Utilization of Optimization for Design of Morphing Wing Structures for Enhanced Flight

DISSERTATION

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Abstract

Conventional aircraft control surfaces constrain maneuverability. This work is a comprehensive study that looks at both smart material and conventional actuation methods to achieve wing twist to potentially improve flight capability using minimal actuation energy while allowing minimal wing deformation under aerodynamic loading. A continuous wing is used in order to reduce drag while allowing the aircraft to more closely approximate the wing deformation used by birds while loitering. The morphing wing for this work consists of a skin supported by an underlying truss structure whose goal is to achieve a given roll moment using less actuation energy than conventional control surfaces. A structural optimization code has been written in order to achieve minimal wing deformation under aerodynamic loading while allowing wing twist under actuation. The multi-objective cost function for the optimization consists of terms that ensure small deformation under aerodynamic loading, small change in airfoil shape during wing twist, a linear variation of wing twist along the length of the wing, small deviation from the desired wing twist, minimal number of truss members, minimal wing weight, and minimal actuation energy. Hydraulic cylinders and a two member linkage driven by a DC motor are tested separately to provide actuation. Since the goal of the current work is simply to provide a roll moment, only one actuator is implemented along
the wing span. Optimization is also used to find the best location within the truss structure for the actuator. The active structure produced by optimization is then compared to simulated and experimental results from other researchers as well as characteristics of conventional aircraft.
Dedication

This document is dedicated to my family.
Acknowledgments

I would like to thank Dr. Washington and Dr. Walter for their help and guidance and for their patience.
Vita

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Table of Contents

Abstract................................................................................................................................. ii
Dedication .............................................................................................................................. iv
Acknowledgments............................................................................................................... v
Vita........................................................................................................................................ vi
List of Variables/Symbols................................................................................................. x
List of Abbreviations/Acronyms......................................................................................... xiv
List of Tables ....................................................................................................................... xv
List of Figures ..................................................................................................................... xvii
Chapter 1: Introduction ........................................................................................................ 1
Chapter 2: Literature Review .............................................................................................. 4

  Morphing wings for mission adaptation ............................................................................ 4

  Morphing wings addressing other issues .......................................................................... 8

  Morphing wings for flight control .................................................................................... 9

  Synopsis, advantages and disadvantages of morphing wing research .......................... 18

Chapter 3: Aerodynamic Theory for Lift and Pressure Distribution .................................. 21
List of Variables/Symbols

\( a = \) Cylinder radius, m
\( a = \) Element length in x direction, m
\( A = \) Beam element cross sectional area, m\(^2\)
\( b = \) Wing span, m
\( b = \) Element length in y direction, m
\( B = \) Finite element matrix
\( c = \) Chord, m
\( c = \) Joukowski transform coefficient, m
\( c = \) Stiffness, N/m
\( c = \) Cooling rate for simulated annealing
\( c_\ell = \) Non-dimensional lift coefficient
\( c_m = \) Non-dimensional moment coefficient
\( \vec{c}_{o_j} = \) Unit vector along undeflected member \( j \)
\( C = \) System compliance, m\(^5\)/N
\( d_j = \) Deflection of member \( j \)
\( D = \) Electric displacement, C/m\(^2\)
\( D = \) State space size
\( E = \) Young's modulus

\( E = \) Electric field, V/m

\( E = \) Energy (cost)

\( F = \) Complex potential, \( m^2/s \)

\( G = \) Finite element stiffness matrix, N/m

\( f = \) Axial force along member, N

\( F = \) Externally applied force, N

\( I_y, I_z = \) Area moment of inertia, \( m^4 \)

\( J = \) Polar moment of inertia, \( m^4 \)

\( k_{ij} = \) Stiffness for force in the i direction and nodal displacement in the j direction, N/m

\( L = \) Length of beam element, m

\( L = \) Length of piezoelectric stack, m

\( m = \) Radial displacement of cylinder center in zeta plane, m

\( M = \) Moment, Nm

\( n = \) Number of layers in piezoelectric stack

\( n = \) Iteration number

\( p_j = \) Nodal displacement in the j direction, m

\( p = \) Pressure, N/m\(^2\)

\( p = \) Probability of energy (cost) change

\( q = \) Volumetric flow rate into the system, \( m^3/s \)

\( s_E = \) Compliance under zero electric field, \( m^2/N \)

\( S = \) Strain
\( T = \) Stress, Pa
\( T = \) Temperature
\( T = \) Effective stiffness matrix
\( u = \) Fluid velocity in the x direction, m/s
\( u = \) Actual nodal displacement, m
\( U = \) Free stream velocity, m/s
\( v = \) Fluid velocity in the y direction, m/s
\( v = \) Nodal displacement during ideal twisting, m
\( V = \) Voltage across piezoelectric material layer, V
\( V = \) Fluid volume, m\(^3\)
\( w = \) Nodal displacement under aerodynamic loading, m
\( x = \) Position in the x direction, m
\( y = \) Position in the y direction, m
\( y = \) Position along wing span, m
\( \gamma = \) degree of perturbation in simulated annealing
\( z = \) Position in the z direction, m
\( z = \) Complex position in the z plane, m

Greek:
\( \alpha = \) Geometric angle of attack, rad
\( \alpha_{L=0} = \) Angle of attack at zero lift, rad
\( \delta = \) Angular location of cylinder center in zeta plane, rad
\( \varepsilon_T \) = Permittivity under zero stress, F/m

\( \Gamma \) = Circulation, m²/s

\( \phi \) = Velocity potential, m²/s

\( \Psi \) = Stream function, m²/s

\( \omega \) = Complex flow velocity, m/s

\( \theta \) = Angle in zeta plane, rad

\( \nu \) = Poisson’s Ratio

\( \zeta \) = Complex position in zeta plane, m

\( \zeta_0 \) = Complex location of cylinder center in zeta plane, m
List of Abbreviations/Acronyms

CFD = Computational Fluid Dynamics
DC = Direct Current
FEA = Finite Element Analysis
HECS = Hyper-Elliptic Cambered Span
LE = Leading Edge
MAV = Micro Aerial Vehicle
MFC = Macro Fiber Composite
MPE = Mutual Potential Energy
SA = Simulated Annealing
SE = Strain Energy
SMA = Smart Memory Alloy
TE = Trailing Edge
UAV = Unmanned Aerial Vehicle
List of Tables

Table 1. Nodal deflections of passive test structures......................................................... 62
Table 2. Nodal deflections for test structures with piezoelectric elements......................... 63
Table 3. Nodal deflections for test structures with DC motor actuation .......................... 63
Table 4. Effect of increasing the number of ribs ............................................................... 84
Table 5. Effect of nodal density on the final design ............................................................ 86
Table 6. Effect of structure type on the final cost .............................................................. 87
Table 7. Effect of termination number of iterations without improvement ...................... 88
Table 8. Optimization routines .......................................................................................... 89
Table 9. Effect of optimization routine execution order ..................................................... 89
Table 10. Effect of varying optimization overall termination criteria .............................. 90
Table 11. Effect of varying cooling parameter c ................................................................. 91
Table 12. Optimization settings based on trials ................................................................. 92
Table 13. Optimization orders tested .................................................................................. 94
Table 14. Effect of optimization order ............................................................................... 94
Table 15. Effect of nodal density ....................................................................................... 96
Table 16. Initial case study results ...................................................................................... 108
Table 17. Case study results based on shell theory .......................................................... 112
Table 18. Torque required to achieve a 20 degree twist angle ....................................... 112
Table 19. Effect of varying nodal density ................................................................. 114
Table 20. Effect of number of ribs ........................................................................... 115
Table 21. Effect of structure type ........................................................................... 115
Table 22. Effect of support structure material ......................................................... 116
Table 23. Effect of skin material .............................................................................. 116
Table 24. Effect of re-seeding and variable cooling rate ........................................... 118
List of Figures

Figure 1. Multiple mission morphing aircraft by Neal and Robertshaw [9]................. 5
Figure 2. Multiple mission morphing aircraft by Galantai........................................ 6
Figure 3. Actuation structure used by Galantai. ....................................................... 7
Figure 4. Multiple mission design by Hall et al....................................................... 8
Figure 5. Morphing aircraft by Poonsong................................................................. 9
Figure 6. Morphing wing design by Elzey et al....................................................... 10
Figure 7. Actuation structure for Elzey et al............................................................. 10
Figure 8. Morphing aircraft design by Henry......................................................... 11
Figure 9. Morphing aircraft design by Kota et al..................................................... 13
Figure 10. Morphing aircraft design by Abduhrahim............................................ 14
Figure 11. Close up view of twisting mechanism used by Abduhrahim..................... 15
Figure 12. Twisting wing developed by Good.......................................................... 16
Figure 13. Underlying mechanism used by Good to twist wing.............................. 16
Figure 14. Lift comparison between wing rotation and the use of trailing edge flaps ....... 23
Figure 15. Moment required for wing or trailing edge flap rotation ....................... 24
Figure 16. Locating the point corresponding to the trailing edge \( \zeta \) plane .............. 28
Figure 17. Streamlines around the airfoil, angle of attack = 10°.............................. 29
Figure 18. Velocity potentials around the airfoil, angle of attack = 10°..................... 30
Figure 19. Pressure distribution along the length of the airfoil, angle of attack = 10° .... 30
Figure 20. Pressure distribution around the wing, angle of attack = 10°.......................... 31
Figure 21. Pressure distribution around the airfoil, angle of attack = -10°.................... 31
Figure 22. Pressure distribution around the airfoil, angle of attack = 0°.......................... 32
Figure 23. Velocity vectors around the airfoil, angle of attack = 0°.......................... 34
Figure 24. Pressure distribution around the airfoil, angle of attack = 0°.......................... 35
Figure 25. Velocity vectors around the airfoil, angle of attack =5°.......................... 36
Figure 26. Pressure distribution around the airfoil, angle of attack = 5°.......................... 37
Figure 27. Velocity vectors around airfoil, angle of attack = -5°.......................... 38
Figure 28. Pressure distribution around the airfoil, angle of attack = -5°.......................... 39
Figure 29. Member Deflection under Loading .......................................................... 45
Figure 30. Hydraulic Cylinder System ........................................................................ 48
Figure 31. Two member linkage driven by a DC motor .............................................. 49
Figure 32. Example structure ...................................................................................... 50
Figure 33. Two welded pairs connected by a revolute joint ....................................... 52
Figure 34. Test structure A ......................................................................................... 61
Figure 35. Test structure B .......................................................................................... 61
Figure 36. Test structure C .................................................................................................. 62
Figure 37. Simulated annealing structure ........................................................................ 65
Figure 38. Dependence of perturbation on temperature ................................................. 67
Figure 39. Overall optimization scheme ........................................................................ 71
Figure 40. Typical deformation under actuation using Equation 106 ......................... 73
Figure 41. Wing deformation under actuation using Equation 108 ......................... 77
Figure 62. Case 2: actuation energy required to achieve wing twist .................................. 120

Figure 63. A Simple Two Dimensional Structure to Illustrate Global Matrix Packing . 135
Chapter 1: Introduction

The design of aircraft using conventional control surfaces constrains maneuverability [1]. Many researchers are investigating different designs with flexible control surfaces – morphing wings. Morphing is defined as the ability “to change character or form” [2]. Mor phing wings are able to change shape or flight characteristics along their spar or chord [3]. Nature, in particular, birds, whose wings are flexible control surfaces actuated by muscles, serve as motivation for the ideas surrounding morphing wings.

Birds can morph their wings in order to vary flight performance. They can also adjust their wings for different goals such as loitering, diving, or high speed maneuvers [4]. Swifts spend almost their entire lives in flight, eating, mating, and even sleeping while flying. They sweep their wings to chase insects and extend them to loiter or sleep [5]. Bald eagles can fly at altitudes up to 10000 feet and are able to soar using wind currents and thermal updrafts. They are capable of 65 miles per hour during level flight and can attain dive speeds up to 200 miles per hour [6]. They are capable of carrying loads up to half their weight, a feat advanced cargo aircraft, such as the C-17, are barely capable of [7]. Humans have long pursued this kind of performance. Aircraft with morphing wings are a step towards the goals for high flight performance and shifting between wing configurations for either maneuverability or loitering.
Research on morphing wings can be broken down into two categories. The first attempts to alter the wing planform for mission adaptation so that one aircraft is capable of loitering and high speed maneuvering. The second category attempts wing modification for flight control rather than using discreet control surfaces. The objective of this research is to develop a morphing wing for flight control. In particular, flight control using wing twisting is investigated. Wing twisting offers the possibility of greater flight control and can potentially attain the same roll moment as conventional aircraft with less actuation energy. However, this can only be realized if the structural resistance of the wing to twisting is sufficiently small while retaining enough structural stiffness for small deformation under aerodynamic loading. If this can be accomplished, the aircraft would have reduced drag since the control surface is continuous rather than discreet. Reduction of drag would enable a longer flight range.

In this research twisting wings are designed for small unmanned aerial vehicles (UAVs) with wingspans varying from one to 15 meters. For these UAVs, the wing shape is supported by a truss structure. Most of the members of the truss structure are passive. The members may be fixed to one another or may be able to rotate freely relative to one another. In order to reduce weight, only one actuator will be used for each wing. The actuator types considered in this research include piezoelectric, hydraulic, and a two member linkage driven by a geared DC motor.

This thesis is divided into eight chapters. A review of related literature is covered in the second chapter. The reviewed literature includes wings designed to alter their shape depending on the current flight regime, wings designed to morph for flight control, and
flapping wing aircraft. The third chapter presents the aerodynamic theories used in this work. Thin airfoil theory is used to show the potential benefit of morphing wings. Complex potential flow theory is presented since it is used to assess the pressure distribution about the wing during the optimization process. The fourth chapter briefly identifies the models used for the actuators and the finite element equations used to predict wing deflection. The fifth chapter describes optimization procedures and process. The optimization results are covered in the sixth chapter with results from case studies presented in the seventh chapter. Conclusions and suggestions for future work are drawn in the eighth and final chapter.
This chapter covers some of the work by other researchers working on morphing aircraft. Most of the work can be categorized as either adaptation to multiple missions or flight control. However, some of the work done cannot be placed into either category. This chapter will first cover morphing wings used for mission adaptation. The second portion of this chapter covers morphing that is used to address other issues. The third part discusses morphing wings for flight control. The last portion of the chapter gives a brief summary of the research work on morphing wings along with a discussion of the advantages and disadvantages.

Morphing wings for mission adaptation

Much of the work on morphing aircraft has been conducted in order to tailor it to its flight goal, such as loitering and high speed maneuvering. Hong et al. proposed an Unmanned Aerial Vehicle (UAV) design that changes from a loiter configuration with high aspect ratio and low sweep angle wings to a killer/hunter configuration with a low aspect ratio and high sweep angle using revolute joints [8]. In order to assess the aeroelastic properties of their design, they developed a model to estimate the aerodynamic forces and moments on the aircraft and then used a finite element model for
the structural analysis. Neal and Robertshaw presented a design capable of varying both the wing sweep angle using revolute joints and the planform area using telescoping wings in order to change from a loiter configuration to a killer configuration (Figure 1). A model was constructed and tested in a wind tunnel. It was shown that varying the wingspan and sweep angle affects the center of gravity, lift, and drag [9].

![Diagram](image)

**Figure 1.** Multiple mission morphing aircraft by Neal and Robertshaw [9].

Jha and Kudva developed a review of how wing geometry affects aircraft performance and a survey of morphing aircraft [10]. Stubbs designed a kinematic mechanism allowing a morphing wing to change its shape from a conventional wing design more suitable during take-off and landing to a new Hyper-Elliptic Cambered Span wing configuration that may lead to increased stability, control, and improved aerodynamic efficiency, during flight [11]. The work by Galantai allows wing morphing without
hydraulics or servomotors (Figures 2 and 3). He used shape memory alloy (SMA) to morph the wing. The design consists of ribs connected by an SMA actuated spar structure. Thermal energy is needed to power the SMA while the wing morphs, but it is not needed once the desired wing shape has been achieved. The wing is capable of changing the sweep angle and dihedral angle continuously along the length of the wing [12].

Figure 2. Multiple mission morphing aircraft by Galantai.
Hall *et al.* have developed a micro-unmanned aerial vehicle (MAV) capable of varying the sweep angle of the wings (Figure 4). The intended use for the MAV is a flying chemical sensor platform. In order to accomplish this task it must be able to efficiently fly to the potentially hazardous area and then loiter in the area to monitor the chemical concentrations present. An aerodynamic model using CFD was developed in order to investigate how lift and drag change as the sweep angle was varied. They found that the drag coefficient could be varied by nearly 60% by varying the sweep angle [13].
Figure 4. Multiple mission design by Hall et al.

*Morphing wings addressing other issues*

In an effort to reduce sonic boom Wintzer et al. rotated the entire wing such that the sweep angle on the port and starboard sides were equal in magnitude, but opposite in sign [14]. Gano and Renaud proposed a morphing wing design in which the fuel was stored in bladders within the wings. As the fuel was consumed, the wings shrink, reducing the drag on the wings. Using their model they estimated that their morphing wings would be able to fly 22% further than fixed wings [15]. Joshi et al. developed the best wing configurations for various mission roles as a guide for morphing aircraft designers [16]. Poonsong split the wing profile into multiple rigid sections connected by revolute joints (Figure 5). The shape of the wing profile was then controlled using pneumatic actuators. They constructed both a rigid wing and morphing wing for wind tunnel testing. The morphing wing was capable of a 5 degree rotation at the leading and trailing edge without inducing discontinuities in the skin [17].
The goal of the second type of morphing aircraft is to achieve flight control by shaping the wing. Kudva et al. used shape memory alloy to adjust the leading and trailing edges of the wing in order to control lift and drag. They were able to achieve a deflection rate of 80 deg/s and attain a maximum deflection of 20 deg. Due to the continuous deformation of the morphing wing, it achieved a 17% improvement of the roll moment coefficient over a conventional wing [18]. Zheng and Ramaprian used piezoelectric ceramic benders to control the wing profile and therefore its lift and drag [19]. Elzey et al. presented a wing profile built like a vertebrae (Figures 6 and 7). The wing profile was altered using shape memory alloy that connected the outer edges of the vertebrae [20].
Figure 6. Morphing wing design by Elzey et al.

Figure 7. Actuation structure for Elzey et al.
Henry presented a design with telescoping wings that could be used to change the mission role if the right and left wing lengths were equal or for roll control if the right and left wing lengths were mismatched (Figure 8). The center of gravity also shifted due to the wing span mismatch. This created a roll moment that opposed the one induced by the lift disparity. However, the roll moment induced by lift differential overshadowed the moment induced by the shift in the center of gravity [21].

Figure 8. Morphing aircraft design by Henry.

Bourdin et al. proposed a design in which the cant angle of the winglets could be varied in order to control the roll moment. The rigid wing and the winglets were connected at a revolute joint driven by servos. If the left winglet is kept at zero cant angle while the right winglet is rotated to a positive cant angle of 90 degrees, then the left wing, in effect, has a larger planform area. The greater lift produced by the left side results in a roll moment in the clockwise direction (viewed from the cockpit). Also, a force directed toward the cockpit is present due to the vertical winglet on the right wing. Since, in this
case, the winglet is behind the aircraft’s center of gravity, the yawing moment produced aids the turn resulting from the roll moment [22]. In an attempt to take advantage of the refined designs available due to the process of biological evolution, McCowan has studied the way marine and avian animals morph. They presented a wing similar in concept to Bourdin’s. However, in this case the cant angle varied continuously near the wing tip and their goal was to reduce the trailing vortices and therefore drag [23]. Kota et al. used compliant mechanisms to amplify the low displacement, high force available from piezoelectric stack actuators to deflect the leading and trailing edge of wings in order to alter lift and drag (Figure 9). He also applied a moving coil device and piezoelectric stacks amplified by a compliant mechanism to actuate small plate above the wing at high frequencies to displace the high velocity free stream air toward the low velocity boundary layer air in order to control flow separation [24].
A number of researchers have used SMA wires in order to bend the trailing edge of the wing [25,26], while others have used torque tubes [27] or a macro fiber composite actuators (MFC) [28,29] in order to bend the trailing edge of the wing. The previously mentioned design by Neal and Robertshaw was also able to induce wing twist using planetary gears and a pneumatic actuator. His aircraft was able to achieve 20 degrees of twist in either direction [9]. Abduraham constructed a UAV with membrane morphing wings (Figures 10 and 11). Actuation of the wings was achieved by using a servo to vary the tension on wires between the fuselage and trailing edge of the wing. When the tension was increased, the wing twisted, changed its cant angle, angle of attack, and camber. When one wing is curled, its lift increases, causing a roll moment. The
aircraft was flight-tested and performed well under most conditions. However, it was sensitive and difficult to control under high loading conditions (steep turns and high pitch angles). Also, when the critical angle of attack was exceeded by one wing and that wing stalled, the airplane would roll in the opposite direction. The unidirectional wing curl offered by this design was incapable of producing a roll moment sufficient to return the aircraft to level flight. Abduhrahim constructed another UAV with membrane wings that could be twisted using a torque tube. This design was also flight-tested and was found to have good flight characteristics and performed well over a wide variety of flight conditions. The second design also had the advantage of producing a roll moment with significant pitch or yaw, making control easier for some maneuvers [30].

Figure 10. Morphing aircraft design by Abduhrahim.
Good designed an aircraft capable of varying its wingspan, wing twist, sweep angle, tail span, and the camber of the tail. Redundant combinations of these variables are available for a particular flight condition. The values of these variables are chosen so as to minimize the drag. Structural optimization was performed on the tail using the method of asymptotes, commercial finite element software, and thin airfoil theory. The tail was able to deflect over 4 degrees using one actuator. Pneumatic rotary actuators with torque amplification using planetary gears are used to twist the wings. The flexible portion of wing is made of polyethylene foam (Figures 12 and 13). A torque tube transmits the torque to the rib at the end of the wingspan. The wings were able to twist +/- 20 degrees [31].

Figure 11. Close up view of twisting mechanism used by Abduhrahim.
Clingman and Ruggeri used SMA, piezoelectric, and electromagnetic actuators to twist aircraft wings using a torque tube. Preloaded springs were applied to the torque tube in
order to reduce the aerodynamic loading. A model was fabricated and tested. The preloaded springs reduced the necessary actuator torque by half [32]. Popov et al. used a flexible skin along top of wing with embedded pressure sensors. SMA actuators were used to deform the flexible skin in order to achieve the desired airfoil shape [33]. Vos et al. applied the concept of a hollow tube, split along its length. The split allowed more twist by an externally applied moment, but also diminished load carrying capacity since the tube’s bending stiffness was also reduced. However, twist along the length of the tube can also be induced by applying opposing forces parallel to and along the split. Additionally, the mechanism causing the opposing forces acts as a stiffener so that the load carrying capacity is not diminished. Vos et al. used this concept on the wing skin and were able to achieve 15 degrees of twist in either direction [34].

It is possible to morph the aircraft for any flight condition encountered. However, energy is required for the actuation authority. Namgoong et al. included the aerodynamic pressure acting on the wing during the optimization process. They showed that inclusion of the aerodynamic pressure becomes more important when the wing is flexible and the wing needs to be flexible for morphing [35]. However, this ignores the fact that structures can be designed to be relatively inflexible under external loading while moving with ease under actuation.

Optimization can be an important tool during the design of the wing’s support structure. Inman et al. have been working on morphing wings since they offer the enhancement of flight performance and stealth properties over conventional wings. Their goal is to minimize the actuation energy required to achieve the same flight characteristics as a
conventional aircraft. They intend to both twist the wing and change the camber of the Lambda wing as a method of flight control. They have been working on making a relatively simple (compared to FEA and CFD), but accurate model that can be used for optimization. They have used more complex models (such as CFD) to verify the accuracy of their model. They have shown that they can achieve a greater roll moment with morphing wing than conventional wing at 15 degrees over an extended speed range [36-38]. Rotinat-Libersa and Friconneau used optimization software developed for use in MATLAB in order to design the support structure of their wing [39]. Many researchers in trying to design compliant structures use topology optimization [40-44]. Hetrick and Kota optimized their structure by maximizing the efficiency of the compliant mechanism [45]. A few others have used the ratio of mutual potential energy to strain energy as their objective function. Mutual potential energy and strain energy represent the deflection due to actuation and aerodynamic loading, respectively [46,47].

Synopsis, advantages and disadvantages of morphing wing research

The work mentioned above is interesting, has been well done, and has significant potential. However, each of the design methods has its advantages and disadvantages. Limitations of each design type will be covered in the following. Telescoping wings allow both mission adaptation and roll control. But if the wings are collapsed, then they have more structural support than is required and are heavier than necessary. This additional weight can be minimized and perhaps the wings are only collapsed for brief high speed maneuvers. When they are extended, one side must be
shortened to cause a roll moment. This means that the roll moment is always accompanied by a decrease in lift. So this concept requires a choice between extra weight and a roll moment accompanied by lift decrease.

Fixed winglets are often used to reduce induced drag. So their additional weight is offset by this drag reduction. Winglets for morphing wings can be designed to have variable cant angle. As the cant angle is brought toward zero degrees (flat), the lift increases along with the induced drag. A roll moment is induced when the cant angle for one side is varied while the other remains fixed. If the winglet length is increased, the roll moment accompanied by the differential lift increases. However, there is a limit to practical winglet length and so a limit to the roll moment available. Even with this restriction, this remains an interesting method of flight control.

Some researchers have worked on trialing edge and leading edge deformation generally for a continuous wing. This does allow a reduction of drag compared to conventional wings. However, a given roll moment is achieved with lower a actuating moment by twisting the entire wing than by rotating a leading edge or trailing edge discrete control surface.

Some of the twisting wing designs presented above, such as the vertebrae-like design, require a robust structural member to support the ribs along the length of the wing in order to resist deformation under aerodynamic loading. The design by Good et al required only a small central spar for their small aircraft. Other designs for small aircraft required only a reinforced membrane with no additional structure for wing support. However, small aircraft with longer wing spans and larger aircraft require a support
structure for their wings. This work addresses the design of the wing support structure that is capable of limiting deformation under aerodynamic loading while providing little resistance to wing twist under actuation. The next chapter presents the theory used to predict the lift and actuating moment on a wing in addition to the theory used to predict the pressure about the wing.
Chapter 3: Aerodynamic Theory for Lift and Pressure Distribution

The aerodynamic theory used for this research is presented in this chapter. Prandtl’s lifting line theory is used to show that twisting wings have the potential to offer the same roll moment as conventional control surfaces using less actuation force. Complex potential flow theory is used to predict the pressure distribution about the wing in order to predict the wing deflection under aerodynamic loading. First complex potential flow theory is presented. Then it is used to describe the flow about an airfoil. Finally, the flow predicted using complex potential flow theory is compared to the flow predicted by computational fluid dynamics.

Prandtl’s Theory

Prandtl’s lifting line theory is used to assess the lift produced by a wing [48, 49].

\[
\alpha(y_o) = \alpha_{L=0}(y_o) + \frac{r(y_o)}{\pi uc(y_o)} + \frac{1}{4\pi U} \int_{-b/2}^{b/2} \frac{(dr/dy)}{y_o-y} dy. \tag{1}
\]

In this equation the term on the left side of the equation is the geometric angle of attack. The first two terms on the right side of the equation represent the effective angle of attack and the last term is the induced angle of attack. This equation accounts for the reduction of the wing’s angle of attack due to the downwash induced by the trailing vortices caused
by the pressure differential between the top and bottom surfaces of the wing. A
transformation of variables can be applied to the position along the length of the wing

\[ y = -\frac{b}{2} \cos(\theta) \]  

The circulation can be represented by the odd terms of a Fourier series since the
circulation is zero at the ends of the wing

\[ \Gamma(\theta) = 2bU \sum_{n=1}^{N} A_n \sin(n\theta) \]  

With these substitutions, the lifting line equation becomes

\[ \alpha(\theta) = \alpha_{L=0}(\theta) + \frac{2b}{\pi c(\theta)} \sum_{n=1}^{N} A_n \sin(n\theta) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta)}{\sin(\theta)} \]  

Once the Fourier coefficients are realized, the lift, trailing vortex drag, and moment on
the wing can be found.

With this equation the viability of flight control by wing twisting can be tested. Some
initial calculations were performed using two-dimensional aerodynamic models.

Prandtl’s lifting line theory is used to estimate the aerodynamic lift on the wing and the
moment required to rotate the wing or flap to a certain angle. The non-dimensional lift
predicted by the lifting line theory is given by:

\[ c_\ell = 2\pi \alpha \]  

The non-dimensional moment required to rotate the wing to an angle, \( \alpha \), is given by

\[ c_m = \frac{\pi}{2} (\cos(\alpha) - 1)\alpha \]
Similar equations are found to estimate the lift and moment required to rotate a leading edge (LE) or trailing edge (TE) flap. These equations allow a comparison to be made between twisting of the entire wing and rotation of only a control surface. Figure 14 indicates that lift generated by rotating the entire wing is greater than that generated by rotation of a flap as one would expect.

Figure 14. Lift comparison between wing rotation and the use of trailing edge flaps

Figure 15 shows the moment required to rotate the wing or flap to a certain angle. Notice that the moment required to rotate the wing is less than that required to rotate a shorter control surface for small angles. This occurs since the wing is rotated about its center of
pressure. This means that the potential lift produced by rotation of the entire wing is less expensive than that produced by the flaps.

![Graph showing moment coefficient vs. angle for different flap lengths](image)

Figure 15. Moment required for wing or trailing edge flap rotation

This shows that there is potential merit for wing twisting rather than rotation of only the flaps. However, the lift created at the end of the wings contributes more significantly to roll motion due to the longer moment arm. Also, wing rotation would require a strong, heavy hinge where the wing is attached to the fuselage. So rather than wing rotation, wing twisting is proposed.
Complex Potential Flow Theory

The airflow around the wing is studied using complex potential flow theory. Potential flow theory assumes the flow is incompressible, inviscid, and irrotational.

Continuity and irrotational flow are described by:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \]  \hspace{1cm} (7)

respectively. Here \( u \) is the velocity in the horizontal direction and \( v \) is the velocity in the vertical direction. The streamlines and velocity potentials are defined by the following equations:

\[ u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \quad u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \] \hspace{1cm} (8)

Combining these definitions and equations given above, the flow can be described using streamlines and velocity potentials:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \] \hspace{1cm} (9)

where \( \psi \) is the streamline function and \( \phi \) is the velocity potential. Position in the \( z \) plane using complex Cartesian coordinates may be written as:

\[ z = x + iy \] \hspace{1cm} (10)
If $F$ is the complex potential given by

$$F(z) = \phi + i \psi$$  \hspace{1cm} (11)

then the complex velocity is given by:

$$\omega(z) = \frac{dF}{dz} = u - iv$$  \hspace{1cm} (12)

where $u$ and $v$ are the fluid velocity components in Cartesian coordinates and $u_r$ and $u_\theta$ are the fluid velocity components in polar coordinates.

In the following analysis, the cylinder is given in the $\zeta$ plane rather than the $z$ plane. So we need to change the $z$’s given above in the complex potentials to $\zeta$’s. Also, the circle is displaced from the origin and the uniform flow is at an angle $\alpha$ to the horizontal, the complex potential (now in the $\zeta$ plane). This simply changes the constants used to ensure that the streamline along the surface of the cylinder is zero. For this case the complex potential is given by:

$$F(\zeta) = U \left( (\zeta - \zeta_o) e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{\zeta - \zeta_o} \right) + i \frac{\Gamma}{2\pi} \ln \frac{\zeta - \zeta_o}{a}$$  \hspace{1cm} (13)

where $a$ is the radius of the cylinder and $\Gamma$ is the strength of the circulation. Note that changing the angle of the flow is equivalent to changing the angle of attack of the wing.
Uniform flow around an airfoil

The Joukowski transform is a conformal mapping given by $\zeta = f(z)$. This means that, given a complex potential in the $\zeta$ plane, the complex potential in the $z$ plane is given by the same complex potential after making use of the substitution, $\zeta = f(z)$. Complex velocities in the $z$ plane may be found by application of the chain rule:

$$\omega(z) = \frac{dF(z)}{dz} = \frac{dF(\zeta)}{d\zeta} \frac{d\zeta}{dz} = \frac{d\zeta}{dz} \omega(\zeta)$$  \hspace{1cm} (14)

The result is that the flow around an inconvenient shape (such as an airfoil) in the $z$ plane can be found by analyzing flow around a cylinder in the $\zeta$ plane.

The stagnation point near the rear of the airfoil is located at the rear tip since the momentum of the fluid does not allow it to whip around the sharp edge at the rear. This is shown in Figure 16 in both the $z$ plane and the $\zeta$ plane. The Kutta-Joukowski law handles this problem by adding circulation around the airfoil such that the stagnation point occurs at the trailing edge of the airfoil. This law sets the required value of the strength of the circulation and determines the lift on the airfoil. In order to set the circulation, the location of the trailing edge, $z_1$, must be found. The Joukowski transform is written as:

$$z = \zeta + \frac{c^2}{\zeta}.$$  \hspace{1cm} (15)

If $z_1$ is purely real, then $\zeta_1$ must be purely real (the parameter $c$ is real). The corresponding point in the $\zeta$ plane, $\zeta_1$, is given by...
\[ \zeta_1 = ae^{i\alpha} + me^{i\delta} = (a \cos \theta_p + m \cos \delta) + i(a \sin \theta_p + m \sin \delta) \quad (16) \]
\[ \zeta_1 = a \cos \theta_p + m \cos \delta \quad (17) \]
\[ a \sin \theta_p = -m \sin \delta \quad (18) \]

Figure 16. Locating the point corresponding to the trailing edge \( \zeta \) plane

Now that the location of the trailing edge is known, the velocity at that point is set to zero to force it to be a stagnation point. This means the complex velocity at this point, in either plane, must be zero. Taking the derivative of the complex potential with respect to \( \zeta \) yields:

\[ \omega(\zeta) = U \left( e^{-i\alpha} - \frac{a^2 e^{i\alpha}}{(\zeta - \zeta_o)^2} \right) + \frac{\Gamma}{2\pi} \frac{1}{\zeta - \zeta_o} \quad (19) \]

After some manipulation, it is found that in order for this to be zero at \( \zeta_l \), the following must be true:

\[ \Gamma = -4\pi U a \sin(\theta_p - \alpha) \quad (20) \]
With this information the complex velocities in the $\zeta$ plane may be found and from these the complex velocities in the $z$ plane can be found. The pressure distribution around the wing is then found using Bernoulli’s equation. The results using complex potential flow theory are shown in Figures 17 through 22 show the level of detail available using complex potential flow theory. Figures 17 and 18 show the streamlines and velocity potentials for the entire flow field about the wing. The pressure along the wing chord for both the top and bottom surfaces is shown in Figure 19. Figures 20, 21, and 22 show the pressure distribution about the wing for an angle of attack of 10, -10, and 0 degrees, respectively.

![Figure 17. Streamlines around the airfoil, angle of attack = 10°.](image-url)
Figure 18. Velocity potentials around the airfoil, angle of attack = 10°

Figure 19. Pressure distribution along the length of the airfoil, angle of attack = 10°
Figure 20. Pressure distribution around the wing, angle of attack = 10°

Figure 21. Pressure distribution around the airfoil, angle of attack = -10°
Next the accuracy of the model for the fluid flow around the airfoil must be considered. The Joukowski transformation is used to estimate the pressure distribution along the surface of the airfoil. This pressure distribution is then applied to the wing during finite element analysis in order to find the deformation of the wing. However, the Joukowski transformation uses complex potential flow theory and therefore assumes that the fluid flow can be adequately described using an inviscid, incompressible, laminar flow model. These assumptions allow the Navier-Stokes equations to be simplified considerably and allow quick estimation of the pressure distribution along the surface of the airfoil. At moderately high Reynolds numbers, the ratio of inertial forces to viscous forces is high and the assumption of inviscid flow becomes valid. The Mach number is the ratio of
inertial forces to elastic forces. So when this dimensionless number is small, the elastic forces are large compared to the inertial forces. Under these circumstances the flow can be thought of as incompressible.

While the flow around airfoils generally results in a high Reynolds number, the Mach number may not be small. For low flight speeds the use of complex potential flow theory remains valid. However, this is not the case at higher flight speeds. In order to assess the loss of accuracy due to the assumptions made by using complex potential flow theory, computational fluid dynamics (CFD) is used to estimate the pressure distribution around the airfoil. This CFD analysis is performed in ANSYS using the same NACA airfoil used in the previous analysis. The pressure distribution around the airfoil using CFD is expected to be more accurate but also more time-consuming. The pressure distributions obtained using CFD and complex potential flow theory are then compared in order to discern any inaccuracies incurred by the additional assumptions. The following CFD results were obtained assuming incompressible, turbulent, viscous flow. Eventually, the assumption of incompressible flow will be eliminated. This has not been done thus far simply due to time constraints. At present the pressure distributions are simply visually compared. Soon this comparison will additionally be made numerically by finding the aerodynamic moment due to the pressure distributions. The relation between the aerodynamic moments and the angle of attack for each method can then be compared.

The velocity vectors and pressure distribution for a 0° angle of attack are shown in Figures 23 and 24. Comparing the pressure distributions in figures 22 and 24, the same basic features are found. In both cases lower pressure is found along the upper surface of
the wing where the velocity increases as it is forced to go around the wing. This lower pressure region occurs toward front of the wing. A smaller low pressure zone is also predicted along front portion of the lower surface of the wing. Both models also predict a high pressure zone at the nose of the airfoil in the vicinity of the stagnation point. Complex potential flow theory predicts a higher pressure zone along the rear portion of the lower surface of the airfoil. This is not apparent from the CFD results.

Figure 23. Velocity vectors around the airfoil, angle of attack = 0°
Figure 24. Pressure distribution around the airfoil, angle of attack = 0°

The next two figures show the velocity vectors and pressure distribution around the wing for an angle of attack of 5°. Notice that both complex potential flow theory (Figure 20) and CFD (Figure 26) predict the same trends. For a positive angle of attack, there is an increased pressure along the lower surface of the airfoil and a lower pressure along the upper surface. In each case this effect is more prevalent along the front of the airfoil.
Figure 25. Velocity vectors around the airfoil, angle of attack =5°
Figures 27 and 28 show the velocity vectors and pressure distribution around the wing for an angle of attack of -5°. Comparing the pressure distribution using complex potential flow theory for a negative angle of attack shown in Figure 21 to the pressure distribution obtained using commercial software shown in Figure 28, the same trends are apparent. For a negative angle of attack, there is an increased pressure along the upper surface of the airfoil and a lower pressure along the lower surface. In each case this effect is more prevalent along the front of the airfoil.
Figure 27. Velocity vectors around airfoil, angle of attack = -5°
The preceding comparison shows the validity of applying complex potential flow theory in this situation. The pressure distribution predicted by complex potential flow theory can now be applied to a structural finite element model of the airfoil in order to assess its deformation under aerodynamic loading. The next chapter covers the finite element theory used in this analysis and the model used for the actuators.
In this chapter the finite elements implemented to model the wing structure are presented. In this work the matrix displacement approach is used for analysis of the skin and the passive and active beam members. The stiffness matrix for truss members is also presented since this is used for elements with limited extension. Since this theory is well established, its presentation is brief [50, 51]. Another method of generating the stiffness equations that is used to describe the behavior of actuated members is discussed. Following this the equations used to describe actuator behavior is presented. The next portion of the chapter shows how to handle different types of joints and moments applied to only one of the members at a joint. Then the method of solution is presented for both unlimited and limited actuator extension. Finally, the solution using the code developed here is compared to the solution from commercial software.

Elements Developed by the Matrix Displacement Approach

The shape functions for the skin were obtained by assuming a linear variation of displacement along the element dimensions and applying the boundary conditions. The stiffness matrix was then acquired by applying Castigliano’s theorem of virtual work to the strain energy expressions and inclusion of the shape functions. The resulting force
The deflection equation is given below using local coordinates with the x and y axes along the element sides. In this equation the forces, $F_i$, are related to the nodal displacements, $p_j$, by stiffness, $k_{ij}$.

\[
\begin{bmatrix}
\ddot{F}_1 \\
\ddot{F}_2 \\
\ddot{F}_3 \\
\ddot{F}_4
\end{bmatrix} =
\begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} \\
k_{12}^T & k_{22} & k_{14} & k_{24} \\
k_{13} & k_{14}^T & k_{11} & k_{12} \\
k_{14}^T & k_{24} & k_{12}^T & k_{22}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix}
\]

(21)

where

\[
\ddot{F}_i = \begin{bmatrix}
\ddot{F}_{xi} \\
\ddot{F}_{yi} \\
\ddot{F}_{zi}
\end{bmatrix}, \quad \ddot{p}_i = \begin{bmatrix}
\ddot{u}_i \\
\ddot{v}_i \\
\ddot{w}_i
\end{bmatrix}, \quad k_{11} = \begin{bmatrix}
c_1 & c_2 & 0 \\
c_2 & c_3 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad k_{12} = \begin{bmatrix}
c_4 & c_5 & 0 \\
c_5 & c_6 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(22)

\[
k_{22} = \begin{bmatrix}
c_1 & -c_2 & 0 \\
-c_2 & c_3 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad k_{13} = \begin{bmatrix}
\frac{-1}{2}c_1 & -c_2 & 0 \\
-c_2 & -\frac{1}{2}c_3 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad k_{14} = \begin{bmatrix}
c_7 & c_5 & 0 \\
-c_5 & c_8 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(23)

\[
k_{24} = \begin{bmatrix}
\frac{-1}{2}c_1 & c_2 & 0 \\
c_2 & -\frac{1}{2}c_3 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(24)

\[
c_1 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ 2 \frac{b}{a} + (1 - v) \frac{a}{b} \right\}, \quad c_2 = \frac{1}{8} \frac{Et}{1-v^2} \left\{ 2v + (1 - v) \right\}
\]

(25)

\[
c_3 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ 2 \frac{a}{b} + (1 - v) \frac{b}{a} \right\}, \quad c_4 = \frac{1}{12} \frac{Et}{1-v^2} \left\{ -4 \frac{a}{b} + (1 - v) \frac{a}{b} \right\}
\]

(26)

\[
c_5 = \frac{1}{8} \frac{Et}{1-v^2} \left\{ 2v - (1 - v) \right\}, \quad c_6 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ \frac{a}{b} - (1 - v) \frac{b}{a} \right\}
\]

(27)

\[
c_7 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ 2 \frac{b}{a} - (1 - v) \frac{a}{b} \right\}, \quad c_8 = \frac{1}{12} \frac{Et}{1-v^2} \left\{ -4 \frac{a}{b} + (1 - v) \frac{a}{b} \right\}
\]

(28)
and $a$ is the element length in the x-direction and $b$ is the length in the y direction.

The same procedure is used to develop the stiffness matrix for the beam elements. The force deflection relationship is given below using local coordinates with the x axis along the length of the beam. In this equation the forces, $F_i$, and moments, $M_i$, are related to the nodal displacements, $p_j$, and rotations, by stiffness, $k_{ij}$.

\[
\begin{bmatrix}
\bar{F}_1 \\
\bar{F}_2 \\
\bar{M}_1 \\
\bar{M}_2
\end{bmatrix} = 
\begin{bmatrix}
k_{11} & -k_{11} & k_{13} & k_{13} \\
-k_{11} & k_{11} & k_{23} & k_{23} \\
k_{13} & k_{23} & k_{33} & k_{34} \\
k_{13} & k_{23} & k_{34} & k_{33}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]

(29)

where

\[
\bar{F}_i = \begin{bmatrix}
\bar{F}_{xi} \\
\bar{F}_{yi} \\
\bar{F}_{zi}
\end{bmatrix}, \quad \bar{p}_i = \begin{bmatrix}
\bar{u}_i \\
\bar{v}_i \\
\bar{w}_i
\end{bmatrix}, \quad \theta_i = \begin{bmatrix}
\theta_{xi} \\
\theta_{yi} \\
\theta_{zi}
\end{bmatrix}, \quad k_{11} = \begin{bmatrix}
c_1 & 0 & 0 \\
0 & 12c_2 & 0 \\
0 & 0 & 12c_3
\end{bmatrix}, \quad k_{13} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 6Lc_2 \\
0 & -6Lc_3 & 0
\end{bmatrix}
\]

(30)

\[
k_{23} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -6Lc_2 \\
0 & 6Lc_3 & 0
\end{bmatrix}, \quad k_{33} = \begin{bmatrix}
c_4 & 0 & 0 \\
0 & 4L^2c_3 & 0 \\
0 & 0 & 4L^2c_2
\end{bmatrix}, \quad k_{34} = \begin{bmatrix}
-c_4 & 0 & 0 \\
0 & 2L^2c_3 & 0 \\
0 & 0 & 2L^2c_2
\end{bmatrix}
\]

(31)

\[
c_1 = \frac{EA}{L}, \quad c_2 = \frac{EI_y}{L^2}, \quad c_3 = \frac{EI_z}{L^3}, \quad c_4 = \frac{JG}{L}
\]

(32)
The above force deflection relationship may also be written in global coordinates after transformation as shown below. This is detailed in the appendix. This form, shown below, will be helpful in later explanations [50, 51].

\[
\begin{bmatrix}
F_1 \\ F_2 \\ M_1 \\ M_2
\end{bmatrix} =
\begin{bmatrix}
G_A & G_B \\ G_C & G_D
\end{bmatrix}
\begin{bmatrix}
p_1 \\ p_2 \\ \theta_1 \\ \theta_2
\end{bmatrix}
\] (33)

The force displacement equations for bar elements may also be developed using the displacement formulation. A force balance at the nodes results in the following equation.

\[
\sum F = 0 = \vec{F}_i + \sum_j f_j \vec{c}_{o_j}
\] (34)

or

\[
\vec{F}_i = -\sum_j f_j \vec{c}_{o_j}
\] (35)

where

\[
\vec{F}_i = \text{externally applied force at node } i
\]

\[
f_j = \text{axial force along member } j \text{ connected to node } i, \text{ tension is positive}
\]

\[
\vec{c}_{o_j} = \text{unit vector along undeflected member } j
\]

This can be put into matrix form as written below.

\[
F = -Bf
\] (36)
Notice that if member \( j \) has nodes \( n \) and \( m \), then the \( B \) matrix will have non-zero entries in column \( j \) in rows \( n \) and \( m \) (Figure 29). The constitutive equations can be written for each passive member as shown below.

\[
F_j = k_j d_j = \frac{E_j A_j}{L_j} d_j
\]  

(37)

where

- \( k_j = \) stiffness of member \( j \)
- \( d_j = \) deflection of member \( j \)
- \( E_j = \) modulus of elasticity for member \( j \)
- \( A_j = \) cross sectional area of member \( j \)
- \( L_{o_j} = \) initial length of member \( j \)
- \( L_j = \) length of deformed member \( j \)
- \( \vec{L}_{o_j} = \vec{c}_{o_j} L_{o_j} = \) vector along undeformed member \( j \) from node \( m \) to node \( n \)
- \( \vec{L}_j = \vec{c}_j L_j = \) vector along deformed member \( j \) from node \( m' \) to node \( n' \)

Finally, the compatibility equations relating the member deflections to the nodal displacements can be written. In order to develop the compatibility equations, it is helpful to consider member \( j \) with nodes \( m \) and \( n \) before and after deflection.
After loading, nodes m and n move to m' and n' along vectors $\vec{u}_n$ and $\vec{u}_m$. The member deflection is given below.

$$d_j = L_j - L_{o_j} \tag{38}$$

Using vector addition, the deformation of member j can be written as:

$$\vec{L}_{o_j} + \vec{u}_n = \vec{L}_j + \vec{u}_m \tag{39}$$

These equations can be combined to solve for the member deflection in terms of the nodal displacements.

$$d_j = \vec{c}_{o_j} \cdot \vec{u}_n - \vec{c}_{o_j} \cdot \vec{u}_m \tag{40}$$

This can be re-written in matrix form.

$$d = -B^T u \tag{41}$$
Combining the force balances, constitutive equations, and compatibility equations, results in the following [52].

\[ f = -BF = -Bkd = BkB^Tu = Gu \]  \tag{42}

**Actuators**

The force deflection equations for active members use a modified form of the displacement formulation given above. The constitutive equations for piezoelectric material are given below.

\[ S = s_E T + d^T E \] \tag{43}

\[ D = dT + \varepsilon_T E \] \tag{44}

where

- \( S \) = strain
- \( T \) = stress
- \( E \) = electric field
- \( D \) = electric displacement
- \( s_E \) = compliance under zero electric field
- \( \varepsilon_T \) = permittivity under zero stress

Applying this equation to a one dimensional structure, a piezoelectric stack of \( n \) layers, the equation can be written as follows.
\[ F = kx + rV \]  
\[ k = \frac{A}{s_{33}L} \]  
\[ r = \frac{nAd_{33}}{s_{33}L} \]  

where

\( F \) = actuation force  
\( x \) = actuator displacement  
\( V \) = voltage across piezoelectric material layer  
\( k \) = piezoelectric stiffness  
\( r \) = piezoelectric constant  
\( A \) = cross sectional area of piezoelectric material  
\( L \) = length of piezoelectric stack  
\( n \) = number of layers in piezoelectric stack  
\( s_{33} \) = piezoelectric compliance in the 3 direction  
\( d_{33} \) = piezoelectric charge constant in the 3 direction

The model for the hydraulic cylinder systems is shown in the figure below. This model accounts for fluid viscosity and flow restrictions with fluid resistance, \( R \). Any compliance in the fluid or structural flexibility of the piping is accounted for by the compliance, \( C \).
Using conservation of mass and assuming incompressible flow, the volumetric flow rate into the system, $q$, can be related to the cylinder pressure, and therefore piston force, and the flow rate into the cylinder, which is related to the motion of the piston as given in the following equations.

$$q - Ad = C\dot{p}_1$$  \hspace{1cm} (48)

$$F = Ap_1$$  \hspace{1cm} (49)

where

$q =$ volumetric flow rate into the system

$p_1 =$ cylinder pressure

$F =$ piston force

$d =$ piston displacement

$A =$ cross sectional area of the hydraulic cylinder

$C =$ system compliance
These equations can be re-arranged into the same form as the constitutive equations for the piezoelectric material if we make the following definitions and assume that the compliance is constant.

\[ k = \frac{A^2}{c} \]  \hspace{1cm} (50)

\[ r = \frac{A}{c} \]  \hspace{1cm} (51)

The last actuator considered is a two member linkage driven by a DC motor shown in the figure below. The relationship between the torque provided by the DC motor and the applied voltage is assumed to be linear. In this figure, member a has ends at nodes 1 and 2 and member b has ends at nodes 2 and 3. The DC motor at node 2 is attached to members a and b such that equal and opposite torques are applied to each member.

Figure 31. Two member linkage driven by a DC motor
Since the DC motor applies torques directly to members a and b, the same matrix equation given for passive members may be applied here. One modification must be made, however, since the moment is applied only to members a and b, not the rest of the members connected at nodes 1 and 3. This modification is given below.

Allowing Different Connection Types

In order to allow greater deflection for an applied actuator force, it is desired to allow revolute joints with one degree of freedom and spherical joints with three degrees of freedom. In order to preserve the airfoil shape and resist deflection under aerodynamic loading, welds with zero degrees of freedom are used. An additional consideration must be made for the two bar linkage driven by a DC motor. The simple two dimensional example structure shown below allows consideration of these complications. In this schematic, the nodes are numbered, the members are lettered, and the joints types are indicated.

\[ J_w = \text{weld joint} = 0 \text{ dof} \]
\[ J_r = \text{revolute joint} = 1 \text{ dof} \]
\[ J_s = \text{spherical joint} = 3 \text{ dof} \]

Figure 32. Example structure

The global structural equations is written below.
\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix} = 
\begin{bmatrix}
G_A \\
G_C
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix} + 
\begin{bmatrix}
G_B \\
G_D
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4
\end{bmatrix}
\]

(52)

At node 4, member D does not allow rotation while member B does allow rotation. There is no way to apply both of rules at the same time to the global matrix equations. Instead, these rules must be applied to the individual matrix equation for each member.

Next consider the matrix equation for member A given below.

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2
\end{bmatrix} = 
\begin{bmatrix}
G_{11} & G_{12} & G_{13} & G_{14} \\
G_{21} & G_{22} & G_{23} & G_{24} \\
G_{31} & G_{32} & G_{33} & G_{34} \\
G_{41} & G_{42} & G_{43} & G_{44}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix} + 
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]

(53)

Node one is fixed. It cannot be displaced and member A is welded to it. In order to enforce this, \(p_1\) and \(\theta_1\) could be set to zero, or equivalently, the stiffness coefficients in columns one and three could be set to zero. This means that force \(F_I\) and moment \(M_I\) are reactions and there is no need to solve for them. So we can set the coefficients of rows one and three to zero.

Next consider member B whose individual matrix equation is given below.

\[
\begin{bmatrix}
F_2 \\
F_4 \\
M_2 \\
M_4
\end{bmatrix} = 
\begin{bmatrix}
G_{11} & G_{12} & G_{13} & G_{14} \\
G_{21} & G_{22} & G_{23} & G_{24} \\
G_{31} & G_{32} & G_{33} & G_{34} \\
G_{41} & G_{42} & G_{43} & G_{44}
\end{bmatrix}
\begin{bmatrix}
p_2 \\
p_4 \\
p_4 \\
\theta_2
\end{bmatrix} + 
\begin{bmatrix}
\theta_1 \\
\theta_4
\end{bmatrix}
\]

(54)
Node four is fixed, so we can set the coefficients of column two to zero to enforce this condition. Notice that when the individual matrix equation for member B is incorporated into the global matrix equation, the deflection at node four, $\theta_4$, this forces members B and D to have the same rotation. This problem can be handled two different ways. The first way to allow different rotations from members B and D is to separate the moment equations for the two members.

To detail this a bit, consider the situation depicted in the figure above. Members A and B are welded together and members C and D are welded together, but the welded pairs are connected by a revolute joint. The force deflection matrix equations in simplified form are given below for each member.

For member A:

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]

(55)

For member B:
\[
\begin{bmatrix}
F_1 \\
F_3 \\
M_1 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
B_{11} & B_{12} & B_{13} & B_{14} \\
B_{21} & B_{22} & B_{23} & B_{24} \\
B_{31} & B_{32} & B_{33} & B_{34} \\
B_{41} & B_{42} & B_{43} & B_{44}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_3 \\
\theta_1 \\
\theta_3
\end{bmatrix}
\] (56)

For member C:

\[
\begin{bmatrix}
F_1 \\
F_4 \\
M_1 \\
M_4
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{23} & C_{24} \\
C_{31} & C_{32} & C_{33} & C_{34} \\
C_{41} & C_{42} & C_{43} & C_{44}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_4 \\
\theta_1 \\
\theta_4
\end{bmatrix}
\] (57)

For member D:

\[
\begin{bmatrix}
F_1 \\
F_5 \\
M_1 \\
M_5
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & D_{13} & D_{14} \\
D_{21} & D_{22} & D_{23} & D_{24} \\
D_{31} & D_{32} & D_{33} & D_{34} \\
D_{41} & D_{42} & D_{43} & D_{44}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_5 \\
\theta_1 \\
\theta_5
\end{bmatrix}
\] (58)

Since members A and B are welded together, they rotate together at \(\theta_{1,1}\). Since members C and D are welded together, they rotate together at \(\theta_{1,2}\). The force balance at node 1 can be written as follows.

\[
F_1 = A_{11}p_1 + A_{12}p_2 + A_{13}\theta_{1,1} + A_{14}\theta_2 + B_{11}p_1 + B_{12}p_3 + B_{13}\theta_{1,1} + B_{14}\theta_3 \\
C_{11}p_1 + C_{12}p_4 + C_{13}\theta_{1,2} + C_{14}\theta_4 + D_{11}p_1 + D_{12}p_5 + D_{13}\theta_{1,2} + D_{14}\theta_5
\] (59)

In order to keep the moments applied to members A and B separate from the moments applied to members C and D, the moment equations are split up into the following two equations.
Each additional equation also contains an additional unknown variable, so the matrix equations are solved in the usual manner.

The second way to allow different rotations for connected members is to solve the individual equations for the unconstrained degrees of freedom and use these equations to eliminate that degree of freedom from the individual matrix equations. In this case, that means that the individual matrix equation for member B will have a column of zeros in the fourth column such that when it is introduced into the global matrix equation, the rotation of member B can differ from that of member D.

The last consideration that needs to be addressed occurs when a moment is applied to some, but not all, members connected at a node. Since the moment is not applied to all members connected to that joint, the moment is brought to the right hand side of the equation, resulting in the following matrix equation that assumes linear behavior.

\[
\begin{bmatrix}
F \\
M
\end{bmatrix} = \begin{bmatrix}
G_A & G_B \\
G_C & G_D
\end{bmatrix} \begin{bmatrix}
p \\
\theta
\end{bmatrix} + \begin{bmatrix}
G_p & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
p \\
\theta
\end{bmatrix} + \begin{bmatrix}
R \\
0
\end{bmatrix} V + \begin{bmatrix}
A_B \\
A_D
\end{bmatrix} M_a
\]

where

\( F = \) nodal force vector

\( M = \) nodal moment vector

\( p = \) nodal displacement vector
\( \theta \) = nodal rotation vector

\( G_A, G_B, G_C, G_D \) = beam stiffness matrix

\( G_I \) = skin stiffness matrix

\( R \) = actuation matrix

\( V \) = actuator vector

\( A_B, A_D \) = moment matrix

\( M_0 \) = moment applied to specific member

**Matrix solution with unlimited extension**

The force balance equations for the wing structure is given above. The matrix equation is written in this way to reduce the size of the matrices for a given set of nodes. This allows the program to handle a larger number of nodes and reduces the calculation time for matrix inversion. However, with the stiffness matrix split in this manner, there can be rows and columns of zeros. This can occur in \( G_B, G_C, \) and \( G_D \) if all the members connected to a particular node are of the spherical type. All of the matrices can have rows and columns of zeros if the system is two dimensional. The most complicated case occurs when \( G_A \) is full and the number of columns of zeros in \( G_D \) is not equal to the number of columns of zeros in \( G_B \). The number of rows of zeros in \( G_B \) is always greater than or equal to the number of zeros in \( G_A \) since all joints transmit axial forces, but not necessarily moments. The number of columns of zeros in \( G_B \) is always equal to or less than the number of columns of zeros in \( G_D \) since \( k_{13} \) and \( k_{23} \) has a column of zeros while \( k_{23} \) and \( k_{33} \) do not. The matrix equations and the nodal displacements are re-ordered such
that the columns and rows of zeros are at the right and bottom of the matrices. The force balance equations may then be re-written as follows if the truss structure stiffness and plate stiffness are combined: $G_A = G_A + G_p$.

$$F = G_A p + G_B \theta + RV + A_B M_A$$  \hspace{1cm} (63)$$

$$M = G_C p + G_D \theta + A_D M_A$$  \hspace{1cm} (64)$$

or

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} G_{A11} & G_{A12} & G_{A13} \\ G_{A21} & G_{A22} & G_{A23} \\ G_{A31} & G_{A32} & G_{A33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} G_{B11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$  \hspace{1cm} (65)$$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} G_{C11} & 0 & 0 \\ 0 & G_{D11} & G_{D12} \\ 0 & G_{D21} & G_{D22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & G_{D21} & G_{D22} \\ 0 & G_{D31} & G_{D32} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} A_{D11} & A_{D12} & A_{D13} \\ A_{D21} & A_{D22} & A_{D23} \\ A_{D31} & A_{D32} & A_{D33} \end{bmatrix} \begin{bmatrix} M_{A1} \\ M_{A2} \\ M_{A3} \end{bmatrix}$$  \hspace{1cm} (66)$$

This can be simplified to the following equation from which the nodal displacements may be solved.

$$F_p = T p + RV + A_p M_a$$  \hspace{1cm} (67)$$

where
\[ F_p = \begin{bmatrix} F_1 - G_{B11}L_1^{-1}M_p \\ F_2 \\ F_3 \end{bmatrix} \]  

(68)

\[ T = \begin{bmatrix} G_{A11} - G_{B11}L_1^{-1}G_{C11} & G_{A12} & G_{A13} \\ G_{A21} & G_{A22} & G_{A23} \\ G_{A31} & G_{A32} & G_{A33} \end{bmatrix} \]  

(69)

\[ A_p = \begin{bmatrix} A_{B11} - G_{B11}L_1^{-1}J_1 & A_{B12} - G_{B11}L_1^{-1}J_2 & A_{B13} - G_{B11}L_1^{-1}J_3 \\ A_{B21} & A_{B22} & A_{B23} \\ A_{B31} & A_{B32} & A_{B33} \end{bmatrix} \]  

(70)

\[ M_p = M_1 - G_{D12}G_{D22}^{-1}M_2 \]  

(71)

\[ L_1 = G_{D11} - G_{D12}G_{D22}^{-1}G_{D21} \]  

(72)

\[ J_1 = A_{D11} - G_{D12}G_{D22}^{-1}A_{D21} \]  

(73)

\[ J_2 = A_{D12} - G_{D12}G_{D22}^{-1}A_{D22} \]  

(74)

\[ J_3 = A_{D13} - G_{D12}G_{D22}^{-1}A_{D23} \]  

(75)

*Matrix solution with limited extension*

The maximum extension or contraction of hydraulic cylinders is limited by the cylinder length. If the predicted extension exceeds this limit, then the extension is set to the limit and the resulting nodal displacements are calculated. This is accomplished by splitting the stiffness matrix into two matrices. One matrix for active members and one for passive members.
\[ F = G_{Aa}p + G_{Ap}p + G_{b}\theta + RV + A_{b}M_{A} \]  \hspace{1cm} (76)

\[ M = G_{c}p + G_{D}\theta + A_{D}M_{A} \]  \hspace{1cm} (77)

where

\( G_{Aa} = \) stiffness matrix for active members (hydraulic cylinders)

\( G_{Ap} = \) stiffness matrix for passive members

The hydraulic cylinders are attached using spherical joints, so their stiffness is purely axial. This axial stiffness can be written as follows.

\[ G_{Aa}u = (-BK)(-B^{T})u = Hd \]  \hspace{1cm} (78)

where

\[ H = -BK \]  \hspace{1cm} (79)

\[ d = -B^{T}u = Au \]  \hspace{1cm} (80)

\[ A = -B^{T} \]  \hspace{1cm} (81)

Since the term \( Hd \) does not show up in the moment equations, it is unaltered when the moment equations are combined with the force equations. The definitions for \( F_{p} \), \( T \), and \( A_{p} \) are given above.

\[ F_{p} = Tp + RV + A_{p}M_{A} + Hd \]  \hspace{1cm} (82)

The applied forces and moments are combined.
\[ F_c = F_p - A_p M_a = Tp + RV + Hd \] (83)

Next the nodal displacement and actuation vectors are ordered such that exceeded length members are at the bottom.

\[
\begin{bmatrix}
F_{c1} \\ F_{c2}
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
p_1 \\ p_2
\end{bmatrix}
+ \begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2
\end{bmatrix}
+ \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
d_1 \\ d_2
\end{bmatrix}
\] (84)

\[
\begin{bmatrix}
d_1 \\ d_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
p_1 \\ p_2
\end{bmatrix}
\] (85)

Using the known applied actuation, \( V_1 \), and the known exceeded length member set to the limit, the unknown applied actuation can be found using the following equations.

\[ V_2 = D_2^{-1}(F_{c1} - C_1 C_2^{-1} F_{c2} - D_3 d_2 - D_4 A_{22}^{-1} d_2 - D_1 V_1) \] (86)

where

\[ B_1 = H_{11} A_{11} + T_{11} \] (87)

\[ B_2 = H_{11} A_{12} + T_{12} \] (88)

\[ B_3 = H_{21} A_{11} + T_{21} \] (89)

\[ B_4 = H_{21} A_{12} + T_{22} \] (90)

\[ C_1 = B_1 - B_2 A_{11}^{-1} A_{21} \] (91)

\[ C_2 = B_3 - B_4 A_{22}^{-1} A_{21} \] (92)
\[ D_1 = R_{11} - C_1 C_3^{-1} R_{21} \quad (93) \]

\[ D_2 = R_{12} - C_1 C_3^{-1} R_{22} \quad (94) \]

\[ D_3 = H_{12} - C_1 C_3^{-1} H_{22} \quad (95) \]

\[ D_4 = B_2 - C_1 C_3^{-1} B_4 \quad (96) \]

**Solution Verification**

The finite element code was verified by comparing the predicted deflections. First, simple passive structures were tested and are shown in Figures 34 – 36. In each case, the fixed nodes are shown in blue, while the free nodes are shown in magenta. A force is applied to the structure and, while the deflections predicted by the finite element code and the commercial software were compared for each free node, the deflections of only one node is presented. A symmetric load was applied, so only the vertical nodal displacement is presented. The nodal displacements predicted by the commercial software are virtually identical to the estimate from the code developed here as shown in Table 1.
Figure 34. Test structure A

Figure 35. Test structure B
Next, one member of the passive test structures was replaced by a piezoelectric actuator. Again the nodal displacements for only one node are shown. In this case, the loading is not symmetric, so the nodal displacement in all three directions is presented in Table 2. Again, there is good agreement between the displacement predicted by the commercial software and the code developed here.
The final comparison used the same test structures with one member replaced by a two-bar linkage driven by a DC motor as previously shown. Again, the nodal displacements predicted by the code developed for this work and the commercial software were in agreement indicating the veracity of the code developed here. The results are shown in Table 3.

Table 2. Nodal deflections for test structures with piezoelectric elements

<table>
<thead>
<tr>
<th>case</th>
<th>this code</th>
<th></th>
<th></th>
<th>abaqus</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
<td>z</td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>A</td>
<td>0.0003659</td>
<td>0.0006337</td>
<td></td>
<td>0.0003659</td>
<td>0.0006337</td>
</tr>
<tr>
<td>B</td>
<td>0.0000000</td>
<td>-0.0008637</td>
<td></td>
<td>0.0000000</td>
<td>-0.0008637</td>
</tr>
<tr>
<td>C</td>
<td>0.0000607</td>
<td>-0.0010160</td>
<td>0.0000000</td>
<td>0.0000607</td>
<td>-0.0010158</td>
</tr>
</tbody>
</table>

Table 3. Nodal deflections for test structures with DC motor actuation

<table>
<thead>
<tr>
<th>case</th>
<th>this code</th>
<th></th>
<th></th>
<th>abaqus</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
<td>z</td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>A</td>
<td>-0.039284</td>
<td>0.039284</td>
<td></td>
<td>-0.039284</td>
<td>0.039284</td>
</tr>
<tr>
<td>B</td>
<td>0.111100</td>
<td>0.000000</td>
<td></td>
<td>0.111111</td>
<td>0.000000</td>
</tr>
<tr>
<td>C</td>
<td>0.062650</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.062652</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
In this chapter the techniques used for structural optimization for the wing are presented. Initially, gradient based optimization techniques were implemented, but it was quickly apparent that the numerous local minima made this method inconvenient. Simulated Annealing was then implemented due to its simple coding and its ability to jump out of local minima and approach the global minimum.

During the actual process of annealing, material is heated to a temperature high enough such that the atoms are in a random state and are able to re-arrange. The material is then cooled slowly enough so that the Boltzmann distribution collapses and the atoms reach a minimum energy state (i.e. a global minimum). The same principle was applied to optimization by Cerby and Kirkpatrick et al. [53, 54]. For this method, the probability of decreasing the energy state, $p$, is given by the following equation.

$$p = \exp \left( -\frac{\Delta E}{T} \right)$$  \hspace{1cm} (97)

Notice that for a given energy (cost) change, the probability of accepting a solution with a greater cost in order to escape a local minimum decreases as the temperature decreases.

The program structure, shown in Figure 37, is outlined as follows. A random initial solution is chosen and its cost is computed and from this information, the initial
temperature is calculated by setting the initial probability, $p_o$, to 0.8 and using the following equation.

$$T_o = -\frac{\Delta E}{\ln(p_o)} = -\frac{\Delta E}{\ln(0.8)}$$  \hspace{1cm} (98)
The solution is then perturbed and a new solution is found. The cost of that solution is found and the solution is accepted if the probability of the energy (cost) change, $p$, is greater than a random value. Equation 97 indicates that solutions that cause a cost increase are more likely to be accepted at higher temperatures. This procedure is repeated as the temperature is slowly decreased causing a corresponding decrease in the probability that a new energy state is accepted.

Lester Ingber’s Adaptive Simulated Annealing uses a cooling schedule given in the following equation [55].

$$T = T_0 \exp\left(-cn^{1/D}\right)$$  \hspace{1cm} (99)

This equation indicates the temperature $T$ at iteration $n$ for a given state space size $D$.

Notice that $c$ is a control parameter that affects the rate at which the temperature decreases. In this procedure, the temperature affects both the acceptance probability and the perturbation neighborhood. The equation for regulating perturbation follows

$$y = \text{sign} \left(u - \frac{1}{2}\right) T \left[\left(1 + \frac{1}{T}\right)^{|2u-1|} - 1\right]$$ \hspace{1cm} (100)

where $u$ is a random number between 0 and 1. The perturbed solution is then given in Equation 101.

$$A_{new} = A_{old} + y(A_{max} - A_{min})$$ \hspace{1cm} (101)
Figure 38 shows that at high temperatures, the relationship between random number, $u$, and perturbation, $y$, is roughly linear, indicating that all values of $y$ are equally probable. This means that all perturbations within the range of variable A are equally probable. However, at lower temperatures the relationship is nonlinear, indicating that while the perturbation still varies from -1 to 1, smaller values of $y$, and hence smaller perturbations of variable A, are much more likely. So as the temperature decreases, smaller perturbation are more likely [55].

Figure 38. Dependence of perturbation on temperature

At the high temperatures present during the initial portion of the search, Equations 100 and 101 state that the search region is large but as the temperature decreases the search region becomes smaller. Equation 97 states that at high temperatures the ability to climb uphill and accept solutions with a higher cost is more likely, but as the temperature
decreases, this becomes less likely. So the control parameter, temperature, describes both the perturbation neighborhood and ability to climb uphill and so may be thought of as an acceptable move “bubble” about the current solution. At the initial high temperatures, since the perturbation range is large, many solutions are explored and are likely to be accepted since the probability, \( p \), is high. As the temperature decreases, a smaller region is explored and the current accepted solution becomes stuck in valleys. The ability to escape these valleys decreases as the temperature decreases. Because of this initial exploration phase and a decreasing ability to climb hills, the solution eventually becomes stuck very close to a global minimum.

In this work, because of the number of optimization routines, Ingber’s adaptive simulated annealing scheme was altered in order to achieve an “optimal” solution more quickly. The temperature was lowered much more quickly using the following equation.

\[
T = T_o \exp(-cn)
\]  \hspace{1cm} (102)

The drawback is that the global minimum is not achieved. Since the temperature is lowered so quickly, it is more likely to get stuck in a local minimum. In an attempt to avoid this, the system is re-annealed by running an optimization routine numerous times as outlined below. Because the temperature now drops so quickly, it is important to ensure that large perturbations are likely to occur while the initial optimization iterations are performed. In order to accomplish this, the temperature is set to \( T_i \) at iteration \( n_i \) using Equation 103.

\[
T_1 = T_o \exp(-cn_1)
\]  \hspace{1cm} (103)
At temperatures above 1, the relationship between $y$ and $T$ is nearly linear, but when the
temperature drops below 1, large deviations from a linear relationship begins to show.
Because of this, $T_i$ is set to 1 after 200 iterations. In order to maintain an initial high
probability of acceptance for solutions with an increased cost, Equation 97 is altered to
the following equation.

$$p = \exp \left( -\frac{\Delta E}{aT} \right)$$  \hspace{1cm} (104)

Parameter $a$ is found by setting $p = p_o$ to 0.8 at $T_o$.

**Optimization routines**

Structural optimization is achieved by incorporating a few different techniques. These
can be broken down into two categories: discrete and continuous variables. These
methods all use simulated annealing and are described in the following paragraphs,
beginning with the discrete variables.

A truss is used to provide structure and actuation for the wing. The first technique moves
the active members around within this truss structure in order to find the best location for
the actuator. In this method, one active member and one passive member are chosen
randomly within the structure and their roles are switched. The second technique
removes members from the truss structure if that member is not beneficial to either wing
twisting or resistance to deflection under aerodynamic loading. Any members may be
considered for removal except those that are active and those that support the wing skin.
Another set of code adds a passive member between existing nodes where no member
currently exists. Once the new member has been chosen, an existing active member is also randomly selected. Then three tests are performed. First the cost of the structure with the new passive member is evaluated. Then the roles of the randomly selected active and passive members are switched and the cost is evaluated with positive and negative actuation. Then the best design of the three is chosen. Another set of code removes nodes along with all the members connected to that node. The last discrete set of code changes the joint type of randomly selected individual members from its current type to a randomly selected type.

The techniques for continuous variables are described next. The first one varies the depth of the beam while maintaining the cross sectional area of the beam. The second varies the cross sectional area of the beams making up the truss structure. The third set of code varies the location of any interior node in the structure. Another application varies the degree of actuation. The last method is only used when DC motors actuate the structure.

Referring to Figure 31, member a is set perpendicular to the axis through nodes 1 and 3 such that the DC motor can rotate an equal amount clockwise and counterclockwise. This set of code varies the length of the member a in the two bar linkage powered by the dc motor.

The overall optimization scheme is shown in Figure 39. After the initial model is constructed, simulated annealing is applied to find the best actuator location. This is run first since the active structure tends to adapt itself to the current actuator location. The other optimization routines are then run until there are no further improvements. Finally, nodal elimination is attempted to reduce the complexity of the structure. This routine is
run last and less often since nodal construction was not performed and there are rarely any nodal removals made.

Figure 39. Overall optimization scheme
Cost function

The optimization routine provides the method of achieving the optimal solution. The cost function provides a mathematical description of the optimal solution. Therefore describing the optimal solution mathematically is very important. The initial cost function used in this work is based on the work by other researchers [56-59].

\[
    f = -\frac{MPE}{SE} = -\frac{v^T Tu}{w^T Tw} \tag{105}
\]

where:

\( T \) = effective stiffness matrix
\( v \) = nodal displacement during ideal twisting
\( u \) = actual nodal displacement
\( w \) = nodal displacement under aerodynamic loading

Without external loading the simplified equation from chapter 4 may be written as follows.

\[
    0 = Tu + RV \tag{106}
\]

Minimizing this cost function occurs when MPE is maximum and SE is at a minimum. Minimizing SE occurs when the deflection under aerodynamic loading is minimized. MPE is at a maximum when the actuator force direction matches ideal twisting deformation. However, this cost function does not attempt to preserve airfoil shape. Typical results are shown in Figure 40. The airfoil shape is not preserved under actuation.

72
The cost function was then altered to the following equation.

\[ f = -\frac{v^T u}{w^T w} \]  \hspace{1cm} (107)

This equation again attempts to minimize deformation under aerodynamic loading when the denominator is a minimum. The numerator is maximized when the actual deformation matches the ideal deformation, but it simply strives for the actual deformation to be large at the same nodes and directions that have large deformation under ideal deformation. It is not necessary to preserve the airfoil shape.

After a number of trials, the following cost function yielded good results.

\[ f = \sum_{i=1}^{N} c_i e_i \]  \hspace{1cm} (108)

and for the first five individual errors:

\[ e_i = \frac{p_i^T p_i}{N} \]  \hspace{1cm} (109)
where:

\[ c_i = \text{weighting coefficient} \]

\[ e_i = \text{individual error} \]

\[ N = \text{number of ungrounded nodes} \]

The first individual error minimizes deformation of the surface nodes under aerodynamic loading.

\[ p_1 = m_u w \quad (110) \]

where:

\[ w = \text{nodal displacement under aerodynamic loading} \]

\[ m_u = \begin{cases} 1 & \text{for ungrounded surface nodes} \\ 0 & \text{for other nodes} \end{cases} \]

The second individual error measures the degree of difference between the actual and ideal deformation by setting:

\[ p_2 = (u - v)m_u \quad (111) \]

The third individual error attempts to enforce a linear variation of twist along the length of the wing. This error is applied to each node.

\[ p_3 = \theta_{\text{end}} n_{z_i} - \theta_i \quad (112) \]

where

\[ \theta_{\text{end}} = \text{angle of attack at the wing tip} \]
\[ \theta_i = \text{angle of attack at rib } i \]

\[ n_{zi} = \text{rib } i \text{ (numbered from the fuselage to the wing tip) } \]

\[ n_z = \text{number of ribs along the length of the wing} \]

The fourth and fifth individual error measurements emphasize a linear variation of the twist angle at the leading and trailing edge along the length of the wing.

\[ p_4 = \theta_{LE,end} \frac{n_{zi}}{n_z} - \theta_{LE,i} \] (113)

\[ p_5 = \theta_{TE,end} \frac{n_{zi}}{n_z} - \theta_{TE,i} \] (114)

\[ \theta_{LE,end} = \text{angle of attack for the node at the leading edge at the wing tip} \]

\[ \theta_{LE,i} = \text{angle of attack for the node at the leading edge at rib } i \]

\[ \theta_{TE,end} = \text{angle of attack for the node at the trailing edge at the wing tip} \]

\[ \theta_{TE,i} = \text{angle of attack for the node at the trailing edge at rib } i \]

The sixth individual error attempts to minimize the number of members in order to minimize the complexity of the wing.

\[ e_6 = N_m = \text{number of members} \] (115)

The seventh individual error attempts to minimize the weight of the wing structure by summing the cross sectional area of the members.

\[ e_7 = \sum_{i=1}^{N_m} A_i \] (116)
The last individual error is a measure of the deviation between the actual and desired angle of attack, $\alpha$.

$$e_8 = (\theta_{\text{end}} - \alpha)^2$$  \hspace{1cm} (117)

*Effect of method of calculation of twist angle*

Even after the improved results using the revised cost function given by Equation 108, some problems remained. Note that at this time only the first, second, and eighth error terms existed in the cost function (Equations 110, 111, and 117). The weighting coefficients were set to $c_i = [1 \ 1 \ 100]$ in order to emphasize the importance of achieving the desired rotation. When $c_3$ was set to lower values, the resulting structure exhibited only small twist. While most of the optimized structural deformations now looked good, Figure 41 shows one of the designs where the actuated deformation is poor. In this case, the airfoil shape is not preserved under actuation.
Some investigation showed that the calculated rotation angle for the individual nodes was the culprit. The angle of rotation for each node was found using Equation 118. The overall angle of rotation was taken to be the average of these individual angles.

\[
\Delta \theta = \theta_{new} - \theta_{old} = \tan^{-1}\frac{y_{new}}{x_{new}} - \tan^{-1}\frac{y_{old}}{x_{old}}
\]  

However, a problem occurs at the split where the angle changes by 360°. In Matlab, this split occurs between -180° and 180°. So if a node originally at 178° rotates to -178°, the calculated angle is -356° rather than the actual 4° rotation. To fix this, the rotation angle was calculated using Equation 119 through 121. This eliminated the problem with the split and gave results similar to that shown in Figure 42. However, while most of the designs portrayed the behavior shown in Figure 42, this was not always the case. One of the atypical results is shown in Figure 43.

Figure 41. Wing deformation under actuation using Equation 108
\[ \Delta \theta = \tan^{-1}\left(\frac{m_n - m_o}{1 + m_n m_o}\right) \]  

(119)

where

\[ m_o = \tan(\theta_o) = \frac{y_{\text{old}}}{x_{\text{old}}} \]  

(120)

\[ m_n = \tan(\theta_n) = \frac{y_{\text{new}}}{x_{\text{new}}} \]  

(121)

Figure 42. Typical wing deformation under actuation using Equations 108 and 118
The cost function was then updated to include the third error term, given by Equation 112, in an effort to enforce a linear variation of the twist along the length of the wing. The deformation improved slightly (shown in Figure 44), but not enough. Its weighting coefficient was also set to 100 in order to emphasize the importance of a linear variation of twist along the length of the wing.

Figure 43. Atypical wing deformation under actuation using Equations 108 and 118
During the calculation of error three, both the average translation and average rotation are calculated. Using this information, the undeformed airfoil was translated and rotated by the calculated average values and compared graphically to the actual deformation predicted by finite element analysis (FEA). The two were close but did not match well enough. It was decided to use the least squares technique to calculate the average translation and rotation. In order to accomplish this, the nodal displacement for each node was broken down into three parts: translation of the whole airfoil, rotation of the whole airfoil, and translation of each individual node relative to the structure.
Using this method, the translated and rotated airfoil was graphically compared to the nodal displacements predicted by FEA and found to match well. The optimized designs produced were improved. The fourth and fifth error terms were added to ensure a linear variation along the leading and trailing edge of the airfoil. Each term had a weighting coefficient of 100. Typical results are shown in Figure 45.

Figure 45. Typical wing deformation under actuation
Chapter 6: Results from Investigatory Optimization Runs of the Wing Truss Structure

In this chapter the results from the preliminary runs are discussed. The optimization code is able to change numerous aspects of the wing structure. However, it is not designed to change the number of ribs along the length of the wing, nor is it able to remove surface nodes or members along the surface. In addition, there are a number of user controlled options such as the type of actuator used. Because of this several trials were performed to find the effect these have on performance. Other trials were made to assess the best optimization parameters.

During the initial runs it was noticed that a large number of improvements occurred when the routine began but that few improvements occurred later. Because of this observation and because there are a number of optimization routines to run, it was decided to incorporate a termination criteria that would allow a short stay in any routine before moving to the next. The termination criteria was based on setting the maximum number of iterations that occurred without an improvement. The following tests were performed using this termination criteria for the individual optimization routines.
Results using termination based on limiting iterations without improvement

The wing structure used in this work consists of ribs along the length of the wing formed by a truss structure. These ribs are then connect to one another by a truss structure. This is shown in Figure 46 where the passive truss members are shown in purple, the active truss member is green, and the skin is yellow.

The first trials explored the effect of changing the number of ribs along the wing. There were ten runs made at each setting. As shown in Table 4 and Figure 47, the average cost increased as the number of ribs increased. The cost increases due to the increase in the number of members and the increased amount of material (errors 6 and 8, given by Equations 115 and 116). However, the cost also increases due to difficultly in producing a linear variation of twist along the length of the wing using only one actuator. Table 4 also shows error 8 which indicates the difference between the actual and desired twist.
angle. The deviation between the actual and desired twist angle increases as the number of ribs increases. The last column shows the number of acceptable designs based solely on a low error \( \varepsilon \) (less than 1). These trends are expected since it is much easier to have a linear variation of the twist angle with one section (two ribs) than with 12 sections (13 ribs). However, the number of ribs is dependent on the length of the wing, aerodynamic loading, and the acceptable amount of skin deflection.

<table>
<thead>
<tr>
<th>number of ribs</th>
<th>number of members, average</th>
<th>cost, average</th>
<th>cost, uncertainty (95%)</th>
<th>( \varepsilon_s ), average</th>
<th>( \varepsilon_s ), # &lt; 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td>8.577</td>
<td>0.452503</td>
<td>0.00023</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>66</td>
<td>39.83</td>
<td>22.43919</td>
<td>0.000724</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>82</td>
<td>44.49</td>
<td>12.62517</td>
<td>0.04119</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>101</td>
<td>151.9</td>
<td>53.72672</td>
<td>0.7893</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>212.5</td>
<td>56.15162</td>
<td>1.139</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>132</td>
<td>394.5</td>
<td>155.2932</td>
<td>5.872</td>
<td>6</td>
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<tr>
<td>7</td>
<td>159</td>
<td>384.2</td>
<td>80.68665</td>
<td>4.814</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>189</td>
<td>479</td>
<td>201.8597</td>
<td>9.817</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>205</td>
<td>641.6</td>
<td>131.6165</td>
<td>14.48</td>
<td>0</td>
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<tr>
<td>10</td>
<td>226</td>
<td>667.9</td>
<td>94.0629</td>
<td>14.79</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>255</td>
<td>681</td>
<td>66.9313</td>
<td>15.55</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>276</td>
<td>705.3</td>
<td>120.4577</td>
<td>17.38</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4. Effect of increasing the number of ribs
The effect of nodal density was investigated in the second set of trials. The airfoil was broken up into three sections along the chord. Then the number of columns of nodes in each section was specified. This was done to allow an increased nodal density at the leading edge where the airfoil curvature is high and the airfoil height is greater than at the trailing edge. The results from these runs are shown in Table 5. These results were repeated five times at each setting. When there is only one rib the results indicate a slight advantage to a lower nodal density. The cost is 8.6 at the minimum nodal density and increases to 14 when there are two columns of nodes in each section. However, further increase in the nodal density does not result in a significant increase in cost. When the

Figure 47. Effect of the number of ribs
number of ribs is increased to two, the effect of nodal density is less apparent, but it is clear that there is no advantage to increasing the nodal density. The presence of a non-zero standard deviation indicates that the global minimum has not been achieved. The large standard deviations indicate that the final costs achieved are far from the global minimum. Due to this fact, along with the lack of a clear trend, indicates that this trial should be repeated.

<table>
<thead>
<tr>
<th>number of ribs</th>
<th># of columns of nodes in each section</th>
<th># of members</th>
<th>cost, average</th>
<th>cost, uncertainty (95%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>40</td>
<td>8.609</td>
<td>1.97</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>130</td>
<td>14.12</td>
<td>0.93</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>177</td>
<td>16.23</td>
<td>2.53</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>303</td>
<td>14.14</td>
<td>3.02</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>77</td>
<td>53.42</td>
<td>52.91</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>179</td>
<td>116.3</td>
<td>202.42</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>334</td>
<td>59.74</td>
<td>25.15</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>450</td>
<td>73.18</td>
<td>22.74</td>
</tr>
</tbody>
</table>

Table 5. Effect of nodal density on the final design

Three different structure types were investigated. In the first type of structure, only spherical joints are allowed. In the second structure type, the members along the outer surface of the ribs are welded to each other in order to attempt to maintain the airfoil shape. The joints connecting all other members are spherical. The third structural type also welded together the surface members in the ribs, but the joints connecting any other member could be spherical, revolute, or a weld. In the third set of trials, the structure type was varied to see its effect on the final cost. Table 6 shows the results from these trials. The results with only one rib indicate a slight improvement in cost when the
surface members are welded together and another slight improvement in cost when the joint type of the other members are allowed to vary. However, the degree of improvement is within the uncertainty of the cost estimate. When the structure type is varied with five ribs, the results again are unclear and further investigation is required.

<table>
<thead>
<tr>
<th>number of ribs</th>
<th>actuator type</th>
<th>structure type</th>
<th>cost, average</th>
<th>cost, uncertainty (95%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Hydraulic</td>
<td>1</td>
<td>9.659</td>
<td>0.90</td>
</tr>
<tr>
<td>1</td>
<td>Hydraulic</td>
<td>2</td>
<td>9.046</td>
<td>0.72</td>
</tr>
<tr>
<td>1</td>
<td>Hydraulic</td>
<td>3</td>
<td>8.805</td>
<td>1.16</td>
</tr>
<tr>
<td>5</td>
<td>Hydraulic</td>
<td>1</td>
<td>388.1</td>
<td>189.84</td>
</tr>
<tr>
<td>5</td>
<td>Hydraulic</td>
<td>2</td>
<td>392.5</td>
<td>230.62</td>
</tr>
<tr>
<td>5</td>
<td>Hydraulic</td>
<td>3</td>
<td>190.8</td>
<td>35.16</td>
</tr>
<tr>
<td>5</td>
<td>DC motor</td>
<td>1</td>
<td>37.7</td>
<td>8.97</td>
</tr>
<tr>
<td>5</td>
<td>DC motor</td>
<td>2</td>
<td>48.45</td>
<td>25.39</td>
</tr>
<tr>
<td>5</td>
<td>DC motor</td>
<td>3</td>
<td>36.19</td>
<td>14.04</td>
</tr>
</tbody>
</table>

Table 6. Effect of structure type on the final cost

Termination for an individual optimization routine for these sets of trials is solely based on \( n \), the number of iterations without improvement. If \( n \) iterations occur without any improvement in the cost function, that individual optimization routine is terminated and the next one is started. The next set of trials examines the effect of varying this number, \( n \), and the results are shown in Table 7. Not surprisingly, increasing \( n \) allows more random solution perturbations and evaluations before termination and therefore decreases the final cost. The standard deviation of the cost also decreases since the final cost is closer to a global minimum.
<table>
<thead>
<tr>
<th>number of iterations without improvement</th>
<th>cost, average</th>
<th>cost, uncertainty (95%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>76.49</td>
<td>38.94</td>
</tr>
<tr>
<td>20</td>
<td>76.98</td>
<td>31.22</td>
</tr>
<tr>
<td>30</td>
<td>70.23</td>
<td>23.42</td>
</tr>
<tr>
<td>40</td>
<td>53.16</td>
<td>12.27</td>
</tr>
<tr>
<td>50</td>
<td>48.02</td>
<td>14.61</td>
</tr>
<tr>
<td>60</td>
<td>41.18</td>
<td>8.17</td>
</tr>
</tbody>
</table>

Table 7. Effect of termination number of iterations without improvement

The initial runs were repeated multiple times and were found to have a large standard deviation in the final cost. This indicates that the final designs were not close to the global minimum. To try to improve the results, some runs were performed to test whether the order in which the routines were executed affected the final cost. The optimization routines are given in Table 8 and the results are given in Table 9. The optimization routines inside the inner loop are executed every time. The optimization routines under the heading “outer loop” are executed inside the outer loop after the inner loop has been executed two times without improvement. The final design is presented when the outer loop has been executed two times without improvement. The best optimization results occurred when the routine to add and remove members were in both the inner and outer loops. However, the difference was not substantial. There was only one rib for these tests. When this optimization routine ordering was used for wings with five ribs, the uncertainty was still large.
<table>
<thead>
<tr>
<th>optimization number</th>
<th>optimization routine</th>
<th>shorthand</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Vary beam area</td>
<td>geo</td>
</tr>
<tr>
<td>2</td>
<td>Add members</td>
<td>addmem</td>
</tr>
<tr>
<td>3</td>
<td>Remove members</td>
<td>delmem</td>
</tr>
<tr>
<td>4</td>
<td>Vary nodal locations (topology)</td>
<td>top</td>
</tr>
<tr>
<td>5</td>
<td>Vary actuation amount</td>
<td>actA</td>
</tr>
<tr>
<td>6</td>
<td>Vary actuator location</td>
<td>actL</td>
</tr>
<tr>
<td>7</td>
<td>Vary beam depth</td>
<td>dim</td>
</tr>
<tr>
<td>8</td>
<td>Vary joint type</td>
<td>joint</td>
</tr>
<tr>
<td>9</td>
<td>Vary DC motor moment arm length</td>
<td>Lvary</td>
</tr>
<tr>
<td>10</td>
<td>Nodal deletion</td>
<td>nod</td>
</tr>
</tbody>
</table>

Table 8. Optimization routines

<table>
<thead>
<tr>
<th>Inner loop</th>
<th>Outer loop</th>
<th>Cost, average</th>
<th>Cost, uncertainty (95%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>geo, actA, actL, dim, joint</td>
<td>mem, addmem</td>
<td>9.829</td>
<td>1.12</td>
</tr>
<tr>
<td>geo, actA, actL, dim, joint</td>
<td>mem, addmem</td>
<td>8.841</td>
<td>1.17</td>
</tr>
<tr>
<td>geo, actA, actL, dim, joint, mem, addmem</td>
<td>mem, addmem</td>
<td>9.713</td>
<td>2.39</td>
</tr>
<tr>
<td>geo, actA, actL, dim, joint, mem, addmem</td>
<td>mem, addmem</td>
<td>7.931</td>
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</tr>
<tr>
<td>geo, actA, actL, dim, joint, mem</td>
<td>mem, addmem</td>
<td>9.641</td>
<td>1.55</td>
</tr>
<tr>
<td>geo, actA, actL, dim, joint, addmem</td>
<td>mem, addmem</td>
<td>8.096</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Table 9. Effect of optimization routine execution order

In an effort to reduce the uncertainty in the final cost and to best approach the global minimum, another set of trials were conducted to test two termination criteria. The first termination criterion that was varied was the number of iterations without improvement. The second variation either executed the outer optimization loop two times regardless of cost improvement or executed the outer loop until no improvement in cost was found over \( n \) iterations. The results in Table 10 show that there is a cost benefit when \( n \) is large and the outer loop is repeated until no cost improvement is found after \( n \) iterations. However, the uncertainty in the cost is still large.

89
n (quit after n times through without improvement) & outer loop - reset if f(new)<f(old) & cost, average & cost, uncertainty (95%) & $e_s$, average & $e_s$, # < 1 \\
\hline
2 & no & 213.6 & 125.11 & 1.899 & 7 \\
2 & yes & 249.1 & 71.60 & 1.859 & 3 \\
5 & no & 207.1 & 74.11 & 1.217 & 6 \\
5 & yes & 191.3 & 67.12 & 1.268 & 8 \\
10 & no & 233.9 & 79.47 & 1.356 & 5 \\
10 & yes & 145 & 66.58 & 0.6894 & 8 \\
\hline

Table 10. Effect of varying optimization overall termination criteria

Finally, the individual optimization termination criteria and the cooling schedule were varied to test their effect on the final cost. The cooling schedule was changed from Equation 99 to Equation 102. The results from these trials are shown in Table 11, where the entries left unhighlighted were terminated by the temperature approached machine epsilon and those highlighted in light gray were terminated because there were 300 iterations in a row without improvement in the cost. The results show that actuator location (actL), actuation amount (actA), and dimensional (dim) optimization were independent of the cooling rate. For geometric optimization (geo), as the cooling parameter $c$ was decreased, the final cost also decreased until the cooling parameter was set to 0.015. At this point, the cost increased significantly since there were 300 iteration in a row without improvement during the beginning of the run. The lowest cost is achieved when $c$ is set to 0.02. The final cost decreased at slower cooling rates for topological optimization (top) until the $c$ was set to 0.01. At this setting, the cooling rate was too slow relative to the stopping criteria since no improvement was found after 300 iterations close to the onset of optimization. When optimizing the moment arm length
(Lvary) for DC motor actuators, lower costs were obtained for slower cooling rates. The final cost for optimization by member removal and member addition tends to decrease as the cooling rate slows. But at the slower cooling rates it is more likely to terminate the routine due to a lack of improvement over 300 iterations. The final cost for optimization by changing the joint type decreases as the cooling rate decreases.

<table>
<thead>
<tr>
<th>c</th>
<th>geo</th>
<th>actA</th>
<th>dim</th>
<th>top</th>
<th>Lvary</th>
<th>actL</th>
<th>delmem</th>
<th>addmem</th>
<th>joint</th>
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</thead>
<tbody>
<tr>
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<td>1049</td>
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<td>3300</td>
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<td>216.8</td>
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</tr>
<tr>
<td>1</td>
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<td>1932</td>
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<td>3977</td>
<td>1855</td>
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<td>3973</td>
<td>1863</td>
<td>2671</td>
<td>2077</td>
<td>1175</td>
<td>385.9</td>
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</tr>
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<td>280</td>
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<td>1800</td>
<td>2671</td>
<td>2095</td>
<td>1009</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>1053</td>
<td>1057</td>
<td>3973</td>
<td>2224</td>
<td>2671</td>
<td>2077</td>
<td>605.8</td>
<td>1080</td>
<td>3641</td>
</tr>
</tbody>
</table>

Table 11. Effect of varying cooling parameter c

Based on these results, the cooling rate parameter, c, was chosen for each optimization routine as shown in Table 12. There is a slight discrepancy between the optimal cooling rate parameter according to the results shown in Table 11 and that chosen and shown in Table 12. This is due to the incorporation of Equations 103 and 104 after the above trials were run. This affected the values for the cooling rate slightly, but the trends shown in Table 11 remain. In addition to the cooling rate parameter, c, Table 12 shows the final
cost achieved. The optimization routines are categorized based on these values. The routines capable of affecting the cost greatly, labeled as “patient”, will be given more opportunity to do so, while those that have little effect, labeled as “impatient”, are given less opportunity. This has been done in an effort to achieve the final design in an acceptable amount of time.

<table>
<thead>
<tr>
<th>routine</th>
<th>c</th>
<th>final cost</th>
<th>category</th>
<th>jc</th>
<th>ic</th>
<th>Tc</th>
</tr>
</thead>
<tbody>
<tr>
<td>actL</td>
<td>0.2</td>
<td>2077</td>
<td>impatient</td>
<td>floor(300/iter)</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
<tr>
<td>addmem</td>
<td>0.2</td>
<td>39</td>
<td>patient</td>
<td>300</td>
<td>1000</td>
<td>1.00E-100</td>
</tr>
<tr>
<td>delmem</td>
<td>0.4</td>
<td>661</td>
<td>patient</td>
<td>300</td>
<td>1000</td>
<td>1.00E-100</td>
</tr>
<tr>
<td>actA</td>
<td>0.2</td>
<td>1057</td>
<td>impatient</td>
<td>60</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
<tr>
<td>top</td>
<td>0.05</td>
<td>1800</td>
<td>impatient</td>
<td>60</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
<tr>
<td>geo</td>
<td>0.04</td>
<td>28</td>
<td>patient</td>
<td>300</td>
<td>3000</td>
<td>1.00E-250</td>
</tr>
<tr>
<td>Lvary</td>
<td>0.1</td>
<td>2671</td>
<td>impatient</td>
<td>60</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
<tr>
<td>dim</td>
<td>0.2</td>
<td>3977</td>
<td>impatient</td>
<td>60</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
<tr>
<td>joint</td>
<td>0.05</td>
<td>3642</td>
<td>impatient</td>
<td>60</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
<tr>
<td>nod</td>
<td>0.5</td>
<td>4024</td>
<td>impatient</td>
<td>60</td>
<td>500</td>
<td>1.00E-50</td>
</tr>
</tbody>
</table>

Table 12. Optimization settings based on trials

At this point, the termination criterion was also altered based on the above results. The new termination criteria forces at least ic iterations and stops when either the temperature drops below Tc or the number of concurrent iterations without improvement exceeds jc. The variables ic, jc, and Tc are given smaller values for “impatient” optimization routines and larger values for “patient routines.”

These results also allow a better interpretation of the results in Table 6. For the structures with one rib, the final designs must be close to the global minimum since the uncertainty is low. The trend shows a decreasing final cost with increasing structure type. This makes sense since welding the surface members of the rib should help maintain the airfoil
shape during twisting for structure types 2 and 3. The additional option of selectable joint types for the other members should be helpful as well. However, Tables 11 and 12 show that this effect is small and this is reflected by the results in Table 6 for structures with one rib. The structures with more than one rib did not approach the global minimum based on the large uncertainty values. This makes interpretation of the results more difficult. For the hydraulically actuated structures, the final cost for structure types 1 and 2 are similar, while it is significantly lower for structure type 3. This could be due to a poor estimation of the mean due to the small sample size (10 samples). But it is likely that the final cost is lower because of the added optimization routine for varying joint types. While this routine has very little effect on the cost directly, it makes frequent, but negligible, changes to the cost. These changes cause the inner optimization loop to repeat, causing additional evaluations in more productive optimization routines. For the DC motor actuated structures, the optimization routine for joint types and the optimization routine for varying the moment arm length for the DC motor linkage both have little direct effect on the final cost (see Table 12) but both readily make tiny changes causing the inner loop to repeat, and again additional evaluations for more productive routines.

Results using the new termination criteria
The trials testing the effect of optimization order was repeated with the new termination criteria and cooling schedule. The optimization orders tested are shown in Table 13. Cases 1 and 2 have the “patient” routines followed by the “impatient” routines, while
cases 3 and 4 reverse this order. Cases 1 and 3 add members prior to deleting them, while cases 2 and 4 reverse this. The results from these trials are shown in Table 14 and Figure 48. The difference between the final costs are not substantial and the uncertainty is large compared to the difference between the cases, but having the patient routine run prior to the impatient routines yielded better results. The lowest cost was for case 2 which deleted members before adding them.

<table>
<thead>
<tr>
<th>order</th>
<th>optimization case</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>actL</td>
</tr>
<tr>
<td>2</td>
<td>addmem</td>
</tr>
<tr>
<td>3</td>
<td>delmem</td>
</tr>
<tr>
<td>4</td>
<td>geo</td>
</tr>
<tr>
<td>5</td>
<td>top</td>
</tr>
<tr>
<td>6</td>
<td>actA</td>
</tr>
<tr>
<td>7</td>
<td>Lvary</td>
</tr>
<tr>
<td>8</td>
<td>dim</td>
</tr>
<tr>
<td>9</td>
<td>joint</td>
</tr>
<tr>
<td>10</td>
<td>nod</td>
</tr>
</tbody>
</table>

Table 13. Optimization orders tested

<table>
<thead>
<tr>
<th>Optimization case</th>
<th>Number of members</th>
<th>Cost, average</th>
<th>Cost, uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>199</td>
<td>13.9</td>
<td>1.445</td>
</tr>
<tr>
<td>2</td>
<td>190</td>
<td>13.36</td>
<td>1.04</td>
</tr>
<tr>
<td>3</td>
<td>219</td>
<td>14.05</td>
<td>1.413</td>
</tr>
<tr>
<td>4</td>
<td>179</td>
<td>14.46</td>
<td>1.634</td>
</tr>
</tbody>
</table>

Table 14. Effect of optimization order
Figure 48. Effect of optimization routine order

Since the initial trials testing nodal density were inconclusive, they were repeated using the new termination criteria and cooling schedule. A set of trials were made using a single rib and another set of trials were made using two ribs. The results are presented in Table 15 and Figure 49. With either a single rib or two ribs, an increasing nodal density result in an increased cost. Also note that increasing the nodal density results in an increased uncertainty indicating that it is more difficult to attain the global minimum using the impatient cooling schedule implemented. An increased nodal density does not offer any advantage using the current optimization scheme.
Table 15. Effect of nodal density

<table>
<thead>
<tr>
<th>nodal density test case</th>
<th>number of ribs</th>
<th># of division in each section</th>
<th>cost, average</th>
<th>cost, uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5.415</td>
<td>0.1059</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>8.766</td>
<td>1.708</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>7.775</td>
<td>2.36</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
<td>8.442</td>
<td>2.212</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>14.7</td>
<td>5.74</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2</td>
<td>21.44</td>
<td>3.689</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>3</td>
<td>31.16</td>
<td>5.346</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>4</td>
<td>35.78</td>
<td>7.701</td>
</tr>
</tbody>
</table>

Figure 49. Effect of nodal density on average cost

Summary

Due to the numerous investigations presented above, the results are summarized here.

The initial runs, using a quick termination and slow cooling rate, tested the number of
ribs along the wing, nodal density, the effect of structure type, termination criterion, and routine order. As the number of ribs was increased, the cost and its uncertainty also increased. Varying the nodal density, structure type, and optimization routine order had little effect on the cost and its uncertainty. Changing the termination criterion by increasing the number of iterations without cost improvement allowed, resulted in a decrease in cost and its uncertainty. However, the resulting improvement was not enough to achieve good designs.

In an effort to improve understand and obtain better results, the optimization routines were tested individually while varying the cooling rates. The results showed that lower costs were obtained when the cooling rate was slowed down (though the cooling rate was faster than that used in the initial runs). It was also apparent that the routines that affected the location of the structural material had the greatest effect on the final cost. These effective routines added and deleted members and varied the beam cross-sectional area. The routines that were less effective included actuator location, actuation amount, nodal location, structure type, and nodal deletion. The termination criteria for the effective routines was made more lenient than for the less effective routines.

Using the settings shown in Table 12 for each optimization routine, the entire set of optimization routines were run to test the effect of optimization routine order and nodal density. The order in which the routines were run had a negligible effect on the final cost. Note that this does not include the optimization routine that chooses the best location for the actuator. This routine is run prior to the others, since the design seems to get evolve based on the actuator location and changes in actuator location after the first
time through the routines is rare. As the nodal density was increased, the cost and its uncertainty also increased. More importantly, the optimization code now produced good designs with acceptable costs.

The case studies run in chapter seven use the cooling rate and termination criteria shown in Table 12. Optimization order case 2, shown in Table 13, was used for all subsequent test unless otherwise stated. The nodal density was kept as low as possible for these runs.
In this chapter the designs obtained using the code developed in this work are compared to the published work of others. The goal of this work is to develop designs able to resist deformation under aerodynamic loading and capable of wing twist with little actuation energy. It is assumed that all acceptable designs allow only small deformation under aerodynamic loading. Therefore, the metric used for comparison is the ratio of wing twist to actuation energy. Unfortunately, only a few of the researchers developing twisting wings have reported both the twist produced along with the actuation energy required to obtain it. Two researchers reported enough information in order to allow a comparison to be made.

In the following paragraphs, the pertinent details of the designs by other researchers are given. The skin thickness is first estimated using beam theory. This theory is presented along with the resulting designs for Cases I and II. Following this the skin thickness is estimated using shell theory. This theory is briefly presented along with the designs obtained using shell theory to approximate the skin thickness. Finally, comparisons between the designs in this work and the designs made by other researchers are then made.
Designs by other researchers

The first design, herein called Case I, by a researcher that reported enough information for a comparison was by Good whose aircraft has a wing span of 1.5 m, attains airspeeds up to 22.3 m/s, and is capable of a wing twist angle of 20 degrees. The torque was generated using a pneumatic rotary actuator capable of providing 1.76 Nm. This torque was insufficient to achieve the desired wing twist, so it was geared to provide 12 Nm. The torque required to generate the wing twist was not reported, but it must be between 1.76 and 12 Nm, which at least yields maximum and minimum values, respectively, for the ratio of wing twist to actuation energy. In this design, the structure supporting the skin for the twisting portion of the wing consists of a single spar supporting a rib at the wing tip. The rotary actuator applies a torque to the single spar causing the rib at the wing tip to twist (see Figures 12 and 13) [31].

The other designers, Raither et al., developed an aircraft with a wingspan of 7.5 m and an airspeed of 41 m/s, which was capable of a 12.6 degree wing twist [60]. This design is herein called Case II. The torque was generated by aerodynamic pressure. In this design, shown in Figure 50, the structure supporting the skin consists of a hollow rectangular beam running the length of the wing. This beam is heated in order to reduce its modulus of elasticity and allow twisting of the wing. Two aluminum structures, bonded to the front and rear portions of the beam create the leading and trailing edges. The torque required to generate the wing twist was measured, enabling a true comparison to the current work.
The code developed in this work attempts to generate an acceptable design by varying the location and amount of passive material comprising the truss, location of actuators, and degree of actuation. Significant skin deformation will affect the airfoil shape between the ribs and degrade the flight capability of the aircraft. Nonetheless, the current code does not attempt to optimize the number of ribs, skin material, or skin thickness. So these
variables must be set prior to optimization. The variables are set based on skin
deformation during aerodynamic loading and an approximate analysis of anticipated wing
twist. This is described in more detail in the following paragraphs.

In order to establish acceptable values for skin thickness, number of ribs and the skin material, a simple estimate was used. The skin deformation was estimated based on beam theory. The maximum pressure seen by the airfoil was applied to a simply supported beam with a length equal to the distance between the ribs and the same thickness as the skin. The following equation gives the maximum deflection, \( y_{\text{max}} \), as a function of beam length, \( L \), thickness, \( h \), and force per unit length, \( w \):

\[
y_{\text{max}} = \frac{5wl^4}{384EI} = \frac{15pl^4}{96EI^3} \quad (122)
\]

where

\( E \) = modulus of elasticity

\( I \) = area moment of inertia

\( w \) = force per unit length

\( p \) = maximum pressure seen by wing

Setting the maximum skin deflection to be 5% of the distance between the ribs, the relationship between rib separation, skin thickness, and skin material is obtained and is shown in Figure 51 for the Case I, and in Figure 52 for the Case II. Alternatively, knowing the wing length, the relation between the number of ribs and skin thickness can be calculated.
Figure 51. Relation between skin thickness and rib separation to yield 5% skin deflection for case study 1

Figure 52. Relation between skin thickness and rib separation to yield 5% skin deflection for case study 2
The approximate analysis for wing twist was based on simple torsion theory and the equation governing this behavior is as follows:

\[ T = \frac{GJ}{L} \theta \]  

(123)

where

\( \theta \) = wing twist  
\( T \) = applied torque  
\( L \) = wing length  
\( G \) = shear modulus of elasticity  
\( J \) = polar moment of inertia

The wing and skin act like two springs that are physically in parallel. Because of this, the combined stiffness is the addition of each individual stiffness as shown in Equation 124.

\[ \theta = \frac{TL}{(GJ)_{skin} + (GJ)_{truss}} \]  

(124)

The polar moment of inertia for the truss structure is crudely approximated in Equation 125, where it is assumed that each rib section has a polar moment of inertia given by \( J_o \) and that adding additional rib sections increases the polar moment of inertia of the entire truss structure, \( J_{truss} \), in the following manner:

\[ J_{truss} = nJ_o \]  

(125)

where
\( n = \text{number of rib sections} \)

The polar moment of inertia of the skin can be represented by the polar moment of inertia of a hollow tube in a manner similar to the use of the radius of gyration.

\[
J_{\text{skin}} = \rho^2 (\pi r_o^2 - \pi r_i^2) = 2\pi \rho^3 t_{\text{skin}} \tag{126}
\]

where

\( \rho = \text{radius of midsurface of tube} \)

\( r_o = \text{outer radius of tube} \)

\( r_i = \text{inner radius of tube} \)

\( t_{\text{skin}} = \text{skin thickness} \)

Incorporating these into Equation 126, the following equation is obtained.

\[
\theta = \frac{TL}{E_{\text{skin}} 2\pi \rho^3 t_{\text{skin}} + \frac{E_{\text{truss}}}{2(1+\nu_{\text{truss}})} n J_o} \tag{127}
\]

Equation 127 shows that, for the truss structure, as the number of ribs is increased, the wing twist decreases. This assertion is reinforced by the results obtained in chapter 6. This equation also shows that as the skin thickness is increased, the wing twist decreases. However, in order to limit skin deformation to 5% of the rib spacing, Figure 51 shows that the number of ribs and the skin thickness are inversely related. This suggests that the optimal number of ribs and skin thickness can be approximated by optimization using Equations 122 and 127. In this work, the number of ribs was chosen based on

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105
computational speed of the optimization code and Equations 122 and 127, whose combination is as follows:

\[
\theta = \frac{T_L}{\pi \rho^3 L / (15 \rho^2)} \left(\frac{1}{1 + v} \right)^{1/3} E_{\text{skin}}^{2/3} \left( \frac{E_{\text{truss}}}{2(1 + v_{\text{truss}})^2} \right)^{\frac{n}{3}}
\]  

(128)

where

\( s = \) percent skin deflection allowed relative to rib spacing

Equation 128 states that wing twist increases as the skin’s modulus of elasticity decreases as is expected. A couple of different number of ribs was then investigated by running trials. For a given number of ribs, Equation 122 was used to calculate the required skin thickness.

Results based on beam theory approximation

For the Case I, trials were made with 10 and 20 ribs in order to investigate the effect of varying the number of ribs. Equation 128 indicates that wing twist increases as the skin modulus decreases. However, the wing weight must also be taken in consideration. Therefore, Teflon was chosen as the skin material in order to limit the skin thickness and wing weight. The effect of the number of ribs was investigated in chapter 6. However, the skin thickness used for the trials in chapter 6 was only 0.1 mm, making the skin stiffness small. Now that the skin thickness is greater (Table 16), those trials should be repeated. The results show that the designs using DC motor actuation produced low costs. The wing deflection under aerodynamic loading was very small and the desired
twist angle was achieved. However, the results using hydraulic actuators suffered. While the designs were able to achieve minimal wing deflection under aerodynamic loading, there was also very little deflection when actuated. In this case, this is due to the fact that the setting for maximum fluid volume was set too low. Increasing this setting fixed the problem.

For Case II, the number of ribs was set to 7. The skin material was chosen to be aluminum in order to limit skin thickness and therefore wing weight. The results of these tests are shown in Table 16. The cost of the optimized design indicates a problem. The deflection under aerodynamic loading was very small, but so was the deflection when actuated. Increasing the actuation volume helped only minimally. The skin, with a 4.7 mm thickness, was too stiff for the actuator to twist. While a material with a lower modulus of elasticity could have been used, its thickness would be too great, resulting in heavy wings. The use of Equation 122 accounts for the bending stiffness of the material, but yields a conservative estimate for the skin thickness required as it does not account for the additional stiffness due to the skin’s axial stiffness and due to the curvature of the skin. In the next section, shell theory is used to provide a less conservative estimate of the required skin thickness.
In order to obtain a less conservative estimate of the skin thickness, shell theory was used. To get a quick estimate, instead of using the actual airfoil shape, the equation for the deflection of a simply supported cylinder was used [61].

\[
w = A \left[ 1 - \frac{2\sin(a)\sinh(\alpha)}{\cos(2\alpha)\cosh(2\alpha)} \sin(\beta x) \sinh(\beta x) - \frac{2\cos(a)\cosh(\alpha)}{\cos(2\alpha)\cosh(2\alpha)} \cos(\beta x) \cosh(\beta x) \right]
\]

(129)

where

\[
A = \frac{pa^2}{Eh}
\]

(130)

\[
\beta^4 = \frac{3(1-v^2)}{a^2 h^2}
\]

(131)

\[
\alpha = \frac{1}{2} \beta \mu
\]

(132)

\[p = \text{applied pressure}\]
\( a \) = radius of curvature of cylinder

\( E \) = skin modulus of elasticity

\( \nu \) = Poisson’s ratio

\( h \) = skin thickness

\( L \) = rib spacing

The deflection halfway between the ribs is given by the following equation.

\[
w = A \left[ 1 - \frac{2 \cos(\alpha) \cosh(\alpha)}{\cos(2\alpha) \cosh(2\alpha)} \right]
\]  

(133)

In order to apply this equation for the skin deflection, the pressure distribution about the airfoil and its curvature was computed. The maximum deflection of the skin was found by using the point along the airfoil where the product, \( pa^2 \), was greatest. Using this equation, the results shown in Figures 53 and 54 were generated. It is desired to limit the skin deflection to 1 mm.
Figure 53. Skin deflection using shell theory for case study 1

Figure 54. Skin deflection using shell theory for case study 2
Results based on shell theory approximation

For Case I, Teflon was chosen as the skin material and the thickness was set to 1.7 mm. This is the same material chosen for Case I using beam theory and the thickness is very close, and a little higher, than that predicted using beam theory. However, the skin thickness is a little higher because the rib spacing is greater. The results using either hydraulic or DC motor actuation have a low cost, have very little wing deflection under actuation, and achieve the desired angle. This is true whether there are 2 or 4 ribs along the wing. These results, shown in Table 17, are similar to those obtained using beam theory to predict the required skin thickness. Table 18 shows the torque required to achieve the desired 20 degree wing twist when the structure is driven using a hydraulic actuator. One design requires 5.3 Nm while the other requires 211 Nm. This indicates that a new term, shown in Equation 134, is needed for the cost function that minimizes the torque required, $T$.

$$e_9 = c_9 T$$  \hspace{1cm} (134)

However, the first design has very little deformation under aerodynamic loading, as shown in Figure 55, and requires a small actuation energy to achieve the desired 20 degree twist angle. The deformation when actuated is shown in Figures 56 and 57. The actuation energy required to achieve a given twist angle is shown in Figure 58.
<table>
<thead>
<tr>
<th>Case study</th>
<th>Actuator type</th>
<th>Flight speed (m/s)</th>
<th>Number of ribs</th>
<th>Skin thickness (m)</th>
<th>Chord (m)</th>
<th>Cost (Nm)</th>
<th>Angle error (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>hydraulic</td>
<td>22.3</td>
<td>2</td>
<td>0.0017</td>
<td>0.3556</td>
<td>19.6</td>
<td>1.99E-05</td>
</tr>
<tr>
<td>1</td>
<td>hydraulic</td>
<td>22.3</td>
<td>4</td>
<td>0.0017</td>
<td>0.3556</td>
<td>23</td>
<td>0.007</td>
</tr>
<tr>
<td>1</td>
<td>DC motor</td>
<td>22.3</td>
<td>2</td>
<td>0.0017</td>
<td>0.3556</td>
<td>8.57</td>
<td>6.40E-05</td>
</tr>
<tr>
<td>1</td>
<td>DC motor</td>
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<td>4</td>
<td>0.0017</td>
<td>0.3556</td>
<td>21.5</td>
<td>0.008</td>
</tr>
</tbody>
</table>

Table 17. Case study results based on shell theory

<table>
<thead>
<tr>
<th>Case study</th>
<th>Actuator type</th>
<th>Wing twist (degrees)</th>
<th>Actuation energy, this work (Nm)</th>
<th>Actuation energy, other (Nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>hydraulic</td>
<td>20</td>
<td>5.317</td>
<td>12</td>
</tr>
<tr>
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<td>hydraulic</td>
<td>20</td>
<td>21.1</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 18. Torque required to achieve a 20 degree twist angle

Figure 55. Case 1: deformation under aerodynamic loading

Figure 56. Case 1: Deformation when actuated
For Case II, nylon was chosen as the skin material and the thickness was set to 4 mm.

The thickness is similar to that predicted using beam theory, but the rib spacing is greater.
and a material with a lower modulus of elasticity is used. Case II proved to be more challenging since the wing was narrow and long. This made it difficult to minimize deformation under aerodynamic loading by stiffening the structure which, in turn, made it difficult to achieve the required twist. Initial trials resulted in costs between 100 and 1000 and the resulting design either showed excessive deformation under aerodynamic loading or very little twist when actuated. Some trials seemed to converge to relatively high values (around 1000). At least part of the problem seemed to be due to the optimization settings. Trials were again run in order to determine the effect of these settings. The first set of trials sought to establish the effect of nodal density on the cast. The results are shown in Table 19. Examining each trial revealed too much deflection under aerodynamic loading. The first trial shows a case where the cost was stuck near 1000. While only the sparsest nodal density (the first trial) yielded extremely poor results, this set of trials does not indicate a clear pattern.

<table>
<thead>
<tr>
<th># of division in each section</th>
<th>cost</th>
<th>angle error</th>
<th>actuation energy</th>
<th>angle desired</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(1) n(2) n(3)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 1 1</td>
<td>1057</td>
<td>31.30900</td>
<td>0.35</td>
<td>12.7</td>
</tr>
<tr>
<td>1 2 1</td>
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<td>0.02700</td>
<td>2.59</td>
<td>12.7</td>
</tr>
<tr>
<td>2 2 1</td>
<td>120</td>
<td>0.00020</td>
<td>32.7</td>
<td>12.7</td>
</tr>
<tr>
<td>1 3 1</td>
<td>99</td>
<td>0.00500</td>
<td>8.72</td>
<td>12.7</td>
</tr>
<tr>
<td>2 3 1</td>
<td>140</td>
<td>0.00015</td>
<td>288.9</td>
<td>12.7</td>
</tr>
</tbody>
</table>

Table 19. Effect of varying nodal density
Next the number of ribs was varied to see its effect on cost. The results are shown in Table 20. When the number of ribs was less than three, extremely poor results were obtained. However, this set of trials does not indicate a clear pattern. These trials were repeated with a higher nodal density, but similar results were obtained.

<table>
<thead>
<tr>
<th>number of ribs</th>
<th>wing length</th>
<th># of divisions in each section</th>
<th>cost</th>
<th>angle error</th>
<th>actuation energy</th>
<th>angle desired</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(m)</td>
<td>n(1)</td>
<td>n(2)</td>
<td>n(3)</td>
<td></td>
<td>(Nm)</td>
</tr>
<tr>
<td>1</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>917</td>
<td>0.083</td>
</tr>
<tr>
<td>2</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1063</td>
<td>31.204</td>
</tr>
<tr>
<td>3</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>132</td>
<td>0.213</td>
</tr>
<tr>
<td>4</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>112</td>
<td>0.026</td>
</tr>
<tr>
<td>5</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>145</td>
<td>0.123</td>
</tr>
<tr>
<td>6</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>153</td>
<td>0.09</td>
</tr>
<tr>
<td>7</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>291</td>
<td>0.859</td>
</tr>
</tbody>
</table>

Table 20. Effect of number of ribs

The structure type was also varied to see its effect on the cost. The results are shown in Table 21 and indicate that there is no advantage to the more complex structure types.

<table>
<thead>
<tr>
<th>structure type</th>
<th>cost</th>
<th>angle error</th>
<th>actuation energy</th>
<th>angle desired</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Nm)</td>
<td>(degrees)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>97.8</td>
<td>0.001</td>
<td>0.95</td>
<td>12.7</td>
</tr>
<tr>
<td>2</td>
<td>607</td>
<td>16.84</td>
<td>21.6</td>
<td>12.7</td>
</tr>
<tr>
<td>3</td>
<td>756</td>
<td>12.379</td>
<td>1.69</td>
<td>12.7</td>
</tr>
</tbody>
</table>

Table 21. Effect of structure type
In order to create conditions under which the optimization routine may succeed in obtaining an acceptable design, the support structure material was varied. The results, shown in Table 22, indicates that a steel structure is able to maintain small deformation under aerodynamic loading while allowing deformation under actuation.

<table>
<thead>
<tr>
<th>passive material</th>
<th>cost</th>
<th>angle error</th>
<th>actuation energy</th>
<th>angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>aluminum</td>
<td>917</td>
<td>0.083</td>
<td>63090</td>
<td>12.7</td>
</tr>
<tr>
<td>steel</td>
<td>93.3</td>
<td>2.57E-06</td>
<td>19.96</td>
<td>12.7</td>
</tr>
</tbody>
</table>

Table 22. Effect of support structure material

The skin material was varied next to see its effect on the cost. The results, shown in Table 23, do not indicate a clear pattern.

<table>
<thead>
<tr>
<th>skin elastic modulus (Pa)</th>
<th>skin thickness (m)</th>
<th>skin weight/area (kg/m^2)</th>
<th>cost</th>
<th>angle error</th>
<th>actuation energy (Nm)</th>
<th>desired angle (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00E+06</td>
<td>0.01</td>
<td>12</td>
<td>90.42</td>
<td>0.037</td>
<td>42.58</td>
<td>12.7</td>
</tr>
<tr>
<td>3.40E+06</td>
<td>0.01</td>
<td>9.5</td>
<td>67.96</td>
<td>0.006</td>
<td>3.587</td>
<td>12.7</td>
</tr>
<tr>
<td>5.00E+08</td>
<td>0.0032</td>
<td>7.04</td>
<td>80.48</td>
<td>0.004</td>
<td>7.369</td>
<td>12.7</td>
</tr>
<tr>
<td>2.50E+09</td>
<td>0.0006</td>
<td>0.672</td>
<td>75.07</td>
<td>0.009</td>
<td>5.066</td>
<td>12.7</td>
</tr>
<tr>
<td>6.90E+10</td>
<td>0.0005</td>
<td>1.35</td>
<td>88.9</td>
<td>0.594</td>
<td>41.14</td>
<td>12.7</td>
</tr>
</tbody>
</table>

Table 23. Effect of skin material

Finally, the individual errors and overall cost were observed for several trials and a few observations became clear. First, there were two cases when running an optimization routine that had no effect. The first occurred when adding or deleting members. After a
few members were added at high temperatures when accepting a solution that causes a
cost increase is more probable, further member addition was not helpful. This was solved
by setting the current solution to the best solution at a lower temperature where only
changes causing an improvement were accepted. This process was called re-seeding.
Another case when an optimization routine does not affect the cost occurs when the
initial high temperature allows acceptance of a solution with a large cost decrease. If the
subsequent rate of cost decrease is too slow, then the current solution may not reach the
best solution prior to termination of the individual optimization routine. Re-seeding may
also be an advantage in this case. Second, the rate of cost decrease varies as the
temperature is decreased. In an effort to maintain higher rates of cost decrease, it might
be an advantage to slow the cooling rate when the rate of cost decrease is high. The
results of the trials to test these ideas are shown in Table 24. Comparing trials 1 and 2
shows that re-seeding when the cost remains constant decreases the cost. Comparing
trials 1 and 3 and trials 2 and 4 shows that a variable cooling rate lowers the final cost.
However, re-seeding when the rate of cost decrease was too slow resulted in the most
significant improvement. Note that re-seeding does limit the exploration of other regions
to find a global optimum. But this technique is only implemented when it is likely that an
optimization routine will have no effect prior to its termination.
The final cost for trial 5 in Table 24 was low and its deflection under aerodynamic loading was small, with about one centimeter wing tip displacement, as shown in Figure 59. Figures 60 and 61 show the deformation under actuation. The desired wing twist is achieved without excessive deformation under aerodynamic loading. Figure 62 shows that an actuation energy of about 25 Nm is required to achieve the desired wing twist of 12.7 degrees. This is significantly lower than the 1500 Nm required to twist the wing in the work by Raither *et al.* However, the skin and underlying support structure has significant resistance to twisting. The same is true for the design developed here, but the underlying structure is designed to be able to twist.

Table 24. Effect of re-seeding and variable cooling rate

<table>
<thead>
<tr>
<th>trial</th>
<th>re-seed when flat?</th>
<th>re-seed when too slow</th>
<th>cooling rate</th>
<th>cost</th>
<th>angle error</th>
<th>actuation energy (Nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>no</td>
<td>no</td>
<td>constant</td>
<td>176</td>
<td>0.665</td>
<td>0.56</td>
</tr>
<tr>
<td>2</td>
<td>yes</td>
<td>no</td>
<td>constant</td>
<td>143</td>
<td>0.623</td>
<td>10.25</td>
</tr>
<tr>
<td>3</td>
<td>no</td>
<td>no</td>
<td>variable</td>
<td>160</td>
<td>1.88</td>
<td>19.21</td>
</tr>
<tr>
<td>4</td>
<td>yes</td>
<td>no</td>
<td>variable</td>
<td>105</td>
<td>0.909</td>
<td>20.99</td>
</tr>
<tr>
<td>5</td>
<td>yes</td>
<td>yes</td>
<td>variable</td>
<td>36.2</td>
<td>0.005</td>
<td>24.53</td>
</tr>
</tbody>
</table>

Figure 59. Case 2: deformation under aerodynamic loading
Figure 60. Case 2: deformation under actuation

Figure 61. Case 2: deformation under actuation, 12.7 degree wing twist
Summary

This chapter began by presenting the designs published by other researchers used as case studies in this work. Initially, beam theory was used to estimate the skin thickness. This was presented along with the designs. The costs for Case I designs were acceptable, but the cost for the Case II design was not. Shell theory was then used to approximate the skin thickness and was presented next. The resulting designs for Case I were acceptable. But this was not true for the Case II designs. Because of this, the nodal density, number of ribs, structure type, and skin material were varied. However, these variations did not reduce the cost significantly. When the support structure material was changed from aluminum to steel, there was a significant reduction in the cost. But the resulting design still wasn’t good enough. Next, a variable cooling rate and re-seeding were added during

Figure 62. Case 2: actuation energy required to achieve wing twist
the optimization routine and an acceptable design was obtained. The final designs were compared to the designs made by other researchers, showing that the code developed in this research produces improved wing designs.
Chapter 8: Conclusions and Future Work

The research in this thesis examined smart material and conventional actuation to achieve wing twist and potentially improve flight capability using minimal actuation energy while allowing minimal wing deformation under aerodynamic loading. Many researchers have presented designs for morphing wing aircraft and, in particular, twisting wings. However, there hasn’t been research devoted to the design of the support structure using optimization and a comprehensive comparison of smart material and conventional aircraft.

In the first two chapters of this thesis, the motivation for this work is presented along with the work done by other researchers. In Chapter 3, the theory used to estimate the pressure distribution about the wing was presented. The predicted pressure distribution about the wing predicted using this theory compared well with that predicted using commercial computational fluid dynamics software. The finite element theory used in this work was presented in Chapter 4. The finite element portion of the code was tested against results from commercial programs and they yielded the virtually matching results. Simulated annealing was presented in Chapter 5 along with the routines used to perform the structural optimization. The cost function used to evaluate the structure was also
given. In Chapter 6, initial runs were performed to establish the effect that various factors such as structure type, number of ribs, and optimization order have on the final cost. The optimization portion of this work functioned properly and the cost function developed accurately described the behavior of the wing under aerodynamic loading and actuation. The code developed in this work is able to attain designs capable of wing twist under small actuation energy. Comparisons to the designs produced by others, show that the designs developed here required less actuation energy to achieve the same wing twist, as shown in Chapter 7. However, it should be noted that the required torque in this work is a theoretical prediction, while the torque reported in the works by others has been verified experimentally.

Overall this work provides a method of design for the underlying structure and actuation of twisting wings. This study was comprehensive, investigating both smart materials and conventional actuators. Twisting wings have been shown to potentially reduce the actuation required to achieve a given roll moment compared to conventional fixed wings with discrete control surfaces. The wing designs obtained were able to attain a given twist angle using less actuation energy than other designs. Also, because the wings are continuous rather than using discrete control surfaces, there is reduced drag. Note that the optimization procedure in this work is best described as “impatient,” meaning that an exhaustive search for the global minimum was not allowed. Instead a design hopefully near the global minimum was accepted since it could be obtained more quickly.
**Future Work**

The future work can be broken into two categories: design and experimental measurements/verification. Future work in the design category is focused on optimization procedures. The optimization routines used in this study are run individually. Optimization of these parameters simultaneously may or may not be an advantage, but it should be tested. One advantage of running the optimization routines separately is the natural inclusion of re-annealing. If the routines are run simultaneously, re-annealing should be incorporated artificially. Another consideration would be to separate different types of wing deformation and penalize each differently. For example, deformation of the wing in the vertical direction, while not sought, is not unexpected, especially in long, thin wings like those in Case II. This type of deformation could be penalized less than unintended twisting along the wing or an airfoil shape change. Also, it should be tested as to whether replacing the weighting constants in the cost function with variables whose values change based on constraints or based on the degree to which a design goal is met. Additional optimization routines varying skin thickness and the number of ribs, and the material used for the skin or individual structural members should also be incorporated. Finally, the design code should be made robust such that it is capable of handling wing design for a variety of conditions without user interaction.

Optimization has been performed using linear equations to predict nodal displacements. This does not account for changes in the magnitude and direction of forces as the structure deforms. More accurate techniques should be used to assess nodal displacement such as iterative evaluation of the linear equations as the structure moves or the use of
nonlinear equations. If nonlinear equations are used, then nonlinear equations for the actuators should be used as well.

Also, the weight of the wings designed should be compared to the weight of other designs since this is very important for any aircraft. In this work, the designs presented are compared to other designs of twisting wings. It would also be interesting to compare the designs in this work to conventional aircraft. Two comparisons would have to be made. First the weight of the twisting wings needs to be compared to the weight of the conventional wings. Secondly, the actuation energy required to produce a given roll moment for the designs developed in this work should be compared to that in conventional aircraft.

One of the first things that needs to be done before construction of the wing is to calculate the sensitivity the final cost has for deviations of the nodal locations, member area, and joint stiffness and friction. The more sensitive the cost is to any deviation from the optimal design, the more exacting and difficult it will be to construct. The constructed wing can then be tested to see how much actuation energy is needed to achieve the desired wing twist. The joints can be constructed using large displacement compliant joints developed by Trease et al. [62]. Finally, the lift and drag should be assessed experimentally in a wind tunnel for the twisting wings.
References

Appendix: Details for FEA Equations

The matrix equations given in Chapter 4: Elements Developed by the Matrix Displacement Approach for the finite element analysis are taken from published sources [50, 51]. However, in order to provide a more complete description of the current research, the equations used for finite element analysis are presented below in more detail. The finite element equations used for the skin are presented first, followed by those for the frame members and then the truss members.

The finite element equations for the skin elements are given in Chapter 4 but are also repeated here for convenience. They relate the externally applied forces, $F_i$, to the nodal displacements, $p_i$. These relations are obtained using strain energy and virtual work.

\[
\begin{bmatrix}
\tilde{F}_1 \\
\tilde{F}_2 \\
\tilde{F}_3 \\
\tilde{F}_4
\end{bmatrix} =
\begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} \\
k_{12} & k_{22} & k_{14} & k_{24} \\
k_{13} & k_{14}^T & k_{11} & k_{12} \\
k_{14} & k_{24}^T & k_{12} & k_{22}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix}
\]

(A.1)

where

\[
\tilde{F}_i = \begin{bmatrix} \tilde{F}_{xi} \\ \tilde{F}_{yi} \\ \tilde{F}_{zi} \end{bmatrix}, \quad \tilde{p}_i = \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{bmatrix}, \quad k_{11} = \begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_3 & 0 \end{bmatrix}, \quad k_{12} = \begin{bmatrix} -c_5 & c_6 \\ 0 & 0 \end{bmatrix}
\]

(A.2)
\[ k_{22} = \begin{bmatrix} c_1 & -c_2 & 0 \\ -c_2 & c_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_{13} = \begin{bmatrix} -\frac{1}{2}c_1 & -c_2 & 0 \\ -c_2 & -\frac{1}{2}c_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_{14} = \begin{bmatrix} -c_7 & c_5 & 0 \\ c_5 & c_8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (A.3)

\[ k_{24} = \begin{bmatrix} -\frac{1}{2}c_1 & c_2 & 0 \\ c_2 & -\frac{1}{2}c_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (A.4)

\[ c_1 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ 2 \frac{b}{a} + (1-v) \frac{a}{b} \right\}, \quad c_2 = \frac{1}{8} \frac{Et}{1-v^2} \left\{ 2v + (1-v) \right\} \] (A.5)

\[ c_3 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ 2 \frac{a}{b} + (1-v) \frac{b}{a} \right\}, \quad c_4 = \frac{1}{12} \frac{Et}{1-v^2} \left\{ 4 \frac{b}{a} + (1-v) \frac{a}{b} \right\} \] (A.6)

\[ c_5 = \frac{1}{8} \frac{Et}{1-v^2} \left\{ 2v - (1-v) \right\}, \quad c_6 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ a - (1-v) \frac{b}{a} \right\} \] (A.7)

\[ c_7 = \frac{1}{6} \frac{Et}{1-v^2} \left\{ 2 \frac{b}{a} - (1-v) \frac{a}{b} \right\}, \quad c_8 = \frac{1}{12} \frac{Et}{1-v^2} \left\{ 4 \frac{a}{b} + (1-v) \frac{b}{a} \right\} \] (A.8)

The equations for beam elements using beam energy and virtual work are also repeated here. These equations relate externally applied forces, \( F_i \), and moments, \( M_i \), to the nodal displacements, \( p_i \), and nodal rotations, \( \theta_i \).

\[
\begin{bmatrix}
F_1 \\
F_2 \\
\bar{M}_1 \\
\bar{M}_2
\end{bmatrix}
= \begin{bmatrix}
k_{11} & -k_{11} & k_{13} & k_{13} \\
-k_{11} & k_{11} & k_{23} & k_{23} \\
k_{13} & k_{23} & k_{33} & k_{34} \\
k_{13} & k_{23} & k_{34} & k_{33}
\end{bmatrix}
\begin{bmatrix}
p_{1} \\
p_{2} \\
\bar{\theta}_1 \\
\bar{\theta}_2
\end{bmatrix}
\] (A.9)

where

\[
\bar{F}_i = \begin{bmatrix} \bar{F}_{i1} \\ \bar{F}_{i2} \\ \bar{F}_{i3} \end{bmatrix}, \quad \bar{p}_i = \begin{bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \end{bmatrix}, \quad \theta_i = \begin{bmatrix} \theta_{xi} \\ \theta_{yi} \end{bmatrix}, \quad k_{11} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 12c_2 & 0 \\ 0 & 0 & 12c_3 \end{bmatrix}, \quad k_{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6Lc_2 & 0 \\ -6Lc_3 & 0 \end{bmatrix}
\]
\[ k_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -6Lc_2 \\ 0 & 6Lc_3 & 0 \end{bmatrix}, \quad k_{33} = \begin{bmatrix} c_4 & 0 & 0 \\ 0 & 4L^2c_3 & 0 \\ 0 & 0 & 4L^2c_2 \end{bmatrix}, \quad k_{34} = \begin{bmatrix} -c_4 & 0 & 0 \\ 0 & 2L^2c_3 & 0 \\ 0 & 0 & 2L^2c_2 \end{bmatrix} \]

(A.11)

\[ c_1 = \frac{EA}{L}, \quad c_2 = \frac{EI_y}{L^3}, \quad c_3 = \frac{EI_z}{L^3}, \quad c_4 = \frac{JG}{L} \]

(A.12)

Looking at the simple case where there are only nodal forces and displacements, Equation A.9 may be re-written as follows.

\[ \bar{F}_i = K\bar{p}_i \]

(A.13)

In order to obtain the global stiffness matrix for the entire structure, the individual equations must be combined using a common global coordinate system. A transformation matrix is used to rotate the elements from local coordinates to global coordinates.

\[ \bar{p}_i = T_t p_i \]

(A.14)

\[ \bar{F}_i = T_t F_i \]

(A.15)

where

\[ T_t = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \]
The transformation matrix is comprised of the direction cosines: \( l, m, n \) for each of the unit vectors. Applying these transformations to Equation A.13 results in the following equation:

\[
F_i = T_t^T K T_t p_i = G p_i
\]  
(A.17)

Applying this procedure to Equation A.9 results in the following equation:

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2
\end{bmatrix} = 
\begin{bmatrix}
T_t & 0 & 0 & 0 \\
0 & T_t & 0 & 0 \\
0 & 0 & T_t & 0 \\
0 & 0 & 0 & T_t
\end{bmatrix}^T
\begin{bmatrix}
k_{11} & -k_{11} & k_{13} & k_{13} \\
-k_{11} & k_{11} & k_{23} & k_{23} \\
k_{13}^T & k_{23}^T & k_{33} & k_{34} \\
k_{13} & k_{23} & k_{34} & k_{33}
\end{bmatrix}
\begin{bmatrix}
T_t & 0 & 0 & 0 \\
0 & T_t & 0 & 0 \\
0 & 0 & T_t & 0 \\
0 & 0 & 0 & T_t
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]  
(A.18)

or

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2
\end{bmatrix} = 
\begin{bmatrix}
G_{11} & G_{12} & G_{13} & G_{14} \\
G_{21} & G_{22} & G_{23} & G_{24} \\
G_{31} & G_{32} & G_{33} & G_{34} \\
G_{41} & G_{42} & G_{43} & G_{44}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]  
(A.19)

The equations for the beam element can also be written in the following form:

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2
\end{bmatrix} = 
\begin{bmatrix}
G_A & G_B \\
G_B & G_C \\
G_C & G_D
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]  
(A.20)

where

\[
G_A = 
\begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\]  
(A.21)
\[ G_B = \begin{bmatrix} G_{13} & G_{14} \\ G_{23} & G_{24} \end{bmatrix} \]  \hspace{1cm} (A.22)

\[ G_C = \begin{bmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{bmatrix} \]  \hspace{1cm} (A.23)

\[ G_D = \begin{bmatrix} G_{33} & G_{34} \\ G_{43} & G_{44} \end{bmatrix} \]  \hspace{1cm} (A.24)

This transforms the individual stiffness equations for each member from local coordinates to global coordinates. Once the stiffness equation for the individual member has been obtained in global coordinates, it must be split up and placed in the stiffness equation for the entire structure using a force balance and compatibility equation at each node. This results in a global stiffness matrix. To illustrate this point, consider the two dimensional structure shown in Figure 63.

![Figure 63. A Simple Two Dimensional Structure to Illustrate Global Matrix Packing](image)

Applying Equation A.19 to each member, the individual stiffness equations can be written for each member in global coordinates.
In order for the structure to remain intact, each member end must have the same
displacement. A force balance is then written at each node stating that the externally
applied force at each node must balance the member forces. This forms the global
stiffness matrix as shown below.

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2
\end{bmatrix} = \begin{bmatrix}
G_{A11} & G_{A12} & G_{A13} & G_{A14} & p_1 \\
G_{A21} & G_{A22} & G_{A23} & G_{A24} & p_2 \\
G_{A31} & G_{A32} & G_{A33} & G_{A34} & \theta_1 \\
G_{A41} & G_{A42} & G_{A43} & G_{A44} & \theta_2
\end{bmatrix}
\]

(A.25)

\[
\begin{bmatrix}
F_2 \\
F_3 \\
M_1 \\
M_2
\end{bmatrix} = \begin{bmatrix}
G_{B11} & G_{B12} & G_{B13} & G_{B14} & p_2 \\
G_{B21} & G_{B22} & G_{B23} & G_{B24} & p_3 \\
G_{B31} & G_{B32} & G_{B33} & G_{B34} & \theta_2 \\
G_{B41} & G_{B42} & G_{B43} & G_{B44} & \theta_3
\end{bmatrix}
\]

(A.26)

\[
\begin{bmatrix}
F_1 \\
F_3 \\
M_1 \\
M_2
\end{bmatrix} = \begin{bmatrix}
G_{C11} & G_{C12} & G_{C13} & G_{C14} & p_1 \\
G_{C21} & G_{C22} & G_{C23} & G_{C24} & p_3 \\
G_{C31} & G_{C32} & G_{C33} & G_{C34} & \theta_1 \\
G_{C41} & G_{C42} & G_{C43} & G_{C44} & \theta_3
\end{bmatrix}
\]

(A.27)

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} = \begin{bmatrix}
G_{A11} + G_{C11} & G_{A12} & G_{C12} & G_{A13} + G_{C13} & G_{A14} & G_{C14} & p_1 \\
G_{A21} & G_{A22} + G_{B11} & G_{B12} & G_{A23} & G_{A24} + G_{B13} & G_{B14} & p_2 \\
G_{C21} & G_{B21} & G_{B22} + G_{C22} & G_{C23} & G_{B23} & G_{B24} + G_{C24} & p_3 \\
G_{A31} + G_{C31} & G_{A32} & G_{C32} & G_{A33} + G_{C33} & G_{A34} & G_{C34} & \theta_1 \\
G_{A41} & G_{A42} + G_{B31} & G_{B32} & G_{A43} & G_{A44} + G_{B33} & G_{B34} & \theta_2 \\
G_{C41} & G_{B42} & G_{B43} + G_{C42} & G_{C43} & G_{B43} & G_{B44} + G_{C44} & \theta_3
\end{bmatrix}
\]

(A.28)
The above provides the details for the finite element equations used for the passive members of the structure.

Next the details for the active members of the structure are presented. Some of the following is presented in Chapter 4, but is repeated here for convenience. A force balance at the nodes results in the following equation.

\[ \sum F = 0 = \vec{F}_i + \sum f_j \vec{c}_{o_j} \]  \hspace{1cm} (A.29)

or

\[ \vec{F}_i = -\sum f_j \vec{c}_{o_j} \]  \hspace{1cm} (A.30)

where

- \( \vec{F}_i \) = externally applied force at node i
- \( f_j \) = axial force along member j connected to node i, tension is positive
- \( \vec{c}_{o_j} \) = unit vector along undeflected member j

This can be put into matrix form as written below.

\[ F = -Bf \]  \hspace{1cm} (A.31)

Notice that if member j has nodes n and m, then the B matrix will have non-zero entries in column j in rows n and m. The constitutive equations can be written for each passive member as follows:

\[ F_j = k_j d_j = \frac{E_j A_j}{L_j} d_j \]  \hspace{1cm} (A.32)
After loading, nodes m and n move to m’ and n’ along vectors $\vec{u}_n$ and $\vec{u}_m$. The member deflection is given as follows:

$$d_j = L_j - L_{oj}$$  \hspace{1cm} (A.33)

Using vector addition, the deformation of member j can be written as:

$$\vec{L}_{oj} + \vec{u}_n = \vec{L}_j + \vec{u}_m$$  \hspace{1cm} (A.34)

Assuming small deflections, the unit vector for the undeformed and deformed members remain unchanged and equation A.34 can be written as follows.

$$\vec{c}_{oj}(L_{oj} - L_j) = \vec{u}_m - \vec{u}_n$$  \hspace{1cm} (A.35)

Combining equations A.33 and A.35, the equations can be written in terms of the nodal displacements in the following manner:

$$d_j = \vec{c}_{oj} \cdot \vec{u}_n - \vec{c}_{oj} \cdot \vec{u}_m$$  \hspace{1cm} (A.36)

Equation A.36 can be re-written in matrix form as:

$$d = -Cu$$  \hspace{1cm} (A.37)

Notice that if member j has nodes n and m, then the C matrix will have non-zero entries in row j in columns n and m. In fact, the C matrix is the transpose of the B matrix, allowing Equation A.37 to be written as follows:

$$d = -B^T u$$  \hspace{1cm} (A.38)
Combining the force balances, constitutive equations, and compatibility equations, results in the following:

\[ f = -BF = -Bkd = BkB^T u = Gu \]  

(A.39)

Notice that the B matrix performs two roles. First, just as the transformation matrix, T, did for each element in Equations A.14 and A.15, the B matrix rotates the stiffness equations from the local coordinates to the global coordinates. Second, the B matrix also performs the task of splitting up the stiffness matrix for each individual matrix and placing it into the global stiffness matrix by virtue of the placement of the direction cosines within the B matrix. This provides details for the finite element equations used to describe the behavior of the active members if the joints are spherical. If the joints are not spherical, then the stiffness matrix given in Equation A.9 can be split up into the following equation:

\[
\begin{bmatrix}
\vec{F}_1 \\
\vec{F}_2 \\
\vec{M}_1 \\
\vec{M}_2
\end{bmatrix} =
\begin{bmatrix}
k & -k & 0 & 0 \\
-k & k & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\vec{p}_1 \\
\vec{p}_2 \\
\vec{\theta}_1 \\
\vec{\theta}_2
\end{bmatrix} +
\begin{bmatrix}
k_{11} & -k_{11} & k_{13} & k_{13} \\
-k_{11} & k_{11} & k_{23} & k_{23} \\
k_{13}^T & k_{13}^T & k_{33} & k_{34} \\
k_{13}^T & k_{13}^T & k_{34} & k_{33}
\end{bmatrix}
\begin{bmatrix}
\vec{p}_1 \\
\vec{p}_2 \\
\vec{\theta}_1 \\
\vec{\theta}_2
\end{bmatrix}
\]  

(A.40)

where

\[
\vec{F}_i = \begin{bmatrix}
\vec{F}_{xi} \\
\vec{F}_{yi} \\
\vec{F}_{zi}
\end{bmatrix},
\vec{p}_i = \begin{bmatrix}
\vec{u}_i \\
\vec{v}_i \\
\vec{w}_i
\end{bmatrix},
\vec{\theta}_i = \begin{bmatrix}
\theta_{xi} \\
\theta_{yi} \\
\theta_{zi}
\end{bmatrix},
k_{11} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 12c_2 & 0 \\
0 & 0 & 12c_3
\end{bmatrix},
k_{13} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 6Lc_2 \\
0 & -6Lc_3 & 0
\end{bmatrix}
\]  

(4.41)
\[ k_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -6Lc_2 \\ 0 & 6Lc_3 & 0 \end{bmatrix}, \quad k_{33} = \begin{bmatrix} c_4 & 0 & 0 \\ 0 & 4L^2c_3 & 0 \\ 0 & 0 & 4L^2c_2 \end{bmatrix}, \quad k_{34} = \begin{bmatrix} -c_4 & 0 & 0 \\ 0 & 2L^2c_3 & 0 \\ 0 & 0 & 2L^2c_2 \end{bmatrix} \] (A.42)

\[ k = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (A.43)

This representation allows the use of the individual transformation matrix, T, for the second stiffness matrix and the use of the B matrix for transformation and placement for the first stiffness term in Equation A.40. With this representation actuated members are not restricted to spherical joints.