DYNAMICS OF HYPOID AND BEVEL GEARED ROTOR SYSTEMS

DISSERTATION

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By

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This dissertation research is primarily focused on the development of an analytical multiple-degrees of freedom vibration model for analyzing hypoid or bevel geared rotor systems with emphasis on automotive applications. The focuses are on the modeling of the gear meshing mechanism that forms the basis of the dynamic formulation, and development of the basic understanding of the generation of gear whine response related to the vibratory characteristics of the geared rotor system. The following key issues are specifically addressed in this dissertation:

1. The spiral bevel and hypoid gear tooth surfaces in three-dimensional space are defined analytically and generated numerically by the simulation of the cutting process. A general algorithm is built on separated blocks. The method includes a modified roll generation scheme. The generated tooth surface forms the basis for the tooth contact analysis that leads to the formulation of a suitable gear mesh model.

2. Tooth contact analysis is developed to predict the unloaded kinematic transmission error and the theoretical contact path of idealized hypoid gear designs. Furthermore, an exact spatial mesh position and force vector theory is explicitly derived.

3. A generic three-dimensional coupled translation-rotational vibratory model for analyzing hypoid and bevel geared rotor systems is developed. The mesh formulation is based on the geometric characteristics and mechanics of contact cells on the gear
tooth surface. In this case, the formulation applies the loaded transmission error as the primary excitation source, and includes the effects of gear backlash, friction, load, time-varying mesh position and force vectors. The mesh force coupling properties are explicitly represented by the time-varying and load-dependent mesh characteristic vectors.

4. The modal properties and force transmissibility are computed using a reduced linear time-invariant dynamic model by assuming stationary mesh stiffness and load vectors. The effect of pinion offset and spiral angle on the system vibratory response is analyzed. The force transmissibility through the gear mesh interface is investigated to examine the effectiveness of specific combinations of design parameters.

5. A translation-torsional coupled non-linear time-varying mesh dynamic model based on the loaded contact analysis of hypoid gears is studied numerically. The effects of the gear backlash non-linearity and friction force are also included. In addition, the loaded transmission error and load dependent mesh stiffness are integrated into the dynamic analysis. This study investigated the effect of time varying mesh vectors on the cross-axis gear vibration for the first time.

6. An experiment was conducted to provide a glimpse of the complex gear noise and vibration phenomena. The measured response is also used to partially verify the proposed model.
Dedicated to my family
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Major Field of Study

Mechanical Engineering
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NOMENCLATURE

c  damping coefficient
C  hypoid pinion offset
[C_δ]  compliance function
[C]  damping matrix
D_{1,2}  pinion and gear outer diameter at large end
[D_γ(ω)]  spectral mesh stiffness matrix
e  transmission error vector
e_L  loaded transmission error
E  pinion offser
E_M  pinion cutter offset
E_0  column vector of initial separations of tooth elements
F_w  face width
{f}  forcing vector
F  forcing term
g  direction cosine vector for friction force
[G]  stiffness transfer function matrix
G_{q,q}^{(l)}  transfer function element
h  direction cosine vector for normal force
H_z  gear head-cutter vertical setting
[H]  compliance matrix
k  stiffness term
i_c  pinion cutter tilt angle
J_c  swivel angle
i_j,k  triad of unit vectors of a coordinate system
I  mass moment of inertia term
[K]  stiffness matrix
m  mass
M_{1,2}  torques on pinion and gear
[M]  mass matrix
[M_{ij}]  transformation matrix from coordinate i to j
n  surface normal vector
N_{1,2}  pinion and gear tooth number
\( \gamma_{m2} \quad \text{gear machine root angle} \\
\kappa \quad \text{normal vector coefficient for representing different mesh conditions} \\
\omega \quad \text{frequency} \\
\varepsilon \quad \text{initial separation of tooth element} \\
\xi \quad \text{modal damping ratio} \\
\bar{\xi}_{(1,2)} \quad \text{position of a point on tooth surface in the face cone coordinate} \\
\zeta \quad \text{element of direction cosine} \\
\lambda_u \quad \text{rotational direction radius about} \ u\text{-axis} \\
\Lambda_{1,2} \quad \text{column vector of rotational direction radius} \\

\textbf{subscripts} \\
b \quad \text{bearing or blade} \\
t \quad \text{cutter} \\
c \quad \text{concave side of gear teeth} \\
\delta \quad \text{mesh interface} \\
E \quad \text{driver} \\
f \quad \text{fixed coordinate system or friction} \\
g,G \quad \text{gear} \\
l,L \quad \text{indices for local coordinate systems} \\
n \quad \text{coordinate of normal vector} \\
o \quad \text{external load} \\
O \quad \text{load} \\
p,P \quad \text{pinion} \\
r \quad \text{modal number} \\
v \quad \text{convex side of gear teeth} \\

\textbf{superscripts} \\
l \quad \text{gear member} (l = 1,2) \\
m \quad \text{pitch point}
CHAPTER 1

INTRODUCTION

This dissertation research is concerned with the development of an analytical multiple-degree of freedom dynamic model for analyzing the hypoid and bevel geared rotor systems, and the associated gear meshing mechanism. The foci are on the modeling of the hypoid gear mesh interface, and formulation of the three-dimensional coupled translation-rotational vibration model. Also, this research study seeks a better understanding of the generation of the gear whine phenomenon due to the vibratory characteristic of the hypoid geared rotor system.

1.1 Literature Review

In the past couple of decades, extensive analytical, computational and experimental studies have been performed to analyze the dynamics of parallel axis geared rotor systems as evident from the vast number of publications found in the open literature. In general, research efforts on parallel geared rotor system dynamics have been made in the following areas.
(a) Stress, dynamic load, load distribution, transmission error and tooth profile predictions and optimizations (Tavakoli and Houser, 1986; Velex and Maatar, 1996; Welbourn, 1979; Lin et al., 1988; Vijayakar, 1987).

(b) Force coupling and transmissibility analysis at the gear mesh interface (Blankenship and Singh, 1995; Kubo, 1978; Umezawa et al., 1984, 1985)

(c) Linear and non-linear multi-degrees of freedom dynamic modeling and analysis of generic geared rotor system (Özgüven and Houser, 1988; Kahraman and Singh, 1990, 1991; Blankenship and Singh, 1995; Raclot and Velex, 1999).

(d) Modal analysis of planetary gear train (Kahraman, 1994; Lin and Parker, 1999)

While parallel axis geared rotor system dynamics has been extensively investigated and its field is relatively mature, research on the dynamics of nonparallel axis geared systems such as bevel and hypoid gears is quite sparse. Previous research has not led to the same level of in-depth studies, even though many investigations have been performed to improve the design and manufacture of these types of power transmission components. Only a few experimental and semi-empirical publications related to dynamics have been reported in the open literature. Thus far, most of the current research work in this area includes

(a) Synthesis of machine tool and cutter settings to manufacture higher precision pinion and gear profiles to obtain the desirable loaded and unloaded transmission error and contact patterns (Litvin et al., 1981, 1989, 1991, 1998;

(b) Stress, load distribution and loaded transmission error predictions (Krenzer, 1981; Gosselin et al., 1995; Mark, 1987; Tilak, 1990; Vijayakar, 1987, 1991; Bible et al., 1995; Chen et al., 1996)

(c) Torsional vibration model for simulating hypoid or bevel gear pairs based on empirical mesh position and force vector (Kiyono, 1981; Rautert and Kollmann, 1989; Donley et. al., 1992)

It is generally known that the gear kinematic transmission error is the primary source of vibratory energy excitation that produces tonal noise in most geared applications. In the case of hypoid gears, previous engineering efforts have been directed towards the synthesis of machine tool and cutter settings to manufacture higher precision pinion and gear profiles, and optimization of mesh patterns by applying tooth contact analysis. Hypoid or bevel gear tooth stress, load distribution and loaded transmission error were computed based on finite element analysis (Gosselin et al., 1995), with a surface integral technique using Simplex type of contact algorithm (Vijayakar, 1987), and by using much simplified ‘slice’ deflection model (Krenzer, 1981). Sugimoto et al. (1991) experimentally investigated the effect of the tooth contact and alignment error of the hypoid assembly on loaded transmission error using measured actual tooth surface coordinates. Similarly, extensive research has been done in the areas of kinematic optimization of tooth surface (Fong and Tsay, 1992; Shibata et al., 1997) and identification of optimal machine settings (Litvin et al., 1981, 1989, 1991, 1998; Fong
and Tsay, 1991, 1992; Gosselin et al., 1989, 1998; Lin et al., 1996, 1997; Wang and Ghosh, 1994; Simon, 1996; Kubo et al., 1997) based on desired tooth contact pattern and transmission error. There are also a few publications on issues related to lubricated contact and thermal problem of hypoid gears (Simon, 1980; Chittenden et al., 1985; Chao and Cheng, 1987). However, only a few experimental works for measuring transmission error and noise source in bevel and hypoid gear pair were done (Pitts, 1972; Milenkovic et al., 1983, Poling, 1999).

Due to the complexity of tooth geometry and meshing behavior of spiral bevel and hypoid gears, relatively few research activities have been carried out to study the system dynamic aspects of non-parallel gearing. The above-mentioned research activities are mostly directed towards the reduction of transmission error and computation of optimal tooth contact bearing patterns to produce more durable and quieter gears. Theoretical and experimental studies on the dynamics of hypoid geared rotor systems are extremely sparse. It has long been realized that some axle gear noise problem are not only caused by the quality of the gear design and manufacturing parameters, but also result from structure resonance behavior and the sensitivity to the driveline component design. For instance, Remmers (1971) used a lumped mass-spring model to simulate the rear axle driveline and developed a test system to identify the pinion resonance spectrum of bevel and hypoid gears, in which mesh stiffness is assumed to be infinitely rigid. Kiyono et al. (1981) derived an analytical model for bevel gears by neglecting friction-induced effect and studied the stability of a two degrees of freedom translational vibratory model by assuming a sine pattern for the line of action vector. Terauchi et al. (1980, 1981) studied the dynamic load, torque variation and shaft bending vibration for straight bevel gears.
experimentally and identified resonant natural frequencies. Nakayashiki et al. (1983) proposed a torsional model for rear axle driveline and showed that the gear whine is largely affected by the stiffness of supporting bearings. Rautert and Kollmann (1989) developed a lumped mass-spring model to simulate the dynamic forces of bevel gears with compliant shafts using a constant mesh force coupling vector. Abe and Hagiwara (1990) demonstrated experimentally that axle gear noise of a specific case could be reduced by modifying driveline vibration modes through addition of side-flanges to the output shaft, although no detailed explanation was given on how this was to be accomplished. Hirasaka et al. (1991) developed an experimental method to estimate the dynamic mesh force of a hypoid gear pair, and predicted that lowering propeller shaft stiffness could reduce gear mesh force significantly. Donley et al. (1992) proposed an approximate hypoid gear dynamic model that is used in the context of a general-purpose finite element method. Their formulation of the gear mesh is only approximate and relies on prior knowledge of the pitch point, because it is based on bevel gear mesh equivalence theory.

Most of the above-mentioned models on the dynamics of hypoid or bevel gears are based on either experimental or semi-empirical approach. The gear mesh characteristics that describe the spatial mesh force vector and contact position are only approximately represented. None provides a comprehensive formulation and computational study of the bevel and hypoid geared rotor system dynamics. Also, an exact analytical gear mesh source excitation model for hypoid and bevel gears remains un-formulated. The previous hypoid/bevel dynamic modeling practice simply extends the previous parallel axis gear pair concept of stationary mesh line-of-action to represent the
excitation (Remmers, 1971; Nakayashiki et al., 1983; Rautert and Kollmann, 1989; Donley et al., 1992). The effect of time-varying mesh position and force vector problem, which is an inherent characteristics of hypoid gearing, was not addressed or discussed at all. Accordingly, this dissertation intends to address these issues in greater depth and would provide a general framework for gear mesh excitation models in non-parallel gear pairs.

As indicated above, the meshing position and normal force vectors for bevel and hypoid gears possess time and spatial-varying characteristics. Moreover, the friction force generated at the mesh interface can possibly produce effects of 3-dimensional oblique force and moment reactions on the gear members, which may lead to internal parametric excitation, similar to the phenomenon seen in parallel gearing problems (Ikeda and Muto, 1981; Rebechechi and Crisp, 1983; Lida et al., 1985; Hochmann, 1997). Gear backlash resulting in clearance type non-linearity at the mesh can lead to tooth separation or backside collision. However, it is still unclear when and if these effects are important, since virtually no research has been accomplished previously in the area of hypoid dynamics. On the other hand, nonlinear dynamic problems in spur or helical gears have been studied extensively by numerous researchers (Shaw and Holmes, 1983; Choi and Noah, 1988; Ö zgüven and Houser, 1988; Comparin and Singh, 1989; Kahraman and Singh, 1990, 1991(a), (b), Kahraman and Blankenship, 1996; Hochmann, 1997). Some of these studies include the effects of non-linear contact stiffness, gear backlash or bearing clearance, and friction. For instance, Hochmann (1997) examined the role of friction excitation in spur gears by studying a 2 degrees of freedom torsional model in the form of the Meissner equation (Richards, 1983). Most of the solution techniques depend on
approximate formulations such as piecewise linear theory, harmonic balance method and numerical integration, as closed-form analytical solution is not yet possible.

In spite of the fact that much is known about nonlinear effects in parallel axis gears, their formulations are not directly applicable here due to the unique nature of the contact position vector and line-of-action, which vary with time. Hence, the recent approaches used in parallel axis gears must be modified, which requires further investigation from basic principles applicable to non-parallel cases.

1.2 Motivation

As one can see from the above discussions, very few publications are available in the open literature on the system dynamics of bevel and hypoid geared rotor systems, even though many research investigations have been directed at the dynamics of parallel axis cases. Thus far, an integrated and exact dynamic model for hypoid geared rotor system does not exist. Also, many chronic axle noise problems seen in automotive systems continue to persist due to the lack of a comprehensive way to deal with the design concerns effectively. This stems partly from the lack of understanding of the hypoid/bevel gear noise generation and transmission problem. Current remedies rely mostly on trial-and-error semi-empirical scheme that is costly and often ineffective. Furthermore, the dynamic characteristics of time-varying mesh vectors was not addressed at all in previous studies. Accordingly, this dissertation intends to address these issues in greater depth and would provide a general framework for gear mesh excitation models in non-parallel gear pairs.
Therefore it should be of great interest to develop a generic gear mesh formulation and an analytical lumped parameter model for analyzing the dynamics of bevel and hypoid geared rotor system reliably.

1.3 Scope and Objectives

The focus of the present study is to address the tonal noise generation and transmission problems in right-angle hypoid and bevel gear drives by analyzing the oscillatory motion behavior about known quasi-steady state condition. Gear bodies are assumed to be rigid except for the localized elastic compliance of the gear teeth in mesh. Further, the energy equivalent viscous damping representation is assumed. The emphasis is on developing a basic understanding of the recurring hypoid gear whine problem. Dynamical effect of sliding friction at the mesh interface will also be studied.

Two approaches will be applied to establish the contact position vector and associated tooth surface normal vector or line-of-action. One method is based on the concept of pitch cone designs, and it leads to a constant position vector and line-of-action direction. The other is based on a 3-dimensional loaded tooth contact analysis applying the Contact Analysis Program Package (CAPP) enveloped by Vijayakar (1987) which can be used to formulate a time-varying mesh position and line-of-action vector. These vectors form the basis for developing a set of generalized 3-dimensional, multi-degrees of freedom dynamical equations to describe the coupled rotational-translational vibratory motion of hypoid gears. The proposed model will be capable of simulating numerous operating conditions, including drive and coast cases. It also incorporates the net
characteristics of the driveline system flexibility from the driver to the load component. The model will be applied to perform a series of parametric studies to demonstrate its versatility and salient features.

Therefore the primary objective of this dissertation research is to develop an analytical modeling framework for studying the vibration characteristics of hypoid and bevel geared rotor systems. Here, the bevel gear pair system will be treated as a special case of the hypoid gear with zero pinion offset. This can be achieved with a specialized formulation even though their kinematic meshing mechanisms are quite distinct. To accomplish the above-mentioned objective, the following specific tasks are proposed:

a) Establish the coordinate systems for the generation of hypoid and bevel gears, and develop a 3-dimensional tooth generation method by simulation of the manufacturing process. This is the basis for subsequent analysis applying the finite element method to perform a more precise gear tooth contact analysis, which can be simplified to obtain a time-invariant mesh vector for linear vibration calculations.

b) Simulate tooth contact analysis to obtain unloaded transmission error and contact path. Furthermore, the time-invariant hypoid gear pair meshing model based on pitch cone designs will be derived.

c) Develop a generalized equations of motions for the hypoid and bevel geared rotor systems incorporating the effects of off line-of-action friction forces and moments, time-varying mesh, gear backlash with loaded transmission error excitation.
d) Study the model properties and identify the dominant modes and mesh force coupling mechanism.

e) Apply the CAPP code, and study the non-linear time-varying dynamic response of the coupled translation-rotation motions.

f) Perform a component experiment to capture a glimpse of the hypoid gear vibration and noise phenomena. The results will be used to validate key elements of the proposed vibration model of the hypoid gear drive.

g) Development of efficient computational solutions for non-linear and time-varying hypoid gear mesh problems.

1.4 Organization

The generation method for 3-dimensional spiral bevel and hypoid gear surface is presented in Chapter 2. Chapter 3 derives the unloaded tooth contact analysis and mean mesh position and line-of-action vectors, while Chapter 4 develops a generalized 14 degrees of freedom vibration model. The free and forced vibration solutions of the linear time-invariant system are discussed in Chapter 5, and studies on the modal property and force transmissibility are also performed in this chapter. The non-linear time-varying mesh model is solved numerically and the effects of mean load, friction, time-varying mesh are examined in Chapter 6. Chapter 7 presents the experimental results from a component test setup, and the correlation of the prediction of the proposed simulation to measure response spectrum.
CHAPTER 2

GENERATION OF SPIRAL BEVEL AND HYPOID GEAR TOOTH SURFACES

2.1 Introduction

In this chapter, the generation of spiral bevel and hypoid pinion and gear surfaces in three-dimensional space will be presented. There are two commonly manufacturing methods for hypoid gears in industry, namely the single indexing face-milled method with tapered depth tooth (Gleason method) and continuous spiral face-hobbed method with uniform depth tooth (Gleason and Oerlikon method). The present study focuses on the first method. The objective of this chapter is to form the basis for the loaded gear contact simulation based on finite element analysis (Contact Analysis Program Package, or CAPP, Vijayakar, 1987). The formulation derived in this chapter will also be used to conduct unloaded tooth contact simulation and derive hypoid mesh simulation needed in Chapter 3.

2.2 Method of Generation

The tooth cutting process is carried out either by a form-cutting method (such as FORMATE®) or by generation. In the first case, the tooth surfaces are obtained as a copy
of a cutting tool profile. The second method requires the gear teeth of the workpiece to mesh with the teeth of the generating gear that represents the imaginary crown cone. The spiral bevel or hypoid pinion and gear can be generated either by FORMATE® (non-generated) method or generated method. The present study deals with both types of cutting methods. In this case, the gear is produced by a cutting-in process in which the cradle and gear workpiece are fixed, and its surface is a copy of the head-cutter surface. Hence, the concave and convex sides are generated simultaneously by the respective side of the cutter blade. The pinion is generated by the rotation of cradle as it carries the head-cutter. A detailed description of the generation method can be found in References (Baxter, 1961; Gleason Works, 1971, Shtipelman, 1978; Litvin and Gutman 1981; Stadtfeld, 1983; Litvin 1990; Litvin and Zhang, 1991).

2.3 Pinion Tooth Coordinate Generation

The pinion tooth surface is generated as the envelope to the family of cutting tool surfaces. The head-cutter with blades is mounted to the cradle of the cutting machine. The cutter spins about an axis, and at the same time, the pinion blank rotates. The machine cradle with the head-cutter may be imagined as a crown gear that meshes with the pinion being cut. The cradle rotates with angular velocity $\omega_c$, and the pinion workpiece being generated rotates about its own axis with angular velocity $\omega_h$ which depends on $\omega_c$. The concave and convex flanks of the pinion are generated separately. The pinion tooth coordinates can be obtained by simulating this cutting process, as presented next.
2.3.1 Geometry of the Head-cutter

The head-cutter is provided with blades. Here, two coordinate systems denoted by $S_{bl}(X_{bl}, Y_{bl}, Z_{bl})$ and $S_{rt}(X_{rt}, Y_{rt}, Z_{rt})$, which are rigidly connected to the blade and the cutter respectively, are used to describe the surfaces of the blade and the head-cutter, as shown in Figures 2.1 and 2.2. The blade surface consists of four parts: Parts I and II are straight surface corresponding to the working profiles, and Parts III and IV are fillet and root land respectively. The lengths of these 4 parts are denoted as $L_{(1,2,3,4)}$, respectively, and will be determined by the blade geometry parameters.

For convenience, the profiles of the head-cutter are expressed using a general Gaussian surface coordinates $u_1$ and $v_1$, starting from an arbitrary point that is away from the starting point of the Part III, as shown in Figure 2.1. The tool surface profile may be represented by a family of position vector and the corresponding normal vector $(r^{(1)}, n^{(1)})$,

$$r^{(1)}(u_1, v_1) \subset C^1; \frac{\partial r^{(1)}}{\partial u_1} \times \frac{\partial r^{(1)}}{\partial v_1} \neq 0$$

(2.1)

$$N^{(1)} = r^{(1)}_u \times r^{(1)}_v; n^{(1)} = \frac{N^{(1)}}{|N^{(1)}|}$$

(2.2)

Here the superscript (1) in parenthesis refers to the pinion cutter, and $C^1$ refers to the class of function $r^{(1)}$ that possesses continuous derivatives up to the first order. The surface normal vector is $N^{(1)}$ and the corresponding unit normal vector is $n^{(1)}$. 
Figure 2.1: Pinion cutter blade geometry and coordinate $S_{b1}$.

Figure 2.2: Pinion head cutter coordinate system $S_{rl}$. 
The coordinates of the blade surface can be expressed in the coordinate system denoted by \([S_{b1}(x_{b1}, y_{b1}, z_{b1})]\) as shown in Figure 2.1. For the root land (Part IV), it can be shown that,

\[
r_{b1}^{(1)}(u_t) = \begin{bmatrix} R_{ui} \pm u_t \\ 0 \\ 0 \\ 1 \end{bmatrix} ; \quad u_t < \frac{W_{cp}}{2} = L_1
\]

\[
R_{ui} = x_{ci} \pm \frac{W_{cp}}{2}
\]

where \(x_{ci}\) is the location of the fillet-arc center and will be given below, \(W_{cp}\) is the distance from the starting location of \(u_t\) to the fillet-arc beginning and can be an arbitrary. Note that \(x_{ci}\) and \(W_{cp}\) are not defined in the standard cutter geometry and only used here for root definition. The subscribe \((b1)\) in parenthesis denotes the coordinate system that is used for the vector, and the upper sign is for the outer blade (OB) which corresponds to the concave side of the generated pinion and lower sign for the inner blade (IB) that corresponds to the convex side of the pinion tooth. Note that the fourth element is unity for position vector indicating homogenous coordinate.

For the arc fillet (Part III), the coordinate is,

\[
r_{b1}^{(1)}(u_t) = \begin{bmatrix} x_{ci} \pm \rho_i \sin \theta_a \\ 0 \\ -\rho_i (1 - \cos \theta_a) \\ 1 \end{bmatrix} ; \quad L_1 < u_t < L_1 + L_2
\]

where
\[ \varphi_a = \frac{u_1 - L_1}{\rho_1} ; \quad L_2 = \rho_1 \left( \frac{\pi}{2} - |\phi_{BPP}| \right) ; \quad x_i = R_{pp} \mp \frac{\rho_1}{\cos \phi_{BPP}} (1 \mp \sin \phi_{BPP}) \]  

\[ R_{pp} = R_{cp} + D_{m_1} (\cos \phi_{BPP} \tan(\phi_{BP}) - \sin(\phi_{BPP})) ; \quad \phi_{BPP} = \phi_{BP} - \gamma_{pc} \]  

where \( R_{cp} \) is the cutter point radius, \( \rho_1 \) is the blade angle radius, \( \phi_{BP} \) is the blade angle (\( \phi_{BP} < 0 \) for convex, and \( \phi_{BP} > 0 \) for concave side), \( \gamma_{pc} \) the is angle of cutter modification (\( \gamma_{pc} < 0 \) for convex and \( \gamma_{pc} > 0 \) for concave side), and \( D_{m_1} \) is the depth of modification. The upper sign is for concave flank and the lower sign is for convex flank of the pinion tooth.

In the case of the modified working profile (Part II),

\[ r_{b_i}^{(3)}(s_{pp}) = \begin{cases} 
R_{pp} + U_{pp} \sin \phi_{BPP} \\
0 \\
-U_{pp} \cos \phi_{BPP} \\
1 
\end{cases} ; \quad L_1 + L_2 < u_1 < L_1 + L_2 + L_3 \]  

\[ U_{pp} = u_1 - (L_1 + L_2) + \frac{\rho_1 (1 - \sin(\phi_{BPP}))}{\cos(\phi_{BPP})} \]  

\[ L_3 = D_{m_1} - \frac{\rho_1 (1 - \sin(\phi_{BPP}))}{\cos(\phi_{BPP})} \]  

Similarly, for the working profile (Part I)
\[ r_{b1}^{(1)}(S_p) = \begin{cases} \frac{R_{cp} + U_p \sin \phi_{BP}}{i} & ; \quad u_1 > L_1 + L_2 + L_3 \end{cases} \]  
(2.7a)

\[ U_p = u_1 - (L_1 + L_2 + L_3) + D_{m1} \frac{\cos \phi_{BP}}{\cos \phi_{BP}} \]  
if \( D_{m1} \neq 0 \)  
(2.7b)

\[ U_p = u_1 - (L_1 + L_2 + L_3) + \rho_1 (1 - \sin |\phi_{BP}|) \]  
if \( D_{m1} = 0 \)  
(2.7c)

Next, since the head-cutter surface is generated by the rotation of the blades about the \( Z_{11} \) axis, its coordinate can be found by the transformation of the blade coordinate in \( S_{b1} \) of all the 4 parts of the blade profile described above into the cutter-head coordinate \( S_{11} \) as shown in Figure 2.2. Using the parameters \( u_1 \) and \( v_1 \) to locate a point on the cutting head,

\[ r_{11}^{(1)}(u_1, v_1) = [M_{11b1}(v_1)] r_{b1}^{(1)}(u_1) \]  
(2.8)

where the matrix \([M_{11b1}]\) is explicitly given by

\[[M_{11b1}] = \begin{bmatrix} \cos v_1 & -\sin v_1 & 0 & 0 \\ \sin v_1 & \cos v_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  
(2.9)

For instance, it can be shown that, for the working profile of the head-cutter surface, the position vector is
\[
\mathbf{r}_{11}(u_1, v_1) = \begin{bmatrix}
(r_p + U_p \sin \phi_{BP}) \cos v_1 \\
(r_p + U_p \sin \phi_{BP}) \sin v_1 \\
-U_p \cos \phi_{BP} \\
1
\end{bmatrix}
\] (2.10a)

and the corresponding surface normal vector

\[
\mathbf{n}_{11}(u_1, v_1) = \frac{\partial \mathbf{r}_{11}}{\partial u_1} \times \frac{\partial \mathbf{r}_{11}}{\partial v_1} = \begin{bmatrix}
\cos \phi_{BP} \cos v_1 \\
\cos \phi_{BP} \sin v_1 \\
\sin v_1 \\
0
\end{bmatrix}
\] (2.10b)

Hence using \(u_1\) and \(v_1\) along with other machine tool settings, one can completely define the spatial location of a point on the cutting head.

### 2.3.2 Pinion Tooth Surface Coordinate

The pinion tooth surface can be determined as the envelope of the family of the head-cutter surfaces that are generated in the coordinate system rigidly connected to the pinion. A series coordinate transformation from the blade coordinate to the pinion coordinate is necessary to obtain this envelope. Figure 2.3 shows the coordinate systems and their relationships. Here, the references \(S_{m1}, S_c, S_x\), and \(S_w1\) are rigidly connected to the cutting machine, rigidly connected to the cradle, installation of the pinion on the cutter machine, and rigidly connected to the pinion workpiece being cut, respectively. The pinion rotation angle \(\phi_1\) is determined by the cradle rotation angle \(\phi_c\) and the ratio of roll \(R_a\). In the present formulation, the modified roll provision allows modifications of the cradle-to-work speed ratio during generation. Hence, the pinion coordinate position and
the corresponding normal vector can be obtained by transforming the cutter surface in $S_{b_1}$ to the $S_1$:

$$r^{(1)}_{w_1}(u_1, v_1, \phi_c) = [M_{w_1l_1}(v_1, \phi_c)]r^{(1)}_{l_1}(u_1) = [M_{w_1p}][M_{p_1n}][M_{n_1m}][M_{mlc}][M_{clb}][M_{b_1l_1}]r^{(1)}_{l_1}$$

(2.11a)

$$n^{(1)}_{w_1}(u_1, v_1, \phi_c) = [M_{w_1l_1}(v_1, \phi_c)]n^{(1)}_{l_1}(u_1) = [M_{w_1p}][M_{p_1n}][M_{n_1m}][M_{mlc}][M_{clb}][M_{b_1l_1}]n^{(1)}_{l_1}$$

(2.11b)

Figure 2.3: Instalment of the cradle and pinion on the cutting machine.
where the transformation matrix between $S_{w1}$ and $S_p$ is (see Figure 2.3)

$$[M_{w1p}] = \begin{bmatrix}
0 & \cos \phi_i & -\sin \phi_i & 0 \\
0 & \sin \phi_i & \cos \phi_i & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (2.12a)$$

$$\phi_i = R_x (\phi_c - \frac{2C}{2})\phi_c^2 - (\frac{6D}{6})\phi_c^3 - (\frac{24E}{24})\phi_c^4 - (\frac{120F}{120})\phi_c^5 \quad (2.12b)$$

where the coefficients $2C, 6D, 24E$ and $120F$ are the modified roll parameters.

The installation of the pinion in the cutter machine is determined by

$$[M_{pm}] = \begin{bmatrix}
\cos \gamma_m & 0 & \sin \gamma_m & -X_p \\
0 & 1 & 0 & 0 \\
-\sin \gamma_m & 0 & \cos \gamma_m & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (2.12c)$$

where $\gamma_m$ is the machine root angle of pinion cutter setting, and $X_p$ is the machine center to back setting. The intermediate coordinate system $S_n$ in the cutting machine (see Figure 2.3) is related to

$$[M_{nm1}] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{E_M}{X_B} \\
0 & 0 & 1 & -X_B \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (2.12d)$$

where $X_B$ is the sliding base setting, and $E_M$ is the pinion cutter offset, given by

$$X_B = X_{b0} + (VH_1)\phi_c + (VH_2)\phi_c^2 + (VH_3)\phi_c^3 \quad (2.12e)$$
\[ E_M = E_{M0} + (VE_1)\phi_c + (VE_2)\phi_c^2 + (VE_3)\phi_c^3 \]  

(2.12f)

In the above equation, \( E_M \) is positive for a left-hand pinion and negative for a right-handed one, and VH and VE terms are the coefficients of modifications during cutting.

The cradle rotation motion in the cutting machine can be expressed as

\[
[M_{mic}] = \begin{bmatrix}
\cos \phi_c & \sin \phi_c & 0 & 0 \\
-\sin \phi_c & \cos \phi_c & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(2.13)

Moreover, the head-cutter may be tilted in order to achieve different tooth modification as shown in Figure 2.4. The coordinate \( S_{t1} \) is rigidly connected to head-cutter and its relationship to the blade is

\[
[M_{bt1}] = \begin{bmatrix}
\cos(i_c) & 0 & \sin(i_c) & 0 \\
0 & 1 & 0 & 0 \\
-\sin(i_c) & 0 & \cos(i_c) & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(2.14)

where \( i_c \) is the pinion cutter tilt angle.
From Figure 2.5, the installment of the head-cutter $S_b$ in $S_c$ is given by

$$
[M_{cb}] = \begin{bmatrix}
-\sin(J_c \mp Q_c) & \mp \cos(J_c \mp Q_c) & 0 & S_R \cos(Q_c) \\
\pm \cos(J_c \mp Q_c) & -\sin(J_c \mp Q_c) & 0 & -S_R \sin(Q_c) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

(2.15)

where the upper sign applies to a for LH pinion and the lower sign applies to a RH pinion, $J_c$ is the swivel angle, $Q_c$ is the basic cradle angle and $S_R$ is the radial sitting of pinion cutter center.

It is noted that Equation (2.11) contains three variables, cradle position angle $\phi_c$, cutter surface coordinates $u_1$ and $v_1$. Since the cutting-head meshes with the pinion workpiece being cut, the relative velocity between the cradle and the workpiece is perpendicular to the surface normal vector. Thus the equation of meshing holds,
Figure 2.5: Installment of the head-cutter. Note that cradle angle is negative in the right-hand pinion case.
\[ f_1(u_1, v_1, \phi_c) = \frac{\partial r_{w1}^{(1)}(u_1, v_1, \phi_c)}{\partial \phi_c} \cdot n_{w1}^{(1)}(u_1, v_1, \phi_c) = 0 \] (2.16)

Because there are two unknowns \( v_1 \) and \( \phi_c \) if \( u_1 \) is given, two equations must be used to solve them to get the complete description of the tooth surface. In order to develop two other equations, the parameters \( u_1, v_1 \) and \( \phi_c \) are mapped into the pinion workpiece coordinate \( S_{w1} \). Now consider an arbitrary point at the pinion surface with the prescribed position \( \bar{\xi}_1 \) as shown Figure 2.6, in which \( O_{f1} \) is the face apex, \( O_{w1} \) is the crossing point of pinion and gear axes and is coincident with \( O_{\rho} \) in Figure 2.3. The dimensionless position parameter \( \bar{\xi}_1 \) is defined as

\[
\bar{\xi}_1(u_1, v_1, \phi_c) = 1 - \frac{D_1}{F_{w1} \sin \gamma_{f1}} + \frac{2 \cos \gamma_{f1}}{F_{w1}} (z_{w1}(u_1, v_1, \phi_c) + z_{fp}) + \frac{2 \sin \gamma_{f1}}{F_{w1}} \cdot r_1(u_1, v_1, \phi_c)
\]

\[-1 \leq \bar{\xi}_1 \leq 1 \] (2.17)

such that \( \bar{\xi}_1 = 1 \) at pinion heel and \( \bar{\xi}_1 = -1 \) at toe. This provides the second equation

\[ f_2(u_1, v_1, \phi_c) = \bar{\xi}_1(u_1, v_1, \phi_c) - \bar{\xi}_0 = 0 \] (2.18)

In the above equations, \( r_1(u_1, v_1, \phi_c) = \sqrt{x_{w1}^2(u_1, v_1, \phi_c) + y_{w1}^2(u_1, v_1, \phi_c)} \), \( D_1 \) is the pinion outer diameter at large end, \( F_{w1} \) is the face width, and \( \gamma_{f1} \) is the pinion face angle, and \( z_{fp1} \) is the face cone apex beyond the crossing point, and is positive if the apex lies beyond the crossing point and negative if lies between the pinion axis and the crossing
Figure 2.6: Projection of pinion teeth over $X_{w1}$-$Z_{w1}$ plane.

point. Note in Figure 2.6, the heel and toe sections are chosen to be perpendicular to the face cone generatrix. Other types of sections may also be used, such as crowns. By specifying variable $(\mu_1, \xi_1)$, the exact position of a point on a pinion tooth can be located, as shown in Figure 2.7. This can be done after solving the 2 non-linear equations given by Equations (2.16) and (2.18) for unknown $(\phi_e, v_t)$. The Newton-Raphson method was employed to solve this set of equations. The analytical Jacobian matrix must be obtained at every iteration. Therefore, the following derivatives arising from Equation (2.16) are needed,
Figure 2.7: Surface Coordinate $(u_i, \xi_1)$ locates a point on the pinion tooth surface.

\[
\frac{\partial r_{wl}^{(i)}}{\partial \phi_c} = \left[ \frac{\partial M_{wl}}{\partial \phi_c} \right] r_{h}^{(i)}(u_i); \quad \frac{\partial^2 r_{wl}^{(i)}}{\partial \phi_c^2} = \left[ \frac{\partial^2 M_{wl}}{\partial \phi_c \partial v_1} \right] r_{h}^{(i)}(u_i)
\]

(2.19)

\[
\frac{\partial r_{wl}^{(i)}}{\partial u_i} = \left[ M_{wl,i} \right] \frac{\partial r_{h}^{(i)}}{\partial u_i}
\]

(2.20)
\[ n_{wl}^{(1)}(v_1, \phi_c) = \frac{\partial r_{wl}^{(1)} \times \partial r_{wl}^{(1)}}{\| \partial r_{wl}^{(1)} \times \partial r_{wl}^{(1)} \|} = \begin{bmatrix} \frac{\partial y_{wl}}{\partial u_1} \times \frac{\partial z_{wl}}{\partial u_1} - \frac{\partial z_{wl}}{\partial u_1} \times \frac{\partial y_{wl}}{\partial u_1} \\ \frac{\partial z_{wl}}{\partial u_1} \times \frac{\partial x_{wl}}{\partial u_1} - \frac{\partial x_{wl}}{\partial u_1} \times \frac{\partial z_{wl}}{\partial u_1} \\ \frac{\partial x_{wl}}{\partial u_1} \times \frac{\partial y_{wl}}{\partial u_1} - \frac{\partial y_{wl}}{\partial u_1} \times \frac{\partial x_{wl}}{\partial u_1} \\ 0 \end{bmatrix} / \left\| \frac{\partial r_{wl}^{(1)}}{\partial u_1} \times \frac{\partial r_{wl}^{(1)}}{\partial v_1} \right\| \] (2.21)

\[ \frac{\partial n_{wl}^{(1)}(v_1, \phi_c)}{\partial v_1} = \frac{\partial^2 r_{wl}^{(1)}}{\partial u_1^2} \times \frac{\partial r_{wl}^{(1)}}{\partial v_1} + \frac{\partial^2 r_{wl}^{(1)}}{\partial u_1 \partial v_1} \times \frac{\partial r_{wl}^{(1)}}{\partial v_1} - \frac{\partial^2 r_{wl}^{(1)}}{\partial u_1 \partial v_1} \times \frac{\partial r_{wl}^{(1)}}{\partial v_1} \] (2.22)

\[ \frac{\partial n_{wl}^{(1)}(v_1, \phi_c)}{\partial \phi_c} = \frac{\partial r_{wl}^{(1)}}{\partial v_1} \times \frac{\partial^2 r_{wl}^{(1)}}{\partial \phi_c^2} + \frac{\partial r_{wl}^{(1)}}{\partial u_1} \times \frac{\partial^2 r_{wl}^{(1)}}{\partial \phi_c \partial v_1} + \frac{\partial r_{wl}^{(1)}}{\partial v_1} \times \frac{\partial^2 r_{wl}^{(1)}}{\partial \phi_c \partial v_1} \] (2.23)

Suppose \((u_1, \xi_1)\) is specified, the corresponding point \(r_{wl}^{(1)}\) and its outer bound normal \(n_{wl}^{(1)}\) on the pinion tooth surface can be found by substituting \((\phi_c, v_1)\) back into Equation (2.11).

To generate the complete tooth surface, the maximum \(u_1\) at the face cone corresponding to any \(\xi_1\) is needed. The following equation should be satisfied as shown in Figure 2.6:

\[ f(u_1, \xi_1) = \tan \gamma_{f1} \cdot (z_{wl}(u_1, \xi_1) - z_{f1}) - \sqrt{x_{wl}^2(u_1, \xi_1) + y_{wl}^2(u_1, \xi_1)} \] (2.24)
Instead of using the exact derivative of \( r_{w1}^{(1)} = \{x_{w1}, y_{w1}, z_{w1}\}^T \) with respect to \( u_1 \), the numerically approximated derivative is used in solving the above equation since \( u_1 \) is only piecewise smooth. In this case, the Power Hybrid algorithm is employed in which the forward finite difference method is used to get the Jacobin matrix. To solve equation (2.24), solutions for \((v_2, \phi_c)\) and \(r_{w1}^{(1)}\) from Equations (2.16) and (2.18) at an arbitrary point \((u_1, \xi)\) is used to search for the proper \( u_1 \) satisfying Equation (2.24).

### 2.4 FORMATE Gear Tooth Coordinate Generation

Unlike the generation of the pinion, the FORMATE hypoid gear is generated by the form-cutting method, in which the gear surface is a copy of the head cutter surface. The blade and the head-cutter geometry are similar to those of the pinion with some differences in the definition, as shown in Figures 2.8 and 2.9. The coordinate of the blade can be expressed in the blade coordinate \( s_{b2} \). For the root land surface, the position vector is

\[
r_{b2}^{(2)}(u_2) = \begin{bmatrix} R_{u2} \pm u_2 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad u_2 < L_1
\]

(2.25)

\[
L_1 = \frac{W_{cG}}{2} - |R_{cG} - x_{c2}|
\]

(2.26)

where \( R_{u2} \) is mean cutter radius, \( W_{cG} \) is the cutter point width, \( R_{cG} \) is the cutter point radius given by \( R_{cG} = R_{u2} \pm \frac{W_{cG}}{2} \), and \( X_{c2} \) is the location of the arc center of the blade.
Figure 2.8: Gear cutter blade geometry and coordinate system.

Figure 2.9: Gear head-cutter and blade relationship.
corner and given by

\[ X_{c_2} = R_{GG} \pm \frac{\rho_2}{\cos \phi_{BGG}} (1 \pm \sin \phi_{BGG}) \]  \hspace{1cm} (2.27a)

where

\[ R_{GG} = R_{cG} + D_m (\cos \phi_{BGG} \tan \phi_{BG}) - \sin (\phi_{BGG}) \]  \hspace{1cm} (2.27b)

and the upper sign is for concave and lower sign is for convex side of the gear in Equations (2.25)-(2.27), and

\[ \phi_{BGG} = \phi_{BG} - \gamma_{Ge} \]  \hspace{1cm} (2.28)

where \( \phi_{BG} \) is blade angle, \( \phi_{BG} > 0 \) for convex, and \( \phi_{BG} < 0 \) for concave side. Note that the sign convention used for \( \phi_{BG} \) is different from that of the pinion cuter blade angle. Also, \( \gamma_{Ge} \) is the angle of cutter modification, \( \gamma_{Ge} > 0 \) for convex, \( \gamma_{Ge} < 0 \) for concave, and \( \rho_2 \) is the blade angle radius.

In the case of the arc fillet surface, the position vector is

\[ r_{b2}^{(2)}(u_2) = \begin{cases} 
   x_{c_2} \pm \rho_2 |\sin \varphi_b| \\
   0 \\
   -\rho_2 (1 - \cos \varphi_b) \\
   1 
\end{cases}; \quad L_1 < u_2 < L_1 + L_2 \]  \hspace{1cm} (2.29)

where

\[ \varphi_b = \frac{S_2 - L_1}{\rho_2} \]  \hspace{1cm} (2.30a)

\[ L_2 = \rho_2 (\frac{\pi}{2} - \phi_{BGG}) \]  \hspace{1cm} (2.30b)

For the straight line of the modified working profile, the position vector is
\[ r_{b2}^{(2)}(u_2) = \begin{cases} 
    R_{GG} - U_{GG} \sin \phi_{BGG} & \text{if } L_1 + L_2 < u_2 < L_1 + L_2 + L_3 \\
    0 & \text{if } L_1 + L_2 + L_3 < u_2 < L_1 + L_2 + L_3 + L_3 \\
    -U_{GG} \cos \phi_{BGG} & \text{if } L_1 + L_2 + L_3 + L_3 < u_2 \\
    1 & \text{if } u_2 > L_1 + L_2 + L_3 + L_3 
\end{cases} \] (2.31)

\[ U_{GG} = u_2 - (L_1 + L_2) + \frac{\rho_z (1 - \sin |\phi_{BGG}|)}{\cos(\phi_{BGG})} \] (2.32a)

\[ L_3 = D_{m2} - \frac{\rho_z (1 - \sin |\phi_{BGG}|)}{\cos(\phi_{BGG})} \] (2.32b)

where \( D_{m2} \) is the depth of modification for the gear blade. Finally, for the straight line of working part surface,

\[ r_{b2}^{(2)}(S_1, \nu_1) = \begin{cases} 
    R_{cG} - U_{G} \sin \phi_{BG} & \text{if } u_2 > L_1 + L_2 + L_3 \\
    0 & \text{if } L_1 + L_2 + L_3 < u_2 < L_1 + L_2 + L_3 + L_3 \\
    -U_{G} \cos \phi_{BG} & \text{if } L_1 + L_2 + L_3 + L_3 < u_2 \\
    1 & \text{if } u_2 > L_1 + L_2 + L_3 + L_3 
\end{cases} \] (2.33)

\[ U_{G} = u_2 - (L_1 + L_2 + L_3) + D_{m2} \frac{\cos \phi_{BGG}}{\cos \phi_{BG}} \text{ if } D_{m2} \neq 0 \] (2.34a)

\[ U_{G} = u_2 - (L_1 + L_2 + L_3) + \frac{\rho_z (1 - \sin |\phi_{BG}|)}{\cos \phi_{BG}} \text{ if } D_{m2} = 0 \] (2.34b)

The blade surface then is transformed into the coordinate system \( S_{i2} \) by

\[ r_{i2}^{(2)}(S_2, \nu_2) = [M_{i2b2}] r_{b2}^{(2)} \] where the matrix \([M_{i2b2}]\) is given by

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\[
[M_{r1b2}] = \begin{bmatrix}
\cos \nu_2 & -\sin \nu_2 & 0 & 0 \\
\sin \nu_2 & \cos \nu_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.35)

The normal vector for the head-cutter surface can be similarly obtained as the pinion head-cutter case,

\[
n_{r2}^{(2)}(u_2, \nu_2) = \frac{\partial r_2^{(2)}}{\partial u_2} \times \frac{\partial r_2^{(2)}}{\partial \nu_2} = \begin{bmatrix}
\cos \phi_{BG} \cos \nu_2 \\
\cos \phi_{BG} \sin \nu_2 \\
\sin \nu_2 \\
0
\end{bmatrix}
\]

(2.36)

To obtain the geometry of the gear surface, the head-cutter surface has to be transformed to the gear coordinate. Now consider the three coordinate systems shown in Figure 2.10, the coordinate systems \(S_{w2}, S_{m2},\) and \(S_{c2}\) are rigidly connected to the gear work piece, the cutting machine, and the head-cutter, respectively. The gear surface coordinate in \(S_{w2}\) can be obtained by a series coordinate transformation from \(S_{b2}\) to \(S_{w2}\), given by

\[
r_{w2}^{(2)}(u_2, \nu_2) = [M_{w2m2}][M_{m2c2}]r_{r2}^{(2)}
\]

(2.37a)

\[
n_{w2}^{(2)}(u_2, \nu_2) = [M_{w2m2}][M_{m2c2}]n_{r2}^{(2)}
\]

(2.37b)

where

\[
[M_{w2m2}] = \begin{bmatrix}
\cos \gamma_{m2} & 0 & -\sin \gamma_{m2} & 0 \\
0 & 1 & 0 & 0 \\
\sin \gamma_{m2} & 0 & \cos \gamma_{m2} & -X_G \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.38)
Figure 2.10: Installment of head-cutter in the cutting machine and the workpiece coordinate system.

\[
[M_{m2r2}] = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -V_2 \\
0 & 1 & 0 & H_2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(2.39)

where $X_G$ is machine center to back, $H_2$ is vertical setting of the head-cutter, $V_2$ is horizontal setting of the head-cutter which is positive for right-hand gear and negative for left-hand, and $\gamma_{m2}$ is the machine root angle. Thus the gear tooth surface is just copy of the head-cutter surface represented in $S_{w2}$.

Similar to the pinion case, the surface parameters $u_2$ and $v_2$ can also be mapped into the workpiece coordinate $S_{w2}$ as shown in Figure 2.11, and satisfies the prescribed position $\bar{x}_{52}$.
Figure 2.11: Surface coordinate \((u_2, \xi_2)\) determines an exact point on the gear tooth surface.

\[
f(u_2, v_2) = \frac{\xi_2}{D_2} = 1 - \frac{D_2}{F_{w2} \sin \gamma_{F2}} + \frac{2 \cos \gamma_{F2}}{F_{w2}} \cdot (z_{w2}(u_2, v_2) + z_{fp2}) + \frac{2 \sin \gamma_{F2}}{F_{w2}} \cdot r_2(u_2, v_2)
\]

\[-1 \leq \xi_2 \leq 1 \quad (2.40)\]

where \(r_2(u_2, v_2) = \sqrt{x_{w2}^2(u_2, v_2) + y_{w2}^2(u_2, v_2)}\), \(D_2\) is the gear outer diameter at the large end, \(F_{w2}\) is the face width and \(\gamma_{F2}\) is the gear face angle, and \(z_{fp2}\) is the face apex beyond the crossing point. Note \(\xi_2 = 1\) at gear heel and \(\xi_2 = -1\) at toe. By specifying the cutter coordinate \(u_2\) on the blade surface and the position in the tooth face direction \(\xi_2\), the exact position of a point on gear tooth can be located. Using Equation (2.40), \(v_2\) can be
obtained given \( (\mu_2, \xi_2) \) by using the Newton-Raphson method. Similar procedures as in the pinion case were employed here to compute the maximum \( \mu_2 \) for any \( \xi_2 \) for the gear face cone.

2.5 **Synthesis of Tooth Coordinate**

The two flanks of teeth are not the same in geometry, thus the above procedures are to be performed for both the concave and convex sides of pinion and gear using the corresponding machine-tool settings. To achieve the correct tooth thickness, the position vectors at the pitch points of the two sides are to be solved. First, equations similar to Equation (2.24) can be obtained for the pinion and gear pitch cones (Figure 2.12 shows pinion pitch cone), given by

![Pitch cone surface coordinate](image)

Figure 2.12: Pitch cone surface coordinate.
\begin{align}
  f(u_1, \xi_1) &= \tan \gamma_{p1} \cdot (z_{w1}(u_1, \xi_1) - z_{pcp1}) - \sqrt{x_{w1}^2(u_1, \xi_1) + y_{w1}^2(u_1, \xi_1)} \\
  f(u_2, \xi_2) &= \tan \gamma_{p2} \cdot (z_{w2}(u_2, \xi_2) - z_{pcp2}) - \sqrt{x_{w2}^2(u_2, \xi_2) + y_{w2}^2(u_2, \xi_2)} 
\end{align}

(2.41a)

(2.41b)

where \( \gamma_{p1} \) and \( \gamma_{p2} \) are pitch angles of the pinion and gear respectively, and \( z_{pcp1} \) and \( z_{pcp2} \) are the distances from the crossing point to the pitch apexes of the pinion and gear respectively. By setting \( \xi_1 = 0 \) and \( \xi_2 = 0 \), the corresponding surface coordinates \( u_1 \) and \( u_2 \) can be obtained using the similar procedures described in Sections 2.3 and 2.4. Thus, the position vector and normal vector at the pitch point can be obtained. In the method adopted here, the concave side is fixed, and the convex side rotates a certain angle to the position with the proper circular tooth thickness. Figure 2.13 shows the procedure for achieving the proper tooth thickness for the pinion and gear. The rotation angle \( \Delta \phi_i \) is determined by the coordinate of the pitch point. Note that the conventions used in defining the tooth side here are different for pinion and gear, and they are illustrated in the figure. Thus for a left-hand (LH) pinion and right-hand (RH) gear configuration, the forward drive corresponds the engagement between the concave flank of the pinion and the convex flank of the gear, and vice versa for the RH pinion and LH gear. Figure 2.14 shows examples of the simulated LH pinion and RH gear 3-dimensional surface. The machine and cutter settings can be identified from the Gleason special analysis file given in Appendix A. The finite element grids used in the contact analysis (CAPP) is shown in Figure 2.15.
2.6 Summary

The spiral bevel and hypoid gear tooth surfaces in three-dimensional space are defined analytically and generated numerically by the simulation of the cutting process of both generated and FORMATE pinion and gear. The fundamental gear meshing equations are obtained for both generated and non-generated gears. A general and robust algorithm is built on separated blocks, thus further employing new designs of cutters can be easily incorporated. The method also includes a modified roll generation scheme. The generated tooth surface forms the basis for the tooth contact analysis that leads to the formulation of a suitable gear mesh model. The geometry of 3-dimensional tooth surface is the basis for the proposed 3-dimensional tooth contact mechanics model, which is being used in the Contact Analysis Program Package (CAPP).
Figure 2.13: Tooth profiles at $\xi = 0$ and the procedure to achieve proper tooth thickness.
Figure 2.14: Pinion and gear are generated and oriented into the mesh.
Figure 2.15: Mesh grids used in the 3-dimensional contact analysis (CAPP).
CHAPTER 3

KINETIC SIMULATION OF GEAR MESH

3.1 Introduction

In Chapter 2, the generation method for 3-dimensional spiral bevel and hypoid
gear surfaces was introduced. This forms the basis for tooth contact analysis using the
CAPP code. Since an explicit representation of the tooth profile of hypoid gears generally
does not exist, unlike that for spur or helical gears, one has to rely on the numerical space
gearing method to compute the tooth surface and associated geometric properties, as
described in Chapter 2. Although the 3-dimensional contact analysis can provide loaded
transmission error and other valuable mesh information, it is computationally intensive.
Hence, there is still an interest to perform unloaded tooth contact analysis that is
generally more efficient even though it is less accurate. Gleason Works (Baxter, 1961;
Krenzer, 1981), and Litvin and Gutman (1981) developed mathematical models to
simulate tooth contact analysis (TCA) for bevel and hypoid gears based on the
manufacturing process given a complete set of machine and cutter settings. This TCA
analysis assumes one pair of teeth in mesh under no load or lightly loaded condition. For
the purpose of dynamic analysis on the hypoid gears based on the mesh information of
TCA, the numerical simulation of hypoid gear with generated pinion and FORMATE gear is performed to obtain the unloaded kinematic transmission error, contact path, and the effect of machine tool settings. Furthermore, in order to obtain the mean mesh position and line-of-action vectors in terms of the gear blank design data, the pitch cone design process is introduced.

3.2 Mesh Simulation

Principle concepts of the tooth contact analysis for hypoid gears will be developed in the manner compatible to results published by Gleason Works (1981) and Litvin (1981). It involves mathematical descriptions of gear tooth surface in mesh and their contact patterns. The simulation of the idealized meshing process is used to determine the contact location, and the corresponding normal vector, which is the basis for the subsequent dynamic modeling.

Consider three Cartesian coordinate systems fixed to the frame, pinion and gear components respectively, as shown in Figure 3.1. The coordinate system fixed to the frame is assumed inertial and denoted by \( S_f(x_f, y_f, z_f) \). The other two coordinates are given by \( S_{wl}(x_{wl}, y_{wl}, z_{wl}) \), where subscript \( l = 1, 2 \), are rigidly connected to the driving pinion and driven gear respectively, in which both \( Z_{wl} \)-axes define the operating rotational directions. The origin \( O_{wl} \) is the apex of gear \( l \), and \( O_f \) is the crossing point, \( \Delta H \) and \( \Delta Q \) are the axial shifts of the pinion and gear in assembly. The pinion tooth
surface that rotates about $Z_{w1}$ can be represented as a family of curved surfaces in $S_f$ given by

$$r_f^{(1)} = [M_{f_{w1}}]r_{w1}^{(1)} = r_f^{(1)}(u_1, v_1, \phi_c, \phi'_f)$$  \hspace{1cm} (3.1)

where $[M_{f_{w1}}]$ is the coordinate transformation matrix from $S_1$ to $S_{w1}$, $\phi'_f$ is the instantaneous roll angle of the pinion, and $r_{w1}^{(1)}$ is given in Equation (2.11). Here the superscript in parenthesis refers to the gear member, while the subscript indicates the reference coordinate system. The corresponding unit normal vector can be expressed as
\[ n_f^{(1)} = [M_{fw1}] n_w^{(1)} = n_f^{(1)}(u_1, v_1, \phi_1, \phi'_1) \quad (3.2) \]

\[ [M_{fw1}] = \begin{bmatrix}
\cos \phi'_1 & \sin \phi'_1 & 0 & 0 \\
-\sin \phi'_1 & \cos \phi'_1 & 0 & 0 \\
0 & 0 & 1 & \Delta H \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (3.3) \]

Similarly, the mating gear tooth surface that rotates about \( Z_{w2} \) can be defined in \( S_f \) by

\[ r_f^{(2)} = [M_{fw2}] r_w^{(2)} = r_f^{(2)}(u_2, v_2, \phi'_2) \quad (3.4) \]

\[ n_f^{(2)} = [M_{fw2}] n_w^{(2)} = n_f^{(2)}(u_2, v_2, \phi'_2) \quad (3.5) \]

\[ [M_{fw2}] = \begin{bmatrix}
\cos \phi'_2 & -\sin \phi'_2 & 0 & E \\
0 & 0 & -1 & \Delta Q \\
\sin \phi'_2 & \cos \phi'_2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (3.6) \]

where \( \phi'_2 \) is the instantaneous roll angle of the gear, \( r_w^{(2)} \) and \( n_w^{(2)} \) are given in Equation (2.34).

During the process of gear meshing, the working parts of the pinion surfaces and gear surfaces must be in continuous tangency, and the fillets of the pinion and gear are
assumed to be not involved in the contact. Therefore the condition of surface continuous
tangency must be satisfied,

\[ r_f^{(1)}(u_1, v_1, \phi_c, \phi'_c) = r_f^{(2)}(u_2, v_2, \phi'_2) \quad (3.7) \]

\[ n_f^{(1)}(u_1, v_1, \phi_c, \phi'_c) = n_f^{(2)}(u_2, v_2, \phi'_2) \quad (3.8) \]

It should be noted that \( |n_f^{(1)}| = |n_f^{(2)}| = 1 \). Thus the above pair of equations provide a
system of 5 independent scalar equations expressed as

\[ f_j(u_1, v_1, \phi_c, \phi'_c, u_2, v_2, \phi'_2) = 0 \quad j = 1 \sim 5 \quad (3.9) \]

Note that there are 6 unknowns in the above 5 equations if the pinion rotation angle \( \phi'_i \) is
given. But for pinion generation, the parameters \( u_1, v_1 \) and \( \phi_c \) are inter-related by the mesh
equation described below, which is equivalent to equation (2.16).

Since the head cutter surface and the pinion workpiece must be in continuous
contact during the cutting process, the equation of meshing (Litvin, 1989) holds

\[ n_{ri}^{(t)} \cdot v^{(wri)} = 0 \quad (3.10) \]

Here, \( v^{(wri)} \) is the relative velocity of the pinion with respect to the head cutter, and \( n_{ri}^{(t)} \) is
unit normal vector of the head-cutter surface. Equation (3.10) indicates that the relative
velocity along the common tangent point between the head-cutter and the pinion being
cut, which is perpendicular to the common normal between these two surfaces. Equation
(3.10) also holds if $\mathbf{v}^{(w\ell)}$ and $\mathbf{n}^{(l)}_{\ell_1}$ are transformed into the cutting machine coordinate system $S_{m1}$ and they are expressed as $\mathbf{v}^{(w\ell)}_{m1}$ and $\mathbf{n}^{(l)}_{m1}$, respectively.

\[
\mathbf{v}^{(w\ell)}_{m1} = \mathbf{v}^{(w\ell)}_{m1} - \mathbf{v}^{(l)}_{m1} = \omega^{(w)}_{m1} (\mathbf{r}^{(l)}_{m1} - \mathbf{A}^{(l)}_{m1}) - \omega^{(l)}_{m1} \times \mathbf{r}^{(l)}_{m1}
\]  

(3.11)

where $\omega^{(w)}_{m1}$ and $\omega^{(l)}_{m1}$ are vectors of angular velocities of the pinion workpiece and the head-cutter expressed in $S_{m1}$, $\mathbf{r}^{(l)}_{m1}$ is the position vector of contact point in $S_{m1}$, and $\mathbf{A}^{(l)}_{m1}$ is the position of $O_n$ in $S_{m1}$ as shown in Figure 3.2, which is similar to Figure 2.3. Again, the

![Figure 3.2: Derivation of the equation of meshing.](image)

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superscripts ‘\( w1 \)’ and ‘\( t1 \)’ indicate that the vectors are represented in \( S_{w1} \) and \( S_{t1} \), respectively, and the subscript ‘\( ml \)’ indicates that the vectors are represented in \( S_{ml} \). And,

\[
A_{ml} = O_{ml} O_a = \begin{bmatrix} 0 & -E_{ml} & X_B & 1 \end{bmatrix}^T
\]
(3.12)

\[
\omega_{ml}^{(w1)} = -\omega_i \begin{bmatrix} \cos \gamma_{ml} & 0 & \sin \gamma_{ml} & 0 \end{bmatrix}^T
\]
(3.13)

\[
\omega_{ml}^{(t1)} = -\omega_c \begin{bmatrix} 0 & 0 & \frac{1}{R_a} & 0 \end{bmatrix}^T
\]
(3.14)

where \( \omega_i \) and \( \omega_c \) are angular velocities of the pinion workpiece and the cradle, respectively. The pinion surface coordinate \( r_{ml}^{(1)} \) in \( S_{ml} \) is

\[
r_{ml}^{(1)}(u_1, v_1, \phi_c) = [M_{mc}][M_{cb}][M_{bt_i}]r_{li}^{(i)}
\]
(3.15a)

\[
n_{ml}^{(1)}(u_1, v_1, \phi_c) = [M_{mc}][M_{cb}][M_{bt_i}]n_{li}^{(i)}
\]
(3.15b)

Substituting \( r_{li}^{(1)} \) and \( n_{li}^{(i)} \) from Equations (2.8) into Equation (3.15) produces

\[
r_{ml}^{(1)}(u_1, v_1, \phi_c) = [M_{ml1}]
\begin{bmatrix}
  r_{cp} \cos v_1 \\
  r_{cp} \sin v_1 \\
  0 \\
  1
\end{bmatrix}
+ U_p \cdot [M_{ml1}]
\begin{bmatrix}
  \sin \phi_{bp} \cos v_1 \\
  \sin \phi_{bp} \sin v_1 \\
  -\cos v_1 \\
  0
\end{bmatrix}
\]
(3.16)

\[
n_{ml}^{(1)}(u_1, v_1, \phi_c) = [M_{mc}][M_{cb}][M_{bt_i}]n_{li}^{(i)} = \begin{bmatrix} n_x & n_y & n_z & 0 \end{bmatrix}^T
\]
(3.17)
Equation (3.9) with 5 nonlinear algebraic equations can then be solved numerically using a standard iterative method. In this study, a modified Powell-Hybrid algorithm is used, in which a forward-difference approximation to the Jacobian matrix is applied. The solution will determine the relationships between the roll angles of the pinion and gear members, path of contact, and transmission error as well as meshing position and normal vectors for any angular position.

Then, the kinematic transmission error via pinion is expressed as

$$\Delta \phi_2' (\phi') = (\phi_2' - \phi_2'') - \frac{N_1}{N_2} (\phi_1' - \phi_2'')$$  \hspace{1cm} (3.18)

where $\phi_1''$ and $\phi_2''$ are initial roll angles of the pinion and gear at a mean point. The contact path on the gear surface can be found by $r_f^{(2)} (u_2 (\phi'), v_2 (\phi'))$ and the line-of-action is given by $n_f^{(1)} (u_1 (\phi'), v_1 (\phi'), \phi')$.

### 3.3 Pitch Cone Design Process

The CAPP and the above mesh simulation results can readily provide mesh characteristics such as the contact position and normal vectors for the subsequent dynamic modeling. However, it is simpler to use only gear blank design data to get the time-invariant mesh position and normal vector necessary for the dynamic study. This approach does not distinguish between the different cutting methods such as face-milled and face-hobbed.
In pitch cone design, the pitch point and its normal vector, which are essentially the effective or mean mesh position and line-of-action, will be derived considering the basic kinematics of hypoid gearing. This produces the relationship between these vectors and the gear blank design parameters. There are two general approaches in seeking this relationship. One is the Gleason's "trial and error" technique (Wildhaber, 1946; Baxter, 1961; Gleason Works, 1971; Shtipelman, 1978) and the other is the formulation published by Litvin et al. (1990). Both are used here as the basis of the present derivation, but it is extended to include the ability to simulate all possible operating conditions.

The relative motion between a hypoid pinion and gear is similar to a screw type mechanism that translates and revolves about an instantaneous axis. Rotations of the instantaneous axis about the respective axes of the pinion and gear will form two hyperboloids of revolution, which are the kinematic pitch surfaces of the hypoid gear pair. In the design of hypoid gears, however, the surfaces of both pitch cones are used as the pitch surfaces, provided such cones satisfy the following requirements: (i) the axes form the prescribed shaft angle and the shortest linear distance between two axes equals the hypoid offset $E$; (ii) the pitch cones are in tangency at the pitch point where a common contact normal exists; and (iii) the relative velocity coincides with the tangent to the helices of contacting pitch cones (Litvin, 1989; Litvin et al., 1990; Shtilpelman, 1978). The pitch cones are characterized by the spiral angles, denoted by $\beta_p$ and $\beta_G$, and pitch angles, symbolized by $\gamma_p$ and $\gamma_G$, as shown in Figure 3.3, where subscripts $P$ and $G$ refer to the pinion and gear components respectively, and the shaft angle is assumed to be always $90^\circ$. In the subsequent dynamic modeling, the theoretical pitch point denoted by
‘m’ and its corresponding surface normal line segment are assumed to be the mesh point and line-of-action respectively. Their derivations are given next.

By applying the kinematic conditions of hypoid gears described above, the pitch point position vector and its normal line segment can be derived analytically. Consider three distinct coordinate systems defined in Figures 3.3 and 3.4. The first two are represented by $X_p$-$Y_p$-$Z_p$ and $X_G$-$Y_G$-$Z_G$, which are denoted as $S_L$ ($L$=P for pinion and $L$=G for gear). Their origins are situated at the pinion and gear pitch apexes. The third coordinate system $X_f$-$Y_f$-$Z_f$, denoted as $S_f$, is fixed to the reference frame that supports the geared rotor system. Note that $Z_f$ is in the direction that has the shortest distance between the gear rotational axes. Their assumed rotational directions are about $Y_L$. The position vector $R_f$ of ‘m’ in $S_f$ can be expressed as

$$R_f = x_f i_f + y_f j_f + z_f k_f$$  \hspace{1cm} (3.19)

where

$$x_f = R_G \tan \gamma_G - E \sin \gamma_G / \sqrt{\cos^2 \gamma_p - \sin^2 \gamma_G}$$  \hspace{1cm} (3.20a)

$$y_f = R_G \sin \gamma_p / \cos \gamma_G$$  \hspace{1cm} (3.20b)

$$z_f = E - R_G \sqrt{\cos^2 \gamma_p - \sin^2 \gamma_G} / \cos \gamma_G$$  \hspace{1cm} (3.20c)

In the above equations, $R_G$ is the gear pitch cone radius, and $\gamma_p$ and $\gamma_G$ represent the pitch angles of the pinion and gear respectively. The vector in Equation (3.19) can further be normalized into a unit vector given by $r_f = x_m i_f + y_m j_f + z_m k_f$. The vector $r_f$ is related

50
Pitch plane and spiral angles

Figure 3.3: Hypoid gear coordinate systems $S_f$ and $S_l$ ($l=P,G$) used in tooth contact simulation. Distance $E$ defines the pinion offset, $\Delta H$ is the pitch apex $O_P$ beyond cross point $O_F$, and axes $Y_P$ and $Y_G$ represent rotation centerlines for the pinion and gear respectively.
Figure 3.4: Relationship between the local coordinate systems $S_L$ where $L=P, G$, and the fixed coordinate system $S_f$.

...to the other two local coordinate systems in Figure 3.4 via $r_L = [M_{fl}]^{-1}r_f$, where $[M_{fl}]$ the transformation matrix between $S_f$ and $S_l$.

Next, the normal vector can be conveniently defined using an intermediate coordinate system formed by vector $v_P$ and $v_G$, and their corresponding orthogonal vector as suggested by Litvin et al. (1990). It is noted that the common tangents to the tooth longitudinal shapes of the pinion and gear form the spiral angles, $\beta_P$ and $\beta_G$, with the generatrix of its respective pitch cone. The pressure angle $\phi$ is defined in the normal...
plane that is perpendicular to the tooth surface and pitch plane. Applying a series of transformations, \( n_f \) can be expressed in \( S_f \) as

\[
n_f = n_{xf} i_f + n_{yf} j_f + n_{zf} k_f
\]

where the coefficients of orthogonal elements are

\[
n_{xf} = \cos \theta_p \cos \gamma_p \sin \phi - \kappa_1 \sin \theta_p \cos \beta_p \cos \phi + \kappa_2 \sin \gamma_p \cos \theta_p \sin \beta_p \cos \phi \quad (3.22a)
\]

\[
n_{yf} = -\sin \gamma_p \sin \phi + \kappa_2 \cos \gamma_p \sin \beta_p \cos \phi \quad (3.22b)
\]

\[
n_{zf} = \sin \theta_p \cos \gamma_p \sin \phi + \kappa_1 \cos \theta_p \cos \beta_p \cos \phi + \kappa_2 \sin \theta_p \sin \gamma_p \sin \beta_p \cos \phi \quad (3.22c)
\]

and where \( \theta_p = \cos^{-1}(-\sin \gamma_c / \cos \gamma_r) \). The coefficient \( \kappa_1 \) and \( \kappa_2 \) are used to distinguish specific geometrical configuration and/or operating condition of the gear set. Therefore, these 2 coefficients essentially provide generalization to this theory. Similarly, \( n_f \) can also be related to other local coordinate system \( S_L \) by

\[
n_L = [M_{FL}]^{-1} n_f = n_{xL} i_L + n_{yL} j_L + n_{zL} k_L \quad (3.23)
\]

where \( i_L, j_L \) and \( k_L \) are the triad of unit vectors of \( S_L \), and \([M_{FL}]\) is the transformation matrix relating \( S_f \) and \( S_L \). There are four possible cases for a generic pair of hypoid gear set: right-hand (RH) spiral pinion with clockwise (CW) or counter-clockwise (CCW) rotation; and left-hand (LH) spiral pinion with CW or CCW rotation. The rotational direction of the pinion will determine the specific tooth side that engages in mesh. Note that the definition of pinion rotation is based on viewing the motion from the larger end of the pinion. Table 3.1 depicts the coefficients and pressure angles for typical automotive driveline applications. Here \( \phi_c \) and \( \phi_r \) represent the pressure angles of the
concave and convex sides of the pinion respectively. It may be noted that these configurations assume the pinion to be below the gear axis, and it is customary to look at the face of the gear with the pinion on the right in determining the direction of offset.

The above mesh position and normal vector are related to the basic hypoid gear blank design data. The following section presents the procedure in computing these parameters.

<table>
<thead>
<tr>
<th>LH pinion with RH gear</th>
<th>Drive</th>
<th>Coast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward</td>
<td>$\kappa_1=1$, $\kappa_2=1$, $\phi=\phi_c$</td>
<td>$\kappa_1=-1$, $\kappa_2=-1$, $\phi=\phi_v$</td>
</tr>
<tr>
<td>Reverse</td>
<td>$\kappa_1=-1$, $\kappa_2=-1$, $\phi=\phi_v$</td>
<td>$\kappa_1=1$, $\kappa_2=1$, $\phi=\phi_c$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>RH pinion with LH gear</th>
<th>Drive</th>
<th>Coast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward</td>
<td>$\kappa_1=1$, $\kappa_2=-1$, $\phi=\phi_v$</td>
<td>$\kappa_1=-1$, $\kappa_2=1$, $\phi=\phi_c$</td>
</tr>
<tr>
<td>Reverse</td>
<td>$\kappa_1=-1$, $\kappa_2=1$, $\phi=\phi_c$</td>
<td>$\kappa_1=1$, $\kappa_2=-1$, $\phi=\phi_v$</td>
</tr>
</tbody>
</table>

Table 3.1: Normal vector coefficients and pressure angles for different operating conditions in typical automotive driveline applications.

### 3.4. Pitch Cone Design Parameters

The pitch cone design requires observing certain kinematic conditions for proper meshing without undercutting. These conditions will be given here without detailed
derivation (see Wildhaber, 1946; Litvin et al., 1990). The equation of meshing at the pitch point is governed by

$$f_i(\beta_G, \gamma_p, \gamma_G) = \frac{N_2}{N_1} \frac{\cos \beta_G}{\cos \beta_p} \left[ \frac{E \cos \gamma_p}{R_2 \sqrt{\cos^2 \gamma_p - \sin^2 \gamma_G}} - \frac{\cos \gamma_p}{\cos \gamma_G} \right]$$ (3.24)

Next, from the relationship of longitudinal shape of the gear tooth, as shown in Figure 3.3, the following equation is observed

$$f_2(\beta_G, \gamma_p, \gamma_G) = \cos(\beta_p - \beta_G) - \tan \gamma_p \tan \gamma_G$$ (3.25)

One goal of the pitch cone design is to avoid undercutting. This condition depends on the curvature of hypoid gear tooth lengthwise line on the pitch plane and pressure angle, and is expressed as

$$f_3(\beta_G, \gamma_p, \gamma_G) = \frac{\tan \beta_p - \tan \beta_G}{\cos \phi_0} - R_{a2} \left( \frac{- \sin \gamma_p}{R_p \cos \beta_p} - \frac{- \sin \gamma_G}{R_G \cos \beta_G} \right)$$

$$- \tan \phi_0 \left( \frac{\tan \beta_p \cos \gamma_p}{R_p} + \frac{\tan \beta_G \cos \gamma_G}{R_G} \right)$$ (3.26)

where $R_{a2}$ is the gear mean cutter radius, and $\phi_0$ is the limit pressure angle for a face-milled hypoid gear. The function $f_j (j=1,2,3)$ constitutes a set of nonlinear algebraic equations with three unknown variables $\beta_G$, $\gamma_p$, and $\gamma_G$. Given the input data of $E$, $\beta_p$, $D_2$, $R_{a2}$, $N_1$ and $N_2$, this set of non-linear equations can be solved numerically. In this study, a modified Power Hybrid algorithm. It should be noticed from Equations (3.26) and (3.27)
that the selection of variables is subjected to the constraints: \( \tan \gamma_p \tan \gamma_G \leq 1 \) and 
\[ \cos^2 \gamma_p - \sin^2 \gamma_G \geq 0. \]

### 3.5 Case Studies

A hypoid gear set used in a rear axle is applied here as an example to demonstrate the above proposed mesh computation. Table 3.2 is the gear design data including the blank data and machine tool settings for the pinion and gear. The kinematic transmission error obtained by the above simulation is shown in Figure 3.5 along with the result of Litvin and Gutman (1981). The negative transmission error suggests that the gear lag behind its theoretical position. The contact path is mapped into a plane that is formed by the root cone generatrix and the axis perpendicular to this generatrix and passes the mean contact point, and is shown in Figure 3.6. From the patterns of the transmission error and contact path, we can potentially optimize the gear machine setting or design parameters to minimize the transmission error and get the designed contact pattern, and this is left as further study.

Table 3.3 lists the comparison of the mesh position vector and the normal vector obtained using the CAPP code under a light load, tooth contact simulation and pitch cone design approaches. It suggests that the three methods can provide close solution for the mean mesh position and the normal vector. Figure 3.7 is the comparison of the transmission error under unloaded case obtained by the mesh simulation and CAPP. The disparity between the two may result from the difference in geometrical description.
(a) Blank data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>10 (pinion)</td>
</tr>
<tr>
<td></td>
<td>43 (gear)</td>
</tr>
<tr>
<td>Face width (mm)</td>
<td>48</td>
</tr>
<tr>
<td>Pinion offset (mm, below)</td>
<td>31.75</td>
</tr>
<tr>
<td>Mean cone distance (mm)</td>
<td>152.14</td>
</tr>
<tr>
<td>Gear face angle</td>
<td>1.2834</td>
</tr>
<tr>
<td>Gear root angle</td>
<td>1.2322</td>
</tr>
<tr>
<td>Gear addendum (mm)</td>
<td>3.41</td>
</tr>
<tr>
<td>Gear dedendum (mm)</td>
<td>10.42</td>
</tr>
</tbody>
</table>

(b) Gear machine & cutter setting

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine root angle</td>
<td>1.2287</td>
</tr>
<tr>
<td>Machine center to back (mm)</td>
<td>1.270</td>
</tr>
<tr>
<td>Horizontal setting (mm)</td>
<td>85.598</td>
</tr>
<tr>
<td>Vertical setting (mm)</td>
<td>96.177</td>
</tr>
<tr>
<td>Cutter blade angle</td>
<td>0.3927</td>
</tr>
<tr>
<td>Nominal radius (mm)</td>
<td>114.30</td>
</tr>
<tr>
<td>Point width (mm)</td>
<td>3.81</td>
</tr>
</tbody>
</table>

(c) Pinion machine & cutter settings

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutter blade angle</td>
<td>0.3491</td>
</tr>
<tr>
<td>Machine root angle</td>
<td>-0.0226</td>
</tr>
<tr>
<td>Machine center to back (mm)</td>
<td>-4.5847</td>
</tr>
<tr>
<td>Point radius (mm)</td>
<td>108.450</td>
</tr>
<tr>
<td>Basic swivel angle</td>
<td>-0.7046</td>
</tr>
<tr>
<td>Radial setting (mm)</td>
<td>118.513</td>
</tr>
<tr>
<td>Basic cradle angle</td>
<td>1.0614</td>
</tr>
<tr>
<td>Blank offset (mm)</td>
<td>24.542</td>
</tr>
<tr>
<td>Sliding base (mm)</td>
<td>18.242</td>
</tr>
<tr>
<td>Ratio of roll</td>
<td>3.9936</td>
</tr>
</tbody>
</table>

Table 3.2: Hypoid gear design parameters.
Figure 3.5: Static transmission error as function of pinion roll angle calculated from TCA simulation for concave pinion meshing.

Figure 3.6: Projection of contact path on the gear surface at the convex side at gear mean position.
<table>
<thead>
<tr>
<th>Method</th>
<th>$x_f$</th>
<th>$y_f$</th>
<th>$z_f$</th>
<th>$n_{sf}$</th>
<th>$n_{yf}$</th>
<th>$n_{zf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pitch cone equation</td>
<td>-40.31</td>
<td>142.31</td>
<td>2.532</td>
<td>-0.1896</td>
<td>-0.7468</td>
<td>-0.6474</td>
</tr>
<tr>
<td>Tooth contact simulation</td>
<td>-40.32</td>
<td>140.51</td>
<td>2.702</td>
<td>-0.1797</td>
<td>-0.7448</td>
<td>-0.6426</td>
</tr>
<tr>
<td>CAPP prediction</td>
<td>-40.39</td>
<td>139.82</td>
<td>1.341</td>
<td>-0.1818</td>
<td>-0.7520</td>
<td>-0.6326</td>
</tr>
</tbody>
</table>

Table 3.3: Comparison of the position and normal vectors of the effective mesh point obtained from pitch cone equation, tooth mesh simulation and CAPP for meshing between concave side of the pinion and convex side of the gear.

![Graph](image)

Figure 3.7: Comparison of unloaded transmission error obtained by mesh simulation and CAPP prediction.
Figure 3.8: Comparison of contact path on tooth surface for the present simulation and CAPP.
in the two methods. Figure 3.8 shows the contact path comparison of the two methods, which indicates that there are certain differences in the contact locations on the tooth surface. These differences in contact path are thought to cause the differences in the transmission error calculation. Further investigation would be necessary to understand the causes of the difference in these two methods.

3.6 Summary

An algorithm is developed to predict the unloaded kinematic transmission error and contact path, which is essential to the design of the hypoid gear. The mean mesh position vector and the normal vector obtained using the CAPP code, TCA, and pitch cone design approaches are reasonably close. Furthermore, an exact spatial mesh position and force vector are derived explicitly and can be used for face-milled and face-hobbed hypoid gear under different operating conditions such as forward and coast driving conditions.
CHAPTER 4

HYPOID GEARED ROTOR SYSTEM VIBRATION MODEL

4.1 Introduction

In parallel axis gears, the line-of-action is generally time-invariant as defined by the pressure and helix angles, because the normal vectors at the theoretical contact points of each pair of teeth in mesh are co-linear. However, this is not the case in hypoid gears. Due to the inherent geometry, the resultant mesh force vector varies as the hypoid gears rotate over the mesh cycle. This is because the normal lines of action of each pair of teeth in contact are neither co-linear nor parallel. This produces a net force transmission that oscillates periodically. Moreover, tooth modifications in hypoid gears are usually performed in both the longitudinal and profile directions to reduce the sensitivity to errors due to installation and manufacturing processes. This further causes contact position and mesh force vector to perturb more from the theoretical mesh configuration. Similar effect can be expected from the resultant friction excitation that also behaves periodically.

From the open gear literature, one can find only a few analytical studies on hypoid gear vibrations (Remmers, 1971; Pitts, 1972; Kiyono et al., 1981; Nakayashiki, et al., 1983; Abe and Hagiwara, 1990), even though the dynamics of parallel axis gears have been investigated extensively as reported by Özgüven and Houser (1988), Kahraman and Singh (1990), Blankenship and Singh (1995) and Velex and Maatar (1996). Of the few
studies that exist on hypoid gear dynamics, many actually ignored the excitation of transmission error. The most practical model prior to the present study was suggested by Donley et al. (1992) who proposed an approximate hypoid gear mesh formulation for use in the context of a linear time-invariant dynamic finite element representation. Their approximate formulation relies on the assumption of a mean pitch point location and is based on a bevel gear mesh equivalence theory. Also, none of the above hypoid gear dynamics studies clearly defines the mesh coupling precisely, and the models essentially rely on simplified mesh force vectors. Hence, it is quite clear that a more precise hypoid gear mesh formulation, which should be directly related to the gear design parameters, is needed. The model is required to simulate the coupled translation and rotation vibrations simultaneously.

It is widely accepted that the transmission error excitation is the major source of gear whine noise, and many theoretical and experimental studies have been directed towards gaining a better understanding of the fundamental phenomenon. Although unloaded kinematic transmission error is found to be related to the gear noise problem, the loaded transmission error (LTE) is believed to be more closely correlated to gear noise, because of the effects of tooth deflection and load sharing phenomena as described by References (Gosselin et al., 1995; Tavakoli and Houser, 1986; Özcüven and Houser, 1988, 1991). In the present study, the approach introduced by Özcüven and Houser (1988(b)) in the non-linear dynamic model for spur gears is used to derive the model with loaded transmission error excitation.
4.2 Objectives

Based on the meshing kinematics of hypoid gears presented in the previous chapters, a generalized dynamic model that consists of a hypoid gear drive with compliant input and output elements will be formulated. The purpose of this chapter is to develop a new multi-degrees-of-freedom non-linear time-varying (NLTV) mesh model for hypoid gears based on the simulation of loaded gear meshing using a 3-dimensional contact analysis program CAPP. In this analysis, a 3-dimensional elastically coupled lumped parameter model will be developed subject to LTE excitation. Loaded time-varying mesh vectors, time- and spatial-varying mesh stiffness, loaded transmission error, friction forces, and backlash non-linearity will be incorporated into the derivation.

4.3 Gear Mesh Modeling

One of the most difficult tasks associated with 3-dimensional modeling of bevel and hypoid gear sets is to accurately describe the gear surface geometry. A mesh generator for spiral and hypoid gears has been developed based on simulation of the cutting process of both pinion and gear members, as introduced in Chapter 2. When machine and cutter settings are given, the kinematic motion of the manufacturing process can be simulated to obtain the coordinates of convex and concave flanks of teeth. Using the Contact Analysis Program Package (CAPP) that is based on finite element and surface integral methods, a 3-dimensional tooth contact mechanics calculation is performed. The CAPP employs a Simplex type algorithm (developed by Vijayakar, 1987) to simulate the elastic tooth contact problem. From the results generated by this code, the
time-varying mesh position, line-of-action, loaded transmission error, mesh stiffness, and friction load distributions are obtained using the concept of contact cells. As it is shown in Chapter 3, the pitch cone-based time-invariant and linear meshing case that assumes constant mesh position and line-of-action vectors is quite close to the results derived from CAPP. Hence, the pitch cone design mesh can be used to generate the linear time-invariant dynamic model, which simplifies the computational process significantly.

In this approach, the contact areas of the gear teeth are discretized into a series of smaller cells. Each cell contains a localized compliance $c_{ij}$ that is a function of the spatial dimensions, gear mesh position and applied mean torque. The position vector of the contact cell $i$ in the coordinate system $S_l$ ($l=1,2$ for pinion and gear respectively) is $r_i^{(l)} = \{x_i^{(l)}, y_i^{(l)}, z_i^{(l)}\}^T$, and the unit normal vector is given by $n_i^{(l)} = \{n_{ix}^{(l)}, n_{iy}^{(l)}, n_{iz}^{(l)}\}^T$, as shown in Figure 4.1. The projection of the unit normal vector into the tangential direction of rotational motion relative to $S_l$ coordinate system represented by $X_l, Y_l$ and $Z_l$ axes can be expressed as

\begin{align}
\chi_{x}^{(l)} &= n_{ix}^{(l)} \cdot (i^{(l)} \times r_i^{(l)}) = n_{ix}^{(l)} y_i^{(l)} - n_{iy}^{(l)} z_i^{(l)} \\
\chi_{y}^{(l)} &= n_{iy}^{(l)} \cdot (j^{(l)} \times r_i^{(l)}) = n_{iy}^{(l)} z_i^{(l)} - n_{iz}^{(l)} x_i^{(l)} \\
\chi_{z}^{(l)} &= n_{iz}^{(l)} \cdot (k^{(l)} \times r_i^{(l)}) = n_{iz}^{(l)} x_i^{(l)} - n_{ix}^{(l)} y_i^{(l)}
\end{align}

(4.1a) (4.1b) (4.1c)

where $i^{(l)}$, $j^{(l)}$ and $k^{(l)}$ are the triad of unit vectors that defines the coordinate system $S_l$. Hence, the directional cosine of cell $i$ clearly depends on the gear geometry and its actual mesh position. Here, the dimensional mesh parameter $\chi_{iu}^{(l)} (u=x,y,z)$ is referred to as the
directional rotation radius about the \( u \)-axis, which represents the tangential force component at contact point \( i \) for a unit contact force in the normal direction \( n_y^{(i)} \).
The relative sliding velocity between the two meshing cells on the gear surface with respect to the fixed coordinate $S_0$ is given by

$$v_i^{(12)} = v_i^{(1)} - v_i^{(2)} = (\omega^{(1)} - \omega^{(2)}) \times r_i^{(2)} + \omega^{(1)} \times R^{(12)} = \{v_{ix}, v_{iy}, v_{iz}\}^T$$

(4.2)

where $\omega^{(1)}$ and $\omega^{(2)}$ are vectors of angular velocity of gear members 1 (pinion) and 2 (driven gear), which can be expressed explicitly as $\omega^{(1)} = \{0, 0, \omega_1\}^T$ and $\omega^{(2)} = \{0, \omega_2, 0\}$. In the above equation, $R^{(12)}$ is the radius vector from the pinion apex $O_1$ to an arbitrary point on the directed line segment of $\omega^{(2)}$, and given by $R^{(12)} = \{-E, 0, 0\}^T$ where $E$ is vertical mounting distance of the pinion. Furthermore, the relative sliding velocity vector with respect to $S_0$ may be transformed into the local coordinate system $S_l$ by $\{v_i^{(1)}\} = [M_{l0}] \cdot \{v_i^{(12)}\} = \{v_{ilx}, v_{ily}, v_{ilz}\}^T$. Projection of the relative sliding velocity vector in the tangential direction of rotational motion relative to $X_l, Y_l$ and $Z_l$ can be shown to be

$$\tau_{ix}^{(l)} = v_{ilx}^{(l)} - v_{ily}^{(l)} z_i^{(l)}$$

(4.3a)

$$\tau_{iy}^{(l)} = v_{ilx}^{(l)} z_i^{(l)} - v_{ilx}^{(l)} x_i^{(l)}$$

(4.3b)

$$\tau_{iz}^{(l)} = v_{ilx}^{(l)} x_i^{(l)} - v_{ilx}^{(l)} y_i^{(l)}$$

(4.3c)

Parameter $\tau_{ix}^{(l)}$ can be regarded as the tangential friction force component at contact point $i$ for a unit friction force in the sliding direction $v_i^{(l)}$.

The loaded transmission error (LTE) is resulted from the tooth errors and deflections due to the base rotation, bending, shearing and contact deformation.
Assuming that the pinion and gear rotate about their respective Y-axes, their contact regions can be divided into \( N_c \) cells as shown in Figure 4.2. Here, \( N_c \) is dependent on load and angular position. Since the rotations of all the simultaneously contacting cells are the same under load because of load sharing compatibility (Krenzer, 1986; Gosselin et al., 1995; Tavakoli and Houser, 1986; Vijayakar, 1987), the following expression for the equilibrium state of gear relative rotation (also known as the LTE of the pinion assuming fixed gear) can be derived (CAPP),

\[
\Delta \theta_L \{A_i\}^T - [C_\delta]\{p\} - \{E_0\}^T = 0
\]  

(4.4)

where \( \{A_i\} = \{\lambda_{1y}^{(i)} \lambda_{2y}^{(i)} ... \lambda_{N_{cy}}^{(i)}\} \) is a column vector of dimension \( N_c \), which represents the increase in separation between the mating gears at individual cell position due to the angular displacement \( \Delta \theta_L \). The compliance matrix \([C_\delta]\) represents the net compliance due to normal and frictional loads in all cells. The initial gear teeth separation is given by \( E_0^T = \{\epsilon_{01} \epsilon_{0N_c}\}^T \). Multiply \( \{A_i\} [C_\delta]^{-1} \) to both sides of Equation (4.4) and note the moment balance equation

\[
([A_i] - \mu\{T_i\})\{p_i\} = M_1
\]  

(4.5)

where \( M_1 \) is the torque applied on the pinion, \( \mu \) is the friction coefficient, and \( \{T_i\} = \{\tau_{1y}^{(i)} \tau_{2y}^{(i)} ... \tau_{N_{cy}}^{(i)}\} \) is the radius vector of the friction load. One can get the
Figure 4.2: Contacting cells and load distributions at gear surface.
angular displacement deviation or LTE

\[
\Delta \theta_L = \frac{M_1 - (\{A_1\} - \mu(\{T_1\})[C_\delta]^{-1}\{E_0\}^T}{[\{A_1\} - \mu(\{T_1\})][C_\delta]^{-1}\{A_1\}^T}
\]

(4.6a)

Alternately using the geometry and structure in the gear coordinate \(S_2\), the LTE can also be expressed as

\[
\Delta \theta_L = \frac{M_2 - (\{A_2\} + \mu(\{T_2\})[C_\delta]^{-1}\{E_0\}^T}{(\{A_2\} + \mu(\{T_2\})[C_\delta]^{-1}\{A_2\}^T}
\]

(4.6b)

where the output torque \(M_2 = \{A_2\} + \mu(\{T_2\})\{p_i\}\) is a variable, and depends on the instantaneous transmission ratio and friction force. LTE is normally periodic with mesh frequency, and is a function of gear rotation position and applied load. It can be expressed in the Fourier expansion form as

\[
\Delta \theta_L(\theta) = e_0 + \sum_{r=1}^{\infty} (e_r \cos(r \omega_m (\theta - \theta_0)) + (e_n \sin(r \omega_m (\theta - \theta_0))), where \omega_m is fundamental gear mesh frequency and \theta_0 is initial position angle of the pinion.
\]

4.4 Normal Load and Friction Force Distributions

Under loaded condition, several teeth may be in contact simultaneously and the transmitted load is shared among them. The corresponding normal tooth loads and friction forces predicted by CAPP are shown in Figure 4.2. The contact areas are divided into smaller cells, and each cell is subjected to a pair of normal and friction loads. The friction force is assumed to be Coulomb and is computed from \(F_{fi}(t) = p_1 \cdot \mu(t) \frac{\vec{v}_i^{(1)}}{\|\vec{v}_i^{(1)}\|}\).
The stiffness $k_j$ associated with each cell $j$ denotes the force needed at cell $j$ for a unit displacement at cell $i$. This allows for the influence of the cross compliance between the cells to be incorporated in the analysis.

It may be pointed out that the time-varying variables $n_{iu}^{(1)}, v_{iu}^{(1)}, \dot{a}_i^{(1)}$ and $\tau_i^{(1)}$ are direct representations of the mesh characteristic that includes the vectors of the normal force, friction force and their moments. Thus, they implicitly describe the translation-rotational and rotational-rotational force coupling characteristics. Next, these generalized force and moment vectors will be applied in the derivation of the dynamic model.

4.5 Backlash Non-linearity

Most gear pairs generally contain a certain degree of backlash, in order to achieve better lubrication and eliminate interference. The presence of backlash may induce tooth separation and impacts in unloaded or lightly loaded geared drives, and such impacts typically cause intensive vibration and noise problems and larger dynamic loads that negatively affect durability (Dudley, 1984). The gear backlash is considered a non-linear mesh function as shown in Figure 4.3, and its dynamic behavior has been examined by a number of investigators for parallel gears (Comparin and Singh, 1989; Kahraman and Singh, 1990; Özgüven and Houser, 1988). In the present dynamic study for hypoid gears, this type of non-linearity effect will be examined.
4.6 Dynamic Model

Consider a generic driveline system comprising of a hypoid gear pair, an engine inertia and a load element as shown in Figure 4.4. Each gear is modeled as a rigid conical body attached to a torsionally flexible shaft, which is supported by a compliant rolling element bearing represented by a set of discrete stiffness and damping elements (Lim and Singh, 1990). The nominal rotations of the pinion and gear are about $Y_1$ and $Y_2$ respectively. Furthermore, only the torsional coordinates of the engine and load are modeled as their translation coordinates are normally decoupled from those of the gears by design.

To mathematically define the representation of the mesh coupling stiffness and force distributions, the contact areas on the tooth surfaces are discretized into a series of
Figure 4.4: Schematic of a hypoid geared rotor system.
cells as described earlier. The mesh vectors, such as contact position and force vectors, under the dynamic condition are assumed to be the same as those for the static condition. In other words, the dynamic vibratory motions are assumed not to perturb the normal load distribution, friction load distribution, and line of action. This approach has also been used in the previous studies on parallel gear dynamics (Kahraman and Singh, 1990; Özgüven and Houser, 1988(b); Blankenship and Singh 1995).

The net moments acting on the pinion and gear are due to the normal and the friction forces. For the pinion, the moments about the $Y_f$-axis resulting from the normal and friction loads are given by

$$M_{\bar{y}}^{(1)} = \sum_{i}^{N_c} \sum_{j}^{N_c} \lambda_{iy}^{(2)} k_{ij} \delta_j = \sum_{i}^{N_c} \sum_{j}^{N_c} \lambda_{iy}^{(1)} k_{ij} \left( \lambda_{iy}^{(1)} \theta_2 - \lambda_{iy}^{(1)} \theta_1 + \epsilon_{ij} \right) = \{ A_i \} \{ C_\delta \}^{-1} \{ A_2 \}^T \theta_2 - \{ A_1 \}^T \theta_1 + \{ E_0 \}^T$$

respectively, where $\theta_1$ and $\theta_2$ correspond to the angular displacements of the pinion and gear, and the compliance matrix $[C_\delta]$ of dimension $N_c$ is computed using the CAPP program. Hence the total moment acting on the pinion and gear consists of the moment by the normal forces given by Equation (4.7) and the moment by the friction forces given by Equation (4.8). The governing equations for the torsional motion of a gear pair thus can be obtained as
\[ I_1 \ddot{\theta}_1 - (\{A_1\} - \mu(T_1\})[C_d]^{-1}(\{A_2\}^T \theta_2 - \{A_1\}^T \theta_1 + \{E_0\}^T) \\
- (\{A_1\} - \mu(T_1\})[C_d](\{A_2\}^T \dot{\theta}_2 - \{A_1\}^T \dot{\theta}_1 + \{E_0\}^T) = M_1 \]  \tag{4.9a}

\[ I_2 \ddot{\theta}_2 + (\{A_2\} - \mu(T_2\})[C_d]^{-1}(\{A_2\}^T \theta_2 - \{A_1\}^T \theta_1 + \{E_0\}^T) \\
+ (\{A_2\} + \mu(T_2\})[C_d](\{A_2\}^T \dot{\theta}_2 - \{A_1\}^T \dot{\theta}_1 + \{E_0\}^T) = M_2 \]  \tag{4.9b}

where \([C_d]\) is damping matrix. The input torque \(M_1\) is assumed to be constant, while the output torque \(M_2\) is variable and depends on the instantaneous transmission ratio and friction force as well. Substitution of the LTE expressed in Equation (4.6) into Equation (4.9) leads to

\[ I_1 \ddot{\theta}_1 - (\{A_1\} - \mu(T_1\})[C_d]^{-1}(\{A_2\}^T \theta_2 - \{A_1\}^T \theta_1 + \{A_1\}^T \Delta \theta_L) \\
- (\{A_1\} - \mu(T_1\})[C_d](\{A_2\}^T \dot{\theta}_2 - \{A_1\}^T \dot{\theta}_1 + \{A_1\}^T \Delta \dot{\theta}_L) = 0 \]  \tag{4.10a}

\[ I_2 \ddot{\theta}_2 + (\{A_2\} + \mu(T_2\})[C_d]^{-1}(\{A_2\}^T \theta_2 - \{A_1\}^T \theta_1 + \{A_1\}^T \Delta \theta_L) \\
+ (\{A_2\} + \mu(T_2\})[C_d](\{A_2\}^T \dot{\theta}_2 - \{A_1\}^T \dot{\theta}_1 + \{A_1\}^T \Delta \dot{\theta}_L) = 0 \]  \tag{4.10b}

A similar procedure is also to be used to establish the higher order dynamical equations given below.

In order to improve computational efficiency and simplify the modeling process, the equivalent forces and moments will be used in the subsequent dynamic study. Thus the above equations are used to seek the averaged equivalent mesh characteristics as a function of angular position. To do so, consider the resultant normal force \(F_{n_{iu}}^{(i)}\) and friction force \(F_{f_{iu}}^{(i)}\) along the u-axis where \(u = x, y\) or \(z\):

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\begin{align}
F_{\delta i}^{(i)} &= \sum_{i}^{N_i} \sum_{j}^{N_j} n_{i u}^{(i)} k_{j} \delta_{j} = N_{i}^{(i)} [C_{\delta}]^{-1} A_{\delta} = n_{u}^{(i)} W_{0} \\
F_{\mu \nu}^{(i)} &= \sum_{i}^{N_i} \sum_{j}^{N_j} \mu \nu_{i u}^{(i)} k_{j} \delta_{j} = \mu V_{i u}^{(i)} [C_{\delta}]^{-1} A_{\delta} = \mu V_{u}^{(i)} W_{0}
\end{align}  \tag{4.11a}
\tag{4.11b}

where \( \delta_{j} \) is the deformation of cell \( j \), \( A_{\delta} = [\delta_{1} \delta_{2} \ldots \delta_{N_{\delta}}]^{T} \), \( N_{\mu}^{(i)} = \{ n_{1 u}^{(i)} n_{2 u}^{(i)} \ldots n_{N_{\mu} u}^{(i)} \}^{T} \), \( V_{\mu}^{(i)} = \{ v_{1 u}^{(i)} v_{2 u}^{(i)} \ldots v_{N_{\mu} u}^{(i)} \}^{T} \), \( [C_{\delta}]^{-1} A_{\delta} \) is the normal force acting on the gear mesh interface, and \( W_{0} \) is the equivalent static normal load acting on the meshing teeth and depends on the instantaneous transmission ratio. Equation (4.11) gives the averaged normal and frictional loads by summing the loads associated with the individual cells. Thus \( n_{u}^{(i)} \) and \( v_{u}^{(i)} \) are the equivalent normal and frictional force vectors. Similarly, the resultant moments contributed by the normal and friction forces about the \( u \)-axis are

\begin{align}
M_{\delta u}^{(i)} &= \sum_{i}^{N_i} \sum_{j}^{N_j} \lambda_{i}^{(i)} k_{j} \delta_{j} = A_{\mu}^{(i)} [C_{\delta}]^{-1} A_{\delta} = \lambda_{u}^{(i)} W_{0} \\
M_{\mu \nu}^{(i)} &= \sum_{i}^{N_i} \sum_{j}^{N_j} \mu \tau_{i}^{(i)} k_{j} \delta_{j} = \mu T_{\mu}^{(i)} [C_{\delta}]^{-1} A_{\delta} = \mu \tau_{u}^{(i)} W_{0}
\end{align}  \tag{4.12a}
\tag{4.12b}

where \( A_{\mu}^{(i)} = (\lambda_{1 u}^{(i)} \lambda_{2 u}^{(i)} \ldots \lambda_{N_{\mu} u}^{(i)})^{T} \), \( T_{\mu}^{(i)} = (\tau_{1 u}^{(i)} \tau_{2 u}^{(i)} \ldots \tau_{N_{\mu} u}^{(i)})^{T} \). The parameters \( \lambda_{u}^{(i)} \) and \( \tau_{u}^{(i)} \) are the equivalent directional rotation radius of the normal and friction forces, respectively. From Equation (4.5), the total moment acting on the pinion is \( M_{1} \) given by

\[ M_{1} = M_{\delta y}^{(i)} - M_{\mu y}^{(i)} \]  \tag{4.13}

Therefore, from Equations (4.12) and (4.13), the equivalent static normal load is

\[ W_{0} = M_{1} / (\lambda_{u}^{(i)} - \mu \tau_{u}^{(i)}) \]  \tag{4.14}
Thus $W_0$ is also time-varying and a function of the pinion angular rotational position.

Next, consider the pinion and gear member with 6 degrees-of-freedom (DOF) motions. Each coordinate is given by $\mathbf{q}_i(t) = \{x_i, y_i, z_i, \theta_{il}, \theta_{jl}, \theta_{dl}\}^T$, where $x_i$, $y_i$, and $z_i$ are the translation coordinates, and $\theta_{il}$, $\theta_{jl}$, and $\theta_{dl}$ are the angular coordinates. Since the mesh force and friction force are obtained under the quasi-static condition, the dynamic force and moment expressions are further simplified by using the equivalent mesh vectors derived above. The equivalent dynamic normal and friction forces acting on gear member $i$ are given by

$$ F_{\alpha}^{(i)} = \sum_{i} \sum_{j} n_{a}^{(i)} k_{ij} \delta_j = n_{a}^{(i)} k_{n} (h^{(1)} q_2 - h^{(2)} q_1 + \varepsilon_0) \quad (4.15a) $$

$$ F_{\mu}^{(i)} = \sum_{i} \sum_{j} \mu \nu_{a}^{(i)} k_{ij} \delta_j = \mu \nu_{a}^{(i)} k_{m} (h^{(1)} q_2 - h^{(2)} q_1 + \varepsilon_0) \quad (4.15b) $$

respectively, where $\varepsilon_0$ is translational type of the unloaded kinematic transmission error in the direction of the line-of-action. The equivalent dynamic moments due to normal and friction forces are

$$ M_{\alpha}^{(i)} = \sum_{i} \sum_{j} \lambda_{a}^{(i)} k_{ij} \delta_j = \lambda_{a}^{(i)} k_{n} (h^{(1)} q_2 - h^{(2)} q_1 + \varepsilon_0) \quad (4.16a) $$

$$ M_{\mu}^{(i)} = \sum_{i} \sum_{j} \mu \tau_{a}^{(i)} k_{ij} \delta_j = \mu \tau_{a}^{(i)} k_{m} (h^{(1)} q_2 - h^{(2)} q_1 + \varepsilon_0) \quad (4.16b) $$
respectively. Vector $h^{(i)}$ denotes the mesh vector under certain mesh position and applied pinion torque, and is given by $h^{(i)}(r) = \{n_x^{(i)} n_y^{(i)} n_z^{(i)} \lambda_x^{(i)} \lambda_y^{(i)} \lambda_z^{(i)} \}$. It can be noted that $h^{(i)}$ is time-varying and load-dependent.

Under quasi-static condition, the scalar value of $(h^{(1)} q_2 - h^{(2)} q_1)$ from the torsional gear contact analysis, in which $\{q_1\} = \{\theta_1\}$ and $\{q_2\} = \{\theta_2\}$, is essentially equivalent to the transmission error $e_L$ in the mesh force direction. Thus, from equation (4.15a), the averaged mesh stiffness $k_m$ can be shown to be

$$k_m = \frac{W_0}{\lambda^{(i)} \Delta \theta - e_0} = \frac{W_0}{e_L - e_0} \quad (4.17)$$

where $e_L$ is the translation form of LTE in the mesh force direction. Similar expressions of the mesh stiffness are also used by Özgüven and Houser (1988b) and Blankenship (1992). and Singh 1995

The instantaneous $k_m$ is a function of load, tooth errors, tooth modifications and gear rotation position. Accordingly, the system equations of motion (14 DOF) with LTE excitation present are given by

$$I_e \ddot{\theta}_E + k_{n_1}(\theta_E - \theta_1) + c_{n_1}(\dot{\theta}_E - \dot{\theta}_1) = -T_1 \quad (4.18a)$$

$$[M_1]\{\ddot{\theta}_1\} + (h^{(1)^T} - \mu g^{(1)^T}) f(\delta_d - e_L) + [C_{1b}]\{\dot{\theta}_1\} + [K_{1b}]\{q_1\} = \{F_{ext}^{(1)}\} \quad (4.18b)$$

$$[M_2]\{\ddot{\theta}_2\} - (h^{(2)^T} - \mu g^{(2)^T}) f(\delta_d - e_L) + [C_{2b}]\{\dot{\theta}_2\} + [K_{2b}]\{q_2\} = \{F_{ext}^{(2)}\} \quad (4.18c)$$

$$I_o \ddot{\theta}_O + k_{n_2}(\theta_O - \theta_2) + c_{n_2}(\dot{\theta}_O - \dot{\theta}_2) = -T_2 \quad (4.18d)$$
where, $I_E$ and $I_O$ are the mass moments of inertia of the driver and load, $k_i$ and $k_{i^2}$ are the torsional stiffnesses of the input and output shafts respectively; $c_i$ and $c_{i^2}$ are the input and output shaft damping coefficients; $T_1$ and $T_2$ are the mean torques applied to the driver and load; $[M_l]$, $[K_{lb}]$ and $[C_{lb}]$ are mass matrix, equivalent stiffness and damping matrices of shaft-bearing components, and will be given later; and $\{F_{ex}^{(l)}\}$ is the external load acting on the gear member $l$. The dynamic transmission error, denoted by $\delta_d$, is given by

$$\delta_d = h^{(1)}(q_1) - h^{(2)}(q_2)$$  \hspace{1cm} (4.19)

The time-varying and load-dependent vector for friction force is

$$g^{(l)}(t) = \{v_x^{(l)} \quad v_y^{(l)} \quad v_z^{(l)} \quad \tau_x^{(l)} \quad \tau_y^{(l)} \quad \tau_z^{(l)}\}$$  \hspace{1cm} (4.20)

In equation (4.18), the non-linear function $f(\delta_d - e_L)$ describes the elastic dynamic force that depends upon the operational condition, and can be defined as

$$f(\delta_d - e_L) = \begin{cases} W_0 + k_m (\delta_d - e_L) + c_m (\delta_d - \dot{e}_L) & \text{if } W_d > 0 \\ 0 & \text{if } W_d = 0, -b_c < \delta_d < 0 \\ W_0 + k_m (\delta_d - e_L + b_c) + c_m (\delta_d - \dot{e}_L + b_c) & \text{if } W_d < 0, \delta_d < -b_c \end{cases}$$  \hspace{1cm} (4.21)

where the dynamic mesh load $W_d$ is given by

$$W_d = W_0 + k_m (\delta_d - e_L) + c_m (\dot{\delta}_d - \dot{e}_L)$$  \hspace{1cm} (4.22)
4.7 Reduced Linear Model

Suppose the mean load is high enough that the backlash non-linearity is not considered. In this case, there is no loss of contact, and the above equations can be reduced to the form of

\[ [M][\ddot{q}] + [C][\dot{q}] + [K][q] = [F(t)] \]  \hspace{1cm} (4.23)

\[ \{q\} = \{\theta_E \quad q_1^T \quad \theta_o \quad q_2^T\}^T \]

\[ [K] = \begin{bmatrix} K_{11}^{\ast} & K_{12}^{\ast} \\ K_{21}^{\ast} & K_{22}^{\ast} \end{bmatrix} + \begin{bmatrix} K_{b1} & 0 \\ 0 & K_{b2} \end{bmatrix} \]

\[ [M] = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \]  \hspace{1cm} (4.23a)

Furthermore, the sub-matrix \( M_i \) associated with each gear is diagonal based on lumped parameter formulation, and is explicitly given by

\[ [M_1] = \text{diag}[I_E \quad m_p \quad m_p \quad m_p \quad I_{px} \quad I_{py} \quad I_{pz}] \]  \hspace{1cm} (4.24a)

\[ [M_2] = \text{diag}[I_O \quad m_g \quad m_g \quad m_g \quad I_{gx} \quad I_{gy} \quad I_{gz}] \]  \hspace{1cm} (4.24b)

\[ [K_{bl}] = \text{diag}[k_{bly} \quad k_{lx} \quad k_{ly} \quad k_{lt} \quad k_{blx} \quad k_{bly} \quad k_{blz}] \]  \hspace{1cm} (4.24c)

where \( m_p \) and \( m_g \) are pinion and gear mass, respectively, \( I_{pu} \) and \( I_{gu} \) are the rotational mass inertia about the axis-\( u \), \( k_{lu} \) (\( u=x,y,z \)) is the translation stiffness term, and \( k_{blu} \) is
related to the effective torsion \((u=y)\) and bending stiffnesses \((u=x,z)\) of the support (bearing) components. The other sub-matrices can be shown to be

\[
[K^*_n] = \begin{bmatrix} A_i \rightarrow \\ \downarrow \quad K_n \end{bmatrix}
\]

\[
[K^*_1] = \begin{bmatrix} 0 \rightarrow \\ \downarrow \quad K_{11} \end{bmatrix}
\]

\[
[K^*_2] = [K^*_1]^T
\]

\[A_i = [k_{bly} \quad 0 \quad 0 \quad 0 \quad -k_{bly} \quad 0] \]

\[
[K_{11}] = k_m (h^{(1)}T - \mu_g^{(1)}T)h^{(1)}
\]

\[
[K_{12}] = -k_m (h^{(1)}T - \mu_g^{(1)}T)h^{(2)}
\]

\[
[K_{21}] = -k_m (h^{(2)}T + \mu_g^{(2)}T)h^{(1)}
\]

\[
[K_{22}] = k_m (h^{(2)}T + \mu_g^{(2)}T)h^{(2)}
\]

\[
\{F^{(1)}\} = (h^{(1)}T - \mu_g^{(1)}T)(k_m h^{(1)}e_L - W_0)
\]

\[
\{F^{(2)}\} = -(h^{(2)}T + \mu_g^{(2)}T)(k_m h^{(2)}e_L - W_0)
\]

For the purpose of applying the forced response analysis by modal method, the damping matrix \([C]\) is assumed to be proportional viscous type to represent for the net vibratory energy dissipation that occurs within the bearing elements and gear mesh interface. The forcing vector on the right-hand side of Equation (4.23) is represented as
\[
\{ \mathbf{F}(t) \} = \{ \mathbf{F}_{\text{int}}(t) \} + \{ \mathbf{F}_{\text{ext}}(t) \} \tag{4.26a}
\]

\[
\{ \mathbf{F}_{\text{int}}(t) \} = \begin{bmatrix} F_{\text{int}}^{(1)} \\ F_{\text{int}}^{(2)} \end{bmatrix} \tag{4.26b}
\]

\[
\{ \mathbf{F}_{\text{ext}}(t) \} = \begin{bmatrix} F_{\text{ext}}^{(1)} \\ F_{\text{ext}}^{(2)} \end{bmatrix} \tag{4.26c}
\]

In the above equation, the forcing sub-vectors \( \mathbf{F}_{\text{int}} \) and \( \mathbf{F}_{\text{ext}} \) correspond to the internal load due to the harmonically driven transmission error excitation and the external load fluctuation vector applied to the gear member, motor and output load respectively. They are explicitly given by

\[
\mathbf{F}_{\text{int}}^{(l)} = \begin{bmatrix} 0 \\ k_m h_{s}^{(l)} T \end{bmatrix} \tag{4.27a}
\]

\[
\mathbf{F}_{\text{ext}}^{(l)} = \begin{bmatrix} T_{l} & F_{x}^{(l)} & F_{y}^{(l)} & F_{z}^{(l)} & M_{x}^{(l)} & M_{y}^{(l)} & M_{z}^{(l)} \end{bmatrix}^{T} \tag{4.27b}
\]

where the projected transmission error term \( e(t) = \{ e_x, e_y, e_z, e_w, e_{w_x}, e_{w_y}, e_{w_z} \}^{T} \) is the vector of equivalent translation and rotation displacements of the pinion relative to the gear resulting from tooth manufacturing errors and variation in the instantaneous stiffness at the mesh interface. It can be related back to the classical definition of transmission error along the line-of-action by \( e_{l}(t) = h_{s}^{(l)} e(t) \).
4.8 Summary

For hypoid gears, the mesh force vector and its position are shown to be time-varying and load dependent due to the relatively large tooth modifications applied. Furthermore, tooth load sharing, backlash, and friction are the important factors in oscillatory dynamics. Based on the 3-dimensional tooth contact analysis, a generalized translation-rotational 14 DOF gear dynamic model was developed by taking into account of time-varying mesh vectors, friction and gear backlash under loaded transmission error excitation into account, and the quasi-static mesh characteristics are assumed in the formulation. The translation-rotational and rotational-rotational force couplings are precisely expressed by the time-varying and load dependent mesh vectors. This model forms the basis for the comprehensive dynamic studies in the following chapters.
CHAPTER 5

TIME-INVARIANT MODEL SOLUTION AND DYNAMIC COUPLING ANALYSIS

5.1 Introduction

Since the mesh force direction is oblique in space, vibrations of the hypoid gears are thus 3 dimensional. Up to now, there are no public reports on the modeling and solution of this type of gear mesh coupling and dynamics. Few studies in this area are limited to either 2-DOF torsional model (Nakayashiki et al., 1983) or using rather simplified mesh vectors (Kiyono et al., 1981; Donley et al., 1992). Therefore, the dynamic characteristics of the hypoid gears including spiral bevel gears remain unsolved. The new 3-dimensional coupled rotational-translation vibratory model of the hypoid geared rotor system developed in Chapter 4 is linearized here by ignoring friction terms and time-varying mesh in Equation (4.23). This time-invariant model is used to study the fundamental dynamic characteristics of a generic hypoid geared rotor system. From the free vibration results, the unique elastic modes that contribute to the generation of gear mesh induced vibrations are to be identified. The mesh force response function is also analyzed to examine the sensitivity of dynamic coupling and vibratory response to critical
design parameters. Parametric studies are performed to quantify the dependence of vibration modes and response trends to selected design values. The proposed formulation is capable of simulating drive and coast operating cases. The resultant linear vibratory dynamic model will be used to analyze the mesh force coupling phenomenon and the transmissibility of the gear mesh excitation and externally applied loads through the gear mesh interface.

5.2 Generalized Excitation and Force Coupling

From Equation (4.23), the 6-DOF generalized dynamic load vector at the gear mesh interface projected onto the centroid of the pinion (l=1) or gear (l=2) can be expressed as

\[ f^{(l)}_{\text{mesh}}(t) = -[K_{12}]q_2 - [K_{11}]q_1 + [K_{11}]\dot{q}(t) - [C_{12}]\ddot{q}_2 - [C_{11}]\ddot{q}_1 + [C_{11}]\ddot{e}(t) \] (5.1)

while the bearing dynamic force vector is predicted using

\[ f^{(l)}_{b}(t) = -[K_{b}][q] - [C_{b}][\dot{q}] \] (5.2)

Since the steady-state harmonic response is of primary interest here, the frequency response functions can be computed assuming \( q(t) = q \exp(i\omega t) \), where \( \omega \) is the mesh excitation frequency. The undamped free vibration formulation yields the classical eigenvalue problem \([K]\Phi_r = \omega_r^2[M]\Phi_r\), where \( \omega_r \) is the natural frequency and \( \Phi_r \) is the corresponding mode shape normalized with respect to \([M]\) according to \( \Phi_r^T[M]\Phi_r = 1 \).
The modal damping ratio is \( \xi_r = \Phi_r^T [C_r] \Phi_r / 2 \omega_r \). The steady-state forced frequency response of the system is directly computed from \( Q(\omega) = [H(\omega)] F(\omega) \), where \( Q(\omega) \) and \( F(\omega) \) are the frequency spectra of \( q(t) \) and \( F(t) \) respectively, and \( [H(\omega)] \) is the dynamic compliance matrix derived from the modal superposition method \( (i = \sqrt{-1}) \),

\[
[H(\omega)] = \sum_{r=2}^{14} \frac{\Phi_r \Phi_r^T}{(\omega_r^2 - \omega^2 + 2i\xi_r \omega_r)} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}
\]  

(5.3)

In the forced response analysis, the mesh force vector \( f_{mesh}^{(l)} \) in Equation (5.1) can be used to determine the severity of gear tooth loading and vibration transmissibility. The modal frequency response function computed using Equation (5.3) is used to transform the time-dependent mesh force vector \( f_{mesh}(t) \) into the spectral domain by applying the Fourier transform as shown

\[
F_{mesh}^{(l)}(\omega) = -[K_{12}] Q_2(\omega) - [K_{11}] Q_1(\omega) + [K_{11}] E(\omega) \\
- [C_{12}] \dot{Q}_2(\omega) - [C_{11}] \dot{Q}_1(\omega) + [C_{11}] \dot{E}(\omega)
\]  

(5.4a)

where

\[
\begin{bmatrix} Q_1(\omega) \\ Q_2(\omega) \end{bmatrix} = [H] \cdot \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} K_{11} E(\omega) + F_{ext}^{(1)}(\omega) + i \cdot C_{11} \dot{E}(\omega) \\ K_{21} E(\omega) + F_{ext}^{(2)}(\omega) + i \cdot C_{21} \dot{E}(\omega) \end{bmatrix}
\]  

(5.4b)

where \( Q(\omega) \) and \( E(\omega) \) are the complex-valued, Fourier transforms of \( q(t) \) and \( e(t) \) respectively, and \( F_i(\omega) \) is the spectral function of \( F_{ext}^{(i)}(t) \). Direct substitution of Equation (5.4b) into (5.4a) gives
\[ F_{\text{mesh}}^{(i)}(\omega) = [G^{(i)}(\omega)]E(\omega) + [R^{(i1)}(\omega)]F_{\text{ext}}^{(i)}(\omega) + [R^{(i2)}(\omega)]F_{\text{ext}}^{(2)}(\omega) \]  

\[ [R^{(i1)}(\omega)] = [D_{11}][H_{11}] + [D_{12}][H_{21}] \]  

\[ [R^{(i2)}(\omega)] = [D_{11}][H_{12}] + [D_{12}][H_{22}] \]  

\[ [D_{ij}(\omega)] = [K_{ij}] + i\omega[C_{ij}] \quad (i, j = 1, 2) \]  

\[ [G^{(i)}(\omega)] = [D_{11}] - ([D_{11}][H_{11}] + [D_{12}][H_{21}])[D_{11}] - ([D_{11}][H_{12}] + [D_{12}][H_{22}])[D_{21}] \]  

where \([G^{(i)}(\omega)]\) is the mesh related dynamic stiffness transfer function matrix of dimension 6 similar to the lower dimension ones derived by Donley et al. (1990) and Blankenship and Singh (1995) for parallel axis gears. The matrix \([G^{(i)}(\omega)]\) physically describes the dynamic load vector generated within the gear mesh interface due to \(e(t)\), which directly acts on the pinion \((l=1)\) or gear \((l=2)\) body mass centroid. On the other hand, \([R^{(i1)}(\omega)]\) and \([R^{(i2)}(\omega)]\) are the transmissibility transfer function matrices for the dynamic load reaction vector due to external load fluctuations acting on the pinion \((l=1)\) and gear \((l=2)\) respectively via the gear mesh interface, and \([D_{ij}(\omega)]\) is the complex-valued mesh stiffness matrix. Similarly, the dynamic load reaction vector of the pinion bearing supports in the frequency domain form can also be obtained from Equation (5.2) using the same derivation technique, resulting in

\[ F_{b}^{(i)}(\omega) = [G_{b}^{(i)}(\omega)]E(\omega) + [R_{b}^{(i1)}(\omega)]F_{\text{ext}}^{(i)} + [R_{b}^{(i2)}(\omega)]F_{\text{ext}}^{(2)} \]
\[ [G_b^{(l)}(\omega)] = -[D_{bl}][H_{l1}][D_{l1}] + [H_{l2}][D_{21}j] \] (5.6b)

\[ [R_b^{(l)}(\omega)] = -[D_b][H_{l1}] \] (5.6c)

\[ [D_{bl}(\omega)] = [K_{bl}] + i\omega[C_{bl}] \] (5.6d)

where \( F_b^{(i)}(\omega) \) is the complex-valued Fourier transform of \( f_b^{(i)}(t) \), and \( [G_b^{(l)}(\omega)] \) refers to the dynamic reaction forces and moments acting on the pinion \((l=1)\) or gear \((l=2)\) support bearing due to excitation of \( e(t) \). Also \( [R_b^{(i)}] \) in the above expressions characterizes the transmissibility of the external harmonic load fluctuation vectors acting on the driving pinion \( F_{ext}^{(1)} \) and driven gear \( F_{ext}^{(2)} \) to the support bearing load vectors for the pinion \((l=1)\) or gear \((l=2)\) via the gear mesh interface.

Since this dissertation is primarily concerned with the perturbations about the mean steady-state operating condition, it is assumed that the external forcing vector \( F_{ext}(t) = 0 \), and the internal forcing vector \( F_{int}(t) \) is solely due to \( e(t) \). The resultant dynamic model is still relatively general and can be applied to predict dynamic transmission error, mesh force, and bearing reaction forces for a variety of steady-state operating conditions, such as forward drive and coast conditions. The proposed theory can also account directly for the effects of hypoid gear parameters, such as pinion offset and pinion mean spiral angle, on the system vibration response. Note that the change in gear design parameter induces changes in other gear design parameters such as pitch angles and pressure angles of both convex and concave sides of the gear teeth. To the knowledge of the authors, this type of model has never been studied in the past.
Accordingly, the corresponding dynamic transmission error that quantifies the effect of tooth deflection along the effective line-of-action due to vibratory motion of gear bodies is given by Equation (4.19). Furthermore, the scalar dynamic mesh force along the line-of-action is expressed as

\[ F_\delta(t) = k_m(\delta_d(t) + e_L(t)) + c_m(\delta_d(t) + e_L(t)) \]  \hspace{1cm} (5.7)

Note that the magnitude of \( F_\delta \) generally plays a major role in determining the sensitivity of a transmission design to gear noise, and is frequently used as a design metric. Such design concept has been applied in previous studies on spur and helical gears (Kahraman and Singh, 1990; Blankenship and Singh, 1995; Özgüven and Houser, 1988) and is being extended here to the hypoid case. In addition, the dynamic bearing reaction loads are examined to understand the vibratory energy transmitted into the housing and surrounding structure. This transmissibility function is often directly proportional to radiated gear noise levels.

### 5.3 Modal Properties

A baseline automotive hypoid gear set that is defined in Table 3.2 and Table 5.1 is used in the subsequent dynamic analysis. First, the eigensolution corresponding to the free vibration problem is used to identify the critical elastic modes that directly affect gear mesh force. Since the severity of the system dynamic response is directly associated with the amplitude of the mesh displacement vector, a modal index \( T_r = \Phi_r^T \{ [0, h^{(1)}] \{0, h^{(2)}] \}^T \) is defined to account for the relative modal displacement
projection along the mesh force direction. Its magnitude can be used to differentiate the modes that affect dynamic response significantly. The predicted $T_r$ values corresponding to the critical out-of-phase gear pair torsional modes that are classically most susceptible to the excitation of $e(t)$ for the baseline system are given in Table 5.2. Here, $T_r$ is normalized in such a manner as to ensure that the maximum value is unity within the same modal set. An illustration of this class of critical modes is given in Figure 5.1. The result shows that $T_r=1.0$ corresponds to the out-of-phase gear pair torsional mode with a superimposed pinion yaw motion. Thus this mode is expected to be very sensitive to $e(t)$. A complementary technique for analyzing the severity and nature of the vibration modes can also be applied here, which relies on the computation of the modal strain energy ratio of each elastic element in the geared rotor system model. Using this technique, the critical structural components that contribute to the mode in question may be identified. Figure 5.2 shows the distributions of modal strain energy ratio for the 5 dominant out-of-phase gear pair torsion modes. The translation or torsion compliances tend to affect the lower modes, while the bending stiffness controls the higher order ones. Also, the structural modes possessing higher percentage of mesh strain energy density actually generate higher response peaks, which will be evident from the response spectra presented in the next section. It is also observed from Figure 5.2 that the specific modes with larger $T_r$ values possess generally higher mesh strain energy ratio, and hence further validating the correlation between $T_r$ and the effectiveness of the modal mesh in storing vibratory energy.
| Mass moment of inertia of the driver (kg m²) | 0.0055 | Mass moment of inertia of the pinion (kg-m²) | 0.0083 |
| Mass moment of inertia of the load (kg m²) | 0.1 | Mass moment of inertia of the gear (kg-m²) | 0.5233 |
| Mass of the pinion (kg) | 11.48 | Mass of the gear (kg) | 49.53 |
| Torsional stiffness of shafts (Nm/rad) | 1.0E4 (pinion) 5.0E5 (gear) | Shaft-bearing bending stiffness (Nm/rad) | 1.0E6 (pinion) 8.0E6 (gear) |
| Axial support stiffness (N/m) | 1.0E8 | Lateral support stiffness (N/m) | 3.8E8 |
| Mesh stiffness (N/m) | 6.0E8 | Pinion type | L.H. |

Table 5.1: Baseline system parameters for a typical automotive hypoid gear set.

<table>
<thead>
<tr>
<th>Mode number, ( r )</th>
<th>( T_r )</th>
<th>( \omega_r ) (Hz)</th>
<th>Mode description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.013</td>
<td>206.3</td>
<td>Out-of-phase torsion, pinion and gear axial translations</td>
</tr>
<tr>
<td>4</td>
<td>0.021</td>
<td>351.8</td>
<td>Out-of-phase torsion, pinion axial translation and gear transverse motion</td>
</tr>
<tr>
<td>8</td>
<td>0.084</td>
<td>799.8</td>
<td>Out-of-phase torsion, gear yaw motion, and pinion axial and vertical translations</td>
</tr>
<tr>
<td>12</td>
<td>0.367</td>
<td>1419.7</td>
<td>Out-of-phase torsion, pinion yaw motion, and pinion axial and vertical translations</td>
</tr>
<tr>
<td>14</td>
<td>1.000</td>
<td>3120.8</td>
<td>Out-of-phase torsion, pinion yaw motion</td>
</tr>
</tbody>
</table>

Table 5.2: Modal indices for the critical class of out-of-phase torsion gear pair modes.

91
Figure 5.1: Critical class of out-of-phase gear pair torsional mode shapes.
Figure 5.1-continued

(c) Mode 8

(d) Mode 12

continued
Figure 5.1-continued

(e) Mode 14
1, 2 - pinion and gear translation compliances; 3, 4 - pinion and gear rotational compliances; 5, 6 - pinion and gear bending compliances; 7 - mesh compliances).

Figure 5.2: Modal strain energy distributions for the 5 critical baseline system modes.
5.4 Mesh Force Transmissibility

Previous studies of hypoid gear dynamics ignored the effect of bending load transmissibility and included only the torsional component (Remmers, 1971; Abe and Hagiwara, 1990; Hirasaka et al., 1991). The present theory that incorporates the exact dynamic mesh force transfer function matrices, given by \([G^{(1)}(\omega)]\), \([R^{(1)}(\omega)]\) and \([R^{(12)}(\omega)]\), in the gear mesh reaction load vector formulation provide a direct means to examine these effects more thoroughly. In this analysis, \([G^{(1)}(\omega)]\) is a symmetric matrix of dimension 6 with 21 distinct terms where each element \(G_{q_i,q_j}^{(1)}\) represents the \(q_i\)-th component of the dynamic load vector related to the mesh coupling acting on the pinion \((l=1)\) or gear \((l=2)\) due to the \(q_j\)-th component of projected transmission error excitation. On the other hand, \([R^{(k)}(\omega)]\) \((k=1,2)\) is a non-symmetric matrix of dimension 6 with 36 distinct terms. Each term denoted by \(R_{q_i,q_j}^{(k)}\) defines the \(q_i\)-th component of mesh related dynamic load vector acting on the pinion \((l=1)\) or gear \((l=2)\) due to the \(q_j\)-th component of external load fluctuation on the pinion \((k=1)\) or gear \((k=2)\). Also note that the subscripts \(q_i\) and \(q_j\) can take the value of \(x, y, z, \theta_x, \theta_y\) and \(\theta_z\). Figure 5.3 shows the frequency response functions of the pinion lateral shear, axial force, bending moment and torque response components of \([G^{(1)}(\omega)]\) per unit torsion component of transmission error excitation. It should be pointed out that other elements of \([G^{(1)}(\omega)]\) not shown here are also comparable in magnitude with similar spectral content that arises from the contributions of the critical modes described in Figure 5.1. The relative amplitudes of these resonance peaks in fact correlate extremely well with the results of \(T_r\). In addition,
the cross coupling terms of \( G^{(1)}(\omega) \) are nearly as significant as its diagonal terms. This implies that all 6 components of projected transmission error excitation contribute strongly to the dynamic response of the system, and the analysis actually proves that the hypoid gear dynamic problem must be treated by the inclusion of a complete 3-dimensional 6 DOF representation of the gear body. In contrast, the simplified mesh force vector used in the previous studies (Remmers, 1971; Nakayashiki, et al., 1983; Abe and Hagiwara 1990; Hirasaka et al., 1991) can only be used to account for \( G_{x-\theta_y}^{(1)}(\omega) \), \( G_{\theta_x-\theta_y}^{(1)}(\omega) \) and the diagonal terms \( G_{\theta_x-\theta_x}^{(1)}(\omega) \). Hence, the mesh force transmissibility calculations are limited to simply the vertical translation \( z \), torsion \( \theta_y \) and bending \( \theta_x \) coordinates, and the equally critical lateral \( x \), axial \( y \) and bending \( \theta_x \) components cannot be predicted.

Figures 5.4-5.9 show the force and moment transmissibility functions of \( R_{u-z}^{(1)}(u=x,y,z) \) and \( R_{\theta_x-\theta_x}^{(1)} \) respectively. The former type of functions is essentially associated with the translation mesh force components acting on the pinion center of mass, while the latter set of terms is related to the rotational moments. By comparing the transfer functions in Figures 5.4-5.9, it can be seen that the torque excitation corresponding to the \( \theta_y \) coordinate frequently produces the highest resonant peaks compared to the other excitations, especially in low frequency range, due to coupling provided by the gear mesh. Also, the axial mesh force component is more sensitive to external axial force and torque fluctuations compared to the pitching-type external bending moment associated with \( \theta_x \) coordinate as seen in Figures 5.4-5.6, as evidenced
by higher $R_{y-y}^{(11)}$ and $R_{y-\theta_y}^{(11)}$ in Figures 5.4-5.6, and low $R_{y-\theta_z}^{(11)}$ in Figures 5.7-5.9. In fact, the external bending moment is observed to be nearly uncoupled from many of its other coordinate counterparts. This is evidenced from its small effect on the pinion bending moment $R_{\theta_z-\theta_z}^{(11)}$, and reaction torque $R_{\theta_y-\theta_z}^{(11)}$ in Figures 5.7-5.9, and on translation force components of mesh reaction force vector $R_{u-\theta_z}^{(11)}$ in Figures 5.4-5.6. It is also noticed the vertical mesh force response is more sensitive to the pinion yaw-type bending moment excitation corresponding to $\theta_z$ coordinate as characterized by $R_{z-\theta_z}^{(11)}$, at high frequency in Figure 5.6. Furthermore, the external drive torque fluctuation about $Y$-axis affects the low frequency resonance $r=4$ significantly as depicted by $R_{q-\theta_z}^{(11)}$ in Figures 5.4-5.9, while the bending moment about $Z$-axis affects high frequency mode such as mode 14. Meanwhile, other external load fluctuations with the exception of the bending moment about $X$-axis also contribute significantly to the response from mid-frequency modes 8 and 12.

### 5.5 Parametric Analysis

The effect of pinion spiral angle $\beta_p$ on the dynamic response solely due to transmission error excitation $\epsilon(t)$ will be investigated to demonstrate the effectiveness of the proposed model for use in parametric design studies. Pinion spiral angle directly affects the directional vectors of the mesh force and contact position as shown analytically in Equation (3.22), and thus the structure stiffness matrix $[K]$ of the system as shown in Equation (4.25). Note that varying the spiral angle also requires re-computation
Figure 5.3: Mesh related pinion dynamic load transfer function with the amplitude normalized to the mean mesh stiffness (Key: \( G_{x-\theta}^{(1)} \); \( G_{y-\theta}^{(1)} \); \( G_{x-\theta}^{(1)} \); \( G_{y-\theta}^{(1)} \)).
Figure 5.4: Frequency response functions of $[R^{(11)}(\omega)]$ corresponding to the lateral components of mesh force vector acting on the pinion center of mass due to external driving load fluctuations.
Figure 5.5: Frequency response functions of $[R_{y-y}^{(11)}](\omega)$ corresponding to the axial components of mesh force vector acting on the pinion center of mass due to external driving load fluctuations.
Figure 5.6: Frequency response functions of $[R^{(11)}(\omega)]$ corresponding to the vertical components of mesh force vector acting on the pinion center of mass due to external driving load fluctuations.
Figure 5.7: Frequency response functions of $[R^{(11)}(\omega)]$ corresponding to the rotational components about X-axis of mesh force vector acting on the pinion center of mass due to external driving load fluctuations.
Figure 5.8: Frequency response functions of $[R^{(1)}(\omega)]$ corresponding to the torsional components about Y-axis of mesh force vector acting on the pinion center of mass due to external driving load fluctuations.
Figure 5.9: Frequency response functions of $[R^{(1)}(\omega)]$ corresponding to the rotational components about Z-axis of mesh force vector acting on the pinion center of mass due to external driving load fluctuations.
of other gear parameters, such as the gear spiral angle $\beta_n$, pitch angles and pressure angle.

The effect of $\beta_p$ on the amplitude of the dynamic mesh force along the line of action $F_\delta$ defined by Equation (5.7) is shown in Figure 5.10. A uniform softening of the system modes with increasing $\beta_p$ is generally seen, primarily due to incremental reduction in the mesh coupling strength. This resulted in shifting of the critical modes lower by approximately 10% per 5° increase in $\beta_p$. The spectral results also show an opposing frequency dependency effect on the two most critical modes. For mode 12 in the vicinity of 1400 Hz, the amplitude response reduces with increasing $\beta_p$. On the other hand, the higher mode 14 near 3100 Hz actually becomes more sensitive to higher $\beta_p$. This behavior is typical of a geared rotor dynamic system possessing frequency ranges with high and low sensitivities to transmission error excitation, which is also observed in spur and helical gear cases (Kahraman 1993).

The effect of pinion spiral angle $\beta_p$ on bearing force transmissibility can be predicted as well. Figure 5.11 shows the harmonic component of the pinion bearing reaction force in the axial direction. In this case, a systematic increase in this component of bearing force over a broad frequency range is noticed, due to either an increase in $F_\delta$ or coupling between the line of action and axial coordinate. It is worthwhile to note that the previous simplified mesh models are unable to predict this axial or the lateral bearing reaction load accurately, which can be just as critical to the structure-borne gear noise transmissibility in the system.
Thirdly, Figures 5.12-5.13 illustrate the effect of \( \beta_p \) on selected two transfer functions of \([G^{(1)}(\omega)]\). The transfer function \( G^{(1)}_{x}\theta_y(\omega) \), the lateral component of mesh force vector acting on the pinion centroid due to the torsion element of the projected transmission error excitation is given by Figure 5.12. The result shows a frequency-wide uniform reduction in amplitude, primarily due to lower coordinate coupling as the spiral angle is increased. The other transfer function \( G^{(1)}_{y}\theta_y(\omega) \) in Figure 5.13 corresponding to the axial component of mesh force on the pinion depicts a trend similar to \( F_\delta \) shown in Figure 5.10, as expected. These results illustrate the feasibility of applying the proposed theory in predicting dynamic response associated with the complete 6 DOF coordinate of the hypoid geared rotor system, and the potential of tuning fundamental hypoid gear design parameters to lower undesirable and sometimes harmful vibration responses for targeted frequency range.

Next, the present formulation is used to study the experimental phenomenon observed by Nakayashiki et al. (1983) where the dynamic response of the forward drive and coast operating conditions are found to be different due to the effect of change in the gear tooth engagement side. Figure 5.14 illustrates the predicted differences in the dynamic mesh force spectra of the forward drive and coast operating conditions for the baseline system defined in Table 5.1. The analysis shows that the drive operating condition actually produces higher amplitudes of peak response for the highest resonance frequency shown, but lower levels for other resonance frequencies compared to the response of the coast condition. These variations are primarily due to the differences in
Figure 5.10: Effect of pinion spiral angle $\beta_p$ on net dynamic mesh force response along the line of action per $\mu$m TE.
(Key: 41°, 46°, 51°).
Figure 5.11: Effect of pinion spiral angle $\beta_p$ on pinion axial bearing force due to a unit magnitude of transmission error excitation.

(Key: $-41^\circ$; $-46^\circ$; $-51^\circ$).
Figure 5.12: Effect of pinion spiral angle $\beta_p$ on $G_{x-\theta,1}^{(1)}$

(Key: \(-\ldots-41^\circ\), \(-\ldots-46^\circ\), \(-\ldots-51^\circ\)).
Figure 5.13: Effect of pinion spiral angle $\beta_v$ on $G_{y-\theta}$,

(Key: \text{---} 41^\circ; \text{----} 46^\circ; \text{------} 51^\circ).
Figure 5.14: Total mesh force spectra of the forward drive and coast operating conditions (Key: —— drive; ———- coast).
the gear mesh coupling characteristics that cause alterations in the resultant system stiffness matrix. This is clearly unpredictable by the simpler gear mesh formulation described earlier.

The pinion offset is a critical parameter in the hypoid drive design. Driveline manufacturers are attempting to change $E$ in newer axle designs (Griffith, 1990), but possess little knowledge about the dynamic consequences. The hypoid offset can directly affect the dynamic coupling between the gear pair, which determines the stiffness matrix given by Equation (4.25), in addition to altering the contact pattern and $e(t)$. Accordingly, this analysis is particularly directed towards examining the extent of which the dynamic response of a typical driveline design having a certain overall torsional stiffness is influenced by $E$. It may be noted that variations in $E$ will induce modifications in other related gear design parameters such as mean spiral angle, pitch angles and pressure angles of both convex and concave sides of the gear teeth, as shown in Equations (3.24)-(3.26). These chain effects are included in the proposed computation.

For this purpose, a hypoid gear set used in a passenger car is used as a baseline design, as given in Table 5.3. First, the free vibration characteristics are analyzed. The effect of $E$ on the resonant frequencies was examined within 400-1400 Hz, which is the typical range for hypoid gear whine problems. Figure 5.15 shows $\omega_r$ as a function of $E$. It is shown that the selected variation in $E$ only changes $\omega_r$ by about 10% of their baseline value. Hence, the prediction of natural frequencies using one offset value may be used to approximate $\omega_r$ of other $E$ values in the frequency range of interest assuming other gear parameters remain about the same when $\omega_r$ accuracy of is not a major concern. Also it
may be noted that \( \omega_e \) generally reduces monotonically when \( E \) increases as a result of the system becoming more compliant.

Table 5.4 gives the \( T_r \) values of several flexible modes of interest for selected values of \( E \). The first 2 modes are not shown here since the first mode is a rigid body motion and the second one is a lower frequency mode corresponding to the overall drivetrain compliance, which is of no interest in the present study. The numerical result shows that out-of-phase torsional motion of the gear pair coupled with pinion translational displacement tends to dominate the resonant amplitude of forced response spectrum, as evident from the relatively large value of \( T_r \) for \( r=6 \) and 7. This observation is also consistent with results of the normalized dynamic mesh force response spectra \( F_\delta(\omega) \) as shown in Figure 5.16. Also by changing \( E \) from 20 mm to 50 mm, the magnitude of \( T_6 \) decreases due to reduction in modal coefficients of \( \theta_{y_i} \) and \( y_i \). This resulted in lower \( F_\delta \) as shown in Figure 5.16. On the other hand, \( T_3 \) tends to increase with increasing \( E \) due to higher modal coefficients of \( \theta_{y_i} \) and \( y_i \), which also resulted in larger response of \( F_\delta \) at this particular resonant frequency. It is worth noting that the gear lateral motions are essentially uncoupled from other modal deformations and are not affected by \( e(t) \) as evident from \( T_5=0 \). Also, pure bending modes only occur at much higher frequencies, which are of less importance to this problem and hence are not tabulated here. In addition, it is quite clear from Figure 5.16 that the effect of \( E \) on \( F_\delta(\omega) \) is frequency-dependent and primarily controlled by the modal coefficients of \( \theta_{y_i} \) and \( y_i \). In particular, higher values of \( E=50 \) mm will tend to increase the amplitude response in the frequency range of 300-500 Hz, but at the same time lowers \( F_\delta(\omega) \) around 750 Hz.
relative to the dynamic response of the baseline design in which $E=35$ mm. For a smaller value of $E=20$ mm, a higher magnitude of $F_\delta(\omega)$ was found around 800 Hz but generally lower level of $F_\delta(\omega)$ for $\omega \leq 800$ Hz. The above results may be explained by noting that the mode shape actually transforms in a gradual manner when $E$ is changed. For instance, when $E$ is varied from 20 mm to 50 mm, gear lateral motion of mode $r=6$ starts to form and grow to a point where it is large enough to cancel some of the torsional displacements at the gear mesh. This resulted in lower response for $F_\delta(\omega)$. On the other hand, for modes $r=3$ and $r=4$, which correspond to resonant peaks around 500 Hz and 600 Hz respectively, increasing $E$ reduces the gear vertical motion, and at the same time amplifies the out-of-phase torsional displacements. The reduction in gear vertical motion deteriorates its effectiveness in canceling $F_\delta(\omega)$, which is primarily induced by the torsional displacements at the mesh. Therefore, there is an increase in $F_\delta(\omega)$ for $\omega \leq 650$ Hz when $E$ increases. Finally, note that $E$ actually produces no significant effect on the magnitude of $F_\delta(\omega)$ within $900 \leq \omega \leq 1200$ Hz, except a shift in the resonant frequency to a higher value when $E$ is reduced. This implies that the dynamic stiffness in this frequency range simply increases with decreasing offset.

Finally, Figure 5.17 and 5.18 show the predicted dynamic torsional moment on the pinion and bearing reaction loads induced by $F_\delta(\omega)$ for 3 values of $E$. The characteristic trends of $F_\delta$ and these dynamic moment and loads are similar except for the gear bearing force response at $E=20$ mm, where it consistently produces the lowest amplitude response for the modes shown in the frequency range of interest. Hence from
this calculation, it is evident that the frequency response characteristics of the dynamic bearing reaction loads are also relatively sensitive to the choice of $E$.

5.6 Summary

The new gear mesh coupling formulation based on the exact gear geometry and kinematic relations have been used to model the time-invariant response of the hypoid geared rotor system. Free and forced vibrations in the presence of transmission error excitation are examined, and the critical classes of pinion-gear out-of-phase torsion modes that affect the undesirable vibration response are identified. The proposed system model also readily provides the complete gear mesh transfer functions and force transmissibility spectra, which is used to characterize the nature of the system vibratory behavior including the dynamic coupling and sensitivity of the vibratory response to critical design parameters. The analysis suggests that all 6 degrees of freedom representations of the gear bodies must be included in the dynamic model to accurately account for the response due to transmission error excitation and external load fluctuations. The out of phase gear torsional motion is exclusively coupled with other vibration modes such as translations and rotational motions about other non-nominal rotational axes. The results also show new mesh coupling effect from lateral, axial, and bending coordinates, which was not modeled in the simpler gear mesh representations utilized in previous studies. It is shown that even though the hypoid offset generally has little effect on the predicted natural frequencies, it clearly generates a significant frequency-dependent effect on the frequency response functions of the dynamic mesh.
force and bearing reaction loads, which are shown to be controlled by the modal coefficients of the angular and translational displacements of the gear pair.
<table>
<thead>
<tr>
<th>Number of pinion teeth</th>
<th>13</th>
<th>Equivalent inertia of pinion (kg·m²)</th>
<th>2.4E-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of gear teeth</td>
<td>46</td>
<td>Equivalent inertia of gear (kg·m²)</td>
<td>3.1E-2</td>
</tr>
<tr>
<td>Pinion spiral angle (degree)</td>
<td>50.42</td>
<td>Inertia of engine (kg·m²)</td>
<td>0.122</td>
</tr>
<tr>
<td>Gear pitch angle (degree)</td>
<td>70.05</td>
<td>Inertia of load (kg·m²)</td>
<td>1.569</td>
</tr>
<tr>
<td>Pinion pitch angle (degree)</td>
<td>18.656</td>
<td>Equivalent mass of pinion (kg)</td>
<td>5.82</td>
</tr>
<tr>
<td>Gear pitch diameter (mm)</td>
<td>205</td>
<td>Equivalent mass of gear (kg)</td>
<td>21.78</td>
</tr>
<tr>
<td>Gear face width (mm)</td>
<td>30.5</td>
<td>Torsional stiffness of pinion shaft (Nm/rad)</td>
<td>8.0E3</td>
</tr>
<tr>
<td>Pinion offset (mm)</td>
<td>35</td>
<td>Torsional stiffness of gear shaft (Nm/rad)</td>
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</tr>
<tr>
<td>Cutter radius (mm)</td>
<td>95</td>
<td>Torsional support stiffness (Nm/rad)</td>
<td>2.0E8 (pinion)</td>
</tr>
<tr>
<td>Mesh stiffness (N/m)</td>
<td>2.0E8</td>
<td>Axial support stiffness (N/m)</td>
<td>2.0E8 (pinion)</td>
</tr>
<tr>
<td>Pinion type and rotation direction</td>
<td>LH</td>
<td>Lateral support stiffness (N/m)</td>
<td>3.5E8 (pinion)</td>
</tr>
<tr>
<td></td>
<td>CW</td>
<td></td>
<td>4.0E8 (gear)</td>
</tr>
</tbody>
</table>

Table 5.3: Design parameters for a typical automotive hypoid gear set for offset study.
<table>
<thead>
<tr>
<th>r</th>
<th>Mode shape descriptions</th>
<th>$E=20$ mm</th>
<th></th>
<th>$E=35$ mm</th>
<th></th>
<th>$E=50$ mm</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$T_r$</td>
<td>$\omega_r$</td>
<td>$T_r$</td>
<td>$\omega_r$</td>
<td>$T_r$</td>
<td>$\omega_r$</td>
</tr>
<tr>
<td>8</td>
<td>Out-of-phase torsional and pinion lateral motion</td>
<td>0.1152</td>
<td>1320</td>
<td>0.1075</td>
<td>1176</td>
<td>0.0851</td>
<td>1081</td>
</tr>
<tr>
<td>7</td>
<td>Out-of-phase torsional and pinion motion in $x$, $y$, and $z$ directions</td>
<td>0.2082</td>
<td>1136</td>
<td>0.2118</td>
<td>1017</td>
<td>0.1948</td>
<td>937</td>
</tr>
<tr>
<td>6</td>
<td>Out-of-phase torsional and pinion axial motion</td>
<td>0.2431</td>
<td>835</td>
<td>0.2182</td>
<td>781</td>
<td>0.1747</td>
<td>751</td>
</tr>
<tr>
<td>5</td>
<td>Gear lateral motion</td>
<td>0.0</td>
<td>680</td>
<td>0.0</td>
<td>682</td>
<td>0.0</td>
<td>684</td>
</tr>
<tr>
<td>4</td>
<td>Out-of-phase torsional and gear motion in $x$, $y$, and $z$ directions</td>
<td>0.0937</td>
<td>636</td>
<td>0.1251</td>
<td>625</td>
<td>0.1286</td>
<td>625</td>
</tr>
<tr>
<td>3</td>
<td>Out-of-phase torsional and gear axial motion</td>
<td>0.0429</td>
<td>520</td>
<td>0.0870</td>
<td>495</td>
<td>0.1081</td>
<td>467</td>
</tr>
</tbody>
</table>

Table 5.4: Effect of $E$ on modal properties.
Figure 5.15: Predicted effect of pinion offset on natural frequency $\omega_r$. 
Figure 5.16: Dynamic mesh force response spectra per unit micrometer of transmission error for 3 pinion offsets.
Figure 5.17: Torsional moment response of pinion due to a unit micro-meter of transmission error excitation.
Figure 5.18: Resultant bearing reaction loads for the pinion and gear components normalized to a unit micro-meter of transmission error.
CHAPTER 6

NON-LINEAR TIME-VARYING DYNAMIC ANALYSIS

6.1 Introduction

In Chapter 4, a generalized 3-dimensional 14-DOF model was developed and its time-variant and load dependent coefficients are obtained by quasi-static gear contact analysis (CAPP). The model includes the effects of friction and backlash type non-linearity as well as time-varying mesh position and line-of-action. The effective gear mesh stiffness and transmission errors caused by tooth errors and elastic deformation are also spatial and load dependent due to the variation in load sharing among adjacent teeth in mesh. Moreover, the off line-of-action friction force generated at the mesh interface exerts a force and moment on the gear members and thus is a potential internal excitation. All the above mesh characteristics make the present study rather distinct from those for parallel gear dynamics, and no public study is known to address these unique issues.

The nonlinear dynamic problem in spur or helical gears in which gear backlash is present has been extensively studied by some researchers (Özgüven and Houser, 1988; Comparin and Singh, 1989; Kahraman and Singh, 1990, 1991(a), (b), Kahraman and Blankenship, 1996; Hochmann, 1997). There are generally three approaches in solving
the governing equations (i) Numerical integration methods, such as those used by Kucukay (1984), Lin et al. (1988), Ö zgüven and Houser (1988b), Umezawa et al (1984) Kahraman and Singh (1991) for single or higher DOF gear systems with backlash. This method can readily solve large classes of non-linearity problems. But in some situations such as lightly damping system, the solution near the jump frequency would depend on the choice of initial conditions (Kahraman and Singh, 1990); (ii) Piecewise linear technique used by Shaw and Holmes (1983) and Wang (1978) for two and three DOF torsional models with backlash. This technique cannot predict several non-linear phenomena such as subharmonic and chaotic responses (Galhoud et al., 1987); (iii) Single or multiple-term harmonic balance method for clearance type non-linearity. This technique is widely used in constructing analytical solutions and criteria for occurrence of tooth separation and back-collision (Comparin and Singh, 1989; Kahraman and Blankenship, 1996; Kim and Noah, 1991).

In the previous gear dynamic studies on parallel or right angle gears, the contact position and line-of-action are usually assumed as time-invariant. While this treatment may be quite reasonable in consideration of the small variation of gear output torque in some cases, the other dynamic consequences have not been investigated thoroughly. Due to considerable amount of tooth modifications sometimes applied to achieve less sensitivity to various gear tooth errors and mounting misalignment, the equivalent tooth contact position and line-of-action may produce a noticeable effect on the system dynamic response. On the other hand, the friction forces generated at the mesh interface produce oblique forces and moment reactions on the gear members that may lead to insignificant internal excitation, similar to the behaviors seen in parallel axis gearing.
problems (Ikeda and Muto, 1981; Rebbechi and Crisp, 1983; Lida et al., 1985; Hochmann, 1997) and spiral bevel gears (Handschuh and Kicher, 1996). However, for hypoid gears, the relative sliding between mating teeth is more uniform and does not reverse direction after passing the pitch point (line). Thus it may be anticipated that the friction would have less effect on structure-borne related gear noise. Nevertheless, even though the friction does not reverse its direction in one gear mesh cycle, the frictional torque can vary. Thus, it is still of interest to investigate the role of friction in hypoid gear dynamics. Moreover, since the present model can be completely used for spiral bevel gear dynamic analysis, in which the friction force direction does reverse its direction after passing pitch line, this study provides a tool for the further investigation for spiral bevel gear cases.

In this chapter, a 10 DOF non-linear time-varying (NLTV) lumped parameter model for hypoid gear pair with torsional and translational motions is studied. The numerical method based on the 5th-6th order Runge-Kutta integration routine with adaptive size will be used to compute the steady-state vibratory solution. An existing loaded tooth contact analysis tool (CAPP) which is based on finite element and surface integral methods is used to predict the required gear mesh parameters. The major objectives of this chapter are to provide a realistic procedure for modeling geared rotor systems and obtain an improved understanding of coupled torsion-translational vibrations. The assumptions applied to this study are: (i) pinion and gear bodies only have angular torsional motion; (ii) gear bodies are rigid with compliant mesh along the time-varying lines-of-action; (iii) theoretical tooth surfaces are generated by 3-dimension
mesh simulation; (iv) mesh position, line-of-action and friction force direction are obtained for a quasi-static case; (v) and friction coefficient is assumed to be a constant.

6.2 Formulation

The model consists of the hypoid gear pair connected to the load and engine possessing torsional coordinate only. The pinion and gear are allowed to rotate about their centerlines and translate in 3 orthogonal directions which are supported by the compliances of the shafts and bearings. Thus, the total DOF of the system is 10 rather than the previous 14 DOF developed in Chapter 4. It is pointed out from the study of the force transmissibility in Chapter 5 that the gear torsional motion is usually coupled with other vibration modes such as translations and rotational motions about other non-nominal rotational axes. The rotational motions about non-nominal rotational axes (i.e., X- and Z- axes in Figure 4.4) will not be included in the present study. However, this should not imply that they are not important, rather at the present time this dissertation focuses primarily on the torsional and translational vibration couplings. Another reason for using 10 DOF is to achieve reasonable convergent time for steady state solution.

The general displacement vector of each gear is given by \( \mathbf{q}_i(t) = [x_i, y_i, z_i, \theta_i]^T \), where \( x_i, y_i \) and \( z_i \) are the translational displacements and \( \theta_i \) is the angular displacement about the respective \( Y \)-axis as shown in Figure 4.4. This results in a 10 DOF system that is semi-definite due to the rigid body torsional motion. In order to make it positive-definite, the following transformation is applied

\[
\mathbf{u}_i(t) = \theta_i - \theta_E
\]  

(6.1a)
\[ u_2(t) = \lambda_2 \theta_2 - \lambda_1 \theta_1 \]  
\[ u_3(t) = \theta_2 - \theta_0 \]  

where $\lambda_1 = \lambda_y^{(1)}$ and $\tau_1 = \tau_y^{(1)}$. Then, the system equations of motion can be rewritten in terms of the relative quantities reducing the order of the system to 9. From Equation (4.18), the following equations are obtained:

\[
[\ddot{u}_1] + \frac{(\lambda_1 - \mu \tau_1)}{I_1} f(\delta_d - e_L) + k_{\lambda_1} (\frac{1}{I_E} + \frac{1}{I_1}) [u_1] + c_{\lambda_1} (\frac{1}{I_E} + \frac{1}{I_1}) [\dot{u}_1] = -\frac{T_1}{I_E} \tag{6.2a}
\]

\[
[\ddot{u}_2] - \frac{(\lambda_2 - \mu \tau_1)}{I_1} f(\delta_d - e_L) - \frac{\lambda_2 k_{\lambda_2}}{I_1} [u_1] + \frac{\lambda_2 c_{\lambda_2}}{I_1} [\dot{u}_1] + \frac{\lambda_2 c_{\lambda_2}}{I_2} [\dot{u}_3] = 0 \tag{6.2b}
\]

\[
[\ddot{u}_3] - k_{\lambda_2} (\lambda_2 + \mu \tau_2) f(\delta_d - e_L) + (\frac{1}{I_2} + \frac{1}{I_2}) k_{\lambda_2} [u_3] + \frac{\lambda_2 c_{\lambda_2}}{I_2} [u_3] + (\frac{1}{I_2} + \frac{1}{I_2}) c_{\lambda_2} [\dot{u}_3] + \frac{\lambda_2 c_{\lambda_2}}{I_2} [\dot{u}_3] = -\frac{T_2}{I_o} \tag{6.2c}
\]

\[
[M_1][\ddot{U}_1] + (h^{(1)})^T - \mu g^{(1)} f(\delta_d - e_L) + [C_{1b}][\dot{U}_1] + [K_{1b}][U_1] = 0 \tag{6.2d}
\]

\[
[M_2][\ddot{U}_2] - (h^{(2)})^T + \mu g^{(2)} f(\delta_d - e_L) + [C_{2b}][\dot{U}_2] + [K_{2b}][U_2] = 0 \tag{6.2e}
\]

where

\[
[M_1] = \begin{bmatrix}
m_1 & m_i \\
m_i & m_1 \\
m_i & m_i
\end{bmatrix} \tag{6.3a}
\]
\[ [U_1] = \{u_4 \; u_5 \; u_6 \}^T = \{x_1 \; y_1 \; z_1 \}^T \tag{6.3b} \]

\[ [U_2] = \{u_7 \; u_8 \; u_9 \}^T = \{x_2 \; y_2 \; z_2 \}^T \tag{6.3c} \]

\[ \mathbf{h}^{(i)}(t) = \{n_x^{(i)} \; n_y^{(i)} \; n_z^{(i)} \} \tag{6.3d} \]

\[ \mathbf{g}^{(i)}(t) = \{v_x^{(i)} \; v_y^{(i)} \; v_z^{(i)} \} \tag{6.3e} \]

The nonlinear function \( f(\delta_d - e_L) \) is given in Equation (4.21), and other parameters have already been defined in Chapter 4. Note that gear backlash, time-varying mesh characteristic and off line-of-action friction force are taken into account in this NLTV model.

### 6.3 Computational Analysis and Discussions

The solutions for the above non-linear time-varying vibration equations are obtained by numerical integration using the 5/6th order Runge-Kutta routine with adaptive size. The result provides the time domain steady-state response. As part of the solution scheme, the second order differential equations must be casted in the state-space form generally given by \( \dot{u}_i = f_i(u_1, u_2, \ldots, u_{18}) \), where \( i = 1, 2, \ldots, 18 \). The proposed computational procedure is shown in Figure 6.1. For comparison purpose, the corresponding linear model is also solved using the modal summation method described in Chapter 5. The LTE calculated from CAPP is used as input in the simulation. For a specific mesh position, the dynamic load can be computed from Equation (4.22).
Machine and cutter settings for hypoid pinion and gear (Table 3.2).

Gear tooth surface and finite element mesh generations based on the simulation of cutting process.

Loaded gear contact analysis applying CAPP program.

Calculations of normal and friction load distributions, equivalent mesh, such as LTE, $k_m$, $n_u^{(1)}$, $A_{v}^{(l)}$, $r_{u}^{(l)}$, etc.

Determination of the system parameters: mass, inertias, and shaft- bearing compliance.

The 5/6th order Runge-Kutta integration routine with adaptive size.

Frequency or order domain analysis for steady-state response.

Figure 6.1: Flow chart of the proposed computational procedure.
Negative dynamic load indicates tooth separation. When this happens, the backside collision is checked using Equation (4.21).

First, LTE and $k_m$ are computed under different loading conditions. The present study found that torque applied on the pinion significantly modifies the shape of the loaded transmission error, as shown in Figure 6.2(a). In the unloaded case, TE is parabolic in shape and the gear lags the pinion. With increasing applied torque, the LTE flattens due to more cells coming into contact. Different load levels will also change the contact positions and tooth numbers in mesh. The Fourier coefficients of LTE are shown in Figure 6.2 for two loading conditions. Under light load, LTE is dominated by the first harmonic. On the other hand, more significant higher mesh harmonics are observed under heavy load. Figure 6.3 illustrates that the mean mesh stiffness is load dependent, and is a function of contact position, load and tooth modification.

Next, the free vibration of the corresponding linear model with time-invariant mesh stiffness and force vectors is analyzed. A typical hypoid gear set in Tables 3.2 and 5.1 is used as an example. Three types of linear modes are identified: (i) out-of-phase gear torsional mesh coupled with translational motions of pinion and/or gear; (ii) in-phase gear torsional mesh coupled with translational motions of pinion and/or gear; and (iii) pure translation motions of pinion and/or gear body. The predicted modes and their natural frequencies are provided in Table 6.1 for three input pinion torque levels. Modes 5 and 8 are pure translations that are decoupled from the mesh coupling coordinate. Thus, their corresponding natural frequencies are independent of load or mesh stiffness. On the other hand, the natural frequencies corresponding to the modes with strong gear mesh dependency such as modes 7 and 9 vary more with load. Note that the natural frequency
(a) Loaded transmission error under different loads (in-lbf).

(b) 1000in-lbf

(c) 4500 in-lbf

Figure 6.2: Loaded transmission error and Fourier coefficients of loaded transmission error under two different pinion torques.
Figure 6.3: Load dependent mean mesh stiffness under 4 different pinion torques.

will also be affected by the change in mesh position and line-of-action, which in turn are load-dependent.

6.3.1 Dynamic Characteristic of Time-varying Mesh Vectors

First, the mesh characteristic vectors such as $\lambda_u^{(l)}$, $n_u^{(l)}$ for normal force, and $\tau_u^{(l)}$ and $v_u^{(l)}$ for friction force are computed from Equations (4.11) and (4.12). Then, the effect of these time-varying mesh characteristic vectors on the gear dynamics is investigated. Figures 6.4 and 6.5 shows that the mesh characteristic vectors, which include normal and
<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Mode Description</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1000 in-lbf case</td>
</tr>
<tr>
<td>In-phase torsional and translations</td>
<td>$2 \left( Y_1 - Y_2 - \theta_E \right)$</td>
<td>222.4</td>
</tr>
<tr>
<td>Pure translations</td>
<td>$5 \left( Z_2 \right)$</td>
<td>427.4</td>
</tr>
<tr>
<td></td>
<td>$8 \left( X_1 \right)$</td>
<td>887.9</td>
</tr>
<tr>
<td>Out-of-phase torsional mesh and translations</td>
<td>$1 \left( Y_1 - Y_2 - \theta_E \right)$</td>
<td>204.1</td>
</tr>
<tr>
<td></td>
<td>$3 \left( Y_1 - X_2 - Z_2 - \theta_E - \theta_o \right)$</td>
<td>342.7</td>
</tr>
<tr>
<td></td>
<td>$4 \left( Y_1 - X_2 - Z_2 - \theta_o \right)$</td>
<td>391.2</td>
</tr>
<tr>
<td></td>
<td>$6 \left( Y_1 - X_2 - Z_2 - \theta_o \right)$</td>
<td>436.6</td>
</tr>
<tr>
<td></td>
<td>$7 \left( Z_i - Y_1 - X_2 - Z_2 \right)$</td>
<td>786.0</td>
</tr>
<tr>
<td></td>
<td>$9 \left( Z_1 - Y_1 \right)$</td>
<td>1450.0</td>
</tr>
</tbody>
</table>

Table 6.1: Classification of linear mode shapes.
off line-of-action friction direction vectors, and directional rotation radii as functions of time (mesh position) and load. It is found that under light load, these mesh characteristic vectors have more variations than those under higher load due to the fact that more cells or teeth are brought into contact as shown in Figure 6.4(a). As also shown in Figures 6.6 and 6.7, when the gears rotate through the mesh under load, the tooth numbers periodically varies between 1 pair and 2 pairs. The equivalent mesh and friction force vectors are shown to change more rapidly at the mesh positions at which contact tooth numbers change.

Since all the mesh characteristic vectors affect the instantaneous forces and moments acting on the pinion and gear and vary with mesh frequency, they can be regarded as an internal excitation source. To understand their implications in the dynamics of the system, the NLTV model neglecting friction force is used to compute the dynamic responses of mesh force, pinion and gear bearing forces under steady state condition. Figures 6.8 and 6.9 show the comparison of the dynamic mesh force and bearing force spectra predicted using the time-varying and time-invariant mesh vectors under relatively high pinion torque, such as that there are no tooth separations. Note that the frequency response magnitude is represented by the half of maximum peak-to-peak values in the steady state forced response time history. The results show that the time-varying mesh characteristic vectors produce an increase in the response amplitude, especially in the lower frequency range. The participating resonant peaks are identified as modes 1, 3 and 7 shown in Table 6.1, which are out-of-phase torsional motion coupled with translation motions. The resonant peaks consist of primary resonances in the vicinity
of natural frequencies and super-harmonics which are caused by higher order terms of LTE. To demonstrate this behavior, the constant mesh stiffness is used to exclude the possibility of generating super-harmonics by the higher order of \( k_m \) (Blankenship and Kahraman, 1996). Figure 6.9 shows the forced frequency responses under 4500 in-lbf pinion torque. The resonances at 209 and 340 Hz are identified to be the modes 1 and 3, respectively, as described in Table 6.1. The dominant resonance at \( f_m = 900 \text{ Hz} \) of the mesh force response is the super-harmonic of \( \frac{f_g}{2} \), which is caused by the 2nd harmonics of LTE when it coincides with the primary resonance (mode 9) at \( f_g \).

The forced response in the time domain with the time varying and time-invariant mesh vectors are also investigated. Figure 6.10 shows the mesh loads at different mesh positions for the time-varying and time-invariant mesh vectors at the third natural frequency of 344.4 Hz, the FFT calculation shown in Figure 6.11 suggests that the time-varying mesh vectors increase the magnitude of the forced response. Figure 6.12 further shows the lighter load case of 1000 in-lbf. It contains rich harmonic contents that are excited by the mesh force response in the off resonant regime. In the resonant vicinity, the forced response is dominated by pure tone harmonics as indicated in Figure 6.13. In the case where there is tooth separation, the time-varying mesh vectors also affect the jump frequencies of the response, as shown in Figure 6.14 for the pinion bearing response.
Figure 6.4: Directional rotation radius of quasi-static equivalent normal force under forward drive condition. The tooth numbers in mesh are also shown for 4500 in-lbf (forward drive, solid line for 4500 in-lbf, dotted line for 1000 in-lbf).
Figure 6.5: Mesh characteristic vectors under forward drive condition under different torques (in-lbf).
Figure 6.6: Load sharing in one mesh cycle at 4500 in-lbf pinion torque.

(a) at roll angle $-16^0$          (b) at roll angle $3^0$

Figure 6.7: Load distributions at two mesh positions.
Figure 6.8: Comparison of the frequency response functions of the dynamic mesh force for time-varying (TV) and time-invariant (TI) mesh vectors, given 4500 lbf-in of input torque. The sinusoidal TE is used.
Figure 6.9: Comparison of the frequency response functions for the case of the time-varying (TV) and time invariant (TI) problem given 4500 in-lbf of input torque. Note that the linear time-invariant response (LTI) is also shown.
Figure 6.10: Dynamic mesh loads for one mesh cycle at the resonant frequency of 340 Hz for the case of time-varying and time-invariant mesh vectors at 4500 in-lbf.

Figure 6.11: FFT spectra of the mesh loads for the above cases shown in Figure 6.10.
Figure 6.12: Dynamic mesh force for one mesh cycle at the off-resonance frequency of 483 Hz at 1000 lbf-in for the case of the time-varying (TV) and time invariant (TI) mesh vectors.
Figure 6.13: Dynamic mesh force for one mesh cycle near the resonant frequency of 740 Hz at 1000 lbf-in for time-varying (TV) and time-invariant (TI) mesh vectors.

Figure 6.14: Comparison of the dynamic pinion bearing force function for the case of 2000 in-lbf of input torque.
6.3.2 Forced Response Characterization and Load Effect

It is known that any particular tooth modification aimed to reduce gear noise is designed for a certain range of operating load (Tavakoli and Houser, 1986; Krenzer, 1981, Gosselin, et al., 1995). To investigate the forced response for different loading cases, four load scales are used to study the effect of the mean torque applied on the pinion, and they are 1000, 2000, 4500, 7000 in-lbf. Figure 6.15 shows the mesh force response under 3 torques with the same mesh stiffness, which is artificially assumed in order to investigate the effect of magnitude of transmission errors on the response. The result suggests that the forced response amplitude under light load, which is corresponding to higher magnitude of transmission error as shown in Figure 6.2, be higher than that under heavy load condition. The mesh and pinion bearing force spectra for 3 different input torques are shown in Figures 6.16 and 6.17 respectively using their respective mesh stiffness. Under lightly loaded conditions (1000 in-lbf), tooth separation occurred between 1250 Hz and 1830 Hz. This produced the classical jump phenomenon, where the frequency response is discontinuous in the vicinity of the resonant frequency. In this case, it is noted that the upper branch was produced by decreasing the rotational speed, while the lower branch was formed by slowly increasing the rotational speed. This non-linear behavior is analogous to the phenomena of a softening spring. It may be noticed from Figure 6.16 that a higher TE under a lighter load may not show larger responses, since the load also changes the mesh stiffness value (Figure 6.3), which in turn will contribute to the amplitude of the forced response. Figure 6.18 shows the time
Figure 6.15: Dynamic force under different loads without friction effect. The mesh stiffness $3 \times 10^8$ N/m is used for the 3 cases. 

( $\leftrightarrow$ 1000 in-lbf; $\rightarrow$ 4500 in-lbf; $\longrightarrow$ 7000 in-lbf)
Figure 6.16: Dynamic mesh force under different loads without friction effect (······ 1000 in-lbf; 4500 in-lbf; 7000 in-lbf).
Figure 6.17: Pinion bearing force at different loads without friction effect (--- 1000 in-lbf; - - 4500 in-lbf; --- 7000 in-lbf).

Figure 6.18: Mesh force at the jump frequency of 1283 Hz (---) and 1230 Hz ( - - - ) under light load 1000 in-lbf.
history functions for the mesh force before and after the jump. It was noticed that tooth load was zero when the separation occurred, and no back-collision was observed.

The tooth separation is not observed for input loads 4500 and 7000 in-lbf, in which the teeth always maintain continuous contact and the system behaves much like a linear time-invariant system in spite of the backlash present. The reason that separation occurs at light load rather than heavy is because the LTE of the present system is larger at the light load than at the other 2 higher loads. It is interesting to note that the resonant frequencies shifted to lower values at the light load. This is due to the fact that the mesh stiffness is lower than that at high load, as pointed out previously. Carefully examination of the frequency response functions of the mesh force and bearing force shown in Figures 6.16 and 6.17, it is noticed that modes shown up in the mesh force response may not be the same as those in the bearing force reaction. For example, at load 4500 in-lbf, mesh force response has the resonance at mesh frequency of $f_m$=890 Hz. But for bearing force response, this resonance did not show up in the same frequency, rather it showed up at 800 Hz, which is mode 7. To explain this phenomena, two cases based on the linear model were simulated. The first case used a sinusoidal LTE (fundamental LTE) and constant mesh stiffness, and the second used the first three harmonic spectrum LTE and the constant mesh stiffness. The mesh force responses from these two cases are shown in Figure 6.19(a). It was found that the fundamental LTE excited the mode 7 ($f_t$=799.7 Hz), and 2nd harmonics of LTE excited the mesh force resonance at $f_m$=890 Hz, which is $\frac{f_s}{2}$ or $2f_m$ super-harmonics, as explained earlier. However, this second harmonic of TE did not excite the same resonance in the pinion bearing force response, as evidenced in
Figure 6.19(b). This result suggests that a commonly used linear model with sinusoidal TE would result in loss of super-harmonics or modes.

To investigate the load effect on the non-linear forced frequency response, further simulations were done under two relatively light loads. Figure 6.20 shows the frequency response functions of the dynamic transmission error for 1000 and 2000 in-lbs along with the corresponding linear time-invariant solutions. It is noted that the jump frequencies are dependent on torque due to the loaded dependent mean mesh stiffness used. The primary resonant modes are 1, 3, 7 and 9, which are the out-of-phase torsional motion coupled with translation motions of the gear pair. The resonances around 480 Hz for 1000 in-lbf case and 580 Hz for 2000in-lbf case are not the primary modes associated with free vibration modes, rather they are the super-harmonics generated by the higher harmonics of LTE. To justify this observation, FFT spectrum of the time trace of the response at 2000 in-lbf is illustrated in Figure 6.21. It shows that even though the system was being operated at $f_m = 580$ Hz, the response was clearly dominated by the $3f_m$ harmonic component, since the 3rd LTE harmonic hits the 9th natural frequency.

6.3.3 Off-line-of-action Friction Excitation

For hypoid gears, the relative sliding between mating teeth is more uniform than that for spiral bevel gears. Moreover, the relative sliding does not reverse direction as seen in spur or spiral bevel gears when the gear passes through the pitch line, as evidenced from the mesh vector components shown in Figures 6.4 and 6.5. Thus, it may be anticipated that the friction would have less effect on structure-borne related gear
(a) Dynamic mesh force response.

(b) Dynamic pinion bearing force response.

Figure 6.19: Dynamic mesh force and pinion bearing force response due to sinusoidal and periodic TE excitation (with 4500 in-lbf load, no friction, time-varying mesh vector and constant mesh stiffness).
Figure 6.20: Dynamic transmission error vs. load with constant mesh stiffness and without friction effect. The corresponding linear solutions are also shown. The super-harmonics result from the 3rd harmonics of LTE excitations.
Figure 6.21: Spectrum of DTE at running frequency of 580 Hz for 2000 in-lbf case.
noise. But even though the friction does not reverse its direction in one gear mesh cycle, the frictional torque varies due to the variations of the magnitude of the friction force and direction at gear mesh interface. These variations are indicated by the mesh vector components such as \( r^{(i)}_w \) and \( v^{(i)}_w \) \((u=x,y,z)\). To investigate the friction effect on the dynamic behavior, the simulation was made for the assumed high friction coefficient of 0.1. Figure 6.22 shows that the friction does not pose severe influence on the forced response, and it usually affects the forced response at low frequency range. From the time history of the dynamic TE and mesh force shown in Figure 6.23, it can also be seen that the mean values of the responses are lower if the friction effect is considered than that without the friction effect. This is resulted from the lower normal load that occurs if friction is included, leading to lower mean values of the dynamic transmission error and mean mesh force. The variation of friction induced torque acting on the pinion is shown in Figure 6.24.

It is pointed out that the present model is also capable of simulating the dynamic behavior of spiral bevel gears. In this case, sliding friction direction will reverse when the meshing passes through the pitch line (Handschu and Kicher, 1996). Thus it is expected that the friction will have shuttle force type of excitation, like that found in spur or helical gears (Hochmann, 1997). The investigation on friction excitation for spiral bevel gears is left for further study.
Figure 6.22: Effect of friction force for the response spectra under
dynamic mesh force and pinion bearing force. (4500 in-lbf load with
constant mesh stiffness assumption).
Figure 6.23: Comparison of dynamic transmission error and mesh force at 512 Hz for one mesh cycle (4500 in-lbf load with friction coefficient of 0.1).
Figure 6.24: Friction induced torque on pinion for one mesh cycle at 512 Hz (with 4500 in-lbf load, constant mesh stiffness, and time-varying mesh vectors)

6.3.4 Effect of Operation Conditions

Hypoid gear tooth profile is not symmetric, as discussed in Chapter 2. Some experimental data showed that the dynamic responses in terms of the amplitude and resonant frequency operated under the two flanks of teeth are difference (Nakayashiki, et al., 1983). There are two operating situations for the hypoid gears: one is the forward drive with the concave flank side of the pinion meshing with convex side of the gear, and the other is forward coast with the convex side of the pinion meshes with concave side of the gear. The previous dynamic models with employing simplified mesh vectors are unable to interpret the difference in the forced vibration (Pitts, 1972; Kiyono et al., 1981;
Abe and Hagiwara, 1990). The present model is used to study the effect of the above operating conditions on the vibration of hypoid gears. As it was discussed in Chapter 5, the difference in dynamic response under the two operating conditions is due to the fact that the mesh force vectors change directions, along with additional variations in the mesh stiffness and transmission error. The present model is capable of taking into account all of these mesh property variations. Figure 6.25 indicates that under coast condition, the mesh force usually has lower resonant frequencies, but higher resonant peak in the higher speed range, while lower peak in lower speed range, which agrees with trends found in the test results to be discussed in the next chapter.

Next, the overall vibration of the system in frequency or mesh order domain is to be investigated. The 3-dimensional waterfall simulation of speed sweep is conducted. The Fourier Transform (DFT) method is performed to obtain the frequency domain results. This simulation can provide the vibration levels with several mesh harmonics. Figures 6.26 and 6.27 show the speed sweep waterfall plots for the dynamic pinion bearing load and mesh force in the frequency order at 4500 in-lbf of input torque. It is shown that the response peaks for the first mesh order correspond to the damped resonant frequencies for the first mesh order like the one shown in Figure 6.16. Note that the first harmonic usually dominates the whole vibration spectra significantly more than other higher harmonics.

6.4 Summary

A translation-torsional coupled non-linear time-varying mesh dynamic model based on the loaded contact analysis of hypoid gears is studied numerically. The gear
backlash non-linearity and the friction force are also included in this generalized model. The loaded transmission error and the load dependent mesh stiffness are integrated into the dynamic analysis. The resonant modes are identified for primary resonance and super-harmonics. This study investigated for the first time the effect of time varying mesh vectors on the gear dynamics. Under lightly loaded hypoid geared system, tooth separation was noticed and the jump phenomenon is seen in the frequency response function. Friction effect is not strong since the relative sliding force does not reverse direction even though it is periodical. Both forward drive and coast operating cases are also realistically simulated.
Figure 6.25: Comparison of mesh force spectra for forward drive and coast operation conditions with 4500 in-lbf of load, no friction effect, constant mesh stiffness and time-varying mesh vectors.

Figure 6.26: Waterfall plot of the dynamic pinion bearing force response at 4500 in-lbf of pinion torque.
Figure 6.27: Speed sweep waterfall for mesh force at 4500 in-lbf of input torque.
CHAPTER 7

EXPERIMENTAL STUDY

7.1 Introduction

Hypoid gears are commonly used in the rear axle components of some automobiles, trucks, and off-highway equipment. The noise generated from this type of gear typically occurs at tooth mesh frequency, and is one of the predominant noise sources, which can lead to undesirable annoyance problem. Its frequency range is normally between 300-1000 Hz for typical driving conditions of about 40-70 mph vehicle speed, and the spectrum is characterized by one or more distinct tonal peaks. This chapter presents the experimental vibration and noise results for a rear axle carrier consisting of a hypoid gear set, housing components and other coupling accessories and supporting elements. The objective is to quantify the relationships between radiated noise and vibration spectra, and validate of the proposed model. The correlation between measured data and computer simulation will also be presented.

7.2 Test Description

To better quantify the hypoid gear vibration and noise problem, a relatively noisy truck carrier component was tested in a hemi-anechoic chamber. The carrier was driven
by an electric motor with its output connected to an identical pair of absorbing
dynamometers. The overall test setup is shown in Figures 7.1-7.3. Both sound pressure
level (SPL) and vibrations were measured at selected points in the setup. Specifically, a
tri-axial accelerometer (B&K type 4326) was attached to the pinion bearing neck to
measure axial, lateral and vertical vibrations. In addition, three microphones (B&K type
2235) were placed on the side and front locations of the carrier to record the radiated
sound pressure response. The locations of the microphones were situated distant enough
from the carrier to be located in the far field. The microphone’s output voltages were
converted into sound pressure level (dB) by scaling the measured voltages according to
SPL = 20Log_{10}(V / V_{ref}). The test gears were lubricated with oil and shielded from the
vibrations of the input and output parts by massive flywheels. The experiment was
conducted for the following operating condition:

(1) Speed-sweep up and down from 980 rpm (35 mph) to 1960 rpm (70 mph).

(2) Constant speeds from 980 rpm to 1960 rpm with 140 rpm increment and holding
time of 40 seconds.

(3) Two input torque levels were applied on the pinion shaft: 100 ft-lbf and 500 ft-lbf.

(4) Forward and coast drive to allow mating of both tooth flanks.

The transducer outputs were amplified with signal conditioning amplifiers and
recorded with two 8-channel digital data tape recorders using a 24 kHz sampling
frequency setting. This provided a 10 kHz usable bandwidth. Note that 2 of the 8
Figure 7.1: Schematic of the experimental setup and measurement locations.
Figure 7.2: Front view of the experimental setup. The tri-axial accelerometer located in the pinion bearing nose. Three microphones were in the west (W), east (E) and south (S), respectively.
Figure 7.3: Test layout for a truck carrier component (Courtesy of Meritor Automotive).
channels were used for the tachometer pulse trains of the input and output shafts (120 pulse/rev). The rest of the channels are used for accelerometer and microphone signals. The measured data was then digitized into a workstation through a front end (HP 3565 model). A frequency divider circuit was also used to decrease the input shaft tachometer frequency such that it would be usable with the 8192 Hz sampling frequency applied in the digitization process. The flowchart in Figure 7.4 summarizes the overall signal processing procedure.

7.3 Results Analysis

The gear set was initially operated in speed sweep condition from 980 rpm (35 mph) to 1960 rpm (70 mph) with step size of 140 rpm. Figure 7.5 shows typical waterfall plots of the measured dynamic behavior of the carrier component tested. It is observed that the first mesh harmonic (12th shaft order) is clearly dominant in all of the acoustical and acceleration spectra as expected in the case of a noisy gear set. This is also evident from the order tracking results comparing the response functions of the first, second and third mesh harmonics, as shown in Figure 7.6. It can be clearly identified from the waterfall function of Figure 7.5 and order tracking function of Figure 7.6 that the resonance points occur at approximately 202, 264 and 315 Hz when excited by the fundamental mesh at the pinion torque 500 ft-lbf. These resonance points actually correspond to 38, 47, and 57 mph vehicle speeds respectively. It is also noted that for the speed range tested here, the second mesh harmonic starts low enough in frequency to be able to excite the 315 Hz resonance frequency. On the other hand, the higher resonance frequency at 600 Hz can only be excited by the second and third mesh harmonics.
Figure 7.4: Flowchart of signal processing and analysis procedure.
By comparing the fundamental mesh vibrations of the pinion bearing coordinates for the case of 500 ft-lbf of input mean torque load as shown in Figure 7.7, it can be seen that the axial vibration is significantly lower than in the vertical and lateral motions. Moreover, the resonant peaks differ by some degree for each given coordinates. This variation in resonant frequencies may be related to the differences in the effective stiffness, mass or inertia values of the vibratory system contributing to these 3 spectra. Similar characteristics are observed in Figures 7.8 and 7.9, that show measured vertical acceleration at the pinion bearing and sound pressure responses for 100 ft-lbf of pinion input mean torque. In addition, the levels depicted in the SPL spectrum shown Figure 7.10 at the three locations are quite similar to the ones observed in the pinion bearing vibration spectrum. This suggests that the pinion bearing motion, in particular the vertical coordinate, can be reasonably used as an indication of the generated whine level of the hypoid geared drive. This correlation was also reported in Remmers (1971) experimental study. Also, it may be pointed out that similar results and trends are also seen in the power spectra of the SPL and vibration spectra corresponding to approximately 1400 rpm of steady state cruise shown in Figures 7.11 and 7.12. Here, one can clearly see that the relative noise peaks and pinion vertical vibration peaks are very similar.
(a) Vertical acceleration

(b) Sound pressure level

Figure 7.5: Waterfall plot of the vertical acceleration at the pinion bearing nose and sound pressure at the west location for speed sweep condition from 980 rpm with 1960 rpm with 500 ft-lbf input torque under forward drive condition.
Figure 7.6: Order tracking of the first, second and third mesh harmonics corresponding to 500 ft-lbf pinion input mean torque.
Figure 7.7: Fundamental mesh vibrations of the pinion bearing coordinates for 500 ft-lbf of input mean torque on the pinion side.

Figure 7.8: Fundamental mesh vibrations of the pinion bearing coordinates for 100 ft-lbf of input mean torque on the pinion side.
Figure 7.9: Order tracking response of the first, second and third mesh harmonics of the pinion bearing vertical coordinate and external SPL (west side) response for 100 ft-lbf of input mean torque.
Figure 7.10: Comparison of SPL at different locations for the first mesh order (100ft-lbf, drive condition)
Furthermore, the measured vibration and sound pressure response clearly exhibit both amplitude and frequency modulations as evidenced by the existence of strong side bands spaced at multiples of pinion shaft frequency clustered around each mesh harmonics as shown in Figure 7.13. The presence of side bands at spacing of pinion shaft frequency is not unusual in vibration signal generated by geared system. It is related to the geometric irregularities in the rotating members such as tooth spacing errors, variation of the rotational speed, and bolt hole distortions (Remmers, 1971, Houser and Singh, 1999).

To investigate the effect of input mean torque load on vibration response, the motion at the pinion bearing was measured for two torque values. The resultant vibration spectra are shown in Figure 7.14. In this figure, one can see that increasing torque from 100 ft-lbf to 500 ft-lbf will essentially shift the natural frequencies higher. However, the peaks do not have obvious change in magnitude, except around 210 Hz, where the smaller mean torque load produces a 5 dB higher vibration peak compared to the 500 ft-lbf case. The frequency shift is most likely caused by the higher mesh stiffness at higher load, as indicated in Figure 6.3 earlier.

Finally, Figure 7.15 shows the fundamental mesh vibration spectrum under drive and coast operating conditions. Since the drive and coast flanks engage different tooth sides in mesh, the mesh force direction changes, reversing the axial bearing force direction. This resulted in the change in the system stiffness property, as shown analytically by Equation 6.2.
Figure 7.11: Power spectra of sound pressure response measured at the west side location for steady state speed of about 1400 rpm subjected to 100 ft-lbf of input mean torque load.

Figure 7.12: Power spectrum of vertical vibration at pinion neck location for steady state speed of about 1400 rpm subjected to 100 ft-lbf of input mean torque load.
Figure 7.13: Evidence of strong side band centered about the fundamental mesh $f_m$.

Figure 7.14: Effect of input mean torque load on the vertical vibration spectrum of the fundamental mesh component.
Figure 7.15: Fundamental mesh vibration spectra measured in drive and coast operating conditions.
7.4 Correlation of Experimental and Simulation Results

The proposed theory is used to predict dynamic mesh force and bearing reaction loads. Here, the NLTV model described in Chapter 6 was applied with time-invariant gear mesh vectors and stiffness. Table 7.1 shows the gear geometry data. The hypoid gear set was manufactured with a face-hobbed method. Since the mesh generator does not exist for the face-hobbed hypoid gear at the present time, mesh stiffness and transmission error are estimated from the Gleason Load Tooth Contact Analysis (LTCA) instead. Figure 7.15 shows the trend comparison of the dynamic mesh force response spectrum computed from the simulation study and the SPL response measured from the experiment described earlier. The comparison suggests that the mesh force response is closely related to the overall noise level of the hypoid gears and can be used as the noise indicator, as suggested by many gear dynamic investigators (Houser and Singh, 1999). Figures 7.16 and 7.17 show a comparison of order-tracked experimental data and predicted response functions of the pinion nose vibration. The prediction appears to be quite reasonable for the vertical and lateral coordinates but is less accurate for the axial one. In spite of the discrepancy, the overall correlation is reasonably good. The error seen in the simulated and measured resonant frequency is found to be approximately 5~11%. This discrepancy may be due to the following reasons:

(a) The limitations of lumped parameter model used since the flexibilities of the shaft and other components such as flanges, housing and the like are not considered in the simulation.
(b) Discrepancies in mass and inertia distributions.

(c) Bearing stiffness, mesh stiffness and damping uncertainties.

(d) Potential misalignment in the assembly, which is clearly not considered in the model.

(e) Nonlinear effects such as gear backlash, bearing clearance and friction are not used in the prediction due to the unavailability of these properties.

(f) Other factors associated with higher orders of mesh stiffness and transmission error are not applied as they are not clearly known or understood.

Figure 7.19 shows the mesh force transmissibility of the pinion bearing force in the axial, lateral and vertical directions computed based on Equation (5.6). It clearly indicates that the pinion vertical bearing load is larger than those in the other two directions, with axial reaction lowest. All the three modes at 159, 237 and 322 Hz are characterized by out of torsional motion of the gear pair associated with translational motions. For instance, the mode at 322 Hz is the axial and vertical motion of the pinion and the vertical and lateral motions of the gear. The mode at 237 Hz is the axial and vertical motions of the pinion and the vertical and lateral motions of the gear. The mode at 159 Hz is pinion axial and gear lateral motion. Thus it suggests that the dominant modes are usually characterized by axial and vertical motions of the pinion, which are consistent with that shown in Table 5.2.
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Table 7.1: Design parameters of the tested hypoid geared drive.
Figure 7.16: Comparison of the simulated dynamic mesh force and measured radiated noise corresponding to the fundamental mesh component.
Figure 7.17: Comparison of measured and predicted vibration spectra corresponding to the vertical coordinate of the pinion bearing. The results of the two test carriers that are identical are shown.
Figure 7.18: Comparison of measured and predicted vibration spectra corresponding to the pinion bearing coordinates. The results of the two test carriers that are identical are shown.
Figure 7.19: Force transmissibility function $[G_i^{(l)}]$ for the pinion bearing in axial, lateral and vertical directions excited by transmission error.
7.5 Summary

The sound pressure and vibration response were measured for a relatively noisy hypoid geared carrier used in a truck application for 2 different pinion torques under drive and coast conditions. The dominant vibration and noise peaks are clearly observed at the gear mesh harmonics. Also, the analysis shows that the vertical motion of the pinion bearing location can be used as a relatively good indicator of radiated noise level. The input mean torque loads applied did not show an obvious effect on the vibration amplitude, but cause a clear shift in the resonant frequencies. The test results were simulated by applying the proposed dynamic model that includes translation-torsional motions of pinion and gear. The simulation showed that the predicted trend in dynamic mesh force closely matches the overall noise spectrum. Additionally, the vibrations of the pinion bearing coordinates show reasonably good agreement with the predicted response.
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 Conclusions

1. A generic algorithm for the generation of spiral bevel and hypoid gear surface in three-dimensional space is developed. Hypoid gear tooth surfaces are represented analytically and generated numerically by the simulation of the cutting process. This generated tooth surface is the basis for the proposed 3-dimensional tooth contact mechanics model, which is being performed using the Contact Analysis Program Package (CAPP).

2. An efficient method is developed to predict unloaded kinematic transmission error and contact path. Furthermore, a new mesh position and force vectors are derived explicitly and can be used for face-milled and face-hobbed hypoid gears under different operating conditions such as forward and coast driving conditions.

3. A new generalized translation-rotational 14 DOF gear dynamic model has been developed for modeling hypoid and spiral bevel geared rotor system. The model incorporates the load-dependant time-varying mesh characteristic vectors due to tooth load sharing and tooth profile modifications, backlash non-linearity as well as off
line-of-action friction forces. Based on the 3-dimensional tooth contact analysis results, the quasi-static mesh characteristics that describe the translation-rotational and rotational-rotational force couplings are developed for the dynamic formulation.

4. The new gear mesh coupling formulation based on the exact gear geometry and kinematic relations have been used to model the time-invariant response of the hypoid geared rotor system. Free and forced vibrations in the presence of transmission error excitation are examined, and the critical pinion-gear out-of-phase torsion modes that affect the undesirable vibration response are identified. The proposed system model also readily predicts the complete gear mesh transfer functions and force transmissibility spectra. These results are used to characterize the nature of the system vibratory behavior including the dynamic coupling and sensitivity of the vibratory response to critical design parameters.

5. The three-dimensional representations of the mesh vectors, normal and friction forces, and moments generated at the mesh interface are developed. It is believed that the present approach is a major improvement over the previous dynamic models which do not consider actual magnitudes and positions of the time-varying load distributions. The effect of time-varying mesh vectors on the gear oscillatory motions is investigated. Tooth separation and the occurrence of jump phenomenon in the frequency response functions were studied. Friction effect is found not significant since the relative sliding motion does not reverse direction even though it changes periodically over one mesh cycle. Both forward drive and coast operating cases are realistically simulated. The results are presented in time, spatial, and frequency domains. The proposed theory is also capable of explaining the observed vibration
behavior under forward drive and coast conditions, which have eluded researchers in the past.

6. The sound pressure and vibration response under drive and coast conditions were measured for a specific hypoid gear set used in a truck application. The analysis shows that the vertical motion of the pinion bearing location can be used as a relatively good indicator of the radiated noise level. The test results were simulated by applying the proposed dynamic model that includes the translation-torsional coupling motions of the pinion and gear members. The simulation showed that the predicted trend in the dynamic mesh force closely matches the overall noise spectrum. Additionally, the measured vibrations of the pinion bearing coordinates show reasonably good agreement with the predicted response.

8.2 Recommendations for Future Work

1. Perform a comprehensive parametric investigation to study the effects of cutter design, modified roll parameters on root stress, load distributions, contact pattern, and transmission errors using the proposed 3-dimensional tooth surface simulation. Investigate the reason that causes the transmission error difference in CAPP and the kinematic mesh simulation.

2. Examine the sensitivity of the contact characteristics due to misalignment of the angular deviation between the pinion and gear axes and axial or vertical variation of the pinion position, and tooth profile modifications.
3. Perform a non-linear vibration simulation to include all 6 coordinates of each gear. This will yield a 14-DOF non-linear time-varying mesh model. This is expected to provide a more thorough insight in the couplings of translation-rotation forces. Also non-linear friction force can be modeled using an elastic hydrodynamic fluid model.

4. Develop the analytical solutions for a 2-DOF torsional model to better understand the occurrence of tooth separation and back-collision using the method of harmonic balance. The stability issues related to the time-varying mesh vectors and non-linearity can also be studied using the Floquet theory.

5. Include true multi-body dynamic behavior in the simulation to examine the coupling effects of the dynamic transmission error excitation and vibratory characteristics of the gear bodies.
APPENDIX A

The following provides the Gleason special analysis file for generation of a hypoid gear set. Note for confidential concern, the data shown may not be actual values. Only those used in Chapter 2 are listed. Comments in parentheses are added here for easy identification.

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 3  J.           -0.721000 (swivel angle)
 4  EM.          0.973210 (blank offset)
 5  ROOT ANGLE   -0.021688 (machine root angle)
 6  DELTA XP     -0.170512 (machine center to back)
 7  DELTA XB     0.728262 (sliding base)
 8  CALC RCP     4.265000 (cutter point radius)
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<td>24E</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>120F</td>
<td>0.000000</td>
</tr>
<tr>
<td>3</td>
<td>VH</td>
<td>0.000000</td>
</tr>
<tr>
<td>4</td>
<td>VH2</td>
<td>0.000000</td>
</tr>
<tr>
<td>5</td>
<td>VH3</td>
<td>0.000000</td>
</tr>
<tr>
<td>6</td>
<td>VE</td>
<td>0.000000</td>
</tr>
<tr>
<td>7</td>
<td>VE2</td>
<td>0.000000</td>
</tr>
<tr>
<td>8</td>
<td>VE3</td>
<td>0.000000</td>
</tr>
<tr>
<td>9</td>
<td>Q</td>
<td>1.01984</td>
</tr>
<tr>
<td>10</td>
<td>THETA O</td>
<td>1.017778</td>
</tr>
<tr>
<td>15</td>
<td>RE</td>
<td>0.040000</td>
</tr>
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</table>

**ITEM RECORD 20 PIN. IB FINISHING**

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>4</td>
<td>INP. BLD ANG</td>
<td>-0.436332</td>
</tr>
<tr>
<td>5</td>
<td>INP. PT RAD</td>
<td>4.820000</td>
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**ITEM RECORD 21 GEAR IB FINISHING**

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>S</td>
<td>5.14389(radial setting)</td>
</tr>
<tr>
<td>2</td>
<td>I</td>
<td>0.000000(tilt angle)</td>
</tr>
<tr>
<td>3</td>
<td>J</td>
<td>0.212019(swivel angle)</td>
</tr>
<tr>
<td>4</td>
<td>EM</td>
<td>0.000000(blank offset)</td>
</tr>
<tr>
<td>5</td>
<td>ROOT ANGLE</td>
<td>1.213463(machine root angle)</td>
</tr>
<tr>
<td>6</td>
<td>DELTA XP</td>
<td>0.051023(machine center to back)</td>
</tr>
<tr>
<td>7</td>
<td>DELTA XB</td>
<td>0.000000(sliding base)</td>
</tr>
<tr>
<td>8</td>
<td>CALC RCP</td>
<td>4.500000(point radius of blade)</td>
</tr>
<tr>
<td>9</td>
<td>TIP BLD ANG.</td>
<td>0.392699(blade angle)</td>
</tr>
<tr>
<td>13</td>
<td>BASIC RA</td>
<td>0.000000(ratio of roll)</td>
</tr>
<tr>
<td>14</td>
<td>2C</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>15</td>
<td>6D</td>
<td>0.000000(modified roll parameter)</td>
</tr>
</tbody>
</table>

**ITEM RECORD 22 GEAR IB FINISHING**

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>24E</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>2</td>
<td>120F</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>3</td>
<td>VH</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>4</td>
<td>VH2</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>5</td>
<td>VH3</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>6</td>
<td>VE</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>7</td>
<td>VE2</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>8</td>
<td>VE3</td>
<td>0.000000(modified roll parameter)</td>
</tr>
<tr>
<td>9</td>
<td>Q</td>
<td>0.843002(initial cradle angle)</td>
</tr>
<tr>
<td>15</td>
<td>RE</td>
<td>0.075000(blade edge radius)</td>
</tr>
</tbody>
</table>

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ITEM RECORD  23 GEAR IB FINISHING
  4  INP. BLD ANG  0.392699 (blade angle)
  5  INP. PT RAD.  4.500000 (nominal radius of blade)

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ITEM RECORD  24 GEAR IB FINISHING
  1  V. ........ 3.765665 (vertical setting)
  2  H. ........  3.187263 (horizontal setting)
  3  DELTA XG .. 0.064232 (machine center to back)
  4  ROCT ANGLE . 1.219383 (machine root angle)
  9  BLADE ANGLE . 0.392699 (blade angle)
 10  WG ........  0.150000 (point width)

5

ITEM RECORD  25 GEAR OB FINISHING
  1  S. ........  5.14389
  2  L. ........  0.000000
  3  J. ........  0.212019
  4  EM ........  0.000000
  5  ROOT ANGLE . 1.213463
  6  DELTA XP ..  0.051023
  7  DELTA XN ..  0.060000
  8  CALC RCP .  4.500000
  9  TIP BLD ANG. -0.392699
 13  BASIC RA ..  0.000000
 14  2C ..........  0.000000
 15  6D ..........  0.000000

26

ITEM RECORD  26 GEAR OB FINISHING
  1  24E. ........  0.000000
  2  129F ........  0.000000
  3  VH ..........  0.000000
  4  VH2 ..........  0.000000
  5  VH3 ..........  0.000000
  6  VE ..........  0.000000
  7  VE2 ..........  0.000000
  8  VE3 ..........  0.000000
 15  RE ..........  0.075000

27

ITEM RECORD  27 GEAR OB FINISHING
  4  INP. BLD ANG -0.392699
  5  INP. PT RAD.  4.500000

28

ITEM RECORD  28 GEAR OB FINISHING
  1  V. ........  3.765665
  2  H. ........  3.187263
  3  DELTA XG ..  0.064232
  4  ROCT ANGLE . 1.219383
  9  BLADE ANGLE. -0.392699
 10  WG ........  0.150000
BIBLIOGRAPHY


195


Hochmann, D., 1997, “Friction Force Excitation in Spur and helical Involute Parallel Axis Gearing,” *Ph.D. Dissertation*, The Ohio State University, Columbus, OH.

Houser, D.R. and Singh, R., 1999, *Gear Noise Short Course Notes,* The Ohio State University, Columbus, OH.


Poling, G. R., 1999, Hypoid Gear Test Rig Transmission Error Measurement Enhancement And Topics In Root Stresses of Several Gear Types, *MS Thesis*, Ohio State University, Columbus, OH.


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Tilak, S., 1990, “Design, Analysis and Inspection of Spiral Bevel and Hypoid Gears Using Tooth Contact Analysis Programs,” *M.Sc. Thesis*, The Ohio State University, Columbus, OH.


