiber coatings to tailor the interphase, interphase mechanical properties remain difficult to measure and interpret. The complexity of such interphases will become even greater if the properties are altered during manufacturing or in service by chemical reaction between the constituents, or if micromechanical defects occur.

Recently we developed two approaches to determine the effective elastic moduli of the interphases in composites. The interphase is considered to be a thin layer of bulk coating material between the fiber and matrix with distinct properties. Two ultrasonic techniques for interphase characterization are considered: wave scattering (attenuation) and phase velocity. The basic idea of the proposed methods is illustrated schematically in Fig. 1.3. One technique measures the frequency dependence of the ultrasonic attenuation in the composite \([68, 11]\). By modeling the scattering of ultrasonic waves from the fiber-matrix interphase the attenuation data can be related to the interphase elastic properties by accounting for scattering losses from multiple fibers. The second approach is based on the measurements of ultrasonic phase velocities to find the composite elastic moduli, which are used to determine the interphasial moduli via micromechanical analysis \([79, 7, 8, 9, 10]\). Other potential ultrasonic methods for the fiber-matrix interphase characterization
include measurement of local reflectivity of single fiber-matrix interface [83] and measurement of leaky guided waves along individual fibers [120].

12.2 Fatigue Damage in Metal Matrix Composites

The samples used in this work are $[0/90]_2$, SiC/Ti-15V-3Cr-3Al-3Sn metal matrix composites [127, 128]. The composite has an 8-ply symmetry layup as shown in Fig. 12.1 made by hot isostatic pressing of a foil/fiber/foil layup with thickness about 1.7mm. The matrix is metastable $\beta$ titanium alloy Ti-15V-3Cr-3Al-3Sn (Ti-15-3, weight ratio) and the SiC fiber is SCS-6 by TeXtron. The fiber volume fraction of the composite is 35 percent and the composite density is 4.18 g/cm$^3$. To simulate the effect of heat treatment during material processing, the as-received samples were heat-treated at temperatures above (815 °C) and below (540 °C) the matrix $\beta$ transus. The heat treated samples were first fatigued to failure under different stress ranges to obtain their S/N curves. Based on the S/N curves, two stress-controlled fatigue tests ($\Delta \sigma$ equals 50 or 70 percent of the ultimate strength, $R = \sigma_{\text{min}}/\sigma_{\text{max}} = 0.1$) were selected for ultrasonic damage assessment using samples with different heat treatments. The fatigue cycling frequency is 10 Hz.

Ultrasonic bulk wave velocity and attenuation measurements were performed on samples prior to fatigue as well as at different stages of fatigue (with step size 0.1 of the fatigue life) [11]. The fatigued samples were obtained from Laboratory
Figure 12.1: Schematic diagram of the fatigue damage for [0/90] cross-ply SCS-6/Ti-15-3 fatigue tensile samples.
of Dr. Soboyejo, Materials Science and Engineering Department, The Ohio State University. As an example the ultrasonically measured attenuation spectra (in Np/mm) of the same sample measured at different fatigue stages are shown in Fig. 12.2(a) (fatigue stress level at 70% of ultimate stress). One sees from the figure that at frequencies below the high frequency dynamic noise limit there is a significant increase of attenuation as fatigue progresses. The high frequency dynamic noise is mostly due to wave interference between neighboring fibers and laminae. Fig. 12.2(b) shows the dependence of the attenuation on the fatigue life cycle at a fixed frequency $f = 13$ MHz. One sees that the attenuation increases linearly at a significant rate before 50% of the fatigue life cycle due to the extension of partial debonding in the $90^\circ$ laminae (with fibers perpendicular to the loading direction) as shown in Fig. 12.1. Considering the increase of attenuation as fatigue progresses, one concludes that attenuation is sensitive to fatigue-induced damage on the fiber-matrix interface at the early stages of fatigue life cycle. Then the attenuation increases steadily but at a smaller rate until 70% of the fatigue life cycle, mainly due to fiber bonding occurring in $0^\circ$ laminae. Finally the attenuation increases rapidly again, probably due to matrix cracking. To compare with the phase velocity approach, we also plot in Fig. 12.2(b) the dependence of $C_{33}$ of the composite on the fatigue life cycle which was determined from velocity measurements at 5 MHz. One sees that the rate of increase of attenuation and that of the drop in $C_{33}$ are similar.
Figure 12.2: Ultrasonic results for a [0/90] cross-ply SCS-6/Ti-15-3 fatigue tensile sample (heat treated at 815° for 10 hours) measured in the through thickness direction. (a) The attenuation spectra of the sample at different stages of fatigue life; (b) the attenuation dependence on fatigue life cycles at a fixed frequency \( f = 13 \text{ MHz} \) (solid circles). The dependence of the through-thickness modulus \( C_{33} \) on the fatigue life cycle is also plotted here for comparison (open circles).
Fig. 12.2 (continued)

---

**ATTENUATION VALUES AT** \( f = 13 \text{ MHz} \)**

**MEASURED BY A 20 MHz TRANSDUCER**

**C\text{33}** **VALUES OBTAINED FROM VELOCITIES**

**MEASURED BY A 5 MHz TRANSDUCER**

---

**ATTENUATION \( \alpha \) (Np/cm)**

**ELASTIC CONSTANT C\text{33} \text{(GPa)}**

**FATIGUE CYCLE \( N/N_t \)**

(b)
The strong dependence of measured attenuation spectra on fatigue damage and the calculated scattering cross-section from a single fiber with an interphase of variable stiffness in Chapter IX indicate a possibility of using attenuation data for quantitative assessment of interfacial damage. Multiple scattering must be considered in this composite to calculate the attenuation due to the high fiber fraction (35%). Significant additional effort is needed to account for the partial debonding of the fibers, inhomogeneous distributions of damage among different fiber/matrix interfaces, the effect of $0/90^\circ$ fiber orientation in cross-ply composites and the periodical laminae lay up and fiber distribution in each laminate.
CHAPTER XIII

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

13.1 Summary and Conclusions

This dissertation presents a development of theoretical and experimental techniques for ultrasonic scattering from imperfect plane and cylindrical interfaces with application to composites. The results achieved in this study are summarized as follows.

For plane interfaces, the second order asymptotic B.C. give solutions which, in addition to being accurate and consistent, satisfy energy balance and uniqueness, and give zero scattering from an interfacial layer with properties equal to those of the substrates. The second order asymptotic B.C. have the same simplicity as the first order B.C. and approximate the exact solution over a greater frequency range. They are simpler than the exact solution, since there is no need to describe the wave behavior inside the interfacial layer. A drastic improvement in the prediction of transmitted field is achieved using the second order B.C. model. It has been shown
that retention of coupling and inertial terms in the asymptotic models greatly improves the quality of the approximation for oblique wave incidence. When the incident angle is very close to the interface normal, the second order B.C. can be replaced by the stiffness-mass model. In that case, the anisotropic coupling compliance should still be kept if the incident plane is not a plane of symmetry and the interfacial anisotropy is significant.

For a thin anisotropic layer between two identical semispaces the second order B.C. can be decomposed for both symmetric and antisymmetric cases into stiffness-type relations described by $3 \times 3$ matrices. Analytical solutions for wave scattering and interface wave localization by an off-axis orthotropic interfacial layer between identical isotropic semispaces are given. These can be applied to predict the effect of interface anisotropy on wave scattering and also to predict the effect of interface anisotropy on interface mode dispersion. The characteristic equations for interface modes relate explicitly to the interphase elastic moduli and can be utilized for moduli determination from interface wave velocity measurements. It has been shown that retention of coupling and mass terms in the asymptotic models greatly improves the accuracy of the approximation, which is especially critical for interface waves.

For a fractured plane interface with arbitrary crack orientation the generalized spring B.C. are introduced. The spring stiffness elements can be obtained from those defined for the plane of symmetry. An additional coupling shear spring ap-
pears in the B.C. The generalized shear springs depend on the angle of deviation of the incident plane from the plane of symmetry and can be used for anisotropic interfaces between dissimilar anisotropic solids. Analytical solutions for wave scattering and dispersion equations for interface modes on such an interface described by the generalized spring B.C. (including mass terms) are given. They can be applied to predict the effect of interface imperfection orientation on wave scattering and interface mode dispersion. It has been demonstrated that mode conversions to SH waves and as a result angle changes in polarization of the scattered shear waves are observed.

For cylindrical interphasial layers, we have obtained a transfer matrix representation which allows a systematic treatment of wave scattering from a $N$-phase multilayered cylindrical system. The same transfer matrix can also be obtained by solving a matrix differential equation for the elastic field in a cylindrical medium with radially-dependent properties. The effect of each intermediate cylindrical layer is described by a transfer matrix which relates the stresses and displacements on the outer and inner boundaries of the layer. The total transfer matrix, which relates the stresses and displacements in the surrounding medium and the core, is obtained in the form of products of intermediate transfer matrices. The scattering coefficients are found by solving a system of $4 \times 4$ boundary condition equations on the outer boundary of the multilayered cylinder. The advantage of the transfer matrix formulation lies in its natural application to multilayered fibers and inter-
phases. Also there is no need to find wave solutions in each of the cylindrical layers between the surrounding medium and the core.

The first and second order asymptotic B.C models are derived to represent a thin fiber-matrix interphase by asymptotically expanding the transfer matrix for the interphase layer. The new notion of the spring B.C. model, where springs connect the fiber and matrix through interphase gap, is introduced. The model is investigated as an effective representation of a thin fiber-matrix interphase. The spring B.C. model with interphase gap is compared with those when the interphase gap is filled by matrix or fiber material. It is shown that although the second order asymptotic B.C give the best overall results, the spring approximation with springs connecting the interphase gap is sufficient for a thin \((h/r_f < 0.05, k_{ij}^i, h \ll 1)\) interphase which is softer than the fiber. For even thinner and softer interphases the spring B.C. model with gap filled with fiber material may also be used.

Further it is demonstrated that an equivalent homogeneous fiber with springs between those of the matrix and the fiber can be used to describe scattering in the low frequency range. Numerical examples show that scattering is significantly affected by the interphase stiffnesses and thus can be used for interphase characterization. The minimum of the scattering cross-section versus interphase stiffness is shown to be due to matching of stiffness between the effective fiber (actual fiber plus interphase) and the matrix. It is also shown that for a compliant fiber-matrix interphase there exist resonance peaks in the wave scattering cross-section, and
these resonances can be analyzed by the spring model at very low frequencies.

For applications to composites we have performed a theoretical and experimental study of fiber scattering-induced wave attenuation and dispersion in a unidirectional random fiber composite. For waves propagating normal to the fibers, it is found that the attenuation data are close to those predicted by the independent scattering model and Waterman-Truell solutions at low frequencies. This shows that the multiple scattering effect in this composite is insignificant due to close impedance matching between fibers and matrix. On the other hand the wave velocity data are very close to those predicted by the modified Waterman-Truell models where composite cylinders are considered either in a composite medium or in a dynamic effective medium. This indicates that wave attenuation is mostly a local effect and can be described by scattering models where the fiber is considered immersed in the unbound matrix, while wave dispersion is affected more by multiple scattering and should be described by models where a composite cylinder (fiber with matrix layer) is considered immersed in a composite or effective medium. It has also been found from both experimental and simulated data that dispersion normal to the fibers is small, so the ultrasonic wave velocities can be used to determine composite moduli without needing correction for dynamic effects. Finally it is shown numerically that both longitudinal and shear wave velocities are sensitive to the interphase stiffnesses and can be used for determination of interphase transverse moduli.
We have also developed a transfer matrix approach for the composite cylinder model to predict wave dispersion along the fibers in a unidirectional multiphase composite. Numerical examples show that this model gives the same zero-frequency velocity limit as that found from the composite axial elastic moduli determined by the generalized self-consistent model. It is found that wave dispersion is affected moderately by the interphase stiffness. It is also found that the composite axial Young's modulus is much less affected by the interphase stiffness than the composite axial elastic modulus. We have also shown that dynamic corrections are not mandatory when the velocity measured by low frequency ultrasound is used to determine the composite axial elastic moduli, but is necessary if it is used to determine the interphase axial Young's modulus. Unlike for waves propagating perpendicular to the fibers where the dispersion is found to be insignificant, for waves propagating along the fibers the dispersion is found from both numerical and experimental data to be significant. The wave velocity data show a continuous decrease as frequency increases. Such behavior is predicted at low frequencies by both the composite cylinder model and the approximate continuum mixture theory, but the exact composite cylinder solution has better agreement. At low frequencies ($f < 10$ MHz) wave attenuation in the fiber direction is much smaller than perpendicular to the fiber direction. At higher frequencies the attenuation in the fiber direction increases rapidly accompanied by strong dispersion.

Finally ultrasonic velocity and attenuation measurements have been performed
on high-temperature composites under fatigue, showing that ultrasonic waves are very sensitive to fatigue-induced damage on the fiber-matrix interphase, and thus can be used for interphase characterization.

13.2 Recommendations for Future Work

I hope that this study has helped build a good foundation for theoretical and experimental studies of ultrasonic characterization of solid-solid interfaces. There remain many areas to be explored for future work. For wave propagation in composites the next logical step is to consider oblique wave scattering from a fiber in a solid and wave propagation characteristics in an arbitrary direction in composites. Significant additional effort is needed to account for the nonhomogeneous interphase condition of the fiber-matrix interphases and the effect of crossply lay-up in cross-ply composites.
Appendix A

THE SECOND ORDER B.C. IN THE MATERIAL COORDINATE SYSTEM

The second order B.C. in the material coordinate system (Fig. 3.2) \((x^0, y^0, z)\) are

\[
\begin{align*}
  u_{x^0} - u_{x^0}' &= \frac{h}{C_{65}^0} \left( \frac{\sigma_{xx^0} + \sigma_{zz^0}}{2} - ikh \cos(\varphi) \right) \frac{u_z + u_z'}{2} \\
  u_{y^0} - u_{y^0}' &= \frac{h}{C_{44}^0} \left( \frac{\sigma_{xy^0} + \sigma_{yy^0}}{2} + ikh \sin(\varphi) \right) \frac{u_z + u_z'}{2} \\
  u_z - u_z' &= \frac{h}{C_{33}^0} \left( \frac{\sigma_{zz} + \sigma_{zz}'}{2} - ikh \cos(\varphi) \right) \frac{u_z + u_z'}{2} + ikh \sin(\varphi) \frac{C_{33}^0 \sigma_{y^0} + u_{y^0}'}{2} \\
  \sigma_{xx^0} - \sigma_{xx^0}' &= -\omega^2 M_{sv}^0 \frac{u_{x^0} + u_{x^0}'}{2} - \omega^2 M_c^0 \frac{u_{y^0} + u_{y^0}'}{2} - ikh \cos(\varphi) \frac{C_{13}^0 \sigma_{zz} + \sigma_{zz}'}{2} \\
  \sigma_{xy^0} - \sigma_{xy^0}' &= -\omega^2 M_c^0 \frac{u_{x^0} + u_{x^0}}{2} - \omega^2 M_{sh}^0 \frac{u_{y^0} + u_{y^0}}{2} + ikh \sin(\varphi) \frac{C_{23}^0 \sigma_{y^0} + \sigma_{y^0}'}{2} \\
  \sigma_{zz} - \sigma_{zz}' &= -\omega^2 M_n \frac{u_z + u_z'}{2} - ikh \cos(\varphi) \frac{\sigma_{zz} + \sigma_{zz}'}{2} + ikh \sin(\varphi) \frac{\sigma_{xy^0} + \sigma_{xy^0}'}{2}.
\end{align*}
\] (A.1)

where

\[
M_n = \rho_0 h, \quad M_{sv}^0 = M_n \left( 1 - \frac{Q_{11}^0}{\rho_0 V^2} \cos^2(\varphi) - \frac{C_{66}^0}{\rho_0 V^2} \sin^2(\varphi) \right);
\]

\[
M_{sh}^0 = M_n \left( 1 - \frac{Q_{22}^0}{\rho_0 V^2} \sin^2(\varphi) - \frac{C_{66}^0}{\rho_0 V^2} \cos^2(\varphi) \right),
\]

\[
M_c = -M_n \left( \frac{Q_{12}^0}{\rho_0 V^2} + \frac{C_{66}^0}{\rho_0 V^2} \cos(\varphi) \sin(\varphi) \right);
\]
and

\[ Q_{ij}^0 = \frac{C_{ij}^0 C_{33}^0 - C_{i3}^0 C_{j3}^0}{C_{33}^0}. \]

At normal incidence, (A.1) reduces to the stiffness-mass model:

\[
\begin{align*}
    u_x - u_x' &= \frac{h}{C_{55}^0} \frac{\sigma_{xx} + \sigma_{xx}'}{2} \\
    u_y - u_y' &= \frac{h}{C_{44}^0} \frac{\sigma_{yy} + \sigma_{yy}'}{2} \\
    u_z - u_z' &= \frac{h}{C_{33}^0} \frac{\sigma_{zz} + \sigma_{zz}'}{2} \\
    \sigma_{xx} - \sigma_{xx}' &= -\omega^2 M_n \frac{u_x + u_x'}{2} \\
    \sigma_{yy} - \sigma_{yy}' &= -\omega^2 M_n \frac{u_y + u_y'}{2} \\
    \sigma_{zz} - \sigma_{zz}' &= -\omega^2 M_n \frac{u_z + u_z'}{2}.
\end{align*}
\]

(A.2)

Note that compared to (3.46) the B.C. (A.2) have no coupling terms with factors analogous to the coupling compliance \( S_c \). However equivalent mode coupling will be predicted because the incident and scattered fields must be decomposed into components in the material coordinate system thus introducing terms equivalent to those predicted by B.C. (3.46).
Appendix B

DERIVATION OF EQUATION (3.86)

For matrix $A$

\[
- \frac{ikh}{2} A = \begin{bmatrix} N & S \\ M & N^T \end{bmatrix}, \tag{B.1}
\]

where the elements of $N$ are pure imaginary, and $M$ and $S$ are real symmetric matrices, the following equality holds [42]

\[
J \left( -\frac{ikh}{2} A \right) J^T = \left( \frac{ikh}{2} A \right)^H. \tag{B.2}
\]

Using equalities (3.81) and (B.2) we can rewrite the left side of the equation (3.86) as

\[
= \left( I + \frac{ikhA}{2} \right)^{-1} \left( I - \frac{ikhA}{2} \right)^H \left( I + \frac{ikhA}{2} \right)^{-1} J^T \left( I - \frac{ikhA}{2} \right)^{-1} J^T
\]

\[
= \left( I + \frac{ikhA}{2} \right)^{-1} \left( I - \frac{ikhA}{2} \right)^H \left[ \left( I - \frac{ikhA}{2} \right)^H \right]^{-1} \left( I + \frac{ikhA}{2} \right)^H \tag{B.3}
\]

\[
= \left( I + \frac{ikhA}{2} \right) \left( I - \frac{ikhA}{2} \right)^{-1} \left( I + \frac{ikhA}{2} \right)^{-1} \left( I - \frac{ikhA}{2} \right)^H.
\]
Appendix C

DERIVATION OF SYSTEMS OF BOUNDARY EQUATIONS FOR DETERMINATION OF $R^{(a,s)}$ IN (4.27)

Let us assume unit amplitudes for the longitudinal and SV and SH transverse waves simultaneously incident onto the interface for the decomposed antisymmetric and symmetric elastic fields. The reflected field (omitting the common phase term $e^{ikz}$) in the upper semispace can be written as

$$
\tilde{u}_l = \tilde{P}_l e^{-ik_1 \cos(\theta_1)z} + [R_{ll}^{(a,s)} + R_{Vl}^{(a,s)} + R_{Hl}^{(a,s)}] \tilde{P}_r e^{ik_1 \cos(\theta_1)z},
$$

$$
\tilde{u}_{SV} = \tilde{P}_{Vl} e^{-ik_1 \cos(\theta_1)z} + [R_{lV}^{(a,s)} + R_{VV}^{(a,s)} + R_{HV}^{(a,s)}] \tilde{P}_r e^{ik_1 \cos(\theta_1)z},
$$

$$
\tilde{u}_{SH} = \tilde{P}_{Hl} e^{-ik_1 \cos(\theta_1)z} + [R_{lH}^{(a,s)} + R_{VH}^{(a,s)} + R_{HH}^{(a,s)}] \tilde{P}_r e^{ik_1 \cos(\theta_1)z},
$$

where $\tilde{P}_l = \tilde{n}$, $\tilde{P}_H = (0,1,0)^T$ and $\tilde{P}_V = \tilde{n} \times \tilde{P}_H$ are the polarization vectors of the incident or reflected longitudinal and transverse (SH and SV) waves. The wave normal $\tilde{n} = (\cos \theta, 0, -\sin \theta)^T$, where $\theta$ is the propagation angle. Thus the displacement field at the top surface $z = 0$ is simply

$$
u_x = \sin(\theta_i)(R_{ll}^{(a,s)} + R_{Vl}^{(a,s)} + R_{Hl}^{(a,s)} + 1) - \cos(\theta_i)(R_{lV}^{(a,s)} + R_{VV}^{(a,s)} + R_{HV}^{(a,s)} - 1),$$

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\[ u_x = (R_{\mu\mu}^{(\mu, \nu)} + R_{\nu\nu}^{(\mu, \nu)} + R_{\mu\nu}^{(\mu, \nu)} + 1), \]  
\[ u_z = \cos(\theta_i)(R_{\mu\mu}^{(\mu, \nu)} + R_{\nu\nu}^{(\mu, \nu)} + R_{\mu\nu}^{(\mu, \nu)} - 1) + \sin(\theta_i)(R_{\mu\nu}^{(\mu, \nu)} + R_{\nu\mu}^{(\mu, \nu)} + R_{\mu\nu}^{(\mu, \nu)} + 1). \]  

It is advantageous to define nondimensional parameters \( \alpha = \sin(\theta_i) = k/k_i; \xi = V_i/V_t, \) \( V_t \) and \( V_i \) being respectively the longitudinal and transverse wave velocities in the semispace; \( \beta = \cos(\theta_i) = \sqrt{1 - \alpha^2} \) and \( \gamma = \xi \cos(\theta_i) = \sqrt{\xi^2 - \alpha^2} \). Then we can write equations (C.2) in matrix form as

\[ (u_x, u_y, u_z) = [D(-\beta, -\gamma) + D(\beta, \gamma)R^{(\mu, \nu)}](1, 1, 1)^T, \]  

where \( R^{(\mu, \nu)} \) are the scattering matrices defined by Eq. (4.27), and

\[ D(\beta, \gamma) = \begin{bmatrix} \alpha/\xi & 0 & -\beta \\ 0 & i & 0 \\ \gamma/\xi & 0 & \alpha \end{bmatrix}. \]  

Here \( D(\beta, \gamma) \) describes the displacements due to waves propagating upwards, and \( D(-\beta, -\gamma) \) describes those due to waves propagating downwards.

Using the stress-displacement relations

\[ \sigma_{xx} = \mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}), \quad \sigma_{xy} = \mu \frac{\partial u_y}{\partial z}, \quad \sigma_{zz} = \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu) \frac{\partial u_z}{\partial z}, \]  

where \( \lambda \) and \( \mu \) are the Lamé constants of the semispace, we can write the stresses at the top surface \( z = 0 \) as

\[ (\sigma_{xx}, \sigma_{xy}, \sigma_{zz}) = i\mu k_i[S(-\beta, -\gamma) + S(\beta, \gamma)R^{(\mu, \nu)}](1, 1, 1)^T \]  

where

\[ S(\beta, \gamma) = \begin{bmatrix} 2\alpha\gamma/\xi & 0 & (\alpha^2 - \beta^2) \\ 0 & \beta & 0 \\ -(\alpha^2 - \beta^2)/\xi & 0 & 2\alpha\beta \end{bmatrix}. \]
S(β, γ) describes the stresses due to waves propagating upwards, and S(−β, −γ) describes those due to waves propagating downwards.

Using the symmetric (4.15) and antisymmetric (4.3) stiffness matrices to relate the stresses (C.6) and displacements (C.3) on the interface and dividing both sides of the resulting equation by μk_t, we obtain

\[ \left[ \frac{K_{(A,S)}}{\mu k_t} D(\beta, \gamma) - iS(\beta, \gamma) \right] R_{(a,s)}^{(a,s)} = - \left[ \frac{K_{(A,S)}}{\mu k_t} D(\beta, \gamma) - iS(\beta, \gamma) \right] \]  \hspace{1cm} (C.8)

where C_{ij}, ρ_0 and h are the elastic constants, density and thickness of the interfacial layer. Thus the scattering matrices \( R_{(a,s)}^{(a,s)} \) are found as

\[ R_{(a,s)}^{(a,s)} = - \left[ \frac{K_{(A,S)}}{\mu k_t} D(\beta, \gamma) - iS(\beta, \gamma) \right]^{-1} \left[ \frac{K_{(A,S)}}{\mu k_t} D(\beta, \gamma) - iS(\beta, \gamma) \right]. \]  \hspace{1cm} (C.9)
Appendix D

ELEMENTS OF SCATTERING MATRICES
$R^{(a,s)} (4.27)$

Other elements for $R^a$ are

\[
R^a_{HH} = -\frac{\Delta^a(-\beta, -\gamma, -C_{44}, -C_{45}^2, -\bar{h})}{\Delta^a(\beta, \gamma, C_{44}, C_{45}^2, \bar{h})} \tag{D.1}
\]
\[
R^a_{VV} = -\frac{\Delta^a(-\gamma, -C_{44}, -C_{45}^2, -\bar{h})}{\Delta^a(\gamma, C_{44}, C_{45}^2, \bar{h})} \tag{D.2}
\]
\[
R^a_{VH} = -\frac{4\alpha\beta C_{45}}{\Delta^a} \frac{\bar{h}}{\mu} \{ -2iW + \left( \frac{\rho_0}{\rho} - 2\alpha^2 \right) \gamma \bar{h} \} \tag{D.3}
\]
\[
R^a_{V\bar{V}} = -\frac{4\alpha\beta\xi}{\Delta^a} \frac{\bar{h}}{\mu} \{ 4i \frac{C_{44}}{\mu} W + 2i \frac{C_{44} C_{55} - C_{45}^2}{\mu^2} \frac{\rho_0}{\rho} (\rho_0 + \beta^2 - 3\alpha^2) \}
+ \left[ 2W + \frac{C_{55}}{\mu} (\frac{\rho_0}{\rho} + \beta^2 - 3\alpha^2) \right] \beta \bar{h} \} \tag{D.4}
\]

and those for $R^s$ are

\[
R^s_{HH} = \frac{N^s}{\Delta^s} \tag{D.5}
\]
\[
R^s_{VV} = \frac{N^s(-\beta)}{\Delta^s(\beta)} \tag{D.6}
\]
\[
R^s_{VH} = \frac{4\alpha^2 \beta}{\Delta^s} \frac{\bar{h}}{\mu^2} \{ 2C_{16} C_{33} - C_{13} C_{36} - 2C_{36} \gamma \}
+ i \left( \frac{C_{16}}{\mu} W + \frac{C_{16} C_{13} - C_{36}(C_{11} - \rho_0 V^2)}{\mu^2} \alpha^2 \right) \bar{h} \} \tag{D.7}
\]
\[
R_{\nu l} = \frac{4\alpha\beta \xi}{\Delta_s} \hbar \{2[-2W - \frac{C_{33}}{\mu} \frac{\rho_0}{\rho} q_1 + \frac{C_{13}}{\mu} (1 - 4\alpha^2)]\beta \\
+ i2\frac{C_{66} - \rho_0 V^2}{\mu} \left( \frac{C_{13} - C_{33} \frac{\rho_0}{\rho} q_1}{\mu^2} (C_{66} - \rho_0 V^2) - C_{16} C_{36} \right) \\
+ \left[ \frac{-4C_{66} - \rho_0 V^2}{\mu^2} + \frac{4C_{16} C_{36} - C_{13}(C_{66} - \rho_0 V^2)}{\mu^2} \right] \alpha^2 \hbar \} \\
+ \frac{2C_{13} C_{16} C_{36} - C_{36}^2 (C_{11} - \rho_0 V^2) - C_{16}^2 C_{33}}{\mu^3} \alpha^2 \hbar \}.
\]

(D.8)

Here

\[
N^* = 8i \frac{C_{33}}{\mu} \beta \gamma + 4[\Delta_r \beta - 2 \frac{C_{13}}{\mu} \alpha^2 \beta a_2 + \frac{C_{33}}{\mu} \frac{\rho_0}{\rho} (a_1 q_1 \beta - q_2 \gamma)] \hbar \\
- 2i \left\{ \frac{C_{66} - \rho_0 V^2}{\mu} \Delta_r \alpha^2 - \frac{C_{11} - \rho_0 V^2}{\mu^2} \alpha^2 \beta^2 + \frac{C_{16} C_{36} - C_{13}(C_{66} - \rho_0 V^2)}{\mu^2} a_2 \alpha^4 \\
+ \left[ \frac{C_{33}}{\mu} \frac{\rho_0}{\rho} (C_{11} - \rho_0 V^2) \frac{C_{13}^2}{C_{33} \mu} - \frac{C_{16}^2 C_{36} - 2C_{13} C_{16} C_{36} + C_{13}^2 C_{66}}{\mu^3} \alpha^2 a_1 \right] \hbar^2 \\
- \frac{C_{16}^2 - (C_{11} - \rho_0 V^2)(C_{66} - \rho_0 V^2)}{\mu^2} \alpha^4 \beta \hbar^3 \right\}.
\]

(D.9)

For the elements without mode conversion, given by Eqs. (4.29, 4.32, D.1, D.2, D.5, D.6), the numerators and denominators are the same except for sign changes in some terms. For example \(\Delta^{(a,a)}(-\beta, -\hbar)\) is obtained by replacing \(\beta\) and \(\hbar\) by \(-\beta\) and \(-\hbar\) in expressions (4.31) and (4.34) for \(\Delta^{(a,a)}\) (4.31, 4.34), not forgetting the terms in \(a_1, a_2\) and \(\Delta_r\).
Appendix E

RECURSIVE SUBSTITUTION METHOD FOR CALCULATION OF $E_1^c$, $\nu_1^c$ and $\kappa^c$

Here we briefly review the recursive substitution method [79, 80, 82, 126] for determination of the following composite moduli: axial Poisson's ratio ($\nu_1^c$), transverse bulk ($\kappa^c$) and axial Young's modulus ($E_1^c$). For two-phase composites these moduli were obtained in terms of the fiber and matrix properties by Hashin and Rosen [125] using the composite cylinders assemblage (CCA) model and later by Christensen and Lo [129] using the generalized self-consistent (GSC) model:

$$\kappa^c = \kappa^m + (1 - v^m)/\left[1/(\kappa^f - \kappa^m) + v^m/(\kappa^m + G_t^m)\right]; \quad (E.1)$$

$$\nu_1^c = \nu_1^m v^m + \nu_1^f (1 - v^m) + \frac{(\nu_1^f - \nu_1^m)(1/\kappa^m - 1/\kappa^f)v^m(1 - v^m)}{v^m/\kappa^f + (1 - v^m)/\kappa^m + 1/G_t^m}; \quad (E.2)$$

and

$$E_1^c = E_1^m v^m + E_1^f (1 - v^m) + \frac{4(\nu_1^f - \nu_1^m)^2 v^m(1 - v^m)}{v^m/\kappa^f + (1 - v^m)/\kappa^m + 1/G_t^m}. \quad (E.3)$$

In the above equations the superscripts $f$ and $m$ denote the fiber and matrix properties, $v^m$ is the matrix volume fraction, and $G_t$ is the transverse shear modulus.
To extend the above expressions to a composite with multilayered interphase, Hashin [82] and Qiu and Weng [126] used a recursive substitution method, replacing the fiber moduli \((E^f_i, \nu^f_i \text{ and } \kappa^f)\) in the above expressions by the equivalent properties of a multiphase (multilayered) fiber. This is possible since, due to symmetry, the equations for the composite and the composite cylinder coincide. The intermediate equivalent moduli of the layered fiber are [79, 80]

\[
(\kappa^e)_i = \kappa^f + (1 - v^i)/(1/[(\kappa^e)_{i-1} - \kappa^f] + v^i/(\kappa^f + G^f_i)); \tag{E.4}
\]

\[
(\nu^e)_i = \nu^f_i v^i + (\nu^e_{i-1})(1 - v^i) + [((\nu^e)_i - \nu^f_i)/(1/\kappa^f - 1/(\kappa^e)_{i-1}) v^i(1 - v^i)]/v^i/(\kappa^e)_{i-1} + (1 - v^i)/\kappa^f + 1/G^f_i; \tag{E.5}
\]

and

\[
(E^e_i)_i = E^f_i v^i + (E^e_{i-1})(1 - v^i) + 4[(\nu^e)_{i-1} - \nu^f_i]^2 v^i(1 - v^i)/v^i/(\kappa^e)_{i-1} + (1 - v^i)/\kappa^f + 1/G^f_i. \tag{E.6}
\]

Here

\[
v^i = 1 - (r_{i-1}/r_i)^2, \quad 2 \leq i < N - 1; \tag{E.7}
\]

\(r_{i-1}\) and \(r_i\) are the inner and outer radii of the \(i\)th phase, and \(\kappa^i, \nu^i, E^i, G^i\) and \(v^i\) are the corresponding elastic moduli and volume fraction. \((\kappa^e)_{i-1}, (\nu^e)_{i-1} \text{ and } (E^e_i)_{i-1}\) are the equivalent moduli of the inner \(i - 1\) phases. Following this approach, the effective composite moduli of an \(N\)-phase composite are determined using Eqs. (E.1-E.3) where the outermost phase becomes the matrix and the equivalent properties of the inner \(N - 1\) phase fiber replace the fiber moduli (i.e. \(\kappa^f, \nu^f, E^f_i\) are substituted by \((\kappa^e)_{N-1}, (\nu^e)_{N-1}, (E^e_{i})_{N-1}\)). Note that the outer radius of the matrix is defined by the matrix volume fraction \(v^m\): \(r_N = r_{N-1}/\sqrt{1 - v^m}\).
The generalized self-consistent model for determination of the composite transverse shear modulus [129] has been extended to composites with multilayered fibers and details can be found in [79, 80].
Bibliography


