DISCRETE SYSTOLIC INEQUALITIES

DISSERTATION

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By

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ABSTRACT

Gromov’s systolic estimate, first proved in [2], is considered one of the deepest results in systolic geometry. It states that, for an essential Riemannian $n$-manifold $M$, the length of the shortest noncontractible loop, or systole, of $M$, denoted $\text{sys}_{\pi_1}(M)$ satisfies

$$\text{sys}_{\pi_1}(M) \leq C_n \sqrt[2n]{\text{Vol}(M)},$$

where the constant $C_n$ only depends on $n$ and not on $M$.

We will prove a discrete version of related theorems for triangulated surfaces. The argument involves creating a special Riemannian metric on a triangulated surface whose total volume is close to the number of facets in the triangulation. This metric then allows one to convert Riemannian geodesics to homotopic edge paths of controlled length. The proof of the analogous inequality in the case of a triangulated triangles then follows easily from these facts.

We then apply our discrete version to facts about triangulations of orientable surfaces. Given a triangulated, orientable, closed surface with $x$ 2-simplices, we can ask how many 3-simplices are required to “fill” the triangulation: that is, produce a triangulated 3-manifold whose boundary triangulation is the triangulated surface with which we started. Our method produces such a 3-manifold with no more than $O(x \log^2 x)$ simplices.

We will also prove that a discrete version of this inequality implies the Riemannian
version. The proof of this fact involves creating a triangulation of a Riemannian manifold that is in some sense aware of the geometry of the manifold. We embed $M$ in $\mathbb{R}^m$ using the Nash Embedding Theorem [5] and use an argument of Whitney’s [6] to produce a triangulation whose simplices are large in volume relative to their edges in the metric of $\mathbb{R}^m$. By working on a small enough scale, one obtains information about the geometry of the simplices embedded in $M$ in the induced path metric. Since $M$ is isometrically embedded, this gives the result. Again, once this obtained, proving that a discrete systolic inequality implies a Riemannian one is quite simple.

Finally, we present Whitney’s argument and the necessary modifications needed to obtain our triangulation theorem for embedded submanifolds of $\mathbb{R}^m$. 
To my parents.
VITA

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CHAPTER 1
INTRODUCTION

Gromov’s systolic estimate, first proved in [2], is considered one of the deepest results in systolic geometry. It states that, for an essential Riemannian $n$-manifold $M$, the length of the shortest noncontractible loop, or systole, of $M$, denoted $\text{sys}_{\pi_1}(M)$ satisfies

$$\text{sys}_{\pi_1}(M) \leq C_n \sqrt{\text{Vol}(M)},$$

where the constant $C_n$ only depends on $n$ and not on $M$. There related systolic inequalities for surfaces as well.

In Chapter 2, we prove that Gromov’s systolic estimates for surfaces also hold in the case of triangulated surfaces. The first step in this proof is to put a piecewise Euclidean metric on the triangulated surface. Each triangle will have the geometry of a equilateral Euclidean triangle of area one. We then remove singularities of the metric, which occur only at vertices. This is done by interpolating between the singular metric away from each vertex to a nonsingular Riemannian metric close to each vertex. We then show that the size of the neighborhoods determines the total difference in area between our smooth metric and the original singular one. By letting this size go to zero, we get that the total area is close to the number of triangles in the original triangulation.

Once this metric is obtained, we use the fact that it is nearly piecewise Euclidean to produce from every closed geodesic a freely homotopic edge loop in the triangulated
manifold. The geometry of the simplices and the ability to change the neighborhoods where the metric is not piecewise Euclidean allows us to obtain a bound on the length of the resulting edge path in terms of the length of the original geodesic.

After proving a general lemma, the above results easily imply discrete versions of two results of Gromov: an estimate for surfaces with infinite fundamental group, along with another for surfaces of high genus.

We then use the discrete systolic inequalities for surfaces to prove two facts about triangulations of surfaces. The first is that, for a triangulated, orientable surface of bounded genus, there is a 3-dimensional triangulated manifold that fills the surface with $O(x)$ simplices, where $x$ is the number simplices in the triangulation of the original surface. The second is that a triangulated, orientable surface of unknown genus has a 3-dimensional triangulated filling manifold with $O(x \log^2 x)$ simplices, where $x$ is again the number of simplices in the original surface.

The proof of both of these involves a cut-and-cone procedure: one cuts the triangulated surface along a short, homologically nontrivial loop to obtain a surface of smaller genus with two circle boundary components. Coning off the resulting boundary gives a closed surface of genus one less than before. By inducting on the genus of the surface, the above procedure eventually yields a triangulated 2-sphere. This can be coned off to obtain a triangulated 3-ball. By gluing the 3-ball back to together in the opposite order the cuts were made, we obtain a triangulated 3-manifold whose boundary is the triangulation we started with. The discrete systolic estimates allow us to obtain an upper bound for the number of simplices in this 3-manifold in terms of the genus and number of simplices in the original triangulation. This upper bound then easily implies the desired facts about surfaces.

In Chapter 3, we show that a discrete systolic inequality implies a Riemannian version. The method of proof used is to make a special type of triangulation of
a Riemannian manifold; one that is aware of the geometry of the manifold. More precisely, we prove that every Riemannian manifold has a triangulation where each simplex is not too small in volume relative to its maximum side length.

By the Nash Embedding Theorem [5], we may assume $M$ is an isometrically embedded $n$-dimensional submanifold of $\mathbb{R}^m$. Using an argument of Whitney [6], we show existence of a triangulation $T$ of $M$ contained inside a tubular neighborhood of $M$ whose simplices are nearly parallel to tangent planes of $M$. The tubular neighborhood projection $\pi^*$ induces a homeomorphism from the $T$ to $M$. At a small enough scale, $\pi^*$ is bi-Lipschitz on simplices of $T$, so the geometry of simplices of $T$ implies facts about the geometry of simplices in $\pi^*(T) = M$. Combining all these facts produces a triangulation of $M$ with the desired properties. We then show that these facts easily imply a Riemannian systolic inequality from a discrete version.

Finally, in Chapter 4, we present Whitney’s argument and the modifications needed to get our triangulation $T$ from above. Whitney’s argument is much more complicated than more modern methods of producing smooth triangulations of manifolds; however, the geometry of all of the simplices involved in the construction is computed throughout the procedure. This allows us to conclude very specific facts concerning the geometry of simplices of $T$: namely, if $\sigma$ is an $n$-simplex of $T$, then the ratio

$$\frac{\text{Vol}_n(\sigma)}{(\text{diam} \sigma)^n}$$

is bounded from below by a constant that only depends on $n$, and not on $M$ and its embedding in $\mathbb{R}^m$, which is precisely what is required in Chapter 3.
1.1 Notation used in this paper

Suppose $X$ is a metric space, with $S, T \subset X$. We define the distance between $S$ and $T$ by

$$\text{dist}(S, T) = \inf \{d(s, t) : s \in S, t \in T\}.$$ 

Also, we define the $r$-neighborhood of $S$ in $X$, denoted $U_r(S)$ to be

$$U_r(S) = \{x \in X : \text{dist}(x, S) < r\}.$$ 

Let $M$ be a manifold. Throughout the paper, all manifolds are smooth. If $M$ has a Riemannian metric $g$, we also assume $g$ is smooth. The Riemannian length of a path $\gamma$, denoted $\ell_g(\gamma)$, will refer to the length of $\gamma$ with respect to the metric $g$. The volume of $M$, denoted $\text{Vol}_g(M)$ will refer to the volume according to the Riemannian metric $g$. If $M$ is two-dimensional, we will typically refer to the volume of $M$ as the area of $M$, or $\text{Area}_g(M)$. The $g$ subscript may be omitted if it is clear which metric we are referring to. The systole of a Riemannian manifold $M$, denoted $\text{sys}_{\pi_1}(M)$, is the length of the shortest non-contractible loop in $M$. The homological systole of a Riemannian manifold $M$, denoted $\text{sys}_{H_1}(M)$, is the length of the shortest homologically nontrivial loop in $M$.

We will assume familiarity with the definition of a simplicial complex. A simplicial complex will refer to the particular arrangement of simplices in the complex and also the geometric realization of the complex. If $T$ is a simplicial complex, the $k$-skeleton of $T$, denoted $T^k$, will refer to the subcomplex of $T$ consisting of all simplices of dimension at most $k$. A facet of a triangulation is a simplex of maximal dimension. For any triangulation $T$, the notation $|T|$ will refer to the number of facets in the triangulation. In the case of a triangulated manifold, this will be used as a discrete analogue of volume (or area in the 2-dimensional case).

All triangulated manifolds are assumed to be smooth. If $p$ is an edge path in the
triangulated manifold $T$, the \textit{discrete length} of $p$, denoted $\ell_{\text{dis}}(p)$, will be the number of edges in $p$. The \textit{discrete systole} of a triangulated manifold $T$, denoted $\text{dsys}_{\pi_1}(T)$, will refer to the discrete length of the shortest non-contractible edge loop in $T$. The \textit{discrete homological systole}, denoted $\text{dsys}_{H_1}(T)$, is defined analogously.

Throughout this paper, the term \textit{surface} will mean a closed, orientable 2-dimensional manifold. If $T$ is a triangulated surface, a \textit{filling} of $T$ is a triangulated 3-manifold with boundary $M$ so that the restriction of the triangulation to the boundary is the triangulated surface $T$.

Let $S \subset \mathbb{R}^m$. By a \textit{secant vector} in $S$, we mean any $v = t(x - y)$ where $x, y \in S$ and $t \in \mathbb{R}$. We define a \textit{secant simplex} of $S$ to be the convex hull (in $\mathbb{R}^m$) of an affinely independent collection of points in $S$.

We now define the Hausdorff measure of a metric space. Let $X$ be a metric space and suppose $\{S_i\}_{i \in I}$ is a countable cover of $X$. Then we define the $d$-weight of $\{S_i\}$ to be

$$w_d(\{S_i\}) = \sum_i (\text{diam } S_i)^d.$$ 

For any $\epsilon > 0$, we may define

$$\mu_{d, \epsilon} = \inf \{w_d(\{S_i\}) : \text{diam}(S_i) < \epsilon \text{ for each } i\}$$

where the infimum is over all countable covers of $X$ of sets of diameter less than $\epsilon$. Finally, the \textit{$d$-dimensional Hausdorff measure} of $X$ is

$$\mu_d(X) = C(d) \cdot \lim_{\epsilon \to 0} \mu_{d, \epsilon}(X)$$

where $C(d)$ is a constant that depends only on $d$.

If $\sigma$ is an $n$-simplex in $\mathbb{R}^m$, we define its \textit{fullness} to be

$$\Theta(\sigma) = \frac{\text{Vol}_n(\sigma)}{(\text{diam } \sigma)^n}.$$
where $\text{Vol}_n$ is $n$-dimensional Hausdorff volume in $\mathbb{R}^m$. We also note that the diameter, $\text{diam} \sigma$, is the length of the longest side in this case.

Let $P$ be an affine plane in $\mathbb{R}^m$. We have the orthogonal projection of $\mathbb{R}^m$ onto $P$, denoted $\pi_P$. A point $p \in P$ defines a vector space

$$V_p(P) = \{v \in \mathbb{R}^m : p + v \in P\}.$$ 

There is also an obvious norm on $V_p(P)$ (which depends on $p$). For $P$ as above, the norm will be the one on $V_p(P)$. We note that $V_p(P)$ does not depend on which $p$ is chosen, and when there is no ambiguity, we will denote this vector space $V(P)$. We will frequently identify $P$ with $V(P)$ when there is no ambiguity.

Suppose $P_1, P_2$ are affine subspaces of $\mathbb{R}^m$ and $f : P_1 \to P_2$ is an affine map. Then for any $p \in P_1$, $f$ induces a linear transformation

$$V(f) : V_p(P_1) \to V_{\pi_{P_2}(p)}(P_2).$$

Again, this map does not depend on the choice of $p$ and is uniquely determined by $f$. We may then consider $V_p(P_1)$ and $V_{\pi_{P_2}(p)}(P_2)$ as linear subspaces of $\mathbb{R}^m$ and likewise consider $V(f)$ as a map of a subspace of $\mathbb{R}^m$ into $\mathbb{R}^m$. Thinking of $V(f)$ in this way allows us to speak of $|v - f(v)| = |w - V(f)(w)|$ for $v = w + p \in P_1$ (and depends on $p$). Throughout this section, expressions such as $|v - f(v)|$ for $v \in P_1$ will always be interpreted this way.
CHAPTER 2

DISCRETE SYSTOLIC INEQUALITIES

If one wishes to have a discrete version of Gromov’s systolic estimate in the setting of triangulated manifolds, the natural analogue of the length of a path is the number of edges it contains. Likewise, the numbers of facets in the triangulation would correspond to the volume of the manifold. In this chapter, we prove that such a result holds in the case of surfaces and provide an application of such a result.

2.1 A special metric on a triangulated surface

In this section, we will construct a Riemannian metric on a triangulated surface whose area is approximately the number of triangles in the triangulation. We will then relate the length of Riemannian geodesics with edge paths in the triangulation. More precisely, our goal is the following theorem.

Theorem 2.1.1. Let $T$ be a triangulated surface. Then for any $\epsilon > 0$, there exists a Riemannian metric $g_\epsilon$ on $T$ with the following properties:

1. $|\text{Area}_{g_\epsilon}(T) - |T|| < \epsilon$

2. If $\gamma$ is a closed $g_\epsilon$-geodesic in $T$, then there is an edge loop $p$, freely homotopic to $\gamma$, so that

$$\ell_{\text{dis}}(p) \leq \frac{2}{\sqrt{3}} \cdot \ell_{g_\epsilon}(\gamma).$$  \hspace{1cm} (2.1.1)
Proof. Let $T$ be a triangulated surface and let $\epsilon > 0$. We now describe a Riemannian metric $g_\epsilon$. Declare each triangle to be a flat, equilateral triangle with area 1. Call this singular metric $h$. Note that $h$ is smooth everywhere except possibly at vertices, where too many or too few triangles may meet. Suppose $v$ is a vertex that has $t_v$ triangles meeting at it. Then the metric $h$ near $v$ is given in polar coordinates centered at $v$ by the formula

$$h = dr^2 + \left(\frac{t_v}{6}\right)^2 r^2 d\theta^2,$$  \hspace{1cm} (2.1.2)

which is singular at $r = 0$ if $t_v \neq 6$. For a small $\delta > 0$, let $B_\delta(v)$ be the closed ball of radius $\delta$ in the $h$-metric about $v$. We then have a smooth function $\phi^\delta_v : [0, \infty) \to \text{ConvHull}\{1, 6/t_v\}$ with the following properties:

- For $x \leq \delta/3$, $\phi^\delta_v(x) = 1$.
- For $x \geq 2\delta/3$, $\phi^\delta_v(x) = t_v/6$.

This allows us to define a new, nonsingular metric $h_v^\delta$ on $B_\delta(v)$ by the formula

$$h_v^\delta = dr^2 + \phi^\delta_v(r)^2 r^2 d\theta^2.$$  \hspace{1cm} (2.1.3)

Note that, for $x \in B_\delta(v)$ with $r \leq \delta/3$, $h_v^\delta(x)$ is the standard Euclidean metric, and for $x$ with $r \geq 2\delta/3$, $h_v^\delta(x)$ is the metric $h$.

Let $h^\delta$ be the metric on $T$ given by the formula

$$h^\delta(x) = \begin{cases} 
  h(x) & \text{if } x \in B_{3\delta/4}(v) \text{ for some vertex } v \\
  h_v^\delta(x) & \text{if } x \notin B_{3\delta/4}(v) \text{ for any vertex } v.
\end{cases}$$

Note that this metric is smooth on all of $T$, since $h$ and $h_v^\delta$ agree on an open set containing $\cup_v \partial B_{3\delta/4}(v)$. 

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Note that $B_\delta(v)$ is the same in either metric, since, by formulas (2.1.2) and (2.1.3), $h$ and $h_\delta$ have the same radial geodesics with respect to $v$. If $A$ is the area of $B_\delta(v)$ in the metric $h$ and $A_\delta$ is the area of $B_\delta(v)$ in the metric $h_\delta$, we have that

$$|A - A_\delta| = \left| \int_{B_\delta(v)} dV_h - \int_{B_\delta(v)} dV_{h_\delta} \right|$$

$$= \left| \int_0^{2\pi} \int_0^\delta (t_v/6)rdrd\theta - \int_0^{2\pi} \int_0^\delta rdrd\theta \right|$$

$$\leq \int_0^{2\pi} \int_0^\delta |t_v/6 - 1|rdrd\theta \leq \pi\delta^2|t_v/6 - 1|,$$

which goes to zero as $\delta \to 0$. So if we are given an $\varepsilon > 0$, for $\delta$ sufficiently small and with

$$\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2} > \delta > 0,$$

(2.1.4)

the Riemannian surface area of $T$ with the metric $h_\delta$ is within $\varepsilon$ of the number of triangles in our triangulation $T$. Let this metric be $g_\varepsilon$.

This proves part (1) of Theorem 2.1.1.

We wish to find an edge path homotopic to a Riemannian geodesic $\gamma$ on our triangulated surface that increases the length in a controlled way. If all of our triangles were flat with area 1, the intersection of $\gamma$ with one of these triangles is either a Euclidean geodesic segment connecting different sides of the triangle, a vertex of the triangle, or an edge of the triangle. We will produce an edge path by sending each of the endpoints of the segment to the closest vertex and connecting those vertices by the appropriate edge; we do nothing if the original segment was an edge or vertex of the triangle. If we choose an orientation for each edge in our triangulation, we can use the orientation to determine which vertex to go to when a segment goes through a midpoint of a side in a consistent way. The resulting edge path is homotopic to the original geodesic.
What is the length of the shortest segment inside a triangle that can be sent to an edge in this way?

Consider Figure 2.1. The segment labeled $s$ is the shortest line segment between the closed half-side $a$ and the closed half-side $b$. Up to symmetry, the shortest segment that can be sent to an edge is one that begins in $a$ and ends in $b$. The length of segment $s$ is a lower bound for the length of any such segment.

Thus no geodesic segment that gets sent to an edge can be shorter than $s$. Since these triangles are flat and equilateral of area 1, a simple calculation shows that the segment $s$ is of length $\frac{\sqrt{3}}{2}$.

Now, in our case, we no longer have Euclidean triangles, as small neighborhoods around vertices have a metric that isn’t flat. However, as long as the distance from a midpoint to one of these neighborhoods is larger than the length of $s$, the segment $s$ will remain the same length and still be the shortest segment that can be sent to an edge. We can guarantee this by making $\delta$ less than the difference between half the
length of a side of the triangle and the length of the segment \( s \), which we have done in (2.1.4).

Let \( \gamma \) be a geodesic segment contained in a triangle. Let \( f(\gamma) \) be the edge path that \( \gamma \) is sent to. If \( f(\gamma) \) is an edge, then \( \gamma \) must be of length at least \( \frac{\sqrt{3}}{2} \) and \( f(\gamma) \) is of discrete length 1. If \( f(\gamma) \) is a vertex, then \( f(\gamma) \) has length 0. In both cases,

\[
\ell_{\text{dis}}(f(\gamma)) \leq \frac{2}{\sqrt{3}} \ell_{g_\epsilon}(\gamma).
\]

Notice that we can break up any geodesic \( \gamma \) into a concatenation of segments that are each contained in the triangles of \( T \). Since the edge paths we get are compatible with one another, the concatenation of the edge paths we get is freely homotopic to the concatenation of geodesic segments \( \gamma \).

Thus, for any closed geodesic \( \gamma \), there exists a closed edge path \( p \), freely homotopic to \( \gamma \), so that

\[
\ell_{\text{dis}}(p) \leq \frac{2}{\sqrt{3}} \ell_{g_\epsilon}(\gamma),
\]

which proves part (2) of Theorem 2.1.1.

\[\square\]

## 2.2 Discrete analogues of Riemannian systolic inequalities

We will prove a very general lemma which will allow us to prove our results:

**Lemma 2.2.1.** Let \( T \) be a triangulated surface and let \( P_1, \ldots, P_N \) be free-homotopy-invariant properties a loop in \( T \) can satisfy. Suppose that, for each \( \epsilon > 0 \), there is a closed geodesic \( \gamma_\epsilon \) on the Riemannian surface \((T, g_\epsilon)\) (cf. Theorem 2.1.1) so that \( \gamma_\epsilon \) satisfies properties \( P_1, \ldots, P_N \) and

\[
\ell_{g_\epsilon}(\gamma_\epsilon) \leq C \sqrt{\text{Area}_{g_\epsilon}(T)}.
\]  

(2.2.1)
Then there is an edge loop \( p \) on \( T \) so that \( p \) satisfies properties \( P_1, \ldots, P_N \) and

\[
\ell_{\text{dis}}(p) \leq \frac{2}{\sqrt{3}} C \sqrt{|T|}. \tag{2.2.2}
\]

**Proof.** Let \( \epsilon > 0 \) and let \( \gamma_\epsilon \) be the noncontractible closed geodesic promised in the hypothesis of the lemma. By Theorem 2.1.1, there is an edge loop \( p_\epsilon \) freely homotopic to \( \gamma_\epsilon \) such that

\[
\ell_{\text{dis}}(p_\epsilon) \leq \frac{2}{\sqrt{3}} \ell_{g_\epsilon}(\gamma_\epsilon). \tag{2.2.3}
\]

Combining inequalities (2.2.1) and (2.2.3), we have that

\[
\ell_{\text{dis}}(p_\epsilon) \leq \frac{2}{\sqrt{3}} \frac{\ell_{g_\epsilon}(\gamma_\epsilon)}{2} \leq \frac{2}{\sqrt{3}} C \sqrt{\text{Area}_{g_\epsilon}(T)} \leq \frac{2}{\sqrt{3}} C \sqrt{|T|} + \epsilon. \tag{2.2.4}
\]

Thus we have a collection of edge loops \( p_\epsilon \) in \( T \) that satisfy (2.2.4), all of which have properties \( P_1, \ldots, P_N \). Since the set of edge loops in \( T \) is finite, there must be an edge loop \( p \) with properties \( P_1, \ldots, P_N \) whose length satisfies (2.2.2).

**Corollary 2.2.2.** Let \( T \) be a triangulated surface with infinite fundamental group. Then

\[
\text{dsys}_{\pi_1}(T) \leq \frac{4}{3^{3/4}} \sqrt{|T|}.
\]

Corollary 2.2.2 follows from Lemma 2.2.1 and the following theorem:

**Theorem 2.2.3** (Cor 5.2.B [2]). For every compact Riemannian surface \( \Sigma \) with an infinite fundamental group, one has

\[
\text{sys}_{\pi_1}(\Sigma) \leq \frac{2}{\sqrt{3}} \sqrt{\text{Area}(\Sigma)}.
\]

**Corollary 2.2.4.** Let \( T \) be a triangulated surface of genus \( g > 0 \). Then

\[
\text{dsys}_{\pi_1}(T) \leq K \frac{\log g}{\sqrt{g}} \sqrt{|T|}.
\]

Corollary 2.2.4 follows from Lemma 2.2.1 and the following theorem:
Theorem 2.2.5 (Thm 2.C [3]). Let $\Sigma$ be an orientable Riemannian surface of genus $g \geq 2$. Then we have that

$$\text{sys}_{H_1}(\Sigma) \leq K^{\frac{\log g}{\sqrt{g}}} \sqrt{\text{Area}(\Sigma)},$$

(2.2.5)

where $\text{sys}_{H_1}(\Sigma)$ is the homological systole of $\Sigma$.

2.3 Applications of discrete systolic inequalities

In this section, we will prove the following theorems

**Theorem 2.3.1.** Let $T$ be a triangulated surface of genus at most $g$. Then there exists a filling $T'$, that is a triangulated 3-manifold with boundary $M$, so that the restriction of $M$ to its boundary is the triangulated surface $T$, and we have that

$$|M| \leq C_g |T|,$$

where $C_g$ only depends on $g$, and not on the particular surface.

**Theorem 2.3.2.** Let $T$ be a triangulated surface. Then there exists, $M$, a filling of $T$, so that

$$|M| \leq C |T| (\log |T|)^2,$$

where $C$ does not depend on the particular surface.

The proof of both of these theorems will be very similar. The main idea is to cut the surface along a short homologically nontrivial edge loop. This will yield a surface of smaller genus with two boundary components. We then cone off the boundary components to get a surface of genus one less than the original surface (See Figure 2.2). We iterate this procedure until the surface is a 2-sphere, in which case we perform modified coning-off procedure to get a triangulated 3-ball. By gluing the 3-ball along all of the cuts in the reverse order, we obtain a triangulated 3-manifold with the desired properties.
2.3.1 The cut-and-cone procedure

Suppose we are given a triangulated surface $T$.

If $T$ is a 2-sphere, we perform a special coning off of $T$: For each simplex $\sigma$ of $T$, we will triangulate the prism $\sigma \times I$ where $I$ is the unit interval, in the manner of Hatcher [4]. Here we imagine $\sigma \times \{1\}$ to be on the surface of the sphere $T$, and $\sigma \times \{0\}$ to be on the interior of the sphere. We first order the vertices of $T$. Suppose the vertices of $\sigma \times \{1\}$ are $\{v_0, v_1, v_2\}$, where the indices represent the order of the vertices. Let the corresponding vertices of $\sigma \times \{0\}$ be $\{w_0, w_2, w_3\}$. Then the simplices $v_0v_1v_2w_2$, $v_0v_1w_1w_2$, and $v_0w_0w_1w_2$ triangulate $\sigma \times I$, and if we do this for each simplex of $T$, adjacent simplices will have consistent triangulations. Finally we cone off $T \times \{0\}$ to get a triangulated 3-ball $M$ which has two layers: the center, which is a coned
off copy of $T$ and the exterior shell, which is our triangulated $T \times I$. It satisfies the statement of both theorems with $|M| = 4|T|$.

If $T$ has genus $g \geq 2$, by Theorem 2.2.4, there exists a homologically nontrivial edge loop $p'$ so that
\[
\ell_{\text{dis}}(p') \leq K \frac{\log g}{\sqrt{g}} \sqrt{|T|}.
\] (2.3.1)

Our homotopy may have introduced self-intersections not present in the original geodesic loop; this can be remedied by breaking up $p'$ into loops $p'_1, \ldots, p'_n$, each of which do not self-intersect. Because $p'$ is homologically nontrivial, there must be some $p'_i$ which is homologically nontrivial. Let $p = p'_i$. Note that $p$ still satisfies (2.3.1). We may now cut $T$ along $p$ to get a connected surface of genus $g - 1$ with two boundary components. We may then cone off the two boundary components to get a triangulated surface $T_{(1)}$ of genus $g - 1$. Note that
\[
|T_{(1)}| \leq |T| + 2\ell_{\text{dis}}(p) \leq |T| + 2K \frac{\log g}{\sqrt{g}} \sqrt{|T|} \leq \left( \sqrt{|T|} + K \frac{\log g}{\sqrt{g}} \right)^2.
\]

Suppose, inductively, that we have triangulated surfaces
\[
T, T_{(1)}, \ldots, T_{(n)}
\]
where $n \leq g - 1$, $T_{(i)}$ is obtained from $T_{(i-1)}$ by the above cut-and-cone procedure, and
\[
|T_{(i)}| \leq \left( \sqrt{|T_{(i-1)}|} + K \sum_{k=g-(i-1)}^{g} \frac{\log k}{\sqrt{k}} \right)^2.
\]

If $n < g - 1$, then $T_{(n)}$ has genus $g - n \geq 2$, so by Theorem 2.2.4, there exists a homologically nontrivial edge loop $p_{(n)}$ so that
\[
\ell_{\text{dis}}(p_{(n)}) \leq K \frac{\log(g - n)}{\sqrt{g - n}} \sqrt{|T_{(n)}|}.
\]
We may cut $T(n)$ along this path and cone off the boundaries to get a triangulated surface $T(n+1)$ with genus one less than the genus of $T(n)$ so that

$$|T(n+1)| \leq |T(n)| + 2K \frac{\log (g - n)}{\sqrt{g - n}} \sqrt{|T(n)|}$$

$$\leq \left( \sqrt{|T|} + K \sum_{k=g -(n-1)}^{g} \frac{\log k}{\sqrt{k}} \right)^2 + 2K \frac{\log (g - n)}{\sqrt{g - n}} \left( \sqrt{|T|} + K \sum_{k=g -(n-1)}^{g} \frac{\log k}{\sqrt{k}} \right)$$

$$\leq \left( \sqrt{|T|} + K \sum_{k=g -n}^{g} \frac{\log k}{\sqrt{k}} \right)^2 .$$

If $n = g - 1$, then $T(n) = T(g-1)$ is a torus and we may apply Corollary 2.2.2 to get a noncontractible edge loop $p'$ so that

$$\ell_{dis}(p') \leq \frac{4}{3^{3/4}} \sqrt{|T(g-1)|} .$$

As before, we may have introduced self-intersections. Since $T(g-1)$ is a torus, $p'$ must be homologically nontrivial and the same argument as above gives a non-self-intersecting loop $p$ that satisfies (2.3.2). Cutting and coning along $p$ gives us a triangulated 2-sphere $T(g)$ such that

$$|T(g)| = |T(g-1)| + 2\ell_{dis}(p)$$

$$\leq |T(g-1)| + 2 \frac{4}{3^{3/4}} \sqrt{|T(g-1)|}$$

$$\leq \left( \sqrt{|T|} + K \sum_{k=2}^{g} \frac{\log k}{\sqrt{k}} \right)^2 + 8 \frac{2}{3^{3/4}} \left( \sqrt{|T|} + K \sum_{k=2}^{g} \frac{\log k}{\sqrt{k}} \right)$$

$$\leq 5 \left( \sqrt{|T|} + K \sum_{k=2}^{g} \frac{\log k}{\sqrt{k}} \right)^2 .$$

We may then cone off this 2-sphere in the special manner above to get a triangulated 3-manifold $M(g)$ so that the restriction of $M(g)$ to its boundary is $T(g)$ and

$$|M(g)| = 4|T(g)| \leq 20 \left( \sqrt{|T|} + K \sum_{k=2}^{g} \frac{\log k}{\sqrt{k}} \right)^2 .$$
By gluing together $M_{(g)}$ along the cuts in the reverse order, we obtain a triangulated 3-manifold with boundary $M$ which is a filling of $T$ and

$$|M| \leq 20 \left( \sqrt{|T|} + K \sum_{k=2}^{9} \frac{\log k}{\sqrt{k}} \right)^2$$

$$= 20 \left( \sqrt{|T|} + K \sum_{k=2}^{7} \frac{\log k}{\sqrt{k}} + K \sum_{k=8}^{9} \frac{\log k}{\sqrt{k}} \right)^2$$

$$\leq 20 \left( \sqrt{|T|} + C' + \int_{7}^{9} \frac{\log x}{\sqrt{x}} \, dx \right)^2$$

$$\leq 20 \left( \sqrt{|T|} + C' + 2\sqrt{g} \log g \right)^2.$$

2.3.2 Proofs of theorems

Proof of Theorem 2.3.1. Suppose $T$ is a triangulated surface of genus at most $g$. After gluing together the triangulated 3-sphere that we obtain from the procedure above, we have $M$, a filling of $T$, so that

$$|M| \leq 20 \left( \sqrt{|T|} + C' + 2\sqrt{g} \log g \right)^2$$

$$\leq 20 \left( \sqrt{|T|} + C'_g \right)^2$$

$$\leq C'_g |T|$$

for a suitable constant $C'_g$.

Proof of Theorem 2.3.2. Suppose $T$ is a triangulated surface of genus $g$. After gluing together the triangulated 3-sphere that we obtain from the procedure above, we will have $M$, a filling of $T$, so that

$$|M| \leq 20 \left( \sqrt{|T|} + C' + 2\sqrt{g} \log g \right)^2.$$  \hspace{1cm} (2.3.3)

The number of edges in $T$ is $(3/2)|T|$. Thus if $|v(T)|$ is the number of vertices of $T$, we know that the Euler characteristic $\chi(T)$ satisfies

$$2 - 2g = \chi(T) = |v(T)| - \frac{|T|}{2}.$$
Solving for $g$ then gives that

$$g = \frac{-|v(T)|}{2} + \frac{|T|}{4} + 1 \leq \frac{|T|}{4}. \tag{2.3.4}$$

Combining (2.3.3) and (2.3.4), we can conclude that

$$|M| \leq 20 \left( \sqrt{|T|} + C' + 2\sqrt{g \log g} \right)^2$$

$$\leq 20 \left( \sqrt{|T|} (1 + \log |T|) + C'' \right)^2$$

$$\leq 20 \left( |T| (1 + \log |T|)^2 + 2 C'' \sqrt{|T|} (1 + \log |T|) + C'^2 \right)$$

$$\leq C|T| (\log |T|)^2$$

for some suitable $C$. \hfill \Box
CHAPTER 3
DERIVING RIEMANNIAN SYSTOLIC INEQUALITIES

Suppose we have a discrete systolic inequality

$$d_{sys_{n_1}}(T) \leq C_n \sqrt{|T|}$$

(3.0.1)

and we want to derive a version for Riemannian manifolds. Let \((M, g)\) be a Riemannian manifold and let \(T\) be a triangulation of \(M\). If \(T\) satisfied the conditions

- length of an edge of \(T \leq E\),
- volume of a facet of \(T \geq v\),

then we can easily get that

$$sys_{n_1}(M) \leq \frac{E}{v^{1/n}}C_n(\text{Vol}_g(M))^{1/n}.$$  

To get a true systolic inequality, the constant in this inequality should depend only on \(n\) and not on \(M\); thus, we need \(E/v^{1/n}\) to be bounded above in a way that only depends on \(n\). The purpose of this section is to produce such a triangulation for a Riemannian manifold. More precisely, we will prove the following theorem:

**Theorem 3.0.3.** For every integer \(n \geq 2\), there exists a constant, \(F_n\), with the following property: Let \(M^n\) be any compact Riemannian manifold of dimension \(n\). Then there exists a triangulation of \(M\) so that if \(v\) is the smallest volume of a facet...
in the triangulation (according to \( M \)) and \( E \) is the length of the longest edge in the triangulation, then
\[
\frac{E}{v^{1/n}} \leq F_n.
\]

### 3.1 Proof of main theorem

In this section we prove Theorem 3.0.3. We do this by using the following theorem, which will be proved in Chapter 4.

**Theorem 3.1.1.** Let \( M \) be a compact \( n \)-dimensional smooth Riemannian submanifold of \( \mathbb{R}^m \). Then there is an \( n \)-dimensional simplicial complex \( T \subset \mathbb{R}^m \) with the following properties:

1. Each simplex of \( T \) is a secant simplex of \( M \)
2. \( T \) is contained in a tubular neighborhood of \( M \). The projection \( \pi^* \) from this neighborhood onto \( M \) induces a homeomorphism \( \pi^*: T \to M \).
3. If \( \sigma \) is a simplex of \( T \), then it’s fullness, \( \Theta(\sigma) = \text{Vol}_n(\sigma)/\text{diam}_{\mathbb{R}^m}(\sigma)^n \), is bounded below by \( \Theta_{n,m}^* \), which depends only on the dimensions of the manifold and the ambient space.
4. For any \( n \)-simplex \( \sigma \) of \( T \), point \( q \in \sigma \) and tangent vector \( v \in T_q \sigma \), we get that
\[
|\pi_{\pi^*(q)}(v)| \geq \frac{1}{2}|v|,
\]
where \( \pi_{\pi^*(q)} \) is the orthogonal projection onto the tangent plane \( P_{\pi^*(q)} \).
5. If \( L \) is the length of an edge in \( T \), then
\[
C_{n,m} \bar{L} \leq L \leq \bar{L}
\]
for some positive \( \bar{L} \) and constant \( C_{n,m} \) depending only on \( n \) and \( m \).
Let $M$ be an $n$-dimensional Riemannian manifold. By the Nash Embedding theorem, $M$ embeds isometrically into $\mathbb{R}^m$, where $m$ depends only on $n$. Thus we may consider the case where $M$ is Riemannian submanifold of $\mathbb{R}^m$. Then applying Theorem 3.1.1, we get a simplicial complex $T$ so that the tubular neighborhood projection $\pi^*$ induces a homeomorphism from $T$ to $M$.

We will proceed in two parts. The first part will consist of showing that the restriction of $\pi^*$ to any $n$-simplex of $T$ is a bi-Lipschitz mapping of metric spaces. The second part will use this fact to relate the geometry of $T$ with the geometry of $\pi^*(T)$ to prove the main theorem.

### 3.1.1 $\pi^*$ is bi-Lipschitz on every $n$-simplex of $T$

Let $\sigma$ be an $n$-simplex of $T$ and suppose $x_1, x_2 \in \sigma$. Let $p_1 = \pi^*(x_1)$ and $p_2 = \pi^*(x_2)$.

Suppose $p$ is a unit-speed geodesic in $\sigma$ from $x_1$ to $x_2$ so that $|x_1 - x_2| = \ell(p)$. For every $t \in [0, \ell(p)]$, $p'(t)$ is a tangent vector in $\sigma$, so by Theorem 3.1.1 and Section 4.2,

$$1 = |p'(t)| \geq |p'(t)_n| = |\pi \pi^*(p'(t))| \geq \frac{1}{2} |p'(t)| = \frac{1}{2}. \tag{3.1.3}$$

Now $\pi^* \circ p$ is a path on $M$ from $p_1$ to $p_2$, and since $M$ is an isometrically embedded submanifold of $\mathbb{R}^m$,

$$d_M(p_1, p_2) \leq \ell(\pi^* \circ p). \tag{3.1.4}$$
Combining 4.2.1 and 3.1.3, we get that
\[
\ell(p) = \int_0^{\ell(p)} |D\pi_{p,t}'_1(t)| \, dt \\
= \int_0^{\ell(p)} |D\pi_{p,t}'_2(t)| \, dt \\
\leq \sqrt{3/2} \int_0^{\ell(p)} |p'(t)| \, dt \\
\leq \sqrt{3/2} \int_0^{\ell(p)} dt \\
= \sqrt{3/2}\ell(p).
\]

Combining the above with 3.1.4 gives that
\[
d_M(p_1, p_2) \leq \sqrt{3/2}|x_1 - x_2|. \tag{3.1.5}
\]

Now suppose \(\gamma\) is a unit-speed geodesic in \(M\) from \(p_1\) to \(p_2\) so that \(d_M(p_1, p_2) = \ell(\gamma)\). Then \((\pi^*)^{-1} \circ \gamma\) is a piecewise smooth path in \(T\) from \(x_1\) to \(x_2\). We may take a partition of the interval \([0, \ell(\gamma)]\) into
\[0 = a_0 < a_1 < a_2 < \cdots a_N = \ell(\gamma),\]
so that for each \(i\), \((\pi^*)^{-1} \circ \gamma([a_i, a_{i+1}]) \subset \sigma_i\) where \(\sigma_i\) is an \(n\)-simplex of \(T\). Let \(\gamma|_{[a_i, a_{i+1}]} = \gamma_i\) and let \((\pi^*)^{-1} \circ \gamma(a_i) = b_i\).

Then \((\pi^*)^{-1} \circ \gamma_i\) is a path in \(\sigma_i\) and for every \(t \in [a_i, a_{i+1}]\), \((D\pi^*)^{-1}(\gamma_i'(t))\) is a tangent vector in \(\sigma_i\). So for every \(t \in [a_i, a_{i+1}]\), (3.1.3) gives that
\[
|(D\pi^*)^{-1}(\gamma_i'(t))| \leq 2||D\pi^*(\gamma_i'(t))||_h. \tag{3.1.6}
\]
Suppose \(D\pi^*(v) = w\) for \(v \in T_p\sigma_i\). Then \(w = D\pi^*(v_h)\) and 4.2.1 gives that
\[
|v_h| \leq \sqrt{3/2}|w|.
\]

Using the above, we get that, for any \(t \in [a_i, a_{i+1}]\),
\[
||D\pi^*(\gamma_i'(t))||_h \leq \sqrt{3/2}|\gamma_i'(t)|. \tag{3.1.7}
\]
Combining 3.1.6 and 3.1.7 gives that

\begin{align*}
|b_i - b_{i+1}| &\leq \ell((\pi^*)^{-1} \circ \gamma) \\
&= \int_0^{a_{i+1} - a_i} |(D\pi^*)^{-1}(\gamma'_i(t))| \, dt \\
&\leq 2 \int_0^{a_{i+1} - a_i} |[(D\pi^*)^{-1}(\gamma'_i(t))]_a| \, dt \\
&\leq 2\sqrt{3/2} \int_0^{a_{i+1} - a_i} |\gamma'_i(t)| \, dt \\
&\leq 2\sqrt{3/2}d_M(\gamma(a_i), \gamma(a_{i+1})).
\end{align*}

Since \(\gamma\) is a minimizing geodesic, \(\sum d_M(\gamma(a_i), \gamma(a_{i+1})) = d_M(p_1, p_2)\). So

\begin{equation}
|x_1 - x_2| \leq \sum_{0}^{N+1} |b_i - b_{i+1}| \leq 2\sqrt{3/2} \sum_{0}^{N+1} d_M(\gamma(a_i), \gamma(a_{i+1})) = 2\sqrt{3/2}d_M(p_1, p_2).
\end{equation}

Combining (3.1.5) and (3.1.8) gives that, for any \(x_1, x_2 \in \sigma\),

\begin{equation}
\frac{1}{2\sqrt{3/2}}|x_1 - x_2| \leq d_M(\pi^*(x_1), \pi^*(x_2)) \leq \sqrt{3/2}|x_1 - x_2|.
\end{equation}

3.1.2 The geometry of \(\pi^*(T)\)

By the previous section, if \(e\) is an edge of the complex \(T\), then

\begin{equation}
\frac{1}{2\sqrt{3/2}}\ell(e) \leq \ell(\pi^*e) \leq \sqrt{3/2}\ell(e).
\end{equation}

We also need the following fact.

**Lemma 3.1.2** (Proposition 1.7.8(4), [1]). Let \(X\) and \(Y\) be metric spaces and let \(\mu_d\) be \(d\)-dimensional Hausdorff measure. Suppose \(f : X \to Y\) is a \(C\)-Lipschitz map. Then \(\mu_d(f(X)) \leq C^d \cdot \mu_d(X)\).

**Proof.** Let \(\{S_i\}\) be any cover of \(X\). Then \(\{f(S_i)\}\) is a cover of \(f(X)\). Since \(f\) is \(C\)-Lipschitz, \(\text{diam } f(S_i) \leq C \text{ diam } S_i\). This means that

\[w_d(\{f(S_i)\}) \leq C^d \cdot w_d(\{S_i\})\]
and thus $\mu_{d,C}(f(X)) \leq C^d \cdot \mu_{d,C}(X)$. So $\mu_d(f(X)) \leq C^d \cdot \mu_d(X)$. □

Let $\sigma$ be an $n$-simplex of $T$. Because $M$ is isometrically embedded, the $n$-dimensional Hausdorff volume on $M$ coming from the distance in $\mathbb{R}^m$ is the same as the volume on $M$ induced by its Riemannian metric. Using the theorem above for the map $\pi^*: \sigma \rightarrow \pi^*(\sigma)$ then gives that

$$\left(\frac{1}{2\sqrt{3/2}}\right)^n \text{Vol}_n(\sigma) \leq \text{Vol}_M(\pi^*(\sigma)) \leq (\sqrt{3/2})^n \text{Vol}_n(\sigma). \quad (3.1.11)$$

### 3.1.3 Proof of Theorem 3.0.3

Let $\sigma$ be the simplex in $M$ for which $\text{Vol}_M(\sigma)$ is minimal among all simplices in $\pi^*(T)$. Let $E$ be the length of the longest edge in $\pi^*(T)$. If $L$ is the length of longest edge in $\sigma$, then

$$E^n \leq \frac{L^n}{C_{n,m}}. \quad (3.1.12)$$

By (3.1.9), we have that

$$L \leq \sqrt{3/2} \text{diam } (\pi^*)^{-1}\sigma). \quad (3.1.13)$$

Using (3.1.12), (3.1.13) and (3.1.11), we get that

$$\frac{\text{Vol}_M(\sigma)}{E^n} \geq \frac{C_{n,m}^{n} \text{Vol}_M(\sigma)}{L^n} \geq \frac{C_{n,m}^{n} \text{Vol}_n(\pi^*)^{-1}(\sigma)}{2^n(3/2)^n L^n} \geq \frac{C_{n,m}^{m} \text{Vol}_n(\pi^*)^{-1}(\sigma)}{2^n(3/2)^n (\text{diam}(\pi^*)^{-1}\sigma)^n} \geq \frac{C_{n,m}^{m}}{3^n} \Theta_{n,m}.$$

Since $m$ depends only on $n$, we have proved Theorem 3.0.3, where

$$F_n = \frac{3}{C_{n,m}^{n/m} \Theta_{n,m}^{1/n}}.$$
3.2 A Riemannian systolic inequality from a discrete one

We prove the following theorem:

**Theorem 3.2.1.** Suppose there is a class of closed manifolds, $\mathcal{M}$, with the following property. For every triangulated $n$-manifold $T \in \mathcal{M}$, we have that

$$\text{dsys}_{\pi_1}(T) \leq C_n (|T|)^{1/n}, \quad (3.2.1)$$

where $C_n$ depends only on the dimension $n$, not on the particular manifold or triangulation. Then, for every Riemannian $n$-manifold $M \in \mathcal{M}$, we have that

$$\text{sys}_{\pi_1}(M) \leq K_n (\text{Vol}(M))^{1/n},$$

where $K_n$ only depends on $n$.

**Proof.** Suppose we have an $n$-dimensional Riemannian manifold $M$ with metric $g$. Let $T$ be the triangulation of $M$ promised by Theorem 3.0.3. Let $p$ be an edge loop in the triangulation $T$ whose length is $\text{dsys}_{\pi_1}(T)$. Letting $E$ be the length of the longest edge of $T$, we have that

$$\ell_g(p) \leq E \ell_{\text{dis}}(p). \quad (3.2.2)$$

Letting $V$ be the smallest volume of a simplex of $T$, we have that

$$\text{Vol}_n(M) \geq V |T|. \quad (3.2.3)$$

Let $\gamma$ be a geodesic of minimal length that is freely homotopic to $p$. Then $\ell_g(\gamma) \leq \ell_g(p) \leq E \ell_{\text{dis}}(p)$. Therefore,

$$\text{sys}_{\pi_1}(M) \leq K_n (\text{Vol}(M))^{1/n},$$

where $K_n$ only depends on $n$. 

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\( \ell_g(p) \). Combining this with (3.2.1), (3.2.2), and (3.2.3), we get that

\[
\ell_g(\gamma) \leq \ell_g(p)
\]

\[
\leq E\ell_{\text{dis}}(p)
\]

\[
\leq EC_n(|T|)^{1/n}
\]

\[
\leq \frac{E}{V^{1/n}} C_n (\text{Vol}_n(M))^{1/n}
\]

\[
\leq \frac{C_n}{\Theta_n^{1/n}} (\text{Vol}_n(M))^{1/n}.
\]

\[\square\]
CHAPTER 4

WHITNEY’S TRIANGULATION PROCEDURE

Our goal is to prove Theorem 3.1.1, which we restate here for convenience.

**Theorem 3.1.1.** Let $M$ be a compact $n$-dimensional smooth Riemannian submanifold of $\mathbb{R}^m$. Then there is an $n$-dimensional simplicial complex $T \subset \mathbb{R}^m$ with the following properties:

1. Each simplex of $T$ is a secant simplex of $M$.

2. $T$ is contained in a tubular neighborhood of $M$. The projection $\pi^*$ from this neighborhood onto $M$ induces a homeomorphism $\pi^*: T \to M$.

3. If $\sigma$ is a simplex of $T$, then its fullness, $\Theta(\sigma) = \text{Vol}_n(\sigma)/\text{diam}_{\mathbb{R}^m}(\sigma)^n$, is bounded below by $\Theta_{n,m}$, which depends only on the dimensions of the manifold and the ambient space.

4. For any $n$-simplex $\sigma$ of $T$, point $q \in \sigma$ and tangent vector $v \in T_q\sigma$, we get that

\[ |\pi_{\pi^*(q)}(v)| \geq \frac{1}{2}|v|,\]

where $\pi_{\pi^*(q)}$ is the orthogonal projection onto the tangent plane $P_{\pi^*(q)}$.

5. If $L$ is the length of an edge in $T$, then

\[ C_{n,m} \bar{L} \leq L \leq \bar{L} \quad (4.0.1)\]

for some positive $\bar{L}$ and constant $C_{n,m}$ depending only on $n$ and $m$. 

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This chapter parallels the procedure outlined by Whitney [6] to prove smooth manifolds are triangulable. The idea is to cubulate $\mathbb{R}^m$ and take the cubulation’s barycentric subdivision. We then move the vertices to ensure $M$ is far from the $(m - n) - 1$-skeleton of the complex. Using the poset of intersections of simplices of dimensions $(m - n), \ldots, m$ gives us a simplicial complex that sits inside a tubular neighborhood of $M$. We then prove the tubular neighborhood projection induces a diffeomorphism onto $M$.

We are using Whitney’s construction with a slightly different purpose in mind: to produce triangulations whose fullness only depends on the dimension of the manifold in question. This means some minor modifications need to be made, mostly relating to the choosing of certain quantities.

### 4.1 Notation used in this chapter

If $M^n \subset \mathbb{R}^m$ is a smooth, embedded submanifold of $\mathbb{R}^m$ and $p \in M$, we may identify the tangent space $T_p M$ with an affine $n$-plane $P_p \subset \mathbb{R}^m$, whose points are of the form $p + v$ where $v \in T_p M$. Since $P_p$ is affine subspace of $\mathbb{R}^m$, we have the orthogonal projection map $\pi_p : \mathbb{R}^m \to P_p$. Note that there are two ways to specify a projection map: if $P$ is an affine subspace of $\mathbb{R}^n$, $\pi_P$ is the orthogonal projection onto that affine subspace, while if $x \in M$, $\pi_x$ is the orthogonal projection onto $P_p$. We note that $\pi_p |_{M} : M \to P_p$ is regular at $p$, so there exists a neighborhood $U \subset M$ of $p$ so that $\pi_p : U \to \pi_p(U)$ is a diffeomorphism. For $\xi > 0$, let

$$P_{p,\xi} = U_\xi(p) \cap P_p.$$ 

For $\xi$ sufficiently small, $\pi_p^{-1}(P_{p,\xi}) \cap M \subset U$ and we let

$$M_{p,\xi} = \pi_p^{-1}(P_{p,\xi}) \cap M.$$
If $\sigma^n$ is an $n$-simplex in $\mathbb{R}^m$, it spans an affine $n$-plane, the plane of $\sigma$, which we will denote by $P(\sigma)$.

Let $P$ be an affine plane in $\mathbb{R}^m$. We have the orthogonal projection of $\mathbb{R}^m$ onto $P$, denoted $\pi_P$. A point $p \in P$ defines a vector space

$$V_p(P) = \{v \in \mathbb{R}^m : p + v \in P\}.$$

There is also an obvious norm on $V_p(P)$ (which depends on $p$). For $P_p$ as above, the norm will be the one on $V_p(P)$. We note that $V_p(P)$ does not depend on which $p$ is chosen, and when there is no ambiguity, we will denote this vector space $V(P)$. We will frequently identify $P$ with $V(P)$ when there is no ambiguity.

For affine subspaces $P, P'$ of $\mathbb{R}^m$, define the independence of $P$ and $P'$ to be

$$\text{ind}(P, P') = \inf\{|v - \pi_P(v)| : v \in P', |v| = 1\},$$

where $\pi_P : \mathbb{R}^m \to P$ is the orthogonal projection onto $P$. Note that this quantity is symmetric, and does depend on any particular choices made; in fact, it only depends on $V(P)$ and $V(P')$. We also note that $\text{ind}(P, P') = 0$ if the planes have a vector in common and $\text{ind}(P, P') = 1$ if and only if the planes are orthogonal.
4.2 Tubular neighborhoods and horizontal tangent vectors

Suppose $M$ is an embedded submanifold of $\mathbb{R}^m$. Let $U$ be a tubular neighborhood of $M$ in $\mathbb{R}^m$. Then the tubular neighborhood projection map $\pi^*: U \to M$ is a Riemannian submersion. Thus $TU \cong TU_v \oplus TU_h$, the horizontal and vertical parts of $TU$ induced by $\pi^*$.

Note that for any point $q \in U$, the space $T_qU_v$ is equal to the kernel of the derivative of the projection map $D\pi^*(q) = \pi^*(q)$. So if $w \in T_qU$ and $w = w_h + w_v$, then $|w_h| = |\pi^*(q)w|$.

Now, for any $p \in M$, the map

$$D\pi^*: T_pU_h \to T_pM$$

is the identity. Thus for any $\epsilon > 0$, there is a smaller tubular neighborhood $U' \subset U$ so that, for any $q \in U'$, the map

$$D\pi^*: T_qU'_h \to T_{\pi^*(q)}M$$

has the property that, for any $w \in T_qU'_h$,

$$\frac{1}{\sqrt{3/2}}|w| \leq |D\pi^*(w)| \leq \sqrt{3/2}|w|. \tag{4.2.1}$$

4.3 Quantities used in the proof

First, some remarks on the barycentric subdivision of a cube in $\mathbb{R}^m$ with side length $h$. This breaks up the cube into $2^m m!$ simplices, where the longest edge is one connecting a corner of a cube to its center, which is of length $h\sqrt{m}/2$. Thus each $m$ simplex has volume $h^m/2^m m!$ and diameter $h\sqrt{m}/2$, which gives a fullness of $1/m!m^{m/2}$. So any barycentric subdivision of a cubical subdivision of $\mathbb{R}^m$ forms a triangulation of $\mathbb{R}^m$ where the fullness of each simplex is $2\Theta_0 = 1/m!m^{m/2}$. Let $N$ be the maximum number of simplices in any star of such a triangulation.
Let $\rho^*$ be given by Lemma A.4.

By Lemma A.1, we may choose $\rho_0 < 1/4m^{1/2}$ and $\rho_0 \leq 2\rho^*/m^{1/2}$ so that for any $m$-simplex $\sigma = p_0 \cdots p_m$, if $\Theta(\sigma) \geq 2\Theta_0$, and $|q_i - p_i| \leq \rho_0 \text{diam} \sigma$, then $\tau = q_0 \cdots q_m$ is a simplex, with $\Theta(\tau) \geq \Theta_0$.

We choose $\rho_1$ as in Lemma A.5 so that

$$\frac{4}{\rho_0 \sqrt{2}} > \rho_1 > 0,$$

where $s = m - n \geq 1$ is the codimension of the embedding. We then define the following constants, which depend only on $n$ and $m$:

$$\rho = \rho_0 \rho_1 / 4, \quad \alpha_r = \rho^r \rho_0 \rho_1 / 2, \quad \alpha = \alpha_{s-1} / 4,$$

$$\beta = \Theta_0 \alpha / m^{1/2} N, \quad \Theta_1 = \beta^n / 2^n, \quad \gamma = (n - 1)! \Theta_1 \beta / 2.$$

The choice of $\rho_1$ in (4.3.1) ensures that $\rho < 1/2$, $\alpha_{r+1} < \alpha_r$, and $\alpha < 1/4$.

Choose $\rho_0' \leq 1/4$ by Lemma A.1 using $n$, $\Theta_1$, $\Theta_1 / 2$ in place of $r$, $\Theta_0$, $\epsilon$. Then let

$$\lambda = \inf \{ \alpha \gamma / 128, \rho_0' \alpha \beta / 8 \}. $$

Using the notation of Section 4.2, we may also choose $\delta_0$ in Theorem A.6 so that $U^* \subset U'$.

We then choose $\xi_0$ by Lemma A.7.

Now choose $\xi_1 \leq \xi_0$ in Lemma A.8. Finally, we define more constants:

$$\xi = \inf \{ \xi_1, \alpha \delta_0 / 3 \lambda \}, \quad \delta = \xi / 8, \quad h = 2 \delta / m^{1/2},$$

$$a = 2 \alpha \delta, \quad b = \beta \delta, \quad c = \gamma \delta.$$

4.4 The complexes $L$ and $L^*$

First, we let $L_0$ be a cubical subdivision of $\mathbb{R}^m$ with cubes of side length $h$ and let $L$ be the barycentric subdivision of $L_0$. Then each edge of $L$ has length at least $h/2$ and the $m$-simplices have diameter $\delta$. 

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Suppose \( L \) has vertices \( \{p_1, p_2, \ldots \} \). We are going to construct a new triangulation of \( \mathbb{R}^m, L^* \), whose \((s-1)\)-skeleton is sufficiently far away from \( M \). The vertices of this new complex will be \( \{p_1^*, p_2^*, \ldots \} \). We will choose the vertices so that

\[
|p_i^* - p_i| < \rho_0 \delta
\]

for all \( i \). By Lemma A.3 and the definition of \( \rho_0 \), we get a new triangulation of \( \mathbb{R}^m \).

Since \( \rho_0 < 1/4m^{1/2} \), we get that \( \rho_0 \delta < h/8 \). The diameter of any simplex of \( L \) is at least \( h/2 \), so any simplex \( \tau \) of \( L^* \) will have that

\[
diam(\tau) \geq h/2 - 2\rho_0 \delta
\]

\[
> h/2 - h/4
\]

\[
= h/4.
\]

Because the diameter of each simplex of \( L \) is at most \( \delta \), we get that

\[
diam(\tau) < \delta + 2\rho_0 \delta
\]

\[
< \delta + 2\frac{1}{4m^{1/2}} \delta
\]

\[
< 2\delta.
\]

Combining the above, we get that

\[
h/4 < diam(\tau) < 2\delta. \tag{4.4.2}
\]

By choice of \( \rho_0 \) and (A.1), we get that, for all simplices \( \tau \) of \( L^* \) of dimension at least 1,

\[
\Theta(\tau) \geq \Theta_0. \tag{4.4.3}
\]

Suppose \( p_1^*, \ldots, p_{i-1}^* \) have been found so that the complex \( L_{i-1}^* \) with these vertices satisfies

\[
dist(M, \tau^r) > \alpha_r \delta \text{ for all } \tau^r \in L^*, r \leq s - 1. \tag{4.4.4}
\]
We wish to find $p_i^*$ so that (4.4.4) holds for $L_i^*$.  

**Case 1.** If $\text{dist}(M, p_i) \geq 3\delta$, then we set $p_i^* = p_i$. By (4.4.2) we get that $\text{dist}(M, \tau) > \delta \geq \alpha_r \delta$ and (4.4.4) holds for $L_i^*$.

**Case 2.** If there is a point $p \in M$ with $|p - p_i| < 3\delta$, let $P_0 = P_p$ be the tangent plane of $M$ at $p$. Let $\tau'_1, \ldots, \tau'_\nu$ be the simplexes of $L_{i-1}^*$ of dimension at most $s - 2$ so that $\tau_j = p_i^* \tau'_j$ will be a simplex of $L_i^*$. Since the star of any vertex of $L$ or $L_i^*$ cannot have more than $N$ simplices, we have that $\nu < N$. For $j \geq 1$, let $P_j$ be the affine plane spanned by $\tau'_j$ and $P_p$. Its dimension is at most $(s - 2) + n + 1 = m - 1 < m$. Let

$$Q_j = U_{\rho_0 \delta}(p_i) \cap U_{\rho_1 \rho_0 \delta}(P_j), \text{ for } j = 0, \ldots, \nu. \tag{4.4.5}$$

Note that each $Q_j$ is (possibly strictly) contained in the part of the ball $U_{\rho_0 \delta}(p_i)$ between a pair of parallel $(m - 1)$-planes each at distance $\rho_1 \rho_0 \delta$ from $P_j$, and at distance $2\rho_1 (\rho_0 \delta)$ from each other. Thus, by the definition of $\rho_1$, we have that

$$\text{Vol}(Q_j) < \text{Vol}(U_{\rho_0 \delta}(p_i)) / N.$$ 

The total volume of the sets $Q_0, \ldots, Q_\nu$ is less than the volume of the ball $U_{\rho_0 \delta}(p_i)$; thus we can find a point $p_i^*$ so that (4.4.1) holds and we have that

$$\text{dist}(p_i^*, P_j) > \rho_1 \rho_0 \delta, j = 0, \ldots, \nu. \tag{4.4.6}$$

Claim:

$$\text{dist}(\tau'_j, P_0) > 2(\alpha_{r-1}) \delta / 3 \text{ if dim}(\tau'_j) = r - 1 \text{ and } r < s. \tag{4.4.7}$$

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Since $\tau_j'$ is in $L_{i-1}^*$, we have that $\text{dist}(\tau_j', M) > \alpha_{r-1}\delta$. Note that

$$\lambda \leq \frac{\rho_0\alpha\beta}{8} \leq \frac{\alpha\beta}{32} = \frac{\alpha}{32} \frac{\Theta_0\alpha}{m^{1/2}N} = \frac{\alpha_{s-1}}{128} \frac{1}{m} \frac{1}{m^{1/2}N}\frac{1}{\alpha} \leq \frac{\alpha_{s-1}}{24} \leq \frac{\alpha_{r-1}}{24}$$

so we have that $\lambda \xi = \lambda 8\delta < \alpha_{r-1}\delta/3$. By (A.11), $P_{p,\xi} \subset U_{\lambda\xi}(M)$. Using this and (4.4.4) gives that

$$\text{dist}(\tau_j', P_{p,\xi}) > \alpha_{r-1}\delta - \lambda\xi \geq 2\alpha_{r-1}\delta/3.$$

so (4.4.7) holds for $P_{p,\xi}$ in place of $P_p$. Recalling that $|p - p_i| < 3\delta$ and $\text{diam}(p_i\tau_j') < 2\delta$, we have

$$\text{dist}(\tau_j', P_p \setminus P_{p,\xi}) > \text{dist}(p, P_p \setminus P_{p,\xi}) - 3\delta - 2\delta \geq 3\delta$$

and the claim is proved.

Using Lemma A.11, we get that

$$\text{dist}(\tau_j, P_p) \geq \text{dist}(\tau_j', P_p) \text{dist}(p_i^*, P_j)/\text{diam}(\tau_j) \quad (4.4.8)$$

$$> (2\alpha_{r-1}\delta/3)(\rho_1\rho_0\delta)/2\delta \quad (4.4.9)$$

$$= 4\alpha_{r-1}\rho\delta/3 = 4\alpha_r\delta/3. \quad (4.4.10)$$

Using the above and (A.11) gives

$$\text{dist}(M_{p,\xi}, \tau_j) > 4\alpha_r\delta/3 - \alpha_r\delta/3 = \alpha_r\delta.$$
Applying (A.8), (4.4.1), and (4.4.2), we get

\[ \text{dist}(M \setminus M_{p,\xi}, \tau^r) \geq \text{dist}(M \setminus M_{p,\xi}, p) - |p - p_i| - |p_i - p^*_i| - \text{diam}(\tau^r) \]
\[ > \xi - 3\delta - \rho_0\delta - 2\delta \]
\[ > \delta. \]

Thus we get (4.4.4) for \( \tau^r = \tau_j, j \geq 1 \). Using \( j = 0 \) in (4.4.6) and (A.11) then gives

\[ \text{dist}(p^*_i, M_{p,\xi}) \geq \text{dist}(p^*_i, P_p) - \lambda\xi \]
\[ > \rho_1\rho_0\delta - \alpha_0\delta/3 \]
\[ > \alpha_0\delta. \]

Again by (A.8) and (4.4.1), we get

\[ \text{dist}(M \setminus M_{p,\xi}, p^*_i) \geq \text{dist}(M \setminus M_{p,\xi}, p) - |p - p_i| - |p_i - p^*_i| \]
\[ > \xi - 3\delta - \rho_0\delta \]
\[ > 2\delta > \alpha_0\delta. \]

So (4.4.4) holds for \( \tau^r = p^*_i \) and (4.4.4) holds in all cases. In particular, if \( L^{s-1} \) is the \((s-1)\)-skeleton of \( L^s \), then

\[ \text{dist}(M, L^{s-1}) > \alpha_{s-1}\delta = 4\alpha\delta = 2a. \] (4.4.11)

### 4.5 The intersections of \( M \) with \( L^s \)

**Claim 1.** For any \( p \in M \) and \( r \)-simplex \( \tau^r \) of \( L^s \), we have that

\[ \text{dist}(P_p, \tau^r) > a \text{ if } \tau^r \subset U_{7\delta}(p), r \leq s - 1. \] (4.5.1)

**Proof.** By the above,

\[ \text{dist}(P_p - P_{p,\xi}, \tau^r) \geq \text{dist}(P_p - P_{p,\xi}, \{p\}) - 7\delta \]
\[ > \xi - 7\delta = \delta > a. \]
We also have that $P_{p, \xi} \subset U_{\lambda \xi}(M)$ and $\lambda \xi < a$; using (4.4.11) gives

$$\text{dist}(P_{p, \xi}, t^r) \geq \text{dist}(P_{p, \xi}, L^{s, \lambda})$$
$$\geq \text{dist}(M, L^{s, \lambda}) - \lambda \xi$$
$$> 2a - a = a.$$

So (4.5.1) is true in all cases. \qed

**Claim 2.** If $M$ intersects $t^r$, $p \in M$, and $t^r \subset U_{7\delta}(p)$, then $P_p$ intersects $t^r$.

**Proof.** If $p' \in M \cap t^r$, then by (A.8), $p' \in M_{p, \xi}$ since $|p' - p| < 7\delta$. By (A.11), $\text{dist}(p', P_p) < \lambda \xi < a$. Let $t^t$ be a face of smallest dimension of $t^r$ with $\text{dist}(t^t, P_p) \leq a$. By (4.5.1), $t \geq s$ and $\text{dist}(P_p, t^t) < \text{dist}(P_p, \partial t^t)$. Lemma A.9 then gives that $P_p$ intersects $t^t$. \qed

**Claim 3.** If $r = s$ in the previous part, and $P(t^s)$ is the affine plane spanned by $t^s$, then

$$\text{ind}(P_p, P(t^s)) > \frac{a}{2\delta} = \alpha. \quad (4.5.2)$$

**Proof.** This follows immediately Lemma A.9, (4.5.1), and (4.4.2). \qed

**Claim 4.** If $p \in M$, $t^r \subset U_{7\delta}(p)$, and $P_p$ intersects $t^r$, then $r \geq s$, and $M_{p, \xi}$ intersects $t^r$.

**Proof.** Let $t^t$ be the face of smallest dimension of $t^r$ such that $\text{dist}(P_p, t^t) \leq a$. By (4.5.1) and Lemma A.9, $t = s$ (hence $r \geq s$), $P_p$ has a point $p'$ in $t^s$, and (4.5.2) holds. Let $\pi'$ be the projection into $P_p$ along planes parallel to $t^s$. By Lemma A.10, $\pi'(M_{p, \xi})$ covers $P_{p, \eta}$, where

$$\eta = (1 - \lambda/\alpha)\xi = (1 - \lambda/\alpha)8\delta > 7\delta.$$
Since $\tau^r \subset U_7(p)$, $|p' - p| < 7\delta$, and there is a $p^* \in M_{p,\xi}$ with $\pi'(p^*) = p'$. Thus $p^* - p' \in P(\tau^s)$ and $p^* \in P(\tau^s)$ since $p \in P(\tau^s)$. By (A.13),

$$|p' - p^*| < \lambda\xi/\alpha \leq \rho'_0\beta\alpha < \beta\delta < a.$$ 

Since $p' \in \tau^s$, (4.5.1) shows that $p^* \in \tau^s$. \hfill $\square$

Claim 5. $M$ intersects any $\tau^s$ in at most one point.

Proof. Suppose $p, p'$ are distinct points of $M$ both contained in $\tau^s$. By (A.8), $p' \in M_{p,\xi}$, since diam $\tau^s < 2\delta < \xi$. So $p' - p = v$ is a secant vector in $M_{p,\xi}$, and by (A.9), $|v - \pi_p v| \leq \lambda|v|$. Since $v$ is parallel to $P(\tau^s)$, (4.5.2) gives that $|v - \pi_p v| > \alpha|v| > \lambda|v|$, a contradiction. \hfill $\square$

Claim 6. If $M$ intersects $\tau^r = q_0 \cdots q_r$, then for each $k$, $M$ intersects some $s$-face of $\tau^r$ containing $q_k$.

Proof. Take $p \in M \cap \tau^r$. Let $\tau^t$ be the face of smallest dimension of $\tau^r$ containing $q_k$ which $P_p$ intersects. Suppose $t > s$. Lemma A.9 says $P_p$ cannot only intersect the interior of $\tau^t$, so it must intersect $\partial\tau^t$. Letting $\tau^{t-1}$ be the face of $\tau^t$ opposite the vertex $q_k$, our choice of $\tau^t$ means that $P_p$ must intersect $\tau^{t-1}$. Again by Lemma A.9, $P_p$ cannot intersect any face of $\tau^{t-1}$ of dimension larger than $s$ in only its interior, so $P_p$ intersects some $s$-face of $\tau^{t-1}$. By Claim 3 and Lemma A.9, $P_p$ must meet $P(\tau^s)$ at a positive angle in exactly one point, so it must intersect the interior of $\tau^t$, which means it intersects $\partial\tau^t \setminus \tau^{t-1}$, a contradiction. So $t = s$, and by Claim 4, $M$ also intersects $\tau^s$. \hfill $\square$

4.6 The complex $K$

For every simplex $\tau$ of $L^*$ that intersects $M$, we will choose a point $\psi(\tau)$ inside $\tau$, and create a simplicial complex which is the order complex of the poset of faces of
simplices of $L^*$ that intersect $M$. (Note that this produces a complex of dimension $n$).

If a simplex $\tau^s$ intersects $M$, it does so at a single point; let $\psi(\tau^s)$ be that point. If $\tau^r$ intersects $M$, for $r > s$, let $\tau_1^s, \ldots, \tau_k^s$ be the $s$-faces of $\tau^r$ that intersect $M$ (which exist by the previous section). Then let

$$\psi(\tau^r) = \frac{1}{k} \sum_{0}^{k} \psi(\tau_i^s). \quad (4.6.1)$$

For $\tau^s = q_0 \ldots q_s$ intersecting $M$, we get that

$$\mu_k > 2\alpha, 0 \leq k \leq s, \text{ where } \psi(\tau^s) = \sum \mu_i q_i. \quad (4.6.2)$$

Let $\tau_k$ be the $(s-1)$-face opposite $q_k$. Let $A_k$ and $A'_k$ be the altitudes from $q_k$ and $\psi(\tau^s)$ respectively to $P(\tau_k)$. Since $\psi(\tau^s) \in M$, (4.4.2) and (4.4.2) give that

$$\mu_k = A'_k/A_k > 2\alpha/2\delta = 2\alpha.$$

Also, if $\tau^r = q_0 \ldots q_r$ intersects $M$, then

$$\mu_k > 2\alpha/N, 0 \leq k \leq r, \text{ where } \psi(\tau^r) = \sum \mu_i q_i. \quad (4.6.3)$$

Given $k$, let $\tau^s$ be an $s$-face of $\tau^r$ containing $q_k$, which intersects $M$ by Claim 6 of the previous section. By (4.6.2), the barycentric coordinate $\mu'$ of $\psi(\tau^s)$ corresponding

Figure 4.2: The complex $K$
to \( q_k \) is at least \( 2\alpha \). By (4.6.1), \( \mu_k \) is the average of at most \( N \) barycentric coordinates, one of which is \( \mu' \). Thus, (4.6.3) holds.

Since \( K \) is an order complex, each simplex \( \sigma \) of \( K \) has a natural order. Let \( \text{alt}(\sigma) \) be the altitude from the last (highest dimension) vertex of \( \sigma \). We wish to show that

\[
\text{alt}(\sigma^r) \geq rb. \tag{4.6.4}
\]

Let \( \sigma^r = \psi(\tau_0) \cdots \psi(\tau^r) \) be a simplex of \( K \), with vertices in ascending order. Let \( \sigma^{r-1} \) be the \((r-1)\)-face of \( \sigma^r \) opposite \( \psi(\tau_r) \). Then \( \sigma^{r-1} \) lies in \( \tau_{r-1} \).

If \( \dim(\tau_r) = t \geq r \), Lemma A.2, (4.6.3), and (4.4.2) give

\[
\text{alt}(\sigma^r) \geq \text{dist}(\psi(\tau_r), \partial \tau_r)
\geq t! \Theta(\tau_r) \text{diam}(\tau_r) \inf\{\mu_0, \ldots, \mu_r\}
\geq r! \Theta_0(h/4)(2\alpha/N)
\geq rb.
\]

Because \( \text{Vol}_r(\sigma^r) = \text{alt}(\sigma^r) \text{Vol}_{r-1}(\sigma^{r-1}) \), induction gives \( \text{Vol}_r(\sigma^r) \geq b^r \). Thus, by (4.4.3) and the above,

\[
\Theta(\sigma) \geq b^r/(2\delta^r) = \beta^r/2^r \geq \beta^n/2^n = \Theta_1. \tag{4.6.5}
\]

### 4.7 Embedding of simplices in \( M \)

**Claim 7.** If \( \sigma \) is a simplex of \( K \) and \( \sigma \subset U_{6\delta}(p) \) for \( p \in M \) then

\[
\sigma \subset U_{\lambda\xi}(P_{p,\xi}). \tag{4.7.1}
\]

**Proof.** Let \( \sigma \) be a simplex of \( K \) with \( \sigma \subset U_{6\delta}(p) \) for \( p \in M \). Suppose \( \sigma \) is contained in the simplex \( \tau \) of \( L^* \). Then \( \tau \subset U_{8\delta}(p) \) since \( \text{diam}(\tau) < 2\delta \). By (A.11), each \( \psi(\tau_{s}^j) \) is in \( U_{\lambda\xi}(P_{p,\xi}) \). Since every vertex \( v \) of \( \sigma \) is in the convex hull of the points \( \psi(\tau_{s}^j) \) and
$U_{\lambda \xi}(P_{p,\xi})$ is convex, each vertex is in $U_{\lambda \xi}(P_{p,\xi})$. So the claim is true for every vertex $v$ and thus it holds for all of $\sigma$.

Combining the above claim and (A.7) then gives that

$$K \subset U_{2\lambda \xi}(M) \text{ and } |\pi^*(q) - q| < 4\lambda \xi \text{ if } q \in K.$$  \hspace{1cm} (4.7.2)

**Lemma 4.7.1** \hspace{1cm} (IV, 21a [6]). Let $\sigma = p_0 \cdots p_n$ be an $n$-simplex of $K$ (with vertices in increasing order) and let $p'_0, \ldots, p'_n$ be any points such that

$$|p'_i - p_i| \leq \lambda \xi / \alpha, i = 0, \ldots, n.$$ 

Then $\sigma' = p'_0 \cdots p'_n$ is a simplex in $U^*$, and $\pi^*$ embeds $\sigma'$ in $M$.

**Proof.** First, note that $\lambda \xi / \alpha \leq \rho'_0 \beta \delta / 8 = \rho'_0 b$, $\Theta(\sigma) \geq \Theta_1$, and diam($\sigma$) $\geq b$. By choice of $\rho'_0$, we get that

$$\Theta(\sigma') \geq \Theta_1 / 2.$$ \hspace{1cm} (4.7.3)

By (4.7.2) and the definition of $\xi$, we get that $\sigma' \subset U_\eta(M)$, where

$$\eta = \lambda \xi / \alpha + 2\lambda \xi \leq 3\lambda \xi / 2 \leq \delta_0.$$ \hspace{1cm} (4.7.4)

So $\pi^*$ is defined on $\sigma'$. Let $q \in \sigma'$ and suppose $q \in Q_\sigma^*$ where $p \in M$; thus $\pi^*(q) = p$. By (A.7), (4.7.4) and the definition of $\lambda$,

$$|p - q| \leq 2(3\lambda \xi / \alpha) < \delta.$$ 

Also, $\lambda \xi / \alpha < \delta$, so $\sigma \subset U_{4\delta}(p)$ since diam($\sigma$) $< 2\delta$. So (4.7.1) implies that $\sigma \subset U_{\lambda \xi}(P_p)$ and thus $\sigma \subset U_{2\lambda \xi/(\alpha(P_p))}$. (4.6.4) gives that $|p_i - p_0| \geq b$, thus

$$|p'_i - p'_0| \geq b - 2\lambda \xi / \alpha \geq b - 2\rho'_0 b \geq b/2.$$ \hspace{1cm} (4.7.5)

By Lemma A.3, if $u$ is a unit vector in $P(\sigma')$, we get that

$$|u - \pi_p u| \leq \frac{2(2\lambda \xi / \alpha)}{(n-1)!(\Theta_1/2)(b/2)} = 64\lambda / \alpha \gamma \leq 1/2.$$ \hspace{1cm} (4.7.6)
Since \( Q_p^* \) is normal to \( P_p \), \( u \) cannot be in \( Q_p^* \). Also \( \pi^* \) maps any nonzero vector in \( \sigma' \) at \( q \) to a nonzero vector and thus \( \pi^* \) is regular at \( q \). If \( q' \) is any other point of \( \sigma' \), letting

\[
 u = \frac{q' - q}{|q' - q|}
\]

in the above gives that \( q' \notin Q_p^* \), thus \( \pi^*(q) \neq \pi^*(q') \).

\[ \square \]

### 4.8 The complexes \( K_p \)

For \( p \in M \), let \( L_p^* \) be the subcomplex of \( L^* \) containing simplices which touch \( \bar{U}_{\delta \delta}(p) \), together with their faces. Then

\[
 L_p^* \subset U_{\delta \delta}(p). \tag{4.8.1}
\]

Let \( K''_p \) be the complex in \( P_p \) formed by the intersections of \( P_p \) with the simplices of \( L_p^* \) and let \( K'_p \) be the barycentric subdivision of \( K''_p \). By the discussion in Section 4.5, \( P_p \) intersects a simplex of \( L_p^* \) if and only if \( M \) does. Let \( K_p \) be the subcomplex of \( K \) containing all simplices with vertices \( \psi(\tau) \) for each simplex \( \tau \) in \( L_p^* \). Then there is a one-to-one correspondence \( \phi_p \) of the vertices in \( K_p \) onto the vertices of \( K'_p \) which defines a simplicial mapping \( \phi_p \) that is an isomorphism of \( K_p \) onto \( K'_p \).

**Claim 8.** If \( q \in K_p \), then

\[
 |\phi_p(q) - q| < \lambda \xi / \alpha. \tag{4.8.2}
\]

**Proof.** Suppose \( q = \psi(\tau^s) \) for some \( \tau^s \) in \( L_p^* \). Then \( v = q - \phi_p(q) \) is in \( \tau^s \). Using (A.11) and (4.5.2) gives that

\[
 \lambda \xi > \text{dist}(q, P_p) = |q - \pi_p(q)| = |v - \pi_p(v)| \geq \alpha |v|,
\]

which implies (4.8.2). Since \( \phi_p \) is affine, if \( q = \psi(\tau^r) \) for \( r > s \), then \( q \) and \( \psi_p(q) \) are the same averages of a set of points, each pair of which satisfies (4.8.2); thus (4.8.2)
holds for \( q \) as well. Since (4.8.2) holds for all vertices of \( K_p \), it holds for all points of \( K_p \).

Claim 9.

\[
K \cap U_{2\delta}(p) \subset K_p. \quad (4.8.3)
\]

Proof. Let \( q \in \sigma \) where \( \sigma = \psi(\tau_0) \cdots \psi(\tau^r) \) is a simplex in \( K \) such that \( |q - p| < 2\delta \). Then \( \tau^r \subset U_{4\delta}(p) \), so \( \tau^r \) is in \( L^*_p \); thus every \( \tau_i \) is in \( L^*_p \). So every vertex of \( \sigma \) and thus all of \( \sigma \) is contained in \( K_p \). 

Choose an orientation for \( P_p \) and use this to orient all \( n \)-simplices of \( K'_p \). This makes \( K'_p \) an oriented \( n \)-psuedomanifold with boundary; (4.8.1) and the definition of \( L^*_p \) then give that

\[
K'_p \subset U_{6\delta}(p), \quad \partial K'_p \subset P_p - U_{4\delta}(p). \quad (4.8.4)
\]

Define a mapping \( \pi^*_p \) of \( K'_p \) into \( P_p \) as follows: Each \( q \in K_p \) is in a unique \( Q^*_p(q) \), so \( p' = \pi^*(q) \). By (4.8.1), \( |q - p| \leq 6\delta \) and by (4.7.2), \( |p' - q| < 4\lambda \xi < \delta \); thus \( |p - p'| < \xi \). By Lemma A.12, \( P_{p'} \) intersects \( P_p \) in a unique point, which we call \( \pi^*_p(q) \).

Claim 10. If \( q \in K_p \), then

\[
|\pi^*_p(q) - q| < 6\lambda \xi. \quad (4.8.5)
\]

Proof. By (4.7.2), this reduces to showing that \( |v| < 2\lambda \xi \) where \( v = p' - \pi^*_p(q) \). Since \( \lambda < 1/2 \), (A.15) gives that \( |\pi_p(v)| \leq |v|/2 \). From (A.10) we have that \( |v - \pi_p(v)| < \lambda \xi \), so \( |v| < |v|/2 + \lambda \xi \) and the claim is proved.

4.9 Proof of Theorem 3.1.1

Let \( p \in M \) and choose an orientation of \( P_p \), which gives an orientation of both \( K'_p \) and \( K_p \). We prove the following lemmas.

Lemma 4.9.1 (IV, 23a [6]). \( \pi^*_p \) is a simplex-wise positive mapping of \( K_p \) into \( P_p \).
Proof. Let $\sigma$ be an $n$-simplex of $K_p$. Let

$$\phi_{p,t}(q) = (1-t)q + t\phi_p(q) \text{ for } q \in \sigma, \quad \sigma_t = \phi_{p,t}(q).$$  \hfill (4.9.1)

$\phi_{p,t}$ is affine in $\sigma$ since $\phi_p$ is and $\sigma_t$ is a simplex. Suppose $\sigma = q_0 \cdots q_n$. For any $t \in [0,1]$, let $q_{it} = \phi_{p,t}(q_i)$; thus $\sigma_t = q_{0t} \cdots q_{nt}$. By (4.8.2), $|q_{it} - q_i| < \lambda \xi / \alpha$. By Lemma 4.7.1, $\pi^*$ embeds $\sigma_t$ in $M$.

Claim 11. $\pi^*_p$ embeds $\sigma_t$ into $P_p$.

Proof. We know that $\pi^*$ embeds $\sigma_t$ into $M$. So each point $q \in \sigma_t$ is in a unique $Q^*_p$, so $\pi^*(q) = p'$. By (4.8.1), we know that $|p - q| \leq 6\delta$. By (4.7.2), $|p' - q| < 4\lambda \xi < \delta$. Thus $|p - p'| < \xi$ and Lemma A.12 implies that $P^*_p$ intersects $P_p$ in a single point, which we call $\pi^*_p(q)$. So $\pi^*_p$ is defined on $\sigma_t$.

Now, $\pi^*$ embeds $\sigma_t$ into $M$. So if $q$ and $q'$ are distinct points of $\sigma_t$, $\pi^*(q) \neq \pi^*(q')$. But $\pi^*_p(q)$ is the unique intersection of $P^*_p$ with $P_p$ and likewise for $\pi^*_p(q')$. So $\pi^*_p(q) \neq \pi^*_p(q')$ and thus $\pi^*_p$ embeds $\sigma_t$ into $P_p$. \hfill $\square$

Since $\sigma_1$ is in $P_p$, $\pi^*_p$ is the identity on $\sigma_1$ and thus $\det(D\pi^*_p) > 0$ on $\sigma_1$. Because $\pi^*_p$ is embeds each $\sigma_t$ into $P_p$, $\det(D\pi^*_p)$ is never zero on $\sigma_t$ for all $t$, thus $\det(D\pi^*_p) > 0$ on $\sigma_0 = \sigma$. \hfill $\square$

Define the set $R_p$ to all $q \in K_p$ such that $\pi^*_p(q) \in P_{p,3\delta}$.

Lemma 4.9.2 (IV, 23b [6]). For each $p \in M$, the map

$$\pi^*_p : R_p \to P_{p,3\delta}$$

is bijective.
Proof. Let \( \sigma_1 \) be an \( n \)-simplex of \( K'_p \) containing \( p \) and let \( \sigma_0 \) in \( K_p \) be such that \( \phi_p(\sigma_0) = \sigma_1 \). If \( p_0 \) is the barycenter of \( \sigma_0 \), then by (A.2), (4.6.5), (4.6.4), and (4.3.3),

\[
\text{dist}(p, \partial \sigma_0) \geq n! \Theta(\sigma) \text{diam}(\sigma) \frac{1}{n+1} \\
\geq n! \Theta_1 nb \frac{1}{n+1} \\
\geq (n-1)! \Theta_1 b \frac{1}{2} \\
\geq \gamma b = c.
\]

Combining this with (4.8.2) then gives that

\[
\text{dist}(\phi_p(p_0), \partial \sigma_1) > c - 2\lambda \xi / \alpha = c'.
\]  

(4.9.2)

Let \( q \in K_p \setminus \sigma_0 \). Since \( \phi_p \) is an isomorphism, (4.9.2) gives that \( |\phi_p(q) - \phi_p(p_0)| > c' \).

By the definition of \( \lambda \) and the fact that \( \alpha < 1 \), we get that

\[ 4\lambda \xi / \alpha + 12\lambda \xi < 16\lambda \xi / \alpha \leq \gamma \delta = c. \]

Then by (4.8.2) and (4.8.5),

\[
|\pi^*_p(q) - \pi^*_p(p_0)| \\
\geq |\phi_p(q) - \phi_p(p_0)| - |\phi_p(q) - q| - |\pi^*_p(q) - q| - |\pi^*_p(p_0) - p_0| - |\phi_p(p_0) - p| \\
> c' - 2\frac{\lambda \xi}{\alpha} - 2(6\lambda \xi) > c - \left( \frac{4\lambda \xi}{\alpha} + 12\lambda \xi \right) > 0,
\]

which means \( \pi^*_p(q) \neq \pi^*_p(p_0) \). So \( p^* = \pi^*_p(p_0) \) is covered exactly once by simplices of \( K_p \) via \( \pi^*_p \). Also,

\[
|p^* - p| \leq |\pi^*_p(p_0) - \phi_p(p_0)| + \text{diam}(\sigma_1) \\
< 6\lambda \xi + \lambda \xi / \alpha + 2\delta + 2\lambda \xi / \alpha \\
< 72\lambda \delta / \alpha + 2\delta < 3\delta,
\]

so \( p^* \in P_{p,3\delta} \).

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Now using (4.8.4), (4.8.2), (4.8.5) and the fact that \( \lambda \xi / \alpha + 6 \lambda \xi < \delta \), we get that

\[
\text{dist}(\pi^*_p(\partial K_p), p) \geq \text{dist}(\partial K_p, p) - 6 \lambda \xi
\]

\[
= \text{dist}(\phi^{-1}_p(\partial K'_p), p) - 6 \lambda \xi
\]

\[
\geq \text{dist}(\partial K'_p, p) - 6 \lambda \xi - \lambda \xi
\]

\[
> 4 \delta - \delta = 3 \delta.
\]

So

\[
\pi^*_p(\partial K_p) \subset P_p \setminus \bar{U}_{3\delta}(p).
\]  \hspace{1cm} (4.9.3)

Combining Lemma 4.9.1 and Lemma A.13 concludes the proof. \qed

**Proof of theorem 3.1.1.** Let \( p \in M \), then by Lemma 4.9.2, there is \( q \in K_p \) with \( \pi^*_p(q) = p \). Thus \( \pi^*(q) = p \) and \( \pi^* \) is onto. Suppose \( \pi^*(q') = p \) for \( q' \in K \). By (4.7.2), we have that \( |q' - p| < 4 \lambda \xi < \delta \). Using (4.8.3), \( q' \in K_p \) and applying (4.8.5) gives that

\[
|\pi^*_p(q') - p| \leq |\pi^*_p(q') - q'| + |q' - p| < 6 \lambda \xi + \delta < 3 \delta,
\]

so \( q' \in R_p \). By Lemma 4.9.2, \( q' = q \). So \( \pi^* \) is bijective. Since \( \pi^* \) is a bijective map between compact Hausdorff spaces, it is a homeomorphism. From Theorem A.6, we know that \( \pi^* \) is smooth. By the proof of Lemma 4.7.1, we know \( \pi^* \) always has positive Jacobian. Therefore, \( \pi^* \) is a diffeomorphism.

Now, using Lemma 4.7.1, (4.7.2) and the fact that \( 4 \lambda \xi < \lambda \xi / \alpha \) gives that the simplicial complex in \( \mathbb{R}^m \) whose vertices are \( \pi^*(v) \) for each vertex \( v \) of \( K \) is still homeomorphic to \( M \) via \( \pi^* \). Call this simplicial complex \( T \). Then every simplex of \( T \) is a secant simplex of \( M \). By (4.7.3), we get that every simplex of \( T \) has fullness at least \( \Theta_1/2 \), a number which depends only on \( n \) and \( m \).

Let \( v \in T_q \sigma \) for \( q \in \sigma \) with \( \sigma \) a simplex of \( T \). Then 4.7.6 gives that

\[
|\pi^*_{\pi^*(q)}v| \geq |v| - |v - \pi^*_{\pi^*(q)}v| \geq |v| - \frac{1}{2}|v| = \frac{1}{2}|v|.
\]
By (4.7.2) and (4.7.5),

$$\beta \delta / 2 = b / 2 \leq \text{length of an edge in } T \leq 2\delta + 8\lambda \xi \leq 3\delta.$$ 

If $\bar{L} = 3\delta$ and $C_{n,m} = \beta / 6$, we know that $C_{n,m}$ depends only on $n$ and $m$ and thus (4.0.1) holds in $T$. \qed
Here we present many facts used in the proof of Whitney’s triangulation theorem. Detailed arguments can be found in Whitney’s book [6].

We first discuss some properties of the fullness of a simplex.

The first property is that fullness of a simplex implies fullness of all of its faces. That is, if \( \sigma^k \) is a face of the simplex \( \sigma^r \), then

\[
  r! \Theta(\sigma^r) \leq k! \Theta(\sigma^k). \tag{A.1}
\]

Fullness can be also be nearly preserved if the vertices of the simplex are not moved too much.

**Lemma A.1** (IV, 14c [6]). Given \( r, \Theta_0 > 0 \), and \( \epsilon > 0 \), there is a \( \rho_0 > 0 \) with the following property. Take any simplex \( \sigma = p_0 \cdots p_r \) with \( \Theta(\sigma) \geq \Theta_0 \), and take points \( q_0, \ldots, q_r \) with \( |q_i - p_i| \leq \rho_0 \text{diam}(\sigma) \). Then \( \sigma' = q_0 \cdots q_r \) is a simplex, and \( \Theta(\sigma') \geq \Theta_0 - \epsilon \).

**Lemma A.2** (IV, 14b [6]). For any \( r \)-simplex \( \sigma = p_0 \cdots p_r \) and point \( p = \mu_0 p_0 + \cdots \mu_r p_r \in \sigma \),

\[
  \text{dist}(p, \partial \sigma) \geq r! \Theta(\sigma) \text{diam}(\sigma) \inf \{\mu_0, \ldots, \mu_r\}. \tag{A.2}
\]

**Lemma A.3** (IV, 15c [6]). Let \( \pi \) be the orthogonal projection onto a plane \( P \). Let
\[ \sigma = p_0 \cdots p_r \text{ be a simplex, and suppose that } \sigma \subset U_\zeta(P) \text{ and } |p_i - p_0| \geq \delta > 0 \text{ for all } i > 0. \text{ Then for any unit vector } u \text{ in } P, \]
\[ |u - \pi u| \leq 2\zeta/(r - 1)!\Theta(\sigma)\delta. \]

Now we list some results used to pick various quantities in the proof.

**Lemma A.4** (App. II, 16a [6]). Given the integer \( m \), there is a number \( \rho^* > 0 \) with the following property. Let \( K_0 \) be a subdivision of \( \mathbb{R}^m \) into cubes of side length \( h \), and \( K \) be the barycentric subdivision of \( K_0 \), with vertices \( p_0, p_1, \ldots \). For each \( i \), let \( p_i' \) be a point with
\[ |p_i' - p_i| \leq \rho^* h. \quad (A.3) \]
Let \( f \) be the simplex-wise affine mapping of \( K \) into \( \mathbb{R}^m \) defined by \( f(p_i) = p_i' \). Then \( f \) is a one-to-one map from \( K \) onto \( \mathbb{R}^m \), and the simplexes \( f(\sigma) \) form a simplicial subdivision of \( \mathbb{R}^m \).

**Lemma A.5.** Let \( N \) be a natural number. Then there is a \( \rho_1 \) with the following property: For any ball \( B \) in \( \mathbb{R}^m \) of any radius \( a \), let \( B' \) be the part of \( B \) between any two parallel \((m - 1)\)-planes whose distance apart is less than \( 2\rho_1 a \) apart. Then we have that
\[ \text{Vol}(B') < \text{Vol}(B)/N. \quad (A.4) \]

**Proof.** Let \( \rho_1 \) be such that
\[ 0 < \rho_1 < \frac{\text{Vol}(\mathbb{R}^m)}{2N \text{Vol}(\mathbb{R}^{m-1})}, \]
where \( \mathbb{R}^k \) is the \( k \)-dimensional unit ball. Suppose two \((m - 1)\)-planes are some distance
Let $B'$ be the volume of the region between them contained in $B$. We then have that
\[
\text{Vol}(B') < d \text{Vol}(\mathbb{B}^{m-1})a^{m-1} < 2\rho_1 \text{Vol}(\mathbb{B}^{m-1})a^m < \text{Vol}(\mathbb{B}^m)a^m/N = \text{Vol}(B)/N.
\]

**Theorem A.6.** (IV, 10A [6]) Let $M$ be a smooth, compact submanifold of $\mathbb{R}^m$. For each $p \in M$, let $P^*_p$ be the s-plane in $\mathbb{R}^m$ normal to $M$. Then there is a positive $\delta_0 > 0$ with the following properties.

Set
\[
Q^*_p = P^*_p \cap U_{\delta_0}(p_0).
\]

The $Q^*_p$ fill out a neighborhood $U^*$ of $M$ in a one-to-one way. Set
\[
\pi^*(q) = p \text{ if } q \in Q^*_p.
\]

This is a smooth mapping of $U^*$ onto $M$, and
\[
|\pi^*(q) - q| \leq 2 \text{dist}(q, M), \quad q \in U^*.
\]

**Lemma A.7** (IV, 8a [6]). Let $M$ in $\mathbb{R}^m$ be compact. Then there is a $\xi_0 > 0$ such that $M_{p,\xi_0}$ is defined for all $p \in M$. Moreover,
\[
\text{dist}(p, M \setminus M_{p,\xi}) \geq \xi \text{ if } \xi \leq \xi_0.
\]

**Lemma A.8** (IV, 8b,c [6]). Let $M$ in $\mathbb{R}^m$ be compact. Then for any $\lambda > 0$ there is a $\xi_1 > 0$ with the following property. For any $p \in M$ and any vector $v$ tangent to $M_{p,\xi_1}$,
\[
|v - \pi_p v| \leq \lambda |\pi_p v| \leq \lambda |v|.
\]
Moreover, we have the above inequality for any secant vector $v$ to $M_{p,\xi_1}$ and we get the following results

\[|p' - \pi_p(p')| < \lambda \xi, \quad p' \in M_{p,\xi}, \quad \xi \leq \xi_1,\]  
\[M_{p,\xi} \subset U_{\lambda \xi}(P_{p,\xi}), \quad P_{p,\xi} \subset U_{\lambda \xi}(M_{p,\xi}), \quad \xi \leq \xi_1.\]  

(A.10)

We now present some facts about planes.

**Lemma A.9** (App. II, 14a [6]). If $\tau$ is an $s$-cell and $P$ is an $n$-cell in $\mathbb{R}^m$ where $s + n \geq m$ and $\text{dist}(P, \tau) < \text{dist}(P, \partial \tau)$, then $s + n = m$, $P$ intersects $\tau$ in exactly one point and $\text{ind}(P, P(\tau)) > \text{dist}(P, \partial \tau) / \text{diam}(\tau)$.

**Lemma A.10** (IV, 11a [6]). Given $M$ in $\mathbb{R}^m$, let $\lambda$ and $\xi_1$ be as in Lemma A.8. Take $p \in M$ and let $P'$ be an $(m - n)$-plane such that

\[\text{ind}(P_{p}, P') \geq \lambda' > \lambda.\]  

(A.12)

Then $\pi'$, the projection into $P_{p}$ along $P'$, is an embedding of $M_{p,\xi_1}$ into $P_{p}$. We also have that

\[|\pi'(q) - q| < \lambda \xi / \lambda' \quad \text{if} \quad q \in M_{p,\xi} \cap \xi \leq \xi_1\]  
\[P_{p,c} \subset \pi'(M_{p,\xi}), \quad c = (1 - \lambda / \lambda') \xi, \quad \xi \leq \xi_1.\]  

(A.13)

(A.14)

**Lemma A.11** (App. II, 14b [6]). Let $P^*$ be a plane in $\mathbb{R}^m$, let $P$ be a plane in $P^*$, let $Q$ be a closed set in $P^*$, let $p^*$ be a point of $\mathbb{R}^m$ not in $Q$, and let $Q^*$ be the join of $p^*$ and $Q$. Then

\[\text{dist}(Q^*, P) \geq \text{dist}(Q, P) \text{dist}(p^*, P^*) / \text{diam}(Q^*).\]

**Lemma A.12** (IV, 10a [6]). Take $\lambda$ and $\xi_1 \leq \xi_0$ as in Lemmas A.7 and A.8 and suppose $\lambda < 1$. Take any $p, p' \in M$ with $|p - p'| < \xi_1$. Then $P^*_{p'}$ intersects $P_p$ in a unique point, and if $v \in P^*_{p'}$, then

\[|\pi_p v| \leq \lambda |v|.\]  

(A.15)
Finally, a fact about simplicial complexes.

**Lemma A.13** (App. II, 15a [6]). *Let* $K$ *be an* $n$-*dimensional simplicial complex where each* $(n-1)$-*simplex is a face of one or two* $n$-*simplices. Let $f$ *be simplex-wise positive on* $K$. *Then for any connected open subset* $R$ *of* $\mathbb{R}^n \setminus f(\partial K)$, *any two points of* $R$ *not in* $f(K^{n-1})$ *are covered the same number of times. If this number is one, then* $f$, *considered in the open subset* $R' = f^{-1}(R)$ *of* $K$ *only, is bijective onto* $R$. 
BIBLIOGRAPHY


