ON SPECIAL VALUES OF PELLARIN’S $L$-SERIES

DISSertation

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By

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ABSTRACT

We consider various methods for explicitly computing the special values of Pellarin’s $L$-series in several indeterminates, both at the positive integers and negative integers. Our premier result is the explicit calculation, in terms of the Goss-Carlitz zeta values, of the rational functions appearing in the paper by Pellarin in which these $L$-series were first introduced, as well as an explicit generalization of his result to $s \leq q$ indeterminates. We draw applications to new divisibility results for the numerators of the Bernoulli-Carlitz numbers by degree one irreducible polynomials and to explicit generating series and recursive relations for Pellarin’s series.

On the negative integers side, we deepen some work of Goss on the special polynomials associated to Pellarin’s series. In particular, we show that when certain parameters are fixed, the logarithmic growth in the degrees of the aforementioned special polynomials that is common in this area holds in great generality. We study the natural action of Goss’ group of digit permutations on a sub-class of these special polynomials and show that their degrees are an invariant of the action of Goss’ group. Finally, we include a computation of the coefficients of the measure whose moments are the special polynomials associated to Pellarin’s $L$-series in one indeterminate.

The Wagner series for Pellarin’s evaluation character $\chi_t$ will play a central role in nearly all of the results of this dissertation. We will show that the Wagner series for $\chi_t$, an object arising in the study of the completions of our ring of integers at the finite places, will also have meaning at the infinite place. In fact, at the infinite
place, this Wagner series is none other than Anderson’s generating function for the Carlitz module, and we shall see that Pellarin’s $L$-series, the Wagner series for $\chi_t$ and Anderson’s generating function are inextricably tied.
to Chelsea Christine and Noah Benjamin
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CHAPTER 1
INTRODUCTION

Definitions and Notation

We set the following notation for the remainder of this dissertation.

- Let $\mathbb{F}_q$ be the finite field with $q$ elements of characteristic $p$.

- Let $\theta$ be an indeterminate and $A := \mathbb{F}_q[\theta]$, $K := \mathbb{F}_q(\theta)$ be the ring of polynomials and field of rational functions over $\mathbb{F}_q$, respectively.

- Let $K_\infty := \mathbb{F}_q((\theta^{-1}))$ be the ring of formal Laurent series over $\mathbb{F}_q$.

- Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of $K_\infty$ equipped with the absolute value $|\cdot|$, normalized so that $|\theta| = q$.

- Let $\iota$ be a fixed root in $\mathbb{C}_\infty$ of the polynomial $x^{q-1} + \theta \in A[x]$.

- Let $A_+, A_+(d)$ denote the monic elements of $A$ and the monic elements in $A$ of degree $d$, respectively.

- For $j \geq 1$, let $[j] = \theta^{q^j} - \theta$.

- Let $D_0 := 1$ and for all $j > 0$, let $D_j := (\theta^{q^j} - \theta^{q^{j-1}}) \cdots (\theta^{q^1} - \theta)(\theta^{q^j} - \theta)$ be the product of all monic polynomials of degree $j$. 
• Let $\ell_0 := 1$ and for $j > 0$, let $\ell_j := (\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^j})$. Observe that $L_j := (-1)^j \ell_j$ is the least common multiple of the set of polynomials of degree $j$.

• We shall say a non-negative integer $k$ written as $k = \sum k_i b^i$ is written in base $b$ if $0 \leq k_i < b$ for all $i \geq 0$. We call the number $k_i$ appearing in the base $b$ expansion of $k$ its $i$-th digit. We define the $b$-length of a non-negative integer to be the sum of its digits when written in base $b$:

$$l_b(k) := \sum_{i \geq 0} k_i.$$

We will work mostly in base $q$, and if $l(\cdot)$ is encountered, the reader may assume it is the $q$-length function $l_q(\cdot)$.

• For $x$ a rational number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x$.

• Let $\log_b(\cdot)$ denote the classical logarithm in base $b$.

1.1 Carlitz’ First Results

Definition 1.1.1. Let $d, k$ be non-negative integers. The power sum of degree $d$ and exponent $k$ is defined as an element of $A$ by the formula

$$S_d(k) := \sum_{a \in A_+(d)} a^k. \quad (1.1.1)$$

Through an analysis of the generating functions for the power sums $S_d(k)$, Carlitz [7] was lead to the discovery of an explicit transcendental constant $\tilde{\pi} \in K_{\infty}(\iota)$ and a function $e_C(z) \in K\{\{z\}\}$ such that

$$\tilde{\pi}^{-1} e_C(\tilde{\pi} z) = z \prod_{a \in A} \left( 1 - \frac{z}{a} \right),$$

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where the prime indicates that the product is taken over the non-zero elements of \(A\).

This product is easily seen to define an entire function of \(z\) on \(\mathbb{C}_\infty\). Later we briefly sketch how the function \(\pi^{-1}e_C(\pi z)\) gives rise to generating series for the Carlitz zeta values

\[
\zeta(l(q - 1)) := \sum_{a \in A_+} a^{-l(q-1)}
\]

which represent well defined elements of \(K\) for all positive integers \(l\). These sums give rise to elements in \(K\) that contain interesting arithmetic information about \(K\), and we still have a great deal to learn about them. On the one hand, while much was known about the numbers \(\zeta(l(q - 1))\), Pellarin’s series - the main topic of this dissertation - has been instrumental to the further understanding of these numbers through new approaches, and on the other, in the final chapter of this dissertation we shall see that both the power sums \(S_d(k)\) as well as the zeta values \(\zeta(l(q - 1))\) play a crucial role in obtaining our explicit formulas for Pellarin’s series.

We now guide the reader through the theory relevant to the special values of the Goss zeta function for the ring \(A\) in hopes that the relevance of our results in later chapters will be more easily seen.

**The Carlitz Module**

**Definition 1.1.2.** Let \(L\) be an \(\mathbb{F}_q\)-algebra. Define the ring of twisted power series over \(L\), written \(L\{\{z\}\}\), to be the set of power series of the form \(\sum_{i \geq 0} l_i z^q^i\) with coefficients \(l_i\) in \(L\), with ordinary addition of power series, and with product given by composition.

We will denote by \(L\{z\}\) the subring of \(L\{\{z\}\}\) whose elements consist of those series such that all but finitely many coefficients equal zero. We call \(L\{z\}\) the ring of twisted polynomials over \(L\).
Definition 1.1.3. The Carlitz module $C$ is the $\mathbb{F}_q$-algebra monomorphism from $A$ into the twisted polynomial ring $A\{z\}$ determined by

$$\theta \mapsto C_\theta(z) := z^q + \theta z.$$ 

Given any $A$-algebra $R$, the twisted polynomial ring $A\{z\}$ acts on the underlying additive group of $R$ via evaluation. The Carlitz module determines a new $A$-module structure on $R$ via the monomorphism $C$. In this way we may also think of the Carlitz module as a functor from $A$-algebras to $A$-modules, and from this point of view the Carlitz module plays the role for function field arithmetic that the multiplicative group functor plays classically. See Rosen [25], Chapter 12 or Goss [16], Chapter 3 for how deep this analogy actually goes.

The Fundamental Period of the Carlitz Module

Definition 1.1.4. We refer to Carlitz’ constant

$$\bar{\pi} = -\iota^q \prod_{j=1}^{\infty} \left(1 - \theta/\theta^q\right)^{-1}$$

as the fundamental period of the Carlitz module.

Note that this constant is unique up to the choice of $\iota$.

Anderson and Thakur discovered the form for $\bar{\pi}$ given above, see [1]. Carlitz‘ student Wade was the first to prove the transcendence of this constant [31], [32].

The Carlitz Exponential

Definition 1.1.5. We refer to the formal twisted power series

$$e_C(z) = \sum_{j \geq 0} D_j^{-1} z^q^j \in K\{\{z\}\},$$

as the Carlitz exponential.
Theorem 1.1.6 (Carlitz). For all $z \in \mathbb{C}_\infty$ the following equality holds,

$$z \prod_{a \in A} (1 - \frac{z}{a}) = \frac{\tilde{\pi}}{i} \sum_{j \geq 0} D_j^1(\tilde{\pi}z) q^j.$$

The fundamental property of the Carlitz exponential is that it is the unique entire $\mathbb{F}_q$-linear function with derivative equal to 1 that takes the linear action of multiplication by elements of $A$, for any $A$-algebra $R \subseteq \mathbb{C}_\infty$, to the action of the Carlitz module. This is summarized by the equality

$$e_C(\theta z) = C_\theta(e_C(z)). \quad (1.1.2)$$

The analogy with Lie groups is the reason we refer to this function as an exponential.

The Carlitz Logarithm

Definition 1.1.7. The Carlitz logarithm $\log_C$ is defined as the formal twisted power series inverse of the Carlitz exponential function.

Using the functional equation (1.1.2) one may find the coefficients of the Carlitz logarithm explicitly. We have,

$$\log_C(z) = \sum_{j \geq 0} \ell_j^{-1} z q^j.$$

One may easily check that the radius of convergence of this power series is $|\tilde{\pi}|$.

1.1.1 The Carlitz-Euler Relations

In analogy with the special values of the Riemann zeta function at the positive integers, Carlitz studied the following elements of $K_\infty$.

Definition 1.1.8. For positive integers $k$, define the $k$-th Carlitz zeta special value as an element of $K_\infty$ by,

$$\zeta(k) := \sum_{a \in A_+} a^{-k}.$$
Carlitz used the function $e_C$ to prove the following beautiful theorem, in perfect analogy with Euler’s theorem for the special values of the Riemann zeta function at the positive even integers.

**Theorem 1.1.9** (The Carlitz-Euler relations). Suppose $k$ is a positive integer such that $k \equiv 0 \mod (q - 1)$. Then

$$\zeta(k)/\tilde{\pi}^k \in K.$$  

*Proof.* The essential idea is to examine the logarithmic derivative of $\tilde{\pi}^{-1}e_C(\tilde{\pi}z)$ and to expand about the origin. Since $e_C$ has coefficients in $K$, the result follows. See [7] for the complete proof.

**Remark 1.1.10.** Chang and Yu have determined the exact transcendence degree over $K$ of the field $K(\tilde{\pi}, \zeta(1), \ldots, \zeta(n))$ for all $n \geq 1$, [11]. Their calculation implies that the only $K$-algebraic relations among $\tilde{\pi}, \zeta(1), \ldots, \zeta(n)$ are those coming from the trivial relations $\zeta(k)^{p^n} = \zeta(p^nk)$ and the Carlitz-Euler relations.

**Remark 1.1.11.** L. Taelman has proven an analog of the Herbrand-Ribet theorem relating the numerators of the rational numbers $\zeta(k)/\tilde{\pi}^k$ occurring in the last theorem to the vanishing and non-vanishing of certain $\chi$ parts of his finite class module associated to the integral extension $A[e_C(\tilde{\pi}/P)]$. Here $P \in A$ is a monic irreducible polynomial, and $\chi$ ranges over the characteristic $p$ valued characters of the Galois group of the extension $K(e_C(\tilde{\pi}/P))/K$. See [28] for the precise statement.

### 1.1.2 Carlitz’ von-Staudt Theorem

It proved useful to Carlitz to normalize power series in positive characteristic with certain “factorials” now bearing his name.
The Carlitz Factorial

For \( j \geq 0 \), recall that we have defined \( D_j \) as the product of all monic polynomials of degree \( j \). The Carlitz factorials are now defined for all non-negative integers by a process known as the digit principle, see [12].

**Definition 1.1.12.** For non-negative integers \( k = k_d q^d + \cdots + k_1 q + k_0 \) written in base-\( q \) with \( k_d \neq 0 \). Define the \( k \)-th Carlitz factorial by

\[
\Pi(k) := \prod_{i=0}^{d} D_i^{k_i}. \tag{1.1.3}
\]

These are factorials for the ring \( A \) in the very general sense discovered by Bhargava [4]. They may be interpolated to functions from \( \mathbb{Z}_p \rightarrow K_{\infty} \), and this interpolation is referred to as the *arithmetic gamma function*. For the definition of this function and some of its connections with \( \tilde{\pi} \), see [16] Section 9.4. The transcendence and algebraic independence of the special values of the arithmetic gamma function (even in combination with Carlitz zeta values) is well understood, see [10].

As an example of the factorial like properties of these numbers, we recall one such property that we will use later. For a proof, see [16] Section 9.1, and for many more factorial-like properties of these elements, see [4].

**Proposition 1.1.13.** Let \( j, k \) be non-negative integers. Then \( \Pi(j) \Pi(k) \) divides \( \Pi(j+k) \).

\[ \square \]

The Bernoulli-Carlitz numbers

**Definition 1.1.14.** The Bernoulli Carlitz numbers are defined by the generating series relation,

\[
\frac{z}{e_C(z)} := \sum_{k \geq 0} BC(k) \frac{z^k}{\Pi(k)}. \tag{1.1.4}
\]

**Remark 1.1.15.** Since \( \frac{\lambda z}{e_C(\lambda z)} = \frac{z}{e_C(z)} \) for all \( \lambda \in \mathbb{F}_q^\times \), we observe that \( BC(k) = 0 \) if \( k \not\equiv 0 \mod (q-1) \).
As mentioned above, for $k$ divisible by $q - 1$ we have
\begin{equation}
\zeta(k) = \frac{\bar{\pi}^k BC(k)}{\Pi(k)}.
\end{equation}

**Theorem 1.1.16** (Carlitz’ von-Staudt. [16] Theorem 9.2.2). Suppose $q = p^m > 2$.

Let $k = k_0 + k_1 p + \cdots + k_d p^d$ be a positive multiple of $q - 1$ written in base-$p$ (i.e. $0 \leq k_i < p$ for all $i \geq 0$). Suppose that $(p - 1)m$ divides $l_p(k) := k_0 + k_1 + \cdots + k_d$, and let $h := l_p(k) / ((p - 1)m)$. Suppose $q^h - 1$ divides $k$. Then the denominator of $BC(k)$ (in reduced form) is the product of all monic irreducible polynomials in $A$ of degree $h$.

Otherwise, $BC(k)$ is an element of $A$.

**Remark 1.1.17.** For the case $q = 2$, see [16] Remark 9.2.3 where references to the correct statement are given.

We shall use this theorem later in Theorem 4.3.5 in combination with our explicit calculation of Pellarin’s $L$-series in $q - 1$ indeterminates to give a lower bound on divisibility of the Bernoulli-Carlitz numbers by degree one primes of $A$.

### 1.2 Goss’ Extension of Carlitz’ zeta values

In order to discuss Goss’ extension of Carlitz’ zeta values to the negative integers, we must first discuss some basic properties of the power sums $S_d(k)$. We will examine their generating series and some well known results about them due to Carlitz.

#### 1.2.1 Generating series for the $S_d(k)$

**Definition 1.2.1.** Let $A(0) = \{0\}$, and for $d \geq 1$ define $A(d) := \{a \in A : \deg(a) < d\}$. The sets $A(d)$ are all $\mathbb{F}_q$-vector spaces and give a filtration of $A$.

**Definition 1.2.2.** For $d \geq 0$ define the polynomial $e_d(z) := \prod_{a \in A(d)} (z - a) \in A[z]$. 

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One knows that a polynomial whose roots form an $\mathbb{F}_q$-vector space is $\mathbb{F}_q$-linear, see [16]. Hence, for all $d \geq 0$ the polynomials $e_d$ are $\mathbb{F}_q$-linear. Carlitz computed their coefficients explicitly.

**Lemma 1.2.3** (Carlitz). Let $d$ be a non-negative integer. Then

$$e_d(z) = \sum_{i=0}^{d} \frac{D_d}{D_i \ell_{q^{-i}}} z^q^i. \tag{1.2.1}$$

**Proof.** See [16], Chapter 3. \qed

**Power sums and $\mathbb{F}_q$-vector spaces**

We shall be interested in power sums of the form

$$S_V(k) := \sum_{\lambda \in V} \lambda^k,$$

where the elements $\lambda$ come from some finite-dimensional $\mathbb{F}_q$-vector space $V$ (or an affine set obtained by linearly shifting $V$, e.g. $A_{+}(d) = \theta^d + A(d)$). As we have remarked above, any polynomial of the form $\prod_{\lambda \in V} (z - \lambda)$ is $\mathbb{F}_q$-linear. A basic technique for us is to examine the logarithmic derivative of some $\mathbb{F}_q$-linear polynomial (or more generally, power series) and to expand it about either the origin or the point at infinity on the projective line over $\mathbb{C}_\infty$. These techniques are well-known, and we do not develop them in full generality. Rather, we apply them in the specific instances related to our interests. The following proposition is an example of this.

**Proposition 1.2.4.** For all $d \geq 1$ and $z \in \mathbb{C}_\infty$ sufficiently large,

$$\frac{D_d}{\ell_d e_d(z - \theta^d)} = \sum_{k \geq 0} z^{-(k+1)} S_d(k). \tag{1.2.2}$$

**Proof.** By the previous lemma, the derivative with respect to $z$ of $e_d(z - \theta^d)$ equals $D_d/\ell_d$. Hence the left side equals the logarithmic derivative $e_d(z - \theta^d)'/e_d(z - \theta^d)$...
of \( e_d(z - \theta^d) \). Further, one sees easily that \( e_d(z - \theta^d) = \prod_{a \in A_+(d)} (z - a) \), so that the left side of the equality to be proved equals \( \sum_{a \in A_+(d)} (z - a)^{-1} \). The expansion of this function at \( \infty \) on the projective line over \( \mathbb{C}_\infty \) equals the right side of the equality to be proved.

**Corollary 1.2.5.** Let \( d \geq 1 \). The power sums \( S_d(k) \) vanish whenever \( 0 \leq k < q^d - 1 \), and \( S_d(q^d - 1) = D_d/\ell_d \).

**Proof.** The order of vanishing at \( \infty \) of the left side of the previous proposition is \( q^d \). Hence the same must be true of the right.

For the second claim, one multiplies both sides of the previous proposition by \( z^{q^d} \) and lets \( z \to \infty \). Noting that \( e_d(z - \theta^d) \) is monic finishes the proof.

The previous proof is due originally to Carlitz, see [7]. We obtain a strengthening of this classical vanishing result through an analysis of the power sums arising in connection with Pellarin’s \( L \)-series, see Proposition 3.3.1.

**Special polynomials for Goss’ zeta**

Now we know that when \( k \) is fixed the power sums \( S_d(k) \) vanish for \( d \) sufficiently large, we make the following definition.

**Definition 1.2.6.** For non-negative integers \( k \) we define the \( k \)-th special polynomial associated to Goss’ zeta function by

\[
z(x, -k) := \sum_{d \geq 0} x^{-d} S_d(k) \in A[x^{-1}].
\] (1.2.3)

There are many interesting properties of these special polynomials. As Corollary 1.2.5 implies, the degree in \( x^{-1} \) of the polynomial \( z(x, -k) \) grows logarithmically in \( k \). This should be compared with the growth of the number of zeros of the Riemann zeta function in the critical strip and up to a fixed height. The zeros of \( z(x, -k) \) are
also all simple and lie in $K_{\infty}$. This is a weak version of the Riemann hypothesis for the Goss zeta function, a result which is due in full generality to J. Sheats. See [26].

The specializations at $x^{-1} = 1$ of $z(x, -k)$ give rise to Goss’ zeta values at the negative integers, also known as the Bernoulli-Goss numbers for $A$.

Special values of the Goss zeta at the negative integers

**Definition 1.2.7.** For non-negative integers $k$ we define the $-k$-th special value of the Goss zeta function by

$$\zeta(-k) := z(1, -k) \in A.$$  \hfill (1.2.4)

**Lemma 1.2.8.** Suppose $k$ is a positive integer such that $k \equiv 0(q - 1)$, then $\zeta(-k) = z(1, -k) = 0$.

This result is remarkable in comparison with the vanishing of the Riemann zeta function at the negative even integers. For proofs see [16] or [30]. This result is also implied by our Proposition 3.2.1 below.

These zeta values at the negative integers were also shown by Goss and Sinnott [19] to be directly related to the vanishing and non-vanishing of the irreducible pieces (after decomposition as a Galois module) of the classical class group for extensions obtained by adjoining Carlitz torsion.

### 1.3 Non-archimedean measures

Here we recall the basic notation and theorems regarding non-Archimedean measures in positive characteristic. See [16] Section 8.22 and also Katz’ paper [21] for a more thorough discussion of these things. Our intention is to describe a beautiful result of Thakur and to motivate a result of our own appearing in a later chapter.

Let $v$ be a finite place of $A$, and denote the completion of $A$ at $v$ by $A_v$. Denote the completion of $K$ at $v$ by $K_v$. Write $| \cdot |_v$ the absolute value under which both
$A_v$ and $K_v$ are complete. In order to give a basis for the continuous maps from $A_v$ to itself we follow a construction of Carlitz, again appealing to the so called digit principle.

**Definition 1.3.1.** For $i \geq 0$, let $E_i(z) := e_i(z)/D_i \in K[z]$.

For non-negative integers $k = k_0 + k_1q + k_2q^2 + \cdots$, written in base $q$, define

$$G_k(z) := \prod_{i \geq 0} E_i(z)^{k_i} \in K[z].$$

By construction, $G_k$ is a polynomial of degree $k$ in $z$, and hence any polynomial $f \in K[z]$ of degree $d$ may be expanded as

$$f = \sum_{k=0}^{d} f_k G_k$$

(1.3.1)

for some $f_k \in K$.

The following theorem, due to Carlitz, already makes a strong case for comparing the polynomials $G_k$ with the classical binomial polynomials

$${z \choose k} := \frac{z(z - 1) \cdots (z - (k - 1))}{k!}.$$

**Theorem 1.3.2** (Carlitz [9]). *Let $f \in K[z]$ be a polynomial of degree $d$, expanded as in (1.3.1). Then $f$ maps $A$ to $A$ if and only if $f_i \in A$ for $i = 0, \ldots, d$.***

**Definition 1.3.3.** For all continuous $f : A_v \to A_v$ we define

$$||f||_v = \sup_{x \in A_v} |f(x)|_v.$$

The space of all continuous functions $f : A_v \to A_v$ is complete with respect to the norm $|| \cdot ||_v$.

We also have the following classical theorem of Wagner which should be compared with Mahler’s theorem for continuous functions on the $p$-adic integers.
Theorem 1.3.4 (Wagner [34]). A function $f : A_v \to A_v$ is continuous if and only if there exist coefficients $f_i \in A_v$ such that $f_i \to 0$ as $i \to \infty$ and $f = \sum_{i \geq 0} f_i G_i$ on $A_v$. Further, the coefficient sequence $(f_i)$ determined by $f$ is unique, and

$$||f||_v = \max_{i \geq 0} |f_i|_v.$$

Definition 1.3.5. We call the coefficient sequence $(f_i)_{i \geq 0}$ determined by a continuous function $f : A_v \to A_v$, as in the last theorem, the Wagner coefficients of $f$.

Definition 1.3.6. An $A_v$-valued measure on $A_v$ (or just measure) will be any continuous $A_v$-linear functional on the space of continuous functions $f : A_v \to A_v$, equipped with the norm $|| \cdot ||_v$. We will write $\int_{A_v} f \, d\mu$ (or just $\int f \, d\mu$) for the image of $f$ under $\mu$.

Remark 1.3.7. If a continuous $A_v$-valued function $f$ on $A_v$ is given by a formula $z \mapsto p(z)$ we will sometimes write $\int p(z) \, d\mu(z)$ in place of $\int f \, d\mu$. For example, it will be convenient to write $\int z^n \, d\mu(z)$ rather than giving the function $z \mapsto z^n$ an additional name.

Remark 1.3.8. Our definition may be shown to be equivalent to Goss’ definition [16] 8.22.11 of an $A_v$-valued measure on $A_v$, that is, as a finitely additive $A_v$-valued function on the compact-open subsets of $A_v$.

Definition 1.3.9. For each non-negative integer $k$ and measure $\mu$, we define the $k$-th measure coefficient associated to $\mu$ by the value

$$\mu(k) := \int_{A_v} G_k \, d\mu \in A_v.$$  

Observe that the numbers $\mu(k)$ are bounded in absolute value, as they lie in $A_v$. Clearly then, by continuity any measure $\mu$ is uniquely determined by its associated sequence of measure coefficients.
Remark 1.3.10. The collection of $A_v$-valued measures on $A_v$ forms an algebra under natural definitions of addition and convolution. Given a measure $\mu$ one can associate a unique divided power series to $\mu$ via
\[ p_\mu(z) := \sum_{k \geq 0} \mu(k) \frac{z^k}{k!}. \]
Here the basis elements $\frac{z^k}{k!}$ are to be understood as formal symbols (i.e. not taken mod $p$) and are to be multiplied according to the rule
\[ \frac{z^k}{k!} \frac{z^j}{j!} = \binom{j+k}{j} \frac{z^{j+k}}{(j+k)!}. \]
Goss computed the algebra of such measures, and it was Anderson who observed that the algebra of measures is isomorphic via the map just defined to the algebra of divided power series over $A_v$, see [13].

Now on the power series ring $A_v[[x]]$, we may consider the family of hyper-derivatives $\{\partial_j\}_{j \geq 0}$ which are defined by their action on the powers of $x$ (and then extended $A_v$-linearly to all of $A_v[[x]]$) by the rule
\[ \partial_j x^n = \binom{n}{j} x^{n-j} \]
for all non-negative integers $j, n$. The $\partial_j$ are a good replacement in positive characteristic for the powers of the usual derivative with respect to $x$. One may show that for all $f \in A_v[[x]]$ and for all non-negative integers $j, k$ one has
\[ \partial_j \partial_k f = \partial_k \partial_j f = \binom{j+k}{j} \partial_{j+k} f. \quad (1.3.2) \]
Thus one may consider the $A_v$-algebra generated by the hyper-derivatives with multiplication defined as in (1.3.2) and addition defined in the natural way. Goss was the first to observe in [14] that the algebra of $A_v$-valued measures on $A_v$ is canonically isomorphic to the algebra of hyper-derivatives as just defined via the map
\[ \mu \mapsto \sum_{j \geq 0} \mu(j) \delta_j. \]
This remains an area in need of much development.

**Definition 1.3.11.** For each non-negative integer \( k \) and measure \( \mu \), we define the \( k \)-th moment of \( \mu \) to be the value of the integral \( \int_{A_v} z^k \, d\mu(z) \in A_v \).

**Remark 1.3.12.** An \( A_v \)-valued measure on \( A_v \) is uniquely determined by its moments, since one can reconstruct the measure coefficients from its sequence of moments.

Conversely, not every sequence in \( A_v \) gives rise to an \( A_v \)-valued measure whose moments arise from this sequence. Given a sequence from \( A_v \), one can formally compute the measure coefficients by expanding out \( G_k(z) \) in powers of \( z \) and using one’s desired moment sequence. If all of these measure coefficients lie in \( A_v \), then one obtains a measure from the given sequence.

For any finite extension \( L \) of \( K \), Goss has defined a Dedekind zeta function for the ring of integers of \( L \), and he has shown in Section 3.3.1 of [14] that there exists a unique measure whose moments are the special polynomials associated to this zeta function. The next definition is an instance of his result.

**Definition 1.3.13.** For each \( x \in K_v \) such that \( |x|_v \geq 1 \), let \( \mu_x \) be the \( A_v \)-valued measure whose \( k \)-th moment \( \int_{A_v} z^k \, d\mu_x(z) \) is the value at \( x \) of the \( k \)-th special polynomial \( z(x, -k) \) associated to Goss’ zeta function.

**Theorem 1.3.14** (Thakur, [16] Theorem 8.22.12). For \( x \in K_v \) such that \( |x|_v \geq 1 \), let \( \mu_x \) be as in the previous definition. Then for all \( j \geq 0 \), the \( j \)-th measure coefficient
\( \mu_x(j) \) of \( \mu_x \) is given by \( \mu_x(0) = 1 \), and

\[
\mu_x(j) = \begin{cases} 
(-x^{-1})^l & \text{if } j = cq^l + q^l - 1 \\
(-x^{-1})^l + (-x^{-1})^{l+1} & \text{if } j = cq^l + q^l - 1 \\
0 & \text{for some } l \geq 0 \text{ and } 0 < c < q - 1, \\
\text{otherwise}.
\end{cases}
\]

**Remark 1.3.15.** Notice that the measure coefficients above are independent of the place \( v \) used to compute them!

We will use Thakur’s calculation in Section 3.4 to outline a method for determining the measure coefficients arising from the special polynomials associated to Pellarin’s \( L \)-series. We shall actually compute these coefficients in the special case of Pellarin’s \( L \)-series in one indeterminate.

### 1.4 The Tate Algebra, the Frobenius Endomorphism, and Twisting

The ideas of this section are basic in all applications of Anderson’s theory of \( t \)-motives. Pellarin has discovered how these ideas fit into the realm of \( L \)-series in positive characteristic, and in particular, in [22] he uses these ideas to prove an explicit formula for a value of his \( L \)-series in one indeterminate introduced in that paper. We introduce these ideas here since it is becoming apparent that they hold a basic places in the future of function field arithmetic. Anderson and Thakur’s function \( \omega \) fits perfectly within the ideas developed in this section and is fundamentally related to Pellarin’s \( L \)-series as we shall see in later chapters. The ideas of this section also allow us to quickly and easily compute the Wagner series for the \( \mathbb{F}_q \)-algebra evaluation map.
χ_t : A → \mathbb{F}_q[t] determined by the replacement \( \theta \mapsto t \) which we will study thoroughly in later chapters.

1.4.1 The multivariate Tate Algebra

Let \( I = (i_1, \ldots, i_s) \) be a multi-index. As usual, we will write \( |I| := i_1 + \cdots + i_s, \)

\( a_I := a_{i_1, \ldots, i_s} \), and \( t^I := t_1^{i_1} \cdots t_s^{i_s} \).

**Definition 1.4.1.** For positive integers \( s \) we define the Tate algebra in \( s \)-variables over \( \mathbb{C}_\infty \) to be

\[
T_s := \left\{ \sum_{I \in (\mathbb{N} \cup \{0\})^s} a_I t^I \in \mathbb{C}_\infty[[t_1, \ldots, t_s]] : a_I \to 0 \text{ as } |I| \to \infty \right\}.
\]

The subalgebra \( T_s \) of \( \mathbb{C}_\infty[[t_1, \ldots, t_s]] \) is complete with respect to the norm given

for all \( \phi \in T_s \) by

\[
||\phi|| = \max_{I \in (\mathbb{N} \cup \{0\})^s} |a_I|.
\]  

(1.4.1)

**Remark 1.4.2.** The space \( T_s \) may be equivalently described as those power series in \( \mathbb{C}_\infty[[t_1, \ldots, t_s]] \) that converge on the closed unit polydisc

\[
\mathcal{O}^s := \{(l_1, \ldots, l_s) \in \mathbb{C}_\infty^s : |l_i| \leq 1 \text{ for } i = 1, \ldots, s\}.
\]

One shows that this algebra of functions on the closed unit polydisc in \( \mathbb{C}_\infty^s \) is complete with respect to the sup-norm

\[
||\phi||' := \sup_{(t_1, \ldots, t_s) \in \mathcal{O}^s} |\phi(t_1, \ldots, t_s)|
\]

and that the numbers assigned by the \textit{a priori} different norms \( || \cdot || \) and \( || \cdot ||' \) are actually equal.

We also mention that \( ||\phi_1 \phi_2|| = ||\phi_1|| \cdot ||\phi_2|| \) for all \( \phi_1, \phi_2 \in T_s \).

**Convention 1.4.3.** As often as possible, we shall denote the elements of the Tate algebra by lowercase Greek letters.
1.4.2 The Frobenius endomorphism and Twisting

In an effort to assist the reader, we describe a ring isomorphic to the twisted power series ring whose elements we will let act as linear transformations on subsets of the Tate algebra.

Twisted power series in $\tau$

The ring of twisted power series $\mathbb{C}_\infty \{z\}$ is naturally isomorphic to the non-commutative ring of twisted power series in $\tau$, written $\mathbb{C}_\infty \{\tau\}$. The isomorphism is given by “evaluation”,

$$\tau^i(z) := z^{q^i}.$$

The elements $a = \sum_{i \geq 0} a_i \tau^i$ and $b := \sum_{j \geq 0} b_j \tau^j$ in $\mathbb{C}_\infty \{\tau\}$ are power series in the variable $\tau$ with coefficients in the ring $\mathcal{R}$ subject to usual addition of series and the non-commutative product

$$ab := \sum_{k \geq 0} \sum_{i+j=k} a_i b_j^{q^i} \tau^k. \tag{1.4.2}$$

Convention 1.4.4. Elements of $\mathbb{C}_\infty \{\tau\}$ shall be written in Gothic script.

Example 1.4.5. Important elements in $\mathbb{C}_\infty \{\tau\}$ (whose coefficients actually lie in $K$) include $c_\theta := \theta + \tau$ (and the images of all elements of $A$ under the $\mathbb{F}_q$-algebra map determined by $c_\theta$), the Carlitz exponential $e_C := \sum_{i \geq 0} D_i^{-1} \tau^i$, and the Carlitz logarithm

$$\log := \sum_{i \geq 0} L_i^{-1} \tau^i$$

which is the formal twisted power series inverse of $e_C$.

We obtain a notion of evaluation by twisted polynomials in $\tau$ on elements of the Tate algebra from the following definition.

Definition 1.4.6 (Twisting). The $\mathbb{F}_q$-linear maps $\tau^i : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ defined by $\tau^i(z) := z^{q^i}$ may be extended to $\mathbb{F}_q[t_1, \ldots, t_s]$-linear maps for all $\sum a_i t^i \in \mathcal{T}_s$ and all $i \geq 0$ via the twisting operation

$$\tau^i \left( \sum a_i t^i \right) := \sum \tau^i(a_i) t^i.$$
Thus \( \tau \) is a map of \( \mathbb{F}_q[t_1, \ldots, t_s] \)-algebras. This action extends linearly to all polynomials in \( \tau \) with coefficients in \( \mathbb{C}_\infty \).

**Twisting by \( \mathbb{C}_\infty \{\{\tau\}\} \)**

We wish to allow series such as \( e^C \) and \( \log \) to act on elements of the Tate algebra as well. We need two preliminary definitions.

**Definition 1.4.7.** The *radius of convergence* of an element \( \sum_{i \geq 0} a_i \tau^i \in \mathbb{C}_\infty \{\{\tau\}\} \) is the usual radius of convergence of the image \( \sum_{i \geq 0} a_i z^{q^i} \in \mathbb{C}_\infty \{\{z\}\} \) as a power series.

**Definition 1.4.8.** For \( r > 0 \), we define \( \mathbb{B}_s(r) \) as the \( \mathbb{F}_q[t_1, \ldots, t_s] \)-submodule of \( \mathbb{T}_s \) whose elements consist of those \( \phi \in \mathbb{T}_s \) such that \(|\phi| < r\).

**Definition 1.4.9.** A twisted power series \( a \) over \( \mathbb{C}_\infty \) with radius of convergence \( r > 0 \) gives a well-defined \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear map from \( \mathbb{B}_s(r) \) into \( \mathbb{T}_s \) for all \( \phi = \sum\phi_I t^I \in \mathbb{B}_s(r) \) via

\[
a(\phi) := \sum a(\phi_I) t^I.
\]

**Remark 1.4.10.** Pellarin writes \( E_a(\phi) \) for what we write simply as \( a(\phi) \). We hope that the change of notation, from Roman to Gothic will be enough to clue the reader into the fact that we are using these series in \( \tau \) as operators on \( \mathbb{T}^s \). Also, note that when \( \mathbb{C}_\infty \) is viewed as the subspace of constants in \( \mathbb{T}_s \), the operator action of \( f = \sum f_i \tau^i \) on \( z \in \mathbb{C}_\infty \) is simply evaluation \( f(z) = \sum f_i z^{q^i} \).

**Example 1.4.11.** The operator \( e^C \) associated to the Carlitz exponential has infinite radius of convergence, and hence gives a well-defined action on all of \( \mathbb{T}_s(\mathbb{C}_\infty) \). Further, \( e^C \) is an isometry on \( \mathbb{B}_s(|\pi|) \) with inverse \( \log \).
1.5 Anderson and Thakur’s function $\omega$

The material from this section is a summary of the author’s exposition in [24]. The reader may look there for more explanation and proofs.

As a first, and perhaps most important, example of an element of the Tate algebra $T_1$, we have a function discovered by Anderson and Thakur in connection with the Carlitz module and its tensor powers, see [1]. This function will be of fundamental importance in all that follows.

**Definition 1.5.1.** For $t \in \mathbb{C}_\infty$ such that $|t| < |\theta|$, we define the *Anderson-Thakur function* $\omega(t)$ by the convergent product

$$
\omega(t) := t \prod_{j \geq 0} \left( 1 - \frac{t}{\theta^q} \right)^{-1}.
$$

**Remark 1.5.2.** This function should be thought of as a deformation of the fundamental period of the Carlitz module $\tilde{\pi}$. Indeed, we have

$$
\lim_{t \to \theta} (t - \theta)\omega(t) = -\tilde{\pi}.
$$

In Anderson’s theory of $t$-motives (again see [1]), the function $\omega(t)$ plays the role of the rigid analytic trivialization of the Carlitz motive where its importance arises from the following proposition.

**Theorem 1.5.3** (Anderson-Thakur, [1]). The function $\omega(t)$ generates the rank one $\mathbb{F}_q[t]$-submodule of $T$ of solutions to the $\tau$-difference equation

$$
\tau(X) = (t - \theta)X.
$$

**Definition 1.5.4.** For $a \in A$, we denote by $\chi_t(a)$ the image of $a$ in the Tate algebra obtained from the change of variables $\theta \mapsto t$.

**Corollary 1.5.5** (Pellarin, [22] Lemma 29). For all $a \in A$,

$$
\varepsilon_a(\omega) = \chi_t(a)\omega \in T.
$$
Pellarin used the twisting operation to give another characterization of the function $\omega(t)$. Write $\tilde{\pi}(\theta-t)^{-1}$ for the element of $\mathbb{T}$ given by the power series $\tilde{\pi}\sum_{i \geq 0} \theta^{-(i+1)}t^i$.

**Theorem 1.5.6** (Pellarin, [22] (4) pg. 2058). As elements of the Tate algebra, we have

$$\omega(t) = e_C(\tilde{\pi}(\theta-t)^{-1}) = \sum_{i \geq 0} e_C(\tilde{\pi}\theta^{-(i+1)})t^i = \sum_{i \geq 0} \frac{\tilde{\pi}q^i}{D_i \cdot (\theta^q - t)}.$$

While enough was known to Carlitz to establish the next result, it first appears implicitly in Anderson’s fundamental paper on log-algebraicity, and it was Pellarin who extended these ideas to the realm of operators on the Tate algebra and wrote everything as clearly and plainly as we do now.

**Theorem 1.5.7** (Pellarin, [22] Lemma 31). We have the following equality of operators on $B_1(|\tilde{\pi}|)$,

$$\log = \sum_{d \geq 0} \sum_{a \in A_+ \langle d \rangle} a^{-1}c_a.$$

**Remark 1.5.8.** We draw the reader’s attention to the analogy between the basic roles played by the polynomials $z^n$ for the action of the integers $\mathbb{Z}$ on the multiplicative group of $\mathbb{C}$ and the polynomials $C_a(z)$ for the action of $A$ given by the Carlitz module on the additive group of $\mathbb{C}_\infty$. Both of these actions are uniformized by analytic exponential functions. In the first case, the classical exponential $e^z$ does this, and in the second, the Carlitz exponential $e_C(z)$ plays the role.

With this in mind, take note of the remarkable similarity between the expression for the operator $\log$ on the right of the equality above, and the power series expansion about the origin for the classical logarithm, that is

$$\log(1 - z) = -\sum_{n \geq 0} \frac{x^n}{n}.$$
The Theorems of F. Pellarin

We have now stated enough to formally see (and in fact, everything is rigorously justifiable) the next beautiful theorem of Pellarin. We will refer to this result throughout this dissertation as Pellarin’s formula. Just as the classical logarithm is closely related to the special values of Dirichlet $L$-series at 1, so too is the Carlitz logarithm.

**Theorem 1.5.9** (Pellarin, [22] Theorem 1). *We have the following identity of elements of the Tate algebra,*

$$
\omega(t) \sum_{a \in A_+} \frac{\chi(t)(a)}{a} = \log(\omega(t)) = \tilde{\pi}(\theta - t)^{-1}.
$$

We state one more result discovered by Pellarin through the use of his theory of deformations of vectorial modular forms.

**Theorem 1.5.10.** For $k \equiv 1 \mod (q - 1)$, the series $\frac{\omega(t)}{\tilde{\pi}^k} \sum_{a \in A_+} \frac{\chi(t)(a)}{a^k} \in K_{\infty}[[t]]$ is equal in $\mathbb{T}$ to a rational function in $K(t)$.

The series $\sum_{a \in A_+} \frac{\chi(t)(a)}{a^k}$ are our first examples of special values of Pellarin’s $L$-series in one indeterminate. In the next chapter, we shall introduce similar series in an arbitrary, finite number of indeterminates and state a theorem of Anglès and Pellarin demonstrating just how far rationality results like the one above go. A major result in this dissertation is the solution of Pellarin’s question to determine the explicit rational functions in $K(t)$ that appear in the last theorem stated above. We further generalize this computation to Pellarin’s $L$-series in $s \leq q$ indeterminates which will be introduced in the next chapter.
1.6 The Anderson Generating Function

In order to catch a glimpse of the underlying formalism to which this dissertation gives evidence, we introduce a slight generalization of the function \( \omega(t) \) of the last section. First, we make some preliminary observations.

In the last section we mentioned that \( \omega \) is a generator of the rank one \( \mathbb{F}_q[t] \)-submodule of \( T \) satisfying \( \tau(\omega) = (t - \theta)\omega \). Also, recall that \( \tau \) is multiplicative on \( T \).

**Definition 1.6.1.** For \( i \geq 0 \), define \( b_i(t) \in A[t] \) by the equality

\[
\tau^i(\omega) = b_i(t)\omega.
\]

Explicitly \( b_0(t) = 1 \), and for \( i \geq 1 \)

\[
b_i(t) = \prod_{j=0}^{i-1} (t - \theta^{q^j}).
\]

**Remark 1.6.2.** Observe that both the families of polynomials \( D_i \) and \( \ell_i \) are specializations of the \( b_i(t) \). Indeed, for all \( i \geq 0 \), we have \( b_i(\theta^{q^i}) = D_i \) and \( \ell_i = \tau(b_i(t))|_{t=\theta} \).

Recall from 1.3.1 the definition of the polynomials \( E_i(z) \in K[z] \).

**Theorem 1.6.3.** Let \( a \in A \), and let \( \deg(a) \) denote its degree in \( \theta \). The following equality holds in \( A[t] \),

\[
\chi_t(a) = \sum_{i=0}^{\deg(a)} b_i(t)E_i(a).
\]

**Proof.** In Corollary 1.5.5 we observed that \( c_a(\omega) = \chi_t(a)\omega \) in \( T \). Expanding out the left side we have,

\[
\chi_t(a)\omega = c_a(\omega) = \sum_{i=0}^{\deg(a)} E_i(a)\tau^i\omega = \sum_{i \geq 0} b_i(t)E_i(a)\omega.
\]

Using the fact that \( \omega \in T^\times \), we obtain the desired equality in \( T \). Hence, both members in the equality are polynomials in \( A[t] \) which agree for \( t \) in the closed unit disc in \( \mathbb{C}_\infty \).

We conclude that they are identically equal in \( A[t] \). \( \square \)
1.6.1 The Wagner Series for $\chi_t$ and the Anderson Generating Function

Theorem 1.6.3 shows that the formal series

$$\Xi_t(z) := \sum_{i \geq 0} b_i(t)E_i(z) \quad (1.6.1)$$

stores the information contained in the function $\chi_t$ on $A$. Indeed, upon making the formal replacement $z \mapsto a$ for some $a \in A$ the series on the right becomes $\chi_t(a)$. This formal series may be considered in two different directions. In Section 3.4.1 we consider it as a function of $z$ on the completion $A_v$ of $A$ at a finite place $v$ for those $t$ such that $\chi_t$ extends to a continuous function from $A_v$ to itself. Now we proceed to analyze its $\infty$-adic properties.

**Convention 1.6.4.** In working with elements of the Tate algebra we shall always reserve the variable $z$ for the action of the operators $\tau^i$ given by $\tau^i(z) = z^{q^i}$, and we shall continue with the $\mathbb{F}_q[t]$-linear action of $\tau$. As an example, see the next definition.

**Definition 1.6.5.** The Anderson’s Generating Function for the Carlitz module (AGF) is the function of two variables $t, z \in \mathbb{C}_\infty$ such that $|t| < |\theta|$ and $z$ is arbitrary given by the formula

$$\omega(t, z) := e_C(z(\theta - t)^{-1}) = \sum_{i \geq 0} \frac{z^{q^i}}{D_i \cdot (\theta^{q^i} - t)}.$$

One sees easily that for each fixed $z \in \mathbb{C}_\infty$ the AGF is an element of $\mathbb{T}$. This function has simple poles at $t = \theta, \theta^q, \theta^{q^2}, \ldots$. Observe that the Anderson-Thakur function $\omega(t)$ is given under this formalism by $\omega(t, \tilde{\pi})$. For a more leisurely account of Anderson generating functions for general Drinfeld modules over $A$ and also interesting explicit formulas for them, see [2].

**The major theme**

The author’s first discoveries were all based upon the use of the series $\Xi_t(z)$, and as the reader will notice, the coefficients $b_i(t)$ appear throughout this work. There is a
deep connection between Pellarin’s series and the function $\omega(t)$ and more generally with the AGF $\omega(t, z)$, and it is this connection that explains why the author has obtained the explicit formulas in this dissertation. We record this connection as a moral, first in words, and then in a precise mathematical statement in the result that follows. We shall direct the reader’s attention to this underlying principle at relevant points throughout this dissertation.

The Wagner Series for the character $\chi_t$ is the Anderson Generating function for the Carlitz module.

**Theorem 1.6.6** (Pellarin, Perkins). Let $z \in \mathbb{C}_\infty$. The following equality holds in $\mathbb{T}$:

$$
\sum_{i \geq 0} E_i(z)b_i(t) = \frac{\omega(t, \tilde{\pi}z)}{\omega(t)}.
$$

**Proof.** By definition

$$
\omega(t, \tilde{\pi}z) = e_C(\tilde{\pi}z(\theta - t)^{-1}).
$$

Using Pellarin’s formula $\log(\omega(t)) = \tilde{\pi}(\theta - t)^{-1}$ we see that

$$
\frac{\omega(t, \tilde{\pi}z)}{\omega(t)} = \omega(t)^{-1}e_Cz\log(\omega(t)).
$$

We have an equality of formal twisted power series,

$$
e_Cz\log = \sum_{i \geq 0} E_i(z)\tau^i.
$$

Thus showing $\left(\sum_{i \geq 0} E_i(z)\tau^i\right)(\omega(t)) = \sum_{i \geq 0} E_i(z)\tau^i(\omega(t))$ will finish the proof. This amounts to an interchange of sums that we justify now.
One may easily show, see [13] Lemma 3.1.8, that there exists an explicit constant $c(z)$ depending on $z$ and not on $i$ such that $|E_i(z)| \leq c(z)/|L_i|$. Using this we have

$$
\sum_{i \geq 0} \sum_{j \geq 0} \|E_i(z)e_{C_i(\pi/\theta^{i+1})q^it^j}\| \leq c(z) \sum_{i \geq 0} \sum_{j \geq 0} q^{-\left(\frac{q^i+1}{q-1}\right)} q^{-q^i(\frac{(q-1)-1}{q-1})}
$$

$$
= c(z) q^{\frac{q-1}{q-1}} \sum_{i \geq 0} q^{-q^i} \sum_{j \geq 0} (q^{-q^i})^j
$$

$$
= c(z) q^{\frac{q-1}{q-1}} \sum_{i \geq 0} \frac{q^{-q^i}}{1-q^{-q^i}}
$$

$$
\leq c(z) q^{\frac{q-1}{q-1}} \frac{q}{q-1} \sum_{i \geq 0} q^{-q^i} < \infty.
$$

Similarly, using Goss’ Lemma [13] Lemma 3.1.8 again, one sees that $\Xi_i(z)$ is a uniform limit of polynomial functions on the disc $\mathcal{O}$ and hence resides in $\mathbb{T}$. The argument above shows that $\omega(t, \pi z)/\omega(t)$ and $\Xi_i(z)$ agree on $\mathcal{O}$ and are thus equal in $\mathbb{T}$. 

The proof of the previous theorem was inspired by the methods of Pellarin in [22]. The statement was implicit in the work of the author [23], and we gratefully acknowledge F. Pellarin for bringing the statement to light and allowing us to include it here.

We will see in Section 4.2.2 that Lemma 1.6.6 gives the simplest explanation of the explicit formulas obtained in this dissertation for the rational special values of Pellarin’s $L$-series at the positive integers.
CHAPTER 2
PELLARIN’S L-SERIES

Throughout this chapter $s$ will denote a non-negative integer and $t_1, \ldots, t_s$ independent indeterminates over $\mathbb{C}_\infty$. By convention, any empty product or empty sum shall be taken to be 1 or 0, respectively.

2.1 First Definitions and Properties

Definition 2.1.1. For $t$ an indeterminate (or an element of $\mathbb{C}_\infty$), and for $a \in A$ we shall denote by $\chi_t(a)$ the image of $a$ in $\mathbb{F}_q[t]$ (or $\mathbb{C}_\infty$, respectively) under the $\mathbb{F}_q$-algebra morphism of evaluation at $t$, i.e. $a \mapsto a(t)$.

Special values at the positive integers

Definition 2.1.2. Let $k$ be a positive integer. The $k$-th special value of Pellarin’s $L$-series in $s$ indeterminates is defined by the formal series / Euler product,

$$L(\Pi^s \chi_{t_i}, k) := \sum_{a \in A_+} a^{-k} \chi_{t_1}(a) \cdots \chi_{t_s}(a) = \prod_{\varpi \in A_+ \text{ irreducible}} (1 - \varpi^{-k} \chi_{t_1}(\varpi) \cdots \chi_{t_s}(\varpi))^{-1}.$$

Remark 2.1.3. Clearly, $L(\Pi^s \chi_{t_i}, k)$ converges (in $\mathbb{C}_\infty$) upon specializing the $t_i \in \mathbb{C}_\infty$ such that $|t_i| \leq 1$ for all $i$. For any given $s$ and $k$, the closed unit polydisc $|t_i| \leq 1$ is not the largest domain on which $L(\Pi^s \chi_{t_i}, k)$ converges. We have merely emphasized
this set as it gives a uniform domain on which $L(\Pi^s \chi_{t_i}, k)$ converges for all positive integers $s$ and $k$. We shall discuss the rigid analyticity of these functions in the $t_i$ in Theorem 2.1.8.

**Special values at the negative integers**

To define the special values of Pellarin’s $L$-series at the negative integers we shall need a lemma on the vanishing of certain power sums, just as we did for Goss’ special zeta values at the negative integers.

**Lemma 2.1.4.** Let $k$ be a non-negative integer. Then the power sums

$$S_d(\Pi^s \chi_{t_i}, k) := \sum_{a \in A_+ (d)} a^k \chi_{t_1}(a) \cdots \chi_{t_s}(a)$$

vanish for $d \gg 0$ (depending on $s$ and $k$).

*Proof.* See propositions 3.3.1 and 3.3.3.

**Definition 2.1.5.** Let $k$ be a non-negative integer. We define the $k$-th special polynomial associated to Pellarin’s $L$-series in $s$ indeterminates via

$$z(\Pi^s \chi_{t_i}, x, -k) := \sum_{d \geq 0} x^{-d} S_d(\Pi^s \chi_{t_i}, k).$$

**Remark 2.1.6.** This definition follows in the footsteps of Goss’ original extension of the Carlitz zeta values to a much larger analytic domain. Lemma 2.1.4 guarantees that the series given above is in fact a polynomial in $A[t_1, \ldots, t_s][x^{-1}]$. We will show later that for fixed $s$, the degree in $x^{-1}$ of the special polynomial $z(\Pi^s \chi_{t_i}, x, -k)$ grows logarithmically in $k$. This will be useful information for many applications to follow.

**Definition 2.1.7.** Let $k$ be a non-negative integer, then the special value of Pellarin’s $L$-series at $-k$ is defined by

$$L(\Pi^s \chi_{t_i}, -k) := z(\Pi^s \chi_{t_i}, 1, -k) \in A[t_1, \ldots, t_s].$$

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2.1.1 The Analyticity Theorem of Anglès and Pellarin

We include here a theorem of Anglès and Pellarin on the rigid analyticity in the $t_i$ of the functions $L(\Pi^s \chi_{t_i}, k)$. Their result generalizes the constructions and results of Goss [18] to the case of arbitrarily many indeterminates. Further, their result allows us to make the precise connection with Goss’ zeta values at the negative integers.

For $a \in A_+(d)$, we let $\langle a \rangle := a/\theta^d$. Then $\langle a \rangle$ is a 1-unit in $K_\infty$. Hence, via the binomial series, we may exponentiate $\langle a \rangle$ by elements $y$ in the $p$-adic integers $\mathbb{Z}_p$.

**Theorem 2.1.8** (Anglès and Pellarin, [3] Prop. 32). For all $t_1, \ldots, t_s \in \mathbb{C}_\infty$, $x \in \mathbb{C}_\infty^\times$ and $y \in \mathbb{Z}_p$, the series

$$\sum_{d \geq 0} \sum_{a \in A_+(d)} x^{-d} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \langle a \rangle^y$$  \hspace{1cm} (2.1.1)

converges to a function that is rigid-analytic in $t_1, \ldots, t_s, x$ and continuous in $y$.

As a corollary, for each $y \in \mathbb{Z}_p$ the function in (2.1.1) may be represented by an entire power series in $x^{-1}$ and the $t_i$. Hence the special values of Pellarin’s series at the positive integers are rigid analytic entire functions in the $t_i$. Also, observe that when $y$ is a positive integer, making the change of variables $x \mapsto x\theta^{-y}$ returns our special polynomial $z(\Pi^s \chi_{t_i}, x, -y)$.

2.1.2 The Rationality Theorem of Anglès and Pellarin

The following theorem is due to Anglès and Pellarin [3]. The statement was inspired by the author’s work [24]. It gives the general picture, in arbitrarily many indeterminates, of when Pellarin’s $L$-series is a rational function.

**Theorem 2.1.9** (Anglès, Pellarin [3], Theorem 4). Let $\alpha, s$ be positive integers such
that \( \alpha \equiv s \mod (q-1) \). Let \( \delta \) be the smallest positive integer such that, simultaneously, 
\( q^{\delta} - \alpha \geq 0 \) and \( s + l(q^{\delta} - \alpha) \geq 2 \). Then the formal series

\[
\tilde{\pi}^{-\alpha}L(\Pi^* \chi_{t_1}, \alpha)\omega(t_1) \cdots \omega(t_s) \prod_{i=1}^{s} \prod_{j=0}^{\delta-1} \left(1 - \frac{t_i}{\theta^j}\right) \in K_{\infty}[[t_1, \ldots, t_s]]
\]

is in fact a symmetric polynomial in \( K[t_1, \ldots, t_s] \) whose total degree \( \phi(a, s) \) satisfies

\[
\frac{\phi(a, s)}{s} \leq \left( \frac{s + l(q^{\delta} - \alpha)}{q - 1} \right) - 1.
\]

**Remarks on the proof**

The proof of this theorem follows by first noting that for \( \alpha \) and \( s \) as in the statement
the formal series above is actually a symmetric polynomial in the \( t_i \) with coefficients in 
\( K_{\infty} \). Proving that the coefficients are in fact in \( K \) is achieved by specializing the \( t_i \) at
roots of unity in the algebraic closure of \( \mathbb{F}_q \). When the conditions on \( \alpha \) and \( s \) appearing
in the theorem above are satisfied, these specialized \( L \)-series are the Dirichlet \( L \)-series
defined by Goss times a Gauss-Thakur sum divided by a power of \( \tilde{\pi} \). It is shown
that this specialization lies in an extension of \( K \) obtained by adjoining roots of unity.
This combined with the fact that these series are rigid analytic functions is enough
to ensure the rationality in \( K[t_1, \ldots, t_s] \) of the function under consideration in the
theorem.

It is worth highlighting that Anglès and Pellarin’s method is completely distinct
from the methods used in this dissertation.

### 2.1.3 Deformation of Carlitz-Goss zeta values and Goss \( L \)-values

The purpose of defining these special \( L \)-values is to gain a new perspective on Dirichlet
type Goss \( L \)-values and especially the Carlitz-Goss zeta values at the positive integers.
We shall also see that these series are “inductively related” by the parameter \( s \). Let
us give some examples of how this works.
Example 2.1.10. Let $s = 1$, and let $t = \lambda$ be an element of the algebraic closure in $\mathbb{C}_\infty$ of $\mathbb{F}_q$ with minimal polynomial $\varpi$. Then $\chi_t$ becomes a Dirichlet character from $A$ into $A/\varpi A$, and $L(\chi_t, k)$ becomes a Dirichlet type Goss $L$-value. Specializations of the $t_i$ at roots of unity is a major tool in [3].

Example 2.1.11. Let $1 \leq l \leq s$ and $m_1, \ldots, m_l$ non-negative integers such that $k - \sum q^{m_j} > 0$. Then setting $t_j = \theta^{q^{m_j}}$ for $j = s - l + 1, \ldots, s$ returns the Pellarin special $L$-value in $s - l$ indeterminates $L(\Pi^{s-l} \chi_t, k - \sum q^{m_j})$.

Similarly, now that the analytic continuation of Pellarin’s $L$-series in $s$-indeterminates has been shown in the Theorem 2.1.8 of Anglès and Pellarin, we may allow the positive integers $m_j$ to vary freely, with out the constraint $k - \sum q^{m_j} > 0$, and one obtains all special values (at both positive and negative integers) of Pellarin’s $L$-series in any integer less than $s$ indeterminates. In particular, when $l = s$, one obtains the Goss zeta special values $\zeta(k - \sum q^{m_j})$. We will see later in this dissertation that these $L$-values of Pellarin really do allow us to gain new insight into the behavior of Carlitz-Goss zeta special values.
CHAPTER 3
PELLARIN’S SPECIAL $L$-VALUES AT THE NEGATIVE INTEGERS

The results of this chapter are generalizations of the author’s work in [23]. In [3], Section 3 similar results are considered as a prerequisite for the proof of analyticity of Pellarin’s series in $s$ indeterminates. Outside of the method of specialization to Goss zeta values, which is a major theme in the life of Pellarin’s series anyway, our methods of proof are different. Further, our results concern the special polynomials associated to Goss’ extension of Pellarin’s series and are slightly finer in general than those of [3].

3.1 Recursive Nature

In the section and the next we focus on the special polynomials

$$z(\Pi^s \chi_{t_i}, x, 0) = \sum_{d \geq 0} x^{-d} \sum_{a \in A_+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a).$$

We remind the reader that by specializing some of the variables $t_i$ of $z(\Pi^s \chi_{t_i}, x, 0)$ at $\theta^{q^{j_i}}$ we may deduce results about the whole class of special values of Pellarin’s series at the negative integers.
Proposition 3.1.1. Let $s$ be a non-negative integer. Then $z(\Pi^0 \chi_{t_i}, x, 0) = 1$, and for all $s \geq 1$,

$$z(\Pi^s \chi_{t_i}, x, 0) = 1 - x^{-1} \sum_{d \geq 1} \sum_{b \in A_+(d-1)} \sum_{\lambda \in F_q} \prod_{i=1}^{s} (t_i \chi_{t_i}(b))^{1-j_i} \lambda^{j_i} =$$

$$= 1 + \sum_{d \geq 1} x^{-d} \sum_{b \in A_+(d-1)} \prod_{i=1}^{s} t_i \chi_{t_i}(b) +$$

$$+ \sum_{d \geq 1} x^{-d} \sum_{b \in A_+(d-1)} \sum_{\lambda \in F_q^X} \prod_{i=1}^{s} (t_i \chi_{t_i}(b))^{1-j_i} \lambda^{j_i}$$

\hspace{1cm} (\ast) \hspace{1cm} = 1 + \sum_{d \geq 1} x^{-d} \sum_{b \in A_+(d-1)} \prod_{i=1}^{s} t_i \chi_{t_i}(b) +$$

$$+ \sum_{(j_i) \in \{0,1\}^s} \left( \prod_{i=1}^{s} t_i^{1-j_i} \right) \sum_{d \geq 1} x^{-d} \sum_{b \in A_+(d-1)} \left( \prod_{i=1}^{s} \chi_{t_i}(b)^{1-j_i} \right) \sum_{\lambda \in F_q^X} \lambda^{\sum_{i=1}^{s} j_i}.$$

Using the fact that $\sum \lambda^{\sum_{i=1}^{s} j_i} = -1$ if $\sum j_i \equiv 0 \mod (q - 1)$ and equals zero otherwise, and taking account of the cancellation of $\sum_{d \geq 1} x^{-d} \sum_{b \in A_+(d-1)} \prod_{i=1}^{s} t_i \chi_{t_i}(b)$ in (\ast) and the term arising from the $s$-tuple of all zeros in (\ast\ast) we deduce the claim. \hfill \Box

Corollary 3.1.2. For all non-negative integers $s$, the degree in $x^{-1}$ of $z(\Pi^s \chi_{t_i}, x, 0)$ is at most $\lfloor s/(q - 1) \rfloor$.

Proof. The proof proceeds by strong induction on $s$. By the recursive formula we see immediately that the claim is true for $0 \leq s < q - 1$. Now let $s \geq q - 1$, and
suppose the claim holds for all $0 \leq j < s$. By the recursive formula and induction, the highest power of $x^{-1}$ appearing in $z(\Pi^s \chi_{t_i}, x, 0)$ comes from any $z(\Pi^s \chi_{1-t_i}, x, 0)$ such that exactly $q - 1$ of the $i_l$ are equal to 1, and by induction the degree of this latter special polynomial is then \(\left\lfloor \frac{s - (q-1)}{q-1} \right\rfloor = \left\lfloor \frac{s}{q-1} \right\rfloor - 1\). The extra $x^{-1}$ appearing in the recursive formula shows that the degree of $z(\Pi^s \chi_{t_i}, x, 0)$ is at most $\left\lfloor \frac{s}{q-1} \right\rfloor$.

\[\Box\]

**Remark 3.1.3.** We will see in Theorem 3.3.5 that the degree in $x^{-1}$ of $z(\Pi^s \chi_{t_i}, x, 0)$ actually equals $\left\lfloor \frac{s}{q-1} \right\rfloor$.

### 3.2 Trivial zeros of Pellarin’s series

**Proposition 3.2.1.** Suppose $s$ is a positive integer divisible by $q - 1$. Then

\[z(\Pi^s \chi_{t_i}, x, 0) \text{ vanishes at } x^{-1} = 1.\]

**Proof.** The proof proceeds by induction on $s$ divisible by $q - 1$ using the recursive formula from 3.1.1.

First suppose $s = q - 1$. Then as the recursive formula above shows we have

\[z(\Pi^s \chi_{t_i}, x, 0) = 1 - x^{-1}.\]

Thus we have a simple zero at $x = 1$.

Now suppose $s > q - 1$ is divisible by $q - 1$. As the recursive formula shows, apart from $z(\Pi^0 \chi_{t_i}, x, 0) = 1$, only special polynomials occur whose number of indeterminates is positive, divisible by $q - 1$ and less than $s$. By induction, these all vanish at $x^{-1} = 1$. Hence

\[z(\Pi^s \chi_{t_i}, x, 0) = 1 - x^{-1}z(\Pi^0 \chi_{t_i}, x, 0) - g(x),\]

with $g(x) \in A[t_1, \ldots, t_s][x^{-1}]$ vanishing at $x^{-1} = 1$. 

\[\Box\]
Proposition 3.2.2. Suppose \( s \) is divisible by \( q - 1 \). Then the zero of \( z(\Pi^s \chi_{t_i}, x, 0) \) at \( x^{-1} = 1 \) is simple.

Proof. We argue by contradiction. Let \( s \) be as in the statement, and suppose the claim is false. Since evaluation is a ring homomorphism, our assumptions imply that we have a non-simple zero of

\[
0 \neq z(x, -s) = z(\Pi^s \chi_{t_i}, x, 0)|_{t_1 = \theta, \ldots, t_s = \theta}
\]

at \( x^{-1} = 1 \), a contradiction. \( \square \)

3.3 Logarithmic growth

In the classical setting (over the complex numbers), for a primitive character \( \chi \) of modulus \( m \) one sets \( N(T, \chi) \) to be the number of zeros of \( L(\sigma + it, \chi) \) in the critical strip \( 0 < \sigma < 1 \) such that \( |t| < T \). It is well known that the “average number of zeros” \( N(T, \chi)/T \) grows like a constant multiple of \( \log(T) + \log(m) \).

One of the main results of this section will be the proof that the degree in \( x^{-1} \) of the special polynomials

\[
z(\beta_1, \ldots, \beta_s, x) := \sum_{d \geq 0} x^{-d} \sum_{a \in A_+(d)} \chi_{t_1}(a)^{\beta_1} \cdots \chi_{t_s}(a)^{\beta_s}
\]

equals

\[
\phi(\beta_1, \cdots, \beta_s) := \min_{i \geq 0} \left[ \frac{l_q(p^i \beta_1) + \cdots + l_q(p^i \beta_s)}{q - 1} \right].
\]

Specializing \( t_s \) at \( \theta \) and the remaining \( t_i \) at roots of unity in the algebraic closure of \( \mathbb{F}_q \) we obtain the special value at \( -\beta_s \) for Goss’ Dirichlet \( L \)-function. The growth rates for sum of digit functions \( l_q \) and \( \log_q \) may be easily related, and the formula obtained above for the degree of \( z(\beta_1, \ldots, \beta_s, x) \) is strikingly similar to the classical formula for \( N(T, \chi)/T \). The analogy can be made even tighter by observing that a
primitive character $\chi$ with square-free modulus $a = P_1 \cdots P_{s-1}$ (Here the $P_i$ are the distinct monic irreducible factors of $a$.) may be realized as a map on $A_+$ with values in the algebraic closure of $\mathbb{F}_q$ via $\chi : a \mapsto a(\lambda_1)^{\beta_1} \cdots a(\lambda_{s-1})^{\beta_{s-1}}$ for positive integers $\beta_i$ and roots $\lambda_i$ of $P_i$.

The result stated above gives precise meaning to the statement the degree in $x^{-1}$ of the special polynomials $z(\beta_1, \ldots, \beta_s, x)$ grows logarithmically in the $\beta_i$. Such logarithmic growth of the special polynomials associated to $L$-series of $\tau$-sheaves was proven in great generality by Böckle, see [5]. Goss used this growth to give analytic continuation to these series using non-Archimedean measure theory. We do not pursue this direction here, but direct the interested reader to [15].

For the remainder of this section we only use the sum of base $q$ digits function $l_q$ and omit the subscript $q$.

3.3.1 Some preliminary bounds

In this section we study the power sums $S_d(\Pi^s \chi_{t_1}, k)$ and related special polynomials introduced and studied in the last section. We give a lower bound on $d$, depending on $s$ and $k$, above which these sums vanish. Further, we determine the exact degree in $x^{-1}$ of the special polynomials $z(\Pi^s \chi_{t_1}, x, 0)$. In Section 3.3.3 we draw corollaries for the action of Goss’ group $S_{(q)}$ on the degrees of certain special polynomials obtained by specialization from those of the last several sections.

In this section alone we will make use of the following notation: For $s, k$ non-negative integers, define

$$\phi(s, k) := \left\lfloor \frac{l(k) + s}{q - 1} \right\rfloor.$$
Method One

From Proposition 3.1.1 we acquire a new proof that the special polynomials \( S_d(k) \) vanish for \( d > \lfloor l(k)/(q - 1) \rfloor \). We will deduce this from a slightly more general result about the special polynomials \( z(\Pi^s \chi_{t_i}, x, -k) \) associated to Pellarin’s \( L \)-series. We will give a second proof of this result in 3.3.3 by different means, assuming the vanishing of Carlitz power sums \( S_d(k) \). Here, no assumption on the \( S_d(k) \) will be used.

Proposition 3.3.1. For all non-negative integers \( s \) and \( k \), the degree in \( x^{-1} \) of the special polynomial \( z(\Pi^s \chi_{t_i}, x, -k) \) is at most \( \phi(s, k) \).

Proof. Write \( k = q^{i_1} + \cdots + q^{i_l(k)} \) with \( 0 \leq i_j \leq i_{j+1} \) and \( i_j < i_{j+q} \), so that collecting the powers of \( q \) according to their exponents we have the base \( q \) expansion of \( k \). Letting \( r = s + l(k) \) we have \( z(\Pi^r \chi_{t_i}, x, 0) |_{t_{s+1} = \theta_t^{i_1}, \ldots, t_r = \theta_t^{i_l(k)}} = z(\Pi^s \chi_{t_i}, x, -k) \), and the claim follows from Corollary 3.1.2. \( \square \)

Remark 3.3.2. Taking advantage of the \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear extension of the \( p \)-th power map, we can slightly improve the upper bound on the degree of \( z(\Pi^s \chi_{t_i}, x, -k) \) to \( \min_{i \geq 0} \left\lfloor \frac{s + l(p^k)}{q - 1} \right\rfloor \).

Method Two

We thank G. Böckle for suggesting this simple proof, a version of which appears in the author’s paper [24].

Proposition 3.3.3. Let \( s, k \) be non-negative integers. Then the power sums

\[
S_d(\Pi^s \chi_{t_i}, k) := \sum_{a \in A_+(d)} a^k \chi_{t_1}(a) \cdots \chi_{t_s}(a)
\]

vanish for \( d > \phi(s, k) \).
Proof. This is proved by induction on $s$. When $s = 0$, this follows from a result of Carlitz. See Böckle [6].

Now assume the result holds for some $s \geq 0$. Let $d > \phi(s + 1, k)$. We view $S_d(\Pi^{s+1} \chi_{t_i}, k)$ as a polynomial in $t_{s+1}$ over $A[t_1, \ldots, t_s]$. We shall show that this polynomial vanishes for $t_{s+1} = \theta^m$ for all $m$ sufficiently large, as we now describe.

Denote by $S_d(\Pi^{s+1} \chi_{t_i}, k) \mid_{t_{s+1}=\theta^m}$ the evaluation at $t_{s+1} = \theta^m$. Then

$$S_d(\Pi^{s+1} \chi_{t_i}, k) \mid_{t_{s+1}=\theta^m} = S_d(\Pi^s \chi_{t_i}, k + q^m).$$

By induction the right side above vanishes for $d > \phi(s, k + q^m)$. Now, for all $m > \log_q(k) + 1$ we have $l(k + q^m) = l(k) + 1$, and hence, $\phi(s, k + q^m) = \phi(s + 1, k)$. Thus we have shown for $d > \phi(s + 1, k)$ and $m > \log_q(k) + 1$ the polynomial $S_d(\Pi^{s+1} \chi_{t_i}, k) \mid_{t_{s+1}=\theta^m}$ vanishes. Thus it must be identically zero. This finishes the proof.

\[\Box\]

3.3.2 Exact Degree

In this section and the next we shall use the following obvious lemma without comment.

Lemma 3.3.4. If $j, k$ are two positive integers such that there is no base $q$ carry-over in the sum $j + k$, then $l(j + k) = l(j) + l(k)$.

\[\Box\]

Theorem 3.3.5. For any $s$ positive integers $\beta_1, \ldots, \beta_s$, the exact degree in $x^{-1}$ of the special polynomial

$$z(\beta_1, \ldots, \beta_s, x) := \sum_{d \geq 0} x^{-d} \sum_{a \in A_d(d)} \chi_{t_1}(a)^{\beta_1} \cdots \chi_{t_s}(a)^{\beta_s}$$

equals

$$\phi(\beta_1, \cdots, \beta_s) := \min_{i \geq 0} \left\lfloor \frac{l(p^i \beta_1) + \cdots + l(p^i \beta_s)}{q-1} \right\rfloor.$$
Proof. The upper bound \( \phi(\beta_1, \ldots, \beta_s) \) already follows from similar specializations to those in the proof of Proposition 3.3.1 and the additivity of the \( p \)-th power mapping. It remains to argue that the degree cannot be lower, and this is done by specializing to the one variable case.

We argue by contradiction. Suppose the degree in \( x^{-1} \) is strictly less than \( \phi(\beta_1, \ldots, \beta_s) \). Let \( m_0 = 0 \), and recursively choose a sequence of positive integers \( m_1, \ldots, m_{s-1} \) such that \( q^{m_j} > q^{m_j-1} p^{e-1} \beta_j \). Then for \( i = 0, \ldots, \log_p(q) - 1 \) there is no base \( q \) carry-over in the sum \( p^i \beta_1 + p^i q^{m_1} \beta_2 + \cdots + p^i q^{m_{s-1}} \beta_s \). Hence for these \( i \),

\[
  l(p^i \beta_1) + l(p^i \beta_2) + \cdots + l(p^i \beta_s) = l(p^i (\beta_1 + q^{m_1} \beta_2 + \cdots + q^{m_{s-1}} \beta_s)).
\]

Thus

\[
  \phi(\beta_1, \ldots, \beta_s) = \min_{i \geq 0} \left[ \frac{l(p^i (\beta_1 + q^{m_1} \beta_2 + \cdots + q^{m_{s-1}} \beta_s))}{q - 1} \right].
\]

But, by Böckle [6, Theorem 1.2 (a)], the exact degree in \( x^{-1} \) of the non-zero polynomial

\[
  z(x, -(\beta_1 + q^{m_1} \beta_2 + \cdots + q^{m_{s-1}} \beta_s)) = z(\beta_1, \ldots, \beta_s, x)|_{t_1=\theta, t_2=\theta^{q^{m_1}}, \ldots, t_s=\theta^{q^{m_{s-1}}}}
\]

is \( \phi(\beta_1, \ldots, \beta_s) \), a contradiction. \( \square \)

### 3.3.3 The action of Goss’ group

In his \( \zeta \)-phenomenology paper [17], Goss has introduced a group of digit permutations that appear to have much to say about the symmetries of special zeta values in positive characteristic.

**Definition 3.3.6.** Let \( \rho \) be a permutation of the non-negative integers, and let \( k = \sum k_i q^i \) be a \( p \)-adic integer written in base \( q \), i.e. \( 0 \leq k_i < q \) for all \( i \geq 0 \). Define

\[
  \rho^*(k) := \sum k_i q^\rho(i).
\]

We let \( S(q) \) be the collection of all such *digit-permutations*. 

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Remark 3.3.7. It is easy to see that $S(q)$ stabilizes the non-negative integers, and Goss has shown that the group $S(q)$ acts as continuous automorphisms of the $p$-adic integers that also stabilizes the non-positive integers.

Now we wish to show now that the degrees in $x^{-1}$ of the special polynomials $z(\beta_1, \ldots, \beta_s, x)$ are invariant under a natural action of the group $S(q)$.

Lemma 3.3.8. Let $\rho^* \in S(q)$ and $k$ a non-negative integer. Then for all $i \geq 0$,

$$l(p^i k) = l(p^i \rho^*(k)).$$

Proof. Let $e = \log_p(q)$. It suffices to prove this for $i = 0, 1, \ldots, e - 1$, so let $i$ be one such number in this range. Write $k = \sum k_l q^l$ in base $q$.

Clearly for each $l \geq 0$ we may write $p^i k_l = a_l + b_l q$ with $p^i \leq a_l < q$ or $a_l = 0$, and $0 \leq b_l < p^i$. Hence for all $j, k \geq 0$ there is no base $q$ carry over in the sums $a_j + b_k$. Thus $p^i k = \sum_{l \geq 0} a_l q^l + \sum_{l \geq 0} b_l q^{l+1}$ is the sum of two positive integers written in base $q$, and there is no carry over in this decomposition. Hence $l(p^i k) = \sum_{l \geq 0} (a_l + b_l)$. Similarly, $p^i \rho^*(k) = \sum_{l \geq 0} a_l q^{\rho(l)} + \sum_{l \geq 0} b_l q^{\rho(l)+1}$ is the sum of two positive integers written in base $q$, and there is no carry-over in this decomposition. Hence $l(p^i \rho^*(k)) = \sum_{l \geq 0} (a_l + b_l) = l_q(p^i k)$. \hfill $\square$

Theorem 3.3.9. The degree in $x^{-1}$ of $z(\beta_1, \ldots, \beta_s, x)$ is invariant under the action of $\rho^* \in (S(q))^s$ defined by

$$\rho^*(z(\beta_1, \ldots, \beta_s, x)) := z(\rho_1^* \beta_1, \ldots, \rho_s^* \beta_s, x)$$

for all $\rho^* := (\rho_1^*, \ldots, \rho_s^*) \in (S(q))^s$.

Proof. This is an immediate consequence of the previous result and Theorem 3.3.5. \hfill $\square$
3.4 Measure coefficients for Pellarin’s series

Here we describe the calculation of the non-Archimedean measure whose moments are the special polynomials $z(\chi_t, x, -k)$. We begin by analyzing the Wagner series for the character $\chi_t$ as a function on the completions of $A$ at finite places.

3.4.1 The Wagner Series for $\chi_t$ on $A_v$

We recall that we have chosen $\theta$ as the $\mathbb{F}_q$-algebra generator of $A$, and $\chi_t$ depends on this choice by our definition $\chi_t : \theta \mapsto t$. As we saw in the introduction, the formal series

$$\Xi_t(z) := \sum_{i \geq 0} b_i(t) E_i(z)$$

stores the information contained in the function $\chi_t$ on $A$ in the sense that making the formal replacement $z \mapsto a \in A$ we have $\Xi_t(a) = \chi_t(a)$. We have already studied $\Xi_t(z)$ as a function of $z$ on $\mathbb{C}_\infty$. While reading this section keep in mind Theorem 1.6.6, i.e. $\infty$-adically,

The function $\Xi_t(z)$ is the Anderson Generating Function for the Carlitz module.

In this section we consider $\Xi_t(z)$ as a function of $z$ on a $v$-adic space, for $v$ a finite place of $A$. We shall abuse notation by also writing $v$ for the monic irreducible polynomial from which the place $v$ arises. We shall write $vA$ for the ideal generated by $v$, and $A_v$ for the completion of $A$ at $v$. Finally, we write $K_v$ for the completion of $K$ at $v$ and $|\cdot|_v$ for the absolute value making $K_v$ complete, normalized so that $|v|_v = 1/q$. Our main references for this background material are Goss’ book [16] Chapter 8.22 and Thakur’s paper [29].

The following sets will serve as convenient places to choose $t$ from that make $\chi_t$
continuous on $A$ with respect to $|\cdot|_v$: let $\Theta := \{\theta^{q^i} : i \geq 0\}$ and $\Theta + vA := \{x + \varpi : x \in \Theta$ and $\varpi \in vA\}$.

**Remark 3.4.1.** The set $\Theta + vA$ is by no means the largest choice that provides $t$ that make $\chi_t$ continuous. For example, call a polynomial *affine* if it is of the form

$$
\gamma = \gamma_c + \sum_{i \geq 0} \gamma_i \theta^{q^i}
$$

for some $\gamma_c, \gamma_0, \gamma_1, \cdots \in \mathbb{F}_q$.

If $v$ is an affine irreducible element of $A$, and $\alpha$ is another affine element of $A$ satisfying $(v(1) - v(0) + 1)\alpha(0) + v(0) = \alpha(v(0))$, then one easily shows that $v|v(\alpha)$, where here we are treating $\theta$ as a variable and using parenthesis to denote composition as usual.

**Lemma 3.4.2** ($v$-adic continuity of $\Pi^s \chi_t$). Let $t_i \in \Theta + vA$ for $i = 1, \ldots, s$. Then the map $\Pi^s \chi_{t_i} : A \to A$ given by $a \mapsto \chi_{t_1}(a) \cdots \chi_{t_s}(a)$ is continuous with respect to the absolute value on $A_v$.

*Proof.* Let $t_i$ be as above. Then we have $\chi_{t_i}(v) = v(\theta^{q^j} + \varpi)$ for some $j \geq 0$ and $\varpi \in vA$. Reducing modulo $v$ we see plainly that $\chi_{t_i}(v) \equiv v(\theta)^{q^j} \equiv 0 \mod v$. Thus since evaluation is an $\mathbb{F}_q$-algebra morphism, if $v^n$ divides $a$, then $v^n$ divides $\chi_{t_i}(a)$. Thus $\chi_{t_i}$ is continuous at zero, and continuity in general now follows using the linearity of $\chi_{t_i}$. Finally, the proposition follows by observing that a product of continuous functions is continuous. \hfill \square

Thus Wagner’s Theorem (Theorem 1.3.4 above) tells us that the continuous extension of $\Pi^s \chi_{t_i}$ (here we have fixed $t_i \in \Theta + vA$ for all $i$) to $A_v$ has a Wagner series representation. We now move toward proving that $\Xi_t(z)$ as in (1.6.1) is the Wagner series for $\chi_t$.

**Lemma 3.4.3.** For $t \in \Theta + vA$, the coefficients $b_i(t)$ lie in $A_v$ and tend to zero as $i \to \infty$. 

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Proof. Let \( t \in \Theta + vA \). Then \( t = \theta^j + \varpi \). We will prove the lemma for \( j = 0 \), and the modifications necessary for \( j > 0 \) will be completely evident.

We have \( b_i(t) = \prod_{k=0}^{i-1} (\varpi + \theta - \theta^k) \). Now \( v \) divides \( \theta - \theta^k \) depending on whether \( d = \deg v \) divides \( k \) or not. Hence \( |b_i(t)|_v \leq q^{-\lfloor (i-1)/d \rfloor} \to 0 \) as \( i \to \infty \). \( \square \)

**Theorem 3.4.4.** For \( t \in \Theta + vA \), the series \( \Xi_t(z) \) defined in (1.6.1) gives the continuous extension of \( \chi_t \) to \( A_v \).

**Proof.** Let \( t \in \Theta + vA \). Then the previous lemma shows that the coefficients \( b_i(t) \) of \( \Xi_t(z) \) tend to zero as \( i \to \infty \). Thus by Wagner’s Theorem, \( \Xi_t(z) \) defines a continuous function on \( A_v \). Hence uniformly continuous by the compactness of \( A_v \). This function agrees with \( \chi_t \) on \( A \) which is dense in \( A_v \). This finishes the proof. \( \square \)

### On the differentiability of \( \chi_t \) on \( A_v \) and connection with \( v \)-adic \( L \)-values

Throughout this section, we fix \( t \in \Theta + vA \). Given the Wagner series \( \Xi_t \) for \( \chi_t \) on \( A_v \), one may be tempted to think that \( \Xi_t(z) \) is differentiable in \( z \). If \( \Xi_t(z) \) were differentiable in \( z \) on \( A_v \), with non-zero derivative, then a simple argument using the multiplicativity of \( \Xi_t \) shows it must be the identity function which it is not, see [23] for more details.

Wagner gives the following criterion in terms of a function’s Wagner coefficients for differentiability.

**Theorem 3.4.5** (Wagner [33], Theorem 5.1). Suppose \( f \) is a continuous \( \mathbb{F}_q \)-linear function on \( A_v \) with Wagner series \( \sum A_i E_i(z) \). If \( A_i / \ell i \to 0 \) as \( i \to \infty \), then \( f \) is differentiable everywhere on \( A_v \) with derivative equal to \( \sum_{i \geq 0} A_i / \ell i \).

The theorem of Wagner just stated plus a result from the next chapter allow us to obtain as a corollary the vanishing of the Goss \( (\theta - \lambda) \)-adic special \( L \)-value

\[
L_{\theta-\lambda}(\chi_\lambda, 1) := \sum_{d \geq 0} \sum_{a \in A_+(d)} \frac{\chi_\lambda(a)}{a}.
\]

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Here $\lambda \in \mathbb{F}_q^\times$.

**Remark 3.4.6.** For arbitrary irreducibles $v$, working in the extension of $A_v$ containing the root $\lambda'$ of $v$, it appears that we may obtain the vanishing of $L_v(\chi_{\lambda'}, 1)$ via the same method. We will not use this remark in the sequel and leave the verification to the interested reader.

**Corollary 3.4.7 (Vanishing of Goss’ Dirichlet $L$-values for certain odd characters).**

Let $\lambda \in \mathbb{F}_q^\times$, and let $v = \theta - \lambda$. Then the Goss special $L$-value $L_v(\chi_{\lambda}, 1)$ vanishes.

**Proof.** Later in Theorem 4.1.2 we shall show that we have an equality of polynomials

$$\sum_{a \in A_+(d)} \frac{\chi_t(a)}{a} = \frac{b_d(t)}{\ell_d} \in K[t].$$

Goss has shown that for $t = \lambda \in \mathbb{F}_q^\times$ the sums $\sum_{a \in A_+(d)} \frac{\chi_{\lambda}(a)}{a}$ tend to zero $v$-adically as $d \to \infty$. Thus by Wagner’s theorem above $\Xi_{\lambda}$ is differentiable on $A_v$ with derivative

$$\sum_{d \geq 0} \frac{b_d(\lambda)}{\ell_d} = \sum_{d \geq 0} \sum_{a \in A_+(d)} \frac{\chi_{\lambda}(a)}{a} = L_v(\chi_{\lambda}, 1).$$

Thus since $\Xi_{\lambda}$ is not the identity function, we conclude that $L_v(\chi_{\lambda}, 1)$ must vanish. This agrees with the well known fact that the $v$-adic special Dirichlet $L$-values at 1 vanish for odd characters $\chi$. Our character $\chi_{\lambda}$ is an example of this phenomenon. \[\Box\]

**Remark 3.4.8.** Let $\lambda \in \mathbb{F}_q$ and $v = \theta - \lambda$. As F. Pellarin points out, the sums

$$\sum_{a \in A_+(d)} \frac{\chi_{\lambda}(a)}{a} = b_d(\lambda) \ell_d^{-1}$$

tending to zero $v$-adically as $d \to \infty$ suggests the $v$-adic analytic continuation in the variable $t$ of the formal sum

$$L_v(\chi_t, 1) := \sum_{d \geq 0} \sum_{\substack{a \in A_+(d) \\text{\scriptsize\textit{i}} \in A_+(d) \\text{\scriptsize\textit{a},}\theta - \lambda = 1}} \frac{\chi_t(a)}{a}.$$  

In Corollary 4.1.9 we show that

$$\sum_{a \in A_+(d)} \frac{\chi_{t_1}(a) \cdots \chi_{t_q}(a)}{a} = \ell_d^{-1} (\text{[d]b}_{d-1}(t_1) \cdots b_{d-1}(t_q) + b_d(t_1) \cdots b_d(t_q))$$

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for all $d \geq 1$, and from this it is easy to show that

$$\sum_{e=0}^{d} \sum_{a \in A_v(e)} \frac{\chi_{t_1}(a) \cdots \chi_{t_q}(a)}{a} = \ell_d^{-1} b_d(t_1) \cdots b_d(t_q).$$

Thus if we let $t_i = \lambda$ for $i = 1, \ldots, q - 1$ and $t = t_q \in \Theta + vA$ we learn that

$$L_v(\chi_t, 1) = \lim_{d \to \infty} \ell_d^{-1} b_d(\lambda)^{q-1} b_d(t)$$

exists in $A_v$ and is identically equal to zero.

### 3.4.2 Calculation of the Measure coefficients for $z(\chi_t, x, -k)$

In this section we show, for select $x, t$, the existence of the non-Archimedean measure whose $k$-th moment is the special polynomial $z(\chi_t, x, -k)$. We then go on to calculate the coefficients of the divided power series associated to $\mu_{x,t}$ in two ways. The first technique, while just a sketch, is intended to generalize to any number of characters. The second technique will be a calculation straight from the definitions, following Thakur’s original calculation. We stick to the particular example of the evaluation character $\chi_t$, but it will be clear to the reader that the techniques below extend inductively (but not easily!) to any product of continuous $\mathbb{F}_q$-linear functions on $A_v$.

The next theorem appeared as Theorem 1.3.14. We state it again here for the convenience of the reader.

**Theorem 3.4.9.** For each $x \in K_v$ such that $|x| \geq 1$ there is a unique $A_v$-valued measure $\mu_x$ whose $k$-th moment $\int z^k \, d\mu_x(z)$ is the value at $x$ of the $k$-th special polynomial associated to the Goss zeta function $z(x, -k)$. For all $j \geq 0$ the measure
coefficients $\mu_x(j)$ of $\mu_x$ are given by $\mu_x(0) = 1$, and

$$\mu_x(j) = \begin{cases} 
(-x^{-1})^l & \text{if } j = cq^l + q^l - 1 \\
(-x^{-1})^l + (-x^{-1})^{l+1} & \text{if } j = cq^l + q^l - 1 \\
0 & \text{for some } l \geq 0 \text{ and } c = q - 1, \\
0 & \text{otherwise}. 
\end{cases}$$

\[ \Box \]

**Definition 3.4.10.** Given a continuous function $h : A_v \to A_v$ and measure $\mu$ we write $h \, d\mu$ for the measure which takes the value $\int gh \, d\mu$ for all continuous functions $g : A_v \to A_v$.

Following a tip from Anderson, Goss observed in [13] for continuous additive maps $h : A_v \to A_v$ that what he calls the *abstract Fourier transform on measures* $\mu \mapsto h \, d\mu$ is a derivation on the algebra of measures. The next result is an instance of this.

**Proposition 3.4.11** (Equality of Measures). Let $\mu_x$ be as defined in the previous theorem. Then for each $x \in K_v$ such that $|x| \geq 1$ and $t \in \Theta^+ A_v$, there exists a unique $A_v$-valued measure $\mu_{x,t}$ whose $k$-th moment is the special polynomial $z(\chi_t, x, -k)$. It is given by

$$d\mu_{x,t} = \chi_t d\mu_x.$$  

**Proof.** It suffices to prove that the moments $\int_{A_v} y^k \chi_t(y) \, d\mu_x(y)$ all equal $z(\chi_t, x, -k)$. We will unfold the definition of $z(\chi_t, x, -k)$ in terms of $\mu_x$.

By definition, we have,

$$z(\chi_t, x, -k) := \sum_{d \geq 0} x^{-d} \sum_{a \in A_+(d)} a^k \chi_t(a) \in A[t, x^{-1}],$$

where the degree in $x^{-1}$ is at most $D(k) := \lfloor (l(k) + 1)/(q - 1) \rfloor$ by Prop. 2.1.4.
Hence, we rewrite the integral $\int_{A_v} y^{k} \chi_t(y) \, d\mu_x(y)$ in two parts as
\[
\int_{A_v} y^{k} \sum_{i=0}^{D(k)} b_i(t) E_i(y) \, d\mu_x(y) + \int_{A_v} y^{k} \sum_{i>D(k)} b_i(t) E_i(y) \, d\mu_x(y),
\]
(3.4.1)
and we will show first that $\int_{A_v} y^{k} E_i(y) \, d\mu_x(y)$ vanishes for $i > D(k)$.

Let $i > D(k)$, and define the coefficients $\alpha_{i,j} \in A$ through the equality,
\[
D_i E_i(y) = \sum_{j=0}^{i} \alpha_{i,j} y^q j.
\]
Then (we drop the subscript $A_v$)
\[
\int y^k D_i E_i(y) \, d\mu_x(y) = \sum_{j=0}^{i} \alpha_{i,j} \int y^{k+q^j} \, d\mu_x(y)
\]
\[
= \sum_{j=0}^{i} \alpha_{i,j} \sum_{d\geq 0} x^{-d} \sum_{a \in A_+(d)} a^{k+q^j}
\]
\[
= \sum_{d\geq 0} x^{-d} \sum_{a \in A_+(d)} a^k D_i E_i(a).
\]
Again by 2.1.4 the degree in $x^{-1}$ of the last polynomial (*) above is at most $D(k)$.
Hence, $E_i(a)$ vanishes for all $a \in A_+(d)$ for $0 \leq d \leq D(k)$. Thus the polynomial in
(*) vanishes. This finishes the proof of the vanishing of the second integral in (3.4.1).

Now one easily sees by the same method as above, i.e. using the representation for
the $D_i E_i(y)$ in powers of $y$ and the definition of $\mu_x$, that the first member of (3.4.1)
is exactly $z(\chi_t, x, -k)$. We do not repeat the argument.

In order to calculate the measure coefficients for $\mu_{x,t}$ we shall need to understand
how the Carlitz basis elements behave under multiplication. In theory, the following
lemma should also be useful for inductively calculating the Wagner series for products
of continuous $\mathbb{F}_q$-linear functions.

**Lemma 3.4.12** (The multiplication lemma). Write $l = \sum l_i q^i$ in base $q$. We have two cases:

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1. Suppose \( l_i < q - 1 \) for some \( i \geq 0 \). Then \( G_iE_i = G_{i+q^i} \).

2. Suppose \( l_i = q - 1 \) for some \( i \), and let be \( k \geq 0 \) be such that \( l_{i+j} = q - 1 \) for \( 0 \leq j \leq k \) and \( l_{i+k+1} < q - 1 \). Then

\[
G_iE_i = \sum_{j=0}^{k} \left( \prod_{m=1}^{j} \left[ i + m \right] \right) G_{i+q^i+j-(q-1)\sum_{n=0}^{j} q^{n+i}} + \left( \prod_{m=1}^{k+1} \left[ i + m \right] \right) G_{i+q^i+k+1-(q-1)\sum_{n=0}^{k} q^{n+i}}.
\]

**Proof.** The first claim is immediate from the definitions.

For the second, we shall use the recursive formula

\[
E_i(z)^q = [i + 1]E_{i+1}(z) + E_i(z),
\]

which holds for all \( i \geq 0 \) and can be proven by observing that both sides are polynomials in \( z \) of the same degree that have the same set of zeros and the same value at \( z = \theta^{i+1} \), see e.g. Goss [16] the second proof of Theorem 3.1.5. This also follows from the functional equation for the Carlitz exponential by using that \( E_i(z) \) is the coefficient of \( x^{q^i} \) in \( e_C(z \log_C(x)) \).

Assume \( l_i = q - 1 \) for some \( i \geq 0 \) and let \( k \) be as in case two. We shall prove the desired equality by induction on \( k \). The case where \( k = 0 \) is immediate from (3.4.2). Assume now that \( k \geq 1 \), and that the desired equality holds for all \( 0 \leq j < k \). Then

\[
(*) \ G_jE_i = G_{j-j_i q^i}E_i^q = [i + 1]G_{j-i q^i}E_{i+1} + G_{j-j_i q^{i+1}}.
\]

Now, \( j - j_i q^i \) has \( j_{i+1} = j_{i+1} = \cdots = j_{i+1+k-1} = q - 1 \). Hence, by induction on \( k \) we may apply the formula to \( G_{j-j_i q^i}E_{i+1} \), and \((*)\) becomes

\[
G_{j-(q-1)q^i+q^i} + [i + 1] \sum_{j=0}^{k-1} \left( \prod_{m=1}^{j} \left[ i + 1 + m \right] \right) G_{j-(q-1)q^i+i+1+j-(q-1)\sum_{n=0}^{j} q^{n+i+1}} + [i + 1] \left( \prod_{m=1}^{k} \left[ i + 1 + m \right] \right) G_{j-(q-1)q^i+i+k+1-(q-1)\sum_{n=0}^{k-1} q^{n+i+1+n}}.
\]
Now combining the two terms in the first line and re-indexing the first and second lines finishes the proof.

**Theorem 3.4.13** (Measure coefficients of $\mu_{x,t}$). We have $\mu_{x,t}(0) = 1$, and

$$
\mu_{x,t}(j) = \begin{cases} 
(-x^{-1})^l b_t(t) & \text{(i) if } j = cq^l + q^l - 1 - q^e \\
((x^{-1})^l + (-x^{-1})^{l+1}) b_t(t) & \text{(ii) if } j = q^{l+1} - 1 - q^e \\
(-x^{-1})^l b_t(t) + (-x^{-1})^{l+1} b_{l+1}(t) & \text{(iii) if } j = q^{l+1} - 1 \\
0 & \text{(iv) otherwise.}
\end{cases}
$$

**Remark 3.4.14.** As Goss points out, this theorem highlights the importance of the coefficients $b_i(t)$ of $\Xi_t$, and we draw the attention of the reader again to the relationship between $\Xi_t$ and the Anderson Generating Function.

**First method of calculation**

Here we indicate how the proof of Theorem 3.4.13 will go if one wishes to use Thakur’s theorem.

**First proof (Sketch).** We have the following string of equalities:

$$
\mu_{x,t}(j) := \int_{A_v} G_j \, d\mu_{x,t} = \int_{A_v} G_j \, \chi_t \, d\mu_{\zeta} = \sum_{i \geq 0} b_i(t) \int_{A_v} G_j E_i \, d\mu.
$$

Here the first equality is by definition, the second by Prop. 3.4.11, and the third replaces $\chi_t$ with its Wagner series $\Xi_t$ and uses the continuity of $\mu$. The remainder
of the proof is an elementary calculation of the rightmost member of the string of equalities above following from Thakur’s theorem and Lemma 3.4.12. The full details are left to the interested reader.

**Second method of calculation**

Now we give a full proof of Theorem 3.4.13 straight from the definitions with no reference to Thakur’s theorem. We shall need two small lemmas. The first highlights the immense cancellation that occurs when calculating the measure coefficients in terms of the moments. We use the logarithmic growth in the degrees of the special polynomials associated to Pellarin’s series in the proof.

**Lemma 3.4.15.** Let \( d \geq 0 \) be an integer. Then for integers \( k \) in \( I_{d,1} := I_1 := [q^d, q^{d+1} - q^d - 1] \),

\[
\mu_{t,x}(k) = x^{-d} \sum_{a \in A_{+}(d)} G_k(a) \chi_t(a).
\]

For integers \( k \) in \( I_{d,2} := I_2 := [q^{d+1} - q^d - 1, q^{d+1} - 1] \),

\[
\mu_{t,x}(k) = x^{-d} \sum_{a \in A_{+}(d)} G_k(a) \chi_t(a) + x^{-(d+1)} \sum_{a \in A_{+}(d+1)} G_k(a) \chi_t(a).
\]

**Proof.** Write \( \Pi(k)G_k(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0 \in A[z] \).

For \( k \) in \( I_1 \)

\[
\Pi(k) \mu_{t,x}(k) = \int_{A_p} \Pi(k)G_k(z) \, d\mu_{t,x}
\]

\[
= \sum_{i=0}^{k} a_i z^i \chi_t(x, -i)
\]

\[
= \sum_{i=0}^{k} a_i \sum_{\varepsilon=0}^{d} \sum_{a \in A_{+}(\varepsilon)} a^i \chi_t(a).
\]

It is worth making a special note here that the second sum in the last line only goes up to \( d \) when \( k \in I_1 \) precisely because of the logarithmic growth of the degrees of the
special polynomials obtained in the last section. Interchanging sums and observing that \( G_k(z) \) vanishes on polynomials of degree less than \( d \) gives

\[
\mu_{t,x}(k) = x^{-d} \sum_{a \in A_+(d)} G_k(a) \chi_t(a).
\]

For \( k \) in \( I_2 \), we have the same calculation, except that now we must include the degree \( d + 1 \) monic polynomials in the double sum for \( z(\chi_t, x, -i) \), and the last line becomes

\[
\sum_{i=1}^{k} a_i \sum_{e=0}^{d+1} x^{-e} \sum_{a \in A_+(e)} a^i \chi_t(a).
\]

Again \( G_k(z) \) vanishes on all polynomials of degree less than \( d \), and we obtain

\[
\mu_{t,x}(k) = x^{-d} \sum_{a \in A_+(d)} G_k(a) \chi_t(a) + x^{-(d+1)} \sum_{a \in A_+(d+1)} G_k(a) \chi_t(a).
\]

The second result, just below, is a corollary to Jeong’s Theorem 8 in [20].

**Lemma 3.4.16 (Orthogonality relations).** For \( j < q^m \) and \( l \geq 0 \),

\[
\sum_{a \in A_+(m)} G_j(a) E_l(a) = \begin{cases} (-1)^m & \text{if } j + q^l = q^m - 1 \\
 & \text{or if } l = m \text{ and } j = q^m - 1, \\
0 & \text{if } j + q^l \neq q^m - 1. \end{cases}
\]

**A Second Proof**

*Second proof of Theorem 3.4.13.* Let \( d \geq 0 \), and suppose \( k \in [q^d, q^{d+1} - q^d - 1] \). Let \( k_d \) be the base \( q \) digit of \( q^d \) in \( k \). Observe that by our restriction on \( k \), it necessarily follows that \( 1 \leq k_d \leq q - 2 \). By our first lemma and the definition of the polynomials \( G_j \) we have

\[
\mu_{t,x}(k) = x^{-d} \sum_{a \in A_+(d)} G_{k-k_dq^d}(a) \chi_t(a).
\]
Now replacing $\chi_t$ by its Wagner series we see from the orthogonality relations that

$$\mu_{t,x}(k) = (-x^{-1})^db_\epsilon(t)$$

if $k = q^d + q^d - 1 - q^e$ for some $0 \leq e < d$ or $k = kdq^d + q^d - 1 - q^e$ for some $0 \leq e \leq d$ and $2 \leq k_d \leq q - 1$, and $\mu_{x,t}(k) = 0$ otherwise. This completes the proof of part (i) of the theorem.

Say $k = q^{d+1} - q^d - 1$. Then, using our lemmas,

$$\mu_{x,t}(k) = x^{-d} \sum_{a \in A_+(d)} G_{k-kdq^d}(a)\chi_t(a) + x^{-(d+1)} \sum_{a \in A_+(d+1)} G_k(a)\chi_t(a)$$

$$= ((-x^{-1})^d + (-x^{-1})^{d+1})b_d(t).$$

Now suppose $k \in (q^{d+1} - q^d - 1, q^{d+1})$, and observe that the digit of $q^d$ in $k$ when written in base $q$ necessarily equals $q - 1$. By our first lemma

$$\mu_{t,x}(k) = x^{-d} \sum_{a \in A_+(d)} G_{k-(q-1)q^d}(a)\chi_t(a) + x^{-(d+1)} \sum_{a \in A_+(d+1)} G_k(a)\chi_t(a).$$

By our orthogonality lemma, the first summand is non-zero precisely when $k = (q - 1)q^d + q^d - 1 - q^e$ for some $0 \leq e < d$ or when $k = q^{d+1} - 1$. In the first case the first summand equals $(-x^{-1})^db_\epsilon(t)$. For those $k = (q - 1)q^d + q^d - 1 - q^e$ for some $0 \leq e < d$, the second summand is also non-zero since then $k + q^e = q^{d+1} - 1$, and hence equals $(-x^{-1})^{d+1}b_\epsilon(t)$. Thus for such $k$,

$$\mu_{x,t}(k) = ((-x^{-1})^d + (-x^{-1})^{d+1})b_\epsilon(t).$$

This completes the proof of part (ii) of the theorem.

In the second case, when $k = q^{d+1} - 1$, the first summand equals $(-x^{-1})^db_d(t)$ and the second summand equals $(-x^{-1})^{d+1}b_{d+1}(t)$. This completes the proof of part (iii) of the theorem.

It remains to remark that for $k \in (q^{d+1} - q^d - 1, q^{d+1})$ the second summand is non-zero precisely when the first summand is non-zero. This completes the proof. \[\square\]
CHAPTER 4

EXPLICIT FORMULAE FOR PELLARIN’S SPECIAL

L-VALUES AT THE POSITIVE INTEGERS

In this chapter we calculate various explicit formulae for certain power sums arising in connection with the special values of Pellarin’s L-series at 1 as well as the explicit rational special values of Pellarin’s L-series when the number of indeterminates is at most \( q \).

Our premier result, Theorem 4.2.1, will follow most easily from the observation that the rational special values of Pellarin’s L-series in less than \( q \) indeterminates arise from generating series in a way that is totally analogous to Bernoulli polynomials classically. The following equality, which is due to the author and which holds for \( s < q \), will be made precise in Section 4.2.2,

\[
\prod_{i=1}^{s} \omega(t_i)^{-1} e_C(zL(\chi_{t_i}, 1)\omega(t_i)) = \sum_{a \in A} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z - a}.
\]

Recall that \( e_C \) is the \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear operator arising from the Carlitz exponential, and that by Pellarin’s formula (Theorem 1.5.9) \( e_C(zL(\chi_{t_i}, 1)\omega(t_i)) \) is the AGF for the Carlitz module. Also observe that because \( \chi_{t_i}(0) = 0 \), both sides of the equality above have power series expansions in \( z \) in a non-empty open disc about the origin. It is exactly by comparing coefficients of these power series expansions that one obtains the explicit formulae for the rational values of Pellarin’s series. We expect that there are deep connections with Taelman’s unit module [27] present in the above equality.
some of which are already under investigation by various authors. However it is beyond the scope of this dissertation to introduce them here.

4.1 Explicit formulae for power sums when $s < q, k = 1$

In this section, we shall employ the method of polynomial interpolation, as first exploited by Carlitz over $A$, to calculate the value at $k = 1$ of Pellarin’s $L$-series when the number of indeterminates $s$ is strictly less than $q$. More precisely, we shall calculate the sums

$$S_d(s, -1) := \sum_{a \in A_+^{(d)}} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a} \in K[t_1, \ldots, t_s].$$

Extending our method of proof to $s \geq q$ appears possible using the Multiplication Lemma (i.e. Lemma 3.4.12), but things become complicated quite fast. We do not pursue this here, and we leave the search for a better method to future works.

4.1.1 The general principle of interpolation and twisted power sums

Recall that the normalized linear Carlitz functions are defined for $d \geq 0$ by the equality $E_d(z) := e_d(z)/D_d \in K[z]$.

Lagrange interpolation shows that given any function function $f$ from $A_+(d)$ to an integral domain $B$ that is a $K$-algebra we may interpolate $f$ with the polynomial

$$M_d f(z) := \ell_d \sum_{a \in A_+^{(d)}} f(a) \frac{E_d(z - a)}{z - a} \in B[z].$$

Hence, if we know the coefficients of $M_d f$ then, following an idea of Carlitz, we may re-express the twisted power sums $\sum_{a \in A_+^{(d)}} a^{-1} f(a)$ in terms of these coefficients by merely evaluating $\ell_d^{-1} M_d f(z)$ at $z = 0$.

Similarly, since we know the coefficients of $E_d$ explicitly for all $d \geq 0$ and since $E_d(z - a) = E_d(z - \theta^d)$ for all $a \in A_+(d)$, we may explicitly calculate the sums...
\[
\sum_{a \in A_+} a^{-k} f(a) \text{ by determining the expansion of } \ell_a^{-1} M_a f(z)/E_d(z - \theta^d) \text{ about } z = 0. \text{ While this gives explicit formulas for these power sums they are usually too complicated to use outside of special cases.}
\]

4.1.2 Calculation of \( S_d(s, -1) \)

Recall the formal series
\[
\Xi_t(z) := \sum_{i \geq 0} b_i(t) E_i(z)
\]
defined in (1.6.1) and that by Theorem 1.6.3 this series agrees with \( \chi_t(a) \) for all \( a \in A \) upon the formal replacement \( z \mapsto a \). It follows that
\[
(M_d \chi_t)(z) = b_d(t) + \sum_{i=0}^{d-1} b_i(t) E_i(z).
\]

We now use the general principle of interpolation to calculate the sums \( S_d(s, -1) \).

We will be dealing with products of polynomials expressed in the linear Carlitz basis \( E_i \) with constant terms. Therefore, it is convenient to introduce some notation to ease writing. For \( d \geq 1 \) define
\[
E_j^d(z) = \begin{cases} 
E_j(z) & (0 \leq j \leq d - 1) \\
1 & (j = d) \\
0 & \text{otherwise}.
\end{cases}
\]

Example 4.1.1. For \( d \geq 1 \) we have the following equality of functions on \( A_+ \),
\[
(M_d \chi_t)(z) = \sum_{i=0}^{d} b_i(t) E_i^d(z).
\]

By abuse of notation, we shall often refer to the right side above as a polynomial.

Recall the \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear extension of the Carlitz logarithm
\[
\log = \sum_{j \geq 0} \ell_j^{-1} \tau^j,
\]

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and that $\tau^j \omega(t) = b_j(t) \omega(t)$ for all $j \geq 0$. One of the theoretical benefits of the following result is the connection it suggests between the special values of Pellarin’s series at $k = 1$ and the action of $\log$ on $\omega(t_1) \cdots \omega(t_s)$. Such a connection is currently being made precise by various authors. We shall discuss some immediate benefits after the statement and proof.

**Theorem 4.1.2.** Let $d \geq 1$ be an integer, and let $0 \leq s < q$. Then

$$S_d(s, -1) = \frac{b_d(t_1) \cdots b_d(t_s)}{\ell_d}.$$  

**Proof.** Theorem 1.6.3 shows that $\chi_{t_1}(z) \cdots \chi_{t_s}(z)$ agrees with the polynomial

$$\prod_{i=1}^{s} \left( \sum_{j_i=0}^{d} b_{j_i}(t_i) E_{d}^{j_i}(z) \right)$$

for $z \in A_+(d)$. The degree of this polynomial in $z$ is $s q^d - 1$, which by our assumption on $s$ is strictly less than $|A_+(d)|$. Hence,

$$\prod_{i=1}^{s} \left( \sum_{j_i=0}^{d} b_{j_i}(t_i) E_{d}^{j_i}(z) \right) = \sum_{a \in A_+(d)} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z - a}. \quad (4.1.1)$$

Evaluating at $z = 0$ proves the claim. \hfill $\square$

**Remark 4.1.3.** Again, as we described earlier, the equality (4.1.1) contains enough information to calculate $S_d(s, -k) := \sum_{a \in A_+(d)} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a) a^k}{\ell_d E_d(z - \theta^d)}$ for all $k \geq 1$. One expands both sides of the equality about the origin and compares coefficients. We do not pursue this here because the formulas we obtain in this way, while pretty, are difficult to use. However, see the second proof of Theorem 4.2.2 where we bypass these complications with a closely related idea.

**Application: An elementary proof of Pellarin’s formula for $L(\chi_t, 1)$**

We thank F. Pellarin for allowing us to include here his elementary proof for the formula $L(\chi_t, 1) = \frac{\pi}{(\theta - t) \omega(t)}$ that follows from the explicit formula for $S_d(1, -1)$. In the next section, we give an explicit formula for $L(\Pi^q \chi_t, 1)$ via the same route.
**Theorem 4.1.4 (Pellarin’s Formula).** The following equality holds in $\mathbb{T}$:

$$L(\chi_t, 1) = \frac{\tilde{\pi}}{(\theta - t)\omega(t)}.$$  

**Proof.** It suffices to prove this for $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$. Merely observe that by Theorem 4.1.2 the partial sum $\sum_{e=0}^{d} S_e(1, -1)$ telescopes to $\frac{b_{d+1}(t)}{(t - \theta)\ell_d}$. Now an easy algebraic manipulation makes plain that these partial sums do indeed limit to $\frac{\tilde{\pi}}{(\theta - t)\omega(t)}$ as $d \to \infty$. \hfill $\square$

**4.1.3 Example: Calculation of $S_d(q, -1)$**

As an example of extending the results of the last subsection, we indicate how to calculate $(M_d\chi_{t_1} \cdots \chi_{t_q})(z)$ for all $d$. A clean expression for $S_d(q, -1)$ will follow from this line of thought as will the rational function to which $L(\chi_{t_1} \cdots , \chi_{t_q}, 1)$ gives rise.

**Lemma 4.1.5.** We have the following identity of functions on $A_+(d)$ for all $j \geq 0$:

$$(\mathcal{E}^d_j)^q = [d + 1]E_{j+1}^d + \mathcal{E}_j^d. \quad (4.1.2)$$

**Proof.** We have seen that the equality claimed above holds for the functions $E_j$, i.e. the polynomial identity $E_j^q = [j + 1]E_{j+1}^d + E_j$ holds for all $j$. Thus what remains to be checked is only the equality when $j = d - 1$. In this case we have

$$(\mathcal{E}^d_{d-1})^q = E_{d-1}^q = [d]E_d + E_{d-1} = [d] + E_{d-1} = [d + 1]E_d^d + \mathcal{E}^d_{d-1}$$

as functions on $A_+(d)$. This proves the claim. \hfill $\square$

As a polynomial in $z$ over $K[t_1, \ldots, t_s]$, the polynomial $(M_d\chi_{t_1} \cdots \chi_{t_q})(z)$ is determined by its values on $A_+(d)$. Hence we could use the previous lemma and the interpolation polynomial for the product of characters in one less indeterminate to compute its Wagner expansion. We will not pursue this, but we will use this idea to compute the constant coefficient of $(M_d\chi_{t_1} \cdots \chi_{t_q})(z)$.  

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Let $B^s = \{(j_1, \ldots, j_s) : 0 \leq j_i \leq d \text{ for } i = 1, \ldots, s\}$. Let $D^s = \{(j_1, \ldots, j_s) \in B^s : j_1 = j_2 = \cdots = j_s\}$ and $C^s = B^s \setminus D^s$.

**Proposition 4.1.6.** For all $d \geq 1$ we have the following equality of functions on $A_+(d)$,

$$M_d \chi_{t_1} \cdots \chi_{t_q} = \sum_{j=0}^{d} ([j+1] \mathcal{E}_{j+1}^d + \mathcal{E}_j^d) \prod_{i=1}^{q} b_j(t_i) + \sum_{(j_i) \in C^q} \prod_{i=1}^{q} b_{j_i}(t_i) \mathcal{E}_{j_i}^d.$$  

**Remark 4.1.7.** Observe that $\mathcal{E}_{d+1}^d = 0$, and $\mathcal{E}_{d}^d = 1$ so that the expression on the right side is equal as a function on $A_+(d)$ to a polynomial of degree at most $q^d - 1$ and thus is the unique polynomial interpolating the product $\chi_{t_1} \cdots \chi_{t_q}$ on $A_+(d)$.

**Proof.** Simply multiply out $\prod_{i=1}^{q} \sum_{j_i=0}^{d} b_{j_i}(t_i) \mathcal{E}_{j_i}^d$ and use Lemma 4.1.5. \qed

**Corollary 4.1.8.** For all $d \geq 1$ we have

$$S_d(q, -1) = \ell_d^{-1} \left( [d] \prod_{i=1}^{q} b_{d-1}(t_i) + \prod_{i=1}^{q} b_d(t_i) \right).$$

**Proof.** Recall that

$$S_d(q, -1) = - \left. \frac{(M_d \chi_{t_1} \cdots \chi_{t_q})(z)}{\ell_d E_d(z - \theta^d)} \right|_{z=0}.$$ 

The constant term of $(M_d \chi_{t_1} \cdots \chi_{t_q})(z)$ comes from the coefficients of the functions $\mathcal{E}_{d}^d$ in Prop. 4.1.6. From the way we have split the sum above, our claim is evident. \qed

The next corollary follows by observing that the partial sums $\sum_{e=0}^{d} S_e(q, -1)$ telescope to a quantity whose limit we can understand. That is, the proof is exactly similar to Pellarin’s elementary proof of his formula for $L(\chi_t, 1)$ given earlier in this chapter.

**Corollary 4.1.9.** The following equality holds in $\mathbb{T}_q$,

$$L(\Pi^q \chi_{t_i}, 1) = \tilde{\pi}/\prod_{i=0}^{q} \omega(t_i).$$
Proof. It suffices to check this for \( t_i \in \mathbb{C}_\infty \) such that \(|t_i| \leq 1\) for \( i = 1, \ldots, q \). For such \( t_i \) we have,

\[
L(\Pi^q \chi_{t_i}, 1) = \lim_{d \to \infty} \sum_{e=0}^{d} S_e(q, -1)
\]

\[
= \lim_{d \to \infty} \ell_d^{-1} \prod_{i=1}^{q} b_d(t_i)
\]

\[
= \lim_{d \to \infty} \prod_{j=1}^{d} \left( 1 - \theta/\theta^j \right)^{-1}
\]

\[
= \frac{\pi}{\prod_{i=1}^{q} \omega(t_i)}
\]

\[\square\]

Remark 4.1.10. The author first obtained the corollary above in [24] via a strict analysis of the remainder term appearing in the proof of Theorem 4.2.2. For this approach see Remark 4.2.7.

A different point of view for the calculation 4.1.6

We have seen that the interpolation polynomial

\[
M_d(\chi_t)(z) = b_d(t) + \sum_{j=0}^{d-1} b_j(t) E_j(z)
\]

agrees as a function of \( z \) with \( \chi_t \) on \( A_+(d) \). Thus the product \( \prod_{i=1}^{q} M_d(\chi_{t_i})(z) \) agrees with the product character \( \chi_{t_1} \cdots \chi_{t_q} \) on \( A_+(d) \). However, the degree in \( z \) of this polynomial is too large (it is exactly \( q^d \)) for it to be \( M_d(\chi_{t_1} \cdots \chi_{t_q}) \). The main point is that powers of \( z \) which are too large are coming from the function \( E_{d-1}(z)^q \) obtained after multiplying the above product out. Happily, via the relation \( E_{d-1}(z)^q = [d] E_d(z) + E_{d-1}(z) \) and the fact that \( E_d(z) \) is identically equal to the constant function
1 on $A_+(d)$, we see exactly how to remove the powers of $z$ that are too large, while still interpolating the product of characters. Indeed,

$$\prod_{i=1}^{q} M_d(\chi_{t_i})(z) - (E_d(z) - 1)[d] \prod_{i=1}^{q} b_d-1(t_i)$$

clearly agrees with the product character on $A_+(d)$, and one easily checks that this difference is a polynomial in $z$ of degree strictly less than $|A_+(d)| = q^d$.

The $E^d$ notation is a slick way of disguising all of this, but we thought it would be helpful to the reader to include these remarks. The constant term is also clear from the representation given above.

4.2 Explicit formulae when $s \leq q$ and $k \equiv s \mod (q - 1)$

The main theorem of this section is the explicit calculation of the rational special values of Pellarin’s $L$-series when the number of indeterminates $s$ is at most $q$. Again we use the Wagner series for $\chi_{t_i}$, and we give two distinct calculations - one by “unfolding” Pellarin’s series just from its definition and another using the Anderson generating function. We believe that our explicit results helped inspire the major work of Anglès and Pellarin [3] mentioned above.

**Theorem 4.2.1.** Let $k, s$ be positive integers such that $1 \leq s \leq q$, such that $k \geq s$, and such that $k \equiv s \mod (q - 1)$. The following equality holds in $T_s$,

$$\tilde{\pi}^{-k} L(\Pi^s \chi_{t_i}, k) \prod_{i=1}^{s} \omega(t_i) = \sum_{k_1, \ldots, k_s} \left[ BC(k - \sum_{i=1}^{s} q^{k_i}) \prod_{i=1}^{s} \frac{1}{D_{k_i} \cdot (\theta q^{k_i} - t_i)} \right].$$

Here the sum is over all $s$-tuples of non-negative integers $(k_1, \ldots, k_s)$ such that $k - \sum_{i=1}^{s} q^{k_i} \geq 0$.

We shall prove the following theorem from which the previous theorem immediately follows using the Carlitz-Euler relations (1.1.5) and Pellarin’s formula for $L(\chi_{t}, 1)$, Theorem 4.1.4.
Theorem 4.2.2. Let $s, k$ be positive integers such that $1 \leq s \leq q$, such that $k \geq s$ and such that $k \equiv s \mod (q - 1)$. The following formula holds in $\mathbb{T}_s$,

$$L(\chi_{t_1} \cdots \chi_{t_s}, k) = \sum_{k_1, \ldots, k_s} \left[ \zeta \left( k - \sum_{i=1}^{s} q^{k_i} \right) \prod_{i=1}^{s} \frac{b_{k_i}(t_i)}{D_{k_i}} L(\chi_{t_i}, q^{k_i}) \right], \quad (4.2.1)$$

where the sum is over all combinations of $s$ non-negative integers $k_1, \ldots, k_s$ such that $k - \sum_{i=1}^{s} q^{k_i} \geq 0$.

4.2.1 A first proof of Theorem 4.2.2 by “unfolding”

We copy here word for word the proof given in [24]. While not as easy as the proof involving generating series appearing in the next section, this proof has interest in its own right as it uses such classical results as the vanishing of the Goss zeta function at the negative “even” (divisible by $q - 1$) integers and the logarithmic growth of the special polynomials associated to Goss’ zeta function. This proof also gives the case of $s = q$ indeterminates.

The proof proceeds by grouping the sum over $A_+$ in Pellarin’s series according to degree and for each $a \in A_+(d)$ one replaces each instance of $\chi_i(a)$ by its polynomial interpolation on $A_+(d)$. One can then justify an interchange of sums which results in the desired explicit formulae.

Proof by “unfolding”. Let $t_i \in \mathbb{C}_\infty$ be such that $|t_i| \leq 1$ for $i = 1, \ldots, s$. We break the proof into several lemmas for the convenience of the reader and to highlight how the various conditions on $s$ and $k$ arise. The first lemma uses in a crucial way the well known vanishing of the power sums

$$\sum_{a \in A_+(d)} a^k$$

for positive integers $k, d$ whenever $k \leq q^d - 1$, see Corollary 1.2.5. Its proof gives rise to the condition on $s$ appearing in the main result.
Lemma 4.2.3 (Condition on $s$). Let $1 \leq s \leq q$ and $k \geq s$, then $L(\chi_{t_1}, \ldots, \chi_{t_s}, k)$ equals
\[
\sum_{i_1=0}^{\infty} \cdots \sum_{i_s=0}^{\infty} \sum_{k_s=0}^{i_s} \sum_{k_1=0}^{i_1} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{i_j}} \right) \zeta(k - (q^{k_1} + \cdots + q^{k_s})). \tag{4.2.2}
\]

Proof. We begin by replacing $\chi_{t_i}$ by its Wagner representation and interchanging sums. The equation
\[
\sum_{d=0}^{e} \sum_{a \in A_+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \tag{4.2.3}
\]
becomes
\[
\sum_{d=0}^{e} \sum_{i_1=0}^{d} \cdots \sum_{i_s=0}^{d} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)}{D_{i_j}} \right) \sum_{a \in A_+(d)} \left( a^{-k} \prod_{l=1}^{s} \alpha_{i_l}(a) \right). \tag{4.2.4}
\]
Next we wish to interchange the sum indexed by the variable $d$ with the sums indexed by the variables $i_1, \ldots, i_s$. It is convenient to observe that due to the vanishing of the polynomials $e_i(b)$ whenever $b \in A(i)$ (i.e. whenever $i > \deg_{e_i}(b)$) we may first increase the upper limits of the sums indexed by $i_1, \ldots, i_s$ from $d$ to $e$ without changing anything. Then equation (4.2.4) becomes
\[
\sum_{i_1=0}^{e} \cdots \sum_{i_s=0}^{e} \prod_{j=1}^{s} \frac{b_{i_j}(t_j)}{D_{i_j}} \sum_{d=0}^{e} \sum_{a \in A_+(d)} \left( a^{-k} \prod_{l=1}^{s} \alpha_{i_l}(a) \right). \tag{4.2.5}
\]
Next we replace the polynomials $e_i(z)$ with their series representations
\[
e_{i_j}(z) = \sum_{k_j=0}^{i_j} \alpha_{i_j,k_j} z^{k_j}
\]
and interchange sums. Equation (4.2.5) becomes
\[
\sum_{i_1=0}^{e} \cdots \sum_{i_s=0}^{e} \sum_{k_s=0}^{i_s} \sum_{k_1=0}^{i_1} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{i_j}} \right) \sum_{d=0}^{e} \sum_{a \in A_+(d)} a^{q^{k_1} + \cdots + q^{k_s} - k}. \tag{4.2.6}
\]
Next we insert and remove the tail of Goss zeta special values
\[
\sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \cdots + q^{k_s} - k}
\]
and (4.2.6) becomes
\[ \sum_{i_1=0}^{e} \ldots \sum_{i_s=0}^{e} \sum_{k_1=0}^{i_1} \ldots \sum_{k_s=0}^{i_s} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{ij}} \right) \zeta(k - q^{k_1} + \ldots + q^{k_s}) - (4.2.7) \]
\[ \sum_{i_1=0}^{e} \ldots \sum_{i_s=0}^{e} \sum_{k_1=0}^{i_1} \ldots \sum_{k_s=0}^{i_s} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{ij}} \right) \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \ldots + q^{k_s} - k}. \quad (4.2.8) \]

Now we wish to show that the remainder term (4.2.8) vanishes as \( e \to \infty \). In order to do this, we use the well known fact (see Lemma 1.2.5) that the power sums \( \sum_{a \in A_+(d')} a^{k'} \) vanish for non-negative integers \( k' \) such that \( k' < q^{d'} - 1 \) and \( d' \geq 1 \).

We observe that the maximum power of \( a \) which can occur in the sum above is \( sq^e - k \), and this is less than \( q^{e+1} - 1 \) since \( 1 \leq s \leq q \) and \( k \geq s \). Hence for all possible exponents \( d \) of \( a \) occurring in the sum over \( A_+(d) \) in equation (4.2.8) we have \( \sum_{i=1}^{s} q^{k_i} - k < q^{d'} - 1 \). Hence for all \( e > \log_q(k - s + 1) \) the sums indexed by \( k_1, \ldots, k_s \) are over \( s \)-tuples \( (k_1, \ldots, k_s) \) of non-negative integers such that \( 0 \leq k_j \leq i_j \) for \( j = 1, \ldots, s \) and \( \sum_{i=1}^{s} q^{k_i} - k < 0 \). We indicate this with a prime in the summation over \( k_1, \ldots, k_s \). Explicitly, we will write (4.2.8) as
\[ \sum_{i_1=0}^{e} \ldots \sum_{i_s=0}^{e} \sum_{k_1=0}^{i_1} \ldots \sum_{k_s=0}^{i_s} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{ij}} \right) \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \ldots + q^{k_s} - k}, \quad (4.2.9) \]
where the prime indicates only \( s \)-tuples \( (k_1, \ldots, k_s) \) such that \( 0 \leq k_j \leq i_j \) for \( j = 1, \ldots, s \) and \( \sum_{i=1}^{s} q^{k_i} - k < 0 \) appear, and from now on we suppose \( e > \log_q(k + 1 - s) \).

Now we estimate.
\[ \left| \sum_{i_1=0}^{e} \ldots \sum_{i_s=0}^{e} \sum_{k_1=0}^{i_1} \ldots \sum_{k_s=0}^{i_s} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{ij}} \right) \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \ldots + q^{k_s} - k} \right| \leq \]
\[ \sum_{i_1=0}^{e} \ldots \sum_{i_s=0}^{e} \sum_{k_1=0}^{i_1} \ldots \sum_{k_s=0}^{i_s} \left| \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{ij}} \right| \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \ldots + q^{k_s} - k} \mid \leq \]
\[ \sum_{i_1=0}^{e} q^{-q^{i_1}} \ldots \sum_{i_s=0}^{e} q^{-q^{i_s}} \sum_{k_1=0}^{s} \ldots \sum_{k_s=0}^{s} q^{-k_j \sum_{j=1}^{r} q^{k_j} + 2 \sum_{j=1}^{r} q^{k_j} - q - 1} \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \ldots + q^{k_s} - k} d = \]
\[ \sum_{i_1=0}^{e} q^{-q^{i_1}} \ldots \sum_{i_s=0}^{e} q^{-q^{i_s}} \sum_{k_1=0}^{s} \ldots \sum_{k_s=0}^{s} q^{-k_j \sum_{j=1}^{r} q^{k_j} + 2 \sum_{j=1}^{r} q^{k_j} - q - 1} \frac{q^{(q^{k_1} + \ldots + q^{k_s} - k)(e+1)} - 1}{q^{q^{k_1} + \ldots + q^{k_s} - k}}. \]
Now we use the upper bound \( q^{(q^{k_1} + \cdots + q^{k_s} - k)(e+1)} \leq q^{-(e+1)} \), which holds because, as we have noted, \( \sum_{i=1}^{s} q^{k_i} - k < 0 \) (and in particular is at most \(-1\) since this quantity is an integer) for all indices \( k_i \) appearing in the primed summation. Thus the last line above is less than or equal to

\[
q^{-(e+1)} \sum_{i_1=0}^{e} q^{-q^{i_1}} \cdots \sum_{i_s=0}^{e} q^{-q^{i_s}} \sum_{k_1, \ldots, k_s} \sum_{j=1}^{s} q^{-k_j q^{k_j} + \frac{q^{k_j+1}}{q-1}} \frac{1}{q^{k_1} + \cdots + q^{k_s} - k},
\]

(4.2.10)

Now the sum indexed by \( k_1, \ldots, k_s \) is finite and the upper limit depends only on \( k \) and \( i_1, \ldots, i_s \) but not on \( e \) as described above. We remove the dependence on \( i_1, \ldots, i_s \) by dropping the condition that \( k_j \leq i_j \) for \( j = 1, \ldots, s \). This has the effect of adding more positive terms to the sum indexed by \( k_1, \ldots, k_s \). Thus equation (4.2.10) is maximized by

\[
q^{-(e+1)} \left( \sum_{i_1=0}^{e} q^{-q^{i_1}} \cdots \sum_{i_s=0}^{e} q^{-q^{i_s}} \sum_{k_1, \ldots, k_s} \sum_{j=1}^{s} q^{-k_j q^{k_j} + \frac{q^{k_j+1}}{q-1}} \right),
\]

(4.2.11)

where the double prime signifies summing over all \( s \)-tuples \((k_1, \ldots, k_s)\) such that \( \sum_{i=1}^{s} q^{k_i} - k < 0 \). The sums indexed by \( i_1, \ldots, i_s \) are convergent as \( e \to \infty \), the sum indexed by \( k_1, \ldots, k_s \) is a constant which does not depend on \( e \), and \( q^{-(e+1)} \to 0 \) as \( e \to \infty \). Thus the error term in (4.2.8) vanishes as \( e \to \infty \), and we conclude that

\[
L(\chi_{t_1}, \ldots, \chi_{t_s}, k) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_s=0}^{\infty} \sum_{k_1=0}^{i_1} \cdots \sum_{k_s=0}^{i_s} \left( \prod_{j=1}^{s} \frac{b_{i_j}(t_j)\alpha_{i_j,k_j}}{D_{i_j}} \right) \zeta(k - (q^{k_1} + \cdots + q^{k_s})).
\]

(4.2.12)

The second lemma uses the vanishing of Goss zeta special values for negative even integers (Lemma 1.2.8), i.e. those negative integers divisible by \( q - 1 \), in a crucial way that is totally analogous to how the vanishing of the power sums was used above. This lemma gives rise to the congruence condition on \( k \) appearing in the statement of the main theorem.
Lemma 4.2.4 (Condition on \( k \)). Let \( s, k \) be as in the statement of Theorem 4.2.2, then \( \mathcal{L}(\chi_{t_1}, \ldots, \chi_{t_s}, k) \) equals

\[
\sum_{k_1, \ldots, k_s} \zeta(k - \sum_{i=1}^s q^{k_i}) \left( \prod_{i_1=k_1}^\infty \frac{b_{i_1}(t_1)\alpha_{i_1,k_1}}{D_{i_1}} \right) \cdots \left( \prod_{i_s=k_s}^\infty \frac{b_{i_s}(t_s)\alpha_{i_s,k_s}}{D_{i_s}} \right),
\]

where the sum over \( k_1, \ldots, k_s \) includes all \( s \)-tuples of non-negative integers such that \( k - \sum_{i=1}^s q^{k_i} \geq 0 \).

Proof. We begin with equation (4.2.12), and we observe that by assumption, \( k \equiv s \mod (q - 1) \); hence \( k - (q^{k_1} + \cdots + q^{k_s}) \equiv 0 \mod (q - 1) \). Thus \( \zeta(k - (q^{k_1} + \cdots + q^{k_s})) \) vanishes as soon as \( k - (q^{k_1} + \cdots + q^{k_s}) < 0 \). This limits the upper bound on the sum indexed by \( k_1, \ldots, k_s \) and allows for arguments entirely analogous to those above to show that we may interchange sums.

We conclude the proof with a little algebraic lemma.

Lemma 4.2.5. Let \( j \) be a non-negative integer. We have

\[
\sum_{i=j}^\infty \frac{b_i(t)}{D_i} \alpha_{i,j} = \frac{b_j(t)}{D_j} \mathcal{L}(\chi_t,q^j).
\]

Proof.

\[
\sum_{i=j}^\infty \frac{b_i(t)}{D_i} \alpha_{i,j} = \sum_{i=j}^\infty \frac{b_j(t)\tau^j b_{i-j}(t) (-1)^{i-j} D_i}{D_j L_i^q} \frac{D_j L_i^q}{i-j}
\]

\[
= \frac{b_j(t)}{D_j} \sum_{i=j}^\infty (-1)^{i-j} \tau^j \frac{b_{i-j}(t)}{L_{i-j}}
\]

\[
= \frac{b_j(t)}{D_j} \tau^j \sum_{i=j}^\infty (-1)^{i-j} \frac{b_{i-j}(t)}{L_{i-j}}
\]

\[
= \frac{b_j(t)}{D_j} \tau^j \sum_{i=0}^\infty (-1)^i \frac{b_i(t)}{L_i}
\]

\[
= \frac{b_j(t)}{D_j} \tau^j \mathcal{L}(\chi_t,1)
\]

In the last two lines we use Theorem 4.1.2.

Making the substitution in (4.2.13) finishes the proof of the theorem.
4.2.2 A second proof of Theorem 4.2.2 when $s < q$

In this section we give a proof of Theorem 4.2.2 in the special case where $s < q$ which emphasizes the main theme of this dissertation: The Wagner series for $\chi_t$ is the Anderson Generating Function. This point of view allows us to give generating series for the rational special values of Pellarin’s series and demonstrates a similarity of these rational special values to Bernoulli polynomials classically. See also the remarks in the introduction of this chapter.

**Theorem 4.2.6.** Let $1 \leq s < q$. Let $z \in \mathbb{C}_\infty$ be such that $|z| < 1$. We have the following equality in $T_s$,

$$
\tilde{\pi}^{-1} e_C(\tilde{\pi}z) \sum_{a \in A} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z-a} = \prod_{i=1}^{s} \frac{\omega(t_i, \tilde{\pi}z)}{\omega(t_i)}.
$$

**Proof.** Let $t_i \in \mathbb{C}_\infty$ be such that $|t_i| \leq 1$ for $i = 1, \ldots, s$. We begin with the following equality in $K[t_1, \ldots, t_s, z]$,

$$
\prod_{i=1}^{s} \sum_{j_i=0}^{d-1} b_{j_i}(t_i) E_{j_i}(z) = \sum_{a \in A(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \frac{\ell_d E_d(z)}{z-a},
$$

which holds since both are polynomials in $z$ whose degrees are strictly less than $q^d$ by our assumption on $s$, and both interpolate $\chi_{t_1} \cdots \chi_{t_s}$ for $z \in A(d)$. Now [16] Theorem 3.2.8 shows that $\ell_d E_d(z) \to \tilde{\pi}^{-1} e_C(\tilde{\pi}z)$ as $d \to \infty$, for all $z \in \mathbb{C}_\infty$. Further, for $|z| < 1$ (even $z \in \mathbb{C}_\infty \setminus A$) and $|t_i| \leq 1$ the limit $\lim_{d \to \infty} \sum_{a \in A(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a)/(z-a)$ exists. Thus for $z$ and $t_i$ as above, we have

$$
\lim_{d \to \infty} \ell_d E_d(z) \sum_{a \in A(d)} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z-a} = \tilde{\pi}^{-1} e_C(\tilde{\pi}z) \sum_{a \in A} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z-a}.
$$

Now by Theorem 1.6.6 letting $d$ tend to infinity on the left side of (4.2.14) finishes the proof. \qed

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Proof of Theorem 4.2.2 when \( s < q \). We mentioned in the introduction that

\[
\tilde{\pi}/e_C(\tilde{\pi}z) = \sum_{a \in A} \frac{1}{z - a} = 1/z + \sum_{k \geq 0 \atop k+1 \equiv 0 \mod (q-1)} \zeta(k+1)z^k.
\]

In a similar way we have

\[
\sum_{a \in A} \frac{\chi_t(a) \cdots \chi_t(a)}{z - a} = \sum_{k \geq 0 \atop k+1 \equiv s \mod (q-1)} L(\Pi^s \chi_t, k + 1)z^k.
\]

Now by definition \( \omega(t, \tilde{\pi}z) = e_C(z\tilde{\pi}(\theta - t)^{-1}) \), and using Pellarin’s formula we may write \( \tilde{\pi}(\theta - t)^{-1} = L(\chi_t, 1)\omega(t) \). Thus

\[
\omega(t, \tilde{\pi}z) = \sum_{i \geq 0} D_i^{-1} \tau^i(zL(\chi_t, 1)\omega(t)) = \sum_{i \geq 0} D_i^{-1}L(\chi_t, q^i)b_i(t)\omega(t)z^{q^i}.
\]

Hence Theorem 4.2.6 shows that

\[
\sum_{k \geq 0 \atop k+1 \equiv s \mod (q-1)} L(\Pi^s \chi_t, k + 1)z^k
\]

equals

\[
\left(1/z + \sum_{k \geq 0 \atop k+1 \equiv 0 \mod (q-1)} \zeta(k+1)z^k\right) \prod_{i=1}^{s} \left(\omega(t_i)^{-1} \sum_{j_i \geq 0} D_{j_i}^{-1}L(\chi_{t_i}, q^{j_i})b_{j_i}(t_i)\omega(t_i)z^{q^{j_i}}\right).
\]

Multiplying everything out as power series in \( z \) and comparing the coefficients proves Theorem 4.2.2.

\[\square\]

Remark 4.2.7. In the case \( s = q \) we have

\[
\sum_{a \in A} \frac{\Pi_i^q \chi_t(a)}{z - a} = \frac{\tilde{\pi}}{e_C(\tilde{\pi}z)} + \frac{\tilde{\pi}}{e_C(\tilde{\pi}z)} \prod_{i=1}^{s} \omega(t_i) \frac{\omega(t_i, \tilde{\pi}z)}{\omega(t_i)}.
\]

For the coefficients of the non-zero powers of \( z \), the proof is as above working backwards from Theorem 4.2.2. The constant term \( \tilde{\pi}/\prod_{i=1}^{s} \omega(t_i) \) may be obtained in two
ways. We have already calculated that \( L(\Pi^q \chi_{t_i}, 1) = \tilde{\pi}/\prod_{i=1}^{s} \omega(t_i) \) in this chapter. We may also obtain this from closer examination of the error term in (4.2.8) above. Indeed, the proof of Theorem 4.2.2 goes through in exactly the same way in the case \( s = q \) and \( k = 1 \) except that one has a single non-zero term arising from the remainder, namely

\[
\left( \prod_{j=1}^{q} \frac{b_j(t_j)}{D_e} \right) \sum_{a \in A_{q+1}} a^{q^{e+1}-1}.
\]

Carlitz showed that \( \sum_{a \in A_{q+1}} a^{q^{e+1}-1} = D_{e+1}/\ell_{e+1} \), and we have the recursive formula \( D_e^q = D_{e+1}/(\theta^{q^{e+1}} - \theta) \). Inputting these ingredients and taking the limit as \( e \to \infty \) we obtain \( L(\Pi^q \chi_{t_i}, 1) = \tilde{\pi}/\prod_{i=1}^{s} \omega(t_i) \).

4.3 Applications

4.3.1 Numerators of the Bernoulli-Carlitz numbers

In this section we let \( \lambda \in \mathbb{F}_q \) and \( s = q - 1 \) in Theorem 4.2.1 so that \( \chi^s_{\lambda} \) becomes the trivial character with conductor \( \theta - \lambda \) from \( A \) to \( \mathbb{F}_q \). Doing so allows us to obtain new recurrence relations between the Bernoulli-Carlitz numbers and from that new divisibility results for the numerators of the Bernoulli-Carlitz numbers.

A lemma on the function \( \omega \)

The following lemma, discovered by the author, has its own interest and will be used in the sequel. The reader should compare with similar results that hold for products of Carlitz torsion [25], and products of Thakur’s Gauss sums. See [3] for general results connecting the specializations of the function \( \omega \) at roots of unity with Thakur’s Gauss sums.
Lemma 4.3.1. Let $\overline{\mathbb{F}}_q \subseteq \mathbb{C}_\infty$ be the algebraic closure of $\mathbb{F}_q$. Let $\lambda \in \overline{\mathbb{F}}_q$, and let $f \in A$ be its minimal polynomial. Suppose $f$ has degree $d$. Then

$$
\prod_{i=1}^d \omega(\tau^i \lambda)^{q-1} = (-1)^d f.
$$

Proof. With the notations above we have $f = \prod_{i=1}^d (\theta - \tau^i \lambda)$. It is a basic property of limits that if $a_n \to a$ and $b_n \to b$, then $a_nb_n \to ab$. With this and the continuity of the map $x \mapsto x^{q-1}$ we have

$$
\prod_{i=1}^d \omega(\tau^i \lambda)^{q-1} = \prod_{i=1}^d \left( \lim_{n \to \infty} \prod_{j=0}^n \left( 1 - \frac{\tau^i \lambda}{\theta^q} \right) \right)^{q-1}
$$

$$
= \lim_{n \to \infty} \prod_{i=1}^d \prod_{j=0}^n \left( 1 - \frac{\tau^i \lambda}{\theta^q} \right)^{(q-1)}
$$

$$
= (-\theta)^d \lim_{n \to \infty} \prod_{j=0}^n \prod_{i=1}^d \left( 1 - \frac{\tau^i \lambda}{\theta^q} \right)^{(q-1)}
$$

$$
= (-\theta)^d \lim_{n \to \infty} \prod_{j=0}^n (f(\theta)/\theta^d)^{-q^i(q-1)}
$$

$$
= (-\theta)^d \lim_{n \to \infty} (f/\theta^d)^{1-q^n+1}
$$

$$
= (-1)^d f.
$$

The last equality follows since $f/\theta^d = 1 + g$, where $g \in K$ is such that $|g| < 1$, and hence

$$
\lim_{n \to \infty} (f/\theta^d)^{-q^n+1} = \lim_{n \to \infty} (1 + g)^{-q^n+1} = \lim_{n \to \infty} (1 + g^{q^n+1})^{-1} = 1.
$$

\[ \Box \]

New recursive relations for the Bernoulli-Carlitz numbers

Now we proceed toward the new recursion relations for the Bernoulli-Carlitz numbers.

From the definition of the Bernoulli-Carlitz numbers, by comparing coefficients in the formula

$$
z = e_C(z) \sum_{j=0}^{\infty} \frac{BC(j)}{\Pi(j)} z^j
$$

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one may obtain the recurrence relations \( BC(0) = 1 \) and for \( k \geq 1 \)

\[
BC(k) = - \sum_{q \leq q' \leq k+1} \frac{\Pi(k)}{\Pi(q')\Pi(k+1-q')} BC(k+1-q') \cdot \Pi(k) \Pi(q-j) \Pi(k+1-q-j) \cdot BC(k+1-q-j).
\]

For non-negative integers \( k, j \) let us write \( \binom{k}{j} = \Pi(k) \Pi(k-j) \Pi(j) \), and observe that \( \binom{k}{j} \cdot A \) for all \( j, k \geq 0 \), see e.g. [16] Section 9.1. We now give a new recurrence relation satisfied by the \( BC \)s that will be used in Theorem 4.3.5 to give new divisibility results for the numerators of the Bernoulli-Carlitz numbers.

**Lemma 4.3.2** (Second Recursive Formula). We have \( BC(0) = 1 \), and for any positive integer \( k \equiv 0 \mod (q - 1) \) and any \( \lambda \in \mathbb{F}_q \),

\[
BC(k) = \sum_{k_1, \ldots, k_{q-1}} \left( k - \sum_{i=1}^{q-1} q^{k_i} \right) \frac{BC \left( k - \sum_{i=1}^{q-1} q^{k_i} \right) \cdot (\theta - \lambda)^{k - \sum_{i=1}^{q-1} q^{k_i}}}{(\theta - \lambda)(1 - (\theta - \lambda)^k)},
\]

where the sum is over all \( (q - 1) \)-tuples of non-negative integers \( (k_1, \ldots, k_{q-1}) \) such that \( k - \sum_{i=1}^{q-1} q^{k_i} \geq 0 \).

**Proof.** This is an elementary algebraic calculation made using Theorem 4.2.1, the fact that since \( \chi_{\lambda}^{q-1} \) is the trivial character we have

\[
L(\chi_{\lambda}^{q-1}, k) = \left( 1 - \frac{1}{(\theta - \lambda)^k} \right) \zeta(k),
\]

and Lemma 4.3.1. \( \square \)

**Remark 4.3.3.** This same trick can be played to obtain similar recursive relations for \( L(\chi_{t}, k) \) using our explicit formula for \( L(\Pi^q \chi_{t_i}, k) \). The details are left to the reader.

**Divisibility of the numerators by degree one primes**

**Definition 4.3.4.** Let \( k \equiv 0 \mod (q - 1) \) be a positive integer, and let \( X \) be the set of all \( (q - 1) \)-tuples of non-negative integers such that \( k - \sum_{i=1}^{q-1} q^{k_i} \geq 0 \). Let \( (k_i^*) \in X \)
be such that \( \sum_{i=1}^{q-1} q^{k_i} \geq \sum_{i=1}^{q-1} q^{k_i} \) for all \((k_i) \in X\). We call such a tuple \((k_i^*)\) maximal in \(X\).

**Theorem 4.3.5.** Let \(k, X\) and \((k_i^*)\) be as in the definition above, and let \(\mu^* := k - \sum_{i=1}^{q-1} q^{k_i^*}\). Suppose that \(\mu^* > 2\), then \((\theta q - \theta)\mu^*-2\) divides the numerator of \(BC(k)\).

**Proof.** Using Lemma 4.3.2 we see that \((1 - (\theta - \lambda)^k)BC(k)\) equals

\[
\sum_{k_1, \ldots, k_{q-1}} \left( k - \sum_{i=1}^{q-1} q^{k_i} \right)_A \cdot BC \left( k - \sum_{i=1}^{q-1} q^{k_i} \right) \cdot (\theta - \lambda)^{k-1-\sum_{i=1}^{q-1} q^{k_i}}.
\]

Now \(\mu^* - 1\) is minimal among all the differences \(k - 1 - \sum_{i=1}^{q-1} q^{k_i}\) appearing in the exponent of \((\theta - \lambda)\) on the right above. It follows by Carlitz von-Staudt theorem, see [16], Theorem 9.2.2, that the zeros (in \(\mathbb{C}_\infty\)) of the denominators of the non-zero Bernoulli-Carlitz numbers are simple. Taking account of the possibility that \((\theta - \lambda)\) divides the denominator of some \(BC(j)\) on the right we conclude that \((\theta - \lambda)^{\mu^*-2}\) divides the right side above and hence also divides \(BC(k)\). As \(\lambda \in \mathbb{F}_q\) was arbitrary, this finishes the proof.

**Remark 4.3.6.** 1. Integers \(k\) as in the theorem above are easy to obtain. For example let \(d \geq 2\) and suppose \(k = (q-1)q^d + (q-1)q^{d-1}\). Then \(\mu^* = (q-1)q^{d-1} > 2\), and \((\theta q - \theta)(q-1)q^{d-1-2}\) divides the numerator of \(BC(k)\). In fact, as mentioned in the abstract, it is not difficult to show that as a subset of the positive integers such that \(k \equiv 0 \mod (q-1)\) those with \(\mu^*(k) > 2\) form a set of full natural density. This leads to the question of interpolating the Bernoulli-Carlitz numbers directly, and this question remains interesting and open for general primes. There are no known Kummeresque congruences for these numbers. We hope that the theorem above may give some insight on a path to take toward the question of interpolation in general, and we plan to return to this problem in a future work.
2. Finally, it is reasonable to expect that a recursion similar to that in Lemma 4.3.2 holds for \( j(q-1) \) tuples of non-negative integers for all \( j \geq 1 \) arising in a similar way from a closed form formula for Pellarin’s series in \( j(q-1) \) indeterminates. Perhaps finding these formulas will allow us to obtain more information on the numerators of the \( BC(k) \) for all \( k \geq 1 \). Such information should have interesting consequences for L. Taelman’s new class module.

### 4.3.2 Recursive Relations for Pellarin’s L-Series

Now that we have obtained generating series for Pellarin’s \( L \)-series from Theorem 4.2.6, we may use them to obtain recursive relations among the rational functions occurring in Pellarin’s [22] Theorem 2. For simplicity we will stick to the case of one indeterminate.

**Definition 4.3.7.** Let \( k \equiv 1 \mod (q-1) \). Define

\[
\gamma_k(t) := \bar{\pi}^{-k} L(\chi_t, k) \omega(t).
\]

Recall, we have shown that \( \gamma_k(t) \in K(t) \) for all \( k \) as above. Further, using the \( \tau \) action one observes immediately that

\[
\tau^k(\gamma_1(t)) = \bar{\pi}^{-q^k} L(\chi_t, q^k) b_k(t) \omega(t) = b_k(t) \gamma_{q^k}(t).
\]

Hence, using Pellarin’s Formula \( \gamma_1(t) = (\theta - t)^{-1} \), we obtain the following theorem.

**Theorem 4.3.8.** Let \( k \equiv 1 \mod (q-1) \). If \( k = q^j \) for some \( j \), then

\[
\gamma_k(t) = \frac{-1}{b_{j+1}(t)}.
\]

Otherwise,

\[
\gamma_k(t) = -\sum_{j=1}^{[\log_q(k)]} \frac{\gamma_{k+1-q^j}(t)}{D_j}.
\]
Proof. We handled the case when $k = q^j$ for some $j$ above. When $k \neq q^j$ for any $j$ we must appeal to the generating series relation. From Theorem 4.2.6 we have

$$\sum_{k \geq 1 \mod (q-1)} L(\chi_t, k) z^k = \frac{\sum_{\pi z \omega(t, \pi z)} e_C(\pi z) \omega(t)}{e_C(\pi z) \omega(t)}.$$ 

We clear denominators and change coordinates to obtain

$$e_C(z) \cdot \sum_{k \geq 1 \mod (q-1)} \gamma_k(t) z^k = z \omega(t, z).$$

Explicitly,

$$\sum_{l \geq 1 \mod (q-1)} \frac{l^{[\log_q(l-1)]}}{D_j} \gamma_{l-q^j}(t) z^l = \sum_{j \geq 0} \frac{z^{q^j+1}}{D_j \cdot (\theta^{q^j} - t)}.$$ 

Suppose $l \neq q^j + 1$ for any $j$, then the coefficient of $z^l$ on the left vanishes, and hence

$$\gamma_{l-1}(t) = - \sum_{j=1}^{[\log_q(l-1)]} \frac{\gamma_{l-q^j}(t)}{D_j}.$$ 

This finishes the proof. \qed

Example 4.3.9. As an example, on the next page we list the first ten of these $\gamma_j$ in the case $A = \mathbb{F}_3[\theta]$. With specialization to the Bernoulli-Carlitz numbers in mind, it is convenient to normalize by a Carlitz factorial. The reader may check that after multiplying by $\theta - t$, evaluation at $t = \theta$ in $\Pi(j-1) \gamma_j(t)$ yields the Bernoulli-Carlitz number $BC(j-1)$. It is interesting to observe that for our small list below, the zeros of $\gamma_j$ all lie in $A$.  

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The First Ten Non-zero $\gamma_j$

\[
\begin{align*}
\gamma_1(t) &= \frac{1}{\theta - t} \\
\Pi(2)\gamma_3(t) &= \frac{-1}{(\theta^3 - t)(\theta - t)} \\
\Pi(4)\gamma_5(t) &= \frac{1}{(\theta^3 - t)(\theta - t)} \\
\Pi(6)\gamma_7(t) &= \frac{-1}{(\theta^3 - t)(\theta - t)} \\
\Pi(8)\gamma_9(t) &= \frac{-\theta^6 - \theta^4 - \theta^2}{-(\theta^9 - t)(\theta^3 - t)(\theta - t)} \\
\Pi(10)\gamma_{11}(t) &= \frac{\theta^{15} + \theta^{13} + \theta^{11} - \theta^9 - \theta^7 - \theta^5 - \theta^3 + t}{-(\theta^9 - t)(\theta^3 - t)(\theta - t)} \\
\Pi(12)\gamma_{13}(t) &= \frac{\theta^{15} + \theta^{13} + \theta^{11} + \theta^9 - \theta^7 - \theta^5 - \theta^3 - t}{(\theta^9 - t)(\theta^3 - t)(\theta - t)} \\
\Pi(14)\gamma_{15}(t) &= \frac{\theta^{12} - \theta^{10} - \theta^4 + \theta^2}{-(\theta^9 - t)(\theta^3 - t)(\theta - t)} \\
\Pi(16)\gamma_{17}(t) &= \frac{-\theta^{12} + \theta^{10} + \theta^6 - \theta^4}{-(\theta^9 - t)(\theta^3 - t)(\theta - t)} \\
\Pi(18)\gamma_{19}(t) &= \frac{\theta^{21} - \theta^{19} + \theta^{15} - \theta^{13} + \theta^9 - \theta^7 + \theta^3 - t}{-(\theta^9 - t)(\theta^3 - t)(\theta - t)}
\end{align*}
\]
BIBLIOGRAPHY


