REARRANGEMENTS OF CONDITIONALLY CONVERGENT
INFINITE SERIES

A Thesis
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By
Raymond Justice B.S.
The Ohio State University
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Approved by:

C.W. Richard
Adviser
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INTRODUCTION

Investigations of rearrangements of conditionally convergent infinite series date from the time when Riemann \( \text{(5)} \) first proved the fundamental rearrangement theorem. This theorem proves that any pre-assigned real number can be attained as the sum of a rearrangement of an arbitrary conditionally convergent series of real terms. Theorem 1 in this thesis is a slight generalization of this theorem of Riemann.

Among those who conducted investigations into rearrangements of infinite series are Sierpinski \( (8) \), Agnew \( (1) \), Borel \( (2) \), Levi \( (3) \), Schlomilch \( (7) \), and Pringsheim \( (5) \). Some of the results of the first four men mentioned above are included in this thesis.

A theorem due to Sierpinski states that if a conditionally convergent series of real terms converges to \( A \) then any pre-assigned number less than \( A \) can be attained as the sum of a rearrangement of the original series in which all the negative terms retain their original place and order. Theorem 2 of this paper is a generalization of Sierpinski's theorem.

Agnew proved that the set of permutations of the positive integers which leave unaltered the sum of a conditionally convergent series is a set of the first
category in the space of permutations of the positive integers with Frechet metric. This theorem and the theorem which states that those permutations for which the sequence of the partial sums of the permuted conditionally convergent series is bounded form a set of the second category are to be found on pages 12-24.

Borel defines sufficient conditions that a rearrangement be sum preserving. These are to be found in Theorem 6 below. Necessary and sufficient conditions that a rearrangement be sum preserving are to be found in Theorem 7 which is due to Levi.

The second part of this thesis deals with conditionally convergent series of complex terms. The objective is to prove a theorem analogous to Riemann's theorem for series of real terms. Levy\(^{(4)}\) published an incomplete proof of such a theorem. The first complete proof is that of Steinitz\(^{(9)}\). He proved the theorem for series of \(n\)-dimensional vectors. In this thesis a different proof is given for the case of series of complex terms.

Conditionally convergent series are to be considered in this thesis as convergent series whose series of absolute values diverges. This definition is completely equivalent to the usual definition; namely, a convergent series whose sum can be altered by rearranging its terms. For series of real terms both of these definitions are equivalent to saying that the series converges but the subseries of positive terms and the
subseries of negative terms both diverge.

The generally accepted notation is used throughout except that when the range of an index is from one to infinity no notation is made of this range. Thus, in place of \( \sum_{n=1}^{\infty} a_n \), the notation \( \sum a_n \) is used.
I SERIES OF REAL TERMS

1. Main rearrangement theorem. Riemann himself proved that any preassigned number could be attained as the sum of a proper rearrangement of an arbitrary conditionally convergent series of real terms. The extensions to this theorem incorporated in Theorem I below are direct consequences of the original theorem due to Riemann.

Theorem I. Let $\sum a_n$ be any series of real terms satisfying

1) The subseries $\sum \rho_n$ of positive terms of $\sum a_n$ diverges
2) The subseries $\sum \gamma_n$ of negative terms of $\sum a_n$ diverges
3) $\lim_{n \to \infty} a_n = 0$

Let $\sum \rho_n$ and $\sum \gamma_n$ be divergent subseries of $\sum \rho_n$ and $\sum \gamma_n$, respectively. Then given any two real numbers $x$ and $y$, $x \leq y$, it is possible to rearrange $\sum a_n$ in such a way that $x$ and $y$ are the limit inferior and limit superior, respectively, of the sequence of partial sums of the rearranged series. Furthermore, such a rearrangement can be achieved without altering the order, relative to one another, of all terms of $\sum a_n$ which are not terms of either $\sum \rho_n$ or $\sum \gamma_n$.

Proof: Let $a_1', a_2', \ldots$ denote the terms of $\sum a_n$ which are not terms of either $\sum \rho_n$ or $\sum \gamma_n$. We shall show how
to construct a series of the form
\[ p_{\eta_1} + p_{\eta_2} + \cdots + p_{\eta_j} + a_{\gamma} + q_{\kappa_1} + q_{\kappa_2} + \cdots + q_{\kappa_{j+1}} + \cdots + a_{\gamma} + p_{\eta_j+1} + \cdots + p_{\eta_1} + a_{\gamma} + q_{\kappa_1} + q_{\kappa_2} + \cdots + \]
satisfying the required conditions.

Let \( j \geq 1 \) be the first integer such that \( p_{\eta_1} + p_{\eta_2} + \cdots + p_{\eta_j} = y \). Denote by \( A \) the sum \( p_{\eta_1} + \cdots + p_{\eta_j} + a_{\gamma} \). Choose \( k \) to be the first positive integer such that \( A_{\kappa_1} + q_{\kappa_1} + \cdots + q_{\kappa_{j+1}} \leq x \). Let \( A_{\kappa_1} = A + q_{\kappa_1} + \cdots + q_{\kappa_{j+1}} + a_{\gamma} \). Choose \( j \) to be the first integer greater than \( j \) such that \( A_{\kappa_1} + p_{\eta_{j+1}} + \cdots + p_{\eta_1} \geq y \).

It remains only to show that this process continued yields the desired rearrangement of \( \sum a_n \). Clearly every partial sum ending in \( p_{\eta_j} (q_{\kappa_{j+1}}) \) is greater than or equal to (less than or equal to) \( y(x) \). Furthermore, no partial sum exceeds (is less than) \( y(x) \) by more than the last two terms of that partial sum. Thus \( y \) is the limit superior and \( x \) the limit inferior of the sequence of partial sums of the rearranged series. Clearly all terms not in either \( \sum p_{\eta_j} \) or \( \sum q_{\kappa_{j+1}} \) retain their order with respect to themselves which completes the proof.

**Corollary 1.** If a series satisfies conditions i), ii), and iii), then given any number \( \alpha \) it is possible to rearrange the series, subject to the same restrictions.
on the rearrangement, so that the rearranged series converges to $\alpha$.

Proof: Choose $x = y = \alpha$.

Corollary 2. Given any interval $I: a \leq i \leq b$, it is possible to rearrange a series which satisfies conditions i), ii), iii) so that $I$ is exactly the set of limit points of the sequence of partial sums of the rearranged series.

Proof: Choose $x = a$, $y = b$. Let $p$ be any point of $I$.

Then, for all $i$, between the partial sum ending in $p_{\alpha_i}$, and the one ending in $q_{\alpha_i}$, there is a partial sum that differs from $p$ by less than one of the terms

$$a_{\alpha_{i-1}}, q_{\alpha_{i-1}}, \ldots, q_{\alpha_i}.$$  

Since all of these terms approach zero as $i$ becomes infinite it follows that $p$ is a limit point of the sequence of partial sums of the rearranged series.

A general remark may be made at this point relative to the above corollary. If $\sum a_n$ is any series satisfying

$$\lim_{n \to \infty} a_n = 0$$

then any point between the limit inferior and the limit superior of the sequence of partial sums of $\sum a_n$ is a limit point of that sequence.

2. **Rearrangements involving only positive terms.**

Sierpinski has proved a theorem concerning rearrangements of conditionally convergent infinite series in which the
negative terms retain their order and position. Before proving this theorem, however, it will be of interest to see what effect such rearrangements cannot have on the sum of a conditionally convergent series.

Let \( \sum a_n \) be an arbitrary conditionally convergent series of real terms. Let \( \sum a'_n \) be \( \sum a_n \) with the positive terms arranged in monotone descending order and with the order and position of the negative terms unaltered. Let \( \sum a''_n \) be any arrangement of \( \sum a_n \) again with the position and order of the negative terms unaltered. Now let us compare the partial sums \( S'_n \) of \( \sum a'_n \) and \( S''_n \) of \( \sum a''_n \).

The only terms that are not in both \( S'_n \) and \( S''_n \) are necessarily positive by the type of rearrangements being considered. If there are any terms in \( S''_n \) that are not terms of \( S'_n \), then these terms must be smaller than any positive term of \( S'_n \). Thus \( S''_n \leq S'_n \). Since this inequality holds for every \( n \) we may conclude that

\[
\lim S''_n \leq \lim S'_n
\]

Thus, if \( \lim S'_n = K < +\infty \), the set of numbers that can be obtained as sums of rearrangements of \( \sum a_n \) in which the negative terms retain their position and order is bounded by \( K \). An attempt has been made to prove that this upper bound is really the least upper bound; i.e., that an arrangement of \( \sum a_n \) of the type being considered can be found such that the rearranged series converges to \( K \).

The attempt to prove that this is so has been thus far
unsuccessful. If this could be proved then the following theorem could be extended somewhat.

**Theorem 2(G).** Let $\Sigma a_n$ be an arbitrary conditionally convergent series converging to $A$. Let $\Sigma p_n$ be a divergent subseries of the series of positive terms of $\Sigma a_n$. Let $y \leq x \leq A$ be given. Then there exists a series $\Sigma a'_n$, a rearrangement of $\Sigma a_n$ in which all terms not in $\Sigma a_n$ retain their order and position, such that $y$ is the limit inferior and $x$ the limit superior of the sequence of partial sums of $\Sigma a'_n$.

**Proof:** Let $m = A - y$

\[ M = A - x \]

Then

\[ m \geq M \geq 0 \]

We first rearrange

\[ P: p_1 + p_2 + \cdots + p_n + \cdots \]

into

\[ C: c_1 + c_2 + \cdots + c_n + \cdots \]

in such a way that

1) $\lim (P_n - C_n) = M$

2) $\lim (P_n - C_n) = m$

where

\[ P_n = p_1 + p_2 + \cdots + p_n \]

and
\[ c_n = c_1 + c_2 + \cdots + c_n \]

Let \( c_i = p_i \)

Then

\[ p_i - c_i = 0 \leq M \leq m \]

The choice of all the \( c_i \) must be motivated by the objectives 1) and 2). In order to ensure that no \( p_j \) occurs more than once in \( C \) we require once and for all that each time a term \( c_j \) is chosen as some \( p_k \) that \( p_k \) must not be among those \( p \)-terms previously chosen.

Choose for \( c_2 \) some \( p_j \) which is smaller than one half of \( p_2 \) and less than \( \frac{1}{3^1} \). Then

\[ D_2 = p_2 - c_2 > \frac{p_2}{2} > 0 \]

If \( D_2 < m \), then choose for \( c_3 \) some \( p_i \) which is less than half of \( p_3 \), less than \( \frac{1}{3^2} \), and has a larger index than any previously chosen terms.

\[ D_3 = p_3 - c_3 > \frac{p_3}{2} + \frac{D_2}{2} > 0 \]

Continue choosing \( c_i \) in this manner until \( D_i > m \) is attained. That such a \( D \)-term must occur follows from the divergence of \( P \). As soon as such a \( D \)-term is attained we must alter the manner of choosing the \( c \)-terms in order to obtain (I).

If the term \( p_j \) has not been used choose \( c_{j+1} = p_{j+1} \).

Then

\[ D_{j+1} = D_j \cdot \text{If } p_{j+2} \text{ has not been used put } c_{j+2} = p_{j+2} \]

Again \( D_{j+2} = D_j \). Continue choosing the \( c \)-terms in this manner until a \( p \)-term which has already been used is reached.
Such a term will be reached; for $D_{j_1} > 0$ implies that at least one c-term with an index less than $j_1$ has been put equal to a p-term with an index greater than $j_1$. Moreover, this p-term will be less than $\frac{1}{2}$. When such a term is reached choose for the c-term the first p-term not yet used. Then, if this term occurs at the $j_1 + 1$ place, $D_{j_1 + 1} < D_{j_1} + \frac{1}{2}$.

Continue choosing the c-terms in this manner, either equal to the corresponding p-term or equal to the first unused p-term, until the first term $D_{j_2}$ is attained such that $D_{j_2} \leq M$. Such a term must be attained for if the process is continued long enough eventually $D_{j_2 + \nu} = 0 \leq M$.

For $j_1 < k < j_2$ we certainly have $D_k < D_{j_1} + \frac{\sum_{n=2}^{j_1} \frac{1}{n}}{2} = D_{j_1} + 1$.

At this point we again change the manner of choosing c-terms. Choose for $c_{j_3 + 1}$ some $p_n$ which is less than half of $p_{j_3 + 1}$, less than $\frac{1}{2} p_{j_3 + 1}$, and has an index greater than that of any previously chosen p-term, etc., until a term $D_{j_3}$ is attained satisfying $D_{j_3} > m$. Such a term $D_{j_3}$ must be attained since $P$ diverges.

Now choose c-terms equal to either the corresponding p-terms or the first unused p-term until a $D_{j_\nu}$ term is attained satisfying $D_{j_\nu} \leq M$. For $j_3 < k < j_\nu$,

$$D_k < D_{j_3} + \frac{\sum_{n=2}^{j_3} \frac{1}{n}}{2} = D_{j_3} + \frac{1}{2}.$$  

Now that the method of choosing the c-terms has been outlined it is necessary to verify that this method yields
the desired series $C$. Since $D_{2n} \leq M$ and $D_{2n+1} \geq m$ for all $n$, there are an infinite number of $D$ terms less than or equal to $M$ and an infinite number greater than $m$. Given $\epsilon > 0$ there exists an index $N$ such that for all $n > N, p_n < \epsilon$. Hence for all $j > 2N, D_j > M - \epsilon$. Thus it is proved that $M$ is the limit inferior and the first objective is attained.

It is apparent that all $D$ terms between $D_{2n}$ and $D_{2n+1}$ are less than $m + \epsilon$ for all $n$ large enough so that all $p$-terms greater than $\frac{\epsilon}{2}$ have been used. For all $n$ all $D$-terms between $D_{2n+1}$ and $D_{2n+2}$ are less than $D_{2n+2} - \frac{\inf_{1 \leq i \leq 2N} \epsilon}{2^i}$. Choose $N$ so that $\frac{1}{2N} < \epsilon$. Then for all $n > N$, all $D$-terms between $D_{2n+1}$ and $D_{2n+2}$ are less than $D_{2n+1} + \epsilon < M + \epsilon$ which proves that the second objective is attained.

Since it was stipulated that no $p$-term would be used more than once, it remains only to prove that each $p$-term is used at least once to prove that the above process yields the desired series $C$. This, too, is immediate. For between the terms $a_{j_2i-1}$ and $a_{j_2i}$ there is at least one $c$-term chosen equal to the first unused $p$-term.

We are now able to form the series $\sum a'_n$, which will fulfill all the requirements of the theorem.

In the series $\sum a_n$ replace each term of the series $P$ by the corresponding term of the series $C$, and preserve
without any modification as to place or order all other terms of the series \( \sum a_n \).

For all \( n \)

\[
A_n = a_1 + a_2 + \ldots + a_n = P_{\tau_n} + B_{\delta_n}
\]

where \( P_{\tau_n} \) is the sum of the terms of \( A_n \) which are terms of the series \( P \) and \( B_{\delta_n} \) is the sum of the terms of \( A_n \) which are not terms of the series \( P \).

Similarly,

\[
A_n' = a_1' + a_2' + \ldots + a_n' = C_{\lambda_n} + B_{\delta_n}
\]

\[
A_n' = A_n - P_{\tau_n} + C_{\lambda_n}
\]

\[
A_n' = A_n - (P_{\tau_n} - C_{\lambda_n})
\]

Since \( \lim_{n \to \infty} r_n = r \infty \),

\[
\lim_{n \to \infty} A_n' = \lim_{n \to \infty} \left[ A_n - (P_{\tau_n} - C_{\lambda_n}) \right]
\]

\[
= \lim_{n \to \infty} A_n + \lim_{n \to \infty} \left[ -(P_{\tau_n} - C_{\lambda_n}) \right]
\]

\[
= A - \lim_{n \to \infty} (P_{\tau_n} - C_{\lambda_n})
\]

\[
= A - m = y
\]

\[
\lim_{n \to \infty} A_n' = \lim_{n \to \infty} \left[ A_n - (P_{\tau_n} - C_{\lambda_n}) \right]
\]

\[
= \lim_{n \to \infty} A_n + \lim_{n \to \infty} \left[ -(P_{\tau_n} - C_{\lambda_n}) \right]
\]

\[
= A - \lim_{n \to \infty} (P_{\tau_n} - C_{\lambda_n})
\]

\[
= A - M = x
\]

which completes the proof.

To obtain a rearrangement which converges to \( A' \subseteq A \)
choose $x = y = A'$.

**Corollary 1.** If the series $A = \sum a_n$ is conditionally convergent and $x \leq y \geq A$ are assigned, then leaving unaltered as to place and order all terms not in a subseries of $\sum a_n$ which diverges to $-\infty$ it is possible to obtain a rearrangement of $\sum a_n$ such that the limit superior of the sequence of partial sums of the rearranged series is equal to $x$ and the limit inferior is equal to $y$.

**Proof:** We need only apply the theorem to $-\sum a_n'$ and to $-x \leq -y \leq -A$.

To obtain a rearrangement of the type considered in the theorem which converges to $A' \geq A$, let $x = y = A'$.

3. **Rearrangements considered as points of a metric space.**

With every rearrangement of an infinite series it is possible to associate a permutation of the positive integers, and conversely. Thus, if in a rearrangement of $\sum a_n$, $a_j$ is moved to the $(j + \kappa_j)$ position, $\kappa_j > -j$, then corresponding to this rearrangement is the permutation of the positive integers $j \rightarrow j + \kappa_j$. Hence it is possible to consider rearrangements as sequences of positive integers in which each positive integer appears exactly once, and to consider these sequences as points of a metric space. It is this space in which we are primarily interested. However, since it will be necessary to consider more general spaces we now develop some
properties of these spaces.

Denote by \( (s) \) the space of sequences of real numbers.

If \( x = \{ x_n \} \) and \( y = \{ y_n \} \) are two elements of \( (s) \) we define the distance function

\[
\delta(x, y) = \sum \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}
\]

Then we have immediately

1) \( \delta(x, y) \geq 0 \) and \( = 0 \) if and only if \( x = y \)

2) \( \delta(x, y) = \delta(y, x) \)

In order to prove that this function is a true metric we then need to prove the triangle inequality;

\[
\delta(x, y) + \delta(y, z) \geq \delta(x, z)
\]

First we prove that

\[
\frac{|a \pm b|}{1 + |a \pm b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|}
\]

We know that \( |a \pm b| \leq |a| + |b| \)

Hence

\[
|a \pm b| + |a| \cdot |a \pm b| + |b| \cdot |a \pm b| + |a| \cdot |b| \cdot |a \pm b| \\
\leq |a| + |b| + 2 |a| \cdot |b| + |a| \cdot |a \pm b| + |b| \cdot |a \pm b| \\
+ 2 |a| \cdot |b| \cdot |a \pm b|
\]

or

\[
|a \pm b| \left( (1 + |a|) \cdot (1 + |b|) \right) \\
\leq |a| \cdot (1 + |b|) \cdot (1 + |a \pm b|) + |b| \cdot (1 + |a|) \cdot (1 + |a \pm b|)
\]

Dividing by \( (1 + |a|) \cdot (1 + |b|) \cdot (1 + |a \pm b|) \)

we obtain the desired result

\[
\frac{|a \pm b|}{1 + |a \pm b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|}
\]
Hence
\[
\frac{|a - c|}{1 + |a - c|} \leq \frac{|a|}{1 + |a|} + \frac{|c|}{1 + |c|}
\]
or
\[
\frac{|(a - b) + (b - c)|}{1 + |(a - b) + (b - c)|} \leq \frac{|a - b|}{1 + |a - b|} + \frac{|b - c|}{1 + |b - c|}
\]
Thus, letting \( x_n = a, y_n = b, z_n = c \)
\[
\frac{|x_n - z_n|}{1 + |x_n - z_n|} \leq \frac{|x_n - y_n|}{1 + |x_n - y_n|} + \frac{|y_n - z_n|}{1 + |y_n - z_n|}
\]
for every \( n \). This proves the triangle inequality, and hence \( \rho \) is a true metric.

Let \( E \) denote the subspace of \((s)\) in which each point is a permutation of the positive integers. We will denote by \( S(x, r) \) the open sphere in \( E \) of radius \( r \) and center \( x \); i.e., the set of points in \( E \) whose distance from \( x \) is less than \( r \). We will need the following properties of spheres:

1) If \( y \in S(x, \frac{1}{2^n}) \) then \( x_n = y_n \), \( n = 1, 2, \ldots, N \)

For suppose that \( y \in S(x, \frac{1}{2^n}) \) and \( x_n = y_n \) for some \( n \leq N \), then
\[
\rho(x, y) \geq \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \geq \frac{1}{2^n} \frac{1}{2^n} \geq \frac{1}{2^{n+1}}
\]
which contradicts \( y \in S(x, \frac{1}{2^n}) \)

2) If \( x_n = y_n \), \( n = 1, 2, \ldots, N \). Then \( y \in S(x, \frac{1}{2^n}) \).

For
\[
\rho(x, y) = \sum_{n = N+1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \leq \sum_{n = N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^n}
\]

Theorem 4(1. Let \( \sum a_n \) be an arbitrary conditionally
For $x = \{x_n\} \in E$ let $\sum a_{\lambda_n}$ be that rearrangement of $\sum a_{\lambda}$ determined by $x$. Then for all $x \in E$ except those belonging to a set of the first category

\begin{align*}
\text{l.u.b.} & \quad \sum_{n=1}^{N} a_{\lambda_n} = +\infty \\
\text{N} & \\
\text{and} & \\
\text{g.l.b.} & \quad \sum_{n=1}^{N} a_{\lambda_n} \leq \infty
\end{align*}

Proof: Let $B$ denote the set of $x \in E$ for which

\begin{align*}
\text{l.u.b.} & \quad \sum_{n=1}^{N} a_{\lambda_n} < +\infty \\
\text{N} & \\
\text{and for each positive integer } h \text{ let } B_h \text{ denote the set of } x \in E \text{ for which} & \\
\text{l.u.b.} & \quad \sum_{n=1}^{N} a_{\lambda_n} < h \\
\text{N} & \\
\text{Then } B = B_1 \cup B_2 \cup \ldots & \\
\text{To prove } B \text{ is of the first category in } E \text{ it suffices to prove that } B_h \text{ is non-dense for each } h.
\end{align*}

Suppose $h$ exists such that $B_h$ is not non-dense. Then the closure $\overline{B_h}$ of $B_h$ contains a sphere $S'$ in $E$ with center $x' = \{x'_1, x'_2, \ldots\}$ and radius $r > 0$.

Choose $m$ so great that $\frac{1}{2^n} = \frac{1}{n-m+1} \frac{1}{2^n} < \frac{r}{2}$

Let $x''_n = x'_n$ when $1 \leq n \leq m$ and define $x''_n$ for $n > m$
in such a way that
\[ \sum \xi \cdot x_n = \gamma \]

This is always possible by Theorem 1. Then we have
\[ (x', x'') + r \in \mathbb{E} \] so that \( x'' \in S' \).

Now choose an index \( q \) such that
\[ \sum_{n=1}^{q} \xi \cdot x_n = h \]

Choose \( \delta > 0 \) such that \( x, y \in E \), and \( (x, y) < \delta \)
imply \( x_k = y_k \) for \( k = 1, 2, \ldots, q \). If \( x \in E \) and
\[ (x, x'') < \delta \] that is \( x \in S'' \) where \( S'' \) is the sphere
with center \( x'' \), radius \( \delta > 0 \) then
\[ a \xi_1 + a \xi_2 + \ldots + a \xi_q > h \quad \text{and} \quad x \notin B_h \]
Thus \( B_h \) contains no points of \( S'' \) and consequently
\( B_h \) does not contain \( x'' \in S'' \). But this contradicts
the assumption that \( B_h \) contained \( S' \) which contains \( x'' \).
Hence, \( B_h \) is non-dense for every \( h \), and \( B \) is of the first
category in \( E \).

Similarly by considering sets \( D_h \) of points \( x \) of \( E \)
for which
\[ \text{g.l.b. } \sum_{n=1}^{N} a \xi \cdot x_n > -h \]

it can be shown that the set \( D \) of points \( x \in E \) for which
\[ \text{g.l.b. } \sum_{n=1}^{N} a \xi \cdot x_n > -\infty \]

is a set of the first category.
B ∪ D, the union of sets of the first category, is a set of the first category, and the theorem is proved.

If we can prove that E is of the second category with respect to itself it will follow that the set of \( x \in E \) for which

\[
\text{l.u.b.} \sum_{n=1}^{N} a_n + \infty
\]

and

\[
\text{g.l.b.} \sum_{n=1}^{N} a_n = -\infty
\]

is a set of the second category being the complement with respect to E of a set of the first category. The following theorems will be used to prove that E is of the second category.

We will call two metric spaces S and S* topologically equivalent or homeomorphic if there exists a biunique, bicontinuous mapping from S onto S*.

**Theorem a.** The homeomorphic image of a space of the second category is a space of the second category.

Proof: Let S be a space of the second category and let S* be the homeomorphic image of S under the mapping f. Assume that S* is of the first category. Then

\[ S* = \bigcup \mathcal{G}_n \]

where \( \mathcal{G}_n \) is non-dense in S*, n = 1, 2, ...

and
\[ S = f^{-1}(\bigcup G^*_n) = \bigcup f^{-1}(G^*_n) = \bigcup G_n \text{ where } G_n = f^{-1}(G^*_n). \]

If we can prove that \( G^*_i \) is non-dense for each \( i \), we will have a contradiction.

Let \( U \) be a sphere in \( S \). Let \( U^*_i = f(U) \). Then, since \( U^*_i \) is open, for arbitrary but fixed \( i \), \( U^*_i \) contains a sphere \( U^*_i \) such that \( G^*_i \cap U^*_i = \emptyset \).

\[ f^{-1}(G^*_i \cap U^*_i) = f^{-1}(G^*_i) \cap f^{-1}(U^*_i) = G_i \cap f^{-1}(U^*_i) = \emptyset \]

Since \( f^{-1}(U^*_i) \neq \emptyset \), and is open, \( f^{-1}(U^*_i) \) contains a sphere \( U_i \) such that \( G_i \cap U_i = \emptyset \). But \( U \supseteq f^{-1}(U^*_i) \supseteq U_i \), and since \( U \) was arbitrary, \( G_i \) is non-dense in \( S \) for \( i = 1, 2, \ldots \).

This proves that \( S^* \) is of the second category with respect to itself.

**Definition:** If \( X \) and \( Y \) are metric spaces with metrics \( \rho_x \) and \( \rho_y \) respectively, we define the Cartesian product \( X \times Y \) to be the space of all couples \((x, y)\), \( x \in X \), \( y \in Y \), with metric

\[ \rho \left( (x_1, y_1), (x_2, y_2) \right) = \sqrt{\rho_x(x_1, x_2)^2 + \rho_y(y_1, y_2)^2} \]

**Theorem 5:** If \( f(x), x \in X \) is a continuous function which maps the metric space \( X \) onto a subset of the metric space \( Y \), then \( X \) is homeomorphic to a subset of \( X \times Y \).

**Proof:** Let the homeomorphic mapping be

\[ x \leftrightarrow [x, f(x)] \]

This mapping is clearly biunique and bicontinuous from \( X \).
onto a subset of \( X \times Y \).

**Lemma:** In a complete metric space any descending sequence of closed spheres whose radii approach zero has a non-zero intersection.

**Proof:** Let \( K_1 \supset K_2 \supset \ldots \) be a sequence of closed spheres in a complete metric space \( S \) of radii \( r_1, r_2, \ldots \) and centers \( x_1, x_2, \ldots \), respectively. Then for \( p < q \)

\[
\rho(x_p, x_q) \leq r_p
\]

Since \( S \) is complete and \( r_p \) approaches zero,

\[
x_n \to x_0 \in S.
\]

For arbitrary \( p \)

\[
\rho(x_p, x_0) \leq \rho(x_p, x_q) + \rho(x_q, x_0) \leq r_p + \rho(x_q, x_0)
\]

Furthermore,

\[
\lim_{q \to \infty} \rho(x_q, x_0) = 0
\]

Thus it follows that

\[
\rho(x_p, x_0) \leq r_p \text{ or } x_0 \in S_r \text{ for every } p.
\]

Since \( r_p \to 0 \), \( x_0 \) is clearly the only point common to all the spheres. For if two distinct points \( x_0 \) and \( x'_0 \) with \( \rho(x_0, x'_0) = \delta \geq 0 \) were common to all the spheres then for all \( n \)

\[
r_n \geq \delta > 0
\]

which contradicts \( r_n \to 0 \).
Theorem c. A complete metric space $S$ is of the second category.

Proof: Assume that $S = \bigcup G_n$ where each $G_n$ is non-dense in $S$. Then there exists a sequence of spheres $K_1 \supseteq K_2 \supseteq \ldots$ with radii $r_1, r_2, \ldots$ satisfying
\[
\begin{cases}
K_n \supseteq K_{n+1} \\
K_n \setminus G_n = 0 \\
r_n < \frac{1}{n}
\end{cases}
\]
Let $x_0$ be the point contained in $\bigcap K_n$. But since $K_n \setminus G_n = 0$ for each $n$ the point $x_0 \notin G_n$ for any $n$ which contradicts $S = \bigcup G_n$. Therefore $S$ is not of first category and must be of second.

Theorem d. The Cartesian product of two complete metric spaces is a complete metric space.

Proof: Let $S$ and $S^*$ be complete metric spaces. Let \[\{(x_n, y_n)\} \quad x_n \in S, \quad y_n \in S^*\] be a Cauchy sequence from $S \times S^*$. Since
\[d_S(x_n, x_k) = \sup_{1 \leq m \leq n \leq k} d_S(x_m, y_m), \quad (x_k, y_k)\]
and
\[d_{S^*}(y_n, y_k) = \sup_{1 \leq m \leq n \leq k} d_{S^*}(x_m, y_m), \quad (x_k, y_k)\]
it follows that \{\(x_n\)\} and \{\(y_n\)\} are Cauchy sequences from $S$ and $S^*$. Suppose $x_n \to x$ and $y_n \to y$; then,
\[(x_n, y_n) \to (x, y)\] from the definition of
\[\rho_{S_1 \times S_2}(x_n, y_n), (x, y)\] so that \(S \times S^*\) is complete.

If \(S_1, S_2, \ldots\) are metric spaces with metrics \(\rho_1, \rho_2, \ldots\), we define the Cartesian product \(S_1 \times S_2 \times \cdots\) as the set of all sequences \(\{x_n\}\), \(x_n \in S_n, n = 1, 2, \ldots\). The metric of the Cartesian product space will be

\[\rho(x, y) = \sum \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}\]

where \(x = \{x_n\}_{n=1}^{\infty}, y = \{y_n\}_{n=1}^{\infty}\).

Only the triangle inequality need be verified to prove that \(\rho\) as defined above is a true metric. (The other properties needed for a metric are obviously fulfilled by \(\rho\).) However, when proving that the distance function in (3) is a metric, the only properties of \(|a + b|\) that were used are just those properties that make the absolute value a metric. Hence a proof exactly as that used there can be used here with \(|x_n - y_n|\) replaced by \(\rho_n(x_n, y_n)\) to prove that \(\rho\) is a metric.

Note that if \(\{x^n\}_{n=1}^{\infty}, x^n = \{x^n_i\}_{i=1}^{\infty}\) is a sequence of points in \(S_1 \times S_2 \times \cdots\) which converges to \(x = \{x_i\}\) then,

\[x^n_i \longrightarrow x_i, i = 1, 2, \ldots\]

and conversely.

**Theorem 2.** If \(S_1, S_2, \ldots\) are complete metric spaces then \(S_1 \times S_2, \ldots\) is a complete metric space.
Proof: Let \( \{x^n\} \) be a Cauchy sequence in \( S_1 \times S_2 \times \ldots \)
\[ x^n = \{x^n_i\}_{i=1}^{\infty}, \quad x^n_i \in S_i. \]
Let \( i \) be an arbitrary but fixed index. Given \( \varepsilon > 0 \), such that \( 1 - 2^i \varepsilon > 0 \), there exists an index \( n \) such that for all \( K > n \),
\[ d_S(x^n, x^K) < \varepsilon. \]
Then for all \( K > n \),
\[ \frac{d_S(x^n_i, x^K_i)}{1 + d_S(x^n, x^K)} < 2^i \varepsilon. \]
Hence
\[ d_S(x^n, x^K) < \frac{2^i \varepsilon}{1 - 2^i \varepsilon} < \frac{1}{2^i \varepsilon - 1} \]
and this last number approaches zero with \( \varepsilon \). Thus for each \( i \), \( \{x^n_i\}_{n=1}^{\infty} \) is a Cauchy sequence. Since \( S_i \) is complete, \( (x^n_i)_{n=1}^{\infty} \rightarrow x_i \) for each \( i = 1, 2, \ldots \). Thus,
\[ x^n \rightarrow x \quad \text{or} \quad S_1 \times S_2 \times S_3 \times \ldots \text{ is complete.} \]

Corollary: The space \( \mathcal{M} \) is complete.

Proof: If \( \mathcal{R} \) is the space of real numbers, \( \mathcal{R} \) is certainly complete and \( \mathcal{M} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times \ldots \) is complete.

Corollary: If \( M \) is complete, \( M \times S \) is complete.

Note that any closed subset of a complete metric space is complete and hence by Theorem c is of the second category with respect to itself.

Theorem f. A \( G_\delta \) set in a complete metric space \( M \) is of the second category with respect to itself.
Proof: Let \( Q \) be a \( G_\delta \) set in \( M \). Then

\[ Q = \bigcap G_n ; \quad G_n \text{ open } \quad n = 1, 2, \ldots \]

Let \( F_i = M - G_i \), then \( x \in G_i \) implies \( \rho(x, F_i) > 0 \)
where \( \rho(x, F_i) = \inf_{y \in F_i} \rho(x, y) \)

Let \( f_i(x) = \frac{1}{\rho(x, F_i)} \), \( x \in G_i \)

To each \( x \in Q \) mate the sequence \( \{ f_i(x) \} \) considered as a point \( f(x) \) of the space \( \mathbb{A} \). Since \( \rho(x, F_i) \)
\( x \in G_i \) is continuous and positive, \( f_i(x) \) is continuous; hence, \( f(x) \) itself is continuous.

In the complete metric space \( M \) let \( I \) be the set of points \( [x, f(x)] \). We now prove that \( I \) is closed.

If we assume that \( I \) is not closed, then we have

\[ x_n \to x, \quad f(x_n) \to y, \quad (x, y) \in I. \]

From this it follows that \( x \in Q \), since \( f(x) \) is continuous,

\[ f(x_n) \to f(x) \quad y, \text{ and if } x \in Q, \text{ then } (x, y) \in I. \]

Since \( x \notin Q \) it follows that \( x \in F_i \) for some \( i \). The continuity of \( \rho \) required that

\[ \lim_{n \to \infty} \rho(x_n, F_i) = \rho(x, F_i) = 0 \]

and hence,

\[ \lim_{n \to \infty} f_i(x_n) = \infty \]

Thus, \( \lim f(x_n) \) does not exist and we have a contradiction.

Since \( I \) is a closed subset of a complete metric space, it is itself a complete metric space, and hence, is of the second category with respect to itself.
Since the mapping $x \mapsto [x, f(x)]$ is a homeomorphism from $\mathbb{Q}$ onto $I$, $\mathbb{Q}$ is of the second category with respect to itself.

It is easy to see that the space $E$ is not complete. For consider the sequence

$$x_1 = 1, 2, 3, \ldots$$
$$x_2 = 2, 1, 3, \ldots$$
$$x_3 = 2, 3, 1, \ldots$$
$$x_4 = 2, 3, 4, 1, \ldots$$

This sequence is certainly a Cauchy sequence, but it converges to $2, 3, 4, \ldots$ which is not a point of $E$.

However, since a convergent sequence of positive integers necessarily converges to a positive integer, the space $T$ of sequences of positive integers is certainly a complete subspace of $(\mathbb{Q})$.

The space $E$ is a subspace of the complete space $T$. Hence any convergent sequence of elements of $E$ converges to an element of $T$. Furthermore, the limit of any convergent sequence of elements of $E$ must be a sequence which contains each positive integer at most once. For suppose the limit point $x = \{x_n\}$ is such that $x_j = x_k$, $j \neq k$. Then if $\{x^n\}$ is the sequence of elements of $E$ which converges to $x$ we must have

$$\lim_{n \to \infty} x^n_j = \lim_{n \to \infty} x^n_k = x_j = x_k$$

or
\lim_{n \to \infty} (x^n_j - x^n_k) = 0

which, since \( x^n_j \) and \( x^n_k \) are positive integers, is possible only if \( x^n_j = x^n_k \) for sufficiently large \( n \). But this contradicts the assumption that each element of the sequence \( \{ x^n \} \) is a permutation of the positive integers.

Now let \( x_1, x_2, \ldots \) be any sequence of positive integers containing each positive integer at most once. Clearly the following sequence of permutations of the positive integers converges to \( x_1, x_2, \ldots \):

- \( x' = x_1, 2, 3, \ldots \) 1 and \( x_1 \) interchanged
- \( x'' = x_1, x_2, 3, 4, 5, \ldots \) 1 and \( x_1, 2 \) and \( x_2 \) interchanged
- \( x^n = x_1, x_2, \ldots x_n, n+1, n+2, \ldots \) 1 and \( x_i \) interchanged
- \( 1 = 1, 2, \ldots n \).

Thus we have proved that the closure of \( S \) in \( T \)
is the set \( T' \) consisting of those sequences of positive integers which contain each integer at most once. Let \( T_x \{ x^n = k \} \) denote for each \( n, k = 1, 2, 3, \ldots \) the set of points \( x \in T \) having \( k \) as the \( n \)th "coordinate."

\[ T_x \{ x^n = k \} \text{ is open in } S. \text{ For if } y \in T_x \{ x^n = k \} \text{ the sphere } S(y, \frac{1}{n+k+1}) \text{ contains only points of } T_x \{ x^n = k \}. \]

Let \( T_2 = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} T_x \{ x^n = k \}. \) Then \( T_2 \) is a \( G_\delta \) in \( S \) since the sum of any number of open sets is an open set. Also \( T_2 \) is the subset of \( T \) consisting of
sequences of positive integers in which each positive integer occurs at least once.

Since $T_1$ is a closed subset of the complete metric space $T$, and since

$$E = T_1 \cap T_2$$

$E$ is a $G_6$ set in $T$. Therefore, by Theorem f, $E$ is of the second category with respect to itself.

4. **Rearrangements which leave the sum unaltered.** In this section we will concern ourselves with rearrangements that preserve the sum of conditionally convergent series. Theorem 5 defines a special kind of sum-preserving rearrangement. Theorem 6 gives a sufficient condition that an arbitrary rearrangement preserve the sum of a given conditionally convergent series. Theorem 7 gives a necessary and sufficient condition that a rearrangement be sum-preserving for every conditionally convergent series.

**Theorem 5.** Let $\sum a_{n_j}$ be an arbitrary absolutely convergent subseries of the conditionally convergent series $\sum a_n = A$. Then any rearrangement which preserves the order with respect to themselves of all terms not in $\sum a_{n_j}$ will be sum-preserving.
Proof: Let \( S_\eta \) denote the partial sums of \( \sum a_\eta \) and \( S'_\eta \) the partial sums of \( \sum a'_\eta \), a rearrangement of \( \sum a_\eta \) of the type mentioned in the statement of the theorem.

Then \( S_\eta \) converges to \( A \) and any subsequence of \( S_\eta \) converges to \( A \). With each \( S'_\eta \) associate \( S_{K_\eta} \) where \( S_{K_\eta} \) contains exactly all the terms of \( \sum a_\eta \) not in \( \sum a'_\eta \) which appear in \( S'_\eta \). Such a partial sum exists for every \( S'_\eta \) since the order of all terms not in \( \sum a'_\eta \) is preserved.

Consider the differences \( |S'_\eta - S_{K_\eta}| \). Each of these differences consists only of terms from \( \sum a'_\eta \). These terms may not be in their original order, but since \( \sum a_\eta \) converges absolutely late segments are arbitrarily small in absolute value in any order. Hence, given \( \varepsilon > 0 \), if we choose \( m \) such that \( S_m \) and \( S'_m \) contain

\[
\sum_{j=1}^{N} a_\eta \text{ where } \sum_{j=N+1}^{\infty} |a_\eta_j| < \varepsilon
\]

we are assured that for all \( n > m \) \( |S'_\eta - S_{K_\eta}| < \varepsilon \). Since \( S_{K_\eta} \) converges to \( A \), \( S'_\eta \) converges to \( A \) and the proof is complete.

Theorem 6. (Q) A sufficient condition that a rearrangement of a conditionally convergent series leave the sum unaltered is that the product of the maximum displacement of the terms which precede the \( n \)th term in the original series times the maximum absolute value of the terms which follow the \( n \)th term has the limit zero as \( m \to \infty \).
Proof: Let \( A = a_1 + a_2 + \ldots + a_m + \ldots \) be an arbitrary conditionally convergent series of real terms.

Let \( A' = a'_1 + a'_2 + \ldots + a'_n + \ldots \) be a rearrangement of \( A \). Between the indices \( m \) and \( n \) there is a one to one correspondence such that \( a_m = a'_n \).

Let \( |m - n| = \lambda_m \)

Thus \( \lambda_m \) is the displacement the \( m^{th} \) term of series \( A \) undergoes when being placed in the series \( A' \).

Let \( \lambda_m \) be the largest value that \( \lambda_m \) assumes as \( \lambda \) varies from 1 to \( m \).

Let \( p_m = m + \lambda_m \)

Then as \( m \) increases indefinitely so does \( p_m \). We have, from the definition of \( p_m \),

\[
p_{m+1} - p_m = \lambda_{m+1} + 1
\]

We will now seek to establish some relationship between the partial sums \( A'_m \) of the series \( A' \) and the partial sums \( A_m \) of the series \( A \).

For a fixed integer \( p \) greater than \( p \), it is possible to choose \( m \) such that

\[
p_m < p < p_{m+1}
\]

Then \( p - p_m < p_{m+1} - p_m \leq \lambda_{m+1} + 1 \)

Let \( A'_p = A'_m + \sigma \) where \( \sigma \) is the sum of not more than \( \lambda_{m+1} \) terms of the series \( A' \). \( A'_p \) contains the first \( m \) terms of the series \( A \), for any term which precedes \( a_m \) in the series \( A \) has an index less than \( m \) and a displacement not greater than \( \lambda_m \) and \( p_m \) is defined as
the sum \( m + \lambda \). Also in \( A' \) are \( p_m - m = \lambda \) terms of
the series \( A \) whose indices exceed \( m \). Hence each of the
terms which comprise \( \circ \) has an index in the series \( A \)
at least as great as \( m + 1 \). Let \( \eta_{m+1} \) be the maximum
of the absolute values of the terms of the series \( A \)
which follow the \( m \)th term. Then we have
\[
\circ = \Theta \lambda_{m+1} \eta_{m+1} \quad | \Theta | \leq 1
\]
and
\[
A' = A_m + \Theta \lambda_m \eta_m \quad | \Theta | \leq 1
\]
Thus we have
\[
A' = A_m + \Theta \lambda_m \eta_m + \Theta \lambda_{m+1} \eta_{m+1}
\]
If \( \lim_{m \to \infty} \lambda_m \eta_m = 0 \), then
\[
\lim_{p \to \infty} \lim_{m \to \infty} (A_m + \Theta \lambda_m \eta_m + \Theta \lambda_{m+1} \eta_{m+1})
\]
or
\[
A' = A
\]
since as \( p \) increases so does \( p_{m+1} \) and so does \( m \).

Corollary. For the conclusion of the theorem to hold
it suffices that the product of the absolute value of
the \( m \)th term times the maximum displacement of the terms
which precede the \( m \)th term has the limit 0 as \( m \to \infty \).

Proof: Let \( a_{m'} \) be the term among the terms following
the \( m \)th term which has the largest absolute value.

Then
\[
\eta_m = \eta_{m'} = |a_{m'}|
\]
\[
\lambda_m \leq \lambda_{m'}
\]
Thus
\[
\lambda_m \eta_m \leq \lambda_{m'} \eta_{m'} = \lambda_{m'} |a_{m'}|
\]
Then if \( \lambda_{m'} |a_{m'}| \to 0 \) as \( m \to \infty \)
we have \( \lambda_m \eta_m \to 0 \) as \( m \to \infty \)
from which the corollary follows.
Corollary. For the conclusion of the theorem to hold it suffices that the product of the displacement of the $m$th term times the maximum of the absolute values of the terms which follow the $m$th term has the limit 0 as $m \to \infty$.

Proof: Let $a_\nu$ having displacement $\alpha_\nu$ be the term which has maximum displacement among the terms preceding the $m$th term.

Then $\lambda_\nu = \gamma_\nu = \lambda_m$

Thus $\alpha_\nu \gamma_\nu < \lambda_m \gamma_m$

Let $m$ increase indefinitely. If $\lambda_m$ is bounded obviously $\lambda_m \gamma_m \to 0$. If $\lambda_m$ increases indefinitely so does $\alpha_\nu$ and so does $\lambda_m$. Consequently if $\alpha_\nu \gamma_\nu > 0$ as $\nu \to \infty$, then $\lambda_m \gamma_m > 0$ as $m \to \infty$

from which the corollary follows.

For purposes of the next theorem to be proved consider an arbitrary conditionally convergent series

$$A = a_1 + a_2 + \ldots + a_\nu + \ldots$$

and a permutation $\pi$ of its terms. For every $n$ we subdivide the series $A$ into two parts, $S_n$ consisting of the terms $a_1, a_2, \ldots a_n$, and $R_n$ consisting of the terms $a_{n+1}, a_{n+2}, \ldots$.

Let $\tau_{\pi}$ denote the sum of all the terms which are moved
by the permutation \(\pi\) from \(S_n\) to \(R_n\), and let \(\hat{r}_n\) denote
the sum of all terms moved by \(\pi\) from \(R_n\) to \(S_n\). Then
the permutation \(\pi\) changes each partial sum \(S_n\) to
\(S_n + \hat{r}_n\). Obviously if \(\pi\) is to leave the sum
unaltered we must have

\[\lim_{n \to \infty} (\hat{r}_n \cdot c_n) = 0\]

For every \(n\) we will need to group in a particular manner
the terms moved by \(\pi\) from \(S_n\) to \(R_n\) or from \(R_n\) to \(S_n\).
Let \(n\) be fixed. The terms that are moved by \(\pi\) from
\(S_n\) to \(R_n\) may be thought of in terms of "bunches," the
indices in each bunch forming a gapless set of integers.
Similarly the terms that are moved by \(\pi\) from \(R_n\) to \(S_n\)
comprise a certain number of bunches, not necessarily
the same number of bunches as the terms which are moved
by \(\pi\) from \(S_n\) to \(R_n\). The total number of bunches moved
by \(\pi\) either from \(R_n\) to \(S_n\) or from \(S_n\) to \(R_n\) is the number
we are interested in and we will denote this number by \(k_n\).
We are now read to prove Theorem 7.

**Theorem 7(3).** A necessary and sufficient condition that
a permutation \(\pi\) does not alter the sum of any conditionally
convergent series is that

\[\limsup_{n \to \infty} k_n < \infty\]
Proof: Sufficiency:

We assume that \( \limsup k_n = k < +\infty \)

Let \( A = a_n + a_{n+1} + \ldots \) be any conditionally convergent series. Then from the convergence it follows that given an \( \varepsilon > 0 \) there exists an index \( m \) such that for all \( n > m \) and for all \( p > 0 \)

\[ |a_n + a_{n+1} + \ldots + a_{n+p}| < \frac{\varepsilon}{2k} \]

If we choose \( N \) so large that no term from \( S_n \) is moved by \( \Pi \) to \( R_N \) then we are assured that for all \( n > N \) each bunch moved by \( \Pi \) from \( S_n \) to \( R_n \) contributes an amount less than \( \frac{\varepsilon}{2k} \) to \( |S_n| \) and since \( N > m \) each bunch moved from \( R_n \) to \( S_n \) contributes an amount less than \( \frac{\varepsilon}{2k} \) to \( |I_n| \).

Thus we have

\[ |\gamma_n - \sigma_n| \leq |I_n| + |S_n| \leq k_n \cdot \frac{\varepsilon}{2k} + \alpha_n \cdot \frac{\varepsilon}{2k} \leq \varepsilon \]

Hence (1) is satisfied and leaves the sum unaltered.

Necessity:

We prove necessity by showing that if for the permutation \( \Pi \) \( k_n \) is not bounded, then there are conditionally convergent series whose sum is altered by \( \Pi \).

Let \( \sum a_n \) be an convergent series of positive terms and let \( \sum b_n \) be a divergent series of positive terms with the terms approaching zero monotonically. We denote

\[ \frac{b_j + b_{j+1}}{2} \]

by \( \beta_j' \).

We now construct a series from the \( d_j, \beta_j, \) and \( \beta_j' \).
that converges conditionally and yet has its sum altered by the permutation $\Pi$.

First choose integers $p_j = 1, p_2, p_3, \ldots$ such that

$$\sum_{p_j}^+ \beta_{p_j + 1} + \sum_{p_j}^+ \beta_{p_j + 2} + \ldots + \beta_{p_j + 1} > 1$$

This is possible since $\sum \beta_n$ diverges.

Next choose integers $n_1, n_2, \ldots$ such that for the given permutation $\Pi$ and for $j = 1, 2, \ldots$

(a) $k_{n_j} > p_j 
(b) No term is moved by $\Pi$ from $S_{n_j}$ to $R_{n_{j+1}}$ or from $R_{n_{j+1}}$ to $S_{n_j}$.

We can satisfy condition (a) since $k_n$ is assumed unbounded and we can satisfy condition (b) since $\Pi$ moves only a finite number of terms from $S_{n_j}$ to $k_{n_j}$.

The objective is to construct a convergent series such that for $j = 1, 2, \ldots$

$$| \gamma_{n_j} - \sigma_{n_j} | \geq 1$$

This objective is achieved by setting the first term in each of the first $p_{j+1} - p_j \leq k_{n_j}$ bunches moved by $\Pi$ from $S_{n_j}$ to $R_{n_j}$ or from $R_{n_j}$ to $S_{n_j}$ equal to $\sum_{p_j}^+ \beta_{p_j + 1}$ using the minus sign for bunches moved from $S_{n_j}$ to $R_{n_j}$ and the plus sign for bunches moved from $R_{n_j}$ to $S_{n_j}$ where $i$ ranges from $1$ to $p_{j+1} - p_j$.

If these $\geq$ terms were the only ones moved by $\Pi$, then we would have

$$| \gamma_{n_j} - \sigma_{n_j} | \geq | \gamma_{n_j} | + | \sigma_{n_j} |$$
since all the terms in $\gamma_{\eta_j}$ are positive and all the terms in $\sigma_{\eta_j}$ are negative, and also

$$(2) \quad |\gamma_{\eta_j}| + |\sigma_{\eta_j}| = \beta_{\rho_j} + \cdots + \beta_{\rho_{j+1}}$$

In order to insure that when applied to the final series, $\Pi$ does not satisfy $\lim (\gamma_{\eta_j} - \sigma_{\eta_j}) = 0$ we must exercise caution in filling in all the other places in the series. We know that between any two $\beta$ terms of the same sign there is at least one place unfilled which remains in $R_{\eta_j}$ or $S_{\eta_j}$. In this place, say between the terms $-\beta_{\rho_j} + \nu$ and $-\beta_{\rho_j + \nu + 1}$, $(+\beta_{\rho_j} + \nu$ and $+\beta_{\rho_j + \nu + 1})$ we place $\beta'_{\rho_j + \nu}(-\beta_{\rho_j + \nu})$.

Now all remaining places are filled by $\pm \alpha_i, \pm \delta_2, \ldots$

where the sign $+$ is used if the term is moved by $\Pi$ from $R_{\eta_j}$ to $S_{\eta_j}$ or if the term remains in $R_{\eta_j}$ or $S_{\eta_j}$ and the sign $-$ is used if the term is moved by $\Pi$ from $S_{\eta_j}$ to $R_{\eta_j}$.

With the series constructed in this manner, (2) is satisfied and hence the sum, if it exists, is altered. It only remains to prove that the series thus constructed does converge. This, however, follows easily from the fact that the $\beta$ terms form an alternating series with terms approaching zero, and the $\alpha$ terms form an absolutely convergent series.

**Theorem 8.** (9) *If for a given permutation $\Pi$, $k_\eta$ is not bounded then there exist convergent series which are changed by $\Pi$ into series which do not converge.*
Proof: With one change we can utilize the method of construction used in the proof of the necessity of Theorem 7 to construct a convergent series which is changed by \( \Pi \) into a non-convergent series. Thus, if instead of using \( \beta \) terms for each \( n_j \) we use \( \beta \) terms only for indices of the form \( n_j \), and for indices of the form \( n_{j+1} \) we use \( \alpha \)'s where we had used \( \beta \)'s, in all cases with the sign chosen as before, then we will have

\[
(a) \quad | \tau_{n_j} - \sigma_{n_j} | = | \tau_{n_j} | + | \sigma_{n_j} |
\]

but

\[
(b) \quad \lim_{j \to \infty} | \tau_{n_{j+1}} - \sigma_{n_{j+1}} | = 0
\]

since only \( \alpha \) terms are involved in the \( \tau_{n_{j+1}} \) and \( \sigma_{n_{j+1}} \), and the \( \alpha \)'s form an absolutely convergent series. (a) and (b) together imply that the rearranged series does not converge.

We may consider this theorem as proving that if a permutation \( \Pi \) changes every conditionally convergent series into a convergent series then \( \Pi \) does not alter the sum of any series.
II. SERIES OF COMPLEX TERMS

1. General discussion. If a series of complex terms is considered as composed of two component series, one of imaginary terms and one of real terms, then it is possible to apply, more or less directly, certain of the theorems of Part I to series of complex terms which fulfill certain conditions. For example if \( \sum a_n \) is a series of complex terms whose real parts form a series satisfying the hypothesis of Theorem 1 and whose imaginary parts form an absolutely convergent series converging to \( ib \), then it is possible to rearrange \( \sum a_n \) in such a way that every complex number with imaginary part equal to \( ib \) is a limit point of the sequence of partial sums of the rearranged series. For by Corollary 2 to Theorem 1 the series of real terms may be arranged so that every real number is a limit point of the sequence of partial sums of the rearranged series. This rearrangement also rearranges the series of imaginary parts, but since this series converges absolutely it will still converge to \( ib \).

Other theorems of Part I may be applied similarly to certain types of series of complex terms, but in general, since a rearrangement of the series of the real parts entails the same rearrangement of the series of imaginary parts most of the theorems of Part I are not directly applicable to an arbitrary series of complex
terms.

By "the range of values" of a conditionally convergent series will be meant the set of points obtainable as the sum of a rearrangement of the series. Theorem 1 proves that the range of values of conditionally convergent series of real terms is the real axis.

The above discussion shows that for a complex series satisfying certain conditions the range of values is a line parallel to the real axis. A similar argument would show that the range of values of a series of complex terms whose imaginary series converged conditionally and real series converged absolutely is a line parallel to the axis of imaginaries. This part of the thesis contains a proof of the fact that the range of values of a conditionally convergent series of complex terms is always a straight line or the entire complex plane.

It will be sufficient to prove that the range of values of a conditionally convergent series which converge to zero is either a line or the complex plane. For if \( \sum z_n = Z \) is any conditionally convergent series of complex terms, addition of \(-Z\) to the series gives one that converges to zero. If it is desired to rearrange \( \sum z_n \) so that the rearranged series converges to \( W \), we can arrange \(-Z + \sum z_n\) so that it converges to \( W - Z \). Then upon deleting the \(-Z\) we have a rearrangement of \( \sum z_n \) which converges to \( W \) as required.
It will be necessary in section 2 to represent a complex number \( Z_n = \rho_n e^{i\vartheta_n} \) as the sum of two complex numbers, \( \rho'_n e^{i\varphi} \) and \( \rho''_n e^{i(\vartheta + \pi)} \). Such a representation is always possible for we need merely let
\[
\begin{align*}
\rho'_n &= \rho_n \cos (\vartheta_n - \vartheta) \\
\rho''_n &= \rho_n \sin (\vartheta_n - \vartheta)
\end{align*}
\]

2. The range of values of conditionally convergent series of complex terms. A direction \( \vartheta \) will be called critical with respect to the series \( \sum Z_n = \sum \rho_n e^{i\vartheta_n} \) if

1) For every small positive angle \( \delta \) there are infinitely many \( n_j \) such that \( |\vartheta_{n_j} - \vartheta| < \delta \)

11) If when we set
\[
Z_{n_j} = \rho'_{n_j} e^{i\varphi} + \rho''_{n_j} e^{i(\vartheta + \pi)}
\]
then \( \sum \rho'_{n_j} \) is a divergent series.

Lemma: If \( \vartheta \) is a critical direction with respect to \( \sum Z_n \) then there exists a subseries \( \sum Z_{n_j} \) such that \( \sum \rho'_{n_j} \) diverges and \( \sum |\rho''_{n_j}| \) converges.

Proof: Consider any complex number \( \rho e^{i\varphi} \) where
\[
|\vartheta - \varphi| < |\alpha| \quad \text{where} \quad \alpha \text{ is any acute angle. Then}
\]
\[
|\rho'| \geq |\rho \cos \alpha|
\]
\[
|\rho''| \leq |\rho \sin \alpha|
\]
or
\[
|\varphi''| \leq |\rho'\tan \alpha|
\]
For θ = 1, 2, ..., choose α_θ such that
\[ |\tan \alpha_θ| < \frac{1}{\lambda_θ}\]

Choose distinct terms \( Z_1^{\theta}, Z_2^{\theta}, \ldots, Z_{n_\theta}^{\theta}\)
from \( \sum z_n \) satisfying

1. \[ 0 \leq |\theta_k - \theta| \leq |\alpha_\theta| \quad k = 1, 2, \ldots, n_\theta \]
2. \[ 2 \geq \sum_{\kappa=1}^{n_\theta} |\rho_\kappa^{\theta}| \cdot \cos \alpha_\theta > 1 \]

These conditions can be satisfied since \( \theta \) is critical and \( \rho_n \to 0 \). Then we have that
\[ \sum_{\kappa=1}^{n_\theta} |\rho_\kappa^{\theta}\sin \alpha_\theta| \leq \frac{1}{2^{\theta-1}} \]
and hence

3. \[ \sum_{\kappa=1}^{n_\theta} |\rho_\kappa^{\theta}| \geq 1 \]
and

4. \[ \sum_{\kappa=1}^{n_\theta} |\rho^{\theta}_{\kappa}| \leq \frac{1}{2^{\theta-1}} \]

The subseries \( Z_1^{\theta} + \ldots + Z_{n_\theta}^{\theta} + Z_1^{\theta} + \ldots + Z_{n_\theta}^{\theta} + \ldots \)
fulfills all the requirements. For consider the partial sums of this series whose last term is \( Z_{n_\theta}^{\theta} \).

By (4) and (3) we have that the \( \rho^{\theta} \) terms from the terms of this partial sum have a sum less than
\[ 1 + \frac{1}{2} + \ldots + \frac{1}{2^{\theta-1}} < 2 \]
and the \( \rho^{\theta} \) terms have a sum greater than \( \theta \).

Theorem 9. The range of values of a conditionally convergent series \( \sum z_n = 0 \) is either a line or the complex plane.
Case I. There are three critical directions $\theta', \theta, \theta''$, and no closed half-plane contains all three. In this case the range of values is the complex plane.

Case II. There are two critical directions, $\theta$ and $\theta' + \pi$, and when each $z_n$ is written as $\rho_n' e^{i\theta'} + \rho_n'' e^{i(\theta + \pi)}$, the series $\sum \rho_n''$ converges conditionally. Again the range of values is the complex plane.

Case III. The same as Case II except that the series $\sum |\rho_n''|$ converges. In this case the range of values is the line in the direction $\theta$.

Proof: Case I. Let it be required to arrange $\sum z_n$ so that it converges to the assigned number $re^{i\psi}$.

Certainly it is possible to express re$^{i\psi}$ as a linear combination of two of the three numbers $e^{i\theta'}, e^{i\theta}$, and $e^{i\theta''}$ with negative coefficients, say

$$re^{i\psi} = r e^{i\theta'} + r e^{i\theta''}$$

$$r = 0, r = 0$$

Write each $\rho_n e^{i\phi_n}$ in the form $\rho_n' e^{i\theta'} + \rho_n'' e^{i(\theta + \pi)}$.

Then $\sum \rho_n' - 0 - \sum \rho_n''$ and for both series the convergence is conditional since there is a critical direction closer than $\pi$ to each of the directions $\theta'$ and $\theta' + \pi$. By the lemma there exists a subseries $\sum z_n'$ of $\sum z_n$ such that $\sum \rho_n'$ diverges to $+\infty$ and $\sum |\rho_n''|$ converges.

By Theorem 2 it is possible to arrange $\sum \rho_n'$ in such a way that it converges to $r_1$ and still preserve the order
relative to each other of all terms not in \( \sum \left( \overline{\rho}_{n}^{'\prime} \right) \). By Theorem 5 this rearrangement will not alter the sum of the series \( \sum \left( \overline{\rho}_{n}^{\prime\prime} \right) \).

This arrangement of \( \sum \left( \varphi_{n} \right) \) now converges to \( r_{1} e^{\theta^{'}} \). If \( -r_{1} e^{\theta^{'}} \) is now inserted we obtain a series converging to zero and having critical directions \( \theta^{'}, \Theta^{1}, \Theta^{2} \). By an exactly similar argument we can arrange this series so that it converges to \( r_{2} e^{i\theta^{2}} \). Then upon deleting the \( -r_{1} e^{i\theta^{1}} \) we obtain a rearrangement of \( \sum \left( \varphi_{n} \right) \) which converges to \( r_{1} e^{i\theta^{1}} + r_{1} e^{i\theta^{1}} = r_{1} e^{i\psi} \) as required.

Case II. Let \( r_{1} e^{i\psi} \) be the arbitrarily assigned complex number which is to be obtained as the sum of a rearrangement of \( \sum \left( \varphi_{n} \right) \). As before we can write

\[
\text{re} \varphi = \text{re} \varphi^{1} + \text{re} \varphi^{2} \text{e}^{i(\theta + \pi)}
\]

and

\[
\sum \left( \varphi_{n} \right) = \sum \varphi^{1} + \sum \varphi^{2} \text{e}^{i(\theta + \pi)}
\]

By Theorem 1 the series \( \sum \left( \varphi_{n}^{2} \right) \) can be arranged so that it converges to \( r_{2} \). By the lemma, applied to both of the critical directions, there is a subseries \( \sum \left( \varphi_{n} \right) \) of \( \sum \left( \varphi_{n} \right) \) such that the series \( \sum \left( \varphi_{n} \right) \) has divergent subseries of positive and negative terms while the series \( \sum \left( \varphi_{n} \right) \) converges. By Theorem 1 \( \sum \left( \varphi_{n}^{1} \right) \) can be made to converge to \( r_{1} \) while preserving the order, relative to each other, of all terms not in \( \sum \left( \varphi_{n}^{1} \right) \). By Theorem 5 this arrangement will preserve the sum \( r_{1} \) of the previously rearranged \( \sum \left( \varphi_{n}^{1} \right) \). Thus this arrangement of \( \sum \left( \varphi_{n} \right) \) converges to
\[ n_1 e^{j\theta} + n_2 e^{j(\theta + \psi)} = n e^{j\psi} \text{ as required.} \]

Case III. Let \( re^{j\phi} \) be assigned. Write each \( Z_n \) as \( \rho_n' e^{j\theta} + \rho_n'' e^{j(\theta + \psi)} \) as before. Rearrange \( \Sigma / \rho_n' \) so that it converges to \( r \). This is possible by Theorem 1. Since \( \Sigma / \rho_n'' \neq 0 \) this rearranging of \( \Sigma / \rho_n' \) cannot alter the sum of \( \Sigma / \rho_n'' \). Hence this arrangement converges to \( re^{j\phi} \) as required.

If a series converges conditionally then there is necessarily a minimum of two critical directions. If there are only two they must be opposite. If there are more than two then no 180° angle can contain all of them. Therefore we have completely exhausted the possibilities of the disposition of the critical vectors and the theorem is proved.

**Corollary:** If a series falls into Case I or II, then it is possible to rearrange the series so that every point of the complex plane is a limit point of the sequence of partial sums of the rearranged series.

**Proof:** The set of complex numbers with both real and imaginary parts rational forms a dense denumerable set in the complex plane. Order this set in some fashion; say \( w_1, w_2, \ldots \).

Choose a finite number of terms of \( Z_n \) whose sum is within \( \frac{1}{2} \) of \( w_n \). Then put the first unused term in
the next place. Call this the first "block" of terms. Again choose a finite number of terms which, together with the first block of terms have a sum within $\frac{1}{2}$ of $w_1$. Put the first unused term in the next place. Now choose a finite number of terms which, together with the first two blocks of terms, have a sum within $\frac{1}{3}$ of $w_1$. Repeat this process by choosing terms whose sum, together with the first three blocks of terms, is within $\frac{1}{4}$ of $w_1$. Insert the first unused terms, and choose terms whose sum, together with the first four blocks of terms, is within $\frac{1}{5}$ of $w_1$. Insert the first unused term and then choose terms to obtain a sum within $\frac{1}{6}$ of $w_1$; etc.

This method will lead to a rearrangement of $\sum x_n$ such that each of the numbers $w_1, w_2, \ldots$ is a limit point of the sequence of partial sums of the rearranged series. Every point in the plane is a limit point of some subsequence of $w_1, w_2, \ldots$ and is therefore a limit point of the sequence of partial sums of the rearranged series.
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