UNIVERSAL LOSS PROCESSES IN BOSONIC ATOMS WITH POSITIVE SCATTERING LENGTHS

DISSERTATION

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By

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Abstract

In experiments with trapped ultracold gases, atoms can be lost through inelastic scattering processes. If the atoms have a scattering length that is much larger than the range of their interactions, then the system exhibits universal behavior that does not depend on details of their interactions. Dramatic enhancements in the loss rate are observed at special negative values of the scattering length for which there is a universal molecule at threshold. In some experiments, enhancements of the loss rate have also been observed at other positive values of the scattering length. A mechanism proposed to explain this enhancement is that the losses result from many collisions of an energetic diatomic molecule created by a 3-atom collision. In this thesis, I demonstrate the failure of this mechanism as a viable explanation of the enhancement seen at positive scattering length. I present an alternative explanation for these enhancements in experiments using a Bose-Einstein condensate of atoms. They result from inelastic scattering of universal diatomic molecules in a coexisting condensate of these molecules.
To Ghazal, my azeazam.
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Chapter 1
INTRODUCTION

One of the primary goals of physics is to accurately describe the particles that make up matter and the forces between them. The forces between particles, or equivalently their interactions, determine how they scatter from each other and whether they can be bound together by the forces. Of particular interest in physics is universality, which refers to situations in which systems that are very different at short distances have identical long-distance behavior. When applied to particles and forces, universality refers to particles that have very different structure, such as atoms, molecules, nuclei, hadrons, etc., and very different interactions at short distances, but nonetheless have identical behavior at low energies.

One important aspect of the interactions between particles is their range, which is the distance beyond which the interactions become negligible. Another important aspect is the strength of the interactions. Quantum mechanics requires that particles behave like waves, with a wavelength that is inversely proportional to the momentum. Surprisingly, quantum mechanics allows particles to scatter even if their closest approach is much larger than their range, but only if the strength of the interactions is carefully tuned to near the critical value where two particles are just barely bound. In this situation, the particles exhibit universal behavior at energies low enough that their wavelength is larger than the range. Their behavior at those energies is deter-
mined by a single parameter called the *scattering length*.

One of the best systems to study the universal physics associated with large scattering lengths experimentally is ultracold atoms, which are atoms that are trapped in the middle of a vacuum chamber and cooled to a temperature very close to absolute zero. The low temperature makes the wavelengths of the atoms much larger than their range. It is also possible by adjusting the magnetic field to make the scattering length much larger than the range. In this case, the atoms display universal behavior that is determined by the scattering length. One window into the universal physics of ultracold atoms is loss processes that allow them to escape from the trapping potential. This thesis will discuss these loss processes.

All particles, including atoms, can be classified into two categories: *bosons* and *fermions*. Collections of identical particles have dramatically different behavior depending on whether they are bosons or fermions. For identical bosons, the quantum state must be symmetric under exchange of any two bosons. This implies that any number of identical bosons can occupy the same quantum state. The exchange symmetry constraint for bosons was first introduced by Satyendra Bose in 1924, who showed that the massless particles that make up light, which are called photons, are bosons. Albert Einstein considered the possibility that massive particles, such as atoms, could be bosons. He showed that a gas of non-interacting bosons undergoes a phase transition at sufficiently low temperatures to a state in which a nonzero fraction of the bosons occupy the lowest-energy quantum state. The ground state of such a system in which all the bosons occupy the lowest-energy quantum state, is called a *Bose-Einstein condensate* (BEC). If the BEC has a large spatial extent, the effects of quantum mechanics can become apparent on a macroscopic scale.

For identical fermions, the quantum state must be antisymmetric under exchange of any two fermions. This implies the Pauli exclusion principle, which states that
two identical fermions cannot occupy the same quantum state. The ground state of a many-body system of $N$ noninteracting identical fermions is a Fermi gas in which a single fermion occupies each of the $N$ lowest-energy quantum states.

Ultracold atoms emerged as a new subfield of physics in 1995, when scientists at JILA led by Eric Cornell and Carl Wieman created a Bose-Einstein condensate of rubidium atoms. They were able to obtain this condensate phase by using a combination of laser cooling and evaporative cooling [1]. The laser cooling process involves the detuning of the laser to a frequency just below an excitation energy of an atom. An atom moving in the direction of the laser absorbs a photon from the laser and then re-emits it isotropically. The net effect is a reduction in the atom’s momentum along the direction of the laser. If the atoms are trapped in a vacuum chamber, six lasers can be used to reduce the momentum along each of the three spatial directions, thus cooling the atoms. After laser cooling the atoms as far as possible, the atoms are cooled further by evaporative cooling. The evaporative cooling process involves letting the more energetic atoms escape from the system and then allowing collisions between the remaining atoms to redistribute the energy, so that they become colder. At the end of the evaporative cooling process, the number of atoms that remain is typically $10^4$ to $10^6$. Using this cooling process, a dilute system of bosonic atoms can reach temperatures into the nano-kelvin range, below the critical temperature to form a Bose-Einstein condensate. Very shortly following the JILA experiment, condensates of lithium atoms [2] and sodium atoms [3] were also created.

Cold atoms are not restricted to bosons. There have been experiments with ultracold Fermi gases as well [4]. The fermionic atoms in a Fermi gas can not form a condensate, but a small fraction of the fermions can form pairs that are bosons. These pairs can condense into a superfluid. Ultracold Fermi gases are the center of
many interest in the cold atom community. An interesting aspect of a Fermi gas of atoms with a large scattering length is its large critical temperature for superfluidity.

What makes ultracold atoms a good system to study universal physics is that the scattering length is an experimentally controllable parameter. Its is controlled by taking advantage of a phenomenon called a Feshbach resonance. By adjusting an external magnetic field in the region of the Feshbach resonance, the scattering length can be tuned to any value between $-\infty$ and $+\infty$. In particular, the scattering length can be made larger than the range of interactions, either positive or negative valued.

Universal bound states play a crucial role in the loss processes for ultracold atoms. If the scattering length is positive and large, there is a universal two-atom bound state whose binding energy is set by the scattering length. This two-atom bound state is called the *shallow dimer*, or *dimer* for short. The shallow dimer is not the only universal bound state. In 1970, Vitaly Efimov discovered that when the scattering length is infinitely large, there is an infinite sequence of universal three-atom bound states with an accumulation point at the three-atom threshold [5, 6, 7]. These three-atom bound states are called *Efimov trimers*. At infinitely large scattering length, the Efimov trimers have a geometric spectrum. The ratio between the binding energies of successive Efimov trimers is approximately 1/515. This remarkable phenomenon is characteristic of *discrete scale invariance*. Universal phenomena associated with discrete scale invariance are referred to as *Efimov physics* [8, 9].

Within the past 15 years, Efimov physics has been observed in many cold atom experiments. Efimov physics manifests itself most dramatically as narrow peaks in the loss rate of atoms at certain scattering lengths. The mechanism for these peaks at negative scattering lengths is well known: they appear due to Efimov trimers and other universal bound states crossing the threshold where they become unbound. However in some experiments, narrow peaks in the loss rate have also been observed
at positive scattering lengths. The mechanism for these peaks is not known. A mechanism involving multiple collisions of a dimer with atoms called the avalanche mechanism was proposed to explain these peaks at positive scattering lengths [10]. The goal of this thesis is to demonstrate that the avalanche mechanism fails to describe these peaks and to propose an alternative explanation.

In the following chapters, we present theoretical studies of loss processes involving Efimov physics in $^7\text{Li}$ atoms that were published in Refs. [11] [12]. In Chapter 2, we introduce scattering theory and Efimov physics. In Chapter 3, we present analytic expressions for universal reaction rates for bosonic atoms with large positive and negative scattering lengths as found in Ref. [8]. In Chapter 4, we show that the avalanche mechanism originally proposed by Ref. [10] does not explain the loss features near atom-dimer resonances seen at positive scattering lengths in either a BEC or a thermal gas. We came to this conclusion by using a numerical simulation of the avalanche mechanism. In Chapter 5, we propose an explanation for resonant peaks in loss rates at positive scattering lengths in experiments involving a BEC. We propose that the peaks are caused by inelastic collisions of dimers in a BEC of dimers that coexists with the atom condensate. In this initially pure atom condensate, the small population of dimers are created by a phenomenon called atom-molecule coherence. Appendix A describes an the effective field theory approach that can be used to calculate universal reaction rates. Appendix B presents an analytic parametrization of the dimer-dimer scattering length that was obtained by fitting numerical results by Deltuva [13]. Appendix C presents an estimate of the number of dimers in a dimer condensate produced by ramping a magnetic field close to a Feshbach resonance.
Chapter 2  
Scattering Theory and Efimov Physics

The effects of interactions on a sufficiently dilute many-body system can be generally described using few-body physics, which describes the individual interactions between its constituents. These individual interactions can be explained by using basic scattering theory. Universal behavior arises if the range of the interactions is negligible compared to the scattering length. The universal behavior of three or more identical bosons has some remarkable aspects that are referred to as Efimov physics.

In this chapter, we present some basic physics that is required to discuss systems of bosonic atoms. In Sec. 2.1 we discuss the basic properties of alkali atoms and their hyperfine states, with a focus on $^7\text{Li}$ atoms. In Sec. 2.2 we present basic scattering theory for atoms and explain how the strength of atomic interactions can be tuned experimentally. In Sec. 2.3 we introduce Efimov physics and present quantities that will be important in subsequent chapters.

2.1 Alkali atoms

The types of atoms that are most easily cooled to ultra-low temperatures are the alkali atoms that lie below hydrogen in the periodic table. They are lithium (Li), sodium (Na), potassium (K), rubidium (Rb), and cesium (Cs). In this section, we
review the basic properties of alkali atoms.

The constituents of an atom are protons, neutrons, and electrons. Their electric charges in units of the proton charge are +1, 0, and −1, respectively. The proton and neutron are much heavier than an electron; their masses are larger by a factor of about 1840. The structure of an atom consists of a tiny massive core called the atomic nucleus, surrounded by clouds of electrons that are arranged in shells. The nucleus consists of protons and neutrons. The number of protons in the nucleus determines the element of the atom. For example, lithium (Li) atoms have a nucleus with 3 protons. In an electrically neutral atom, the number of electrons is equal to the number of protons. The total number of protons and neutrons in the nucleus is called the atomic mass number and it determines the isotope of the atom. The isotope is commonly specified by giving the atomic mass number as a pre-superscript. The most common isotopes of lithium are $^6\text{Li}$ and $^7\text{Li}$. The nuclei of these isotopes contain three and four neutrons, respectively. The electronic structure of an alkali atom such as Li consists of closed shells of electrons plus a single electron in the outermost shell. The mass $m$ of a $^7\text{Li}$ atom is approximately seven times that of a proton. A convenient conversion constant for $^7\text{Li}$ atoms is

$$\frac{\hbar}{m} = 9.0519 \times 10^{-5} \text{ cm}^2/\text{s},$$

(2.1)

where $\hbar$ is the reduced Planck’s constant.

### 2.1.1 Hyperfine spin states

An alkali atom in its electronic ground state has multiple spin states. There are two contributions to its spin: the electronic spin $S$ with quantum number $s = \frac{1}{2}$ and the nuclear spin $I$ with quantum number $i$. The $2(2i + 1)$ spin states can be labeled $|m_s, m_i\rangle$, where $m_s$ and $m_i$ specify the eigenvalues of $S_z$ and $I_z$. The Hamiltonian for
a single atom includes a hyperfine term that can be expressed in the form

\[ H_{\text{hyperfine}} = \frac{2E_{\text{hf}}}{(2i + 1)\hbar^2} \mathbf{I} \cdot \mathbf{S}. \]  

(2.2)

This term splits the ground state of the atom into two hyperfine multiplets with energies differing by \( E_{\text{hf}} \). The eigenstates can be labeled by the eigenvalues of the hyperfine spin \( \mathbf{F} = \mathbf{I} + \mathbf{S} \). The associated quantum numbers \( f \) and \( m_f \) specify the eigenvalues of \( \mathbf{F}^2 \) and \( F_z \). The eigenvalues of \( H_{\text{hyperfine}} \) are

\[ E_{f,m_f} = \frac{f(f + 1) - i(i + 1) - \frac{3}{4}E_{\text{hf}}}{2i + 1}. \]  

(2.3)

The two hyperfine multiplets of an alkali atom consist of \( 2i + 2 \) states with \( f = i + \frac{1}{2} \) and \( 2i \) states with \( f = i - \frac{1}{2} \). For example, a \(^7\)Li atom has nuclear spin quantum number \( i = \frac{3}{2} \). The two hyperfine multiplets consist of five states with \( f = 2 \) and three states with \( f = 1 \). The \( f = 2 \) multiplet is higher energy by \( E_{\text{hf}} \). The frequency associated with the hyperfine splitting is \( E_{\text{hf}}/\hbar \approx 803.504 \text{ MHz} \).

In the presence of a magnetic field \( \mathbf{B} = B\hat{z} \), the Hamiltonian for a single atom has a magnetic term. The magnetic moment \( \mu \) of the atom is dominated by the term proportional to the spin of the electron: \( \mu = \mu_S/(\frac{1}{2}\hbar) \). The magnetic moment \( \mu \) of an alkali atom such as Li is approximately that of the single electron in the outermost shell: \( \mu \approx -2\mu_B \), where \( \mu_B \) is the Bohr magneton. The magnetic term in the Hamiltonian can be expressed in the form

\[ H_{\text{magnetic}} = -\frac{2\mu}{\hbar} \mathbf{S} \cdot \mathbf{B}. \]  

(2.4)

If \( B \neq 0 \), this term splits the two hyperfine multiplets of an alkali atom into \( 2(2i + 1) \) hyperfine states. In a weak magnetic field satisfying \( \mu B \ll E_{\text{hf}} \), each hyperfine multiplet is split into \( 2f + 1 \) equally-spaced Zeeman levels \( |f, m_f\rangle \). In a strong magnetic field satisfying \( \mu B \gg E_{\text{hf}} \), the states are split into a set of \( 2i + 1 \) states with \( m_s = +\frac{1}{2} \).
whose energies increase linearly with $B$ and a set of $2i + 1$ states with $m_s = -\frac{1}{2}$ whose energies decrease linearly with $B$. Each of those states is the continuation in $B$ of a specific hyperfine state $|f, m_f\rangle$ at small $B$. It is convenient to label the states by the hyperfine quantum numbers $f$ and $m_f$ for general $B$, in spite of the fact that those states are not eigenstates of $F^2$ if $B \neq 0$. We denote the eigenstates of $H_{\text{hyperfine}} + H_{\text{magnetic}}$ by $|f, m_f; B\rangle$ and their eigenvalues by $E_{f,m_f}(B)$. The two eigenstates with the maximal value of $|m_f|$ are independent of $B$:

$$|f = i + \frac{1}{2}, m_f = \pm (i + \frac{1}{2}); B\rangle = |m_s = \pm \frac{1}{2}, m_i = \pm i\rangle.$$  \hspace{1cm} (2.5)$$

Their eigenvalues are exactly linear in $B$: $E_{f,m_f}(B) = \frac{i}{2i+1}E_{\text{hf}} \mp \mu B$.  \hspace{1cm} (2.6)

If $B \neq 0$, each of the other eigenstates $|f, m_f; B\rangle$ is a linear superposition of the two states $|f = i - \frac{1}{2}, m_f\rangle$ and $|f = i + \frac{1}{2}, m_f\rangle$.

The dependence of the hyperfine energy levels of $^7\text{Li}$ atoms on the magnetic field is illustrated in Fig. 2.1. At $B = 0$, the hyperfine multiplets with $f = 2$ and $f = 1$ are split by $E_{hf}$. The magnetic energy scale $\mu B$ is comparable to the hyperfine splitting $E_{hf}$ when $B$ is about 287 Gauss. At higher magnetic fields, the four $m_s = -\frac{1}{2}$ states decrease linearly with $B$, while the four $m_s = +\frac{1}{2}$ states increase linearly.

**2.2 Low-energy scattering**

In this section, we describe the interactions between two atoms with very low energy and we introduce the concept of the *scattering length*. Then, we explore the universality of 2-body physics for large scattering length.
Figure 2.1: The hyperfine energy levels as a function of the magnetic field for $^7$Li atoms.

2.2.1 Natural scale

According to the wave-particle duality of quantum mechanics, a particle with momentum $p$ behaves like a wave with the de Broglie wavelength $\lambda = \frac{2\pi \hbar}{p}$. In a gas with temperature $T$, the typical momentum of a particle with mass $m$ is $p \approx \sqrt{mk_B T}$, where $k_B$ is Boltzmann’s constant. In ultracold atom experiments, the typical temperatures are lower than a micro-Kelvin ($\mu$K). Thus the de Broglie wavelength of a typical $^7$Li atom in the experiment is greater than $10^5 a_0$, where $a_0$ is the Bohr radius: $a_0 \approx 5.29 \times 10^{-11}$m. Such an atom cannot resolve any structure that is smaller than its wavelength. If the wavelength of the atom is larger than the size of an atom, which is typically a few Bohr radii, it cannot resolve the atomic structure. Thus, the atom can be accurately described as a pointlike particle. If the wavelength of the atom
is larger than the range of the force between atoms, it cannot resolve the details of
the interactions. The force could just as well be replaced by a short-range force with
a suitably adjusted strength or even by a zero-range force. This makes a detailed
description of the forces between atoms unnecessary.

The force between two atoms can be specified by a potential \( U(r) \) which gives
the potential energy as a function of the separation \( r \) of the atoms. The potential
between two neutral atoms is highly repulsive at short distances that are comparable
to the Bohr radius and it is attractive at longer distances. The repulsion between
the outermost electron shells of the two atoms can change the charge distributions
of the shells, making the atoms electrically polarized. This deformation causes an
attractive force between the polarized atoms. This attractive potential is called the
van der Waals potential:

\[
U_{\text{vdW}}(r) = -\frac{C_6}{r^6},
\]

where \( C_6 \) is a constant that is different for each element. The constant \( C_6 \) defines a
length scale called the van der Waals length \( \ell_{\text{vdW}} \):

\[
\ell_{\text{vdW}} = \sqrt[4]{mC_6/\hbar^2}.
\]

This is the distance at which the kinetic energy \( p^2/m \sim \hbar^2/m\ell_{\text{vdW}}^2 \) of a pair of
atoms is comparable to their potential energy \( |U_{\text{vdW}}(\ell_{\text{vdW}})| \sim C_6/\ell_{\text{vdW}}^6 \). For \(^7\text{Li}\) atoms, \( \ell_{\text{vdW}} \approx 65\,a_0 \). The van der Waals length is the natural length scale for the
interaction between neutral atoms with sufficiently low energy. Atoms with de Broglie
wavelengths larger than \( \ell_{\text{vdW}} \) are unable to resolve even the power-law tail of the
interatomic potential. Their interactions can therefore be described by a short-range
potential or even by a zero-range potential.
The constant $C_6$ also determines the van der Waals energy given by

$$E_{\text{vdW}} = \frac{\hbar^2}{m \ell_{\text{vdW}}^2}.$$  \hspace{1cm} (2.9)

This is the typical size of the binding energy of the most weakly-bound diatomic molecules. It also sets the temperature scale $E_{\text{vdW}}/k_B$ below which we consider the atoms to be ultracold. For $^7\text{Li}$ atoms, this temperature is about 6 mK. Since the typical wavelength of ultracold atoms is larger than $\ell_{\text{vdW}}$, they are unable to resolve details of the interaction potential. This makes it possible to describe their interactions accurately by a few parameters.

\subsection*{2.2.2 Two-body scattering}

In this section, we briefly review scattering of a 2-body system and define some of the important scattering parameters at low energy.

Let us consider the scattering of a beam of atoms on a target. Part of the beam is scattered by the target and the remainder of the beam passes through the target unscattered. If a beam travels along the $z$ axis, the wavefunction of the atom in the absence of the target is a plane wave $e^{ikz}$, where $k$ is the wavenumber of the atom, which is determined by its energy: $E = \frac{\hbar^2k^2}{2m}$. The plane wave at a large distance $r$ from the target can be expressed as an infinite sum of incoming and outgoing spherical waves [14]:

$$e^{ikz} \rightarrow \frac{i}{2k} \sum_{l=0}^{\infty} (2l+1) i^l \left[ \frac{e^{-i(kr-l\pi/2)}}{r} - \frac{e^{i(kr-l\pi/2)}}{r} \right] P_l(\cos \theta), \quad (r \rightarrow \infty)$$  \hspace{1cm} (2.10)

where $P_l(\cos \theta)$ is the Legendre polynomial. Assuming that the potential is rotationally symmetric, the scattered waves are azimuthally symmetric. In the presence of the target, the wavefunction $\psi(r)$ of the atom at large distance $r$ can still be decomposed into incoming and outgoing spherical waves. The incoming spherical waves are the
same as for the plane wave. Conservation of probability requires that the outgoing spherical waves have the same amplitude as in the plane wave, but they can differ in phase:

\[
\psi(r) \rightarrow \frac{i}{2k} \sum_{l=0}^{\infty} (2l+1)i^l \left[ \frac{e^{-i(kr-l\pi/2)}}{r} - e^{2i\delta_l} \frac{e^{i(kr-l\pi/2)}}{r} \right] P_l(\cos \theta), \quad (r \to \infty)
\]

(2.11)

where \(\delta_l\) is the phase shift due to the scattering, which depends on the wavenumber \(k\). The asymptotic wavefunction \(\psi(r)\) can be expressed as the sum of the incident plane wave in Eq. (2.10) and an outgoing spherical wave:

\[
\psi(r) \rightarrow e^{ikz} + f_k(\theta) \frac{e^{ikr}}{r}, \quad (r \to \infty)
\]

(2.12)

where \(f_k(\theta)\) is the scattering amplitude:

\[
f_k(\theta) = \sum_{l=0}^{\infty} \frac{2l + 1}{k \cot \delta_l - ik} P_l(\cos \theta).
\]

(2.13)

A convenient observable associated with the scattering probability is the cross section. The number of incident atoms per unit time and unit area is proportional to their velocity times their probability density: \((\hbar/k/m) \times |e^{ikz}|^2 = \hbar/k/m\). Similarly, the number of scattered atoms per unit time and unit area is proportional to

\[
(hk/m) \times |f_k(\theta)e^{ikr}/r|^2 = (hk/m)|f_k(\theta)|^2/r^2.
\]

(2.14)

Taking the ratio of these two quantities and integrating over the surface gives the cross section. The differential cross section for two distinguishable atoms or molecules is therefore given by

\[
\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2.
\]

(2.15)

The differential solid angle is \(d\Omega = 2\pi \sin \theta d\theta\). The cross section \(\sigma\) is obtained by integrating over the scattering angle \(\theta\) from 0 to \(\pi\).
In the case of identical bosons, the expression for the differential cross section is a little different. The indistinguishability of the identical bosons implies that one cannot distinguish between bosons in the initial and final states. Both processes produce the same final state with identical bosons at the angle $\theta$ and $\pi - \theta$. The differential cross section for two identical bosons is therefore given by

$$\frac{d\sigma}{d\Omega} = |f_k(\theta) + f_k(\pi - \theta)|^2.$$  \hspace{1cm} (2.16)

The differential solid angle is $d\Omega = 2\pi \sin \theta d\theta$. The cross section $\sigma$ is obtained by integrating over the scattering angle $\theta$ from 0 to $\pi/2$. If one of the bosons is scattered at an angle $\theta$ greater than $\pi/2$, the other is scattered at the angle $\pi - \theta$ less than $\pi/2$. Thus such a scattering process is already taken into account by integrating over $\theta$ from 0 to $\pi/2$.

If the potential is short-ranged with no power-law tail, it is known that $k^{2l+1} \cot \delta_l$ for small scattering energy $E = \hbar^2 k^2 / m$ can be expanded in powers of $k^2$ [15]:

$$k^{2l+1} \cot \delta_l = \sum_{n=0}^{\infty} c_{l,n} k^{2n}. \hspace{1cm} (2.17)$$

The coefficients $c_{l,n}$ are called effective range parameters. For the S-wave phase shift, the leading terms in the effective range expansion are

$$k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_e k^2 + \cdots, \hspace{1cm} (2.18)$$

where $a$ is the scattering length and $r_e$ is the effective range. The natural magnitudes for the effective range coefficients are determined by the length scale $\ell$ set by the range of interaction. By dimensional analysis, $c_{l,n}$ can be expressed as $\ell^{2n-2l-1}$ multiplied by a dimensionless coefficient. In the absence of an enhancement mechanism, we expect the dimensionless coefficient to be order 1. An example of an enhancement mechanism is a bound state that is very close to the two-atom threshold. If the bound
state is in the S-wave \((l = 0)\) channel, the scattering length \(a\) is large compared to \(\ell\). Upon inserting \(k \cot \delta_l\) from Eq. (2.17) into the scattering amplitude in Eq. (2.13), one can see that that the S-wave \((l = 0)\) term dominates the amplitude at low energy. The higher partial waves \((l > 0)\) are suppressed by \((k\ell)^{2l}\).

The low energy expansions in Eqs. (2.17) and (2.18) express the information about the potential that is relevant at low energy in terms of a few parameters, such as \(a\) and \(r_e\). The coefficients of higher powers of \(k\) in the effective range expansion are less important, because they are suppressed by powers of the energy.

The potential between atoms is not short-ranged, because it has the power-law tail at large \(r\) given by the van der Waals potential in Eq. (2.7). Consequently, the low energy expansion in Eq. (2.17) breaks down [16]. Since the van der Waals potential decreases as a high power of \(r\), some of the terms in the expansion are the same as for a short-range potential. For the S-wave phase shift, the two leading terms in the effective range expansion still have the form in Eq. (2.18). The expansion of \(k \cot \delta_1\) still starts at order \(k^{-2}\), so the P-wave term in the cross section is suppressed by \(k^2\) at low energy. For all higher partial waves \((l \geq 2)\), the expansion of \(k \cot \delta_l\) starts at order \(k^{-4}\), so the corresponding terms in the cross section are suppressed by \(k^4\). Therefore, the S-wave term still dominates at sufficiently low energy [16] [8]. At extremely low energy, the only relevant interaction parameter is the scattering length \(a\).

### 2.2.3 Universality with large scattering length

As discussed in the previous section, in the generic case when the effective range parameters have natural values set by the range \(\ell\), low energy scattering can be treated systematically by expanding in powers of the energy. In this subsection, we discuss the case of an unnaturally large scattering length \(|a| \gg \ell\) and we introduce
the concept of universality.

We consider two particles with a large scattering length $|a| \gg \ell$ and with energy small compared to the scale $\hbar^2/mt^2$ set by the range. For convenience, we will refer to the particles as atoms. We will see that this system has nontrivial properties that are completely determined by the scattering length. We will refer to these properties as universal. This adjective is appropriate because different systems with a large scattering length will have identical low-energy behavior up to one overall length scale that is set by $|a|$. If we insert the expansion in Eq. (2.18) into the S-wave term in the amplitude in Eq. (2.13), we find that the scattering amplitude only depends on the scattering length and the scattering wave-number. Thus, the universal scattering amplitude is:

$$f_k(\theta) = \frac{1}{1 - \frac{1}{a} - ik}.$$  \hfill (2.19)

By inserting Eq. (2.19) into Eq. (2.16) and integrating over the solid angle, we obtain the total cross section for identical bosons:

$$\sigma(k) = \frac{8\pi}{1/a^2 + k^2}.$$  \hfill (2.20)

Thus the cross section for low-energy scattering has a nontrivial form that is completely determined by the scattering length.

If the scattering length $a$ is large and positive, there is a diatomic molecule with universal properties. We will refer to this bound state as the shallow dimer. Quantum mechanics implies that bound states are associated with poles in the scattering amplitude $f(k)$ for complex values of the momentum $k$. If $f(k)$ has a pole on the positive imaginary axis at $k = i\kappa$, then there is a bound state with binding energy $\hbar^2\kappa^2/m$. The scattering amplitude in Eq. (2.19) has a pole at $k = i/a$. If $a > 0$, this pole is in the upper half-plane of the complex variable $k$, so there is a corresponding
bound state. The universal expression for the binding energy of the shallow dimer is

\[ E_D = \frac{\hbar^2}{ma^2} \quad (a > 0). \]  

(2.21)

The typical separation of its constituents is \( a \). In addition to the shallow dimer, there may also be diatomic molecules whose binding energies are of order \( \hbar^2/(m\ell^2) \), or larger. We will refer to them as deep dimers, because they are much more deeply bound than the shallow dimer. A deep dimer has no universal properties. Its binding energy is much larger than that of the shallow dimer. The typical separation of its constituents is order \( \ell \) or smaller, so it is much smaller than the shallow dimer.

The limit of large scattering length \( |a| \gg \ell \) is closely related to the zero range limit \( \ell \to 0 \). The zero range limit can be achieved by taking the range of the interaction potential to zero while simultaneously increasing its depth so that the scattering length remains fixed. The limit is independent of the shape of the potential. In the zero range limit, the universal scattering amplitude in Eq. (2.19) becomes exact up to arbitrarily high energies. The universal expression for the binding energy in Eq. (2.21) also becomes exact. If there are any deep dimers, their binding energies become infinitely large. Since the universal results are the same in both limits, we will sometimes use the phrases large scattering length, zero range, and universal interchangeably.

In the limit \( a \to \pm \infty \), the universal cross section approaches \( 8\pi/k^2 \), which is the maximum value allowed by unitarity. The limit \( a \to \pm \infty \) is therefore called the unitary limit. In this limit, there is no length scale associated with the interactions. Thus the system has a symmetry under scaling the spatial coordinates by an arbitrary positive factor \( \lambda \) and the time by a factor \( \lambda^2 \). This symmetry is called scale invariance.

The scale invariance of the unitary limit manifests itself at finite scattering length by simple scaling behavior under simultaneous scaling of \( a \) and kinematic variables.
For example, when \(a\) and the momentum variable \(k\) are scaled by the factors \(\lambda\) and \(\lambda^{-1}\), the cross section in Eq. (2.20) is changed by a factor \(\lambda^2\): 
\[
\sigma(\lambda^{-1}k; \lambda a) = \lambda^2 \sigma(k; a).
\]

The binding energy in Eq. (2.21) also shows the scaling behavior. When \(a\) is scaled by \(\lambda\), \(E_D\) is changed by a factor \(\lambda^{-2}\): 
\[
E_D(\lambda a) = \lambda^{-2} E_D(a).
\]

This scaling behavior is a general feature of the system with large scattering length. It follows from the fact that the scattering length \(a\) is the only interaction parameter that sets a length scale in the zero-range limit.

Universality is important, because it relates phenomena in various fields of physics. There are examples of systems with large scattering lengths in nuclear physics and high energy physics as well as in atomic physics. In nuclear physics, the best example is the neutron, whose two spin states interact with a large negative scattering length. In high energy physics, a good example is the charm mesons \(D^*\) and \(\bar{D}^0\), which form a very weakly bound state called the \(X(3872)\) and therefore must have a large positive scattering length. A classic example in atomic physics is \(^4\text{He}\) atoms, whose scattering length is about \(+200\ a_0\), which is much larger than the van der Waals length scale \(\ell_{\text{vdW}} \approx 10\ a_0\).

Atomic physics is unique in that it is also possible to tune the scattering lengths of atoms experimentally. This can be accomplished by adjusting the magnetic field near a Feshbach resonance. A Feshbach resonance arises when a diatomic molecule is near the threshold for a pair of atoms. If the diatomic molecule has a magnetic moment that is different from twice that of the atoms, its energy relative to the threshold can be changed by a magnetic field. This also changes the scattering length \(a\) of the atoms. Near the Feshbach resonance, the scattering length can be approximated by
\[
a(B) = a_{bg} \left(1 - \frac{\Delta}{B - B_0}\right).
\]

The scattering length diverges at \(B = B_0\), which is the position of the Feshbach resonance.
Figure 2.2: Dependence of the scattering length, $a$, with respect to the magnetic field near a Feshbach resonance. The shaded region indicates the universal region where $|a|$ is larger than the van der Waals length, $\ell_{vdW}$. In the parametrization in Eq. (2.22), $B_0$ is the position of the Feshbach resonance where $|a|$ diverges, $B_0 + \Delta$ is the position of the zero crossing where $a$ vanishes, and $a_{bg}$ is the background scattering length.

It vanishes at $B = B_0 + \Delta$. Far above or below the resonance, the scattering length approaches $a_{bg}$. The dependence of the scattering length on the magnetic field is illustrated in Fig. 2.2 for the case $\Delta < 0$. By adjusting the magnetic field, the scattering length can be tuned to any desired value. In particular, it can be made infinitely large by tuning $B$ to $B_0$. For $^7$Li atoms in the lowest hyperfine spin state, there is a Feshbach resonance at 736.8 G. For a detailed discussion of Feshbach resonances, the reader is referred to a review article [17].
2.3 Efimov physics

In this section, we describe the energy spectrum of Efimov trimers and universal tetrarners for identical bosons. We identify the scattering lengths at which these universal bound states cross thresholds, because they play an important rule in the loss processes considered in this thesis.

2.3.1 Efimov trimers

Three particles with large scattering length also have universal properties, but they are much more intricate than those for two particles. These universal properties were discovered by Vitaly Efimov in 1970 [5] and developed in a series of subsequent papers [6, 7]. The most dramatic consequence is the existence of a sequence of universal 3-body bound states that are now called Efimov trimers. The spectrum of Efimov trimers is particularly simple and remarkable in the unitary limit in which the scattering length is taken to infinity. There are an infinite number of low-energy bound states whose binding energies are geometrically spaced with an accumulation point at zero energy. In the case of identical bosons, the binding energies of two successive trimers differ by a multiplicative factor of $\lambda_0^2 \approx 515$, where $\lambda_0 = e^{\pi/s_0} \approx 22.7$. The constant $s_0$ is the solution to a transcendental equation:

$$s_0 \cosh(\pi s_0/2) = \frac{8}{\sqrt{3}} \sinh(\pi s_0/6). \quad (2.23)$$

The numerical value of $s_0$ is approximately 1.00624. The spectrum of Efimov trimers in the unitary limit can be expressed as

$$E_T^{(n)} = \lambda_0^{-2n} \frac{\hbar^2 \kappa_*^2}{m} \quad (a = \pm \infty), \quad (2.24)$$

where $n$ is an integer and $\kappa_*$ is the binding wavenumber of the trimer labeled by $n = 0$. Note that any one of the Efimov trimers could be labelled $n = 0$, so $\kappa_*$ is only
defined up to factors of \( \lambda_0 \).

In the 2-body sector, the unitary limit is characterized by scale invariance. However scale invariance requires the binding energies of discrete bound states to be either 0 or \( \infty \). Thus the existence of Efimov trimers indicates that scale invariance is violated in the 3-body sector. However, there is a remnant of that symmetry. The Efimov spectrum in Eq. (2.24) is compatible with discrete scale invariance with the discrete scaling factor \( \lambda_0 \). We will refer to universal phenomena associated with discrete scale invariance in the three-body sector as Efimov physics \([8]\).

In the 2-body sector, the scattering length is the only length scale provided by interactions in the zero-range limit. In the 3-body sector, the discrete spectrum of Efimov states in the unitary limit requires a 3-body parameter that provides another length scale in addition to the scattering length. If there were no such parameter, the system would have continuous scale invariance in the unitary limit. One simple choice for the 3-body parameter is the wavenumber \( \kappa_* \) defined by the spectrum of Efimov trimers in the unitary limit in Eq. (2.24). Discrete scale invariance requires that any dependence of physical quantities on the parameter \( \kappa_* \) must be log-period with discrete scaling factor \( \lambda_0 \). The universal properties in the 3-body sector for identical bosons are determined by the scattering length \( a \) and the 3-body parameter \( \kappa_* \).

The binding energies of three successive Efimov trimers are illustrated in Fig. 2.3 using variables that are particularly well suited to exhibiting the discrete scale invariance. The horizontal axis is the inverse scattering length \( a^{-1} \). The vertical axis is an energy variable \( K \) that also has a dimension (length)\(^{-1}\):

\[
K = \text{sign}(E) \sqrt{m|E|} / \hbar.
\]  

(2.25)

According to the Efimov effect, there are infinitely many trimer states along the
vertical axis \((a = \pm \infty)\) as one approaches the origin from below. Only three of those infinitely many trimers are shown in the figure. The spectrum of Efimov trimers at unitarity given in Eq. (2.24) can be expressed as \(K = -\lambda_0^{-n}\kappa_s\), so the ratio of the positions of two successive trimers on the vertical axis is \(\lambda_0 \approx 22.7\). We have taken the liberty of applying a transformation to the radial coordinate that makes the discrete scaling factor look like a factor of 2.2.

In Fig. 2.3, the horizontal axis corresponds to the threshold for three-atom scattering states. For \(a < 0\), Efimov trimers cross the three-atom threshold at a sequence of scattering lengths \(\lambda_0^{-n}a_\) that differ by the discrete scaling factor. The scattering
length $a_-$ is $1/\kappa_*$ multiplied by a universal multiplicative constant \[18\]:

$$a_- = -1.50763/\kappa_*.$$  \tag{2.26}

The three-atom scattering amplitude at threshold diverges at each of those scattering lengths, because there is an Efimov trimer crossing the 3-atom threshold.

In Fig. 2.3 the straight line approaching the origin at a 45° angle with respect to the horizontal axis represents the threshold for atom-dimer scattering states, which is $K = -1/a$. For $a > 0$, the trimers disappear through the atom-dimer threshold at a sequence of scattering lengths $\lambda_0^n a_*$ that differ by the discrete scaling factor. The scattering length $a_*$ is $1/\kappa_*$ multiplied by a universal multiplicative constant \[8\]:

$$a_* = 0.07076/\kappa_*.$$  \tag{2.27}

The atom-dimer scattering length diverges at each of these scattering lengths because there is an Efimov trimer state crossing the threshold. We will refer to any of the scattering lengths $\lambda_0^n a_*$ at which an Efimov trimer crosses the atom-dimer threshold as an atom-dimer resonance.

### 2.3.2 Universal Tetramers

The universality of atoms with large scattering length is not limited to the 2-atom and 3-atom sectors. In 2004, Hammer, Meissner, and Platter made the first suggestion that universality should extend to the 4-atom sector \[19\]. They presented numerical evidence that there are two 4-body bound states associated with each Efimov trimer, and they made the first calculations of the binding energies of these 4-body bound states for some regions of $1/a$ \[19, 20\]. In 2008, von Stecher, D’Incao, and Greene calculated these 4-body binding energies more accurately and over the entire range of $1/a$ \[21, 22\]. These 4-body bound states are called universal tetramers. The binding
energies of the universal tetramers are determined by $a$ and $\kappa_*$. In the unitary limit, the spectrum of universal tetramers also has an infinite number of low-energy bound states whose binding energies are geometrically spaced with accumulation points at zero energy. The binding energies of two successive shallower tetramers at unitarity differ by the same multiplicative factor as the trimers, $\lambda_0^2$. The binding energies of two successive deeper tetramers at unitarity also differ by that same multiplicative factor, $\lambda_0^2$. The spectrum of universal tetramers in the unitary limit can be expressed as:

$$E^{(n)}_{\text{Tet},1} = \lambda_0^{-2n} \frac{\hbar^2 \kappa_1^2}{m} \quad (a = \pm \infty), \quad (2.28a)$$

$$E^{(n)}_{\text{Tet},2} = \lambda_0^{-2n} \frac{\hbar^2 \kappa_2^2}{m} \quad (a = \pm \infty), \quad (2.28b)$$

where $n$ is an integer and $\kappa_{1*}$ and $\kappa_{2*}$ are the binding wavenumbers of the tetramers labeled by $n = 0$. The binding wavenumbers $\kappa_{1*}$ and $\kappa_{2*}$ are related to $\kappa_*$ by multiplicative universal constants. They were first calculated by von Stecher, D’Incao, and Greene [21]. They were recently calculated by Deltuva with 4 digits of precision [13]:

$$\kappa_{1*} = 2.147 \kappa_*, \quad (2.29a)$$

$$\kappa_{2*} = 1.00011 \kappa_. \quad (2.29b)$$

The binding energies of the two universal tetramers associated with a specific Efimov trimer are illustrated in Fig. [2.4] using the energy variable $K$ defined in Eq. [2.25], which is plotted as a function of the inverse scattering length $1/a$. As for Efimov trimers, there are infinitely many pairs of tetramer states along the vertical axis ($a = \pm \infty$) as one approaches the origin from below. Only three of those infinitely many pairs of tetramers are shown in the figure. The spectrum of the deeper tetramers at unitarity given in Eq. [2.28a] can be expressed as $K = -\lambda_0^{-n} \kappa_{1*}$, so the ratio of
Figure 2.4: Binding energies of three successive pairs of universal tetramers (red curves) for identical bosons. Each tetramer pair is associated with a trimer (dotted green curve). The tetramers whose binding momenta at unitarity are $\kappa_{1*}$ and $\kappa_{2*}$, given in Eqs. (2.28a) and (2.28b), disappear across the 4-atom threshold at $1/a_{1-}$ and $1/a_{2-}$, given in Eqs. (2.30a) and (2.30b). They disappear across the dimer-dimer threshold at $1/a_{1*}$ and $1/a_{2*}$, given in Eqs. (2.31a) and (2.31b).

the positions of two successive deeper tetramers on the vertical axis is $\lambda_0 \approx 22.7$. Similarly, the spectrum of the shallower tetramers at unitarity given in Eq. (2.28b) can be expressed as $K = -\lambda_0^{-n}\kappa_{2*}$, so the ratio of the positions of two successive shallower tetramers on the vertical axis is also $\lambda_0 \approx 22.7$.

In Fig. 2.4, the horizontal axis corresponds to the threshold for four-atom scattering states. For $a < 0$, the universal tetramers disappear through the 4-atom threshold at a sequence of scattering lengths $\lambda_0^{-n}a_{1-}$ and $\lambda_0^{-n}a_{2-}$ that differ by the discrete scaling factor. The universal predictions for the scattering lengths where the tetramers cross the 4-atom threshold were first calculated by von Stecher, D’Incao, and Greene
They were recently calculated by Deltuva with 4 digits of precision \cite{13}:

\begin{align}
a_{1}^- &= 0.4254 \, a_-, \quad (2.30a) \\
a_{2}^- &= 0.9125 \, a_-. \quad (2.30b)
\end{align}

The four-atom scattering amplitude at threshold has a resonance where it nearly diverges at each of those values of the scattering lengths, because there is a universal tetramer crossing the 4-atom threshold.

In Fig. 2.4, the straight line approaching the origin at an approximate angle of 55° with respect to the horizontal axis represents the threshold for dimer-dimer scattering states, which is \( K = -\sqrt{2}/a \). For \( a > 0 \), the universal tetramers disappear through the threshold at sequences of scattering lengths \( \lambda_0^{-n}a_1^* \) and \( \lambda_0^{-n}a_2^* \) that differ by the discrete scaling factor. The universal predictions for \( a_1^* \) and \( a_2^* \) were first calculated by von Stecher, D’Incao, and Greene \cite{22}. They were recently calculated by Deltuva with 4 digits of precision \cite{23}:

\begin{align}
a_{1}^* &= 2.196 \, a_*, \quad (2.31a) \\
a_{2}^* &= 6.785 \, a_. \quad (2.31b)
\end{align}

The dimer-dimer scattering length nearly diverges at each of these scattering lengths, because there is a tetramer state crossing the threshold. We will refer to any of the scattering lengths \( \lambda_0^{-n}a_1^* \) or \( \lambda_0^{-n}a_2^* \), at which a tetramer crosses the 2-dimer threshold as a dimer-dimer resonance.
Chapter 3

LOSS PROCESSES FOR LARGE SCATTERING LENGTHS

The easiest way to observe Efimov trimers and universal tetramers in ultracold atomic gases is through resonant enhancement of loss rates. The atoms are trapped in a potential created by a magnetic field or by laser beams. An atom can escape from the trapping potential if it acquires a kinetic energy that is larger than the depth of the potential through an inelastic scattering process. The escaping atoms can usually not be observed directly, but they can be observed indirectly through the decrease in the number of trapped atoms. By appropriate choice of hyperfine states, one can arrange that two-atom inelastic processes are forbidden by conservation of energy. The dominant loss mechanisms will then be through inelastic scattering processes involving three or more atoms. The short lifetimes of trimers, tetramers, and larger molecules prevent the accumulation of significant populations of these molecules. The only molecule that could have a significant population is the shallow dimer. Thus, the only inelastic collisions we need consider are those involving atoms and dimers. In this chapter, we consider a system of identical bosonic atoms and discuss loss mechanisms that can lead to observable loss features in cold atom experiments.
3.1 Trimer Decay

Efimov trimers are sharp states with the spectrum in Eq. (2.24) only if there are no deep dimers in the 2-body spectrum. If there are deep dimers, an Efimov trimer can decay into an atom and a deep dimer and this gives the trimer a width. The inclusive effects of all the deep dimers can be taken into account by analytically continuing the Efimov parameter $\kappa_*$ to a complex value that is conveniently expressed in the form

$$\kappa_* \rightarrow \kappa_* \exp(i\eta_*/s_0),$$

(3.1)

where $\kappa_*$ and $\eta_*$ are positive real parameters [24]. The parameter $\eta_*$ is called the inelasticity parameter. The particular form in Eq. (3.1) was chosen to simplify the analytic expressions for some few-body reaction rates.

Analytic expression for the binding energies of the Efimov trimer in the unitary limit are given in Eq. (2.24). Making the substitution $\kappa_* \rightarrow \kappa_* \exp(i\eta_*/s_0)$ on the right side of Eq. (2.24), we find that the expression for the binding energies of the Efimov trimers becomes complex. The real part can be interpreted as the binding energy. The imaginary part can be interpreted as half of the decay width $\Gamma_{T}^{(n)}$ of the Efimov resonance. The binding energies and widths of the Efimov trimers at unitarity are [8]

$$E_{T}^{(n)} = \lambda_0^{-2n} \frac{\hbar^2 \kappa_0^2 \cos(2\eta_*/s_0)}{m},$$

(3.2)

$$\Gamma_{T}^{(n)} = \lambda_0^{-2n} \frac{2\hbar^2 \kappa_0^2 \sin(2\eta_*/s_0)}{m}.$$  

(3.3)

Away from unitarity, the trimer binding energy and its width depend on the scattering length as well as $\kappa_*$ and $\eta_*$. Analytic expressions for $E_{T}^{(n)}$ and $\Gamma_{T}^{(n)}$ are not known. However, if $\eta_* \ll 1$, there is a simple relation between the width of the trimer and
the binding energy of the trimer \[8\]:

\[
\Gamma_T^{(n)} \approx \frac{4\eta_s}{s_0} \left( E_T^{(n)} + \frac{\hbar^2}{ma^2} \right). \tag{3.4}
\]

At unitarity where \(1/a^2 = 0\), this is consistent with Eq. (3.2) and Eq. (3.3) expanded to leading order in \(\eta_s\).

### 3.2 Three-body Recombination

We consider atoms with a scattering length that is much larger than the range of their interactions: \(|a| \gg r_0\). In a system consisting of atoms only, the dominant loss process at sufficiently low density is three-body recombination. Three-body recombination is the inelastic scattering of three incoming atoms into a dimer and an atom, illustrated in Fig. 3.1. By conservation of energy, the increase in the kinetic energies of the outgoing dimer and atom must be equal to the binding energy of the dimer. At positive scattering length, the outgoing dimer can be either a deep dimer whose binding energy is of order \(\hbar^2/(mr_0^2)\), or a shallow dimer whose binding energy is \(\hbar^2/(ma^2)\). At a negative scattering length, the outgoing dimer can only be a deep dimer. If the kinetic energy of the outgoing atom or dimer is larger than the depth of the trapping potential, it can escape from the trap. This results in a decrease in the number of trapped atoms.

The loss rate is determined by measuring the number of remaining atoms in the trap as a function of time. The number density of atoms decreases in proportion to the cube of the number density of atoms. The loss rate of atoms due to three-body recombination is

\[
\frac{d}{dt} n_A(r) = -L_3 n_A^3(r), \tag{3.5}
\]

where \(L_3\) is the three-atom loss rate coefficient. In a BEC of identical bosons, Eq. (3.5) is multiplied by \(1/6\) on the right hand side to take into account that the three atoms
Figure 3.1: Three-body recombination process: three atoms collide with small momenta, two of the three bind to form a dimer (which can be either a deep dimer or a shallow dimer), and the dimer and remaining atom recoil with large momenta.

are in the same quantum state. If we assume that three atoms are lost per recombination event, \( L_3 \) can be written in terms of the three-body recombination event rate coefficient \( \alpha \):

\[
L_3 = 3\alpha.
\]  

(3.6)

The event rate coefficient can be decomposed into the contribution from the shallow dimer and from the deep dimer:

\[
\alpha = \alpha_{\text{shallow}} + \alpha_{\text{deep}}.
\]  

(3.7)

3.2.1 Negative scattering length

The three-body recombination rate can be resonantly enhanced by an Efimov trimer near the three-atom threshold. Therefore, the observation of a three-atom loss resonance is indirect evidence for an Efimov trimer. In the case of identical bosons, Efimov trimers approach the three-atom threshold at negative values of \( a \) that differ by the discrete scaling factor 22.7, as illustrated in Fig. 2.3. There will be a three-atom loss resonance at each of these values of \( a \).

Since there are no shallow dimers at negative scattering lengths, the three-body recombination event produces only an atom and a deep dimer. Thus the three-body event rate coefficient has only the contribution from deep dimers: \( \alpha = \alpha_{\text{deep}} \). The
Figure 3.2: Universal rate coefficient for 3-body recombination at threshold for $\eta_\ast = 0.03$ as a function of the negative scattering length. The rate coefficient $\alpha_{\text{deep}}$ for recombination into deep dimers is shown as a solid (red) line. The vertical dotted lines mark the positions of $a_-$ and $22.7\ a_-$. 

energy per particle is small compared to the energy set by the scattering length per recombination event. We can therefore neglect the energies of the three incoming atoms and use the recombination rates at threshold. It is convenient to express $\alpha$ in terms of the scattering length, $a_-$, where an Efimov trimer crosses the three-atom threshold, which is given in Eq (2.26). The event rate coefficient for 3-body recombination into deep dimers is a function of $a$, $a_-$, and $\eta_\ast$ [8]:

\[
\alpha_{\text{deep}} = \frac{4590 \sinh 2\eta_\ast}{\sinh^2(\pi s_0 + \eta_\ast) + \sin^2(s_0 \ln(a/a_-))} \frac{h a^4}{m} \quad (a < 0). \tag{3.8}
\]

This rate coefficient is shown as a function of $a$ in Fig. 3.2 for $\eta_\ast = 0.03$. This rate coefficient has log-periodic modulation of $a^4$ scaling behavior. It has peaks at
scattering lengths \((e^{\pi/s_0})^n a_-\) that become infinitely sharp and narrow in the limit \(\eta_* \to 0\).

### 3.2.2 Positive scattering length

If \(a\) is positive, a three-body recombination event produces either an atom and the shallow dimer or an atom and a deep dimer. The event rate coefficient \(\alpha\) is therefore the sum of the two terms in Eq \((3.7)\). We consider systems of atoms in which the energy per particle is much smaller than the binding energy \(E_D\) of the shallow dimer. We can therefore neglect the energies of the three incoming atoms and use the recombination rates at threshold. There are interference minima in the
three-body recombination rate at positive values of $\alpha$ that differ by powers of $2^{2.7}$. These interference features are not directly related to Efimov trimers, but they are another manifestation of Efimov physics. It is convenient to express $\alpha$ in terms of the position of an interference minimum $a_+$:

$$a_+ = 0.31649/\kappa_+.$$  \hspace{1cm} (3.9)

The event rate coefficients for 3-body recombination into the shallow dimer and into deep dimers are conveniently expressed as functions of $a$, $a_+$, and $\eta_*$ [8]:

$$\alpha_{\text{shallow}} = \frac{128\pi^2(4\pi - 3\sqrt{3})(\sin^2[s_0 \ln(a/a_+)] + \sinh^2[\eta_*]) \hbar a^4}{\sinh^2(\pi s_0 + \eta_*) + \cos^2[s_0 \ln(a/a_+)]} \frac{\hbar a^4}{m},$$  \hspace{1cm} (3.10)

$$\alpha_{\text{deep}} = \frac{128\pi^2(4\pi - 3\sqrt{3}) \coth(\pi s_0) \cosh(\eta_*) \sinh(\eta_*) \hbar a^4}{\sinh^2(\pi s_0 + \eta_*) + \cos^2[s_0 \ln(a/a_+)]} \frac{\hbar a^4}{m}. \hspace{1cm} (3.11)

These rate coefficients are shown as functions of $a$ in Fig. [3.3] for $\eta_* = 0.03$. Both rate coefficients have log-periodic modulation of $a^4$ scaling behavior. For $\alpha_{\text{shallow}}$, there are local minima at scattering lengths near $(e^{\pi/s_0})^n a_+$ which become zeroes in the limit $\eta_* \to 0$. For $\alpha_{\text{deep}}$, the amplitude of the log-periodic modulation is too small to be evident in Fig. [3.3]. In the limit $\eta_* \to 0$, $\alpha_{\text{deep}}$ approaches zero.

### 3.3 Atom-Dimer Inelastic Scattering

In a system of both atoms and dimers, one possible loss process is inelastic atom-dimer scattering, which is sometimes called dimer relaxation. Inelastic atom-dimer scattering is the scattering of an incoming atom plus a shallow dimer into an outgoing atom plus a deep dimer, illustrated in Fig. [3.5]. By conservation of energy, the increase in the kinetic energies of the atom and dimer must be equal to the difference between the binding energies of the incoming and outgoing dimers. If the kinetic energy of the outgoing atom or dimer is larger than the depth of the trapping potential, it can escape from the trap. This results in a decrease in the number of trapped atoms and
Figure 3.4: Universal rate coefficient for atom-dimer scattering at threshold for $\eta = 0.03$ as a function of the positive scattering length. The rate coefficient $\beta$ for inelastic scattering of an atom and a shallow dimer is shown as a solid (red) line. The vertical dotted lines mark the positions of $a_-$ and $22.7 \ a_-$. 

To observe these losses in experiment, this process requires a population of both atoms and shallow dimers. The loss rate of dimers and atoms is proportional to the product of the number densities of the atoms and the dimers:

$$\frac{d}{dt} n_A(r) = \frac{d}{dt} n_D(r) = -\beta n_A(r)n_D(r),$$  \hspace{1cm} (3.12)

where $\beta$ is the relaxation event rate coefficient. The event rate coefficient can be expressed concisely in terms of the atom-dimer scattering length, $a_{AD}$:

$$\beta = -\frac{6\pi \hbar}{m} \text{Im} \ a_{AD}. \hspace{1cm} (3.13)$$
It is convenient to express $a_{AD}$ in terms of the scattering length $a_*$ where an Efimov trimer crosses the atom-dimer threshold, which is given in Eq. (2.27). The atom-dimer scattering length is conveniently expresses as a function of $a$, $a_*$, and $\eta_*:

\begin{equation}
    a_{AD} = (1.46 + 2.15 \cot [s_0 \ln (a/a_*) + i\eta_*]) a. \tag{3.14}
\end{equation}

Inserting Eq. (3.14) into Eq. (3.13), we obtain an explicit expression for $\beta$:

\begin{equation}
    \beta = \frac{20.3 \sinh (2\eta_*)}{\sin^2 [s_0 \ln (a/a_*)] + \sinh^2 \eta_*} \frac{\hbar a}{m}. \tag{3.15}
\end{equation}

The relaxation event rate coefficient is shown as a function of $a$ in Fig. 3.4 for $\eta_* = 0.03$. The relaxation event rate coefficient has log-periodic modulation of a linear scaling behavior in $a$. It has peaks at scattering lengths $(e^{\pi/s_0})^n a_*$ that become infinitely sharp and narrow in the limit $\eta_* \to 0$.

### 3.4 Four-Body Recombination

Recombination of atoms is not restricted to three atoms. Four or more atoms can have inelastic collisions in which molecules are formed [25, 26]. The loss rate of atoms...
due to four-body recombination is

$$\frac{d}{dt} n_A(r) = -L_4 n_A^4(r), \quad (3.16)$$

where $L_4$ is the four-atom loss rate coefficient. At low density and generic scattering lengths, four-atom recombination is suppressed compared to 3-body recombination. This suppression is reflected by the extra power of $n_A$ multiplying $L_4$ in Eq. (3.16), which leads to a dimensionless suppression factor $n_A a^3$. Although four-atom recombination losses are generally suppressed, they can be larger than three-atom recombination losses at certain values of the scattering length. In particular, at the negative scattering lengths $a_1$ and $a_2$ where universal tetramers cross the four-atom threshold, resonant peaks can appear in the loss rate that stand out above the background from 3-body recombination \[27, 28\].

### 3.5 Dimer-Dimer Inelastic Scattering

Another type of low-energy inelastic collision involving four atoms is the inelastic scattering of two shallow dimers. The possible final states are two deep dimers, a deep dimer and a shallow dimer, a deep dimer and two free atoms, and a trimer and a free atom. The processes that involve at least one deep dimer are illustrated in Fig. 3.6. By conservation of energy, the increase in the total kinetic energy must be equal to the difference between the sum of the binding energies of the incoming dimers and the outgoing molecules. If the kinetic energies of the outgoing particles are larger than the depth of the trapping potential, they can escape from the trap. This results in a decrease in the number of trapped atoms.

The low-energy inelastic collision rate can be resonantly enhanced when a universal tetramer is near the dimer-dimer threshold. To observe these losses in experiment, this process requires a population of shallow dimers. The loss rate of dimers is proportional
Figure 3.6: Inelastic dimer-dimer scattering process: two shallow dimers collide with small momenta to form a deep dimer and either a deep dimer, shallow dimer, or two free atoms with large recoil momenta.

to the square of the number density of dimers:

\[
\frac{d}{dt} n_D(r) = -\beta_2 n_D^2(r),
\]

(3.17)

where \( \beta_2 \) is the dimer-dimer event rate coefficient. The event rate coefficient, in the limit where the incoming momenta are negligible, can be described in terms of the dimer-dimer scattering length, \( a_D \):

\[
\beta_2 = -\frac{4\pi \hbar}{m} \text{Im} \, a_D.
\]

(3.18)

The dimer-dimer scattering length has been calculated numerically by Deltuva as a function of \( a \) for \( \eta_* = 0 \). An analytic model for \( a_D \) with \( \eta_* > 0 \) is presented in Appendix B. The resulting approximation for \( \beta_2 \) is shown as a function of \( a \) in Figure B.2 in Appendix B. Like the atom-dimer event rate coefficient \( \beta \), the dimer-dimer event rate coefficient \( \beta_2 \) has a log-periodic modulation of a linear scaling behavior in \( a \). The amplitude of the log-periodic modulation produces local maxima at scattering lengths \( (e^{\pi/s_0})^n a_{1*} \) and \( (e^{\pi/s_0})^n a_{2*} \), which are very narrow in the limit \( \eta_* \to 0 \). We refer to these scattering lengths as dimer-dimer resonances.
3.6 Observations of Loss Features

3.6.1 Efimov trimers

The first discovery of an Efimov trimer in atomic physics was made in August 2007 by a group at the University of Innsbruck led by Rudi Grimm [29]. They observed a resonant enhancement in the three-body recombination rate in a thermal gas of ultracold bosonic $^{133}$Cs atoms with large negative scattering length. They also observed an interference minimum in the recombination rate at a positive scattering length. In a subsequent experiment with a mixture of $^{133}$Cs atoms and dimers, the Innsbruck group observed an atom-dimer loss resonance [30].

Efimov trimers have also been observed using other types of bosonic atoms. In April 2009, the Florence group observed the three-body recombination loss resonances associated with two successive Efimov trimers in a BEC of ultracold $^{39}$K atoms [10]. The ratio of the scattering lengths at these resonances was consistent with the predicted discrete scaling factor of 22.7. In June 2009, a group at Bar-Ilan University observed an Efimov loss resonance and an interference minimum on opposite sides of a Feshbach resonance in a thermal gas of ultracold $^7$Li atoms [31]. In March 2010, this group observed an Efimov loss resonance in a thermal gas consisting of ultracold $^7$Li atoms in a different hyperfine spin state [32]. Since the position of the loss feature agreed with that for the other hyperfine state, they concluded that the short-range physics is nuclear-spin independent.

3.6.2 Universal tetramers

The universal tetramers associated with an Efimov trimer have been observed through four-atom loss processes [19, 20, 21]. In March 2009, the Innsbruck group observed loss features from two universal tetramers associated with an Efimov trimer.
in $^{133}$Cs atoms \cite{33}. In November 2009, the Rice group observed two sets of universal tetramers that are associated with two successive Efimov trimers in $^7$Li atoms \cite{27}. In the reanalysis of the data in February 2012 however, only one of these sets of tetramers remain \cite{28}.

### 3.6.3 More complicated systems

In January 2009, a group at the University of Florence observed three-body recombination loss resonances in a mixture of $^{39}$K and $^{87}$Rb atoms \cite{34}. They can be attributed to heteronuclear Efimov trimers that are composed of either two K atoms and one Rb atom or two Rb atoms and one K atom.

### 3.6.4 Unexplained Loss Features

The subject of this thesis is unexplained loss features that have been observed at positive scattering lengths. In April 2009, the Florence group observed narrow peaks at positive scattering length in a system of ultracold $^{39}$K atoms \cite{10}. For $a < 600 a_0$, they used a BEC of $^{39}$K atoms and for $a > 600 a_0$, they used a thermal gas of $^{39}$K atoms. They observed two narrow peaks at scattering lengths that were near the predicted positions of atom-dimer resonances. The first peak occurs at a small scattering length, $30 a_0$, that is smaller than the van der Waals length for $^{39}$K, $\ell_{v_{\text{dW}}} = 129 a_0$ \cite{8}. It is therefore possible that this feature arises from non-universal physics. However the second peak at a scattering length of about $930 a_0$ that is well into the universal regime. It is near an atom-dimer resonance where an Efimov trimer crosses the atom-dimer threshold.

In November 2009, a group at Rice University studied atom loss features in a BEC of $^7$Li atoms at positive scattering lengths and in a thermal gas of $^7$Li atoms at negative scattering lengths. In the BEC, they observed two narrow peaks at the
predicted positions of atom-dimer resonances, a peak at the position of a dimer-dimer resonance due to a shallow tetramer, and another peak at the position of a dimer-dimer resonance due to a deep tetramer [27]. In February 2012, the data used in this experiment was reanalyzed using a more accurate determination of the scattering length as a function of the magnetic field [28]. One of the peaks near an atom-dimer resonance was not observed in the reanalysis.

In January 2012, the Bar-Ilan group observed a peak in $L_3$ at positive scattering length in a thermal gas of ultracold $^7$Li atoms [35]. They attributed this peak to an atom-dimer resonance.

To explain the loss features that they observed near atom-dimer resonances, the Florence group developed the *avalanche mechanism* in which an energetic dimer from a three-body recombination event collides with atoms in the trap causing them to escape [10]. In Chapter 4, I show that the avalanche mechanism fails to explain the atom-dimer loss features observed in the Florence, Rice, and Bar-Ilan experiments. In Chapter 5, I propose a new dimer condensate mechanism that can explain both atom-dimer loss features and dimer-dimer loss features in experiments using a Bose-Einstein condensate.
Chapter 4

FAILURE OF THE AVALANCHE MECHANISM

The following chapter was posted on the arXiv electronic preprint archive in September 2012 as a paper entitled *Avalanche mechanism for the enhanced loss of ultracold atoms*. It was published in February 2013 in Physical Review A [11]. This chapter demonstrates the failure of the avalanche mechanism as an explanation for the loss features at atom-dimer resonances that have been observed in some experiments.

4.1 Introduction

Particles with short-range interactions and an S-wave scattering length $a$ that is large compared to the range have universal low-energy properties that depend on $a$ but not on other details of the interactions or on the structure of the particles [8]. This universality provides deep connections between various fields of physics, including atomic and molecular, condensed matter, nuclear, and particle physics. It has stimulated dramatic advances in theoretical and experimental few-body physics – particularly in the study of the universal few-body reaction rates of ultracold atoms.

Since particles with large scattering length are essentially indivisible at low energies, we refer to them as *atoms*. In the 2-atom sector, the universal properties are simple. If $a > 0$, they include the existence of a loosely-bound diatomic molecule that we refer to as the *shallow dimer*. In the 3-atom sector, the universal properties are
more intricate. In many cases, including identical bosons, they include the existence of a sequence of universal triatomic molecules called Efimov trimers that were discovered by Efimov in 1970 [5]. In the zero-range limit, the spectrum of Efimov trimers is invariant under discrete scale transformations [7]. For identical bosons, the discrete scaling factor is approximately 22.7. Reaction rates among three low-energy atoms also respect discrete scale invariance [36]. We refer to universal few-body phenomena with discrete scaling behavior as Efimov physics.

Ultracold trapped atoms provide an ideal laboratory for studying Efimov physics, because the scattering length can be controlled experimentally using Feshbach resonances. The simplest probes of Efimov physics are loss features: local maxima and minima in the atom loss rate as functions of the scattering length \( a \). The most dramatic signature of an Efimov trimer is the resonant enhancement of the 3-body recombination rate near a negative value of \( a \) for which there is an Efimov trimer at the 3-atom threshold [37]. The first observation of such a loss feature in an ultracold gas of \(^{133}\text{Cs}\) atoms [29] revealed a line shape consistent with universal predictions [24].

In a mixture of atoms and shallow dimers, a narrow loss feature can also be caused by an Efimov trimer near the atom-dimer threshold. We refer to a scattering length \( a_* \) for which an Efimov trimer is exactly at the threshold as an atom-dimer resonance. For \( a \) near \( a_* \), there is resonant enhancement near threshold of the elastic scattering of an atom and the shallow dimer. There is also resonant enhancement of their inelastic scattering into an atom and a more tightly-bound diatomic molecule, which we refer to as a deep dimer. The release of the large binding energy of the deep dimer gives the outgoing atom and dimer large enough kinetic energies to escape from the trapping potential. The resulting peaks in the atom and dimer loss rates near \( a_* \) were first observed in a mixture of \(^{133}\text{Cs}\) atoms and dimers [30].

There have also been observations of narrow loss features near an atom-dimer
resonance in systems consisting of atoms only. Zaccanti et al. observed a narrow loss peak near the predicted position of $a_*$ in a Bose-Einstein condensate of $^{39}\text{K}$ atoms [10]. They also observed a loss peak in a thermal gas near the next atom-dimer resonance, at a scattering length larger by a factor of about 22.7. Pollack et al. observed a loss peak near the predicted position of an atom-dimer resonance in a Bose-Einstein condensate of $^{7}\text{Li}$ atoms [27]. Machtey et al. observed such a loss peak in a thermal gas of $^{7}\text{Li}$ atoms [35]. These loss features near the atom-dimer resonance are puzzling, because the equilibrium population of shallow dimers is expected to be negligible in these systems. Thus, any losses due to inelastic scattering between an atom and a shallow dimer should be negligible.

Zaccanti et al. proposed an \textit{avalanche mechanism} for the enhancement of the atom loss rate near an atom-dimer resonance in systems consisting of atoms only [10]. Near $a_*$, atom-dimer cross sections are resonantly enhanced near threshold. Each 3-body recombination event produces an atom and a shallow dimer with kinetic energies much larger than that required to escape from the trap. If the atom and dimer both escape, 3 atoms are lost. If the dimer instead scatters inelastically, the scattered atom is also lost, so the number of atoms lost is 4. However the dimer can undergo one or more elastic collisions before ultimately escaping or suffering an inelastic collision, and the scattered atoms may gain enough energy to escape from the trap. The scattered atoms may also undergo elastic collisions, producing still more lost atoms. Thus the dimer could initiate an avalanche of lost atoms. The atom from the recombination event could also initiate an avalanche of lost atoms. Thus the number of atoms lost could be significantly larger than 3. Near $a_*$, the resonant enhancement of the atom-dimer elastic cross section increases the probability for the dimer to undergo an elastic collision and initiate an avalanche. This suggests that there should be an increase in the number of atoms lost per recombination event near $a_*$. If the increase is sufficiently
narrow, it could be observed as a local maximum in the atom loss rate. Zaccanti et al. proposed this avalanche mechanism as an explanation for the loss features near the atom-dimer resonance. They also developed a model for calculating the number of atoms lost that demonstrated the plausibility of the avalanche mechanism [10].

In Ref. [12], we analyzed the avalanche mechanism for atom loss and concluded that it was unable to produce a narrow loss feature near an atom-dimer resonance. In this paper, we present a more thorough analysis of the avalanche mechanism. We use Monte Carlo methods to generate avalanches of atoms that are initiated by recombination events and then calculate the number of atoms lost by averaging over avalanches. We confirm that this number can be significantly larger than the naive value 3. However, instead of a narrow peak in the atom loss rate near $a_*$, the avalanche mechanism produces a broad enhancement whose maximum is at a larger value of $a$.

This paper is organized as follows. In Section 4.2, we summarize the few-body physics that is used in our Monte Carlo model for the avalanche mechanism. In Section 4.3, we describe the experimental inputs that are required in the Monte Carlo model. In Section 4.4, we present the Monte Carlo method for generating avalanches initiated by 3-body recombination events. In Section 4.5, we apply the Monte Carlo model to experiments on $^7\text{Li}$, $^{39}\text{K}$, and $^{133}\text{Cs}$ atoms. In Section 4.6, we discuss the possibility of enhanced atom losses near dimer-dimer resonances.

4.2 Few-body physics

In this section, we summarize the few-body physics that enters into our model for the avalanche mechanism. An avalanche is initiated by a 3-body recombination event in which three low-energy atoms collide to create an atom and a diatomic molecule, which can be either the shallow dimer or a deep dimer. The binding energy of the dimer is released in the kinetic energies of the outgoing atom and dimer. In the
case of the deep dimer, the outgoing atom and dimer have very high energies and therefore small cross sections, so they escape from the trapping potential without any collisions. In the case of the shallow dimer, the outgoing atom and dimer have large cross sections, so they may undergo secondary collisions.

4.2.1 Few-body parameters

We consider identical bosons of mass $m$ with a large positive scattering length $a$. The universal few-body reaction rates associated with the zero-range limit are determined by three parameters [8]:

- the scattering length $a$, which can be controlled experimentally by varying the magnetic field near a Feshbach resonance,

- the atom-dimer resonance $a_*$, or an equivalent Efimov parameter, upon which physical observables can only depend log-periodically with discrete scaling factor $e^{\pi/s_0}$, where $s_0 \approx 1.00624$,

- a dimensionless parameter $\eta_*$, which controls the decay width of an Efimov trimer.

The parameter $\eta_*$ is nonzero only if there are deep dimers that provide decay channels for the Efimov trimer. The alkali atoms used in most cold atom experiments have many deep dimers.

In the negative-$a$ region, the most dramatic loss features are a sequence of narrow peaks in the 3-body recombination rate at the 3-atom resonances $(e^{\pi/s_0})^n a_-$, where $e^{\pi/s_0} \approx 22.694$ is the discrete scaling factor, $n$ is an integer, and $a_-$ differs from $a_*$ by a multiplicative constant:

$$a_- = -21.306 \ a_*.$$  (4.1)
The universal ratio $a_-/a_*$ was obtained with 5 digits of accuracy by dividing a 5-digit result for $a_-\kappa_*$ calculated by Deltuva [13] by a 12-digit result for $a_*\kappa_*$ [8], where $\kappa_*$ is the binding momentum of an Efimov trimer in the unitary limit. The parameters $a_-$ and $a_*$ related by Eq. (4.1) are the scattering lengths at which the same Efimov trimer crosses the 3-atom and atom-dimer thresholds. In the positive-$a$ region, the most dramatic loss features are a sequence of minima in the 3-body recombination rate at $(e^{\pi/s_0})^n a_+$, where $n$ is an integer and $a_+$ differs from $a_*$ by a multiplicative constant:

$$a_+ = 4.4724 \ a_*. \quad (4.2)$$

The ratio $a_+/a_- = -e^{-\pi/2s_0}$ was obtained analytically by Hammer, Helfrich, and Petrov [38]. The ratio $a_+/a_*$ in Eq. (4.2) is obtained by multiplying this analytic result by that in Eq. (4.1). Either $a_-$ or $a_+$ can be used as the Efimov parameter in place of $a_*$. The ratios of the positions of loss features can differ from the universal ratios in Eqs. (4.1) and (4.2) due to nonuniversal effects associated with a nonzero range. Range corrections to the universal ratios have been analyzed by Ji, Phillips, and Platter [39].

### 4.2.2 Two-body observables

The 2-body physics that enters into our model for the avalanche mechanism consists of the binding energy for the shallow dimer and the cross section for atom-atom scattering. The universal binding energy for the shallow dimer is

$$E_d = \frac{\hbar^2}{ma^2}. \quad (4.3)$$

1Some papers follow Ref. [29] in using $a_+$ to denote $e^{-\pi/2s_0} a_+ = 0.93882 \ a_*$. This is the position of a local maximum of $L_3/a^4$ but not of $L_3$, where $L_3$ is the 3-body recombination rate constant. The parameter $a_+$ is preferable as an Efimov parameter because, in the limit $\eta_* \to 0$, it is the position of a zero of $L_3$ as well as $L_3/a^4$. The resulting local minimum of $L_3$ is therefore a robust loss feature.
The universal cross section for the elastic scattering of a pair of identical bosonic atoms with center-of-mass wavenumber $k_{cm}$ is

$$\sigma_{AA} = \frac{8\pi a^2}{1 + a^2 k_{cm}^2}. \quad (4.4)$$

This universal expression is accurate if $k_{cm}$ is much smaller than the inverse range.

Three-body recombination at threshold creates an atom with wavenumber $k = 2/(\sqrt{3}a)$. The center-of-mass wavenumber for its first collision is $k_{cm} = 1/(\sqrt{3}a)$. In the elastic collision of an energetic atom with a stationary atom, the kinetic energy of either outgoing atom is smaller than that of the incoming atom by a factor whose average value is 1/2. Thus the kinetic energies of the outgoing atoms decrease rapidly towards 0 as the avalanche develops. The decreasing kinetic energies imply increasing atom-atom cross sections, although the increase is not dramatic. If the recombination atom has many elastic collisions, its cross section is larger than for its first collision by a factor of about 4/3.

### 4.2.3 Three-body recombination rates

The 3-body recombination event that initiates an avalanche either produces an atom and the shallow dimer or an atom and a deep dimer. We consider systems of atoms in which the energy per particle is much smaller than the binding energy $E_d$ of the shallow dimer. We can therefore neglect the energies of the three incoming atoms and use the recombination rates at threshold. If the system of atoms has number density $n$, the recombination event rates can be expressed as $\alpha n^3$, where $\alpha$ is a rate constant. The universal event rate constants $\alpha_{\text{shallow}}$ and $\alpha_{\text{deep}}$ for 3-body recombination at threshold into the shallow dimer and into deep dimers are conveniently expressed as
Figure 4.1: Universal rate constants for 3-body recombination at threshold for \( \eta_* = 0.03 \) as functions of the scattering length. The rate constant \( \alpha_{\text{shallow}} \) for recombination into the shallow dimer is shown as a solid (blue) line. The rate constant \( \alpha_{\text{deep}} \) for recombination into deep dimers is shown as a dashed (red) line that is almost straight. The vertical dotted lines mark the positions of \( a_* \) and 22.7 \( a_* \).

functions of \( a, a_+ \), and \( \eta_* \) [8]:

\[
\alpha_{\text{shallow}} = \frac{128\pi^2(4\pi - 3\sqrt{3})(\sin^2[s_0 \ln(a/a_+)] + \sinh^2\eta_*) \, \hbar a^4}{\sinh^2(\pi s_0 + \eta_*) + \cos^2[s_0 \ln(a/a_+)]} \frac{\hbar a^4}{m}, \tag{4.5a}
\]

\[
\alpha_{\text{deep}} = \frac{128\pi^2(4\pi - 3\sqrt{3}) \coth(\pi s_0) \cosh \eta_* \sinh \eta_* \, \hbar a^4}{\sinh^2(\pi s_0 + \eta_*) + \cos^2[s_0 \ln(a/a_+)]} \frac{\hbar a^4}{m}. \tag{4.5b}
\]

These rate constants are shown as functions of \( a \) in Fig. 4.1 for \( \eta_* = 0.03 \). Both rate constants have log-periodic modulation of \( a^4 \) scaling behavior. For \( \alpha_{\text{shallow}} \), the log-periodic modulation produces local minima at scattering lengths near \((e^{\pi/s_0})^n a_+\) which become zeroes in the limit \( \eta_* \to 0 \). These minima arise from an Efimov
Figure 4.2: Universal cross sections $\sigma_{AD}^{(el)}$ for elastic atom-dimer scattering (left panel) and $k_{cm}^{(in)}$ for inelastic atom-dimer scattering (right panel) for $\eta_s = 0.03$ as functions of the scattering length for three different energies. The vertical dotted lines mark the positions of $a_*$ and $22.7 \, a_*$. The three curves (in order of increasing cross sections at $a = a_*$) are for the first scattering of the recombination dimer, a typical second scattering, and after many elastic scatterings. In the left panel, the three straight closely-spaced (red) lines are the corresponding elastic cross sections for atom-atom collisions.

interference effect. For $\alpha_{\text{deep}}$, the amplitude of the log-periodic modulation is too small to be evident in Fig. 4.1.

4.2.4 Atom-dimer cross sections

The shallow dimer from 3-body recombination can collide with a low-energy atom in the cloud. Since the energy per particle in the atom cloud is much smaller than $E_d$, we neglect the energy of the atom and use the atom-dimer cross sections for a stationary atom. The collision between the shallow dimer and an atom can be elastic, in which case the diatomic molecule in the final state is the shallow dimer, or inelastic, in which case it is a deep dimer.

The energy dependence of the atom-dimer cross sections is important. In the collision of a dimer with wavenumber $k$ with a stationary atom, the center-of-mass wavenumber $k_{cm}$ is $k/3$. An avalanche is initiated by a recombination event at thresh-
old creating an atom and a shallow dimer with wavenumbers \( k = 2/(\sqrt{3}a) \). In the first collision of the dimer with a stationary atom, its center-of-mass wavenumber is \( k_{cm} = 2/(3\sqrt{3}a) \). The collision energy in the center-of-mass frame is \( E_{cm} = \frac{1}{9} E_d \).

A subsequent elastic collision with a stationary atom decreases the dimer’s kinetic energy by a multiplicative factor whose average value is \( 5/9 \). As the number of elastic collisions of the dimer increases, \( E_{cm} \) decreases rapidly towards 0.

The universal elastic and inelastic atom-dimer cross sections \( \sigma_{AD}^{(el)} \) and \( \sigma_{AD}^{(in)} \) are conveniently expressed as functions of \( a, a_*, \eta_*, \) and the center-of-mass wavenumber \( k_{cm} \). The S-wave elastic atom-dimer cross section is given by

\[
\sigma_{AD}^{(el)} = \frac{4\pi}{|k_{cm} \cot \delta_{AD}(k_{cm}) - ik_{cm}|^2}, \tag{4.6}
\]

where \( \delta_{AD}(k_{cm}) \) is the S-wave phase shift \(^{15}\). The total S-wave cross section can be expressed via the optical theorem as \(^{15}\)

\[
\sigma_{AD}^{(total)} = \frac{4\pi}{k_{cm}} \frac{1}{\text{Im} \left( \frac{k_{cm} \cot \delta_{AD}(k_{cm}) - ik_{cm}}{k_{cm}} \right)}. \tag{4.7}
\]

The S-wave inelastic cross section is obtained by subtracting the elastic cross section from the total cross section:

\[
\sigma_{AD}^{(in)} = \frac{4\pi}{k_{cm}} \frac{-\text{Im}[k_{cm} \cot \delta_{AD}(k_{cm})]}{|k_{cm} \cot \delta_{AD}(k_{cm}) - ik_{cm}|^2}. \tag{4.8}
\]

Efimov’s radial law strongly constraints the dependence of the phase shift on \( a_* \) \(^{8}\), implying that it can be expressed as

\[
ka \cot \delta_{AD}(k) = c_1(ka) + c_2(ka) \cot[s_0 \ln(a/a_*) + \phi(ka) + i\eta_*]. \tag{4.9}
\]

The functions \( c_1(ka), c_2(ka), \) and \( \phi(ka) \) have been determined from the atom-dimer threshold \( k = 0 \) to the dimer-breakup threshold \( ka = 2/\sqrt{3} \) by calculating the phase shifts \( \delta_{AD}(k) \) numerically using an effective field theory \(^{8}\). The results were
parametrized as

\[ c_1(ka) = -0.22 + 0.39 (ka)^2 - 0.17 (ka)^4, \]  
\[ c_2(ka) = 0.32 + 0.82 (ka)^2 - 0.14 (ka)^4, \]  
\[ \phi(ka) = -0.83 (ka)^2 + 0.23 (ka)^4. \] (4.10a)

In the low-energy limit, the cross sections in Eqs. (4.6) and (4.8) are determined by the atom-dimer scattering length \( a_{AD} \), which can be expressed in the form

\[ a_{AD} = (b_1 + b_2 \cot[s_0 \ln(a/a_*) + i\eta_*]) a, \] (4.11)

where \( b_1 \) and \( b_2 \) are universal numerical constants: \( b_1 \approx 1.46, b_2 \approx -2.15 \). In the limit \( \eta_* \to 0 \), \( a_{AD} \) diverges at the atom-dimer resonance \( a = a_* \). The low-energy limit of the elastic cross section is

\[ \sigma_{AD}^{(el)} \longrightarrow 4\pi |a_{AD}|^2. \] (4.12)

The low-energy limit of the inelastic cross section multiplied by \( k_{cm} \) is

\[ k_{cm} \sigma_{AD}^{(in)} \longrightarrow 4\pi |a_{AD}|^2 \text{Im}(1/a_{AD}). \] (4.13)

Both of the cross sections in Eqs. (4.12) and (4.13) have a factor of \( \sin^2[s_0 \ln(a/a_*)] + \sinh^2 \eta_* \) in the denominator that produces a sharp peak at the atom-dimer resonance if \( \eta_* \ll 1 \).

The energy dependence of the universal atom-dimer cross sections is illustrated in Fig. 4.2. The elastic cross section \( \sigma_{AD}^{(el)} \) and the inelastic cross section \( k_{cm} \sigma_{AD}^{(in)} \) are shown as functions of \( a \) for \( \eta_* = 0.03 \) at three different energies. These energies correspond to the first collision of the recombination dimer (\( E_{cm} = \frac{1}{9} E_d \)), a typical second collision (\( E_{cm} = \frac{5}{81} E_d \)), and after many elastic collisions (\( E_{cm} \to 0 \)). For the first collision, the elastic cross section has a broad peak as a function of \( a \) with a local
maximum near 4.34 $a_*$, which is close to the minimum in the 3-body recombination rate: $4.34 \ a_* \approx 0.97 \ a_+$. The inelastic cross section for the first collision increases monotonically with $a$. For the typical second collision of the recombination dimer, the elastic cross section is similar to that of the first collision, but the inelastic cross section has a broad maximum just above $a_*$. After many elastic collisions, both the elastic and inelastic atom-dimer cross sections peak sharply near $a_*$. 

The atom-dimer cross sections in Eqs. (4.6) and (4.8) are the S-wave contributions only. There are also contributions from higher partial waves. In the universal zero-range limit, the higher partial wave contributions to the inelastic cross sections vanish and the higher partial wave contributions to the elastic cross sections are determined only by the scattering length $a$. The contribution from the $L$’th partial wave has the threshold behavior $(E_{cm}/E_d)^L$. The leading contribution is P-wave, and it is suppressed by a factor of $E_{cm}/E_d$. Thus its contribution to the atom-dimer elastic cross section for the first collision of the recombination dimer ($E_{cm} = \frac{1}{9} \ E_d$) is expected to be about an order of magnitude smaller than the atom-atom cross section. In Fig. 4.2, the P-wave atom-dimer cross section would be given by a straight line that is parallel to but significantly lower than the lowest straight line for the atom-atom cross section. It would be completely negligible near an atom-dimer resonance at $a = a_*$, but it would decrease the depths of the minima in the cross section.

4.3 Experimental inputs

In this section, we summarize the experiments that have observed narrow loss features near an atom-dimer resonance. We identify the variables for these experiments that are required as inputs to our Monte Carlo model for the avalanche mechanism.
4.3.1 Loss features near the atom-dimer resonance

The first observation of a loss feature near an atom-dimer resonance was by a group at Innsbruck in 2008 [30]. They used a thermal gas that was a mixture of $^{133}$Cs atoms and shallow dimers composed of those atoms. They observed peaks in the atom and dimer loss rates at a scattering length near $+400 \, a_0$. The peak arises from the resonant enhancement of the inelastic scattering of an atom and a shallow dimer into an atom and a deep dimer due to an Efimov trimer near the atom-dimer threshold. The Innsbruck group has also measured the loss rate for atom clouds consisting of $^{133}$Cs atoms only. They did not observe any loss features near the atom-dimer resonance in systems with atoms only.

There are three experiments that have observed loss features near an atom-dimer resonance in atom clouds that do not contain dimers. The first such experiment was by a group at Florence in 2009 using both a BEC and a thermal gas of $^{39}$K atoms [10]. They measured the 3-body loss rate constant $L_3$ as a function of the scattering length. They observed a peak in $L_3/a^4$ near $-1500 \, a_0$ that can be attributed to an Efimov trimer near the 3-atom threshold. They also observed two local minima in $L_3/a^4$ near $+224 \, a_0$ and $+5650 \, a_0$ that can be attributed to successive Efimov interference minima. Of these three loss features, the highest precision in the determination of $\eta^*$ was obtained from the local minimum near $224 \, a_0$. We therefore use this loss feature to determine the Efimov parameters: $a_+ = 224 \, a_0$ and $\eta^* = 0.043$. Given this value of $a_+$ and the universal ratio in Eq. (4.2), atom-dimer resonances are predicted near $50 \, a_0$ and $1140 \, a_0$. The Florence group observed enhancements in the loss rate near $+30.4 \, a_0$ in a BEC and near $+930 \, a_0$ in a thermal gas, both of which are reasonably close to the predicted atom-dimer resonances. They attributed these loss peaks to the avalanche mechanism of enhanced losses from secondary elastic collisions [10].

Another experiment in 2009 that observed a loss feature near an atom-dimer
resonance was by a group at Rice University using both a BEC and a thermal gas of $^7$Li atoms in the $|1, +1\rangle$ hyperfine state [27]. They observed two peaks in $L_3/a^4$ near $-298 \ a_0$ and near $-6301 \ a_0$ that can be attributed to successive Efimov trimers near the 3-atom threshold. They also observed two local minima in $L_3/a^4$ near $+119 \ a_0$ and $+2676 \ a_0$ that can be attributed to successive Efimov interference features. Of these four loss features, the highest precision in the determination of $\eta_*$ was obtained from the local minimum near $2676 \ a_0$. We therefore use this loss feature to determine the Efimov parameters: $a_+ = 2676 \ a_0$ and $\eta_+ = 0.039$. Given this value of $a_+$, an atom-dimer resonance is predicted at $598 \ a_0$. The Rice group observed an enhancement in $L_3/a^4$ in a BEC near $+608 \ a_0$, which is close to the predicted atom-dimer resonance.

The third experiment that observed a loss feature near an atom-dimer resonance in an atom cloud that contained no dimers was by a group at Bar-Ilan University using a thermal gas of $^7$Li atoms in either the $|1, +1\rangle$ or $|1, 0\rangle$ hyperfine state [32]. For both hyperfine states, they observed a peak in $L_3/a^4$ near $-270 \ a_0$ that can be attributed to an Efimov trimer near the 3-atom threshold and a local minimum in $L_3/a^4$ near $+1170 \ a_0$ that can be attributed to Efimov interference. A more thorough analysis was presented in Ref. [40]. The Efimov parameters determined by fitting $L_3$ are $a_+ = 1260 \ a_0$ and $\eta_+ = 0.188$. Given this value of $a_+$, an atom-dimer resonance is predicted at $282 \ a_0$. In Ref. [35], additional data for $a$ below $220 \ a_0$ were presented, revealing a narrow loss peak in $L_3$ near $+200 \ a_0$, which is reasonably close to the predicted atom-dimer resonance.

### 4.3.2 Experimental variables

The important experimental variables in the measurements of the loss rates of trapped atoms include the following:

- the frequencies $\nu_x$, $\nu_y$, and $\nu_z$ of the harmonic trapping potential, which has the
Table 4.1: Experimental variables for experiments with $^7$Li, $^{39}$K, and $^{133}$Cs atoms: the trapping frequencies $\nu_x$, $\nu_y$, and $\nu_z$, the initial number of atoms $N_0$, the trap depth $E_{\text{trap}}$, the temperature $T$, the holding time $t_{\text{hold}}$, and the range of scattering lengths $a$. In the case of a BEC, we give only an upper bound on the temperature $T$. The upper bound is $\frac{1}{2}T_c$ for the BEC in Ref. [27] and $T_c$ for the thermal gas in Ref. [10].

<table>
<thead>
<tr>
<th>Variable</th>
<th>$^7$Li</th>
<th>$^{39}$K</th>
<th>$^{133}$Cs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_x$</td>
<td>236.0 Hz</td>
<td>1300.0 Hz</td>
<td>75.0 Hz</td>
</tr>
<tr>
<td>$\nu_y$</td>
<td>236.0 Hz</td>
<td>1300.0 Hz</td>
<td>75.0 Hz</td>
</tr>
<tr>
<td>$\nu_z$</td>
<td>4.6 Hz</td>
<td>190.0 Hz</td>
<td>75.0 Hz</td>
</tr>
<tr>
<td>$N_0$</td>
<td>$4.0 \times 10^5$</td>
<td>$3.5 \times 10^4$</td>
<td>$1.3 \times 10^5$</td>
</tr>
<tr>
<td>$E_{\text{trap}}$</td>
<td>0.5 $\mu$K</td>
<td>7 $\mu$K</td>
<td>1.0 $\mu$K</td>
</tr>
<tr>
<td>$T$</td>
<td>&lt;0.105 $\mu$K</td>
<td>1.4 $\mu$K</td>
<td>&lt;0.17 $\mu$K</td>
</tr>
<tr>
<td>$t_{\text{hold}}$</td>
<td>$\sim 0.003$ s</td>
<td>0.01–5 s</td>
<td>1 s</td>
</tr>
<tr>
<td>$a$</td>
<td>500–4000 $a_0$</td>
<td>159–2663 $a_0$</td>
<td>68–372 $a_0$</td>
</tr>
</tbody>
</table>

In Table 4.1, we list the important experimental variables for five experiments with $^7$Li atoms [27, 40], $^{39}$K atoms [10], and $^{133}$Cs atoms [41]. In two of the 5 experiments, the atom cloud was a BEC and in the other three, it was a thermal gas. The experimental variables that are not given explicitly in the references were obtained from private communications with the authors. Different values of the experimental variables were used in different regions of the scattering length. The values listed in Table 4.1 are those that were used in the range of $a$ given in the Table.
The holding time \( t_{\text{hold}} \) is generally chosen to be large enough that a significant fraction of the initial number \( N_0 \) of atoms are lost, so that this fraction can be measured with some precision. The product of \( t_{\text{hold}} \) and a trapping frequency gives the number of periods of the oscillation in that dimension before the atom number is measured. The holding time is not used as an input in the Monte Carlo model for simulating avalanches described in Section 4.4.

We use a simple model for the trap depth that is specified by the single variable \( E_{\text{trap}} \). Atoms and dimers that reach the edge of the atom cloud are assumed to be lost if their kinetic energies exceed \( E_{\text{trap}} \) and \( 2E_{\text{trap}} \), respectively. Equivalently, this model for the trap depth can be expressed as a modification of the trapping potential for the atoms. The potential for a single atom is given by Eq. (4.14) if \( V(x, y, z) < E_{\text{trap}} \) and is equal to the constant \( E_{\text{trap}} \) if \( V(x, y, z) > E_{\text{trap}} \). The trap depth \( E_{\text{trap}} \) is generally substantially larger than the energy per atom \( E/N \).

The number of atoms lost also depends on the dimer binding energy \( E_d = \frac{\hbar^2}{ma^2} \), which depends on the scattering length. If \( E_d < \frac{3}{2}E_{\text{trap}} \), the recombination atom and the recombination dimer both remain trapped. The dimer will eventually scatter inelastically from an atom in the cloud, so the number of lost atoms is 3. If \( \frac{3}{2}E_{\text{trap}} < E_d < 6E_{\text{trap}} \), the recombination dimer is trapped and must ultimately scatter inelastically, but the recombination atom is not trapped. The largest possible number of lost atoms in the avalanche initiated by the recombination atom is the integer part of \( \frac{2}{3} \frac{E_d}{E_{\text{trap}}} \), which can be 1, 2, or 3, depending on \( a \). The dimer could also produce a single lost atom through an elastic collision, and it will eventually suffer an inelastic collision, resulting in the loss of 3 more atoms. Thus the maximum number of lost atoms increases from 3 to 7 as \( E_d \) increases from \( \frac{3}{2}E_{\text{trap}} \) to \( 6E_{\text{trap}} \). If \( E_d > 6E_{\text{trap}} \), neither the atom nor the shallow dimer is trapped, so \( N_{\text{lost}} \) can be as large as \( E_d/E_{\text{trap}} + 3 \). Our simple model for the trap depth implies unphysical
discontinuities in physical observables at $E_d = \frac{3}{2}E_{\text{trap}}$ and $E_d = 6E_{\text{trap}}$. Thus a more elaborate model is probably required to give accurate predictions for the number of lost atoms in a region that includes the interval $\frac{3}{2}E_{\text{trap}} < E_d < 6E_{\text{trap}}$.

### 4.3.3 Number densities

The frequencies $\nu_x$, $\nu_y$, and $\nu_z$, the initial number $N_0$ of trapped atoms, and the temperature $T$ determine the number density $n(x,y,z)$ of the atoms. We consider three simple cases for the system of trapped atoms:

- a Bose-Einstein condensate (BEC) of atoms at 0 temperature in the Thomas-Fermi limit,
- a thermal gas of atoms in the weak-interaction limit at the critical temperature $T_c$,
- a thermal gas of atoms in the weak-interaction limit at a temperature $T$ much larger than $T_c$.

In a BEC of atoms at zero temperature in the Thomas-Fermi limit, the number density depends on the scattering length $a$:

$$n(x,y,z) = \frac{m}{4\pi\hbar^2 a} \max\{\mu(a) - V(x,y,z), 0\}, \quad (4.15)$$

where $\mu(a)$ is the chemical potential, which also depends on $a$:

$$\mu(a) = \frac{\hbar^2}{2m} \left( \frac{15Na}{a_x^2a_y^2a_z^2} \right)^{2/5}. \quad (4.16)$$

The trap lengths $a_x$, $a_y$, and $a_z$ are determined by the trapping frequencies: $a_i = (\hbar/2\pi m\nu_i)^{1/2}$. The energy per atom is just the chemical potential:

$$E/N = \mu(a). \quad (4.17)$$
The critical temperature for Bose-Einstein condensation in the trapping potential is

\[ k_B T_c = \frac{\hbar^2}{m} \left( \frac{N N \zeta(3)}{a_x^2 a_y^2 a_z^2} \right)^{1/3}, \]  

(4.18)

where \( \zeta(3) \approx 1.202 \). For a thermal cloud of trapped atoms above \( T_c \), the appropriate phase-space distribution in the weak-interaction limit is the Bose-Einstein distribution. At the critical temperature, the number density can be expressed in terms of a polylogarithm:

\[ n(x, y, z) = \left( \frac{m k_B T_c}{2 \pi \hbar^2} \right)^{3/2} \text{Li}_{3/2}(\exp(-V(x, y, z)/k_B T_c)). \]  

(4.19)

The energy per atom at \( T_c \) is

\[ E/N = \frac{\pi^4}{30 \zeta(3)} k_B T_c, \]  

(4.20)

which is approximately 2.7012 \( k_B T_c \).

If \( T \) is large enough compared to \( T_c \), the Bose-Einstein distribution can be approximated by the Boltzmann distribution. The number density then reduces to a Gaussian:

\[ n(x, y, z) = \frac{N \lambda_T^3}{8 \pi^3 a_x^2 a_y^2 a_z^2} \exp(-V(x, y, z)/k_B T), \]  

(4.21)

where \( \lambda_T = (2 \pi \hbar^2/m k_B T)^{1/2} \) is the thermal quantum wavelength. The energy per atom is given by the equipartition theorem:

\[ E/N = 3 k_B T. \]  

(4.22)

### 4.3.4 Loss rate and heating rate

The rate at which the local number density \( n(r) \) of atoms in a thermal gas decreases due to 3-body recombination can be expressed as a local differential equation:

\[ \frac{d}{dt} n(r) = - (N_{\text{lost}}(r) \alpha_{\text{shallow}} + 3 \alpha_{\text{deep}}) n^3(r), \]  

(4.23)

58
where $N_{\text{lost}}(r)$ is the average number of atoms lost from a recombination event that creates a shallow dimer at the point $r$. Upon integrating Eq. (4.23) over space, we obtain the rate at which the number $N$ of atoms decreases:

$$\frac{dN}{dt} = - (\langle N_{\text{lost}} \rangle \alpha_{\text{shallow}} + 3 \alpha_{\text{deep}}) \langle n^2 \rangle N,$$

(4.24)

where $\langle n^2 \rangle$ and $\langle N_{\text{lost}} \rangle$ are spacial averages weighted by $n(r)$ and $n^3(r)$, respectively. Equivalently, $\langle N_{\text{lost}} \rangle$ is the number of atoms lost in a single avalanche averaged over the probability distribution for avalanches. In the case of a BEC, the right sides of Eqs. (4.23) and (4.24) should be multiplied by 1/6 to take into account that the 3 atoms undergoing recombination are identical bosons. The loss rate constant $L_3$ is defined to be the coefficient of $-\langle n^2 \rangle N$ in $dN/dt$ for thermal gas:

$$L_3 = \langle N_{\text{lost}} \rangle \alpha_{\text{shallow}} + 3 \alpha_{\text{deep}}.$$

(4.25)

Measurements of $L_3$ in a BEC and in a thermal gas of the same atoms should agree to within experimental uncertainties.

Atoms in the avalanche with kinetic energy less than $E_{\text{trap}}$ can never escape from the trapping potential. Through subsequent elastic collisions, their kinetic energy is ultimately transformed into heat. If most of the heat is deposited near the recombination point, the rate at which a thermal gas of trapped atoms gains heat $Q$ from the avalanche mechanism is

$$\frac{dQ}{dt} = \langle E_{\text{heat}} \rangle \alpha_{\text{shallow}} \langle n^2 \rangle N,$$

(4.26)

where $\langle E_{\text{heat}} \rangle$ is the average amount of energy transformed into heat in a single avalanche. In the case of a BEC, the right side of Eqs. (4.26) should be multiplied by 1/6 to take into account that the 3 atoms undergoing recombination are identical bosons.
The number density profiles in Eq. (4.15) for a BEC and in Eq. (4.21) for a thermal gas are those that would be expected in the absence of atom loss processes, such as 3-body recombination. Loss processes decrease the number $N$ of atoms, allow energy to be carried out of the system by the lost atoms, and also add heat $Q$ to the system. These effects can change the number density and the energy density of atoms as functions of time. In the case of a BEC, the atom loss and heating can generate a thermal cloud inside and surrounding the BEC. If too much energy is added to the system, its temperature can be raised above the critical temperature for Bose-Einstein condensation, in which case the BEC component disappears completely.

In the case of a thermal gas, the atom loss and the heating change the number of atoms $N$ and their total energy. If the thermalization rate is sufficiently fast, the number density can still be approximated by the density profile in Eq. (4.21) with time-dependent $N$ and $T$. To measure the loss rate constant $L_3$, that time dependence must be taken into account. A method for doing this was developed in Ref. [42]. The coupled rate equations for $N$ and $T$ (in the absence of background gas collisions) were expressed in the form

$$\frac{dN}{dt} = -\frac{\gamma N^3}{T^3},$$  

(4.27a)

$$\frac{dT}{dt} = \frac{\gamma(T + T_h)N^2}{3T^3},$$  

(4.27b)

where $\gamma$ and $T_h$ are constants. The solutions to these coupled differential equations depend on $\gamma$ and $T_h$. If $N(t)$ is measured as a function of the holding time $t$, the two parameters can be adjusted to fit that time dependence. The rate constant $L_3$ can then be determined from the fitted value of $\gamma$:

$$L_3 = \left(\frac{\sqrt{3k}}{2\pi m\bar{\nu}^2}\right)^3 \gamma,$$  

(4.28)

where $\bar{\nu} = (\nu_x^{1/3}\nu_y^{1/3}\nu_z^{1/3})$ is the geometric mean of the trapping frequencies.
We can derive the coupled equations for $N$ and $T$ in Eqs. (4.27) from our rate equations for $N$ and $Q$ in Eqs. (4.24) and (4.26). This derivation determines the fitting parameters $\gamma$ and $T_h$ in terms of the quantities $\langle N_{\text{lost}} \rangle$ and $\langle E_{\text{heat}} \rangle$ associated with the avalanche mechanism. The total energy $E$ of the thermal gas in a harmonic trap is $E = 3Nk_BT$. It changes because the recombination event delivers energy to an atom and a dimer, the lost atoms carry away their kinetic energy, and the atoms that are elastically scattered but remain trapped deposit their energy as heat. The average energy of an atom in the thermal cloud is $3k_BT$. Since the recombination probability is proportional to $n^3(x, y, z)$, the incoming atoms in a 3-body recombination event have a smaller average energy $2k_BT$. If all the lost atoms originate near the recombination point, their average energy is also $2k_BT$. Thus the rate of change in the total energy is

$$\frac{dE}{dt} = - (\langle N_{\text{lost}} \rangle \alpha_{\text{shallow}} + 3\alpha_{\text{deep}}) \langle n^2 \rangle N(2k_BT) + \frac{dQ}{dt}.$$ (4.29)

Setting $E = 3Nk_BT$ and using Eqs. (4.24) and (4.26) for $dN/dt$ and $dQ/dt$, we can obtain a rate equation for $T$:

$$\frac{dT}{dt} = \left( \frac{\alpha_{\text{shallow}} \langle E_{\text{heat}} \rangle}{3k_BT} + \frac{\langle N_{\text{lost}} \rangle \alpha_{\text{shallow}} + 3\alpha_{\text{deep}}}{3} \right) \langle n^2 \rangle T.$$ (4.30)

The density-weighted average $\langle n^2 \rangle$ in a thermal gas is

$$\langle n^2 \rangle = \frac{1}{3\sqrt{3}} \left( \frac{N\lambda_T^3}{8\pi^3a_x^2a_y^2a_z^2} \right)^2.$$ (4.31)

Since this is proportional to $N^2/T^3$, the rate equations for $N$ in Eq. (4.24) and $T$ in Eq. (4.30) do have the form given in Eqs. (4.27). The constant $\gamma$ is proportional to the rate constant $L_3$ in Eq. (4.25) in accord with Eq. (4.25). The product of the constants $\gamma$ and $T_h$ in Eq. (4.27b) is determined by $\langle E_{\text{heat}} \rangle$ only:

$$\gamma T_h = \left( \frac{2\pi m\bar{v}^2}{\sqrt{3}k} \right)^3 \frac{\alpha_{\text{shallow}} \langle E_{\text{heat}} \rangle}{k}.$$ (4.32)
In Ref. [42], $k_B T_h$ was interpreted as the energy per lost atom. Combining Eqs. (4.32) and (4.28) with the expression for $L_3$ in Eq. (4.25), we see that $k_B T_h$ is indeed equal to $E_{\text{heat}}$ if $\alpha_{\text{deep}}$ is negligible compared to $\alpha_{\text{shallow}}$.

In Section 4.4, we develop a Monte Carlo model for the avalanche mechanism that can be used to calculate $N_{\text{lost}}$ and $E_{\text{heat}}$. These quantities can also be determined experimentally using the values of $\gamma$ and $T_h$ obtained by fitting the time dependence of $N(t)$. By comparing the calculated and measured values of $N_{\text{lost}}$ and $E_{\text{heat}}$, we could test our Monte Carlo model for the avalanche mechanism and perhaps develop a more accurate description of the loss process.

### 4.4 Monte Carlo method

In this section, we describe our Monte Carlo model for the avalanche mechanism. We also compare it to previous models for the avalanche mechanism.

#### 4.4.1 Approximations

The important energy scales in cold atom experiments include the energy per atom $E/N$, the trap depth $E_{\text{trap}}$, and the dimer binding energy $E_d = \hbar^2/ma^2$, which depends on the scattering length $a$. For a thermal gas with temperature $T$, $E/N$ is $3k_B T$. For a BEC, $E/N$ is equal to the chemical potential $\mu(a)$ given in Eq. (4.16), which depends on $a$. Another relevant energy scale is $2.7 k_B T_c$, which is the energy per atom at the critical temperature. In the case of a thermal gas, $E/N$ must be significantly larger than $2.7 k_B T_c$ in order to use the Boltzmann distribution instead of the Bose-Einstein distribution. In the case of a BEC, $N(2.7 k_B T_c)$ is roughly the heat energy that must be added to change it to a thermal gas. The various energy scales are listed in Table 4.2 for each of the five sets of experimental variables listed in Table 4.1.
Our simple model for the trap depth is described in Section 4.3.2. If an energetic atom or dimer reaches the edge of the cloud, it is lost from the trap if its kinetic energy is greater than $E_{\text{trap}}$ or $2E_{\text{trap}}$, respectively. Otherwise it will follow a curved trajectory that returns to the cloud. A trapped atom that returns to the cloud will eventually thermalize through elastic collisions, transforming its kinetic energy into heat. A trapped dimer that returns to the atom cloud will eventually suffer an inelastic collision that results in the loss of 3 atoms. Before the inelastic collision, it could scatter elastically, transforming some of its kinetic energy into heat, but we ignore that small contribution to the heat.

The trap depth $E_{\text{trap}}$ is usually substantially larger than the energy per particle $E/N$. Otherwise, atoms will be rapidly lost from the trap until most of the atoms have energy smaller than $E_{\text{trap}}$. The energy per particle is more than an order of magnitude smaller than $E_{\text{trap}}$ over most of the range of scattering length for most of the experiments listed in Table 4.1. The exceptions are the $^7$Li BEC experiment at the upper end of the range of $a$, where $E/N$ is about 0.5 $E_{\text{trap}}$ and the $^7$Li and $^{39}$K thermal gas experiments, in which $E/N$ is also about 0.5 $E_{\text{trap}}$. In these cases, our simple model for the trap depth may not be sufficient to calculate the effects of the avalanche mechanism accurately.

As $a$ is increased by adjusting the magnetic field, the dimer binding energy $E_d$ can decrease from much larger than $E_{\text{trap}}$ to much smaller than $E_{\text{trap}}$. However it usually remains much larger than $E/N$. This allows the kinetic energies of the atoms in the cloud to be ignored in few-body reaction rates. For the experiments listed in Table 4.1, $E_d$ is more than an order of magnitude larger than $E/N$ over most of the range of scattering lengths. There are a few exceptions near the upper ends of the ranges of $a$. In the $^7$Li BEC experiment and the $^{39}$K thermal gas experiment, $E_d$ becomes as small as $6 E/N$ at the largest values of $a$. In the $^7$Li thermal gas
Table 4.2: Energy scales in µK for experiments with $^7$Li, $^{39}$K, and $^{133}$Cs atoms: the energy per atom $E/N$, the energy per atom at the critical temperature $2.7\ k_B T_c$, the trap depth $E_{\text{trap}}$, and the range of the dimer binding energy $E_d = \hbar^2/m a^2$. The ranges of $E/N$ for a BEC and the ranges of $E_d$ correspond to the ranges of $a$ given in Table 4.1.

If the atom cloud is a BEC in the Thomas-Fermi limit, the trajectory of an energetic atom inside the BEC is a straight line, because the trapping potential energy of an atom and its mean-field energy add up to the constant chemical potential $\mu(a)$. If the atom flies beyond the edge of the BEC, it follows a curved trajectory determined by the harmonic potential. The trajectory of an energetic dimer is curved even inside the BEC, because its mean-field energy differs from that of a pair of atoms. If the atoms are in a thermal cloud, the trajectory of an energetic atom or energetic dimer is always curved. However if the kinetic energy of the atom or dimer is large enough that it can transfer an energy greater than $E_{\text{trap}}$ to an atom in the cloud through an elastic collision, its trajectory has small curvature, and it can be approximated by a straight line. The kinetic energy of an atom or the dimer can change between scattering points, because the potential energy (and, in the case of a BEC, the mean-field energy) depends on the position in the cloud. However if the kinetic energy of the atom or dimer is large enough that it can transfer an energy greater than $E_{\text{trap}}$ to an atom in the cloud, the change in the kinetic energy is negligible.

The rate equations for $N$ and $Q$ in Eqs. (4.24) and (4.26) were derived from the local rate equation for $n(r)$ in Eq. (4.23). However the avalanche mechanism makes
the loss process partly nonlocal. Some of the atoms that escape from the trapping potential receive their kinetic energy from an elastic collision at a scattering point \((x, y, z)\) that may not be near the recombination point \((x_0, y_0, z_0)\). The distance \(\left[ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \right]^{1/2}\) is not a good measure of the nonlocality, because the length scales set by the trapping potential are different in different directions. A better measure of the nonlocality is the dimensionless distance

\[
\hat{\ell}^2 = \frac{(x-x_0)^2}{a_x^2} + \frac{(y-y_0)^2}{a_y^2} + \frac{(z-z_0)^2}{a_z^2},
\]

which is the square of the number of oscillator lengths separating the recombination point and the scattering point. The local approximation for the loss rate will be valid if \(\langle \hat{\ell}^2 \rangle \ll 1\), where the average is over lost atoms and over avalanches. If there are no collisions, the scattering point coincides with the recombination point and \(\hat{\ell}^2 = 0\).

In general, \(\langle \hat{\ell}^2 \rangle\) depends on the prescription for the average. Since the atoms are identical bosons, a lost atom could be identified with any of the stationary atoms in the chain of previous elastic scatterings or with one of the three incoming atoms in the recombination event. Thus the scattering point \((x, y, z)\) for a lost atom can be taken as its point of last scattering or the recombination point or any of the collision points in between. The atoms composing a dimer that escapes or scatters inelastically could be identified with two of the incoming atoms in the recombination event, but they also could be identified with any of the stationary atoms from which the dimer scattered elastically. One possible prescription is to choose the scattering point \((x, y, z)\) to be the first collision point for any of its ancestors in the binary tree with equal probability. A more reasonable prescription is to choose the scattering point \((x, y, z)\) to be the collision point at which the greatest energy is imparted to the lost atom or to one of its ancestors. With this prescription, the 3 lost atoms from an inelastic atom-dimer collision will be assumed to come from the inelastic collision point. For
those atoms that are lost individually, the most energetic will be assumed to come from the recombination point with $\hat{\ell}^2 = 0$. The other lost atoms will usually be assumed to come from one of the first elastic collisions after the recombination event. This prescription is likely to give $\langle \hat{\ell}^2 \rangle \ll 1$, thus providing some justification for the local approximation.

We now list the most important approximations made in our Monte Carlo model:

- We neglect the energies of the low-energy atoms in the atom cloud.
- We approximate the trajectories of the dimer and the atoms between scattering events by straight lines.
- We take the momentum of an incoming atom or dimer in a collision to be the same as that particle’s outgoing momentum from the previous scattering event.
- When comparing the energy of an atom or dimer to the trap depth, we ignore its potential energy (and, in the case of a BEC, its mean-field energy).
- We make the local approximation that most of the lost atoms come from near the recombination point and also that most of the heat from scattered atoms that are not lost is deposited near the recombination point.

### 4.4.2 Simulating avalanches

The development of an avalanche can be decomposed into discrete steps corresponding to the recombination event and the subsequent scattering events. Given the state of the avalanche immediately before each event, the state immediately after the event has a simple probability distribution. All these simple probability distributions together determine the probability distribution of avalanches. We can generate avalanches with this probability distribution using a Monte Carlo method. At each of the events
in the evolution of the avalanche, we use a random number generator to determine the subsequent state. The simple probability distributions can be generated as follows:

• The position \((x, y, z)\) of the recombination point, whose probability distribution is proportional to \(n^3(x, y, z)\), is determined by three random numbers.

• The outgoing wavevectors \(k\) and \(k'\) for a pair of scattered particles are determined by the incoming wavevectors and two random numbers. In the center-of-momentum frame, the distribution of the wavevectors \(\pm k_{cm}\) is isotropic.

• Whether or not an atom or dimer produced by the recombination event or a scattering event is scattered before it reaches the edge of the atom cloud is determined by whether the scattering probability \(1 - \exp(-\sigma \int n \, d\ell)\) is greater than or less than a random number between 0 and 1. The cross section \(\sigma\) is \(\sigma_{AA}\) if the particle is an atom and \(\sigma_{AD}^{(el)} + \sigma_{AD}^{(in)}\) if it is a dimer. The column density \(\int n \, d\ell\) is calculated by integrating from the position of the recombination or scattering event out to infinity along a straight line in the direction of the wavevector \(k\) of the particle. If the atom or dimer scatters, the same random number is used to determine the position of its scattering event by solving for the length \(\ell\) along the path for which \(1 - \exp(-\sigma \int_0^\ell n \, d\ell)\) is equal to the random number.

• Given that a dimer scatters, it scatters inelastically if the probability \(\sigma_{AD}^{(in)}/(\sigma_{AD}^{(el)} + \sigma_{AD}^{(in)})\) is greater than a random number between 0 and 1. Otherwise, the dimer scatters elastically.

### 4.4.3 Atom loss and heating

The Monte Carlo method described in Section 4.4.2 generates a binary tree. The initial node, which represents the recombination event, has two branches corresponding
to the dimer and the atom. For every elastic scattering event, there is a node with two branches that correspond to the two outgoing particles. Finally there are terminal nodes associated with atoms or dimers whose ultimate fate has been determined. More specifically, the terminal nodes correspond to atoms or dimers that are lost, atoms or dimers that are trapped, and dimers that have inelastic collisions. Each terminal node gives a contribution $\Delta N_{\text{lost}}$ to the number of atoms lost and $\Delta E_{\text{heat}}$ to the heat of the remaining atoms. The conditions for a branch to end at a terminal node and the corresponding values of $\Delta N_{\text{lost}}$ and $\Delta E_{\text{heat}}$ are as follows:

- If an outgoing atom from a scattering event has kinetic energy $E < E_{\text{trap}}$, it remains trapped: $\Delta N_{\text{lost}} = 0$ and $\Delta E_{\text{heat}} = E$.

- If an atom that reaches the edge of the atom cloud has kinetic energy $E > E_{\text{trap}}$, it is lost: $\Delta N_{\text{lost}} = 1$ and $\Delta E_{\text{heat}} = 0$.

- If a dimer has an inelastic collision, both it and the scattered atom are lost: $\Delta N_{\text{lost}} = 3$ and $\Delta E_{\text{heat}} = 0$.

- If a dimer that reaches the edge of the atom cloud has kinetic energy $E > 2E_{\text{trap}}$, it is lost: $\Delta N_{\text{lost}} = 2$ and $\Delta E_{\text{heat}} = 0$.

- If a dimer that reaches the edge of the atom cloud has kinetic energy $E < 2E_{\text{trap}}$, it will return to the cloud and will eventually suffer an inelastic collision: $\Delta N_{\text{lost}} = 3$ and $\Delta E_{\text{heat}} = 0$. We ignore any heat from additional elastic collisions before the final inelastic collision.

The number of terminal nodes in the binary tree is 2 if the cross section and the column density are small enough that there is no scattering. The number of terminal nodes is generally larger in a BEC than in a thermal gas. For the sets of experimental
variables listed in Table 4.1, the number of terminal nodes is sometimes greater than 50.

The quantities $N_{\text{lost}}$ and $E_{\text{heat}}$ for a single avalanche are obtained by adding up $\Delta N_{\text{lost}}$ and $\Delta E_{\text{heat}}$ for all the terminal nodes. Their averages $\langle N_{\text{lost}} \rangle$ and $\langle E_{\text{heat}} \rangle$ are calculated by averaging over many avalanches generated using the Monte Carlo method described in Section 4.4.2. These averages have discontinuities as functions of $a$ at $E_d = \frac{3}{2} E_{\text{trap}}$ and $E_d = 6 E_{\text{trap}}$, which are artifacts of our simple model for the trap depth. Aside from these two points, $\langle N_{\text{lost}} \rangle$ and $\langle E_{\text{heat}} \rangle$ are smooth functions of $a$. The number of avalanches required to get smooth results is particularly large in the region near the interval $\frac{3}{2} E_{\text{trap}} < E_d < 6 E_{\text{trap}}$ in which the recombination dimer is trapped but the recombination atom is not. More than 100,000 avalanches are sometimes required to get smooth results in this region.

### 4.4.4 Previous models

Zaccanti et al. developed a simple probabilistic model for the avalanche process that we will refer to as the Zaccanti model [10]. In the Zaccanti model, the avalanche is reduced to a discrete sequence of dimer scattering events. A variable number of elastic collisions is followed either by the escape of the dimer from the trap or by a final inelastic collision. There is one lost atom for each elastic collision up to a maximum number that is determined by the trap depth $E_{\text{trap}}$. The relative probability for each sequence of scattering events is determined by the mean column density $\langle \int n \, d\ell \rangle$ of the trapped atoms and by the atom-dimer cross sections $\sigma_{AD}^{(\text{el})}$ and $\sigma_{AD}^{(\text{in})}$. The Zaccanti model is greatly simplified in several ways compared to our Monte Carlo model:

- The energy dependence of $\sigma_{AD}^{(\text{el})}$ and $k_{\text{cm}} \sigma_{AD}^{(\text{in})}$ is not taken into account. These cross sections were approximated by their low-energy limits given in Eqs. (4.12) and (4.13), which correspond with the sharply-peaked cross sections in Fig. 4.2.
• The spacial structure of the avalanche is ignored. All 3-body recombination events occur at the center of the cloud. The scattering probabilities are all determined by the mean column density $\langle \int n \, d\ell \rangle$ averaged over directions from the center of the trap.

• The elastic scattering of the atoms is not considered. The atom from the recombination event and the scattered atoms from elastic atom-dimer collisions cannot become trapped by losing energy and they also cannot initiate avalanches of additional lost atoms.

• The random variations associated with S-wave scattering are not taken into account. Each elastic collision decreases the kinetic energy of the dimer by the same multiplicative factor $5/9$.

Zaccanti et al. used their model to calculate the average number $\langle N_{\text{lost}} \rangle$ of lost atoms for their experiment with $^{39}$K atoms [10]. It predicts that $\langle N_{\text{lost}} \rangle$ increases from its background value of 3 to about 13 near the atom-dimer resonance. The resulting prediction for the atom loss rate agreed qualitatively with the loss feature they observed near 30.4 $a_0$. The agreement could be made quantitative by decreasing $\sigma_{AD}^{(el)}$ by a factor of 30. Such a decrease was motivated by the energy dependence of the elastic atom-dimer cross section.

Machtey et al. developed an alternative probabilistic model for the avalanche process in Ref. [43]. They made the same simplifications that were itemized above for the Zaccanti model. They reduced the avalanche to discrete sequences of dimer scattering events whose probabilities are determined by an effective column density and the atom-dimer cross sections, but their probabilities for the sequences of scattering events were different from the Zaccanti model. Another difference was that Machtey et al. never introduced the trap depth $E_{\text{trap}}$ into their model. As a consequence, they
could not calculate $\langle N_{\text{lost}} \rangle$. Instead they used their model to calculate the average number $\bar{N}$ of dimer collisions, which is not observable. Machtey et al. suggested that the maximum of $\bar{N}$ as a function of $a$ might coincide with a local maximum of the atom loss rate.

4.4.5 Improvements in the Model

Our Monte Carlo model for the avalanche mechanism has several significant improvements over the probabilistic models proposed by Zaccanti et al. [10] and by Machtey et al. [43]. There are a number of further improvements that could be made. The approximations made in our Monte Carlo model are itemized at the end of Section 4.4.1. Many of them involve neglecting the energies of the low-energy atoms in the atom cloud. One of these approximations is that the energies of the atoms that undergo 3-body recombination are much less than $E_d$. This allowed us to use the universal rate constants at threshold $\alpha_{\text{shallow}}$ and $\alpha_{\text{deep}}$ in Eqs. (4.5). This approximation could be removed for a thermal gas by using the universal results for the 3-body recombination rates in Ref. [44], which were calculated up to temperatures about 100 times larger than $E_d$. The other low-energy approximations allowed the trajectories of particles between collisions to be approximated by straight lines and the changes in their kinetic energies between collisions to be ignored. If the potential energies of the atoms and the dimer (and, in the case of a BEC, their mean-field energies) are taken into account, their trajectories become curved and their kinetic energies change between collisions in accord with conservation of energy. These improvements would be straightforward to implement in our Monte Carlo model.

Another approximation is that we used a simple model for the trap depth that can be expressed as a change in the trapping potential with a single parameter $E_{\text{trap}}$. This simple model implies unphysical discontinuities in $N_{\text{lost}}$ and $E_{\text{heat}}$ at the scattering
lengths for which $E_d = \frac{3}{2}E_{\text{trap}}$ and $E_d = 6E_{\text{trap}}$. The physics represented by that trap depth is actually much more complicated. Given our simple model for the trap depth, it is not clear that the improvement in accuracy from eliminating the low-energy approximations would be worth the effort.

Finally, our rate equations for $N$ and $Q$ in Eqs. (4.24) and (4.26) are based on the local rate equation for the number density in Eq. (4.23). At each collision point in the avalanche, a low-energy atom is replaced by a high-energy atom that can then propagate through the atom cloud. The local approximation requires that most of the lost atoms come from near the recombination point and also that most of the heat from the scattered atoms that are not lost is deposited near the recombination point. Removing this local approximation would be an enormous complication.

4.5 Results

In this section, we apply our Monte Carlo model for the avalanche mechanism to the experiments with $^{39}$K and $^7$Li atoms in which narrow loss peaks near an atom-dimer resonance have been observed. We also apply it to an experiment with $^{133}$Cs atoms in which such a loss feature has not been observed.

4.5.1 $^{39}$K atoms

In 2009, the Florence group observed peaks in $L_3/a^4$ near $30.4 \, a_0$ in a BEC of $^{39}$K atoms and near $930 \, a_0$ in a thermal gas of $^{39}$K atoms [10]. Both loss features were near the predicted position of an atom-dimer resonance. Since the van der Waals length for $^{39}$K atoms is $139 \, a_0$, the loss feature near $30.4 \, a_0$ is in a nonuniversal region of small scattering length. We therefore focus on larger scattering lengths that are safely in the universal region. Different experimental variables were used in different regions of the scattering length. We consider the experimental variables
used in the two regions listed in Table 4.1. In one region, the atom cloud was a BEC and in the other region, it was a thermal gas. We choose the Efimov parameters that were determined from the local minimum of \( L_3/a^4 \) near 224 \( a_0 \): \( a_s = 1140 \ a_0 \) and \( \eta_s = 0.043 \). For the thermal gas experiment, \( E/N = 3k_BT \) in Table 4.2 is actually smaller than the value of 2.7 \( k_BT_c \) calculated from \( N \). This suggests that the system is very close to the critical temperature, so it might be appropriate to use the number density in Eq. (4.19). We nevertheless use the Boltzmann approximation in Eq. (4.21) for simplicity.

In the left panels of Fig. 4.3, the average number \( N_{\text{lost}} \) of atoms lost and the average heat \( E_{\text{heat}} \) from the avalanche are shown as functions of \( a \) for the two sets of experimental variables for \(^{39}\text{K} \) atoms listed in Table 4.1. (In this section, we omit the angular brackets that denote the avalanche averages of \( N_{\text{lost}} \) and \( E_{\text{heat}} \).) For both the BEC and the thermal gas, \( N_{\text{lost}} \) and \( E_{\text{heat}} \) are shown for a range of scattering lengths that extend over two orders of magnitude. The rate constant \( L_3 \) however was measured using these experimental variables only over the smaller ranges of \( a \) specified in Table 4.1. For both the BEC and the thermal gas, \( N_{\text{lost}} \) has a broad peak with a maximum value near 5. The position of the peak is at 293 \( a_0 \) for the BEC and at 827 \( a_0 \) for the thermal gas. This position is determined by the atom-dimer cross sections and the trap depth, among other things. As illustrated in Fig. 4.2, the cross section for the first elastic scattering of the recombination dimer has a broad peak with maximum at \( 4.34(e^{-\pi/so}a_s) \approx 218 \ a_0 \). The trap depth forces \( N_{\text{lost}} \) to decrease to the naive value 3 when \( E_d = \frac{3}{2}E_{\text{trap}} \), which is near \( a = 1720 \ a_0 \) for the BEC and near \( a = 2220 \ a_0 \) for the thermal gas. Thus the cutoff provided by the trap depth has a strong effect on the position of the peaks in \( N_{\text{lost}} \) and \( E_{\text{heat}} \).

In the right panels of Fig. 4.3, the rate constant \( L_3 \) and the heating rate \( dQ/dt \) are shown as functions of \( a \). The panel for \( L_3 \) in Fig. 4.3 shows the data from the
Figure 4.3: The average number $N_{\text{lost}}$ of atoms lost in an avalanche (upper left panel), the average heat $E_{\text{heat}}$ generated by an avalanche (lower left panel), the rate constant $L_3$ (upper right panel), and the heating rate $dQ/dt$ (lower right panel) as functions of $a$. The system consists of $^{39}\text{K}$ atoms with $a_* = 1140 \ a_0$ and $\eta_* = 0.043$. The vertical dotted line marks the position of $a_*$. The dashed (blue) curves and the solid (red) curves are for the BEC and the thermal gas in Ref. [10], respectively. In the panel for $L_3$, the thin (black) line is the universal prediction for $L_3$ without the avalanche mechanism. It provides a lower bound on the other curves, which cover it up over most of the range of $a$. The (blue and red) dots are data from the Florence group that were measured in a BEC and in a thermal gas, respectively [10].

Florence group [10]. The result for $L_3$ is visibly larger than the universal result without the avalanche mechanism in the region just below the local minimum near $e^{-\pi/s_0}a_+ = 225 \ a_0$ for the BEC and in the region just below $a_* = 1140 \ a_0$ for the thermal gas. Thus the avalanche mechanism can affect the fitted values of the Efimov parameters $a_*$ and $\eta_*$. Both $L_3$ and $dQ/dt$ have local minima near $e^{-\pi/s_0}a_+ = 225 \ a_0$ that arise from Efimov interference. The heating rate $dQ/dt$ in the thermal gas is more than an order of magnitude smaller than in the BEC over most of the range of $a$. 

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Figure 4.4: Same as Fig. 4.3 but for $^7\text{Li}$ atoms with $a_*=282\ a_0$ and $\eta_*=0.039$. The dashed (blue) curves and the solid (red) curves are for the BEC in Ref. [27] and the thermal gas in Ref. [35], respectively. In the panel for $L_3$, the thin (black) line is the universal prediction for $L_3$ without the avalanche mechanism. The (red) dots are data from the Bar-Ilan group that were measured in a thermal gas [35].

### 4.5.2 $^7\text{Li}$ atoms

In 2009, the Rice group observed a peak in $L_3/a^4$ near $608\ a_0$ in a BEC of $^7\text{Li}$ atoms in the $|1, +1\rangle$ hyperfine state [27]. This loss feature is near the predicted position of an atom-dimer resonance. Different experimental variables were used in different regions of the scattering length. The experimental variables used in one of these regions are listed in Table 4.1. The Efimov parameters determined from the narrow loss minimum near $2676\ a_0$ are $a_* = 598\ a_0$ and $\eta_+ = 0.039$. The Rice group has improved the accuracy of the determination of $a$ as a function of the magnetic field and a reanalysis of the data from Ref. [27] is underway [28]. Their new analysis will not have a significant effect on the value of $\eta_*$, but it will shift the value of $a_*$ closer to the value measured by the Bar-Ilan group, which is given below.
In 2010, the Bar-Ilan group observed a local minimum in $L_3/a^4$ in a thermal gas of $^7$Li atoms in the $|1, +1\rangle$ hyperfine state [32]. The Efimov parameters determined by fitting their measurements of $L_3$ are $a_\ast = 282 \ a_0$ and $\eta_\ast = 0.188$. Since the van der Waals length for $^7$Li atoms is $65 \ a_0$, the predicted atom-dimer resonance is safely in a universal region of large scattering length. In 2012, they presented additional data that revealed a narrow enhancement in $L_3$ near $200 \ a_0$, which is near the predicted atom-dimer resonance [35]. The experimental variables used in the measurement of $L_3$ are listed in Table 4.1.

The Rice group and the Bar-Ilan group used the same hyperfine state of $^7$Li, so they should obtain the same Efimov parameters to within experimental errors. Since $\eta_\ast$ is particularly sensitive to the width of the loss minimum at $a_+$, thermal smearing and limited experimental resolution are most likely to lead to an overestimate of $\eta_\ast$. For the Efimov parameters, we will therefore use the value of $a_\ast$ obtained by the Bar-Ilan group but the smaller value of $\eta_\ast$ obtained by the Rice group: $a_\ast = 282 \ a_0$ and $\eta_\ast = 0.039$.

In the left panels of Fig. 4.4, the average number $N_{\text{lost}}$ of atoms lost and the average heat $E_{\text{heat}}$ from the avalanche are shown as functions of $a$ for the two sets of experimental variables for $^7$Li atoms listed in Table 4.1. There are visible discontinuities in $N_{\text{lost}}$ and $E_{\text{heat}}$ at the scattering lengths at which $E_d = \frac{3}{2} E_{\text{trap}}$ and $E_d = 6 E_{\text{trap}}$. These discontinuities are artifacts of our simple model for the trap depth. The number $N_{\text{lost}}$ has a broad peak with a maximum value near 7 for the BEC and near 4 for the thermal gas. The position of the peak is at $905 \ a_0$ for the BEC and at $767 \ a_0$ for the thermal gas. This position is determined the scattering cross sections and the trap depth, among other things. The cross section for the first elastic scattering of the recombination dimer has a broad peak with its maximum at $4.34a_\ast \approx 1220 \ a_0$. The trap depth forces $N_{\text{lost}}$ to decrease to the naive value 3 near $a = 5740 \ a_0$ for the
BEC and near $a = 1530 \ a_0$ for the thermal gas. This cutoff provided by the trap depth has a strong effect on the position of the peaks in $N_{\text{lost}}$ and $E_{\text{heat}}$.

In the right panels of Fig. 4.4, the rate constant $L_3$ and the heating rate $dQ/dt$ are shown as functions of $a$. The panel for $L_3$ in Fig. 4.4 shows the data from the Bar-Ilan group [32]. The curves for $L_3$ have a much more pronounced local minimum at $a_+ \approx 1260 \ a_0$ than the data, because we have used the Efimov parameter $\eta_* = 0.039$ from the Rice experiment [27] instead of the value $\eta_* = 0.188$ obtained by fitting the Bar-Ilan data. For both the BEC and the thermal gas, the result for $L_3$ in the region just below the local minimum near $a_+ = 1260 \ a_0$ is visibly larger than the universal result without the avalanche mechanism. Since the Efimov parameters $a_+$ and $\eta_*$ are sensitive to the position and width of the minimum, their fitted values can be strongly affected by the avalanche mechanism. Note that our Monte Carlo model predicts no peak in $L_3$ near the atom-dimer resonance at $a_* = 282 \ a_0$. Both $L_3$ and $dQ/dt$ have local minima near $a_+ = 1260 \ a_0$ that arise from Efimov interference. The heating rates $dQ/dt$ are similar in the BEC and in the thermal gas.

4.5.3 $^{133}$Cs atoms

The Innsbruck group has studied loss features in thermal gases of $^{133}$Cs atoms in several universal regions of the magnetic field. In 2005, they studied a region of low magnetic field and observed a local minimum of $L_3/a^4$ that can be attributed to Efimov interference at a scattering length near $210 \ a_0$ [29]. Setting $a_+ = 210 \ a_0$, atom-dimer resonances are predicted at $47 \ a_0$ and $1070 \ a_0$. Since the van der Waals length for $^{133}$Cs atoms is $200 \ a_0$, only the higher atom-dimer resonance is in a universal region of large scattering length. The Innsbruck group did not observe any loss feature near $1070 \ a_0$.

In 2011, the Innsbruck group observed three peaks in $L_3/a^4$ in different universal
Figure 4.5: Same as Fig. 4.3, but for $^{133}$Cs atoms with $a_\ast = 1017 \, a_0$ and $\eta \ast = 0.08$. The solid (red) curves are for the thermal gas in Ref. [41]. In the panel for $L_3$, the thin (black) line is the universal prediction for $L_3$ without the avalanche mechanism. The (red) dots are data from the Innsbruck group that were measured in a thermal gas [41].

regions with large negative $a$ at higher magnetic field [41]. They can all be attributed to resonant enhancement from an Efimov trimer near the 3-atom threshold. Two of these loss features are complicated by the presence of a G-wave Feshbach resonance. The Efimov parameters associated with the third loss feature are $a_- = -955 \, a_0$ and $\eta_\ast = 0.08$. The Innsbruck group found that the Efimov parameter $a_-$ has almost the same value in all the universal regions, which suggests that it is determined by the van der Waals length [41]. The universal ratios in Eqs. (4.1) and (4.2) can be used to predict an atom-dimer resonance at $a_\ast = 1017 \, a_0$ and an Efimov interference minimum in $L_3/a^4$ near $a_+ = 200 \, a_0$. The Innsbruck group measured $L_3$ in a universal region of large positive $a$. They observed a local minimum near $270 \, a_0$, which is near the predicted position of $a_+$, but they did not see any loss feature near $a_\ast$. The experimental variables used in this region of positive scattering length are listed in
In the left panels of Fig. 4.5, the average number $N_{\text{lost}}$ of atoms lost and the average heat $E_{\text{heat}}$ from the avalanche are shown as functions of $a$ for this set of experimental variables. There are visible discontinuities in $N_{\text{lost}}$ and $E_{\text{heat}}$ at the scattering lengths at which $E_d = \frac{3}{2}E_{\text{trap}}$ and $E_d = 6E_{\text{trap}}$. These discontinuities are artifacts of our simple model for the trap depth. The number $N_{\text{lost}}$ coincidentally has a peak very close to the atom-dimer resonance $a_*$ but the peak in $E_{\text{heat}}$ is at a higher value of $a$. The average heat $E_{\text{heat}}$ is more than an order of magnitude smaller than in the $^7$Li and $^{39}$K thermal gas experiments described above.

In the right panels of Fig. 4.5, the rate constant $L_3$ and the heating rate $dQ/dt$ are shown as functions of $a$. The increase in $L_3$ from the avalanche mechanism is visible only in the region just above $a_* = 1017$ $a_0$. Both $L_3$ and $dQ/dt$ have local minima from Efimov interference near $a_+ = 4550$ $a_0$. The panel for $L_3$ in Fig. 4.5 also shows the data from the Innsbruck group [41]. There is a local minimum near 200 $a_0$, which is near the predicted value of $a_+$. The deviations between the measurements and the universal predictions at smaller $a$ is not unexpected, because this is a nonuniversal region.

### 4.6 Dimer-dimer resonances

We have used our Monte Carlo model for the avalanche mechanism to show that it cannot produce a narrow loss feature near an atom-dimer resonance. The essential reason is explained by the energy dependence of the elastic atom-dimer cross section, which is illustrated in Fig. 4.2. Since the shallow dimer from 3-body recombination loses energy with each elastic collision, the first few collisions of the dimer are those that are the most likely to knock an extra atom out of the trap. In the first few collisions, the dimer’s kinetic energy is comparable to its binding energy $E_d$, and there
is no dramatic enhancement of the atom-dimer cross section. Rather, the atom-dimer cross section is dramatically enhanced only after many elastic collisions have reduced the dimer’s kinetic energy to much smaller than $E_d$. However, with its kinetic energy degraded, the dimer is much less likely to knock an atom out of the trap.

The universality of atoms with large scattering length is not limited to the 2-atom and 3-atom sectors. In 2004, Hammer, Meissner, and Platter made the first suggestion that universality should extend to the 4-atom sector [19]. They presented numerical evidence that there are two universal tetramers associated with each Efimov trimer, and they made the first calculations of the tetramer binding energies for some regions of $1/a$ [19, 20]. In 2008, von Stecher, D’Incao, and Greene calculated the tetramer binding energies more accurately and over the entire range of $1/a$ [21]. They pointed out that the most dramatic signature of a universal tetramer is the resonant enhancement of the 4-body recombination rate at a negative value of $a$ where the tetramer is at the 4-atom threshold. The loss features from a pair of universal tetramers were first observed by the Innsbruck group using a thermal gas of $^{133}$Cs atoms [33]. They measured the 4-body loss rate constant $L_4$ for one of the tetramers. Universal tetramers have also been observed in a thermal gas experiment with $^7$Li atoms by the Rice group [27]. They measured $L_4$ for both members of one pair of tetramers and for one member of another pair.

A universal tetramer could also produce loss features at positive values of $a$. One possibility is a loss feature at a scattering length at which a tetramer crosses the 2-dimer threshold, which we will refer to as a dimer-dimer resonance. The dimer-dimer elastic and inelastic cross sections are resonantly enhanced near threshold at a dimer-dimer resonance. Each Efimov trimer, with atom-dimer resonance at $a_*$, has associated with it two universal tetramers, with dimer-dimer resonances at larger scattering lengths $a_{1*}$ and $a_{2*}$. The universal predictions for the positions of these
dimer-dimer resonances were first calculated by von Stecher, D’Incao, and Greene [21]. They were recently calculated by Deltuva with 4 digits of precision [23]:

\[
\begin{align*}
a_{*1} &= 2.196\ a_*, \quad (4.34a) \\
a_{*2} &= 6.785\ a_* \quad (4.34b)
\end{align*}
\]

In their experiment with a BEC of $^7$Li atoms, the Rice group observed narrow enhancements in the measured 3-body loss rate constant $L_3$ near $+1470\ a_0$ and near $+3910\ a_0$ [27]. These scattering lengths are near the predicted positions of the two dimer-dimer resonances for a pair of universal tetramers. These loss features are even more mysterious than those near atom-dimer resonances. Three-body recombination can create a shallow dimer with kinetic energy comparable to $E_d$. Four-body recombination can create one or two shallow dimers with kinetic energy comparable to $E_d$. If the equilibrium population of shallow dimers in the atom cloud is negligible, the recombination dimers can only undergo atom-dimer collisions. Thus the resonant enhancement of dimer-dimer cross sections near a dimer-dimer resonance should have no effect on the atom loss rate. Therefore, there is no analog of the avalanche mechanism near a dimer-dimer resonance.

One possible explanation for the narrow loss features near the atom-dimer and dimer-dimer resonances is that the equilibrium population of shallow dimers in the atom cloud is not negligible. The resonant enhancement of the inelastic atom-dimer cross section could then produce an enhanced loss rate near $a_*$. Similarly, the resonant enhancement of the inelastic dimer-dimer cross section could produce enhanced loss rates near $a_{*1}$ and near $a_{*2}$. The number density $n_d$ of the dimers must be much smaller than the number density $n$ of the atoms. In the absence of atom loss processes, the rates of atom-dimer and dimer-dimer collisions would be proportional to $nn_d$ and $n_d^2$, respectively. If $n_d$ is proportional to $n^2$, the atom-dimer and dimer-dimer
collision rates have the same dependence on $n$ as the 3-body and 4-body recombination rates, respectively. Thus the enhanced loss rate near $a_*$ from low-energy inelastic atom-dimer collisions would manifest itself as an apparent enhancement of the 3-body recombination rate. Similarly, enhanced loss rates near $a_{*1}$ and near $a_{*2}$ from low-energy inelastic atom-dimer collisions would manifest themselves as apparent enhancements in the 4-body recombination rate. While an equilibrium population of dimers could explain the existence of loss features at the atom-dimer and dimer-dimer resonances, it can not easily account for them quantitatively. It seems likely that a population of dimers large enough to account for the observed loss features should also have been observed more directly.

Narrow loss features have been observed near atom-dimer resonances in several experiments and near dimer-dimer resonances in the $^7$Li BEC experiment. We have shown that the avalanche mechanism cannot produce a narrow loss feature near an atom-dimer resonance. It also cannot produce any loss of atoms near a dimer-dimer resonance. An equilibrium population of dimers could produce loss features near atom-dimer and dimer-dimer resonances, but a population of dimers large enough to account for the observed loss features should probably have been observed more directly. We suggest that another mechanism that has not yet been identified must be responsible for the loss features that have been observed near atom-dimer and dimer-dimer resonances.
Chapter 5
DIMER CONDENSATE MECHANISM

The following chapter was posted on the arXiv electronic preprint archive in February 2013 as a paper entitled Atom Loss Resonances in a Bose-Einstein Condensate. It was published in July 2013 in Physical Review Letters [45]. This chapter proposes that the loss peaks that have been observed at atom-dimer and dimer-dimer resonances in experiments using a BEC of atoms are caused by inelastic scattering of dimers from a coexisting dimer condensate.

5.1 Atom Loss Resonances in a Bose-Einstein Condensate

Nonrelativistic particles whose scattering lengths are large compared to the range of their interactions exhibit universal low-energy behavior [8]. The universal few-body phenomena can include a spectrum of loosely-bound molecules as well as reaction rates of the particles and molecules. In some cases, including identical bosons, the universal behavior is governed by discrete scale invariance. The universal molecules then include sequences of 3-particle clusters (Efimov trimers) [57, 19, 20, 21, 46], 4-particle clusters (universal tetramers) [19, 20, 21], and clusters with even more particles [46]. The universal reaction rates exhibit intricate resonance and interference features [8].

The technology for trapping atoms and cooling them to extremely low temperatures has made the universal low-energy region experimentally accessible. The use of
Feshbach resonances to control the scattering length $a$ experimentally makes ultracold atoms an ideal laboratory for universal physics. One particularly dramatic probe of universality is the loss rate of atoms from a trapping potential. Resonance and interference effects in few-body reaction rates can produce local maxima and minima in the loss rate as a function of $a$. The most dramatic signature for a universal $N$-atom cluster with $N \geq 3$ is the resonant enhancement of the $N$-atom inelastic collision rate at a negative $a$ where the cluster crosses the $N$-atom threshold and becomes unbound. We refer to such a scattering length as an $N$-atom resonance. The first observations of an Efimov trimer \[29\] and a universal tetramer \[33\] and the first evidence for a universal 5-atom cluster \[47\] were all obtained using a thermal gas of $^{133}$Cs atoms by tuning $a$ to 3-atom, 4-atom, and 5-atom resonances, respectively. An Efimov trimer has also been observed as an enhancement in the loss rate in a mixture of $^{133}$Cs atoms and dimers \[30\] at a positive scattering length $a_*$ where an Efimov trimer crosses the atom-dimer threshold and becomes unbound. We refer to $a_*$ as an atom-dimer resonance. Another dramatic loss feature at positive $a$ is an interference minimum in the 3-atom recombination rate into the shallow dimer, which was also first observed in a thermal gas of $^{133}$Cs atoms \[29\]. Many of these loss features have been subsequently observed in ultracold trapped atoms of other elements \[48\].

There are a few loss features in ultracold atoms that have not yet been related to universal few-body reaction rates. They all appear in systems that were believed to consist of atoms only and no dimers. Narrow enhancements of the loss rate near atom-dimer resonances have been observed in both a Bose-Einstein condensate (BEC) and a thermal gas of $^{39}$K atoms \[10\] and in both a BEC and a thermal gas of $^{7}$Li atoms \[27, 35\]. In a BEC of $^{7}$Li atoms, narrow enhancements of the loss rate have also been observed at positive values of $a$ near two-dimer resonances \[27\], at which universal tetramers cross the dimer-dimer threshold and become unbound. No mechanism has
been proposed for a narrow loss feature at a two-dimer resonance in a system consisting of atoms only and no dimers. One proposed mechanism for a narrow loss feature near an atom-dimer resonance $a_*$ in such a system is the avalanche mechanism, in which the 3-body recombination rate into the shallow dimer is amplified by secondary elastic collisions of the outgoing dimer [10]. It was recently shown that the avalanche mechanism cannot produce a narrow loss feature near $a_*$ [12]. This is a consequence of the universal energy dependence of the atom-dimer cross section. Instead of having a narrow peak at $a_*$, the elastic cross section for the energetic dimer from the recombination event has a broad maximum near $4.3 a_*$. Detailed Monte Carlo simulations of the avalanche process demonstrate that it does not produce any narrow loss features [11].

In this Letter, we propose a new mechanism for narrow loss features near atom-dimer and two-dimer resonances in a Bose-Einstein condensate of atoms. The mechanism is motivated by the phenomenon of atom-molecule coherence, which involves the coherent transfer of atom pairs between an atom condensate and a coexisting dimer condensate. A small condensate of shallow dimers can be produced coherently by the time-dependent scattering length as it is ramped to the large final value where the atom loss rate is measured. The loss features then arise from the resonant enhancement of inelastic collisions involving dimers from the dimer condensate.

The phenomenon of atom-molecule coherence was discovered by Donley et al. in 2002 using a BEC of $^{85}$Rb atoms [49]. In these experiments, a pulse in the magnetic field brought the atoms very close to a Feshbach resonance. The atoms were allowed to evolve at a large constant scattering length for a variable holding time, and then a second pulse took the atoms close to the Feshbach resonance again. Subsequent measurements of the number of atoms revealed three distinct components: a “remnant” BEC, a “burst” of relatively energetic atoms, and “missing” atoms that were not
detected. The numbers of remnant, burst, and missing atoms all varied sinusoidally with the holding time at the frequency associated with the dimer binding energy. That sinusoidal dependence can be explained by a coexisting condensate of shallow dimers created by the first pulse \[50, 51, 52\].

The behavior of atoms and dimers with sufficiently small kinetic and potential energies can be described by a quantum field theory with independent fields for the atoms and dimers. Atom and dimer condensates are described by classical fields \(\psi(r, t)\) and \(d(r, t)\) that are the expectation values of the quantum fields. The quantum field equations can be formulated as coupled integro-differential equations for \(\psi, d\), and an infinite hierarchy of correlation functions for quantum fluctuations. A typical experiment on the atom loss rate in a Bose-Einstein condensate begins with a stable BEC of atoms with a small positive scattering length. This system is described by a static atom condensate \(\psi(r)\), with \(d(r) = 0\) and zero correlation functions. A ramp in the magnetic field produces a time-dependent scattering length with a large final value \(a\). During the ramp, the system is described by time-dependent condensates \(\psi(r, t)\) and \(d(r, t)\) and nonzero correlation functions. At the end of the ramp, the dimer condensate \(d(r, 0)\) will be nonzero, although it could be very small if the ramp is nearly adiabatic. It could presumably be calculated using the methods developed in Refs. \[50, 51, 52\] to describe atom-molecule coherence. We will not attempt to calculate \(d(r, 0)\), but simply take the initial fraction \(f_D\) of the atoms that are bound into dimers in the dimer condensate to be an unknown initial condition. We will ask whether observable loss features can be produced by the dimer condensate for a plausibly small fraction \(f_D\).

During the subsequent holding time, there are transient effects in the atom and dimer condensates and in the correlation functions that will die away. We will assume that after they have died away, the system can be described by coexisting atom and
dimer condensates only and that the correlation functions can be neglected. Atom-molecule coherence will produce oscillations in \( \psi(r, t) \) and \( d(r, t) \) at the frequency \( \hbar/(2\pi ma^2) \) associated with the dimer binding energy. The time-averaged condensates also decrease with time due to loss processes. The coexisting condensates can be described by a low-energy effective field theory for atoms and dimers whose kinetic and potential energies are small compared to the dimer binding energy. The interaction terms in the classical Hamiltonian density include

\[
\mathcal{H}_{\text{int}} = \nu_2 d^*d + \frac{\hbar^2 \lambda_2}{4m} (\psi^*\psi)^2 + \frac{\hbar^2 \lambda_3}{m} (d^*d)(\psi^*\psi) \\
+ \frac{\hbar^2 f_4}{4m} (d^*d)^2 + \frac{\hbar^2 \lambda_3}{36m} (\psi^*\psi)^3 + \ldots \\
\quad (5.1)
\]

The coefficients of the interaction terms are universal functions of \( a \) and the complex Efimov parameter \( \kappa_* \exp(i\eta_*/s_0) \), which can be interpreted as the binding wavenumber of an Efimov trimer in the unitary limit \( a = \pm \infty \) [8]. The small positive parameter \( \eta_* \) takes into account tightly-bound diatomic molecules (deep dimers), which provide decay channels for Efimov trimers. The coefficients of the interaction terms in Eq. (5.1) are constrained by discrete scale invariance to be log-periodic functions of \( \kappa_* \), with discrete scaling factor \( e^{\pi/s_0} \), where \( s_0 \approx 1.00624 \) is a universal constant.

The coefficients in Eq. (5.1) can be determined by demanding that few-body results in the fundamental theory be reproduced by the effective field theory. The coefficient of \( d^*d \) is determined by matching the rest energy of a shallow dimer: \( \nu_2 = -\hbar^2/ma^2 \). The coefficient of the terms that are 4th order in the fields can be determined by matching elastic scattering amplitudes at threshold: \( \lambda_2 = 8\pi a \), \( h_3 = 3\pi a_{AD} \), and \( f_4 = 4\pi a_D \), where \( a_{AD} \) and \( a_D \) are the atom-dimer and dimer scattering lengths. The atom-dimer scattering length \( a_{AD} \) is \( a \) multiplied by a simple log-periodic function of \( a\kappa_* \) [8]. In the limit \( \eta_* = 0 \), that function is real and it diverges at the atom-dimer resonance \( a_* = 0.07076/\kappa_* \). The dimer scattering length \( a_D \) is \( a \)
multiplied by a log-periodic function of $a\kappa_*$ that is complex even if $\eta_*=0$. For $\eta_*=0$, Deltuva has calculated $a_D$ with several digits of accuracy \cite{23}. Its imaginary part has narrow resonant peaks at the two-dimer resonances $a_1* \approx 2.196 a_*$ and $a_2* \approx 6.785 a_*$. These two-dimer resonances were first calculated in Ref. \cite{22}. For $\eta_*>0$, $a_D$ can be obtained by constructing an analytic fit to Deltuva’s results for $a_D/a$ as a function of $a\kappa_*$ and then carrying out the analytic continuation $\kappa_* \rightarrow \kappa_* \exp(i\eta_*/s_0)$. The coefficient of $(\psi^*\psi)^3$ in Eq. (5.1) could be determined by matching the elastic 3-atom scattering amplitude in the low-energy limit. By the optical theorem, its imaginary part is proportional to the 3-atom recombination rate coefficient, $\text{Im} \lambda_3 = -(3m/\hbar)\alpha$, which can be separated into contributions from recombination into the shallow dimer and into deep dimers: $\alpha = \alpha_{\text{shallow}} + \alpha_{\text{deep}}$. They both have the form $\hbar a^4/m$ multiplied by log-periodic functions of $a$ \cite{8}. In the limit $\eta_*=0$, $\alpha_{\text{deep}} = 0$ and $\alpha_{\text{shallow}}$ has an interference zero at $a_+ = 0.31649/\kappa_*$. 

The time dependence of $\psi(\mathbf{r},t)$ and $d(\mathbf{r},t)$ is determined by the classical field equations associated with the interaction Hamiltonian density $\mathcal{H}_{\text{int}}$ in Eq. (5.1). The corresponding number densities are $n_A = \psi^*\psi$ and $n_D = d^*d$. The rates at which the numbers $N_A = \int d^3r n_A$ and $N_D = \int d^3r n_D$ change is determined by the anti-hermitian part of $\mathcal{H}_{\text{int}}$: 

$$
\frac{dN_A}{dt} = \frac{6\pi \hbar}{m} \frac{\text{Im} a_{AD}}{m} \int d^3r n_D n_A - \frac{\alpha}{2} \int d^3r n_A^3, \\
\frac{dN_D}{dt} = \frac{6\pi \hbar}{m} \frac{\text{Im} a_{AD}}{m} \int d^3r n_D n_A + \frac{4\pi \hbar}{m} \frac{\text{Im} a_D}{m} \int d^3r n_D^2.
$$

Since unitarity requires the imaginary parts of $a_{AD}$ and $a_D$ to be negative, these equations imply that $N_A$ and $N_D$ both decrease monotonically with time. This excludes atom-molecule coherence, which involves coherent oscillations between $N_A$ and $N_D$ with angular frequency $E_D/\hbar$. The appropriate interpretation is that the high-frequency variations in the condensates associated with atom-molecule coherence are
not resolved within our low-energy effective field theory. It can at best describe number densities \( n_A(r, t) \) and \( n_D(r, t) \) that are averaged over many periods of the atom-dimer oscillations.

To predict the loss rate of atoms during the holding time, we need initial conditions \( \psi(r, 0) \) and \( d(r, 0) \) that are robust approximate solutions of the classical field equations associated with the hermitian part of \( \mathcal{H}_{\text{int}} \) in Eq. (5.1). At the start of the holding time, the scattering length has its final value \( a \) and there is a specified initial total number of atoms \( N_0 \), an unknown fraction \( f_D \) of which are bound atoms forming the dimer condensate. We consider atoms trapped in a cylindrically symmetric harmonic potential: \( V(r) = \frac{1}{2} m \omega_z^2 [z^2 + \zeta^2 (x^2 + y^2)] \). If \( f_D \) is sufficiently small, the effect of the dimer condensate on the atoms is negligible. If the ramp to the final scattering length is slow enough, the atom condensate remains adiabatically in its ground state. We assume \( N_A a \) is large enough that the kinetic energy of the atoms is small compared to their potential and interaction energies. The atom condensate then has the familiar Thomas-Fermi density profile:

\[
n_A(r) = \frac{m}{4\pi \hbar^2 a} \max\{0, \mu_A - V(r)\}. \tag{5.3}
\]

The chemical potential is determined by the number \( N_A = (1 - f_D) N_0 \) of unbound atoms: \( \mu_A = \frac{1}{2} m \omega_z^2 \bar{a}_z^2 [15 \zeta^2 N_A a / \bar{a}_z]^2 / 5 \), where \( a_z = (\hbar / m \omega_z)^{1/2} \).

We next consider the initial condition for \( d(r, 0) \). Since the initial number of dimers \( f_D N_0 / 2 \) is small compared to \( N_0 \), their self-interaction energy is negligible but the mean-field energy from the atom condensate can be important. Assuming the dimer condensate produced by the ramp of the scattering length is in its ground state, it satisfies the Schroedinger equation

\[
\left[ -\frac{\hbar^2}{4m} \nabla^2 + 2V(r) + \frac{3\pi \hbar^2 \Re a_{AD}}{m} n_A(r) \right] d(r) = \mu_D d(r). \tag{5.4}
\]
The eigenvalue $\mu_D$ is the chemical potential of the dimer. The normalization of the dimer condensate is determined by the condition $\int d^3 r \, d^* d = f_D N_0 / 2$. The total potential energy of the dimer is the sum of the trapping potential $2V(r)$ and the mean-field energy. Its minimum is at the origin if $\text{Re}a_{AD} < \frac{8}{3} a$ and near the edge of the atom condensate if $\text{Re}a_{AD} > \frac{8}{3} a$. In these two cases, the dimer condensate is an ellipsoid centered at the origin and an ellipsoidal shell near the edge of the atom condensate, respectively. The boundaries in $a$ for the ellipsoidal shell region are at $a_*$ and $2.86 \, a_*$.

The data on the loss rate of $^7\text{Li}$ atoms in Ref. [27] has recently been reanalyzed using a more accurate determination of the parameters of the Feshbach resonance [28]. Under the assumptions that the system is a pure BEC of atoms and that the loss rate comes from 3-atom recombination only, the rate coefficient $L_3$ is defined by $dN/dt = -(L_3/6) \int d^3 r \, n^3_A$. The data from Ref. [28] that was measured in a BEC of $^7\text{Li}$ atoms is shown in Fig. 5.1. The initial number of atoms was $N_0 = 4 \times 10^5$. Their fit to the universal prediction $L_3 = 3\alpha$ using an adjustable normalization factor is shown in Fig. 5.1. It determines the Efimov parameters $a_+ = 1402 \, a_0$ and $\eta_* = 0.038$ [27] and requires a normalization factor of approximately 9.2. Unless there is a systematic error that could account for such a large factor, it is incompatible with the universal predictions. The corresponding universal predictions for the resonances are $a_* = 313 \, a_0$, $a_{1*} = 688 \, a_0$, and $a_{2*} = 2127 \, a_0$. Near each of these three resonances, there is a narrow loss feature where the data are significantly higher than the fitted curve. The positions of the loss features reported in Ref. [28] are 426 $a_0$, 919 $a_0$, and 1902 $a_0$. The first two are higher than $a_*$ and $a_{1*}$ by factors of 1.36 and 1.34, while the third is lower than $a_{2*}$ by a factor of 0.89. The shifts in $a_*$ and $a_{1*}$ could be attributed to an inaccuracy in the determination of $a_+$. However the large difference between the predicted ratio $a_{2*}/a_{1*} = 3.090$ and the ratio 2.1 for the observed loss
Figure 5.1: Three-atom recombination rate coefficient $L_3$ as a function of the scattering length $a$. The data points were measured using a BEC of $^7\text{Li}$ atoms [28]. The thick (black) curve is the universal result for $L_3$ with $a_+ = 1402\ a_0$ and $\eta_* = 0.038$. The three vertical lines are the universal predictions for $a_*$, $a_{1*}$, and $a_{2*}$ using $a_+$ as input. The arrows point to the narrow loss features identified in Ref. [28]. The thin solid (green) and dashed (red) lines are $L_3^{\text{eff}}$ from 3-body recombination plus atom-dimer losses and from 3-body recombination plus dimer-dimer losses, respectively. In the regions near $a_*$, $a_{1*}$, and $a_{2*}$, the dimer fractions $f_D$ are $10^{-5}$, $3 \times 10^{-2}$, and $6 \times 10^{-3}$.

features is difficult to understand.

We now consider the effects of a small coexisting dimer condensate. Our initial conditions are determined by $N_0 = 4 \times 10^5$ and an assumed dimer fraction $f_D$, which we expect to be small compared to 1 and to depend strongly on $a$. We take $n_A$ to be the Thomas-Fermi profile in Eq. (5.3). For $n_D = d^* d$, we use a variational approximation for $d$ that reduces the 3-dimensional Schroedinger equation in Eq. (5.4) to a 1-dimensional equation for a single radial variable. The subsequent time dependence
of the total number of atoms $N(t)$ can be obtained by solving Eqs. (5.2). A quantitative comparison with the data would require comparing with the time dependence observed in the experiment. Instead we compare the data for $L_3$ in Fig. 5.1 with an effective rate coefficient $L^{\text{eff}}_3$ determined by the initial loss rate. It is defined by $dN/dt = -(L^{\text{eff}}_3/6) \int d^3r n^3_{A0}$, where the integral is evaluated under the assumption that $f_D = 0$: $\int d^3r n^3_{A0} = (N_0/168\pi^2a_2^4a^2)[15\zeta^2N_0a/a_2]^{4/5}$. In Fig. 5.1 the universal prediction for $L_3$ from 3-atom recombination is an order of magnitude below the data. This allows room for additional contributions from the atom-dimer and dimer-dimer terms in Eqs. (5.2). We would like to determine whether narrow loss features can stand out above the smooth contributions for plausible values of $f_D$.

In Fig 5.1 the terms in $L^{\text{eff}}_3$ are illustrated by two thin curves, which correspond to 3-atom recombination plus atom-dimer losses and to 3-atom recombination plus dimer-dimer losses. The total $L^{\text{eff}}_3$ is approximately equal to the maximum of these two curves. We show these curves near $a_*$, $a_{1*}$, and $a_{2*}$, using different values of $f_D$ chosen to make the narrow loss feature visible: $f_D = 10^{-5}$, $3 \times 10^{-2}$, and $6 \times 10^{-3}$, respectively. The curve near $a_*$ that includes the atom-dimer terms in Eqs. (5.2) has a narrow peak at the atom-dimer resonance. The peak is not symmetric, because the dimer condensate changes from an ellipsoid centered at the origin for $a < a_*$ to an ellipsoidal shell near the edge of the atom condensate for $a > a_*$. The curves near $a_{1*}$ and $a_{2*}$ that include the dimer-dimer term in Eqs. (5.2) have narrow peaks at the two-dimer resonances. The behavior near $a_{1*}$ and $a_{2*}$ is different, because near $a_{1*}$ the dimer condensate is an ellipsoidal shell while near $a_{2*}$ it is an ellipsoid. If $f_D$ is large enough near $a_{1*}$ and $a_{2*}$, the peaks can stand out above the 3-atom recombination and atom-dimer contributions. If $f_D$ is too large near $a_+$, the atom-dimer contribution can fill in the interference minimum from 3-atom recombination. For $\eta_* = 0.038$, there is no longer a local minimum near $a_+$ if $f_D > 2.2 \times 10^{-4}$.
The dimer fractions illustrated in Fig. 5.1 are sufficient to make the atom-dimer and two-dimer resonances stand out in the initial loss rate. Larger fractions would be required to make a significant difference in the integrated loss rates. Nevertheless our results demonstrate that narrow loss features can be produced at the atom-dimer and two-dimer resonances with plausibly small values of the dimer fraction.

Our dimer condensate mechanism provides a plausible explanation for the narrow loss feature at an atom-dimer resonance that was observed in a BEC of $^{39}$K atoms [10] and for the narrow loss features near an atom-dimer resonance and near two two-dimer resonances that were observed in a BEC of $^7$Li atoms [27]. It cannot explain the narrow loss features near atom-dimer resonances that have been observed in thermal gases of $^{39}$K atoms [10] and $^7$Li atoms [27, 35]. These loss features were observed at relatively small scattering lengths, so they could be associated with nonuniversal effects. In a BEC, the dimer fraction $f_D$ would depend sensitively on the detailed form of the ramp that brings the scattering length to its final value $a$. This sensitivity could be exploited to test our mechanism. The fraction $f_D$ could be amplified by pulsing the magnetic field very close to the Feshbach resonance before measuring the loss rate, as in the experiments on atom-molecule coherence [49]. The loss rates at atom-dimer and two-dimer resonances would increase linearly and quadratically with the amplification factor, respectively.

In summary, we have proposed a new mechanism for narrow atom loss features at atom-dimer resonances and at two-dimer resonances in a BEC of ultracold atoms. The positions of these features are determined by universal few-body physics and thus provide additional tests of universality. There could be similar loss features where universal 5-atom clusters cross the atom-dimer-dimer threshold. The strengths of all these loss features are determined by the many-body physics of Bose-Einstein condensates and open a new window into the remarkable phenomenon of atom-molecule
coherence.
Effective field theory (EFT) is a general method for describing the low-energy degrees of freedom of a system using the formalism of quantum field theory [53, 54]. The simplest EFT that can describe particles with a large scattering length is the zero-range model, which describes point particles that interact only through contact interactions. The EFT is particularly useful in exploring universality at large scattering length, because all nonuniversal terms suppressed by $\ell/a$ are set exactly to zero.

In this section, the EFT approach to the 2-atom and 3-atom problem for identical bosons using the zero-range model will be discussed. In this appendix, we set $\hbar = 1$ for simplicity.

In the quantum field theory framework, an atom is annihilated by a quantum field $\psi(r, t)$ and is created by a quantum field $\psi^\dagger(r', t)$ at space-time point $r$ and $t$. Symmetry under the exchange of identical bosons is implemented through equal-time commutation relations:

\begin{align}
[\psi(r, t), \psi(r', t)] &= 0, \quad (A.1a) \\
[\psi(r, t), \psi^\dagger(r', t)] &= \delta^3(r - r'). \quad (A.1b)
\end{align}
The Lagrangian density for a non-relativistic free field is given by

\[
L_{\text{free}} = \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi.
\]  

(A.2)

This is the kinetic term in the Lagrangian density for interacting atoms. It implies that the Feynman propagator for an atom of energy \(E\) and momentum \(k\) is \(i/(E - k^2/2m + i\varepsilon)\).

The interaction terms in the Lagrangian must respect the symmetries of the fundamental interactions. Power counting rules can be developed that indicate the relative importance of all the possible interaction terms [55, 56]. The power-counting rules for nonrelativistic particles with short-range interactions reveal that the most important interaction is a contact interaction between the particles. We will not develop these power-counting rules. Instead, we will simply argue that a contact interaction is a natural choice to describe low-energy atoms because their long wavelengths prevent them from resolving the structure of their interaction potential. The interaction terms of the zero-range model are given by

\[
L_{\text{int}} = -\frac{\lambda_2}{4m} (\psi^\dagger \psi)^2 - \frac{\lambda_3}{36m} (\psi^\dagger \psi)^3,
\]  

(A.3)

where \(\lambda_2\) and \(\lambda_3\) are the coupling constants for the 2-atom and 3-atom contact interactions, respectively. The factors 4 and 36 in the denominators are chosen to cancel symmetry factors associated with permutations of identical bosons. The interaction term in Eq. (A.3) implies that the Feynman rules for the two-atom and three-atom vertices are \(-i\lambda_2/m\) and \(-i\lambda_3/m\), respectively.

All effects of interactions in the two-atom sector can be encoded concisely in a function of a single variable: the transition amplitude \(A(E)\) for the scattering of a pair of atoms with total energy \(E\) in their center-of-momentum frame. For example,
the scattering amplitude $f(E)$ in Eq. (2.19) is given by
\[
\mathcal{A}(E = k^2/m) = \frac{8\pi}{m} f(k). \tag{A.4}
\]

Fig. A.1 shows two diagrammatic equations for the 2-atom transition amplitude. The upper diagrammatic equation shows the perturbative expansion of the amplitude order by order in the coupling constant $\lambda_2$. Because of the large scattering length, these diagrams must be summed to all orders in $\lambda_2$. The lower diagrammatic equation in Fig. A.1 is an alternative way to calculate the amplitude that exploits the recursive nature of the perturbative expansion. The corresponding equation is called the Lippmann-Schwinger integral equation:
\[
i\mathcal{A}(E) = -\frac{i\lambda_2}{m} + \left(\frac{-i\lambda_2}{m}\right) \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{i}{E - k^2/m + i\varepsilon (i\mathcal{A}(E))}. \tag{A.5}
\]

The integral over the momentum $k$ in Eq. (A.5) is ultraviolet divergent. It can be calculated analytically by imposing a ultraviolet momentum cutoff $|k| < \Lambda$. The Eq. (A.5) then reduces to an algebraic equation for $\mathcal{A}(E)$ whose solution is
\[
\mathcal{A}(E) = -\frac{1}{m} \left[ \frac{1}{\lambda_2} + \frac{\Lambda}{4\pi^2} - \frac{1}{8\pi} \sqrt{-mE - i\varepsilon} \right]^{-1}. \tag{A.6}
\]

This amplitude depends explicitly on the momentum cutoff $\Lambda$. It can be independent of the cutoff only if the coupling constant $\lambda_2$ depends implicitly on $\Lambda$ in such a way
that the cutoff dependences in Eq. (A.6) cancel out. The dependence on $\lambda_2$ and $\Lambda$ can be eliminated in favor of a physical quantity, such as the scattering length $a$. The scattering amplitude $f(k)$ defined by Eq. (A.4) is

$$f(k) = \left[ \frac{8\pi}{\lambda_2} + \frac{2\Lambda}{\pi} + ik \right]^{-1}.$$  \hspace{1cm} (A.7)

This has the same dependence on $k$ as the universal scattering amplitude in Eq. (2.19) if $a$ is identified with the following function of $\lambda_2$ and $\Lambda$:

$$a = \frac{1}{8\pi} \left[ \frac{1}{\lambda_2} + \frac{\Lambda}{4\pi^2} \right]^{-1}. \hspace{1cm} (A.8)$$

After using this equation to eliminate $\lambda_2$ from the transition amplitude in Eq. (A.6), we find that it is independent of the ultraviolet cutoff:

$$\mathcal{A}(E) = -\frac{8\pi}{m} \left[ \frac{1}{a} - \sqrt{-mE - i\varepsilon} \right]^{-1}. \hspace{1cm} (A.9)$$

This procedure of removing the cutoff dependence by eliminating $\lambda_2$ in favor of $a$ is called renormalization. The parameter $\lambda_2$ is often referred to as a bare coupling constant while $a$ is the renormalized coupling constant. Since the transition amplitude $\mathcal{A}(E)$ in Eq. (A.9) encodes all physical observables in the 2-body sector, the renormalization of $\lambda_2$ is sufficient to remove all dependence on the ultraviolet cutoff in the 2-atom sector. The fact that the renormalization of the 2-atom transition amplitude is simple and analytic is very useful in the calculation of 3-atom amplitudes.

The transition amplitude for three atoms is much more complicated than that for two atoms. The general amplitude in the center-of-momentum frame is a function of 9 independent variables: three energies and two momentum vectors. However all Efimov physics in the three-atom sector can be encoded in a much simpler function $\mathcal{A}_S(p, k; E)$ called the STM amplitude which is a function of only 3 variables: the total energy $E$ and two relative momenta. The STM amplitude satisfies an integral equa-
tion called the STM equation that was first derived by Skorniakov–Ter-Martirosian [57].

The three-body problem for particles with large scattering length was not understood within the EFT framework until 1999, when important progress was made by Bedaque, Hammer, and van Kolck [58, 59]. They introduced a diatom field \( d \) by changing the interaction Lagrangian in Eq. (A.3) to

\[
\mathcal{L}_{BHvK} = \frac{g_2}{4m} d^\dagger d - \frac{g_2}{4m} \left( d^\dagger \psi^2 + \psi^\dagger d^\dagger \right) - \frac{g_3}{36m} d^\dagger d\psi^\dagger \psi.
\]  

(A.10)

There is no kinetic term for \( d \) in the Lagrangian, so its equation of motion is

\[
d - \psi^2 - \left( \frac{g_3}{9g_2} \right) d\psi^\dagger \psi = 0.
\]  

(A.11)

The 3-atom contact interaction \( (\psi^\dagger \psi)^3 \) in Eq. (A.3) has been replaced by an atom-diatom contact interaction \( d^\dagger d\psi^\dagger \psi \) and by an interaction that allows a transition from a diatom to a pair of atoms and vice versa. The Lagrangian in Eq. (A.10) is equivalent to that in Eq. (A.3). This can be seen in the 2-atom or 3-atom sector simply by eliminating \( d \) using the equation of motion in Eq. (A.11). It is more difficult to show that it is also equivalent in the \( N \)-atom sector with \( N \geq 4 \).

The diatom field trick of Ref. [58] allows the general 3-atom transition amplitude to be reduced to the much simpler transition amplitude for an atom and a diatom. The atom-diatom transition amplitude satisfies an integral equation that is equivalent to the STM equation. Fig. A.2 shows a diagrammatic integral equation for the atom-diatom transition amplitude. The single and the double lines represent atom and diatom fields, respectively. The STM amplitude is the projection of the atom-diatom
transition amplitude onto the S-wave term. The STM integral equation is

$$\mathcal{A}_S(p, k; E) = \frac{16\pi}{ma} \left[ \frac{1}{2pk} \ln \left( \frac{p^2 + pk + k^2 - mE - i\epsilon}{p^2 - pk + k^2 - mE - i\epsilon} \right) + \frac{H(\Lambda)}{\Lambda^2} \right]$$

$$+ \frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \left( \frac{p^2 + pq + q^2 - mE - i\epsilon}{p^2 - pq + q^2 - mE - i\epsilon} \right) + \frac{H(\Lambda)}{\Lambda^2} \right]$$

$$\times \frac{\mathcal{A}_S(q, k; E)}{-1/a + \sqrt{3q^2/4 - mE - i\epsilon}}.$$  \hspace{1cm} (A.12)

where \( p (k) \) is the relative momentum of the incoming (outgoing) atom and diatom in the center-of-momentum frame. We refer the reader to Ref. [8] for the details of the derivation of the STM equation. The dimensionless 3-atom coupling constant \( H(\Lambda) \) is defined by

$$g_3 = -\frac{9g_2^2}{\Lambda^2} H(\Lambda),$$  \hspace{1cm} (A.13)
where $H(\Lambda)$ is a dimensionless log-periodic function of $\Lambda$ that can be approximated by

$$H(\Lambda) \approx h_0 \cos\left[ s_0 \ln(\Lambda/\Lambda_\ast) + \arctan s_0 \right].$$

(A.14)

This renormalization condition defines a renormalized three-body parameter $\Lambda_\ast$. The analytic approximation derived in Ref. [58] was Eq. (A.14) with $h_0 = 1$. In Ref. [60], it was found that the analytic approximation is accurate only to about 10%. It was found however that to a numerical accuracy of about $10^{-3}$, $H$ is given by the expression in Eq. (A.14) with the multiplicative numerical constant $h_0 = 0.879$ [60].

For the practical solution of the STM equation in Eq. (A.12), it is convenient to fix the numerical value of $H(\Lambda)$ and then tune $\Lambda$ to reproduce a three-body observable, such as the binding energy of an Efimov trimer. Given the numerical value of $H(\Lambda)$, one can use Eq. (A.14) to determine $\Lambda_\ast$ although it is not necessary. The use of the approximate expression in Eq. (A.14) with a generic cutoff $\Lambda$ introduces an uncertainty of about $10^{-3}$ associated with renormalization of the 3-atom contact interaction. This uncertainty can be avoided by the very simple choice $H = 0$. In this case, there is no atom-diatom contact interaction and it is the ultraviolet cutoff $\Lambda$ that plays the role of the 3-body parameter.
Appendix B

The Dimer-Dimer Scattering Length

The dimer-dimer scattering length $a_D$ is the low-energy limit of the scattering amplitude for a pair of shallow dimers. In Eq. (5.2), the rate at which atoms are lost due to inelastic collisions of two dimers was expressed in terms of the imaginary part of $a_D$. To calculate the loss rate, we need to know $a_D$ as a function of $a$. This requires the solution of a 4-body problem, which is extremely difficult.

By the optical theorem, the imaginary part of $a_D$ is proportional to the cross section for the inelastic scattering of two shallow dimers. If there are no deep dimers, the only inelastic scattering channels for the two shallow dimers are into an Efimov trimer and a recoiling atom. If there are deep dimers, there are additional inelastic scattering channels into two deep dimers, a shallow dimer and deep dimer, or into a deep dimer and two atoms. The existence of deep dimers makes the 4-body problem much more difficult. Fortunately it is not necessary to solve this more difficult problem. If an accurate analytic parameterization of $a_D$ as a function of $a$ were known for the case of no deep dimers, a good approximation for $a_D$ in the case with deep dimers can be obtained by using the analytic continuation in Eq (3.1). Deltuva has solved the 4-body problem for the dimer-dimer scattering length accurately for the case of no deep dimers [23]. If an accurate parameterization of Deltuva’s results for $a_D$ were
Figure B.1: The absolute value of the real part \( a_D/a \) as a function of \( a/a_x \). The real part of \( a_D \) is negative in the regions where \( |\text{Re}(a_D)|/a \) is increasing. The points joined by straight lines (black) are the results of a numerical calculation by Deltuva. The curve (blue) is from the real part of the fitting function in Eq. (B.1) with \( \eta_* = 0 \).

available, the effects of deep dimers could be taken into account through analytic continuation. In this appendix, we present a simple parametrization that reproduces the most dramatic features of Deltuva’s results for \( a_D \). The effects of deep dimers are then taken into account through analytic continuation. The resulting parametrization should produce the appropriate positions for atom loss features from inelastic dimer-dimer collisions, but it may not describe the shapes of these features accurately.

Deltuva’s numerical results for the real and imaginary parts of \( a_D \) are shown in Figs. B.1 and B.2, respectively, as dots connected by straight line segments. In Fig. B.1, the absolute value of \( \text{Re}(a_D/a) \) is plotted on a log scale as a function of \( a \). The two narrow maxima are near the dimer-dimer resonances where the universal tetramers cross the dimer-dimer threshold. The two narrow minima correspond to
zeros of $\mathrm{Re}(a_{\mathrm{D}})$. It is convenient to specify the positions of these maxima and minima in terms of the scattering length where the atom-trimer threshold crosses the dimer-dimer threshold, $a_x$. The peak on the left located at $a_{1*} = 0.38 \, a_x$ is where the deep tetramer crosses the dimer-dimer threshold. The peak on the right located at $a_{2*} = 0.99 \, a_x$ is where the shallow tetramer crosses the dimer-dimer threshold.

Deltuva gave precise results for the positions of these dimer-dimer resonances, which are given in Eqs. (2.31). Deltuva also gave precise results for the scattering lengths where the real part of $a_{\mathrm{D}}$ crosses zero. The zeros are at $a_{1o} = 0.200 \, a_x$ and at $a_{2o} = 0.951 \, a_x$.

We fit Deltuva’s results for the dimer-dimer scattering length with a linear combination of two cotangents that gives a good approximation to the zeros and poles.
when $\eta_* = 0$:

$$a_D(a) = A \left( \cot [s_0 \log (a/a_{1*}) - i \eta_*] + b \cot [s_0 \log (a/a_{2*}) - i \eta_*] + c \right) a, \quad (B.1)$$

where $A$ is an adjustable real parameter and the constants $b$ and $c$ are determined by the conditions that the real part of $a_D(a)$ vanishes at $a = a_{1o}$ and $a_{2o}$:

$$b = -\frac{r_{21} - r_{11}}{r_{22} - r_{12}},$$
$$c = -\frac{r_{11}r_{22} - r_{21}r_{12}}{r_{22} - r_{12}},$$
$$r_{ij} = \text{Re cot} \left[ s_0 \log (a_{io}/a_{j*}) - i \eta_* \right].$$

The coefficient $A$ was determined to be 0.58 by fitting Deltuva’s results for $\text{Re}(a_D)$ in Fig. B.1 to the expression in Eq. (B.1) with $\eta_* = 0$. The fit to $\text{Re}(a_D)$, which is shown in Fig. B.1 is very good everywhere except near the dimer-dimer resonance $a_{2*}$, where it is still reasonably good. The fit to the imaginary part of $a_D$ is also reasonably good between the dimer-dimer resonances. The fit near the dimer-dimer resonances can be improved by adjusting the values of $\eta_*$. The fit to $\text{Im}(a_D)$ with $\eta_* = 0.001$ is shown in Fig. B.2. The fit to $\text{Im}(a_D)$ is not very good below the first dimer-dimer resonance $a_{1*}$ or above the second dimer-resonance at $a_{2*}$.

The effects of deep dimers on $a_D$ can be taken into account approximately by the analytic continuation $a_{1*} \rightarrow a_{1*} \exp(i \eta_*/s_0)$ and $a_{2*} \rightarrow a_{2*} \exp(i \eta_*/s_0)$. This corresponds to increasing the value of $\eta_*$ in Eq. (B.1). In Fig. B.2 the imaginary part of $a_D$ is plotted for $\eta_* = 0.039$. This is the value used in Fig. 5.1 to illustrate the loss features at dimer-dimer resonances that would be produced by a coexisting dimer condensate. When $\eta_*$ is increased from 0.001 to 0.039, it has the appropriate effect of broadening the resonant peaks and reducing their height. However it remains below Deltuva’s results in the region above $a_{2*}$. The parametrization in Eq. (B.1) is likely to give a good approximation to the loss rate between $a_{1*}$ and $a_{2*}$, but it is likely to
be inaccurate in the region above $a_{2*}$. 


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