NEW ASYMPTOTIC METHODS IN THE STUDY OF
ANALYTIC DIFFERENTIAL AND DYNAMICAL
SYSTEMS

DISSERTATION

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In the first part of the dissertation, we prove Borel summability in nonsingular directions of heatlike equation $u_t = a(z)u_{zz}$ in the complex domain where $a(z)$ is a quartic polynomial and the initial condition is analytic. In the special case $a(z) = z$ we obtain the detailed resurgent structure; even in such a simple case, the structure of the singular manifolds is quite intricate. In the second part of the dissertation, we provide mainly two applications of transseries and Borel summation to interesting problems such as nonelementarity of a function, and analytic factorizability of a linear operator. In the third part, we consider the dynamical system arising in sub-monolayer deposition model, and analyze the asymptotic behavior as $t \to \infty$ of the solution.
to my loving son, One
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CHAPTER 1
INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Recent developments in rigorous asymptotic analysis such as transseries and generalized Borel summation enable us to analyze the behavior of solutions of ordinary differential equations (ODEs) and some classes of partial differential equations (PDEs). Transseries are generalizations of asymptotic expansions closed under most operations in analysis. This closedness of transseries means that a large number of problems admit transseries as formal solutions. Generalized Borel summation was developed for transforming formal transseries solutions into actual solutions, even when transseries expansions are divergent. It allows us to analyze differential equations or functions whose behavior is hard to determine using classical methods.

In this dissertation, we will discuss applications of this asymptotic analysis to three different problems. It is organized in the following way: we provide mathematical preliminaries in Chapter 1. In Chapter 2, we will show the Borel summability of the heat equation with variable coefficients: \( u_t = a(z) u_{zz} \) for two different \( a(z) \), one is \( a(z) = z \), and another is \( a(z) = a \) quartic polynomial. Due to the factorially growing coefficients of the formal solution (Gevrey-one), we use Borel summation to obtain the solution. After a series of transformations and applying contractive mapping principle, we show the existence and uniqueness of the solution for general
initial functions, and can get the closed form for special cases. In Chapter 3, we provide some interesting uses of transseries and Borel summation. We use more detailed definition of transseries that will be given there to show the nonelementarity of the functions, and by analyzing the singularities of Borel transform, we show the factorizability of the differential operators. In Chapter 4, we find the asymptotic behavior of a ODE system derived from the rate equations modelling submonolayer deposition for an arbitrary critical island size.

1.2 Preliminaries

For more detailed statements and proofs, see [15] unless otherwise stated.

1.2.1 Complex analysis

Lemma 1.2.1 (Jordan’s Lemma). [29]

Suppose that

(i) a function \( f(z) \) is analytic at all points \( z \) in the upper half plane \( y \geq 0 \) that are exterior to a circle \( |z| = R_0 \),

(ii) \( C_R \) denotes a semicircle \( z = Re^{i\theta}(0 \leq \theta \leq \pi) \), where \( R > R_0 \)

(iii) for all points \( z \) on \( C_R \), there is a positive constant \( M_R \) such that \( |f(z)| \leq M_R \),

Then, for every positive constant \( a \),

\[
\lim_{R \to \infty} \int_{C_R} f(z)e^{iaz} \, dz = 0.
\]

Theorem 1.2.2 (Hartogs’ theorem). A complex-valued function of \( n \) complex variables \( (x_1, x_2, ..., x_n) \) which has first partial derivatives with respect to each variable \( x_i (i = 1, ..., n) \) is analytic as a function of \( n \) complex variables. That is, if \( f : \mathbb{C}^n \to \mathbb{C} \) is analytic in each \( x_i \) for the rest variables fixed, then \( f \) is analytic in \( (x_1, x_2, ..., x_n) \).[28]
1.2.2 Asymptotic expansions

Definition 1.2.3 (Asymptotic expansion). An asymptotic expansion at \( \infty \) is a formal series \( \sum_{k=0}^{\infty} f_k(t) \) of simple functions \( f_k(t) \) in which each successive term \( f_{k+1}(t) \) is much smaller than its predecessors \( f_k(t) \) as \( t \to \infty \), that is,

\[
f_{k+1}(t) = o(f_k(t)), \quad \text{as } t \to \infty \quad \iff \quad \lim_{t \to \infty} \frac{f_{k+1}(t)}{f_k(t)} = 0.
\]

Definition 1.2.4. A function \( f \) is asymptotic to an asymptotic expansion \( \sum_{k=0}^{\infty} f_k(t) \) as \( t \to \infty \) if

\[
f(t) - \sum_{k=0}^{N} f_k(t) = o(f_N(t)) \quad (\forall N \in \mathbb{N}) \quad \text{as } t \to \infty,
\]

and we write

\[
f(t) \sim \sum_{k=0}^{\infty} f_k(t) \quad \text{as } t \to \infty.
\]

For example,

\[
\int_{x}^{\infty} e^{-s^2} \, ds \sim e^{-x^2} \left( \frac{1}{2x} - \frac{1}{4x^2} + \frac{5}{8x^3} - \ldots \right) \quad \text{as } x \to \infty,
\]

\[
n! \sim \sqrt{2\pi e^{n \ln n - n + \frac{1}{2} \ln n}} \left( 1 + \frac{1}{12n} \right) \quad \text{as } n \to \infty,
\]

\[
\int_{1}^{x} \frac{e^t}{t} \, dt \sim e^x \left( \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \ldots \right) \quad \text{as } x \to \infty,
\]

Note 1.2.5. Asymptoticity can be defined at a point \( t_0 \) as well by replacing \( \infty \) by \( t_0 \). The direction in the complex plane along which \( t_0 \) is approached might be relevant.

Definition 1.2.6 (Asymptotic power series). A function is asymptotic to a series as \( t \to t_0 \), in the sense of power series if

\[
f(t) - \sum_{k=0}^{N} c_k(t - t_0)^k = O((t - t_0)^{N+1}) \quad (\forall N \in \mathbb{N}) \quad \text{as } t \to t_0.
\]

For example,

\[
e^{-\frac{1}{2}} \int_{1}^{\frac{1}{2}} \frac{e^t}{t} \, dt \sim \sum_{k=0}^{\infty} k! z^{k+1} \quad \text{as } z \to 0^+.
\]
1.2.3 Transseries

Both asymptotic series and asymptotic expansions are incomplete in a sense that they cannot distinguish two functions when the difference is too small. For example, \( e^{-\frac{1}{x}} \sim 0 \) (in the sense of power series) as \( x \to 0^+ \), meaning that the operator associating functions to asymptotic power series has nontrivial kernel, and, although asymptotic expansion has zero kernel, the operator \( f \mapsto \mathcal{A}(f) \) which associates to \( f \) its asymptotic expansion is not linear, and so \( \mathcal{A}(f) = \mathcal{A}(g) \) does not imply \( f = g \). For example, both \( \sin z \) and \( \sin z + e^{-\frac{1}{z}} \) have the same asymptotic expansions \( \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \) as \( z \to 0^+ \). Transseries can be thought as generalized formal asymptotic expansions, to be defined later in Chapter 3, involving powers, logs and exponentials, and which are operated with almost as if they were convergent. The restriction “asymptotic”, now meaning that the terms can be ordered decreasingly with respect to \( \gg \), is crucial, so it can distinguish two functions of the above example since the transseries of \( \sin z + e^{-\frac{1}{z}} \) is simply \( \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} + e^{-\frac{1}{z}}, \) as \( z \to 0^+ \), which is different from the one for \( \sin z \).

We present more detailed construction in Chapter 3. For a complete description of transseries, see e.g. [15].

As mentioned earlier, transseries are closed under most operations in analysis, and this implies that a large number of problems admit transseries as formal solutions [15]. For example, \( y'' + y = x^{-2} + y^3 \) has the transseries solutions (formal exponential series solution) of the form:

\[
\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C_k e^{-kx} \tilde{y}_k, \quad (\Re(x) > 0)
\]

where \( \tilde{y}_k, k = 0, 1, \ldots \) are formal power series in \( \frac{1}{x} \).

1.2.4 Laplace transform and Watson's lemma

**Proposition 1.2.7.** If \( F \in L^1(\mathbb{R}^+) \) (meaning that \( |F| \) is integrable on \([0, \infty)\)), then
(i) the Laplace transform \( (\mathcal{L}F)(x) := \int_0^\infty e^{-px}F(p) \, dp \) is analytic in \( \mathbb{H} \) (the right half plane of \( \mathbb{C} \)) and continuous in \( \overline{\mathbb{H}} \).

(ii) \( \mathcal{L}F(x) \to 0 \) as \( x \to \infty \) along any ray \( \{x : \arg(x) = \theta\} \) if \( |\theta| \leq \pi/2 \), and \( ||\mathcal{L}F||_\infty \leq ||F||_1 \).

Note that replacing \( x \) by \( x - |\alpha| \) allows us to work in space of functions with the property that \( F(p)e^{-|\alpha|p} \) is in \( L^1 \).

Lemma 1.2.8 (Uniqueness). Assume \( F \in L^1(\mathbb{R}^+) \) and \( \mathcal{L}F(x) = 0 \) for \( x \) in a set with an accumulation point in \( \mathbb{H} \). Then \( F = 0 \) almost everywhere.

Proposition 1.2.9. (i) Assume \( f \) is analytic in an open sector \( \mathbb{H}_\delta := \{x : |\arg x| < \pi/2 + \delta\}, \delta \geq 0 \) and is continuous on \( \partial\mathbb{H}_\delta \), and that for some \( K > 0 \) and any \( x \in \mathbb{H}_\delta \) we have

\[
|f(x)| \leq K(|x|^2 + 1)^{-1}.
\]

Then \( \mathcal{L}^{-1}f \) is well defined by

\[
F = \mathcal{L}^{-1} f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{pt} f(t) \, dt
\]

and

\[
\int_0^\infty e^{-px}F(p) \, dp = \mathcal{L}^{-1} f = f(x).
\]

We have \( ||\mathcal{L}^{-1}\{f\}||_\infty \leq K/2 \) and \( \mathcal{L}^{-1}\{f\} \to 0 \) as \( p \to \infty \).

(ii) If \( \delta > 0 \), then \( F = \mathcal{L}^{-1} f \) is analytic in the sector \( S = \{p \neq 0 : |\arg p| < \delta\} \).

In addition, \( \sup_S |F| \leq K/2 \) and \( F(p) \to 0 \) as \( p \to \infty \) along rays in \( S \).

Here are some important properties of inverse Laplace transform:

\[
-p(\mathcal{L}^{-1}f) = \mathcal{L}^{-1}(f'),
\]

\[
\mathcal{L}^{-1}(fg) = (\mathcal{L}^{-1}f) \ast (\mathcal{L}^{-1}g)
\]
where
\[(F * G)(p) := \int_0^p F(s)G(p - s) \, ds.\]

Lemma 1.2.10 (Watson’s lemma). Let \( F \in L^1(\mathbb{R}^+) \) and assume \( F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1 + \beta_2 - 1} \)
as \( p \to 0^+ \) for some constants \( \beta_i \) with \( \Re(\beta_i) > 0, i = 1, 2 \). Then, for \( a \leq \infty \),
\[f(x) = \int_0^a e^{-xp} F(p) \, dp \sim \sum_{k=0}^{\infty} c_k \Gamma(k \beta_1 + \beta_2) x^{-k \beta_1 - \beta_2}\]
along any ray in \( \mathbb{H} \).

1.2.5 Borel transform and Borel summation

Definition 1.2.11 (Borel transform). \( \mathcal{L} \) is the formal Laplace transform from \( \mathbb{C}[[p]] \) to \( \mathbb{C}[[x^{-1}]] \) defined by
\[\mathcal{L} \left[ \sum_{k=0}^{\infty} c_k p^k \right] = \sum_{k=0}^{\infty} c_k k! x^{-k-1}.\]
Then the Borel transform, \( \mathcal{B} : \mathbb{C}[[x^{-1}]] \mapsto \mathbb{C}[[p]] \) is the formal inverse of the formal Laplace transform by
\[\mathcal{B} \left[ \sum_{k=0}^{\infty} c_k x^{-k-1} \right] = \sum_{k=0}^{\infty} \frac{c_k}{k!} p^k.\]

Note that this definition can be extended to more general forms:
\[\mathcal{B} \left[ \sum_{k=0}^{\infty} c_k \Gamma(k \beta_1 + \beta_2) x^{-k \beta_1 - \beta_2} \right] = \sum_{k=0}^{\infty} c_k p^{k\beta_1 + \beta_2 - 1}\]
with \( \Re(\beta_i) > 0, i = 1, 2 \).

Definition 1.2.12 (Borel summation). Let a series \( \hat{f} = \sum_{k=0}^{\infty} c_k x^{-\beta k - r} \) with \( \Re(\beta) > 0 \) and \( r > 0 \). Borel summation along \( \mathbb{R}^+ \) consists of three operations, assuming (2) and (3) are possible:

1. Borel transform, \( \hat{f} \mapsto \mathcal{B}\{\hat{f}\} \)

2. Convergent summation of the series \( \mathcal{B}\{\hat{f}\} \) and analytic continuation (denote by \( F \)) along \( \mathbb{R}^+ \).
(3) Laplace transform, \( F \mapsto \int_0^\infty F(p)e^{-xp} \, dp =: \mathcal{L}B\{\hat{f}\} \), which requires exponential bounds on \( F \), defined in some half plane \( \Re(x) > x_0 \).

**Definition 1.2.13** (Directionality of Borel sums). *The Borel sum of a series in the direction \( \phi \) (\( \arg x = \phi \)), \( (\mathcal{L}B)_\phi \hat{f} \) is the Laplace transform of \( \mathcal{B}\hat{f} \) along the ray \( xp \in \mathbb{R}^+ \), that is \( \arg(p) = -\phi \):

\[
(\mathcal{L}B)_\phi \hat{f} := \mathcal{L}_{-\phi} F = \mathcal{L}F(e^{-i\phi}) = \int_0^\infty e^{-isx} F(p) \, dp
\]

**Proposition 1.2.14.** There are important 1-1 correspondences: Let \( S_B \) be the set of Borel summable series along some direction, say along \( \mathbb{R}^+ \).

(i) \( S_B \) is a differential field and so is \( \mathcal{L}B\{S_B\} \).

(ii) \( \mathcal{L}B : S_B \mapsto \mathcal{L}BS_B \) commutes with differential field operations, that is, \( \mathcal{L}B \) is a differential algebra isomorphism.

(iii) Let \( S_c \) be the differential algebra of convergent power series. Then \( \mathcal{L}B \) is the identity on \( S_c \) identifying a convergent power series with its sum.

(iv) In addition, for \( \hat{f} \in S_B \), \( \mathcal{L}B\{\hat{f}\} \sim \hat{f} \) as \( |x| \to \infty \), \( \Re(x) > 0 \).

### 1.2.6 Borel summability of the asymptotic series solution for rank one linear systems of ODEs

(For proofs, more details, and more results, see [23],[15] where the results are given for nonlinear systems.)

Let a linear differential system have the normal form of the following form:

\[
y' = f_0(x) - \tilde{A}y + \frac{1}{x} \tilde{A}y + G(x)y, \quad y \in \mathbb{C}^n
\]  

(1.2.1)

where \( f_0(x) \) is analytic at \( \infty \), \( G(x) = O(x^{-2}) \) is also analytic at \( \infty \), and the linearization \( \tilde{A} \) at \( (\infty, 0) \) is assumed to have nonresonant nonzero eigenvalues \( \lambda_i \) \((i = 1, 2, \ldots, n)\)

(that is, no two of them are on the same direction). We can further normalize so that
\( \hat{A} = \text{diag}(\lambda_i) \) and \( \hat{A} = \text{diag}(\alpha_i) \), and by rescaling \( x \) and reorder the components of \( y \), we can let \( \lambda_i = 1 \), and \( \arg \lambda_i \leq \arg \lambda_j \) if \( i < j \), and lastly, substituting \( y = y_1 x^{-N} \) for some \( N \geq 0 \), we have \( \Re(\alpha_i) > 0 \), \( i = 1, 2, \ldots, n \). Then we have the following results:

(i) After normalization, the general transseries solution of (1.2.1) on \( \mathbb{R}^+ \) is

\[
\tilde{y} = \sum_{k \in \mathbb{N}_0, |k| \leq 1} C_k e^{-k \lambda x} x^k \alpha \tilde{y}_k = \tilde{y}_0 + \sum_{|k|=1} C_{1k} C_{2k} \ldots C_{nk} e^{-k \lambda x} x^k \alpha \tilde{y}_k,
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), and \( \tilde{y}_k \) are formal power series in \( x^{-1} \).

(ii) For nonsingular direction, all \( \tilde{y}_k \) are Borel summable in a common half-plane, of the form \( H_{x_0} = \{ x : \Re(x) > x_0 \} \).

(iii) The Borel sums \( y_k = \mathcal{B} \tilde{y}_k \) are analytic in \( H_{x_0} \).

(iv) The obtained function

\[
y = y_0 + \sum_{|k|=1} C_{1k} C_{2k} \ldots C_{nk} e^{-k \lambda x} x^k \alpha y_k \tag{1.2.2}
\]

is a solution of the differential equation (1.2.1).

(v) Any solution of the differential equation (1.2.1) which tends to zero along \( \mathbb{R}^+ \) can be written, in a direction \( d \) in \( \mathbb{H} \), in the form (1.2.2) for a unique \( C = (C_1, \ldots, C_n) \), this constant depending usually on the sector where \( d \) is (Stokes phenomenon).

Also we have the descriptions for the behavior of \( Y_k = \mathcal{B} \tilde{y}_k \) around the singular points in Borel space. First let \( \mathcal{W} = \{ p \in \mathbb{C} : p \neq \lambda_i, i = 1, 2, \ldots, n \} \) and then define a surface \( \mathcal{R} := \{ \gamma : (0, 1) \rightharpoonup \mathcal{W} : \gamma(0) = 0; \frac{d}{dt} |\gamma(t)| > 0; \frac{d}{dt} \arg(\gamma(t)) \text{ monotonic} \} \). Also let \( m_j = 2 + [\Re(\alpha_j)] \) and \( \alpha'_j = m_j - \alpha_j \). Then we have:

(i) \( Y_0 = \mathcal{B} \tilde{y}_0 \) is analytic in \( \mathcal{R} \cup \{0\} \), and the singularities of \( Y_0 \) are described as follows. For small \( z \), we have

\[
Y_0^+(z + \lambda_j) = \left(z^{\alpha'_j - 1} (\ln z)^{0.1} A_j(z) \right)^{(m_j)} + B_j(z)
\]

where the power of \( \ln z \) is one if and only if \( \alpha_j \in \mathbb{Z} \), and \( A_j, B_j \) are analytic for small \( z \).
(ii) For $|k| = 1$, $Y_k = \mathcal{B}\tilde{Y}_k$ are analytic in $\mathcal{R}$, and for small $z$ there exist

$$
Y_k^\pm (z + \lambda_j) + \left[ z^{k\cdot \alpha + \alpha_j - 1}(\ln z)^{0.1} A_{kj}(z) \right]^{(m_j)} + B_{kj}(z)
$$

where the power of $\ln z$ is 0 if and only if $k \cdot \alpha + \alpha_j \notin \mathbb{Z}$.

### 1.2.7 Stokes phenomenon: An example

(For a full treatment of the Stokes phenomenon, see [21] and also [15] for the connection with transseries.)

A complex function $f(x)$ may have different asymptotic behaviors for $|x| \to \infty$ in different sectors. The change in the asymptotic expansion across boundaries is called the Stokes phenomenon. In this dissertation, we define Stokes phenomenon only for Borel summed series. The change in behavior of a Borel summed series as the direction of summation in $\mathbb{C}$ changes is conveniently determined by suitably changing the contour of integration of the Laplace transform. We illustrate this on a simple case:

$$
f(x) := \int_0^\infty e^{-px} \frac{e^{-p^2}}{1 + p} dp \sim \sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}}.
$$

This asymptotic expansion is valid for $\Re(x) > 0$, and the asymptotic behavior of the analytic continuation for large $x$ along different directions in $\mathbb{C}$ can be gotten by rotating the contour of integration, for example,

$$
f(x) = \int_0^{\infty e^{-i\pi/4}} e^{-px} \frac{e^{-p^2}}{1 + p} dp
$$

which is analytic for $\arg x \in (-\pi/4, 3\pi/4)$, and by continuing this rotation process until $\arg x = \pi - \epsilon$ where we have

$$
f(x) = \int_0^{\infty e^{-i\pi + i\epsilon}} e^{-px} \frac{e^{-p^2}}{1 + p} dp
$$
which is analytic for \( \arg x \in (\pi/2 - \epsilon, 3\pi/2 - \epsilon) \), and, at \( p = -1 \), we can proceed further by collecting the residue:

\[
f(x) = \int_0^{\infty} \frac{e^{px}}{1 + p} dp = \int_0^{\infty} \frac{e^{-px}}{1 + p} dp - 2\pi i e^x
\]
since the right hand side is analytic for \( \arg x \in (\pi/2 + \epsilon, 3\pi/2 + \epsilon) \). By freely proceeding with the analytic continuation in similar steps until \( \arg x = 2\pi \), we get \( f(xe^{2\pi i}) = f(x) - 2\pi i e^x \). This new exponential term is added at \( \arg x = \pi \), a ‘Stokes ray’, and stay smaller than the series until \( \arg x = 3\pi/2 \), an ‘antistokes ray’. The Stokes phenomenon is due to the divergence of the asymptotic series, or equivalently, the existence of the singular point in Borel space.

### 1.2.8 Other results needed

**Definition 1.2.15.** A differential field \( \mathcal{F} \) is a set of functions closed under algebraic operations and differentiation. An elementary extension of a differential field \( \mathcal{F} \), \( E(\mathcal{F}) \) can be thought as the minimal field extending \( \mathcal{F} \) so that \( g \in E(\mathcal{F}) \) implies \( \exp(g) \) and \( \log(g) \) are in \( E(\mathcal{F}) \).

**Theorem 1.2.16 (Liouville’s principle).** Let \( \mathcal{F} \) be a differential field with constant field \( K \). For \( f \in \mathcal{F} \) suppose that the equation \( g' = f \) (i.e. \( g = \int f \)) has a solution \( g \in E(\mathcal{F}) \) where \( E(\mathcal{F}) \) is an elementary extension of \( \mathcal{F} \) having the same constant field \( K \). Then there exist constants \( c_1, \ldots, c_n \in K \) and elements \( u_1, \ldots, u_n, v \in \mathcal{F} \) such that

\[
g = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'.
\]

In other words,

\[
\int f = c_1 \log u_1 + \ldots + c_n \log u_n + v.
\]

For example,

\[
\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C = -\frac{1}{2} i \ln \frac{1 + ix}{1 - ix} + C.
\]
For the proof, see [25]. Liouville’s principle states that elementary antiderivatives, if they exist, must be in the same differential field as the function, plus possibly a finite number of logarithms.
CHAPTER 2
HEAT EQUATION WITH VARIABLE COEFFICIENTS:
BOREL SUMMABILITY

2.1 Introduction and settings

In this chapter we study complex partial differential equations of the form

\[ u_t = a(z)u_{zz} \]  

with the initial condition

\[ u(z, 0) = \sqrt{\pi}f_0(z), \]

where \( a(z) \) is a quartic polynomial and \( f_0(z) \) is analytic in \( \mathbb{C} \). Looking for a formal solution of the type

\[ \tilde{u}(z, t) = \sum_{n=0}^{\infty} f_n(z)t^n, \]  

we get

\[ f_n(z) = a(z)f_{n-1}(z)/n = (a(z)\partial_z^n) f_0(z)/n! \]

Since \( |f_0^{(2n)}(z)| = O((2n)!), \) \( \tilde{u}(z, t) \) is Gevrey-one at zero with respect to the time variable \( t \). The purpose of the chapter is to prove the Borel summability of the formal solution \( \tilde{u}(z, t) \) in its nonsingular directions.

In 1999 Lutz, Miyake and Schäfke [11] considered this problem for the complex heat equation, that is, in the case \( a(z) = 1 \), and provided a necessary and sufficient
condition on the initial condition \( f_0(z) \) for the Borel summability of \( \tilde{u}(z, t) \) in a given direction \( \arg t = \theta \). After the work of Lutz, Miyake and Schäfke, various authors have extended their results and discussed the Borel summability (or, more generally, the multisummability) of formal power series solutions of partial differential equations with constant coefficients. See, for example, [5], [12], [9], [6], [7]. See also [17] for a Gevrey-asymptotics approach to Borel summability in constant-coefficients, inhomogeneous heat equations, and [13], [14], [10] for some related results. More recently Balser and Loday-Richaud [8] dealt with equation (2.1.1) in the linear case \( a(z) = z \), and gave a necessary and sufficient condition on the initial condition \( f_0(z) \) for the Borel summability of \( \tilde{u}(z, t) \). In [8] the problem studied is summability along an arbitrarily fixed direction, uniformly in \( z \). The summability of \( \tilde{u}(z, t) \) in general depends on the variable \( z \) in the case of variable coefficients. In this chapter, fixing an initial condition \( f_0(z) \) to be a simple rational function \( 1/(z^2+1) \) and analyzing the singularity structure of the Borel transform of \( \tilde{u}(z, t) \), we clarify for which \( z \) the Borel summability (in a given direction \( \arg t = \theta \)) of the formal solution \( \tilde{u}(z, t) \) of (2.1.1) does hold. The Borel transform \( B \) is defined as the formal inverse Laplace transform (see [15]); the applicability of the inverse Laplace transform is justified subsequently, by proving that the Laplace transform of this Borel transform exists.

Before starting our analysis, we first write down an integral equation that the Borel transform should satisfy. In view of (2.1.2), it is natural to take the Borel transform of (2.1.1) with respect to \( 1/t \). Substituting \( t = 1/T \) and \( v(z, T) = T^{-1/2}u(z, 1/T) \), we get

\[
-\frac{1}{2T}v(z, T) - v_T(z, T) = \frac{a(z)}{T^2}v_{zz}(z, T). \tag{2.1.4}
\]

The Borel transform of (2.1.4) in \( T \) is

\[
pV_{pp} + \frac{3}{2}V_p = a(z)V_{zz} \tag{2.1.5}
\]
with \( V(z,p) = B\{v(z,T)\}(p) \). The transformations \( s = 2\sqrt{p} \), \( W(z,s) = \frac{8}{2} V(z,p) \) then yield

\[
W_{ss} = a(z) W_{zz}. \tag{2.1.6}
\]

The initial conditions for (2.1.6) are:

\[
W(z,0) = f_0(z) \text{ and } W_s(z,0) = 0. \tag{2.1.7}
\]

The fact that \( W_s(z,0) \) is zero comes naturally from the link between \( W \) and the Borel transform of (2.1.2) (via the change of variables \( s = 2\sqrt{p} \)). We further let \( y(z) = \int_{y_0}^z a(s)^{-\frac{1}{2}} ds \), denote the inverse function \( y^{-1} \) by \( z = \phi(y) \) and let \( g(y,s) = W(\phi(y),s)a(\phi(y))^{-1/4} \), giving

\[
g_{ss} = g_{yy} + n(y)g \tag{2.1.8}
\]

where

\[
n(y) = \frac{a''(\phi(y))}{4} - \frac{3(a'(\phi(y)))^2}{16a(\phi(y))}
\]

with initial conditions \( g_0(y) = f_0(\phi(y))a(\phi(y))^{-1/4} \) and \( g_s(y,0) = 0 \).

By Duhamel’s principle, we obtain the following integral equation:

\[
g(y,s) = \frac{g_0(y+s) + g_0(y-s)}{2} + \frac{1}{2} \int_0^s \int_{y-(s-\tilde{s})}^{y+(s-\tilde{s})} n(\tilde{y}) g(\tilde{y},\tilde{s}) d\tilde{y} d\tilde{s}. \tag{2.1.9}
\]

In what follows we mainly use the integral equation to analyze \( g(y,s) \), which is essentially the Borel transform of \( \tilde{u}(z,t) \) except for some simple multiplicative factors.

In Section 2, as a prototypical case, we study the case \( a(z) = z \) and then discuss the case where \( a(z) \) is a quartic polynomial in Section 3. In Sections 4–6 we employ more detailed method to obtain the full resurgent structure of \( W(z,s) \) in the case \( a(z) = z \).

In the next section, however, we use \( a(z) = z \) only to simplify the notation.
2.2 A prototypical case: \( a(z) = z, \ f_0(z) = \frac{1}{z^2 + 1} \)

We start with the study of one of the simplest cases, \( a(z) = z \). We analyze this example using methods that generalize to any quartic polynomial. The concrete choice \( a(z) = z \) merely simplifies the notation to make the discussion clearer. In §2.4 we return to this case with different methods, to obtain the full resurgent structure.

The main strategy here is to use contractivity methods in suitable norms and a self-contained domain in \( \mathbb{C}^2 \) for the reformulated integral equation (2.1.9) to show uniqueness of an analytic solution with suitable exponential bounds showing its Borel summability. The methods extend in a sense those of [1]. Some of the norms have been introduced earlier [2], [3].

**Integral setting.** Consider the domain \( D = \{(y, s) \in \mathbb{C}^2 : \text{dist}(y + \lambda s, S) > \epsilon \} \) where \( S \) is \( \mathbb{R}^- \) (or any other square root branch cut) together with \( S_0 \), the set of all the singular points of \( g_0(y) \) and \( n(y) \). Let \( B_\delta = \{ f : \text{analytic in } D \text{ such that } ||f|| \leq \delta \} \) when \( \delta = 2||g_0|| \) with the norm \( ||g|| := \sup_{(y, s) \in D} e^{-\nu_1|y| - \nu_2|s|} |g(y, s)| \) for sufficiently large \( 0 < \nu_1 < \nu_2 \). Then, noting that \( y = 2\sqrt{z}, \ n(y) = -\frac{3}{4}y^{-2} \) for \( z_0 = 0 \), and \( g_0 = 16\sqrt{2}/(\sqrt{y}(y^4 + 16)) \), we obtain the integral reformulation

\[
g(y, s) = \frac{g_0(y + s) + g_0(y - s)}{2} - \frac{1}{2} \int_0^s \int_{y-(s-s)}^{y+(s-s)} \frac{3}{4y} g(\tilde{y}, \tilde{s}) d\tilde{y} d\tilde{s} =: \mu(g) \quad (2.2.1)
\]

**Theorem 2.2.1.** In the setting above, we have:

(i) \( \mu \) is a contractive mapping from \( B_\delta \) to \( B_\delta \). Thus (2.2.1) has a unique solution \( g \).

(ii) The solution \( g \) is Laplace transformable in a direction \( \theta \) in \( p \) if \( y \) does not belong to the set \( S^{(\theta)} := \bigcup_{y \in S_0} (\{y\} + e^{i\theta/2} \mathbb{R}) \).

**Note 2.2.2.** The restriction on \( y \) is not merely technical. In \( S \) the Borel transform of the solution typically has singularities except possibly at the zeros of \( a(z) \). (If \( a(z) = z \),
It is shown that \( z = 0 \) is not a singular point; cf. Theorem 2.4.1 (i) below. In general, we believe, but do not prove here, that the points that \( a(z) = 0 \) are not singularities.

**Proof.** (i) The integration paths in (2.2.1) are taken to be straight line segments. Then, for any \((y, s) \in D\), \((\tilde{y}, \tilde{s})\) is also in \(D: \tilde{s} = ts\) for \(0 \leq t \leq 1\), and \(\tilde{y} = y + \tau(s - \tilde{s}) = y + \tau(1 - t)s\) for \(-1 \leq \tau \leq 1\). So dist\((\tilde{y} + \lambda\tilde{s}, S) = \text{dist}(y + (\tau(1 - t) + t\lambda)s, S) > \varepsilon\) since \(-1 \leq \tau(1 - t) + t\lambda \leq 1\).

Letting \(g \in B_\delta\), we get

\[
\|\mu(g)\| \leq \|g_0\| + \sup_{(y, s) \in D} e^{-\nu_1|y| - \nu_2|s|} \left| \int_0^s \int_{g(y(-s - \tilde{s}))}^{g(y(s - \tilde{s}))} - \frac{3}{4y^2} g(\tilde{y}, \tilde{s}) d\tilde{y} d\tilde{s} \right| \tag{2.2.2}
\]

Since each integration is along the line segment connecting the end points, parameterizing in terms of the arc length \(d\tilde{y}\), say \(w\), \(|\tilde{y}|\) becomes concave up. Moreover, letting \(w = 0\) at \(\tilde{y} = y\), \(|\tilde{y}|\) cannot exceed \(|y| + |s - \tilde{s}|\) at both end points \(w = |s - \tilde{s}|\) (as \(|\tilde{y} + \Delta\tilde{y}| \leq |\tilde{y}| + |\Delta\tilde{y}|\)), so, for both integrations, \(|\tilde{y}|\) as a function of \(|d\tilde{y}|\) is bounded by the line segment connecting \((0, |y|)\) and \((|s - \tilde{s}|, |y| + |s - \tilde{s}|)\), and we get:

\[
\int_y^{y - (s - \tilde{s})} e^{\nu_1|\tilde{y}|} d\tilde{y} + \int_y^{y + (s - \tilde{s})} e^{\nu_1|\tilde{y}|} d\tilde{y} \leq 2 \int_0^{\nu_1|y| + (s - \tilde{s})} e^{\nu_1|\tilde{y}|} d\tilde{w} \leq \frac{2}{\nu_1} e^{\nu_1|y| + |s - \tilde{s}|}(2.2.3)
\]

and so the outer integration is bounded by:

\[
\frac{2}{\nu_1} \int_0^{\nu_1|y| + (s - \tilde{s})} e^{\nu_2(\nu_1 + \nu_2|s - \tilde{s}|)} d\tilde{s} = \frac{2 e^{\nu_1|y| + |s - \tilde{s}|}}{\nu_1} \int_0^{\nu_1|y| + |s - \tilde{s}|} e^{-(\nu_2 - \nu_1)|s - \tilde{s}|} d\tilde{s} = \frac{2 e^{\nu_1|y| + |s - \tilde{s}|}}{\nu_1} \int_0^{|s|} e^{-(\nu_2 - \nu_1)(\nu_1 - \nu_2)|s - \tilde{s}|} du \leq \frac{2 e^{\nu_1|y| + |s - \tilde{s}|}}{\nu_1(\nu_2 - \nu_1)}, \tag{2.2.4}
\]

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and since $\epsilon > 0$ is fixed, there exist large enough $0 < \nu_1, \nu_2 - \nu_1$ so that:

$$||\mu(g)|| \leq ||g_0|| + \frac{3||g_0||}{2\epsilon^2 \nu_1 (\nu_2 - \nu_1)} < 2||g_0||$$

(2.2.5)

proving $\mu$ is a map from $B_3$ into $B_3$.

(Note that the second equality of (2.2.4) comes from the fact that $\tilde{s}$ is on the line segment between the origin and $s$, and we can parameterize: $u = |\tilde{s}|$.)

Contractivity follows immediately in the same way:

$$||\mu(g_1) - \mu(g_2)|| \leq \frac{3||g_1 - g_2||}{4\epsilon^2 \nu_1 (\nu_2 - \nu_1)} < ||g_1 - g_2||.$$  

(2.2.6)

and so we have a solution of (2.1.8) in $B_3$.

(ii) For every $y$ outside $S(\theta)$ (see Theorem 2.2.1) and any $\theta$, (by taking the branch cut as $e^{i\theta/2}\mathbb{R}^-$ in the definition of $S$) we find that the function $g$ is well defined along $e^{i\theta/2}\mathbb{R}^+$ in $s$, which corresponds to $p \in e^{i\theta}\mathbb{R}^+$. A sequence of inverse transformations gives the solution of (2.1.5):

$$V(z, p) = \frac{1}{\sqrt{p}} z^{i} g(2\sqrt{z}, 2\sqrt{p}),$$

so for any $T$ with $\Re(T e^{i\theta}) > 0$,

$$\int_0^{\infty e^{i\theta}} e^{-pT} \frac{1}{\sqrt{p}} z^{i} g(2\sqrt{z}, 2\sqrt{p}) dp$$

exists since $g \in B_3$, and

$$\int_0^{\infty e^{i\theta}} \frac{1}{\sqrt{p}} e^{-pT + 2\nu_1 \sqrt{p}} dp$$

exists as well. Therefore, the solution is Laplace transformable in the direction of $\theta$. As an example, we can see that for any $y$ whose imaginary part is different from 0 and $\pm \sqrt{2}$, the solution is well defined and Laplace transformable along $\mathbb{R}^+$. 
Corollary 2.2.3. The formal solution $\tilde{u}(z,t)$ of the equation (2.1.1) with $a(z) = z$ and $f_0(z) = 1/(z^2 + 1)$ is Borel summable in any direction $\theta$ in $t$ (or in $T = 1/t$) if $z$ lies outside the set

$$\left\{ z \in \mathbb{C} \mid \int_{0}^{z} \zeta^{-1/2}d\zeta \in S(\theta) \right\},$$

that is,

$$\bigcup_{z \neq 0, \pm i} \left\{ z \in \mathbb{C} \mid \Im \left( e^{-i\theta/2} \int_{z}^{s} \zeta^{-1/2}d\zeta \right) = 0 \right\}.$$  

Note that in the case where $a(z) = z$ and $f_0(z) = 1/(z^2 + 1)$ we have $S_0 = \{0, 2\sqrt{-1}i\}$. Thus the set (2.2.7) is explicitly described by (2.2.8).

Remark 2.2.4. The first part of the proof only uses the fact that the domain is self-contained, the estimates for $n(y)$ on the given domain, and knowledge of the singularities of $g_0(y)$ and $n(y)$. The second part of the proof does not use special properties of the given function $a(z)$ either, so once we can set up a proper domain with the same norm for different $a(z)$, the existence of a unique solution, and its Laplace transformability are guaranteed as well.

2.3 The case when $a(z)$ is a quartic polynomial

Let $a(z) = a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4$. We restrict the analysis to the generic case when $a(z)$ has four distinct roots. Then, letting $z_0$ be one of those roots, $y = \int_{z_0}^{z} a(s)^{-\frac{3}{2}}ds$ can be inverted and expressed in terms of elliptic functions([16]):

$$z = z_0 + \frac{1}{4} a'(z_0) \left( \mathcal{P}(y; g_2, g_3) - \frac{1}{24} a''(z_0) \right)^{-1} =: \phi(y)$$

where $\mathcal{P}$ is a Weierstrass elliptic function with $g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$ and $g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4$, with corresponding periods $2\omega_1$ and $2\omega_2$.  

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Note 2.3.1. Evidently, \( \phi(y) \) is doubly periodic with the same periods \( 2\omega_1 \) and \( 2\omega_2 \). So are 
\[
n(y) = \frac{a''(\phi(y))}{4} - \frac{3(a'(\phi(y)))^2}{16a(\phi(y))}, \quad g_0(y) = f_0(\phi(y))a(\phi(y))^{-1/4}.
\]

(i) \( \phi(y) \) has poles at \( y \) such that \( \mathcal{P}(y; g_2, g_3) = \frac{1}{24}a''(z_0) \).

(ii) \( n(y) \) has poles at the poles of \( \phi(y) \), and at the values of \( y \) for which \( a(\phi(y)) = 0 \), for example, \( y = 0 \) which corresponds to \( z = z_0 \).

(iii) \( g_0 \) has the singularities at the points where \( \phi(y) \) is a singular point of \( f_0 \), and also at the zeros of \( a(\phi(y)) \) and possibly at the poles of \( \phi(y) \).

These are the points to avoid in the integration of (2.1.9). Let the set \( \tilde{S} \) include all these points together with any possible branch cuts. Let \( \tilde{D} = \{(y, s) \in \mathbb{C}^2 : \text{dist}(y + \lambda s, \tilde{S}) > \epsilon \text{ for all } -1 \leq \lambda \leq 1\} \), and \( \tilde{B}_\delta = \{f : \text{analytic on } \tilde{D}, \text{ and } ||f|| \leq \delta\} \) when \( \delta = 2||g_0|| \) with the norm \( ||g|| := \sup_{(y, s) \in \tilde{D}} e^{-\nu_1|y| - \nu_2|s|}|g(y, s)| \), for sufficiently large \( 0 < \nu_1 < \nu_2 \). Then let 
\[
\mathcal{M}(g) := \frac{g_0(y + s) + g_0(y - s)}{2} + \frac{1}{2} \int_0^s \int_{y-(s-s)}^{y+(s-s)} n(\tilde{y})g(\tilde{y}, \tilde{s})d\tilde{y}d\tilde{s} \tag{2.3.1}
\]
where each integration is along a line segment.

Theorem 2.3.2. (i) \( \mathcal{M} \) is a contractive mapping on \( \tilde{B}_\delta \) (as in the previous example).

(ii) Let \( B_i \) be the union of maximal strips in \( y \) plane, parallel to \( \omega_i \), with the distance to any point in \( \tilde{S} \) greater than \( \epsilon \) for \( i = 1, 2 \). Then for any \( y \) in \( B_1 \cup B_2 \), the solution is Laplace transformable.

Proof. (i) The line segments of integrations are contained in \( \tilde{D} \), so \( \mathcal{M}(g) \) is well defined. Let \( A(\epsilon) = \max_{y \in \tilde{D}} n(y) \), then, by the same calculation as (2.2.5), and by choosing large enough \( \nu_1 \), and \( \nu_2 - \nu_1 \), we get 
\[
||g|| \leq ||g_0|| + \frac{2A(\epsilon)||g_0||}{\nu_1(\nu_2 - \nu_1)} \leq 2||g_0|| = \delta.
\]
(ii) Since the function is doubly periodic and the strips in $B_i$ are parallel to $\omega_i$, any $y$ in the union of these clearly has at least one ray that is not too close to the set $\tilde{S}$. For such $y$, Laplace transformability follows from the same argument as in the previous example. Note that double periodicity generates infinitely many repeated singularities, and the choice of $B_i$s ensures that the path stays within a sufficient distance from the singularities.

**Note 2.3.3.** The same is true for some strips for which the slope is rational relative to the slopes of the fundamental parallelogram. We say that the slope is rational relative to the fundamental parallelogram if the slope coincides with that of the diagonal of some $m \times n$ parallelogram made of fundamental parallelograms.

**Note 2.3.4.** Due to the branch cuts, the Laplace transforms are not (at least not obviously) analytic continuation of each other.

□

**Corollary 2.3.5.** Let $\tilde{S}_0$ be the set of all the singular points described in Note 2.3.1. Then the solution $g$ is Laplace transformable in a direction $\theta$ in $p$ if the line $\{y\} + e^{i\theta/2}R$ passing through $y$ does not meet any point of $\tilde{S}_0$ and further its slope is rational relative to the slopes of the fundamental parallelogram. In other words, the formal solution $\tilde{u}(z,t)$ of the equation (2.1.1) with $a(z) = a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4 = a_0 (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ is Borel summable in a direction $\theta$ in $t$ (or in $T = 1/t$) if the slope of $e^{i\theta/2}R$ is rational relative to the slopes of the fundamental parallelogram and $z$ lies outside the set

$$\left\{ z \in \mathbb{C} \left| \int_{z_0}^{z} a(\zeta)^{-1/2} d\zeta \in \bigcup_{y \in \tilde{S}_0} (\{y\} + e^{i\theta/2}R) \right. \right\} \cup \bigcup_{z \in S_\star} \left\{ z \in \mathbb{C} \left| \Re \left( e^{-i\theta/2} \int_{z_\star}^{z} a(\zeta)^{-1/2} d\zeta \right) = 0 \right. \right\},$$

where $S_\star = \{ z_1, z_2, z_3, z_4, \infty \}$, singular points of $f_0$.
2.4 The example $a(z) = z$ revisited; resurgence

With the choice of $a(z) = z$ and $f_0(z) = \frac{1}{z^2 + 1}$, (2.1.6) becomes:

$$W_{ss} = zW_{zz}$$ (2.4.1)

with the initial conditions $W(z, 0) = \frac{1}{z^2 + 1}$ and $W_s(z, 0) = 0$.

**Theorem 2.4.1.** (i) Eq. (2.4.1) has a solution in closed form in terms of the incomplete elliptic integral of the second kind $E_i$, and the incomplete elliptic integral of the first kind $F_i$. (See (2.5.21), (2.5.25) and the discussion after (2.5.21).) The solution is Laplace transformable along $\mathbb{R}^+$, provided $\mathbb{R}^+$ does not cross the following manifolds:

$$Si = \{(s, z) : s = 2c_1\sqrt{z} + c_2\sqrt{2(1 + c_3)}; c_1, c_2, c_3 \in \{-1, 1\}\}$$ (2.4.2)

(ii) The manifolds (2.4.2) are, in fact, singular ones. More precisely, near $Si$, for nonzero $z$ we have

$$W(z, s) \sim \frac{f_{c_1,c_2,c_3}(\sqrt{z}, s)}{(s - 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)})}$$

for $c_1, c_2, c_3 \in \{-1, 1\}$, where $f_{c_1,c_2,c_3}$ are analytic at each point on the singular manifold.

The point $z = 0$ is special; near $z = 0$ the singularity is of the form

$$\left( s - c_2\sqrt{2(1 + c_3)} \right) E \left( \frac{s + 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}}{s - 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}} \right) + 2c_1\sqrt{z} K \left( \frac{s + 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}}{s - 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}} \right)$$

$$3(s + 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}) \sqrt{s - 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}} + R(s, z)$$ (2.4.3)

where the singularities of $R$ are weaker. Here $E$ is a complete elliptic integral of second kind, $K$ is a complete elliptic integral of first kind. Note that the leading term in (2.4.3) is $O([s - 2c_1\sqrt{z} - c_2\sqrt{2(1 + c_3)}]^{-1})$

(iii) $\mathbb{R}^+$ is a Stokes line only for those $z$ satisfying $\Im \sqrt{z} = \pm \frac{1}{\sqrt{2}}$, $z \neq \pm i$. 

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2.5 Proof of Theorem 2.4.1

(i) Near \( s = 0 \) we have

\[
W(z, s) = \sum_{n=0}^{\infty} \frac{T_n(z)}{n!} s^n
\]

(2.5.1)

Then (2.4.1) and (2.5.1) give

\[
T_{n+2}(z) = z T''_n(z)
\]

(2.5.2)

with \( T_0(z) = 1/(z^2 + 1) \); note that \( T_n = 0 \) for \( n \) odd since \( T_1 = 0 \).

Let \( T_n(z) = \int_0^{\infty} t^n H(t, z) dt \). (We use methods from [4].) Then (2.5.2) gives

\[
t^2 H(t, z) = z H_{zz}(t, z).
\]

(2.5.3)

To get \( H(t, z) \), we inverse Laplace transform in \( z \) denoting \( \tilde{H}(t, q) = \mathcal{L}^{-1}\{H(t, z)\}(q) \).

Then we get \( t^2 \tilde{H} = (q^2 \tilde{H})' \) and obtain

\[
\tilde{H} = a(t) \frac{e^{-\frac{t^2 q}{q^2}}}{q^2}.
\]

(2.5.4)

Inverse Laplace transforming \( T_0(z) \), we have

\[
\mathcal{L}^{-1}\{T_0\}(q) = \int_0^{\infty} \mathcal{L}^{-1} H = \int_0^{\infty} a(t) \frac{e^{-\frac{t^2 q}{q^2}}}{q^2} dt = \sin q.
\]

(2.5.5)

By letting \( t^2 = u \) and \( 1/q = x \), we get

\[
\int_0^{\infty} a(\sqrt{u}) e^{-u x^2} du = x^2 \mathcal{L} \left\{ \frac{a(\sqrt{u})}{2 \sqrt{u}} \right\} = \sin \frac{1}{x}
\]

(2.5.6)

and so,

\[
a(\sqrt{u}) = a(t) = 2 \sqrt{u} \mathcal{L}^{-1} \left\{ \frac{\sin \frac{1}{x^2}}{x^2} \right\} (u) = 2 t \mathcal{L}^{-1} \left\{ \frac{\sin \frac{1}{x^2}}{x^2} \right\} (t^2).
\]

(2.5.7)

These give (see (2.5.1) and (2.5.2) and the discussion following them)

\[
W = \sum_{n=0}^{\infty} \frac{T_{2n}(z)}{(2n)!} s^{2n} = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(st)^{2n}}{(2n)!} H(t, z) dt
\]

\[
= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(st)^{2n}}{(2n)!} a(t) \int_0^{\infty} \frac{e^{-\frac{t^2 q}{q^2}}}{q^2} e^{-u q} du dt
\]

(2.5.8)
\[ W = \frac{\sqrt{\pi} \left( -1 + \frac{i}{2} \right)}{2} \int_{0}^{\infty} (e^{st} + e^{-st}) t^2 \left[ K_1((1-i)\sqrt{2t}) + iI_1((1+i)\sqrt{2t}) \right] e^{-\frac{z^2}{4}} dt \]

\[ + \frac{\sqrt{\pi} \left( -1 + \frac{i}{2} \right)}{2} \int_{0}^{\infty} e^{st} t^2 \int_{0}^{\pi} e^{(1-i)\sqrt{2t}\cos\theta} \cos\theta d\theta \int_{1}^{\infty} e^{-2\sqrt{2t}\tau (\tau^2 - 1)^{1/2}} d\tau d\tau dt \]

\[ + \frac{\sqrt{\pi} \left( -1 - \frac{i}{2} \right)}{2} \int_{0}^{\infty} e^{-st} t^2 \int_{0}^{\pi} e^{(1+i)\sqrt{2t}\cos\theta} \cos\theta d\theta \int_{1}^{\infty} e^{-2\sqrt{2t}\tau (\tau^2 - 1)^{1/2}} d\tau d\tau dt \]

\[ + \frac{\sqrt{\pi} \left( -1 - \frac{i}{2} \right)}{2} \int_{0}^{\infty} e^{st} t^2 \int_{0}^{\pi} e^{(1+i)\sqrt{2t}\cos\theta} \cos\theta d\theta \int_{1}^{\infty} e^{-2\sqrt{2t}\tau (\tau^2 - 1)^{1/2}} d\tau d\tau dt \]

\[ (2.5.14) \]
for $|\arg\sqrt{z}| < \frac{\pi}{2}$. (We note that the domain of (2.5.17) below is contained in this region.)

Our strategy is to integrate each term, using analytic continuation when needed, and to define $W$ as this analytic continuation.

Here, we first look at the first term of (2.5.14),

$$\frac{\sqrt{2}}{\pi} (-1 + i) z \int_0^\infty e^{st^2} \int_0^{\infty} e^{(1-i)\sqrt{2}\cos \theta} \cos \theta d\theta \int_0^\infty e^{-2\sqrt{2}\tau^2} (\tau^2 - 1)^{\frac{1}{2}} d\tau d\theta dt. \quad (2.5.15)$$

Convergence follows from Fubini’s theorem and the final form (2.5.21), so after interchanging the order of integration, this term can be written in the following way:

$$\frac{\sqrt{2}}{\pi} (-1 + i) z \int_1^\infty (\tau^2 - 1)^{\frac{1}{2}} \int_0^\infty \cos \theta \int_0^\infty t^2 e^{(s+(1-i)\sqrt{2}\cos \theta-2\sqrt{2}\tau)t} d\theta d\tau d\tau. \quad (2.5.16)$$

If

$$\Re(s + (1 - i)\sqrt{2}\cos \theta - 2\sqrt{2}\tau) < 0, \quad \forall \tau \in [1, \infty), \theta \in [0, \pi] \quad (2.5.17)$$

then we perform the innermost integral to obtain:

$$\frac{2\sqrt{2}}{\pi} (-1 + i) z \int_1^\infty (\tau^2 - 1)^{\frac{1}{2}} \int_0^\infty \frac{\cos \theta}{(-s - (1 - i)\sqrt{2}\cos \theta + 2\sqrt{2}\tau)^3} d\theta d\tau \quad (2.5.18)$$

$$= \frac{2\sqrt{2}}{\pi} (-1 + i) z \int_1^\infty (\tau^2 - 1)^{\frac{1}{2}} \int_0^\infty \frac{\cos \theta}{(2\sqrt{2}\tau - s)^3} \frac{d\theta}{(1 - \frac{(2\sqrt{2}\tau - s)^3}{s})^3} d\tau d\tau. \quad (2.5.19)$$

The inequality (2.5.17) implies nonzero denominator of the innermost integrand of (2.5.19). Since

$$\int_0^\pi \frac{\cos \theta}{(1 - \cos \theta)^3} d\theta = \frac{a^3}{2} \int_0^{2\pi} \frac{\cos \theta}{(a - \cos \theta)^3} d\theta = \frac{a^3}{2t} \int_{|z| = 1} \frac{1}{z(a - \frac{1}{2}(z + \frac{1}{z}))^2} dz, \quad (2.5.20)$$

integrating and using the residue theorem, we get:

$$\pm \frac{2\sqrt{2}}{\pi} (-1 + i) z \int_1^\infty (\tau^2 - 1)^{\frac{1}{2}} \frac{3\pi(\frac{2\sqrt{2}\tau - s}{\sqrt{2}(1-i)})^4}{(2\sqrt{2}\tau - s)^3 2((\frac{2\sqrt{2}\tau - s}{\sqrt{2}(1-i)})^2 - 1)^2} d\tau$$

$$= \pm \frac{3}{8} i z^{-\frac{1}{2}} \int_1^\infty \frac{(2\sqrt{2}\tau - s)(\tau^2 - 1)^{\frac{1}{2}}}{((\tau - s)^2 - (\frac{2\sqrt{2}\tau - s}{2\sqrt{2}\tau})^2)^\frac{3}{2}} d\tau \quad (2.5.21)$$
which can be integrated noting that, by (2.5.17), the denominator is nonzero for any \( \tau \in [1, \infty) \). (By the inequality (2.5.17), the choice of the sign is valid throughout the domain.) A straightforward but lengthy calculation allows for a representation of (2.5.21) in terms of combinations of incomplete elliptic functions. The expression itself is rather long and not too illuminating, so we omit it.

In fact, (2.5.21) can be written in the following way,

\[
3 \int_{\frac{1}{2}}^{\infty} \frac{(2\sqrt{z} - s)(\tau^2 - 1)^{\frac{1}{2}}}{(\tau - \frac{s}{2\sqrt{z}} + \frac{\sqrt{2}(1-i)}{2\sqrt{z}})^{\frac{1}{2}}(\tau - \frac{s}{2\sqrt{z}} - \frac{\sqrt{2}(1-i)}{2\sqrt{z}})^{\frac{1}{2}}} \, d\tau,
\]

and this can be analytically continued in \( \eta := \frac{s}{2\sqrt{z}} - \frac{\sqrt{2}(1-i)}{2\sqrt{z}} \) and \( \xi := \frac{s}{2\sqrt{z}} + \frac{\sqrt{2}(1-i)}{2\sqrt{z}} \) to the domain

\[
\{ (\eta, \xi) : \eta \neq 1, \xi \neq 1 \}
\]

by deforming the line segment \( \tau \in [1, \infty) \), for example, as shown in Figure 2.1, depending on the direction we are continuing from, and the position of \( \eta \) and/or \( \xi \) on the line segment \( [1, \infty) \). Therefore, the solution can be analytically continued in \( s \) and \( \sqrt{z} \) to the domain

\[
\left\{ (z, s) : \frac{s}{\sqrt{z}} - \frac{\sqrt{2}(1-i)}{2\sqrt{z}} \neq 1 \text{ and } \frac{s}{\sqrt{z}} + \frac{\sqrt{2}(1-i)}{2\sqrt{z}} \neq 1 \right\}
\]

for nonzero \( z \).

Near \( z = 0 \), letting \( \sqrt{z} = \epsilon \) and \( \sqrt{z}\tau = \sigma \), (2.5.21) becomes:

\[
\frac{3i}{8\epsilon^3} \int_{\epsilon}^{\infty} \arg^* \frac{((\frac{\sigma}{\epsilon})^2 - 1)^{\frac{1}{2}}(2\sigma - s)}{((\frac{\sigma}{\epsilon} - \frac{s}{2\sigma})^2 - (\frac{\sqrt{2}(1-i)}{2\epsilon})^2)^{\frac{1}{2}}} \epsilon \, d\sigma
= 12i \int_{\epsilon}^{\infty} \arg^* \frac{(\sigma^2 - \epsilon^2)^{\frac{1}{2}}(2\sigma - s)}{(2\sigma - s)^2 - (\sqrt{2}(1-i))^2} \, d\sigma.
\]

Taking \( \sigma_1 = \sigma - \epsilon \), we get

\[
12i \int_{0}^{\infty} \arg^* \frac{\sigma_1^{\frac{1}{2}}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}(2\sigma_1 + 2\epsilon - s)}{(2\sigma_1 + 2\epsilon - s + \sqrt{2}(1-i))^2(2\sigma_1 + 2\epsilon - s - \sqrt{2}(1-i))^2} \, d\sigma_1.
\]
Figure 2.1: The deformation of contour for analytic continuation

which can be also analytically continued in $\epsilon$ and $s$ to the domain

$$\{(\epsilon, s) : 2\epsilon - s + \sqrt{2}(1 - i) \neq 0 \text{ and } 2\epsilon - s - \sqrt{2}(1 - i) \neq 0\}$$

in the similar way, and this includes $\epsilon = 0$.

The other three terms in (2.5.14) can be found, and analytically continued in a similar way, so the solution of (2.4.1) becomes:

$$\frac{3}{8}iz^{-\frac{1}{2}} \int_{1}^{\infty} \frac{(2\sqrt{z} \tau - s)(\tau^2 - 1)^{\frac{1}{2}}}{((\tau - \frac{s}{2\sqrt{z}})^2 - (\frac{\sqrt{2}(1 + i)}{2\sqrt{z}})^2)^{\frac{1}{2}}} d\tau + \frac{3}{8}iz^{-\frac{1}{2}} \int_{1}^{\infty} \frac{(2\sqrt{z} \tau + s)(\tau^2 - 1)^{\frac{1}{2}}}{((\tau + \frac{s}{2\sqrt{z}})^2 - (\frac{\sqrt{2}(1 - i)}{2\sqrt{z}})^2)^{\frac{1}{2}}} d\tau$$

$$- \frac{3}{8}iz^{-\frac{1}{2}} \int_{1}^{\infty} \frac{(2\sqrt{z} \tau - s)(\tau^2 - 1)^{\frac{1}{2}}}{((\tau + \frac{s}{2\sqrt{z}})^2 - (\frac{\sqrt{2}(1 - i)}{2\sqrt{z}})^2)^{\frac{1}{2}}} d\tau - \frac{3}{8}iz^{-\frac{1}{2}} \int_{1}^{\infty} \frac{(2\sqrt{z} \tau + s)(\tau^2 - 1)^{\frac{1}{2}}}{((\tau - \frac{s}{2\sqrt{z}})^2 - (\frac{\sqrt{2}(1 + i)}{2\sqrt{z}})^2)^{\frac{1}{2}}} d\tau$$

(2.5.25)
and it has the singular manifolds \((z, s)\) such that 
\[ s = 2c_1\sqrt{z} + c_2\sqrt{2}(1 + c_3i), \]
where \(c_1, c_2, c_3 \in \{-1,1\}. \)

Since (2.5.25) is bounded in \(s\), as long as the ray does not meet the singular point, Laplace transformability in \(p\) is clear. Likewise, \(V\) is Laplace transformable in \(p\).

(ii) We now analyze the singularity type. The various cases are very similar; we only look at \((z, s)\) such that 
\[ 2\sqrt{z} - s = \sqrt{2}(1 - i). \]

For fixed \(z \neq 0\), we let \(\tau = \sigma + 1\), and near \(s_0 = 2\sqrt{z} - \sqrt{2}(1 - i)\), we let \(s = s_0 + \delta\).

Then the equation (2.5.21) becomes:
\[
\frac{3}{8}iz^{-\frac{3}{2}} \int_0^\infty \frac{(2\sqrt{z}\sigma + \sqrt{2}(1 - i) - \delta)\sigma^{\frac{1}{2}}(\sigma + 2)^{\frac{1}{2}}}{(\sigma - \frac{\delta}{2\sqrt{z}})^{\frac{3}{2}}(\sigma - \frac{\delta}{2\sqrt{z}} + \frac{\sqrt{2}(1 - i)}{\sqrt{z}})^{\frac{1}{2}}} d\sigma.
\]  
(2.5.26)

By letting \(\rho = \frac{\sigma}{\delta}\), the integral in (2.5.26) equals
\[
\frac{3iz^{-\frac{3}{2}}}{8\delta} \int_0^{\infty} e^{-i\arg \delta} \frac{(2\sqrt{z}\delta\rho + \sqrt{2}(1 - i) - \delta)\rho^{\frac{1}{2}}(\delta + 2)^{\frac{1}{2}}}{(\rho - \frac{1}{2\sqrt{z}})^{\frac{3}{2}}(\rho - \frac{1}{2\sqrt{z}} + \frac{\sqrt{2}(1 - i)}{\sqrt{z}})^{\frac{1}{2}}} d\rho \sim f(\sqrt{z}, \delta),
\]  
(2.5.27)

where \(f(\sqrt{z}, \delta)\) is analytic for small \(\delta\).

When \(z = 0\), the behavior near the singular manifold is quite different.

Taking \(s = -\sqrt{2}(1 - i) + \delta\), (2.5.24) becomes:
\[
12i \int_0^{\infty} e^{i\arg \epsilon} \frac{\sigma_1^{\frac{1}{2}}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}(2\sigma_1 + 2\epsilon - \delta + \sqrt{2}(1 - i))}{(2\sigma_1 + 2\epsilon - \delta)^{\frac{3}{2}}(2\sigma_1 + 2\epsilon - \delta + 2\sqrt{2}(1 - i))^2} d\sigma_1.
\]  
(2.5.28)

Choose an \(\epsilon_1\) near 0 (more restrictions on \(\epsilon_1\) will be mentioned soon). Breaking (2.5.28) into two parts, \(\int_{\epsilon_1}^{\infty} + \int_{\epsilon_1}^{\infty} e^{i\arg \epsilon}\), we get the analyticity of the latter integral at \((\epsilon, \delta) = (0, 0)\).

In the first integral, for small enough \(|\sigma_1|\)
\[
\frac{2\sigma_1 + 2\epsilon - \delta + \sqrt{2}(1 - i)}{(2\sigma_1 + 2\epsilon - \delta + 2\sqrt{2}(1 - i))^2}
\]  
(2.5.29)
can be expressed as a convergent series:

\[
f_0(\epsilon, \delta) + f_1(\epsilon, \delta)\sigma_1 + f_2(\epsilon, \delta)\sigma_1^2 + \ldots \tag{2.5.30}
\]

where

\[
f_0(\epsilon, \delta) = \frac{2\epsilon - \delta + \sqrt{2}(1 - i)}{(2\epsilon - \delta + 2\sqrt{2}(1 - i))^{\frac{3}{2}}}, \quad f_1(\epsilon, \delta) = \frac{-6\epsilon + 3\delta - \sqrt{2}(1 - i)}{(2\epsilon - \delta + 2\sqrt{2}(1 - i))^{\frac{3}{2}}}, \ldots \tag{2.5.31}
\]

and all \( f_i(\epsilon, \delta) \) are analytic in \( \epsilon \) and \( \delta \) near \((\epsilon, \delta) = (0, 0)\).

Using this series, (2.5.28) clearly becomes:

\[
12i \left[ f_0(\epsilon, \delta) \int_0^{\epsilon^i} \frac{\sigma_1^\frac{3}{2}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}}{(2\sigma_1 + 2\epsilon - \delta)^\frac{3}{2}} d\sigma_1 + f_1(\epsilon, \delta) \int_0^{\epsilon^i} \frac{\sigma_1^\frac{3}{2}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}}{(2\sigma_1 + 2\epsilon - \delta)^\frac{3}{2}} d\sigma_1 \right. \\
\left. + f_2(\epsilon, \delta) \int_0^{\epsilon^i} \frac{\sigma_1^\frac{5}{2}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}}{(2\sigma_1 + 2\epsilon - \delta)^\frac{3}{2}} d\sigma_1 + \ldots \right] \tag{2.5.32}
\]

The singularity of the integral multiplying \( f_0 \) of (2.5.32) comes from the singularity of the first integral of the following rewriting:

\[
12i f_0(\epsilon, \delta) \left( \int_0^\infty \frac{\sigma_1^\frac{3}{2}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}}{(2\sigma_1 + 2\epsilon - \delta)^\frac{3}{2}} d\sigma_1 - \int_{\epsilon^i}^\infty \frac{\sigma_1^\frac{3}{2}(\sigma_1 + 2\epsilon)^{\frac{1}{2}}}{(2\sigma_1 + 2\epsilon - \delta)^\frac{3}{2}} d\sigma_1 \right) \tag{2.5.33}
\]

since the second part is an analytic function at \((\epsilon, \delta) = (0, 0)\). Integrating the first integral, we get:

\[
\frac{\delta E(\frac{\delta + 2\epsilon}{\delta - 2\epsilon}) + 2\epsilon K(\frac{\delta + 2\epsilon}{\delta - 2\epsilon})}{3(\delta + 2\epsilon)\sqrt{2\epsilon - \delta}} \tag{2.5.34}
\]

where \( E \) is a complete elliptic integral of second kind, and \( K \) is a complete elliptic integral of first kind.

In Appendix B, we discuss a number of interesting regimes of (2.5.34) that imply the singularity of the first part of (2.5.32) is at most \( O((2\epsilon - \delta)^{-1}) \).
To prove that the singularities of $R$ are weaker, we note that, letting $\sigma_1 = \mu \sigma_2$ with $\mu := \frac{2\epsilon - \delta}{2}$, for each $m = 1, 2, \ldots$, we have:

$$12i f_m(\epsilon, \delta) \int_0^{\epsilon_1} \frac{\sigma_1^{2m+1} (\sigma_1 + 2\epsilon)^{\frac{1}{2}}}{(2\sigma_1 + 2\mu)^{\frac{3}{2}}} d\sigma_1$$

$$= \frac{3i}{\sqrt{2}} f_m(\epsilon, \delta) \mu^{m-1} \int_0^{\epsilon_1} \frac{\sigma_2^{2m+1} (\mu \sigma_2 + 2\epsilon)^{\frac{1}{2}}}{(\sigma_2 + 1)^{\frac{3}{2}}} d\sigma_2. \quad (2.5.35)$$

If we choose $\epsilon_1$ so that $\epsilon_1$ has the same argument as $\mu$,

$$\left| \mu^{m-1} \int_0^{\epsilon_1} \frac{\sigma_2^{2m+1} (\mu \sigma_2 + 2\epsilon)^{\frac{1}{2}}}{(\sigma_2 + 1)^{\frac{3}{2}}} d\sigma_2 \right|$$

$$\leq |\mu|^{m-1} \int_0^{\epsilon_1} \left| \frac{\sigma_2^{2m+1} (\mu \sigma_2 + 2\epsilon)^{\frac{1}{2}}}{(\sigma_2 + 1)^{\frac{3}{2}}} \right| d\sigma_2, \quad m = 1, 2, \ldots \quad (2.5.36)$$

which has, at most, a $\ln \mu$ singularity, and that only if $m = 1$. Otherwise, there is no singularity since every singularity arising from integration is at most $\mu^{-m+1}$ and these cancel out. For arbitrary arg $\epsilon_1$, similar calculation and deformation of contour provides the same result.

Note that higher powers of $\sigma_1$ in (2.5.30) result in weaker singularities near $\sigma_1 = 0$. So, with $2\epsilon - \delta = 2\sqrt{z} - s + \sqrt{2}(1 - i)$, the leading behavior is

$$O\left( |s - 2c_1\sqrt{z} - c_2\sqrt{2}(1 + c_3i)|^{-1} \right).$$

(iii) Note that for any $z$ such that $\Im \sqrt{z} \neq \pm \frac{1}{\sqrt{2}}$, the singular point of $s$ is not real, and so the function is analytic in $s$ in a neighborhood of $\mathbb{R}^+$. When $\Im \sqrt{z} = \pm \frac{1}{\sqrt{2}}$, $W$ has the singular points of $s$ (i) at two nonzero points; $s = \mp 2\sqrt{2} + \sqrt{2}(1 + i)$ and $s = \pm 2\sqrt{2} + \sqrt{2}(1 - i)$ for $\Im \sqrt{z} = \pm \frac{1}{\sqrt{2}}$, $z \neq \pm i$, that agree for $\sqrt{z} = \frac{1 \pm i}{\sqrt{2}}$, and (ii) at two points with one of them the origin; $s = 0$ and $s = 2\sqrt{2}$ for $z = \pm i$. 

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Case (i): Taking Laplace transform of $V(z,p) = \frac{1}{\sqrt{p}}W(z,2\sqrt{p})$ in $p$, and using residue theorem, we get:

$$
\int_0^{\infty} e^{-pt} \frac{W(z,2\sqrt{p})}{\sqrt{p}} \, dp = \int_0^{\infty} e^{-pt} \frac{W(z,2\sqrt{p})}{\sqrt{p}} \, dp \\
- 2\pi i \Sigma \text{Res}(e^{-pt} \frac{W(z,2\sqrt{p})}{\sqrt{p}}) \quad (2.5.37)
$$

where the last summation is about $p = (\frac{\pm 2\sqrt{z} + \sqrt{2}(1+i)}{2})^2$ and $(\frac{\pm 2\sqrt{z} + \sqrt{2}(1-i)}{2})^2$,

and it is:

$$
2\pi i \Sigma \text{Res} = 2\pi i (e^{-T(\frac{\pm 2\sqrt{z} + \sqrt{2}(1+i)}{2})^2} f_1(\sqrt{z}, \pm 2\sqrt{z} + \sqrt{2}(1+i)) \\
+ e^{-T(\frac{\pm 2\sqrt{z} + \sqrt{2}(1-i)}{2})^2} f_2(\sqrt{z}, \pm 2\sqrt{z} + \sqrt{2}(1-i)) \quad (2.5.38)
$$

where $f_1$ and $f_2$ are from $f$ of (2.5.27).

Case (ii): Since $s = p = 0$ is one of the singular points, the integrand has

$$
e^{-pt} \frac{W(z,2\sqrt{p})}{\sqrt{p}} = e^{-pt} \frac{f(\sqrt{z},2\sqrt{p})}{p} \quad (2.5.39)
$$

which cannot be Laplace transformed.
CHAPTER 3
APPLICATIONS OF TRANS SERIES AND BOREL SUMMATION: NONELEMENTARITY OF FUNCTIONS AND FACTORIZABILITY OF DIFFERENTIAL OPERATORS

3.1 Transseries and Nonelementarity

We show that transseries, perhaps surprisingly, can be used to show nonelementarity of a wide range of functions arising in applications. For precise formulations of the following definitions and results, see [19] and [18].

Definition 3.1.1. An elementary function is defined as a function built from a finite number of combinations and compositions of algebraic, exponential and logarithm functions under algebraic operations (+, −, ×, ÷). (Allowing the underlying field to be \( \mathbb{C} \), trigonometric functions become elementary, too.)

For example, \( e^{\sqrt{1+\ln^2 x}} (1+x^2)^{-1}, (\sqrt{x+1})^{\ln x^4+x^3-1}, e^{-1/x^2}/(e^x-x) \) and \( x^\alpha = e^{\alpha \ln x} \)
and thus rational functions are elementary functions over the reals.

Convergent transseries. A fully convergent log-free transseries, say as \( x \to +\infty \) (see for more detail §3.1.2), is inductively defined as a convergent multiseries of the
form

\[ T = \sum_{\vec{k} \geq \vec{M}} c_{\vec{k}} \mu^{-\vec{k}} := \sum_{k_i \geq M_i, i = 1, \ldots, n} c_{k_1, \ldots, k_n} \mu^{-k_1} \cdots \mu^{-k_n} \text{ for some } \vec{M} = (M_1, \ldots, M_n) \in \mathbb{Z}^n, \]

where \( \vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n \) and \( \mu_j \) are the generating transmonomials, of the form \( x^{\alpha_j} \) or \( e^{T_j} x^{\alpha_j} \) where \( T_j \) is a large, positive, convergent transseries, constructed at an earlier stage of the induction. We note that the generators \( \mu_i \) can be rechosen so that small transseries has only negative powers for each \( \mu_i \). A fully convergent transseries as \( x' \to -\infty \) is defined similarly, after the substitution \( x = -x' \). Similarly, transseries as \( x' \to x_0^\pm \) are defined as above, after the substitution \( x = (x - x_0)^{-1} \).

**Note 3.1.2.** Clearly, a fully convergent transseries defines (locally) a function.

### 3.1.1 Results and examples

The main results in this section are the following.

**Theorem 3.1.3.** If any part of the asymptotic transseries at some point (infinity included) of a function over \( \mathbb{R} \) has zero radius of convergence, then this function is nonelementary over \( \mathbb{R} \).

**Corollary 3.1.4.** If a function over \( \mathbb{R} \) has a divergent asymptotic power series at some point (infinity included), then it is nonelementary over \( \mathbb{R} \).

For example, \( \text{Ei}(x) = \int_{-\infty}^{x} s^{-1} e^s ds \) (understood in the principal part sense) is not elementary since then so would \( g(x) = e^{-x} \int_{-\infty}^{x} s^{-1} e^s ds \) be; however, it is easy to see, for instance by integration by parts, or by Watson’s lemma, that \( g(x) \sim \sum_{k=0}^{\infty} k! x^{-k-1} \).

The proofs follow from the following propositions.

**Proposition 3.1.5.** Every elementary function has a fully convergent transseries at any point where it is defined as well as at points of discontinuity and at \( \pm \infty \); the transseries is unique at each such point.
Proposition 3.1.6. If a fully convergent transseries is asymptotic to a formal power series, then the series is convergent.

Note 3.1.7. In fact, by Watson’s lemma and Cauchy’s formula, any function of the form $\mathcal{L}f := \int_0^\infty e^{-xp}f(p)dp$ can only be elementary if $f$ is entire. This is a relatively simple way to check for divergence of the asymptotic series. Indeed, it is known ([23]) that solutions of quite general systems of linear or nonlinear ODEs can be written in terms of such Laplace transforms, and checking (analytically or numerically) that $f$ is not entire is typically not so difficult. For instance $e^{-x}\text{Ei}(x) = \mathcal{L}(1 + p)^{-1}$, $\sqrt{\pi}x^{-1/2}e^{-x}\text{Erf}(\sqrt{x}) = \mathcal{L}\sqrt{1 - p}$ (taken as a Cauchy principal part integral), solutions of Painlevé equations, the Airy functions and many more are seen to be nonelementary.

In other (slightly more technical) words, if a function has any singularity in its Borel plane, then it is not elementary.

Remark 3.1.8. Liouville’s principle also allows for proving nonelementarity of antiderivatives of log-exp functions such as $\int e^t/t$.

3.1.2 Transseries

We present here, for convenience, a summary of the notions used and a brief description of the construction of transseries. For details, see e.g. [4], [24], [22].

We also present simplified construction of fully convergent transseries, much of the simplification coming from the use of Hartog’s theorem.

Transseries, say as $x \to +\infty$, are formal (no convergence is required) expansions in terms of powers, finitely iterated exponentials and logs of ordinal length, which
are asymptotic in the sense that the size of the terms decreases as the ordinal index increases. For example,

\[ e^{e^x} \sum_{n \geq 0} \frac{n!}{x^n} + \sum_{n=0}^{\infty} e^{x-n\sqrt{x}} x^{-n} + \cdots, \quad x \to +\infty \]

where the terms systematically decrease from left to right, and there may be infinitely many other sums, with this property, following these two. Here, \( e^{e^x} \sum_{n \geq 0} \frac{n!}{x^n} \) and \( e^{x-n\sqrt{x}} x^{-n} \) are transmonomials, often denoted by \( \mu \), blocks that cannot be reexpanded asymptotically. Note that \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \), though convergent, is not a transseries as \( x \to \infty \) since any prescribed number of initial terms of the series grow for large enough \( x \) and there is no way to order the series decreasingly once and for all \( x \); \( e^x \) is a transmonomial too. The terms might be analyzable further; e.g. \( e^{e^x} \sum_{n \geq 0} \frac{n!}{x^n} \) is the composition of exp with a large transseries. The inductive construction of transseries makes all this transparent.

In short, transseries and the transmonomials (at +\( \infty \)) can be defined in the following way:

Transseries at level zero is a finitely generated formal sum

\[ \sum_{\vec{k} \geq \vec{M}} a_{\vec{k}} x^{-\vec{k} \cdot \vec{\alpha}} = \sum_{k_i \geq M_i, i=1,\ldots,n} a_{k_1,k_2,\ldots,k_n} x^{-\alpha_1 k_1} \cdots x^{-\alpha_n k_n} \]

where each \( \alpha_i > 0 \), that is, each \( x^{\alpha_i} \gg 1 \) as \( x \to \infty \) (they can be chosen this way) and each \( x^{\alpha_i} \) is called transmonomial at level zero. Notice that the summation is for \( \vec{k} \geq \vec{M} \), and so the set \( x^{-\alpha_1 k_1} \cdots x^{-\alpha_n k_n} \) in the transseries can be well ordered. Collecting every term with \( \vec{k} \cdot \vec{\alpha} = \alpha_1 k_1 + \cdots + \alpha_n k_n \leq 0 \), we get a purely large term \( L \), and similarly, we get constant term \( c \) for zero power, and purely small term \( s \) for the rest. For transseries at level one, we use level one transmonomial of the form: \( e^L x^\alpha \) where \( e^L \) cannot be comparable with any power of \( x \) as \( x \to \infty \): if \( L > 0 \), then \( e^L \gg x^\alpha \gg 1 \), and if \( L < 0 \), then \( e^L \ll x^{-\alpha} \ll 1 \) for any \( \alpha > 0 \), obviously by L’Hospital’s rule. (We define \( L > 0 \) to be \( L > 0 \) for large enough \( x \). Likewise, \( L < 0 \)
is defined to be \( L < 0 \) for sufficiently large \( x \).) For the transmonomial to be purely large, we let \( \mu = e^L x^\alpha \) if \( L > 0 \), and \( \mu = e^{-L} x^\alpha \) if \( L < 0 \), and transseries of level one is of the form

\[
T = \sum_{k \geq \bar{M}} c_k \mu_1^{-k_1} \cdots \mu_n^{-k_n}.
\]

Inductively, we define transmonomial \( e^L x^\alpha \) of level \( n + 1 \) for large transseries \( L \) of level \( n \). Note that \( e^{L_1} x^{\alpha_1} \gg e^{L_2} x^{\alpha_2} \) if both \( L_1 \) and \( L_2 > 0 \) and are large transseries, but the level of \( L_1 \) is higher than the one of \( L_2 \). (By induction: \( L_1 - L_2 > 0 \) and \( \gg 1 \) still with the same level of \( L_1 \), then \( e^{L_1 - L_2} \gg x^{\alpha_2 - \alpha_1} \), and so \( e^{L_1} x^{\alpha_1} \gg e^{L_2} x^{\alpha_2} \).) For non log-free transseries, we define it as \( T \circ \ln_n(x) \).

### 3.1.3 Proof of Proposition 3.1.5

Mainly we use two lemmas whose proofs are based on the following facts: Every transseries \( T \) can be written as \( T = L + c + s = \Dom(T)(1 + s) \) where \( L \) is a purely large transseries, \( c \) is a constant, \( s \) and \( s' \) are purely small transseries, and \( \Dom(T) \) is the dominant transmonomial of \( T \); and Hartog’s theorem.

**Lemma 3.1.9.** Fully convergent transseries (i) form a field, and are closed under applying (ii) the exponential and (iii) the logarithm. (cf. [24])

**Proof.** We only show this for log-free transseries, since it is obvious that composition of a convergent expansion with an iterated log amounts to a change of variables, which does not affect convergence.

(i) To show this is a field, the only nontrivial part is to show the existence of the inverse. Letting a fully convergent transseries \( T = \Dom(T)(1 + s) \) as above, we only need to show that \( (1 + s)^{-1} \) is also a fully convergent transseries since \( \Dom(T)^{-1} \) is already a transmonomial. Since \( s \) is purely small, it can be expressed in the following form:

\[
s = \sum_{k \geq \bar{M} \geq 0} a_k \mu_1^{-k_1} \cdots \mu_n^{-k_n} = \sum_{k \geq \bar{M} \geq 0} a_k \nu_1^{k_1} \cdots \nu_n^{k_n} =: F(\nu_1, \ldots, \nu_n) \quad (3.1.1)
\]

where \( \nu_i = \mu_i^{-1} \) are purely small.
Since $s$ is convergent, $F$ is analytic in $\nu_1, \ldots, \nu_n$ for sufficiently small $|\nu_1, \ldots, \nu_n| < \epsilon$. Taking such a small ball with $|F| < \frac{1}{2}$, $(1 + F)^{-1}$ is still analytic in each $\nu_i$, and by Hartog’s theorem, it is analytic in $(\nu_1, \ldots, \nu_n)$. So $(1 + F)^{-1}$ can be expressed, for some $b_m$, as a convergent series:

$$\sum_{\vec{m} \geq \vec{N} \geq \vec{0}} b_{\vec{m}} \nu_1^{m_1} \cdots \nu_n^{m_n} = \sum_{\vec{m} \geq \vec{N} \geq \vec{0}} b_{\vec{m}} \mu_1^{-m_1} \cdots \mu_n^{-m_n}.$$

The second expression gives a fully convergent transseries, and so is the inverse of $T$.

(ii) $T = L + (c + s) = L + \sum_{\vec{k} \geq \vec{0}} c_{\vec{k}} \mu_1^{-k_1} \cdots \mu_n^{-k_n}$ is a fully convergent transseries. Then, since $\sum_{\vec{k} \geq \vec{0}} c_{\vec{k}} \mu^{-\vec{k}} = \sum_{\vec{k} \geq \vec{0}} c_{\vec{k}} \vec{\nu}^{\vec{k}} = F(\vec{\nu}) = F(\nu_1, \ldots, \nu_n)$ is an analytic function for sufficiently small $|\vec{\nu}|$,

$$e^{c+s} = e^{F(\nu_1, \ldots, \nu_n)}$$

is also analytic in each $\nu_i$, so after all, in $(\nu_1, \ldots, \nu_n)$ by Hartog’s theorem, and can be expressed as the following convergent series:

$$e^{c+s} = \sum_{\vec{k} \geq \vec{0}} \tilde{c}_{\vec{k}} \nu_1^{k_1} \cdots \nu_n^{k_n} = \sum_{\vec{k} \geq \vec{0}} \tilde{c}_{\vec{k}} \mu_1^{-k_1} \cdots \mu_n^{-k_n}.$$

Also $e^L$ is a purely large/small transmonomial depending on the sign, and since $L$ is a fully convergent transseries, then $e^L$ is also a fully convergent transmonomial (of level higher by one); call it $\mu_0$. So

$$e^T = \mu_0 \sum_{\vec{k} \geq \vec{0}} \tilde{c}_{\vec{k}} \nu_1^{k_1} \cdots \nu_n^{k_n} = \sum_{\vec{k} \geq \vec{0}} \tilde{c}_{\vec{k}} \mu_0^{-k_1} \cdots \mu_n^{-k_n}$$

is also a fully convergent transseries.

(iii) Having the same expression as in (i), $\ln T$ can be expressed in the following form:

$$\ln T = \ln(Dom(T)(1 + s)) = \ln(Dom(T)) + \ln(1 + s)$$
For sufficiently small \( (\nu_1, \cdots, \nu_n) \), \(|F|\) is small enough, so \( \ln(1 + s) = \ln(1 + F) \) is an analytic function of each \( \nu_i \), by the same reason as in (i), is an analytic function in \( (\nu_1, \cdots, \nu_n) \), and can be expressed in the form:

\[
\ln(1 + s) = \sum_{m \geq 0} \hat{b}_m \nu_1^{m_1} \cdots \nu_n^{m_n} = \sum_{m \geq 0} \hat{b}_m \mu_1^{-m_1} \cdots \mu_n^{-m_n}
\]

which is convergent.

\( \ln(Dom(T)) \) can be defined inductively. Letting \( Dom(T) = c \mu_1^{-m_1} \cdots \mu_n^{-m_n} \), we have

\[
\ln(Dom(T)) = \ln c - \sum_{i=1}^{n} m_i \ln \mu_i,
\]

so we only need to show that \( \ln \mu \) is fully convergent for any fully convergent transmonomial \( \mu : x^a e^L \) form or \( (x^a e^L) \circ \ln n(x) \) form where \( L \) is purely large.

Since the logarithm breaks up the product, and makes things power-free, all the possible cases are logarithms of (a) \( x \), (b) a logarithm, more precisely, \( \ln_n(x) \), or (c) \( e^L \) where \( L \) does not need to be logarithm-free. First two cases are trivial, and the third case cancels out the exponential function and logarithm function getting a fully convergent transseries \( L \) with one lower level than the original \( e^L \).

So \( \ln T \) is also a fully convergent transseries.

Lemma 3.1.10. Every transseries representation is unique in the sense that the coefficients of any transmonomial \( \mu \) are the same, that is,

\[
\sum_{\ell_i \geq 0} \sum_{\nu_k} a_{k_1, \cdots, k_n} = \sum_{\ell_i \geq 0} \sum_{\nu_k} \tilde{a}_{l_1, \cdots, l_m}.
\]

Proof. Let \( T_1 \) and \( T_2 \) be two different representations of \( f \). Since transseries forms a field, for any nonzero transmonomial \( \mu \), we get two different transseries for \( f \mu^{-1} \) that can be written in the following way:

\[
T_1 \mu^{-1} = L_1 + c_1 + s_1 = L_2 + c_2 + s_2 = T_2 \mu^{-1}
\]
Then $c_1 + s_1 = L_2 - L_1 + c_2 + s_2 \gg 1$ which means $L_2 = L_1$, and similarly, $s_1 = c_2 - c_1 + s_2 \ll 1$, so $c_1 = c_2$ which is equivalent to (3.1.2).

Note that the above two lemmas show that any elementary function can have a fully convergent transseries, and it is, in fact, a unique transseries for it.

**Remark 3.1.11.** We restricted the definition of an elementary function to the real function, but even when it involves complex coefficients, as long as being purely large or purely small make senses (such as $e^{ix} = 1 + ix - x^2/2 - ix^3/6 + ...$), the rest of the proof still works with small modification.

### 3.1.4 Proof of Proposition 3.1.6

Let $T = \sum_{\bar{c} \geq \bar{M}} c_{\bar{c}} \bar{\mu}^{-\bar{k}}$ be a fully convergent transseries, and $\tilde{f}$ be an asymptotic series $\sum_{n=m}^{\infty} a_n x^{-n}$ such that $T \sim \tilde{f}$.

Then $T = \text{Dom}(T)(1 + s) = ce^{L}x^\alpha(1 + s) \sim a_m x^{-m}(1 + s')$ for some small transseries $s$ and small series $s'$, and since $e^{L}$ cannot be asymptotic to any power of $x$ for nonzero large $L$, so we have $L = 0$, $\alpha = -m$, and $c = a_m$. If $T$ is not a log-free transseries, then, instead of $e^{L}x^\alpha$, we use $c(e^{L}x^\alpha) \circ \ln_n x$ for $\text{Dom}(T)$, and by taking $y = \ln_n(x)$, we get $e^{L(y)}y^\alpha = \exp(-m \cdot \exp_{n-1}y)$, so $\alpha = 0$ and $L(y) = -m \cdot \exp_{n-1}y \Leftrightarrow L = -m \ln x$. This gives the same result: $e^L = e^{-m \ln x} = x^{-m}$.

Having the same dominant terms, $T - \text{Dom}(T) \sim \sum_{n=m+1}^{\infty} a_n x^{-n}$, and since $f$ is a series, we can equate inductively getting $T = \sum_{n=m}^{\infty} a_n x^{-n} + \sum_{\bar{c} \geq \bar{M}} c_{\bar{c}} \bar{\mu}^{-\bar{k}}$ where $\sum_{\bar{c} \geq \bar{M}} c_{\bar{c}} \bar{\mu}^{-\bar{k}}$ is smaller than any $x^{-n}$. Since $T$ is a fully convergent transseries, and by Lemma 3.1.10, the convergence of $\tilde{f}$ is obvious.
3.2 Analytic factorizability and Reconstruction of ODEs: In examples

In this section, we use Borel summation to see if the differential equations satisfied by certain functions can be reduced to lower order ones for some solutions, equivalently, whether the equations can be factored nontrivially, and then using similar techniques we show that a generic n-th order analytic ODE can be recovered from just one of its (possibly formal) solutions.

3.2.1 Factorizability of differential equations

Finding an algebraic linear homogeneous differential equation of minimal order for a given function (if possible) is an interesting problem, in the spirit of finding a minimal algebraic polynomial for a given number over a certain field. In this section, we show that factorizability of ODEs can be efficiently determined using transseries solutions and Borel summation for a general class of equations. For example, the modified Bessel equation, or a linear ODE which can be brought to standard form of it is not factorizable, that is, it cannot have any nontrivial solution which satisfies an equation of lower order. Many other examples are used to illustrate the idea.

Throughout this section, we:

- assume that \( Lf = \frac{1}{x^m} \) is an algebraic differential equation whose normal form is of the form (1.2.1). \( m \in \mathbb{Z}^+ \) is chosen so that the solution in Borel plane is singular at \( n \) distinct points. (For example, the Bessel operator has a constant solution in Borel plane for \( Lf = 1/x \), but for \( Lf = 1/x^2 \) the solution is \( -\frac{p}{\sqrt{1-p^2}} \). See Example 1.)

- let \( F(p) \) be the solution in Borel space for this differential equation, and denote
$n$ nonresonant singularities by $p_1$, $p_2$, ..., $p_n$ whose types are proven to be Frobenius type (see [23] and [26]), and

-for each $i = 1, ..., n$, let $h_i(x)$ be the difference between two Laplace transforms of $F(p)$ around the singular point $p_i$, that is, a contour integral $\int_C F(p)e^{-xp} dp$ with $C$ from $\infty e^{i\phi_i}$ to $\infty e^{i\phi_i}$ encircling $p_i$ where arg $p_i = \phi_i$.

**Remark 3.2.1.** (i) In the right hand side of the equation, $1/x^m$ is an accessory term which can be replaced by any analytic function at $\infty$ that gives a solution. Our primary interest is in the homogeneous equation $Lf = 0$, and this accessory term will eventually be removed.

(ii) $h_i(x)$ can be analytically continued by rotating the contour for $\Re(xp) > 0$ since the solution $F(p)$ in Borel space has exponential bound ([23]), and by Jordan’s lemma.

Then we define dependency/independency of singular points in Borel space in the following way:

**Definition 3.2.2.** According to [23], the difference between two Laplace transforms around one singularity $p_i$ has the following form: $h_i(x) = Ce^{-p_i x} x^{\alpha_i} (1 + s_i)$ for a small Borel summable series $s_i$. Borel transforming $s_i$, we get an analytic function at 0, and all the possible singular points are those of $F(p)$ shifted by $-p_i$. We can determine analytically or numerically for each $j \neq i$ whether $p_j - p_i$ is still a singular point of $s_i$. If it is so, then $p_i$ is dependent on $p_j$. Otherwise, $p_i$ is independent of $p_j$.

**Note 3.2.3.** It is possible for $p_1$ to be independent of $p_2$ while $p_2$ is dependent on $p_1$. (See Example 4.)

In practice, Definition 3.2.2 allows for a practical calculation, solely in Borel plane, and the following rephrased one in Lemma 3.2.4 is useful in using the existence of a newly added linearly independent function.
Lemma 3.2.4. Definition 3.2.2 is equivalent to the following: if the analytic continuation of $h_i(x)$ obtained by rotating the contour (from $\infty e^{i \arg p_i}$ to $\infty e^{i \arg p_j}$) towards and then beyond $\infty e^{i \arg p_j}$ (see Figure 3.1) does not collect any new linearly independent function as the contour crosses $p_j$, then the singular point $p_i$ is said to be independent of $p_j$. If it generates an additional linearly independent function when it passes through $p_j$, then $p_i$ is dependent on $p_j$.

![Figure 3.1: The deformation of contour of $h_i(x)$ beyond $p_2$](image)

The proof of Lemma 3.2.4 can be seen in Appendix C.
Theorem 3.2.5. 1) Assume that for each \( i \), \( p_i \) is dependent on the rest, \( \{p_j, j \neq i\} \). Then the homogeneous differential equation \( Lf = 0 \) is not analytically factorizable, that is, there is no \( L_2 \) such that \( L = L_1L_2 \) and \( L_2f = \frac{1}{x^2} \) is of the form (1.2.1). This implies that for any solution of \( Lf = 0 \), the differential equation of minimal order should still be \( Lf = 0 \). (Example 1)

2) Assume that a subset of the singular points of \( F(p) \) can be split from the rest, that is, say, none of \( \{p_i, i = 1, \ldots, k\} \) is dependent on any of \( \{p_i, i = k+1, \ldots, n\} \) for \( k < n \). Then \( Lf = 0 \) is analytically factorizable in the following sense: (a) \( L = L_{n-k}L_k \) where \( L_kf = 0 \) is a \( k \)-th order differential equation whose solutions are \( h_1(x), h_2(x), \ldots, h_k(x) \), (b) in its normal form, all the coefficients of \( L_k \) are analytic at \( \infty \), and (c) eigenvalues for \( L_k \) are \( p_1, p_2, \ldots, p_k \) (so, nonresonant). Equivalently, there exist nontrivial solutions of \( Lf = 0 \) that satisfy a homogeneous differential equation of lower order \( k \) whose normal form with an inhomogeneous term \( \frac{1}{x^2} \) is of the form (1.2.1). (Example 2 and 3)

To prove the theorem, we need the following lemmas.

Lemma 3.2.6. For all \( i = 1, \ldots, n \), the differences of two directional Laplace transforms around \( p_i \), \( h_i(x) \), are solutions of the homogeneous differential equation \( Lf = 0 \), and if \( p_1 \) is dependent on another singular point, say \( p_2 \), then the linearly independent function newly generated by deforming the contour is also a solution of the same homogeneous differential equation \( Lf = 0 \). Moreover, the functions generated for \( p_i \) and \( p_j \) are linearly independent for \( i \neq j \).

Lemma 3.2.7. Let \( f_C(x) \) be the contour integral \( \int_C F(p)e^{-xp} \, dp \) for the contour \( C \) from \( \infty e^{i\phi_1} \) to \( \infty e^{i\phi_1} \) \( (\phi_i = \arg p_i) \), around a singular point \( p_0 \) of \( F(p) \) and its branch cut. If \( f_C(x) \) is a solution of \( L_xf = 0 \) for some linear algebraic differential operator \( L_x \), then the contour integral \( f_D(x) \) for the contour \( D \) from \( \infty e^{i\phi_2} \) to \( \infty e^{i\phi_2} \) around
$p_0$ and its branch cut in a different direction $\phi_2$ also satisfies the same differential equation $L_x f = 0$.

![Figure 3.2: Two newly collected contours](image)

**Proof of Lemma 3.2.6.** It is clear that the difference of two directional Laplace transforms around any singular point is a solution of $Lf = 0$ since any directional Laplace transform is a solution of $Lf = \frac{1}{x^2}$, and the difference of two inhomogeneous solutions is always a solution of the linear homogeneous equation. Also, if $p_1$ is dependent on $p_2$, then, by Lemma 3.2.4, rotating the contour towards, and then beyond $p_2$, we collect a nontrivial function and this collected term is the sum of two contour integrals around $p_2$, one from $\infty e^{i(\theta_2-\epsilon)}$ to $\infty e^{i(\theta_2+\epsilon)}$ and another from $\infty e^{i(\theta_2+\epsilon)}$ to $\infty e^{i(\theta_2-\epsilon)}$.
((1) and (2) in Figure 3.2) where arg $p_2 = \theta_2$ (which cancel out each other if $p_1$ is a pole since the integrand is single valued). As said above, each one is the difference of two directional Laplace transforms around $p_2$, so the solution of $L f = 0$, and if the sum is nontrivial, then this sum is also a nontrivial solution of $L f = 0$. Since the function generated for $p_i$ contains exponential term $e^{-zp_i}$, it is linearly independent of any other functions generated for different $p_j$.

Proof of Lemma 3.2.7. First, by multiplying by the common denominator we can assume that all the coefficients are polynomials of $x$, and then divide the equation by $m$, the highest degree of all the polynomial coefficients, and we get a new operator, and we denote this by $\tilde{L}_x$. (The result can be generalized to the analytic coefficients, but for simplicity of the proof, we use the polynomial coefficients.)

Since the contour integral in any direction for a pole simply equals the integral around the pole, in this case the statement is obvious. So we consider only the branch points. For a branch point (say $p_0$ with arg $p_0 = \phi_1$), $C$ is a contour from $\infty e^{i\phi_1}$ to $\infty e^{i\phi_1}$ around $p_0$ and $\tilde{L}_x f_C(x) = \int_C \tilde{F}(p) \tilde{L}_x e^{-zp} dp = \int_C \tilde{F}(p) e^{-zp} dp = 0$

where the last integral is obtained by repeated integration by parts using the fact that $\Re(xp) > 0$. (Note that $\tilde{F}(p)$ can be a function including antiderivatives as well. See the example below.) Now the contour integral can be reduced to the integral from $p_0$ to $\infty e^{i\phi_1}$ (for integrability at $p_0$, if necessary, we can divide $\tilde{L}_x$ by $x^{m_0}$ for bigger $m_0 \in \mathbb{Z}^+$), and then by taking $q = p - p_0$, we get $\int_C \tilde{F}(p) e^{-zp} dp = e^{-p_0 x} \int_0^{\infty e^{i\phi_1}} G(q) e^{-xq} dq$, and by the injectivity of Laplace transform, $G(q) = 0$. The same calculation can be applied to $f_D$ as long as $\Re(xp) > 0$, and this implies that different contour $D$ from $\infty e^{i\phi_2}$ to $\infty e^{i\phi_2}$ around $p_0$ and its branch cut in a different direction $\phi_2$ also gives a solution to $\tilde{L}_x f = 0$.

For example, let $F(p) = \frac{1}{\sqrt{p^2 - 1}}$. Then $\int_0^\infty F(p) e^{-zp} dp$ is a solution of $f'' +
\[ f' / x - f = -i / x, \text{ and contour integral of the same integrand } f_C(x) = \int_C F(p) e^{-xp} \, dp \]
satisfies \( f'' + f'/x - f =: \tilde{L}_x f = 0 \) where \( \mathcal{C} \) is a contour from \( \infty \) to \( \infty \) around \( p = 1 \) and its branch cut \([1, \infty)\). For any \( x \) in the right half plane, by plugging in and integrating by parts, we get
\[
\tilde{L}_x f_C = \int_C \left( p^2 - \frac{P}{x} - 1 \right) F(p) e^{-xp} = \int_C \left[ (p^2 - 1) F(p) - \int p F(p) \right] e^{-xp} = \int_C C_0 e^{-xp}
\]
and since any constant \( C_0 \) is single valued, we get \( G(q) \equiv 0 \) as we expected, and it implies that for the contour \( \mathcal{D} \) in any other direction, \( \tilde{L}_x f_D = 0 \) for \( x \) with \( \Re(xp) > 0 \), and its analytic continuation.

\[ \Box \]

Proof of Theorem 3.2.5. 1) Assume that the solution \( h_i(x) \) corresponding to \( p_i \) satisfies lower order differential equation \( L_i f = 0 \). Then \( n - 1 \) linearly independent solutions generated by contour deformation beyond \( p_j \) for \( j \neq i \) are also the solutions of the same differential equation \( L_i f = 0 \) by the lemmas 3.2.6 and 3.2.7, and this contradicts the assumption. Since any other solutions of \( L f = 0 \) are linear combinations of \( h_i(x), i = 1, ..., n \), and the contour deformation can generate \( h_i(x) \) for a single \( i \) at a time, the same contradiction is applied.

2) Since \( h_1(x), ..., h_k(x) \) are linearly independent solutions of \( L f = 0 \), we can always factorize \( L = L_{n-k} L_k \) where \( L_k f = 0 \) is a \( k \)-th order differential equation whose solutions are \( h_1(x), ..., h_k(x) \) proving (a) ([30]). We now show (b) and (c) for \( k = 2 \) using a proof that can be extended for \( k > 2 \). Let neither of \( \{p_1, p_2\} \) be dependent on any singular point in \( \{p_i, i = 3, ..., n\} \). Then we have \( h_1(x e^{2\pi i}) = a h_1(x) + b h_2(x), h_2(x e^{2\pi i}) = c h_1(x) + d h_2(x) \) with \( ad - bc \neq 0 \), and the Wronskian \( W(x) \) for \( h_1(x) \) and \( h_2(x) \) becomes \( W(x e^{2\pi i}) = (ad - bc) W(x) \) under full circle analytic
continuation. Let $h_1(x)$ and $h_2(x)$ be the solutions of $y'' + A_1(x)y' + A_0(x)y = 0$, and we can calculate $A_i(xe^{2\pi i})$ obtaining:

$$
\begin{pmatrix}
A_1 \\
A_0
\end{pmatrix}(x) = \frac{1}{W} \begin{pmatrix}
h_2 & -h_1 \\
-h'_2 & h'_1
\end{pmatrix} \begin{pmatrix}
h''_1 \\
h''_2
\end{pmatrix}(x) \Rightarrow \begin{pmatrix}
A_1 \\
A_0
\end{pmatrix}(xe^{2\pi i}) = \frac{1}{W} \begin{pmatrix}
-h_2 & h_1 \\
h'_2 & h'_1
\end{pmatrix} \begin{pmatrix}
h''_1 \\
h''_2
\end{pmatrix}(x).
$$

As mentioned earlier, $h_1(x)$ and $h_2(x)$ are in the form of $e^{-p_1 x} x^{s_1} (1 + s_1)$ and $e^{-p_2 x} x^{s_2} (1 + s_2)$ for Borel summable series $s_1$ and $s_2$, and the above calculation shows that the cancellation of the exponential term occurs, so we get Borel summable series $A_1(x)$ and $A_0(x)$. We need to show these Borel summable series are actually analytic at $\infty$ which is equivalent to show that their Borel transforms have no singularity in Borel space. Assume that Borel transform of $A_i(x)$ ($i = 1$ or 2) has any singular point. We can take Laplace transform for this solution in Borel space, and due to the singularity, we need to collect some exponential term under full circle analytic continuation for $x$. But like $A_1(x)$ and $A_0(x)$, cancellation of all the exponential terms for $A_1(xe^{2\pi i})$ and $A_0(xe^{2\pi i})$ occurs, and this shows that $A_i(x) \rightarrow A_i(xe^{2\pi i})(i = 0, 1)$ cannot collect any exponential term in rotating the contour in Borel space implying that it is singularity free there. Therefore $A_i(x)(i = 0, 1)$ are analytic at $\infty$. The highest order terms for $h'_i$ and $h''_i$ are from the derivative of the exponential terms $e^{-p_1 x}$ and $e^{-p_2 x}$, and so the coefficient of the highest order term for $A_0(x)$ is $-p_1 p_2$ implying the term for the product of two eigenvalues in the equation is $p_1 p_2$. Due to the exponential terms of $h_i(x)$, the eigenvalues for $L_k$ are $p_1$ and $p_2$.

**Note 3.2.8.** As mentioned in the proof of Lemma 3.2.6, the two contour integrals around $p_2$ cancel each other if the singularity at $p_1$ is a pole. So if $p_1$ is a pole, then it is independent of $p_2$, and similarly, all the other singular points, and it implies that $h_1(x)$ satisfies a first order equation. Furthermore, if $F(p)$ is a meromorphic function, then for each $i$, $p_i$ is independent of the rest, so the difference of two directional Laplace transforms $h_i(x)$ for each $i$ satisfies a first order differential equation. (Example 5)
(Example 1) Let $L$ be the modified Bessel operator, $Ly = y'' + \frac{y'}{x} - \left(1 + \frac{1}{x^2}\right)y$.

Then Borel transform of $Ly = \frac{1}{x^2}$ is the following:

$$p^2Y \int_0^p y - \int_0^p \int_0^p y = 1 \iff (p^2 - 1)Y'' + (4p + 1)Y' + Y = 0, Y(0) = Y'(0) = -1$$

whose solution is $-\frac{p}{\sqrt{1 - p^2}}$, and the difference of the Laplace transform around $p = 1$ is $C \int_1^\infty \frac{e^{-xp}}{\sqrt{1 - p^2}} dp$ which cannot be expressed in the form of $D \int_1^\infty e^{-xp}G(p) dp$ where $G(p)$ has a branch point at $p = 1$, but analytic at $p = -1$ due to the injectivity.

Also it cannot be expressed in the form of $D \int_{|p-1|=r} e^{-xp}G(p) dp$ where $G(p)$ has a pole at $p = 1$, and analytic at $p = -1$ because it cannot have different type of singularity at $p = 1$. So the difference of two directional Laplace transform around $p = 1$ cannot eliminate the singularity at $p = -1$.

Since it satisfies the conditions of the first of Theorem 3.2.5, the modified Bessel functions $I$ and $K$ cannot be solutions of any first order algebraic linear differential equation of the form (1.2.1).

(Example 2) Borel transform of $Lf = f'' + (1 + \frac{1}{x})f' - \left(1 + \frac{1}{4x^2}\right)f = \frac{1}{x^2}$ is:

$$(p^2 - 1)F'' + 3pF' + \frac{3}{4}F = 0, F(0) = 0, F'(0) = 0$$

$$\iff F(p) = \frac{1}{\sqrt{p + 1}} - \frac{i}{\sqrt{p - 1}}.$$ 

and the difference of its Laplace transform around $p = 1$ is $2i \int_1^\infty \frac{e^{-xp}}{\sqrt{p - 1}} dp$. The integrand is now analytic everywhere except at $p = 1$, so rotating the contour does not generate any other solutions of the equation, and this implies that for the solution corresponding to $p = 1$, the differential equation $Lf = f'' + (1 + \frac{1}{x})f' - \left(1 + \frac{1}{4x^2}\right)f = 0$ can be reduced to $f' + (1 + \frac{1}{2x})f = 0$. In fact, the same thing can be said for $p = -1$, and it implies that the factorization is commutative:

$$Lf = (\partial + (\frac{1}{2x})\left(\partial + (1 + \frac{1}{2x})\right)) = (\partial + (1 + \frac{1}{2x}))\left(\partial + (-1 + \frac{1}{2x})\right).$$
(Example 3) The Borel transform of
\[ Lf = f^{(4)} + \frac{2f^{(3)}}{x} - \frac{3f''}{x^2} + \frac{3f'}{x^3} - \frac{3f}{x^4} - f = -\frac{2}{x^3} - \frac{2}{x} \]
is \((p^4 - 1)F^{(4)} + 14p^3 F^{(3)} + 51p^2 F'' + 45p F' = 0\), its solution is
\[ \frac{1}{\sqrt{p^2 + 1}} + \frac{i}{\sqrt{p^2 - 1}}. \]
and so \(p = \pm 1\) and \(p = \pm i\) are independent of each other. Letting the operators
\(B_1\) be the modified Bessel operator for \(I(x)\), and \(B_2\) be the one for \(I(ix)\), that is,
\(B_1 f = f'' + f'/x - (1 + 1/x^2)f\) and \(B_2 f = f'' + f'/x + (1 - 1/x^2)f\), we have \(Lf = B_2 B_1 f = B_1 B_2 f\).

(Example 4) Borel transform of
\[ Lf = f'' + \frac{1}{2x} f' - (1 + \frac{1}{2x})f = \frac{i}{x} \]
has the solution
\[ F(p) = \frac{1}{\sqrt{p - 1} (p + 1)} \]
and the difference of two Laplace transforms around \(p = -1\) is \(\int_{|p+1|=1} \frac{e^{-xp}}{\sqrt{p - 1} (p + 1)} = \frac{\sqrt{2\pi} e^{\frac{x}{2}}}{\sqrt{2i} \int_{|p+1|=1} \frac{e^{-xp}}{p + 1}}\) by the residue theorem. The new integrand \(\frac{e^{-xp}}{p + 1}\) has no other singularity in Borel space, so \(p = -1\) is independent of \(p = 1\), meaning that the solution corresponding to \(p = -1\), \(\int_{|p+1|=1} \frac{e^{-xp}}{\sqrt{p - 1} (p + 1)}\), satisfies a lower order differential equation, \(f' - f = 0\). Meanwhile, the difference of two Laplace transform around \(p = 1\) cannot eliminate the pole at \(p = -1\) in the integrand due to the injectivity of the Laplace transform. If we rotate the contour of integration beyond \(p = -1\), it collects contour integrals around \(p = -1\) first from \(\infty e^{ix-\epsilon}\) to \(\infty e^{ix+\epsilon}\), and then another from \(\infty e^{ix+\epsilon}\) to \(\infty e^{ix-\epsilon}\), and since they are on different branches, the sum becomes a nontrivial function \(-4\pi \sqrt{2} e^{x}\). This implies \(p = 1\) is
dependent on \( p = -1 \), and so the corresponding solution cannot satisfy any first order algebraic linear differential equation. In fact, this solution is a scalar multiple of \( \int_1^\infty \frac{e^{-xp}}{\sqrt{p-1(p+1)}} \, dp \), and it can be expressed in terms of error function which is known to be a solution of second order homogeneous differential equation. In short, \( Lf = 0 \) can be factored into \( Lf = (\partial + (1 + \frac{1}{2x}))(\partial - 1)f = 0 \), but no other way since it cannot be reduced to first order differential equation for the other solution.

(Example 5) \( F(p) = \frac{1}{(p-1)^2(p+1)} \) generates \( Ce^{-x}(1 + 2x) \) around \( p = 1 \) and \( e^x \) around \( p = -1 \), and they satisfy \((1 + 2x)f' - (1 - 2x)f = 0 \) and \( f' - f = 0 \), respectively. (So the differential equation \( f'' + \frac{f'}{x} + (\frac{1}{x} - 1)f = 0 \) can be reduced either to

\[
(\partial - \left( \frac{1}{x} + 1 - \frac{2}{2x+1}\right))(\partial + (1 - \frac{2}{2x+1}))f = 0
\]

or

\[
(\partial + \left( 1 - \frac{1}{x}\right))(\partial - 1)f = 0,
\]

which implies that the factorization does not need to be unique.)

**Remark 3.2.9.** (i) Although being a pole automatically eliminates the possibility to have singularities at other points, having branch point does not necessarily result in gaining the new singularities. It all depends on dependency/independency (see Examples 3 and 4).

(ii) If \( F(p) \) can be written as a sum of functions with a disjoint set of singularities (see Examples 2, 3 and 5), then the independency of the singular point is guaranteed since the contour integral of the function with one singular point around another becomes zero. But this does not imply the converse is also true (as seen in Example 4).

(iii) Independency does not guarantee commutative factorization (see Example 5).

(iv) Factorization does not need to be unique (as shown by Example 5).
3.2.2 Reconstruction of the corresponding linear ODE

**Theorem 3.2.10.** Let \( f(x) \) be a solution of the unknown nonresonant linear ODE of the form (1.2.1) with analytic coefficients. Assume that the number of singular points of the Borel transform is \( n \). Then we can reconstruct the linear ODE of order \( n \). Furthermore, the differences of the directional Laplace transform of the Borel transform of \( f(x) \) around the singular points generate all the other solutions.

Note that the ODE might be factorizable as discussed in previous section.

**Proof of Theorem 3.2.10.** Let \( Lf = g(x) \) be the linear ODE to determine where \( g(x) \) is analytic at \( \infty \), \( f_0(x) \) be a known solution of this equation, and \( F_0(p) \) be Borel transform of \( f_0(x) \). By the Lemma 3.2.6, the differences of two directional Laplace transforms of \( F_0(p) \) around any singular point are solutions of the homogeneous differential equation \( Lf = 0 \), and for \( n \) nonresonant singular points, we get \( n \) linearly independent solutions for \( Lf = 0 \). By extending the proof of the second part of Theorem 3.2.5, we can construct an \( n \)-th order differential equation \( Lf = 0 \) with analytic coefficients at \( \infty \), and then plugging in \( f_0(x) \), we can get the inhomogeneous term \( g(x) \).

\[ \square \]

**Note 3.2.11.** The given solution can even be a formal solution as an asymptotic series. This process can generate all the solutions (as illustrated in Example 6).

**Note 3.2.12.** The procedure above, namely solving for the coefficients from the formal solution is best for the rigorous proof purposes. If the singularities are explicit, there are much more efficient ways such as writing the algebraic or differential equation in Borel plane first, and then taking Laplace transform (see Example 7).

(Example 6) \( \tilde{y} = \sum_{k=0}^{\infty} \frac{(2k)!}{x^{2k+1}} \) is given as a formal solution of one unknown linear differential equation. If we take the Borel transform, then it becomes \( Y(p) = \sum_{k=0}^{\infty} p^{2k} = \)
which has two singular points \( p = \pm 1 \) in Borel space. It ensures that the equation should be of order two, and the difference of directional Laplace transform for each singular point generates solutions \( e^x \) and \( e^{-x} \), and by solving linear system with these two solutions, we get the homogeneous equation \( y'' - y = 0 \). The inhomogeneous term can be gotten by plugging in the formal solution \( \hat{y} = \sum_{k=0}^{\infty} \frac{(2k)!}{x^{2k+1}} \) obtaining \( y'' - y = -\frac{1}{x} \).

(Example 7) Consider \( f(x) = \int_0^\infty \frac{e^{-xp}}{\sqrt{p^2 - 1}} \, dp \).

In this case, the function is given in the form of Laplace transform of a function. We let \( F(p) = \frac{1}{\sqrt{p^2 - 1}} \), and then \( F(p) \) satisfies

\[
F'(p) = \frac{-p}{p^2 - 1} F(p) \Leftrightarrow (p^2 - 1)F'(p) = -pF(p).
\]

Taking Laplace transform, we get

\[
f''(x) + \frac{f'(x)}{x} - f(x) = -\frac{i}{x},
\]

and

\[
g(x) = \int_0^\infty \frac{e^{-xp}}{\sqrt{p^2 - 1}} \, dp
\]

for \( \Re(x) < 0 \) (and its analytic continuation) is another independent solution.

Its Borel transform \( 1/\sqrt{p^2 + 1} \) has two singular points, and so we expect that the order of the differential equation is two.
CHAPTER 4

ASYMPTOTICS OF THE SOLUTIONS TO A CLASS OF ODES

4.1 Introduction and the equations

In [31] da Costa, van Rössel and Wattis consider the system of equations

\[
\begin{align*}
\dot{c}_1 &= \alpha - 2c_1^2 - c_1 \sum_{i=2}^{\infty} c_i, \\
\dot{c}_j &= c_1c_{j-1} - c_1c_j, \quad j \geq 2,
\end{align*}
\] (4.1.1)

where \(c_j(t)\) is the concentration of a cluster of \(j\) monomers, and \(\alpha \in \mathbb{R}^+\) is the monomer input rate.

In this thesis we generalize the results of [31], in which there also is a steady influx of monomers. The context is of submonolayer deposition, i.e. of depositing atoms onto a surface, such that the deposited particles can then diffuse and coagulate into clusters. If in modelling this process we do not take the spatial structure into consideration, we are, broadly speaking, in the domain of mean-field models. If we furthermore assume that coagulation and fragmentation rates of clusters are not size-dependent, we are dealing with point islands. It makes sense to say that \(i\) is the critical island size if clusters of size \(2 < j \leq i\) can fragment into monomers while clusters of size \(j \geq n := i + 1\) cannot. If we assume that clusters of all sizes larger or
equal to \( n \) are immobile, a point-island mean-field model would lead to Becker-Döring kinetics, i.e. stable clusters would be able to grow only by addition of monomers.

We calculate the asymptotic expansion as \( t \to \infty \) of rate equations for submonolayer deposition with point islands of arbitrary critical island size \( i \geq 2 \). Assuming that clusters of size \( 1 < j \leq i \) simply do not arise, this system of equations becomes

\[
\begin{align*}
\dot{c}_1 &= \alpha - 2c_1^n - c_1 \sum_{j=n}^{\infty} c_j, \\
\dot{c}_n &= c_j^n - c_1 c_n, \\
\dot{c}_j &= c_1 c_{j-1} - c_1 c_j, \quad j > n.
\end{align*}
\] (4.1.2)

Letting \( Y(t) = c_1(t) \) and (formally) set \( X(t) = \sum_{i=n}^{\infty} c_i(t) \), we have

\[
\begin{align*}
\dot{Y} &= \alpha - nY^n - YX, \\
\dot{X} &= Y^n, \\
\dot{c}_n &= Y^n - Yc_n, \\
\dot{c}_j &= Yc_{j-1} - Yc_j, \quad j > n
\end{align*}
\] (4.1.3)

which is equivalent to (4.1.2) due to the following theorem [31]:

**Theorem 4.1.1.** If \( \sum_{j=n}^{\infty} c_j(0) < \infty \), a solution of (4.1.3) also solves (4.1.2) [31].

The mathematical problem is to find the asymptotic behavior of the nonlinear system

\[
\begin{align*}
\dot{Y} &= \alpha - nY^n - YX \\
\dot{X} &= Y^n \\
X(0) &> 0, Y(0) > 0
\end{align*}
\] (4.1.4)

where \( \alpha > 0 \) and \( 2 < n \in \mathbb{N} \). Note that these are the first two equations of (4.1.3) for all initial conditions in the first quadrant. The asymptotic behaviors of \( c_i(t), i = 1, 2, \ldots \), are quite straightforward once \( X \) and \( Y \) are determined.
4.2 Main results

**Proposition 4.2.1.** For (4.1.4):

(i) The first quadrant $(X,Y) \in \mathbb{R}^2_+$ is positively invariant.

(ii) Let $[0,t_0)$, $t_0 > 0$ be an interval of existence of the solution of the initial value problem (4.1.4) above (such an interval exists by general ODE theorems). Then, on some interval $(t_1,t_0)$ with $t_1 \in (0,t_0)$, $Y$ is increasing.

(iii) The solution of (4.1.4) exists for all $t > 0$.

(iv) There exists $\tau > 0$ such that for all $t > \tau$, $Y$ is monotonically decreasing and $X$ is monotonically increasing.

(v) We have $\lim_{t \to +\infty} Y(t) = 0$ and $\lim_{t \to +\infty} X(t) = +\infty$.

(vi) $\lim_{t \to +\infty} X(t)Y(t) = \alpha$.

**Theorem 4.2.2.** As $X \to \infty$ the function $Y(X)$ is well defined and has an asymptotic series of all orders in $X$ of the form

$$Y \sim \alpha X^{-1} - na^nX^{-n-1} + \alpha^{n+1}X^{-n-3} - n^3\alpha^{2n-1}X^{-2n-1} + \cdots$$

where the subsequent terms in the expansion are unique (but their relative order depends on the value of $n$).

**Proof of Proposition 4.2.1.** (i) The fact that $\dot{Y} = \alpha > 0$ means that the line $Y = 0$ can only be crossed from below. Similarly, $\dot{X} \geq 0$ implies that solutions cannot cross the line $X = 0$.

(ii) Indeed, assume first towards a contradiction that $Y$ had infinitely many subintervals where it increases and infinitely many on which it is decreasing. Since $Y$ is smooth, $Y$ has infinitely many strict maxima and infinitely many strict minima. Let $t_\epsilon$ be a minimum point. We have $\dot{Y}(t_\epsilon) = 0$ and thus

$$\dot{Y}(t_\epsilon) = -\dot{X}(t_\epsilon)Y(t_\epsilon) = -Y(t_\epsilon)^{n+1} < 0$$
a contradiction.

(iii) Since $Y$ is monotonic on $(t_1, t_0)$ bounded below by zero, $\lim_{t \to t_0} Y(t) = L_0 \in [0, +\infty]$ exists. If we had $L_0 = +\infty$, then for some $t_2$, we would have $nY^n > 2\alpha$ on $(t_2, t_0)$, implying, by (1), that $Y$ is strictly decreasing on $(t_2, t_0)$ and thus $Y(t) < Y(t_2)$ on this interval, a contradiction. Therefore

$$\lim_{t \to t_0} Y(t) = L_0 \in [0, +\infty) \quad (4.2.3)$$

By (4.2.3), on $[0, t_0)$ we have $\dot{X} \in [0, L_0^n + a)$ for some $a > 0$ and $X$ is increasing, with bounded derivative. Thus $\lim_{t \to t_0} X(t) = L_1$ exists and $L_1 \in [0, \infty)$. Then $\lim_{t \to t_0} (X(t), Y(t))$ exists, and by general ODE arguments, $(X(t), Y(t))$ extends beyond $t_0$, as claimed. Thus the solution $(X(t), Y(t))$ is global.

(iv) Assume that $Y$ were not eventually monotonic. Then there would be infinitely many subintervals where it increases and infinitely many on which it is decreasing, and this is ruled out as in (ii). Thus $Y$ is eventually monotonic and $\lim_{t \to \infty} Y(t) = L_{[\infty]} \in [0, \infty]$. We claim that $L_{[\infty]} = 0$. Indeed, if $L_{[\infty]} > 0$ we get from $\dot{X} = Y^n$ that $X \to +\infty$ as $t \to \infty$. But then for sufficiently large $t$, we see from (4.1.4) that $\dot{Y} \to -\infty$ and thus for some $t_3$ and all $t > t_3$ we have $\dot{Y} < -1$ (say), and thus $Y \to -\infty$, contradicting the fact that the solution is confined to the first quadrant. Since $Y$ is eventually monotonic, this also implies that $Y$ is eventually decreasing.

(v) By the proof of (iv) $Y$ is eventually decreasing and we have

$$\lim_{t \to \infty} Y(t) = 0 \quad (4.2.4)$$

The fact that $X$ is increasing is manifest in the second equation of (4.1.4), since $Y \geq 0$. Assume, again towards a contradiction that $\lim_{t \to \infty} X(t) = l < \infty$. Then, by (4.2.4) and the fact that $X$ is bounded, we have $\dot{Y}(t) \to \alpha > 0$, and thus $Y$ is eventually increasing, which is incompatible with (4.2.4).
(vi) We now look at the function $w(t) = X(t)Y(t)$. Note that

$$w = \alpha X - nY^{n-1}w - Xw + Y^{n+1}$$  \hspace{1cm} (4.2.5)

Thus, if $\dot{w} = 0$ we have

$$w = \frac{\alpha X + Y^{n+1}}{X + nY^{n-1}}$$  \hspace{1cm} (4.2.6)

If there is an infinite sequence of intervals on which $w$ is increasing and an infinite one in which it is decreasing, let $t_k, k = 1, 2, ..., \text{ be the sequence of points such that }\dot{w}(t_k) = 0$. Since $X \to \infty$ and $Y \to 0$, for any $\epsilon > 0$ and large enough $\tau = \tau(\epsilon)$ we have

$$\alpha - \epsilon < \inf_{t_k > \tau} w(t_k) \leq \sup_{t_k > \tau} w(t_k) < \alpha + \epsilon$$  \hspace{1cm} (4.2.7)

from (4.2.6), and this simply means $\lim_{k \to \infty} w(t_k) = \alpha$.

Since both local maxima and local minima converge to $\alpha$, we have

$$\lim_{t \to \infty} w(t) = \alpha.$$  \hspace{1cm} (4.2.8)

The other possibility is that $w$ is eventually increasing or eventually decreasing. If $w$ is eventually increasing, we see that it must be bounded above, or else we would have $\lim_{t \to \infty} X(t)Y(t) = +\infty$ implying from (4.1.4) and (v) above, that $\lim_{t \to \infty} \dot{Y}(t) = -\infty$ contradicting (v). Thus, whether increasing or decreasing, $\lim_{t \to \infty} w(t) = \lambda \in [0, \infty)$, in which case, from the first of (4.1.4),

$$\lim_{t \to \infty} \dot{Y} = \alpha - \lambda$$  \hspace{1cm} (4.2.9)

The only possibility consistent with (v) above is clearly $\lambda = \alpha$.

\begin{flushright}
$\blacksquare$
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**Note 4.2.3.** Since $Y$ is eventually decreasing and $X$ is eventually increasing, the functions $Y(X)$ and $w(X)$ are well defined at least for large $X$ and, since $X \neq 0$,
$Y(X)$ is smooth on $(X_0, \infty)$ and decreasing. Furthermore, from Proposition 4.2.1 we have

$$w(X) \sim \alpha, \ Y(X) \sim \alpha X^{-1}, \quad \text{as } X \to \infty. \quad (4.2.10)$$

We denote by “$\partial$” derivatives with respect to $X$. Writing $w(X) = \alpha + \epsilon(X)$, by (4.2.10) we have $\epsilon(X) \to 0$ as $X \to \infty$.

Straightforward algebra shows that

$$\epsilon' = \frac{\alpha + \epsilon}{X} - \frac{\epsilon}{(\alpha + \epsilon)^n} X^{n+1} - nX \quad (4.2.11)$$

Let $\beta = \frac{n}{n+2} \in (0, 1)$, $\xi = X^{n+2}$, and $\epsilon(X) = g(\xi)$. Then we get

$$g' = \frac{\alpha + g}{\xi(n+2)} - \frac{(\alpha + g)^{-n}}{2 + n} g - \frac{\beta}{\xi^\beta} \quad (4.2.12)$$

where we substitute $g(\xi) = \xi^{-\beta} h(\xi)$ to get

$$h' = -\beta + \frac{\alpha}{\xi^{1-\beta}(2 + n)} - \frac{(\alpha + (\xi^{-\beta} h)^{-n}}{2 + n} h + \frac{h}{\xi} \left( \beta + \frac{1}{2 + n} \right). \quad (4.2.13)$$

**Theorem 4.2.4.** For all solutions in the first quadrant, $h(\xi)$ has an asymptotic behavior of the following form:

$$h(\xi) \sim \sum_{l,m \geq 0} \frac{a_{l,m}}{\xi^{l(1-\beta)+m\beta}} = -n\alpha^n + \frac{\alpha^{n+1}}{\xi^{1-\beta}} - \frac{n^3 \alpha^{2n-1}}{\xi^\beta} + o(\xi^{-\beta}) \quad (4.2.14)$$

where $a_{l,m}$ can be calculated explicitly order by order by iterating (4.2.16).

We have $g = \frac{h}{\xi^\beta} \to 0$ and $\lambda(\xi) = \frac{(\alpha + (\xi^{-\beta} h)^{-n}}{2 + n} \to \lambda := \alpha^{-n}(2 + n)^{-1}$ since $\epsilon(X) \to 0$, and thus

$$h' = -\beta + \frac{\alpha}{\xi^{1-\beta}(2 + n)} + \frac{h\gamma}{\xi} - \lambda h - \delta(\xi, h) \quad (4.2.15)$$

where $\delta(\xi, h) = \lambda h \left( (1 + \frac{h}{\alpha \xi^\beta})^{-n} - 1 \right)$ and $\gamma = \frac{n}{n + 2}$, and the solution of (4.2.15) must satisfy

$$h(\xi) = G(\xi) + e^{-\lambda \xi} \xi^\gamma \int_{\xi_0}^\xi \delta(s, h) e^{\lambda s} s^{-\gamma} ds \quad (4.2.16)$$
where
\[ G(\xi) = e^{-\lambda \xi \gamma} \left[ C + \int_{\xi_0}^{\xi} (-\beta + \frac{\alpha}{s^{1-\beta}(2 + n)}) e^{\lambda s^{-\gamma}} ds \right]. \tag{4.2.17} \]

**Note 4.2.5.** By L’Hospital’s rule, (4.2.17) implies that \( G(\xi) \to -\beta/\lambda \) as \( \xi \to \infty \).

**Lemma 4.2.6.** \( h(\xi) \) is bounded.

**Proof of Lemma 4.2.6.** (4.2.16) can be rewritten in the following way:
\[ g = h(\xi) e^{-\lambda \xi \gamma} \left[ C + \int_{\xi_0}^{\xi} \frac{h}{s^\beta} \left( (1 + \frac{h}{\alpha s^\beta})^{-n} - 1 \right) e^{\lambda s^{-\gamma}} ds \right]. \tag{4.2.18} \]
Replacing \( h/s^\beta \) by \( g \), taking maximum on \([\xi_0, \infty)\), and using the fact that
\[ e^{-\lambda \xi \gamma} \int_{\xi_0}^{\xi} e^{\lambda s^{-\gamma}} ds \to 1/\lambda \text{ as } \xi \to \infty, \]
we have
\[
\max_{[\xi_0, \infty)} |g| \leq \frac{\max_{[\xi_0, \infty)} |G|}{\xi_0^\beta} + \text{Const}(n, \xi_0) \left( \max_{[\xi_0, \infty)} |g| \right)^2
\]
\[ \iff \max_{[\xi_0, \infty)} |g| (1 - \text{Const}(n, \xi_0) \max_{[\xi_0, \infty)} |g|) \leq \frac{\max_{[\xi_0, \infty)} |G|}{\xi_0^\beta} \tag{4.2.19} \]
for large enough \( \xi_0 \).

Since \( \text{Const}(n, \xi_0) \) can stay the same for any bigger \( \xi_0 \), and \( \max_{[\xi_0, \infty)} |g| \to 0 \) as \( \xi_0 \to \infty \), we can make \( \text{Const}(n, \xi_0) \max_{[\xi_0, \infty)} |g| < 1/2 \) by increasing \( \xi_0 \). Then from (4.2.19) we have:
\[
\max_{[\xi_0, \infty)} |g| \leq \frac{2K}{\xi_0^\beta} \tag{4.2.20}
\]
where \( K := \max_{[\xi_0, \infty)} |G| \). (Since \( G(\xi) \to -\beta/\lambda \), \( |G(\xi)| \) is bounded.) Since
\[
\frac{\max_{[\xi_0, 2\xi_0]} |h|}{(2\xi_0)^\beta} \leq \max_{[\xi_0, 2\xi_0]} |g| \leq \max_{[\xi_0, \infty)} |g| \leq \frac{2K}{\xi_0^\beta}, \tag{4.2.21}
\]
we obtain \( \max_{[\xi_0, \infty)} |h| \leq 2^{\beta+1} K \), and since \( \xi_0 \) can be any bigger number, the inequality can be extended to \( \max_{[\xi_0, \infty)} |h| \leq 2^{\beta+1} K \), proving that \( |h| \) is bounded. \( \square \)
Lemma 4.2.7. \( G(\xi) \) has the following asymptotic form:

\[
G(\xi) \sim -\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{\xi^{1-\beta}} - \frac{\beta \gamma}{\lambda^2 \xi} + O(\xi^{-2+\beta}).
\]

More precisely,

\[
G(\xi) \sim \left[ -\frac{\beta}{\lambda} - \frac{\beta \gamma}{\lambda^2 \xi} + \tilde{G}_1(\xi) \right] + \xi^\beta \left[ \frac{\alpha^{n+1}}{\xi} + \tilde{G}_2(\xi) \right]
\]

where \( \tilde{G}_1, \tilde{G}_2 = O(\xi^{-2}) \in C[[1/\xi]] \).

Proof of Lemma 4.2.7. By integration by parts and L'Hospital's rule, we have

\[
e^{-\lambda s} \int_{\xi_0}^\xi e^{\lambda s} s^b \, ds \sim \frac{1}{\lambda s^b} - \frac{b}{\lambda^2 s^{b+1}} + ..., \text{ as } \xi \to \infty, \tag{4.2.22}
\]

and so we get the above asymptotic form for \( G(\xi) \).

Proof of Theorem 4.2.4. By Lemma 4.2.6, we can let \( |h| < M \) for some \( M \). Since

\[
\left| \left( 1 + \frac{h}{\alpha s^\beta} \right)^{-n} - 1 \right| \leq C \frac{M}{s^\beta}
\]

for \( s > \xi_0 \), we have

\[
\left| e^{-\lambda s} \int_{\xi_0}^\xi \lambda s \left( \left( 1 + \frac{h}{\alpha s^\beta} \right)^{-n} - 1 \right) e^{\lambda s} s^{-\gamma} \, ds \right| \leq C \lambda M e^{-\lambda \xi_0} \int_{\xi_0}^\xi e^{\lambda s} s^{-\gamma-\beta} \, ds
\]

\[
\sim \frac{CM^2}{\xi^\beta} = O(\xi^{-\beta}), \tag{4.2.23}
\]

and so, by (4.2.16) we get

\[
h = G(\xi) + O(\xi^{-\beta}) = -\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{\xi^{1-\beta}} + O(\xi^{-\beta}).
\]

Now let

\[
h(\xi) = -\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{\xi^{1-\beta}} + \frac{a}{c^\beta} + o(\xi^{-\beta}), \tag{4.2.24}
\]
then (4.2.16) becomes

\[
-\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{\xi^{1-\beta}} + \frac{a}{s^\beta} + o(\xi^{-\beta})
= G(\xi) + e^{-\lambda\xi} \int_\xi^{\xi_0} \lambda \left( -\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{s^{1-\beta}} + \frac{a}{s^\beta} + o(s^{-\beta}) \right) \left( -\frac{n}{\alpha s^\beta} \left( -\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{s^{1-\beta}} + \frac{a}{s^\beta} + o(s^{-\beta}) \right) \right)
+ O(s^{-2\beta}) \right) e^{\lambda s^{-\gamma} ds}
= -\frac{\beta}{\lambda} + \frac{\alpha^{n+1}}{\xi^{1-\beta}} + \frac{\beta\gamma}{\lambda^2 \xi^2} + O(\xi^{-2\beta}) + e^{-\lambda\xi} \int_\xi^{\xi_0} \lambda \left( -\frac{n\beta^2}{\lambda^2 \alpha s^\beta} + \frac{2n\beta^2 \alpha^{n+1}}{\lambda \alpha s} + o(s^{-1}) \right)
+ O(s^{-2\beta}) \right) e^{\lambda s^{-\gamma} ds},
\]
(4.2.25)
and so, by (4.2.22) we get \(a = \frac{n\beta^2}{\lambda^2 \alpha \xi^\beta} = -\frac{n^3 \alpha^{2n-1}}{\xi^\beta}\), and replacing \(\beta/\lambda\) by \(n\alpha^n\), we obtain:

\[h(\xi) = -na^n + \frac{\alpha^{n+1}}{\xi^{1-\beta}} - \frac{n^3 \alpha^{2n-1}}{\xi^\beta} + O(\xi^{-1}) + O(\xi^{-2\beta}).\]
(4.2.26)

Once \(n\) is given, the coefficients of the asymptotic series can be inductively calculated by iteration. Note that, without specifying the value of \(n\), we cannot proceed explicitly any further for a general \(n\); since the ordering of the subsequent terms depends on the concrete value of \(n\).

By collecting all the possible exponents of \(\xi\) and simple verification, similar to that of (4.2.25), we get the following asymptotic expansion for \(h(\xi)\):

\[h(\xi) \sim \sum_{l,m \geq 0} \frac{a_{l,m}}{\xi^{l(1-\beta)+m\beta}}\]
(4.2.27)
(Note that (4.2.27) can be well ordered: since \(\beta = -\frac{n}{n+2}\) and \(1-\beta = \frac{2}{n+2}\), for even \(n = 2k\), this sum can be expressed as \(\sum_{m \geq 1} a_{m} \xi^{-\frac{m}{n+2}}\), and for odd \(n\), \(\sum_{m \geq 1} a_{m} \xi^{-\frac{m}{n+2}}\)).

Proof of Theorem 4.2.2. Using (4.2.27) we get

\[w = XY = \alpha + \frac{h(\xi)}{\xi^\beta} = \alpha + \frac{h(X^{n+2})}{X^n}
\sim \alpha - na^n X^{-n} + \alpha^{n+1} X^{-n-2} - n^3 \alpha^{2n-1} X^{-2n} + \cdots\]
(4.2.28)
implying the result. □

Since $X' = Y^n$, we can calculate $X(t)$ and $Y(t)$ asymptotically, using

\[
\int_{X(0)}^{X} \frac{X^n}{(f(X))^n} \, dX = \int_0^t dt, \quad Y(t) = \frac{f(X(t))}{X(t)}
\]

and obtain

\[
X(t) \sim [\alpha^n(n+1)]^{\frac{1}{n+1}} t^{\frac{1}{n+1}} - n^2 \left( \frac{\alpha}{n+1} \right)^{\frac{n-1}{n+1}} t^{-\frac{n+1}{n+1}} + O(t^{-1}) \quad (4.2.29)
\]

and

\[
Y(t) \sim \left[ \frac{\alpha}{n+1} \right]^{\frac{1}{n+1}} t^{-\frac{1}{n+1}} + \left[ \frac{n^2 - n\alpha}{\alpha(n+1)} \right] t^{-1} + O(t^{-\frac{n+1}{n+1}}). \quad (4.2.30)
\]

**Note 4.2.8.** Since $X(t) \to \infty$ as $t \to \infty$, and we have interest in the asymptotic behavior as $t \to \infty$, for small initial value of $X$ we re-choose the sufficiently large initial time $t_0$ so that $X(t_0)$ is also large.
Appendix A. Proofs of (2.5.10) and (2.5.11)

As for (2.5.10), by letting \( q = \frac{t}{\sqrt{z}} p \) and \( x = 2\sqrt{z} t \), the problem is now to show:

\[
\int_0^\infty e^{-\frac{z(p+\frac{1}{2})}{2}} dp = x \int_1^\infty e^{-x\tau(\tau^2 - 1)^{\frac{1}{2}}} d\tau. \tag{A.31}
\]

Note that the contour in \( \tau \) can be deformed back to \([0, \infty)\) by restricting \( z \) for now (say, \(|\arg z| < \frac{\pi}{2}\)). By letting \( r = \frac{1}{p} \), the left side of (A.31) becomes:

\[
\int_0^\infty e^{-\frac{z(p+\frac{1}{2})}{2}} dp = \int_0^\infty e^{-\frac{z}{2}(r+\frac{1}{2})} \frac{1}{2} \left(1 + \frac{1}{p^2}\right) dp := f(x). \tag{A.32}
\]

Integrating by parts, and then splitting the interval of integration, we get:

\[
f(x) = \frac{x}{4} \left[ \int_0^1 + \int_1^\infty \right] e^{-\frac{z}{2}(p+\frac{1}{2})} \left(p - \frac{1}{p}\right) \frac{1 - \frac{1}{p^2}}{2} dp. \tag{A.33}
\]

Substituting \( \tau = \frac{1}{2} \left(p + \frac{1}{p}\right) \) in each integral in (A.33), we obtain:

\[
f(x) = -\frac{x}{2} \int_0^\infty e^{-x\tau(\tau^2 - 1)^{\frac{1}{2}}} d\tau + \frac{x}{2} \int_1^\infty e^{-x\tau(\tau^2 - 1)^{\frac{1}{2}}} d\tau = x \int_1^\infty e^{-x\tau(\tau^2 - 1)^{\frac{1}{2}}} d\tau, \tag{A.34}
\]

completing the proof for (2.5.10).

The proof of (2.5.11) is similar: we first deform the contour for \( \mathcal{L}^{-1} \), pushing both upper half and lower half of the contour to the negative real axis. The integrals cancel each other except for a contour around the origin:

\[
\mathcal{L}^{-1} \left\{ \frac{\sin \frac{1}{x^2}}{x^2} \right\} (t^2) = \frac{1}{2\pi i} \oint \left[ \frac{e^{\frac{1}{2}+xt^2}}{2ix^2} - \frac{e^{-\frac{1}{2}+xt^2}}{2ix^2} \right] dx. \tag{A.35}
\]
Taking $x = \frac{e^{i\pi/4}}{t}y$ in the first term in the bracket, and $x = \frac{e^{-i\pi/4}}{t}y$ in the second term, and adapting (A.32) to the case above, (A.35) becomes:

$$
\frac{e^{i\pi t}}{4\pi} \left[ \oint_{|y|=1} e^{2e^{-i\pi/4}t\frac{1}{2}(y + \frac{1}{y})} dy + i \oint_{|y|=1} e^{2e^{i\pi/4}t\frac{1}{2}(y + \frac{1}{y})} dy \right]. \quad (A.36)
$$

(Note that the contour can be modified to be $|y| = 1$ since the origin is the only singular point.)

On the other hand, substituting $\cos \theta = \frac{1}{2} (y + \frac{1}{y})$ in the integral representation of $I_1$, we obtain:

$$
I_1(x) = -\frac{i}{4\pi} \oint_{|y|=1} e^{\frac{x}{2}(y + \frac{1}{y})} \left( 1 + \frac{1}{y^2} \right) dy = -\frac{i}{2\pi} \oint_{|y|=1} e^{\frac{x}{2}(y + \frac{1}{y})} \frac{1}{y^2} dy. \quad (A.37)
$$

Using (A.37) for both $x = \frac{e^{i\pi/4}}{t}y$ and $x = \frac{e^{-i\pi/4}}{t}y$ and then comparing with (A.36) we obtain (2.5.11).
Appendix B. Interesting regimes of (2.5.34)

Here we illustrate some interesting regimes for (2.5.34).

When \( \delta + 2\epsilon \) is decreasing much faster than \( 2\epsilon - \delta \), letting \( \frac{\delta + 2\epsilon}{\delta - 2\epsilon} = \epsilon_2 \), (2.5.34) becomes:

\[
\frac{\delta(\frac{\pi}{2} + O(\epsilon_2)) + 2\epsilon(\frac{\pi}{2} + O(\epsilon_2))}{3(\delta + 2\epsilon)\sqrt{2\epsilon - \delta}} = \frac{\pi}{6\sqrt{2\epsilon - \delta}} + \frac{(\epsilon_2 + 1)O(1)}{\sqrt{2\epsilon - \delta}} \frac{(1 - \epsilon_2)O(1)}{\sqrt{2\epsilon - \delta}}. \quad (B.38)
\]

When \( \delta - 2\epsilon \) is decreasing comparably fast to \( 2\epsilon + \delta \), but \( \frac{\delta + 2\epsilon}{\delta - 2\epsilon} \to 1 \), (2.5.34) becomes:

\[
\frac{\delta O(1) + 2\epsilon O(1)}{3(\delta + 2\epsilon)\sqrt{2\epsilon - \delta}} = \frac{O(1)}{\sqrt{2\epsilon - \delta}}. \quad (B.39)
\]

When \( \frac{\delta + 2\epsilon}{\delta - 2\epsilon} \to 1 \), which is the singularity of elliptic integral K, letting \( \sigma_1 = (2\epsilon - \delta)\lambda \) the first integral of (2.5.33) becomes:

\[
\int_0^{2\epsilon - \delta} \frac{(2\epsilon - \delta)^{\frac{1}{2}} \lambda^{\frac{1}{2}} (2\epsilon - \delta)^{\frac{1}{2}} (\lambda + \frac{2\epsilon}{2\epsilon - \delta})^{\frac{1}{2}}}{(2\epsilon - \delta)^{\frac{1}{2}} (2\lambda + 1)^{\frac{1}{2}}} (2\epsilon - \delta) d\lambda = \frac{1}{(2\epsilon - \delta)^{\frac{1}{2}}} \int_0^{2\epsilon - \delta} \frac{\lambda^{\frac{1}{2}} (\lambda + \frac{2\epsilon}{2\epsilon - \delta})^{\frac{1}{2}}}{(2\lambda + 1)^{\frac{1}{2}}} d\lambda \quad (B.40)
\]

which is \( O\left(\frac{1}{\sqrt{2\epsilon - \delta}}\right) \) since \( \frac{\epsilon}{2\epsilon - \delta} \to 0 \).

When \( \delta - 2\epsilon \) is decreasing much faster than \( 2\epsilon + \delta \), letting \( \frac{\delta - 2\epsilon}{\delta + 2\epsilon} = \epsilon_3 \), (2.5.34) becomes:

\[
\frac{\delta O(\sqrt{\frac{1}{\epsilon_3}}) + 2\epsilon O(\sqrt{\epsilon_3})}{3(\delta + 2\epsilon)\sqrt{2\epsilon - \delta}} = \frac{(1 + \epsilon_3)O(1)\sqrt{\delta + 2\epsilon}}{6(2\epsilon - \delta)} + \frac{(1 - \epsilon_3)O(\sqrt{\epsilon_3})}{\sqrt{2\epsilon - \delta}}. \quad (B.41)
\]

Note that the leading term is from this case, and it approaches in some sense the \( z \neq 0 \) part of the manifold.
Appendix C. Proof of Lemma 3.2.4

Assume that $F(p)$ has a pole at $p_i$ with multiplicity $n$. Then, after cancellation of the contour from and to $\infty e^{i \arg p}$, we get

$$h_i(x) = C \oint_{|p-p_i|=\varepsilon} F(p)e^{-xp} dp = Ce^{-p_i x} P_{n-1}(x) = \tilde{C}e^{-p_i x} x^{n-1}(1 + s_i)$$

where $P_{n-1}(x)$ is a polynomial of degree $n-1$ by Cauchy’s formula. So $s_i$ and its Borel transform $\mathcal{B}s_i$ is also a polynomial in $\frac{1}{x}$ and $p$, respectively, which implies that $\mathcal{B}s_i$ does not have any singularity. Meanwhile, since the difference between two Laplace transforms $h_i(x)$ becomes an integral only around $p_i$, it does not collect any new linearly independent term when we rotate the contour into any direction.

As shown in 1.2.6 and [23], in the case with power 1 for $\ln z$, $F(p)$ has a logarithmic singular point at $p_i$, that is, $F(p) = \frac{1}{(p-p_i)^n} G_1(p) + \ln(p-p_i) G_2(p) + G_3(p)$ where $G_i(p), i = 1, 2, 3$ are analytic at $p_i$ and possibly singular at $p_j, j \neq i$. Then by the above proof, we get:

$$h_i(x) = \tilde{C}e^{-p_i x} x^{n-1}(1 + s_i) + 2\pi i Ce^{-p_i x} \int_0^{\infty e^{i \arg p_i}} \tilde{G}_2(q)e^{-xq} dq$$

(C.42)

$\tilde{G}_2(q)$ is analytic at $q = p_i$, and so rotating the contour beyond any singular point of $\tilde{G}_2$, $h_i(x)$ collects a new linearly independent term.

Meanwhile, let $\tilde{G}_2(q) = \sum_{k=k_0}^{\infty} a_k q^k$, and

$$2\pi i Ce^{-p_i x} \int_0^{\infty e^{i \arg p_i}} \tilde{G}_2(q)e^{-xq} dq = \tilde{C}e^{-p_i x} x^{-k_0-1}(1 + \tilde{s}_i).$$

Then by differentiating $\tilde{G}_2(q) = \mathcal{B}(x^{-k_0-1}(1 + \tilde{s}_i))$, $(k_0 + 1)$-times, we get $\tilde{G}_2^{(k_0+1)}(q) = \mathcal{B}\tilde{s}_i$, and as we expect, the set of singular points of $\mathcal{B}\tilde{s}_i$ are the same as the set of singular points of $\tilde{G}_2$. 

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If \( F(p) \) has a fractional power at \( p_i \), that is, \( F(p) = (p - p_i)^{-a} A_1(p) + A_2(p) \) where \( \Re(a) > 0 \), \( A_1(p) \) and \( A_2(p) \) are possibly singular at \( p_j \) for \( j \neq i \), and analytic at \( p_i \) with \( A_1(p_i) \neq 0 \), then, after integrating by parts whenever needed and taking \( q = p - p_i \), the difference between two Laplace transforms becomes

\[
h_i(x) = Ce^{-p_i x} \int_0^{\infty e^{i\arg(p_i)}} B(x, q) \frac{e^{-xq}}{q^{a - \Re(a)}} dq
\]

with \( B(x, q) = \sum_{k=0}^{[\Re(a)]} a_k(q)x^k \) where \( a_k(q) \) are linear combinations of \( \tilde{A}_1(q) = A_1(q+p_i) \) and its derivatives (up to \( k \)th order), that is, \( \tilde{A}_1, \tilde{A}_1', ..., \tilde{A}_1^{(k)} \).

For simplicity, we let \([\Re(a)] = 1\), and \( n = 2 \) with \( i = 1, j = 2 \). Then we get

\[
h_1(x) = \tilde{C}e^{-p_1 x} \int_0^{\infty e^{i\arg(p_1)}} [\tilde{A}_1(q)x \frac{e^{-xq}}{q^{a-1}} - \tilde{A}_1'(q) \frac{e^{-xq}}{q^{a-1}}] dq.
\]

The same calculation can be applied to the contour integral \( \int_C F(p)e^{-xp} dp \) where \( C \) is a contour from \( \infty e^{i\theta} \) to \( \infty e^{i\theta} \) \((\theta \neq i \arg p_2)\) encircling \( p_1 \), and we get

\[
\int_C F(p)e^{-xp} dp = Ce^{-p_1 x} \int_0^{\infty e^{i\arg(p_1)}} [\tilde{A}_1(q)x \frac{e^{-xq}}{q^{a-1}} - \tilde{A}_1'(q) \frac{e^{-xq}}{q^{a-1}}] dq.
\]

So getting a new linearly independent term from the deformation of contour in Figure 3.1 is equivalent to getting a new term from rotating \( \theta \) for \( \int_0^{\infty e^{i\theta}} \left[ \tilde{A}_1(q)x \frac{e^{-xq}}{q^{a-1}} - \tilde{A}_1'(q) \frac{e^{-xq}}{q^{a-1}} \right] dq \), and the latter occurs only at singular points of \( \tilde{A}_1(q) \) (by integration by parts).

Meanwhile, we also have \( h_1(x) = \tilde{C}e^{-p_1 x} x L[g](x) \) where

\[
g(q) := B \left[ \int_0^{\infty e^{i\arg(p_1)}} \tilde{A}_1(q) \frac{e^{-xq}}{q^{a-1}} - \frac{1}{x} \tilde{A}_1'(q) \frac{e^{-xq}}{q^{a-1}} dq \right] = \tilde{A}_1(q) \frac{e^{-xq}}{q^{a-1}} - \int_0^q \tilde{A}_1'(t) \frac{e^{-xq}}{t^{a-1}} dt.
\]

Since \( g'(q) = \frac{\tilde{A}_1(q)(1 - a)}{q^a} \) after cancellation, all the nonzero singularities of \( g(q) \) are identical with the singularities of \( \tilde{A}_1(q) \), and since \( B s_1 \) shares all the singularities with \( g(q) \) except 0 where \( B s_1 \) is analytic, the set of all the singularities of \( \tilde{A}_1(q) \) are actually the set of all the singularities of \( B s_1 \), proving the lemma.
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