Tracing Growth of Teachers' Classroom Interactions with Representations of Functions in the Connected Classroom

DISSENMATION

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By

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ABSTRACT

The purpose of this study is to create an empirically based theoretic model of change of the use and treatment of representations of functions with the use of Connected Classroom Technology (CCT) using data previously collected for the Classroom Connectivity in Promoting Mathematics and Science Achievement (CCMS) project.

Qualitative analysis of videotapes of three algebra teachers’ instruction focused on different categories thought to influence teaching representations with technology: representations, discourse, technology, and decisions. Models for rating teachers low, medium, or high for each of these categories were created using a priori codes and grounded methodology. A cross case analysis was conducted after the completion of the case studies by comparing and contrasting the three cases.

Data revealed that teachers’ decisions shifted to incorporate the difference in student ideas/representations made visible by the CCT into their instruction and ultimately altered their orientation to mathematics teaching. The shift in orientation seemed to lead to the teachers’ growth with regards to representations, discourse, and technology.
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CHAPTER 1: INTRODUCTION

During my sophomore year of high school I first encountered handheld graphing technology of the TI-82 calculator. My pre-calculus teacher was fluent in using the graphing calculators and taught us how to use it proficiently. My friends and I used the technology to explore mathematics. It helped me explore my mathematical thinking by presenting many representations of the mathematics. Several times in class we would use the technology to test hypothesis, try different situations or try to generalize what was happening. This technology has benefited me greatly in my mathematical career.

A few years ago I was exposed to the TI-Navigator, a classroom networking device. I was intrigued by it then, but I really saw its potential after I took a weeklong course to learn how to use it. One of the main potentials I saw was the ability to do formative assessment; the other was the potential to do the type of exploration I did in high school but in a public atmosphere.

The exploration usually entailed linking the different representations of functions to gain insight into a particular situation. At the time, I had been a graduate teaching associate in mathematics for 6 years and many of my students had problems connecting information from a graph of a function and the equation of the function. The literature (Cunningham, 2005; Even, 1998; Galbraith & Haines, 2000; Hitt, 1998; Knuth, 2000; Sfard, 1991) points to this difficulty for students to transfer representations with meaning.
as well. Initially, I thought since the TI-Navigator could show multiple representations it could aid students with this difficulty. However, after working on the Classroom Connectivity in Promoting Mathematics and Science Achievement (CCMS) project and seeing teachers use the TI-Navigator in the classroom, it seemed that it could help student representational thinking by more than just displaying multiple representations, but by having the potential to encourage negotiation of meanings of representations.

CCMS is a national research study examining the impact of connected classroom technology on student achievement, self-regulated learning and dispositions of students toward mathematics and science. The research design is a randomized crossover field trial in 118 Algebra 1 classrooms (Owens et al., 2008). The intervention of the CCMS project consisted of providing teachers with Texas Instruments (TI) Navigator™, a wireless classroom device that networks students’ handhelds to the teacher’s computer and professional development supporting implementation of this technology.

**Problem Statement**

Several features of the TI-Navigator have the potential to increase discussion or student use of representations. Quick Poll (QP), Activity Center (AC), Screen Capture (SC), and Learn Check (LC) all can invite students to participate in activities and practice using representations. Through Activity Center and Screen Capture, students can see their own representations as well as other students’ representations allowing for reflection on and discussion of the representations. Activity Center has the ability to move among three major representations, algebraic, graphical, and tabular of the same object. With several students participating in using the AC, alternative representations are likely to
occur. Pursuing explanations and justifications of why or why not particular representations are or are not reasonable can aid students to see representations as tools.

Teachers using formative assessment and classroom discourse can aid student representational thinking. Formative assessment can aid students and teachers by providing them evidence of student understanding. Students using the data collected from formative assessment can reflect on their own representations. Teachers can use the data to see if students need scaffolding to better understand of the role of the representations. Discourse is central for negotiation of meaning of representations and is key for using representations as tools for explanation and justification (Pape & Tchoshanov, 2001). The professional development offered by the CCMS project encouraged the use of formative assessment and classroom discourse.

The technology and pedagogy used by the teachers in the CCMS project could provide an opportunity to aid student representational thinking. The purpose of this study was to develop a model for teacher growth in the usage and treatment of representations, discourse, and technology with continued use of CCT in the classroom. The following question guided the data analysis:

What is the relationship between teacher use of CCT in algebra classrooms and the growth of teacher choice of representations of linear functions as manifested in the classroom discourse?

a. What kinds of representations are used in the classroom by teachers?

b. What is the character of discourse about representations or use of representations by the teacher?

c. For what purpose do teachers rely on multiple representations in classroom discourse?
d. What changes occurred within teachers’ in-the-moment decisions over time with continued use of CCT?

**Rationale**

Much of the history of mathematics is about creating and refining representational systems; much of teaching mathematics is about students learning to work with representational systems and solve problems with them (Lesh, Landau, Hamilton, 1983). Representations are crucial for understanding and using mathematics; they give particular insight into certain situations that may not be attainable otherwise and can be used to analyze the patterns that mathematicians study. Without representation of abstract mathematics, it is unclear how teachers can teach, students can learn, and learning can be demonstrated (Heritage & Niemi, 2006). The National Council of Teachers of Mathematics (NCTM) (2000) created a process standard for representations, signifying its importance claiming that representations are fundamental to how people learn and use mathematical concepts. This process standard calls for student to be able to:

1. Create and use representations to organize, record, and communicate mathematical ideas;
2. Select, apply, and translate among mathematical representations to solve problems; and
3. Use representation(s) to model and interpret physical, social, and mathematical phenomena. (p. 67)

Mathematical objects typically have many different ways to represent them. For example, lines can be represented graphically, algebraically, or by a table. These multiple representations of the same line can give insight into the properties of the line. However, representations, in particular using multiple representations, cause trouble for students
Many students do not realize that points on a graph of a line satisfy the equation of the line and they also prefer to use algebraic solution methods even when graphical methods are easier (Knuth, 2000). When solving problems, students typically stick to one representation and do not attempt to look at others (Even, 1998). Even prospective mathematics teachers have problems translating between different representations of functions while preserving meaning (Hitt, 1998). Other problems encountered by students using representations include: translating among algebraic, tabular, and graphical representations of functions, understanding the correspondence among the representations, and interpreting the graphical and tabular representations (Galbraith & Haines, 2000). The inability to translate among them effectively can hinder student problem solving skills.

Empirical data from various studies indicate that students and prospective teachers have difficulties linking different representations of functions. Even when multiple representations are present, the conclusions drawn from one representation had no bearing on another representation for many students. Knuth (2000) reported that students also had difficulty in moving away from algebraic solutions to the extent that even the suggestion of using graphical information was not taken. Knuth (2000) concluded that part of the reason that algebraic solutions are used by most students is due to teachers’ own preferences. In fact, teachers spent most of the class time on algebraic to graphical translations, while the least amount of time was spent on graphical to numeric, which is what students have the most trouble doing (Cunningham, 2005). Spending more time using different representations appear to help bridge the gap. However, more than just
including different representations in instruction and assessment have to occur to in order for students to develop skills. Thompson (1994) noted that including different representations is not enough unless connections are made among the different representations.

Technology can support representational thinking by allowing students to manipulate the different representations of functions and see their connections. Studies were conducted to examine the impact of handheld graphing technology on student learning: results indicate that graphing technology had a positive impact on students’ ability to link representations and increase their understanding of functions. These students out performed their peers who did not use the technology (Harskamp, Suhre, & Van Streun, 2000; Hollar & Norwood, 1999; D. R. Thompson & Senk, 2001). These results are particularly important.

Classroom response systems (CRS) are becoming more commonplace and are typically used to collect attendance and summative assessment data. The TI-Navigator is an example of such an advanced classroom response system. It connects graphing calculators to the teacher’s computer granting much more potential than just receiving answers to multiple choice questions from students. Fies and Marshall (2006) conducted a review of the literature of classroom response systems and concluded that CRS aided in development of student representational thinking. Among the 15 studies reviewed, 9 reported that students were more engaged and participated in discussions more frequently. Seven studies reported evidence of instructors as being more aware of students understanding and leading more responsive instruction. Another set of seven
studies found that students were more aware of their own understanding. All of these benefits could aid in negotiation of meaning of representations, since more students are encouraged to participate and both the teacher and the students have increased awareness of student understanding.

Developing representational thinking requires intensive social co-construction of meaning of the representations, as well as practicing reasoning with and about representations (Pape & Tchoshanov, 2001). Understanding the concept of function, therefore, requires flexibility to move among its different representations comprised of both internal and external representations (Moschkovich, Schoenfeld, & Arcavi, 1993; Sfard, 1991). Students must be given time to practice communicating and reasoning with representations (Pape & Tchoshanov, 2001); this practice must be intentional. The classroom has to support extensive social negotiation of the meanings of representations (Cobb, Wood, & Yackel, 1993; Pape & Tchoshanov, 2001). A classroom environment that provides opportunities for social negotiation of meaning, reflexive thinking as well as using representations to communicate mathematics is needed to aid student representational thinking. The development of such an environment can be facilitated by the CCT and appropriate pedagogy in their presence.

**Conceptual Framework**

The NCTM process standard calls for students to be able to use representations to solve problems, communicate mathematics, and interpret the world. Representations can be taken to be the external representations produced to help think, communicate and interpret mathematical concepts. They may also be considered the internal
representations that a student might possess. Representation(s) refer to both the act of representing as well as the external, noun form of representation (Pape & Tchoshanov, 2001).

**Internal Representations**

Internal representations are knowledge and structures in memory, as schemas, neural networks, or other forms (Zhang, 1997). In mathematics, internal representations can be considered to be abstractions of mathematical ideas or structured ways of thinking that are developed by and are internal to a learner (Pape & Tchoshanov, 2001). How well a subject is understood can be characterized by a description of a person's internal representations. “A mathematical concept is learned and can be applied to the extent that a variety of appropriate internal representations have been developed, together with functioning relationships among them” (Goldin & Shteingold, 2001, p. 6). Internal representations are needed for understanding mathematical concepts. Therefore, it is desirable to observe a student's internal representations. Unfortunately, one cannot observe a learner’s internal representations directly. Only inferences about the internal representations can be made using interaction with external representations.

**External Representations**

External representations are taken to be the knowledge and structure in the environment, as physical symbols, objects or dimensions as well as external rules and constraints embedded in physical configurations. External representations typically have a sign or configuration of signs, characters, or objects that can stand for something else, such as graphs, pictures, diagrams, words, symbols, and manipulatives. Mathematical
representations cannot be understood in isolation. The concept of “5” cannot be isolated from the concept of “1,” “2,” or “3” (Pape & Tchoshanov, 2001). The representations of mathematics are “part of a much wider system within which meanings and conventions have been established” (Goldin & Shteingold, 2001, p. 1). These representational systems have been refined over centuries and are highly structured. The structure creates rich relationships among different representation within a system. Throughout much of history the representational systems of mathematics have been static, in that they depict one graph, equation, or diagram.

**Connections between Internal and External Representations**

Internal and external representations are intimately connected. They are believed to mutually influence each other. Pape and Tchoshanov (2001) advocate the position that the development of student representational thinking is a two-sided process, externalizing the mental concepts and internalizing of external representations. Social interaction typically is hypothesized to cause an interaction between internal and external representations. Pape and Tchoshanov (2001) argue that “there is a mutual influence between the two forms of representations: the nature of an external representation influences the nature of the internal one, and vice versa” (p. 120). They further claim that the more complex representations can facilitate more complex understanding. Representation is inherently a social activity; students should not be asked to only create external representations but to use them to explain and justify. They consider representational thinking as learner’s ability to “interpret, construct, operate (communicate) effectively with both forms of representations, internal and external,
individually and within social situations (Pape & Tchoshanov, 2001, p. 120)."

Four implications for classroom practice are given for the development of student representational thinking to occur. First, students must be allowed to practice the externalization of internal representations and the internalization of external ones by negotiating their meaning with others and reflecting on their own representations. Second, students need to understand the act of representing as both a process and a product of social activity. By externalization of abstractions in a social setting, meanings are negotiated and students are able to refine their understanding and representations. Third, a variety of techniques or representations should be present while teaching. Developing representational thinking requires multiple representations as well as interaction between internal and external representations. Lastly, students need to see and use representations as tools for thinking, explaining and justifying, not just end products.

Studying classroom use of representations of linear functions requires a framework for describing the type of representations that can occur since functions have their own set of external representations. Three major representations are considered by Moschovich, Schoenfeld, and Arcavi (1993) to include algebraic, graphical, and tabular. Functions have a dual nature. They can be seen as a process acting on a set of inputs to obtain outputs. Functions can also be seen as objects that can be manipulated as a whole. Although not specifically discussed by Moschovich, Schoenfeld, and Arcavi, the process perspective and object perspective of functions can be thought of as part of the internal representation of functions, since they are primarily ways to think about functions. This framework excludes a major representation particularly vital in classroom interactions the
verbal representation. The framework was modified to include the addition of the verbal representation, which also can be seen from either the process or object perspective. For example, the function \( f(x) = 3x+5 \) can be verbally represented as "the function that multiplies the input by 3 and then adds 5," which is describing a process. The same function could be represented as "a line with a slope of 3 and a y intercept of 5," which describes an object. The different combinations of external representations and perspectives can be captured in a two dimensional table, shown below.

The four main representations can be used in conjunction with either the process or object perspective. As stated earlier, the perspectives are differentially useful in different circumstances while none is completely superior to the other. Potential solutions to problems involving functions are presented in the Table 1.1. Students may move within a perspective or even between perspectives. Moving “two dimensionally,” i.e. moving between representations and/or perspectives, in this table gives students flexibility in perception that can aid in solving problems (Moschkovich et al., 1993).

Merging the process-object framework (Moschkovich et al., 1993) with the external internal representations interaction framework (Pape & Tchoshanov, 2001) reveals how representations may interact with these perspectives as shown in Figure 1.1. The four main representations of a line, algebraic, graphical, tabular, and verbal representations (Moschkovich et al., 1993) can be considered to interact with a student’s internal representations and perspectives via practice, negotiation, reflection, interaction (Pape & Tchoshanov, 2001)
Figure 1.1: The proposed relationship between internal and external representations of lines within a student.

A line with a slope of -3 and a y intercept of -2.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>-5</td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
</tr>
</tbody>
</table>
The literature (Moschkovich et al., 1993; Pape & Tchoshanov, 2001; Schoenfeld, 2010a; A. G. Thompson, Philipp, Thompson, & Boyd, 1994) indicates that developing student representational thinking is influenced by many aspects, such as treatment and usage of representations themselves, discourse surrounding representations, technology use, and in-the-moment decisions teachers make in response to learners’ ideas and
feedback. Developing student representational thinking requires not only the use of multiple representations but also discussing the connections between them. Pape and Tchashanov (2001) offered a model of what to observe in classrooms that could aid student representational thinking. The major suggestion of students needing to negotiate meanings of representations is explored further through discourse used in the classroom. The process/object framework can identify the representations present in the classroom. The teacher’s orientation influences in-the-moment decision making (Schoenfeld, 2010a) and could influence technology use. The combination of the following components: representations, discourse, technology, and orientations, form the conceptual model for this study presented in Figure 1.2 and this model guided the coding methodology expounded upon in Chapter 3.

Figure 1.2: Conceptual Model
Chapter Summary

Ample evidence suggests that students have difficulties with understanding representations, particularly representations of functions. Also, evidence establishes that understanding representations is essential for learning mathematics. This chapter proposed a rationale for teachers using connected classroom technology with appropriate pedagogy to aid student understanding of the representations of functions. Finally, it provided a framework for looking at classroom interactions that could aid student representational thinking. Chapter 2 explores the literature on several different aspects of teaching that may influence teachers’ ability to create an environment using CCT that has the potential to improve student representational thinking. Chapter 3 presents the methodology conducted for this study. Chapter 4, 5, and 6 are case studies of investigating individual teacher’s, Mr. L., Ms. A., and Ms. B respectively, instruction over time. Chapter 7 is a cross cases analysis of these three cases. Chapter 8 is dedicated to the discussion of findings and implications of this study as well as a propose model for teacher growth while using technology.
CHAPTER 2: REVIEW OF LITERATURE

Representations are essential for understanding mathematics. This review of the literature explores research and theory of mathematical representations in general. It then focuses on representations of functions and some of the difficulties students have with them. Research on effective mathematical pedagogy, specifically formative assessment and classroom discourse will be reviewed. Literature on how technology might aid student representational thinking will be summarized.

Mathematical Representations

Much of the history of mathematics is about creating and refining representational systems; much of teaching mathematics is about students learning to work with representational systems and solve problems with them (Lesh, Landau, Hamilton, 1983). Representations are crucial for understanding and using mathematics; they provide particular insight into certain situations that may not be attainable otherwise and can be used to analyze the patterns that mathematicians study. Without representation of abstract mathematics, the ability for teachers to teach and students to learn mathematics is unclear (Heritage & Niemi, 2006). The National Council of Teachers of Mathematics (NCTM, 2000) set forth the strand for representations as a process standard, signifying its importance claiming that representations are fundamental to how people learn and use mathematical concepts. Most specifically, this standard calls for students to be able to:
1. Create and use representations to organize, record, and communicate mathematical ideas;
2. Select, apply, and translate among mathematical representations to solve problems; and
3. Use representation(s) to model and interpret physical, social, and mathematical phenomena. (p. 67)

Representations may be characterized as internal or external. Internal representations are knowledge and structures in memory, as schemas, neural networks, or other forms (Zang, 1997). They can be thought to be a structured way of thinking, and can be of different forms. Verbal/syntactic representational systems describe language capabilities. Imagistic systems include visual and spatial configurations, or mental images. This can also include imagined motion, such as hand gestures. Formal notational systems can take place internally, where students mentally manipulate symbol systems. Strategic and heuristic internal representational systems, where students mentally organize problem solving heuristics, can also be thought of structured ways of thinking. External representations are taken to be the knowledge and structure in the environment, as physical symbols, objects or dimensions as well as external rules and constraints embedded in physical configurations. External representations in mathematics include graphs, pictures, diagrams, words, symbols, and manipulatives.

Internal and external representations are thought to be intertwined (Pape & Tchoshanov, 2001). A student can write, draw, speak about an internal representation of a concept; this is called externalization. The student can also create a mental image of an external representation fitting it into their way of thinking.

Students use prior knowledge when encountering new concepts and build new internal representational systems based on preexisting ones. Goldin and Kaput (1996)
describe three main stages for the development of new internal representation systems. First is an inventive-semantic stage where new characters or symbols are introduced and are used to symbolize aspects of a previous representational system. From this previous system meaning is given to the new one. During the second stage the earlier system is used as a template for the structure of the new one. The mapping a student makes from the old system to the new is called **grounding** (De Vega, Glenberg, & Graesser, 2008). Rules for the new system are worked out, using the earlier system as well as meanings in the new one. Finally, the system becomes autonomous, where it can be detached from the previous one. Here the new system can acquire new meanings that are different from or more general than those first assigned. One cautionary comment is that children can quickly take the initial assigned meanings as the real meaning for the new representation. For example, a child might assign to the representational system number as meaning the result after counting objects. This meaning would make it difficult to expand the meaning of number to include negatives or rational numbers.

Pape and Tchoshanov (2001) state that when a student initially attempts to represent something non-standard, the symbolism she invented is based upon previous knowledge, which agrees with Goldin's and Kaput's stages. The initial representations and their development depend on the purpose of creating the representational form, how the form is discussed, and the instructional practices engaging the students. One way to move students toward a formal representational system is by negotiation of shared meanings and refinement through discourse with peers and teachers. Purpose is a major factor shaping the creation and use of representations. Pape and Tchoshanov (2001) argue that
representations must not be thought of as an end product, something to be produced at the end of a task, but rather tools for thinking about concepts, that allow for tracking intermediate ideas and results. Since many mathematical representation systems are highly structured, the desired outcome of student learning is for students to move from their initial non-standard representation system to a formal system. Since our standard representation systems are designed by experts, they embody the experts’ conception of mathematics; these conceptions may be unattainable initially for the non-expert. An external representation only has meaning when the proper internal representations are prepared to give it meaning. Representations need to be built up in the classroom as a social activity that students practice doing, especially if the representations are encouraged to be used for explanation or justification (Pape & Tchoshanov, 2001).

**Representations of Functions**

Functions are typically represented in curriculum in four main ways, which are verbal, numeric, graphical, and algebraic. Each of these representations has advantages and disadvantages (Friedlander & Tabach, 2001). The ability for students to see “how different representational forms become useful tools in relation to the problems and issues of the students, especially in how they express and argue for generalizations” (Smith, 2008, p. 134) is important for student learning. The verbal representation is used in the posing of the problem as well as interpretation along the way. The verbal representation creates meaning for the problem context as well as serves as a tool for solving problems and facilitating general patterns. It also connects mathematics to other domains and everyday life. However, it can be imprecise and create misleading
associations. The numerical representation is a natural one for students to use at the beginning stage of exposure to algebra. Essentially, it is looking at specific cases of the problems allowing for some inference to be drawn. However, the lack of generality can be confining or misleading (Friedlander & Tabach, 2001). The graphical/pictorial are visual representations of functions or problem situations. These can provide insight into the behavior of the situation or function as a whole. However, the graph of a function can be imprecise. For example, a line may look like a 45-degree angle with the horizontal, when it is not. The picture of the situation may be too specific. For example, a diagram may look like the angles are right but they may not be. The graphical/pictorial representation can vary in its utility. The algebraic-symbolic representation can provide a concise, general, precise description of mathematical concepts and patterns. However, even though a representation can provide a general description of a pattern, it is sometimes hard to infer how the pattern behaves as a whole. This representation may also obscure mathematical meaning. Each of the different representations of functions has advantages and disadvantages. However, it has been suggested that working with multiple representations can nullify the disadvantages while boosting the advantages (Kaput, 1992).

A function can describe how one variable y depends on another variable x, and usually thought of as performing several operations on x to get y. But in a more robust mathematical view functions are a set of ordered pairs with certain conditions. Simply stated a function is a set. Functions have a dual nature. Moschovich, Schoenfeld, and Arcavi (1993) describe this nature as the process/object duality of functions. Emphasizing
the dependency of y on x for a function focuses attention on the process aspect of the function linking some y with some x, whereas the set nature of function describes an object can be manipulated as a whole. Understanding both the process and object perspectives of functions is an essential part of learning them (Even, 1990; Schwartz & Yerushalmy, 1992; Sfard, 1991).

The four main representations of functions are typically thought of as external representations; however they can be internal representations as well. A graph can be a student's mental picture of the function, just as it can be a drawing on paper. The process perspective is really a conception of a function; the function is thought of as a process acting on some set. The process perspective is really a structured way of thinking about a function or internal representation. The object perspective, then, is another internal representation of the same concept. It is difficult to determine which perspective is being used and can only be inferred by the actions or explanations using external representations. Each perspective sheds light on the different aspects of functions as well as the different representations of functions, but the combination of a perspective and a representation may be differentially useful depending on circumstance (Moschkovich et al., 1993).

Functions are typically introduced from the process perspective. They are described as machines that change the input to an output. The different representations can be seen from both perspectives. However, some of the representations lend themselves to be seen more naturally, at least initially, from one perspective rather than the other. The way tables are constructed emphasizes the process conception: after choosing values for the
x’s the y-values are constructed from them. Since the table is a snapshot of a few ordered pairs, it can be difficult to see the table as representing an entire object. Algebraic equations can be manipulated on the whole, that is an object. However, many students see equations as something that need to be solved (Even, 1998; Knuth, 2000). This perception evokes the need to complete a process. Verbal representations can describe processes, such as rise over run. They can also categorize things, such as lines, quadratics, etc. Verbally describing a function can be rather difficult; categorizing it can be much simpler. For example \( f(x) = e^{x^2 + 2x - 5} \), describing exactly what is being done to x to get \( f(x) \) would be tedious. However, categorizing the function as an exponential function with a quadratic exponent is not. Even with more routine functions, this categorization is not immediate for students (Moschkovich et al., 1993). The graphical representation is different in that it lends itself to be viewed from the object perspective. A student could easily envision picking up the graph of a line and moving it around and treating the graph as something. However, students struggle to see the lines as a process or as the individual points on a line (Knuth, 2000). In most cases, students find it easier to view the representations from one perspective than the other, either because of the nature or the representation or how the representation has been treated. Despite the graphical representation revealing the object nature of functions, students have a difficult time conceptualizing functions as objects. In fact, Sfard (1991) claims:

There is a deep ontological gap between operational and structural conceptions. ...Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing-static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and manipulate it as a whole, without going into details. ...In contrast, interpreting a notion as a process implies regarding it as a potential rather than an actual entity, which comes into
existence upon request in a sequence of actions. Thus whereas the structural conception is static, instantaneous and integrative, the operational is dynamic, sequential, detailed. (p. 4)

The process that students go through to conceptualize functions as objects as well as a process is long and difficult. It may never be fully realized, since it requires the sudden realization that a process is itself an object (Sfard, 1991).

As Goldin and Kaput (2001) noted, the initial meaning of a new representation system is generally taken as the 'true' meaning of the system. Since functions are initially thought of as a process acting on numbers this can make it difficult to move beyond this representation system. Students have difficulty moving between the process and object perspectives and they have trouble moving between different representations as well.

Smith (2008) developed a framework for introducing functions in early grades using representational thinking. Representational thinking and symbolic thinking are two different kinds of algebraic thinking. Symbolic thinking is related to how one uses symbols without reference to their meaning. Representational thinking is the mental processes for creating meaning for symbolic systems. Smith (2008) believes that representational thinking should be central to the teaching and learning of algebra. He claims representational thinking is under-emphasized, whereas representational systems are taken as given and the focus is on manipulating them instead.

Functional thinking is representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships to generalizations of the relationship (Smith, 2008). Smith (2008) proposes six activities as underlying functional thinking:
1. Engaging in some type of physical or conceptual activity.
2. Identifying two or more quantities that vary in the course of this activity and focusing one’s attention on the relationship of these two variables.
3. Making a record of the corresponding values of these quantities, typically tabular, graphical or iconic.
4. Identifying patterns in these records.
5. Coordinating the identified patterns with the actions involved in carrying out the activity.
6. Using this coordination to create a representation of the identified pattern. (Smith, 2008, pp. 143–144).

Smith (2008) argues that developing contexts that allow students to build functional relationships by coordinating actions within the representational system along with their records is essential for aiding student functional thinking. For linear functions he claims that deriving the constant addition or constant rate of change is essential to grasp lines.

**Student Difficulty with Representations**

Knuth (2000) conducted a study to examine student ability in selecting and moving between algebraic and graphical representations of functions. In particular he wanted to see if students understood the Cartesian Connection: A point is on a graph of a line if and only if it satisfies the equation of that line.

The 178 students in Knuth’s (2000) study were enrolled in a large suburban high school with a majority of the students being Caucasian (88%) from middle class families. The students were from several different courses, ranging from first year algebra to AP calculus. The classes were chosen because the teachers were willing to participate. Knuth characterized the course materials as traditional in nature.

Six questions were created to require students to use the Cartesian Connection in its solution. Each student in the class was given one question during the warm up time. Each
question had both algebraic and graphical representation of the function. They were asked to show their work and explain their thinking. They were also asked to provide an alternate solution method. Knuth (2000) hypothesized that students who understood the connection would choose the easier solution method.

Student responses were coded into two categories: algebraic or graphical. Responses that used the equation as primary means of solution were coded as algebraic, and responses that explicitly used the graph were coded as graphical. Responses that used the graph to support the algebraic solution were coded as algebraic. He also looked at patterns among the responses, such as students converting to slope intercept form. According to Knuth (2000) “More than three fourths of the students chose an algebraic approach as their primary solution method, even in situations in which a graphical approach seemed easier and more efficient than the algebraic approach” (p. 504). Many students did not use or even suggest a graphical solution method. Only about a third of the students used graphical approach for either their primary or alternative solutions. He concluded that many students were not making the Cartesian connection.

Students across many backgrounds did not recognize the fact that points on a line satisfied the equation of a line. They had no trouble seeing the points, since many of them found points to calculate slope. But they missed the intimate connection to the algebraic representation of the line. Students seemed to have a predisposition for using algebraic solutions because it is recognized that teacher preference of algebraic solutions influences student own choices (Knuth, 2000).
Even (1998) explored three factors involved in linking representations of functions. The participants of this study were 162 college mathematics students from eight Midwestern universities. Most were senior math majors and all were prospective mathematics teachers. A questionnaire was created by using non-standard mathematical problems that addressed different aspects of functions. During the first phase, 152 students were administered the open-ended questionnaire. During the second phase, 10 students were administered the same questionnaire and then interviewed. The focus of the interview was on student explanations of what they did on the questionnaire and why. The questionnaires were scored for correctness as well as categorized for different solution approaches. The results were then tallied for each question.

The first question examined the connection between representations. It asked how many solutions a generic quadratic equation had under certain conditions. Only 14% of first phase students answered correctly. The students who solved the problem switched representations referring to a graph. Some students who switched representations did not pay attention to the fact the function was quadratic. Most students (80%) did not look at or attempt to look at a different representation. “Seeing a quadratic expression did not immediately bring to mind the graphic representation of a quadratic function” (Even, 1998, p. 108). Seven of the ten interviewees did not solve this question correctly and were asked if they could solve it with graphs. Three took the hint and were then able to make the representation connection and solve the problem, while four of them did not use the suggestion. It seems symbolic representations dominated their thinking.
Two questions dealt with how changing a graph of a function can influence its symbolic representation. One third of the first phase students were able to solve the easier of the two questions correctly and only 10% got the correct expression for the other more difficult question (Even, 1998). Two methods were used by the students to solve their problems. One was a point-wise approach, checking points on one graph seeing where it goes on the second graph and then trying to come up with an expression. The other approach was global, looking at the differences of the graphs as a whole. For these questions those students that used a global approach were more likely to solve the problem. The global approach wasn't always better. Other questions were solved more successfully by students who used a point-wise approach. Even (1998) concluded that neither approach was totally superior to the other, but in certain circumstances one can be superior to the other.

Hitt (1998) examined 30 mathematics teachers at the secondary level who were beginning a postgraduate course on mathematics education. Questionnaires, C1, C2... C14, were constructed to distinguish between different levels of understanding of the function concept. These also included representations used in teaching to develop the function concept. The questionnaire was designed to determine difficulties in translating representations while preserving meaning. The teachers were given two questionnaires per week for seven weeks, and they had an hour to work on each. Only the most relevant questionnaires were discussed in the results.

Questionnaire C1 had 26 curves, some representing functions, while other curves did not. Teachers were asked if each graph was a function. For a curve that had a loop in it,
all but one said it was not a function. Reasons varied from using ordered pair, more than one image and using vertical lines. However, about a third of the participants incorrectly determined that particular conic sections were functions. Hitt concluded that, “the existence of an algebraic expression associated with a curve led them to abandon their definition of function” (Hitt, 1998, p 128). For an irregularly shaped curve, nine teachers incorrectly identified it as not being a function.

Questionnaires C8 and C9 were designed to explore the phenomena of teachers' tendency to construct functions that were continuous and defined by a single algebraic expression. C9 was designed to examine comparisons of functions. Several of the teachers thought that differences in algebraic expressions led to different functions, such as thinking \( f(x) = \sqrt{4} \) is different than \( g(x) = \frac{3}{8} \). Not one teacher was able to give two distinct functions such that \( f(f(x)) = 1 \) for all \( x \). They were able to find one \( f(x) = 1 \), but for the second they gave another form of \( f(x) = 1 \).

Questionnaire C2 explored sub-concepts of function, such as domain and image set. The teachers had difficulty indentifying these points, except for when an arrow diagram is involved (since the domain and image set was easily seen).

Questionnaires C10 and C11 dealt with converting from a graphic representation to a real context and vice versa. For simple graphs and contexts, the teachers were able to give an appropriate solution. However, when the graph or context was complicated only a few of the teachers were able to give a solution. The most common errors were related to producing an iconic translation. For example, if the real context was going up and down a slide and then being asked to graph speed versus time, one may graph something that
looks like a slide. Another common error included not interpreting the variables involved in an analytic or graphic context (Hitt, 1998).

Grounded representations while desirable, since the representations are imbued with meaning, can impede problem solving (Bieda & Nathan, 2009).

One’s perceptions and reasoning processes can be inappropriately bound to the representations they are grounded to, which can impose superfluous or incorrect constraints on the representations themselves and the strategies that draw upon them, thereby negatively affecting problem-solving performance or transfer (Bieda & Nathan, 2009, p. 638).

Bieda and Nathan (2009) found three behaviors students exhibited when dealing with representations that needed to be modified to solve problems. These behaviors were called physically grounded, spatially grounded and interpretatively grounded. The physically grounded category thought the graph of quantitative data was bound by its physical and numerical limits implying the graph and its data could not go beyond the largest number displayed. The spatially grounded group was able to make predictions beyond the numerical limits of the data assuming a linear model, but was unable to do so when that had to alter the graph’s scale or extend the axis. Interpretively grounded students were able to recover from the other disfluencies by translating to a different but equivalent representation. The authors concluded that the nature of the representations themselves can impede students, but if students can translate to different but equivalent representations they can get around the impasse (Bieda & Nathan, 2009).

Gagatsis & Shiakalli (2004) examined 195 university students in the department of education at the University of Cyprus to identify the participants’ translation ability and to examine how translation ability affects mathematical problem solving. Two tests were
given to the students (Tests A and B). Each test had six problems that involve a direct translation from one representation of a particular function to another, a table completion problem and two word problems. The six direct translation problems were identical between Test A and B, except for the initial representation. Test A started each question with a verbal representation and asked students to translate to a graphical or algebraic one. In Test B, the problem started with the graphical representation and students were asked to translate to algebraic or verbal representations. The two word problems were different for the two tests and asked students to create a function from a given situation.

Gagatsis and Shiakalli (2004) used linear regression to determine if there was a relationship between translation ability and problem solving ability. They used ability to translate from the verbal representation as one independent variable and the ability to translate from the graphical representation as the other. They identified the ability to solve problems by articulating different representations of functions as the dependent variable, and used an implicative method of data analysis to determine implicative relationships among the different tasks of the test. For example, the implication of task 1 implying task 2 indicated that the success of task 1 entailed success at task 2, and the failure of task 2 resulted in a failure of task 1.

Significant correlation was established between translation ability and problem solving ability at the $p < 0.05$ level that explained 53% of the variance (Gagatsis & Shiakalli, 2004). Using the implication analysis, they found no important relationships between the six translation tasks of Test A and Test B. Gagatsis and Shiakalli (2004) concluded that
It is evident that there is no relationship between the two different representations of the concept of function - the verbal and the graphical. The students conceive the two representations of the same concept - the verbal and the graphical - as two different tasks and not as different means of representing the same idea. This indicates that they are unable to recognize an idea embedded in a variety of qualitatively different representational systems, and as a result they do not understand this idea (2004, p. 653).

Translation involving the graphic representation had lower success than translation involving the algebraic representation. Given the same initial representation the success rates to translate to the other two representations were different. The translation from the verbal to the algebraic seemed the easiest for students. The ability to problem solve did not imply ability to translate or vice versa indicating many factors involved in problem solving (Gagatsis & Shiakalli, 2004).

Research indicates that students do not switch representations automatically, even when asked or when it is beneficial (Even, 1998; Hitt, 1998; Knuth, 2000). If students can do one type of translation they may not be able to correctly do a different translation (Gagatsis & Shiakalli, 2004). The inability to correctly translate to different yet equivalent representations can impede mathematical learning and problem solving (Lesh, Post, & Behr, 1987). When students do translate representations, meaning is not always preserved. In some cases, information from one representation is rejected based on the presence of another representation. Symbolic representations and solutions seemed to dominate teachers and students. Solutions using other representations either are not used or dismissed by some students. Cunningham (2005) concluded that this is partially due to the fact that more time is spent on using symbolic representations than graphical or tabular representations making algebraic solutions seem superior to others. Research
indicates that the ability to translate among different representations while preserving meaning is aided by translations both from the source to the target and from the target to the source (Janvier, 1987). However, this is not being done in the classrooms (Cunningham, 2005). Using different representations in solving problems is needed. However, only using different representations is not enough without discussing the deep connections among them (P. W. Thompson, 1994).

Pape and Tchoshanov (2001) claimed that students need to practice the use of multiple representations to develop their understanding of the concept being represented and the representations themselves. Constructing representations of mathematical concepts in a social setting allows students to negotiate meaning of the representations they made as well as the standard representations. The negotiation between students and teachers is essential for the social co-construction thought to be needed for internal and external representations to influence one another (Pape & Tchoshanov, 2001).

Representations can aid student problem solving, communication, and support conclusions. They allow “individuals to track intermediate results, ideas and inferences. ... [Since] external representations embodies [sic] the important relationships presented in data or a word problem, they lighten the cognitive load of the individual” (Pape & Tchoshanov, 2001, pp. 124–125). Using representations as tools for thinking and explanation allows the individual as well as other students to abstract meaning.

**Social Constructivism**

The interaction of internal and external representations can be viewed from the lens of social constructivism that largely builds upon the ideas of Vygotsky. Constructivism
views students not as passively receiving information, but actively constructing knowledge (Von Glasersfeld, 1996). The motivating metaphor of constructivism stems from construction or architecture where structures are built up from pre-existing pieces (Ernest, 2010). “Previously built structures become the content of subsequent constructions. Meanings, structures and knowledge are emergent” (Ernest, 2010, p. 40). Many forms of constructivism use the construction metaphor on the individual level. However, social constructivism based on the work of Vygotsky goes further claiming that the individual and the social realm are intimately entangled (Ernest, 2010). Social constructivism has no metaphor for the individual mind. Instead the metaphor for the mind socially embedded becomes a “person in conversation” since higher thought processes occur as internalized dialogs (Ernest, 2010). Vygotsky (1978) claims that the construction of knowledge is based on prior knowledge through social interaction with others by internalizing their interpretations of what happens during these interactions. He even states that an individual’s higher function appear first on the intermental plane before it develops on the intramental plane. The intermental plane consists of ideas and thinking that are shared during social interactions, while the intramental plane consists of ideas and thinking that are not shared. The distinction between the two planes reveals that mental functions are intertwined between an individual and society. Since mental functions are believed to occur first in the intermental realm and then become internalized to the intramental realm, the mental functions available to an individual are different when they are alone or with others.
The mental functions that are completely developed within a student one calls the *actual developmental level* (Vygotsky, 1978) that are associated with what they can do own their own. Determining what students can do by themselves is where most classroom assessment stops. Since mental functions appear first in the social realm, assessment should focus on what students can do with the assistance of others as well. This leads to the construct of the second level of development: *the zone of proximal development (ZPD)*, “It is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (Vygotsky, 1978, p. 86). The actual development level comprises the mental functions that have already matured, whereas the ZPD is comprised of the mental functions that are in the process of maturing. The actual developmental level can be thought of as mental functions residing in the intramental frame, whereas ZPD can be thought of as mental function within the intermental frame. Vygotsky (1978) shows with evidence that what is in the learner’s ZPD today will become part of the actual development tomorrow.

The difference in the mental functions of the intermental and intramental planes of an individual shows how social constructivism can regard “individual subjects and the realm of the social as indissolubly interconnected” (Ernest, 1996, p. 342). This difference also reveals the importance of language mediating the two planes. Vygotsky (1986) went further claiming that language is not only a product of thought, but what forms thought
that can shape and limit human behaviour and cognition. Appropriate classroom discourse becomes essential to develop student understanding.

Representations are another mediator of the intermental and intramental planes. External representations reside in the domain of the intermental plane, whereas internal representations lie in the intramental plane. Pape and Tchoshanov (2001) discuss how representations can mediate each other through externalization and internalization and can mediate other mental functions through using them as tools for explanation or thinking. Viewing representations as residing in the two planes further emphasize the important roles of classroom discourse and prior knowledge to shape and refine them.

**Classroom Discourse**

Classroom discourse is regarded as pivotal to current reforms in mathematics education (Blanton, Berenson, & Norwood, 2001). The distinction between the intermental plane and the intramental plane emphasizes the importance of classroom discourse shaping learning. Since the concepts, arguments, and goals needed for construction of knowledge lie in the intermental plane, McNair (2000) argues that teachers should try to focus on “moving intramental processes to the intermental plane where they can be argued and tested” (p. 198). He argues that the focus should move from what is missing in student thinking to what is missing from their social interactions. He further claims that the concepts and goals not present socially generally do not become part of the student’s intramental resources and that student learning outcomes are a measure of the quality of discourse.
Two important aspects of classroom discourse include the patterns of interaction within the classroom and the function of people’s talk (Blanton et al., 2001). Obligations felt by teachers and students to enact certain routines during discourse give rise to the patterns of interaction. Blanton et al. (2001) uses Lotman’s (1988) argument that text (discourse) has a dualistic nature to identify the function of discourse. The dualistic nature of text can be seen by its treatment as information to be stored, or by its treatment as “a ‘thinking device’ so that, rather than being interpreted as an encoded message to be accurately received, the speaker’s utterances serve to generate new meaning for the respondent” (Blanton et al., 2001, p. 230). Wertsch and Toma (1995) called the treatment of text as information to be received univocal, whereas text treated as a “thinking device” is called dialogic. Dialogic discourse is evidenced by participants actively negotiating the meaning of the discourse (Blanton et al., 2001). Dialogic discourse would seem to provide students with opportunities to develop their representational thinking.

Teachers can refine classroom discourse by shaping the purpose, structure and goals of the discourse through modeling in the intermental plane what is needed for the student’s intramental processes (McNair, 2000; Morrone, Harkness, D’Ambrosio, & Caulfield, 2004). This refinement is dependent upon the context of the discourse and the beliefs held by the participants. The mathematical task and the representations used then are important to shaping discourse to enhance learning. Mathematical discussions need not only a good task but a mathematical purpose for engaging in discourse (McNair, 2000). Purpose can be changed by explicit directions or implicitly emphasizing certain goals. In fact, “students became willing to engage in meaningful discourse about
challenging mathematics problems because the teacher implicitly communicated to them that they would be successful, not through praise, but by honoring their contributions to classroom discourse” (Morrone et al., 2004, p. 35).

McNair (2000) discussed three frames students use when encountering mathematical tasks. These frames are influenced by the task itself and the beliefs and goals of the student.

Adding structure to a mathematical system of organization, involves a search for patterns and consistency, efforts to generalize and formalize procedures, efforts to make connections within the system, and to develop logical arguments that can be used to prove and to share the results of these efforts. These things are good in mathematics because they provide more structure for the system. These goods, and the pursuit of them, define the beliefs, values, and expectations that are at the core of a mathematics frame. (McNair, 2000, p. 201)

The other frames encountered in the classroom are the calculation and problem frames. Both of these frames focus on finishing the task rather than reflecting on the mathematics behind the problem or calculation. Of the three, the mathematical frame has the most potential to influence student learning. McNair suggests that teachers should explicitly use and discuss mathematical thinking that reveals the mathematical subject, purpose and frame to give the students the proper resources to internalize in their intramental plane. In other words, teachers should model thinking they want students to engage in out loud.

The mathematical frame is similar in nature suggestions for improving student representational thinking to (Pape & Tchoshanov, 2001).

Lau, Singh, & Hwa (2009) show that in a classroom that became more interactive in discourse, learning became more of a process of social negotiation instead of imposition. The students changed from passive learners to ones who were “actively involved in
discussions, clarifying the activity, suggesting ideas, explaining, and justifying their own ideas or providing help to justify or refute others’ ideas” (Lau et al., 2009, p. 322). Lau et al. (2009) suggest that students internalize the questions posed to them by their teachers and begin to ask the same questions themselves. The interactive classroom seems to have conditions ripe for student representational development.

The coordination of a student’s experience and the role of language as mediator is believed to be needed for students to make generalizations (Smith, 2008). “It is through our experience that we initially create meaning for language, but this language then becomes the tool that mediates our experience and ultimately allows for general claims of mathematical certainty” (Smith, 2008, p. 138). Generalizations grow out of an attempt to express ideas to ourselves and to others; symbolization grows out of this attempted expression, which in turn helps to expand and further explore the idea. Conceptualization and symbolization are inseparable (Kaput, 2008). Therefore, discourse has an essential role in shaping representation.

Formative Assessment

Black and William (1998, 2002, 2003, 2006, 2009) have done extensive work in theorizing and developing practical implementation for formative assessment. They state that early work on formative assessment focused on five main practices that had the potential of being effective. These five practices are sharing success criteria with learners, classroom questioning, comment only marking, peer and self assessment, and formative use of summative tests (Black & William, 2009). Since earlier work on formative assessment did not have a theoretical base, it was unclear how the five practices were
connected or if they were an exhaustive list. Black and Wiliam (2009) explained that Wiliam and Thompson (2007) drew on Ramaprasad’s (1983) three key processes of learning and teaching, establishing where students are, where they are going and how to get them there, to provide a theoretical grounding for formative assessment. Black and Wiliam (2009) claim that by applying each of the three processes of learning and teaching to the teacher, student, and peers, formative assessment can be thought of as comprising the following five strategies:

1. Clarifying and sharing learning intentions and criteria for success;
2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;
3. Providing feedback that moves learners forward;
4. Activating students as instructional resources for one another; and
5. Activating students as the owners of their own learning. (p. 8)

The five practices then can be seen as a way of enacting these five strategies.

Black and Wiliam (2009), drawing upon their earlier work (1998) and the Assessment Reform Group (2002), define formative assessment as:

Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited (p. 9).

One key feature of the definition of formative assessment is that learners, peers and teachers are the decision makers rather than only teachers. The definition also reveals the meaning of the probabilistic nature of learning where the outcome is likely to be better or better founded than without the evidence. Formative assessment is intentional in that evidence is intentionally collected to better inform instruction. Formative assessment is concerned with the “creation of, and the capitalization upon, ‘moments of contingency’ in
instruction for the purpose of the regulation of learning processes” (Black & Wiliam, 2009, p. 10). Teachers work becomes less predictable when using formative assessment because of the contingency.

Teachers’ classroom assessment can be *convergent* or *divergent* in nature. Convergent assessment is used to determine if students know, understand, or can do a particular thing, whereas divergent assessment determines what the students know, understand and can do (Torrance & Pryor, 2001). Divergent assessment is less about detailed planning but having open questioning and tasks. If the students raise new ideas, the teacher has to decide to pursue it or bring the conversation back to the original topic. Time and curriculum constraints pressure teachers to use convergent assessment (Torrance & Pryor, 2001). However, the pressure to use divergent assessment is strong when teachers wish to value every contribution, since it can advance learning and minimize rejection (Black & Wiliam, 2009). The notion of divergent assessment aids teachers to understand and use formative practices by raising the awareness of “cuing ‘right answers’ through routine, taken-for-granted teacher-student interactions” (Torrance & Pryor, 2001, p. 628). Teachers in Torrance and Pryor’s study believed both assessment types were useful, but found divergent assessment as “potentially more powerful in fostering the social and intellectual conditions in the classroom which would lead to enhanced learning” (Torrance & Pryor, 2001).

Teachers using formative interaction can base their decisions only on their interpretation of the evidence given by the students, after which, the teacher has to decide upon the best course of action. The “diagnostic in interpreting the student contribution …
and the prognostic in choosing the optimum response: both involve complex decisions, often to be taken with only a few seconds available” (Black & Wiliam, 2009). One important instrumental component is deciding what kind of feedback to provide to the students. Feedback given by the teacher has to be “carefully judged in terms of impact on the process of completing the immediate task at hand, and carrying forward understanding of substance and process to future activities” (Torrance & Pryor, 2001, p. 625). Black and Wiliam (1998) conducted an extensive review of literature on assessment. They found that feedback was crucial to the formative process and could have positive or negative effects on student achievement. Black and Wiliam (2003) struggled with defining good feedback and decided upon a simple characterization “good feedback causes thinking” (p.631) To make the most use out of feedback students should be taught how to interpret it and how to improve their work for the future given the feedback (Sadler, 1998).

Formative assessment has the potential to increase student achievement by enabling the teacher to teach within students’ ZPDs. Teachers are better able to negotiate meanings, since they better know where students are in their learning. Divergent assessment can create more engaging discussion (Torrance & Pryor, 2001). Better negotiation and more engaging discussion are key components thought to increase student representational thinking.

**Technology**

In today’s society, technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning (NCTM, 2000).
It is important to find the potential of technology to increase student understanding in general and in the areas of functions and representations. Technology can “not only help children learn things better, it can also help them learn better things” (Roschelle, Pea, Hoadley, Gordin, & Means, 2000, p. 5). The types of technology discussed here and of interest to the study are digital technologies, such as computers, calculators, and other electronic communication devices.

**Graphing Calculators**

Hollar and Norwood (1999) conducted a study that compared a class using a graphical approach in an algebra class with a more traditional algebra class. The students in the class with the graphical approach were placed there because they had the lowest score on the college's mathematics placement test. The students were given the function test developed by O'Callaghan, which tests the following aspects of conceptual knowledge of functions: (a) modeling real world situations, (b) interpreting a function in terms of a realistic situation, (c) translating among the different representations of functions, and (d) student ability to reify functions. Students in an intermediate algebra curriculum graphical approach did significantly better than those in the non-graphing approach in understanding functions on all four of the previous aspects of conceptual knowledge. However, the lowest score was with reification because that is something that cannot be taught (Hollar & Norwood, 1999).

Other studies were conducted to see what impact handheld graphing technology had on students and found a positive impact on students’ ability to link representations and understand the attributes of functions. In addition, students’ ability was better compared
to those who did not use the technology (Harkskamp, Suhre & Van Streun, 2000, Thompson & Senk 2001).

Graphing technology can aid students in linking the different representations of functions as well as increase their understanding of functions. However, graphing calculators have their drawbacks. Graphing calculators have their own limitations in their capabilities that students using the technology may unaware. Some examples are limited resolution, drawing "asymptotes" when they should not, and the inability to perform certain computations such as numerically integrating certain functions. Further, while their use can inhibit small group work, they work well for whole class discussion (Doerr & Zangor, 1999).

**Classroom Response Systems**

Classroom response systems (CRS) are becoming more common in universities and are typically used to collect attendance and summative assessment data. CRS connect devices that can collect information from students to the teacher’s computer. Typically, CRS can aggregate the data collected and display it to the whole class. The devices can be rather simple to complex, ranging from devices with a single button for “Yes” to graphing calculators or hand held computers. The simpler devices are sometimes called “clickers”. The CRS technology can aid teachers to “engage in such best practices as addressing students prior knowledge, targeting conceptual understanding, motivating and engaging all students, facilitating group discussions, and questioning students and providing frequent feedback” (Roschelle, Penuel, & Abrahamson, 2004, p. 52).
Fies and Marshall (2006) conducted a literature review on the use of CRS in classrooms. While the literature reviewed was limited in scope and rigor, some promising results were found. Nine of the fifteen studies found increased engagement and more participation during discussions. Seven studies found that teachers who used the technology were more aware of students’ thinking. Seven other studies found that students also became more aware of their own thinking (Fies & Marshall, 2006). Roschelle et al. (2004) conducted another review of CRS literature and found the main benefits of using CRS technology are “greater student engagement (16 studies), increased student understanding of complex subject matter (11), increased student interest and enjoyment (7), heightened discussion and interactivity (6), increased student awareness of individual levels of comprehension (5), and increased teacher insight into student difficulties (4)” (2004, p. 52). The studies included in the review were not rigorous enough to draw conclusions and do not distinguish using the technology with the pedagogy that it enables (Fies & Marshall, 2006; Roschelle et al., 2004). Evidence shows however, that the use of CRS technology aids formative assessment by overcoming the hurdle of classroom data management (Roschelle et al., 2004).

Crossgove and Curran (2007) conducted a study to determine if students in university biology courses taught using clicker technology performed better than students taught without the technology. Two courses were investigated, a non-majors biology course and a majors genetics course. During fall 2004, and spring 2005 both courses were taught without using clicker technology. Clicker technology was used during fall 2005, spring 2006, and fall 2006. The clicker technology was used to change the nature of the lectures.
given in the courses. Approximately two to eight clicker questions were given per lecture. These questions were of several types: warm-up questions over the previous day’s materials, conceptual questions over current topics or problems given in the genetics course. If less than 70% of the students were correct on a clicker question, students had to discuss answers with peers and then were re-pollled the question. If less than 70% answered correctly the second time the teacher would explain it in another way and usually during the next lecture a similar question would be asked to check understanding.

Students were given an opinion survey about clickers. The final exam in each course was comprised of multiple choice questions that were broken into clicker questions and non-clicker questions. Clicker questions were questions over topics covered by the use of clickers. Students were also invited to return in four months to take another test to determine knowledge retention.

According to the opinion survey students believed that clickers helped them understand the material (79%), clickers stimulated interaction (71%), and it helped connect ideas together (69%) (Crossgrove & Curran, 2008). In the free response portion of the survey, many students stated they valued instant feedback on their understanding as well as increased participation in lecture. Ten questions on the final exam had to be the same in each class for each semester. No significant difference (Wilcoxon z = -0.652, p=0.515) in student performance on these ten questions was found between students taught in clicker lectures or non-clicker lectures (Crossgrove & Curran, 2008). A more detailed analysis was conducted on the fall 2006 exam, when all sections were taught with clickers. Crossgrove and Curran (2008) compared student performance on clicker
questions and non-clicker questions and found significant difference between them
(F=7.300, d.f. =1, p=0.007). They used Bloom’s Taxonomy to partition the questions and
determine if different types of questions were affected differently and found that across
all types of questions students performed better on questions from topics covered by
clickers (F=11.569, d.f. = 1, p=0.0001).

Four months after the final exam only fifteen students took a test to determine
retention of concepts even with the offer of free food. No significant difference
(Wilcoxon z= -.892, p=0.373) was found between performance of students on clicker
questions on the exam and the test four months later, so they performed about as well as
they did four months prior. However, a significant difference (Wilcoxon z= -2.668, p=
0.008) was found for the performance on non-clicker questions. In other words, the
students performed worse on the non-clicker questions over time. Students were better
able to retain information taught with clickers than information taught without
(Crossgrove & Curran, 2008).

Although Crossgrove and Curran (2008) did not distinguish the use of CRS
technology and the instructional pedagogy used, they did conduct a scientific study on
student performance influenced by CRS use. The teaching techniques surrounding the use
of clickers in this study utilized the tenets of formative assessment. Even with CRS
enhanced formative assessment used only a few times per lecture, student performance
and retention was increased on topics taught using CRS.
**Technology-Enhanced Formative Assessment**

Beatty and Gerace (2009) acknowledged that little research has attempted to distinguish between CRS use and pedagogy. Their goal was to develop pedagogy, called technology-enhanced formative assessment (TEFA), to capitalize on the capabilities of CRS to improve its usage. In developing TEFA, Beatty and Gerace asked themselves “what pedagogical approaches a CRS can aid or enable or magnify, and what the learning impacts of those various approaches are” (2009, p. 147). TEFA was developed to help students learn science and prepare them for future learning. Although TEFA was developed with science teaching in mind, it could be adapted to be used in mathematics and other areas.

Four key principles of TEFA are grounded in educational research (Beatty & Gerace, 2009). These core principles are:

1. Motivate and focus student learning with question driven instruction.
2. Develop students’ understanding and scientific fluency with dialogical discourse.
3. Inform and adjust teaching and learning decisions with formative assessment.
4. Help students develop metacognitive skills and cooperate in the learning process with meta-level communication. (Beatty & Gerace, 2009, p. 152).

Beatty and Gerace determined that TEFA should not be a set of rules, tips, or best practices, but based on general, flexible core principles.

**Question driven instruction (QDI).** QDI positions learning “within students’ encounter with questions – often conceptually rich, meaty, messy, challenging ones – to provide context, motivation, and directions to students’ sense making efforts” (Beatty & Gerace, 2009, p. 153). They explain that questioning should occur during all points of instruction. When introducing material questioning should be used to motivate and give it
context as well as to explore it and connect it to other concepts. During elaboration of concepts, Beatty and Gerace claim that questioning should be used to find gaps in students’ understanding, challenge students enabling them to self assess. Beatty and Gerace claim that using QDI helps students “frame knowledge as ‘stuff to answer questions with’ and learning as ‘figuring out how to answer questions’” (2009, p. 153), rather than knowledge as a bunch of facts and learning as memorizing those facts.

**Dialogical discourse (DD).** Beatty and Gerace (2009) explained that developing student understanding with dialogic discourse means to engage students in discourse with multiple ideas, approaches, and conclusions that are explored and challenged. DD comprises similar tenets of classroom discourse discussed above. DD is intended to expose students to different ways of thinking, promote good argumentation, and promote individual sense making. Beatty and Gerace (2009) claim that within TEFA most of the learning occurs during DD and suggest that creating quality discourse should be a priority for teachers.

**Formative Assessment.** The third principle for TEFA is using formative assessment (FA) to inform and adjust teaching. Beatty and Gerace (2009) use an earlier definition of formative assessment as assessment for learning that is used to adjust teaching (Black, Harrison, Lee, Marshall, & Wiliam, 2002), which is similar to Black and Wiliam’s (2009) definition. They explain that formative assessment is needed in TEFA to provide teachers with current and detailed information about a student thinking and understanding as well as to aid student’ understanding and see their own shortcomings in understanding.
Meta-Level Communication (MLC). MLC within TEFA is intended to help students develop metacognitive skills. Beatty and Gerace include MCL within TEFA “to (a) improve learning by increasing the efficiency of the instructional process, and to (b) improve the learning by promoting and scaffolding student development of more productive learning beliefs, attitudes, and behaviors” (2009, p. 155). Three types of discourse are used in MLC: meta-narrative, communication about purpose and design; metacognitive talk, communication about thinking, learning and knowledge; metacommunicaiton, communication about communication (Beatty & Gerace, 2009).

Synergy among principles. The four principles of QDI, DD, formative assessment and MLC are not independent but highly interconnected; without one the rest unravel (Beatty & Gerace, 2009). FA is used to inform tuning the questions of QDI as well as to provide opportunity for DD. “DD requires a context and focus, and engages students more when they have at least provisionally committed to some position; QDI arranges these. QDI, DD, and FA are all aided by students’ active, well-intentioned, well-informed cooperation; MLC helps cultivate that” (Beatty & Gerace, 2009, pp. 156–157).

Role of technology within TEFA. Using the TEFA pedagogy does not require the use of a CRS, however a CRS greatly aids the principles of TEFA. One of the main benefits of CRS technology is that it allows for real time assessment. CRS provide anonymity and accountability for students and all students are encouraged to participate rather than a few who are called upon. Students actually have to submit an answer, so that they are not allowed to waffle and benefit from choosing sides (Beatty & Gerace, 2009). Student submissions become their answers; students are encouraged to pay more
attention during discussions. The chart created from aggregated student responses can show class agreement, undecidedness, and polarity among of students’ response. CRS can also record student data for further use, which additionally supports FA by helping the teacher see class-wide or individual needs (Beatty & Gerace, 2009).

**CRS and Interactive Representations**

Hegedus and Kaput (2004) were involved in the SimCalc project designed to allow students to move beyond the algebra barrier. The SimCalc project has developed strategies that use the interactive representational affordances of technology (visualization, linking representations to each other and to simulations, importing physical data into the mathematical realm in active ways, graphically editing piecewise-defined functions, etc.), to energize and experientially contextualize existing algebra courses, and to do so in ways that lay the base for more advanced mathematics, particularly calculus (Hegedus & Kaput, 2004, p. 129).

They sought to combine the affordances of representational innovations with the new affordances of connected classroom technology (Hegedus & Kaput, 2004). Hegedus and Kaput came to see classroom connectivity (CC) as a critical means to unleash the long-unrealized potential of computational media in education, because its potential impacts are direct and at the communicative heart of everyday classroom instruction” (Hegedus & Kaput, 2004, p. 130). They believe that classroom connectivity technology is the first true educational technology.

In Hegedus’ and Kaput’s (2004) study, students (n=25) enrolled in their afterschool intervention program performed significantly better ($p< 0.001$) on posttests determining algebraic thinking than their pretest, with high effect size (Cohen’s $d = 1.80sd$). The class was set up in rows with computers, the teacher and teacher’s computer off to the side and
projector display in the front. One key component of instruction during the afterschool intervention was to assign students a unique number for use in activities. The numbers they used were a combination of the row they were in (group number) and the position within the row (Count off number). Students were then attached to a group, while keeping an individual identity. Using the number in dynamic ways created interesting learning (Hegedus & Kaput, 2004). The number may be used to explore relationship of “b” in a line, such as \( Y = 2X + \text{Group Number} \). In a more complex situation students may explore varying multiple parameters, such as \( Y = (\text{Count off number}) X + \text{Group number} \). The effects of the varied parameters were typically discussed in depth.

The SimCalc classroom created a learning environment with more intensity, structure and participation (Hegedus & Kaput, 2004). The use of the identification number gave students responsibility to themselves, their group and the whole class through the constructions of linear functions. The SimCalc classroom allowed students to shift their attention from static, inert representations to dynamic personalized representations (Hegedus & Kaput, 2004).

**Connected Classroom Technology**

Combining the pedagogies of formative assessment and classroom discourse with the CRS technology and interactive representation technology would seem to promote a rich learning environment in general, but particularly useful in developing student representational thinking. Connected Classroom Technology (CCT), for example the TI-Navigator, is a combination of CRS technology and interactive representation technology. CCT is a classroom network device that connects students’ devices to a
computer that the teacher uses. The computer is generally projected onto a screen so the whole class can see. The teacher can send a question from the computer to the students' device, then retrieve all the students’ answers and project them on the screen to discuss. With some CCTs entire quizzes can be sent and retrieved in this manner as well.

The CCMS project investigated the impact of the use of CCT on student algebra achievement by using a randomized cross over trial design (Owens et al., 2008). The project used cohort number *dot* year of study number to differentiate each treatment. For example treatment 1.1 is Cohort 1 during year 1 of the study, where as treatment 2.3 is Cohort 2 during year 3 of the study. The data from Cohort 2 year 1 served as a control group called control 2.1 for contrast with each other treatment group. Hierarchal linear modeling was used to determine if there was any difference between the treatment and control in student algebra achievement. During years two and three of the study the pretest was administered late among many teachers prohibiting an assessment of algebra knowledge prior to that year’s instruction. Two models were used; one that included the pretest scores as a covariate and one that did not. The results are summarized in Table 2.1.
<table>
<thead>
<tr>
<th>Groups compared to Control 2.1</th>
<th>Pretest Controlled</th>
<th>Effect Size</th>
<th>p-value</th>
<th>Point increase</th>
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</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>0.30</td>
<td>0.034</td>
<td>1.85</td>
</tr>
<tr>
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<td>0.23</td>
<td>0.154</td>
<td>1.27</td>
</tr>
<tr>
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<td>No</td>
<td>0.36</td>
<td>0.044</td>
<td>2.59</td>
</tr>
<tr>
<td>Treatment 1.3</td>
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<td>1.34</td>
</tr>
<tr>
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<td>0.080</td>
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</tr>
<tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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</tr>
<tr>
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<td>Treatment 2.3</td>
<td>No</td>
<td>0.19</td>
<td>0.046</td>
<td>1.29</td>
</tr>
</tbody>
</table>

Note. Modified from “Algebra Achievement over Four Years in TI-Navigator\textsuperscript{TM} Connected Classrooms” by Owens, D. T., Pape, S. J., Irving, K. E., Abrahamson, L., Silver, D., Sanalan, V. A., & Morton, B. Table 2.1 Summary of student algebra achievement results over three years.

During years 1 - 3, all treatments produced moderate effect sizes ranging from $ES=0.19$ to $ES=0.36$ with significant increases ($p < 0.05$) or approached significance ($p < 0.1$) according to one of the two models (Irving et al., 2010). The treatments with significant effects increased student algebra scores by 1.07 to 2.65 points on average. Other than the pretest score the only other significant covariate was years of teaching experience. Years of teaching experience increased student scores by 0.06 to 0.21 points per year on average according to most models (Irving et al., 2010). The use of CCT supported with appropriate professional development produced effects comparable to ten to fifteen years of teaching experience.

For a large national randomized control study the moderate effect sizes of the treatments are uncommon (Irving et al., 2010). Irving et al. (2010) argue that the use of a
true control group provide strong evidence that the treatment caused the student achievement increase. They argued further that the treatment was the combination of the technology in the classroom and professional development to support its use and hypothesized that improved formative assessment and classroom discourse can explain the gains in student algebra achievement. Improved formative assessment and classroom discourse should provide more opportunities for the development of student representational thinking.

Pape et al. (2010) conducted additional analysis on the classroom interactions of CCMS teachers who used CCT to investigate how interactional patterns within these classrooms relate to growth in treatment classrooms. Specifically they compared the patterns in the similarities and differences between the treatment and control classrooms, and the relationships between classroom interaction constructs and student achievement.

In the first year of the CCMS project, 33 teachers participated in the classroom observation study. The audio file from the two-day classroom observation was transcribed verbatim. The videotapes and transcripts were coded using a priori codes and then reviewed by another coder to identify discrepancies (Pape et al., 2010). Variables used were standardized to 60 minute class periods. Analyses were conducted to describe the interactional patterns, compare the patterns between treatment and control and examine if these constructs related to achievement.

In examining questioning patterns of teachers, Pape et al. (2010) found most of the questions asked by teachers were recitation type questions that elicited short, lower-order responses from students. These interactions usually formed *Initiate-Evaluate-Response*
(IRE) patterns. The interactions of *uptake*, using student response as momentary focus of discussion, pressing for involvement, explanations and/ or justifications were rare (Pape et al., 2010).

Comparing treatment and control classrooms, Pape et al. (2010) found control teachers asked more questions (*p* = 0.002). However, these questions were more often IRE sequences (*p* = 0.005) that elicited lower-order responses (*p* = 0.001). They concluded that these differences were related to the mathematical comments made by teacher and students. The words per mathematical comments were greater in treatment classroom than control (*p* = 0.015).

To examine the relationship between classroom interaction patterns and student achievement, Pape et al. (2010) first used correlations of constructs and achievement, and then relied on regression analysis to determine the final set of variables explored. They found that the number of high order questions, and the ratio of high order (HO) question to low order (LO) questions were related to the posttest achievement. The ratio of HO to LO question remained positively associated with achievement after accounting for pretest score in the regression model. The numbers of words per comment, number of instances of uptake were positively associated with posttest achievement, whereas the number of IRE episodes was negatively associated with achievement (Pape et al., 2010). Pape et al. (2010) stated that assumptions of causality should be made with caution. The researchers concluded that the use of the CCT interrupted the typical questioning patterns in classrooms. Most classroom interactions required a low cognitive load. They also concluded that the higher order questions had the largest impact on student achievement.
Disrupting the traditional flow of IRE episodes that elicit short lower-order student responses could allow for more substantial negotiation of meanings and potentially allow students to use representations as tools for explanations, and thinking more often. The use of connected classroom technology with appropriate pedagogy seems to offer potential to increase student representational thinking.

Treatment of Technology

Goos et al. (2003) conducted a 3-year longitudinal study to investigate the role of digital technologies in supporting “students’ exploration of mathematical ideas and in mediating their social interactions with teachers and peers” (2003, p. 73). They developed four metaphors for how teachers and students treat technology: technology as “master”, “servant”, “partner”, and “extension of self.”

Goos et al. (2003) claimed that teachers and students who are limited to a narrow range of the technology’s capabilities, due to their knowledge of the technology may be subservient to this technology. Further, they claimed that student subservience may become dependence from lack of mathematical authority to assess if outputs of the technology are reasonable. Teachers could treat technology as master when they lack technical expertise and allow demonstrations from a student “expert,” or when they are reluctant to allow students to use the technology to explore mathematical ideas (Goos et al., 2003).

Goos et al. (2003) asserted that teachers who used technology as a more efficient and reliable replacement to pen and paper calculations, while not altering their usual instruction treated the technology as servant. “That is, technology is a supplementary tool
that amplifies cognitive processes but is not used in creative ways to change the nature of activities” (Goos et al., 2003, p. 78). Students treated technology as servant when they used it for repetitive calculations, checking work, or a quicker way to do computations. Goos and colleagues (2003) argued that technology is a servant from the teachers’ perspective if it merely serves to support their preferred teaching methods, such as a when a teacher elicits student responses with CCT and corrects them as in the same manner without the CCT. However, they noted that teachers who treat technology as master could use technology as an intelligent servant that compliments effective components of their instruction (Goos et al., 2003).

Goos et al. (2003) argued that technology can become a partner when it is used to mediate mathematical classroom discourse, or when it is used to allow student mathematical exploration. For example, “instead of functioning as a transmitter of teacher input, the overhead projection panel can become a medium for students to present and examine alternative mathematical conjectures,” (2003, p. 79). Another example occurs when teachers encourage students to explore differences in their solutions made visible to the class by CCT. In the classrooms observed by Goos et al. (2003), graphing calculators and computer promoted peer discussion in small groups when students gathered to compare screens to emphasize a point or compare work. Some students even developed a rapport with the technology; where responses from the technology prompted a student to self-investigate or to seek help to understand the responses.

Goos and colleagues (2003) declared that the most sophisticated mode, called extension of self, for functioning with technology involved incorporating technological
expertise into a natural part of their mathematical or instructional repertoire. Technology that is an extension of a student’s self allows her to construct mathematical arguments so that technology extends her mathematical prowess (Goos et al., 2003). Teachers who treat technology as an extension of self use it to promote collaborative inquiry and mathematical argumentation among the students.

Goos et al. (2003) found that the teacher in their study could help students move through the different modes of technology treatment by the tasks used and encourage students to consult with one another as well as inviting them to present their findings while the rest of the class was encouraged to offer constructive criticism. Their investigation (2003) demonstrated that digital technology is not neutral, since it can change the way teachers, students and the technology interact. They argued that the most significant challenge lies in the teacher’s ability “in orchestrating collaborative inquiry so that control of the technology, and the mathematical argumentation it supports, is shared with students” (Goos et al., 2003, p. 88). However, if a teacher could meet this challenge then students would seem to be able to develop more understanding of representations if teachers and students treated technology as a partner or as an extension of self.

**Technology, Content, and Pedagogical Knowledge (TPACK)**

Mishra and Koehler (2006) conducted a five year investigation to understand teachers’ development of rich mathematics with the uses of technology while trying to help them with this development. From this investigation, they developed a framework for the types of knowledge, technology, content, and pedagogy, teachers’ need for rich teaching with technology. Mishra and Koehler (2006) claimed that the current notion at
the time was that technology knowledge was viewed as separate from both content and pedagogy knowledge and that the relationship between these types of knowledge did not exist or was considered to be trivial. In contrast, they argued that their view of technology “emphasizes the connections, interactions, affordances, and constraints between and among content, pedagogy, and technology” (Mishra & Koehler, 2006, p. 1025). They further claimed that “quality teaching requires developing a nuanced understanding of the complex relationships between technology, content, and pedagogy, and using this understanding to develop appropriate, context-specific strategies and representations” (Mishra & Koehler, 2006, p. 1029). They stated that to understand how to foster good teaching, researchers need to look at the pairs of the three knowledges: pedagogical content knowledge (PCK), technological content knowledge (TCK), technological pedagogical knowledge (TPK), as well as all three combined technological pedagogical content knowledge (TPCK). Mishra and Koehler (2006) used Shulman’s (1986) PCK framework as a basis for their TPCK framework and defined CK, PK, and PCK in a similar manner. Their model extended definitions to acknowledge the interplay between technology and the other two knowledges for teaching. They define technical knowledge (TK) as knowledge and skills needed to operate standard technologies. TCK was defined as knowledge about how technology and content interact as well as knowledge of how content can be changed with the implementation of technology and TPK was defined as knowledge of the technology’s capabilities in the instructional setting (Mishra & Koehler, 2006). Mishra and Koehler claim that TPCK is an emergent form of knowledge that is more than the sum of its three components and defined TPCK to be
the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students’ prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (2006, p. 1029)

Later TPCK became known as TPACK for ease of pronunciation and to emphasize the notion that technology, content, and pedagogy knowledge form the Total PACKage for teaching (A. D. Thompson & Mishra, 2007).

Bowers and Stephen (2011) interpreted the notion of teachers using technology in “appropriate and responsible ways” to mean using technology to explore mathematical relationships. Bowers’ and Stephen’s (2011) vision of Mishra’s and Koehler’s (2006) TPACK framework to be “to help teachers develop a technological habit of mind oriented toward using advanced computation and communication tools to help students explore and understand the underlying concepts and their relation to the larger world outside of school” (2011, p. 286). Mishra and Koehler left the details of specific knowledge within TPACK to other researchers (Bowers & Stephens, 2011). Bowers and Stephens (2011) concluded that TPACK is not comprised of a list of specific knowledge pieces but rather an orientation that views “technology as a critical tool for identifying mathematical relationships” (2011, p. 290) and the goal of the development of pre- and in-service teachers is to develop this orientation. Bowers and Stephens (2011) and Goos and Benson (2002) found that teachers who were simply given access to, and training with technology did not develop a TPACK orientation. Teachers exhibiting a TPACK
orientation to technology seemed as if they would use technology to critically examine the relationships of representations present to aid the development of student representational thinking. The next section discusses how to potentially develop a TPACK orientation.

**Developing a TPACK Orientation**

Niess et al. (2009) further developed a model proposed by Niess, Sadri, and Lee (2007) for describing teachers’ TPACK development with respect to four areas of instruction: curriculum and assessment, learning, teaching, and access. Niess, Sadri, and Lee’s (2007) proposal for teacher’s development of TPACK emanated from Rogers’ (1995) model for the diffusion of innovation. Niess et al. (2009) stated that Niess, Sadri, and Lee (2007) reframed this process to address mathematics teachers integrating a new technology into their instruction using observations of teachers learning how to use and how to integrate spreadsheets into their mathematics classrooms and found that teachers advanced through five developmental stages:

1. **Recognizing** (knowledge), where teachers are able to use the technology and recognize the alignment of the technology with mathematics content yet do not integrate the technology in teaching and learning of mathematics.
2. **Accepting** (persuasion), where teachers form a favorable or unfavorable attitude toward teaching and learning mathematics with an appropriate technology.
3. **Adapting** (decision), where teachers engage in activities that lead to a choice to adopt or reject teaching and learning mathematics with an appropriate technology.
4. **Exploring** (implementation), where teachers actively integrate teaching and learning of mathematics with an appropriate technology.
5. **Advancing** (confirmation), where teachers evaluate the results of the decision to integrate teaching and learning mathematics with an appropriate technology (Niess et al., 2009).
Niess et al. (2009) claimed that an important caveat for this development model was that progression through the level is not always increasing and that teachers may be at different stages for different technologies for certain content or pedagogies. For example, a teacher may be at the adapting level for using graphing calculators for exploring the relationships between the equation and graph of a line, but be at the recognizing level for using CCT for this topic. Also, this same teacher could be at the accepting level for using the memory of graphing calculators to develop the notion of variable. Niess et al. (2009) described examples for teachers at different levels for using calculators’ capability to compute square roots. They (2009) claim that teachers at the recognizing level realize that calculators can be used to aid mathematical processes. Then teachers at the adapting level will try tasks that mimic what they already have done, potentially using it for comparing calculator results with estimates of square roots. Teachers at the exploring level look for ways to shift to concepts of the square root rather than the procedures to find them. At the advancing level, teachers look for ways to use the calculators for other topics, but explore how other tasks can change as a result of the calculators’ capabilities. Niess et al. (2009) developed indicators for how the different levels apply to four major themes of instruction: curriculum and assessment, learning, teaching, and access and claimed that teachers could be at different levels of incorporating technology into their instruction for different themes. For example, a teacher could be at the exploring level for using technology in class activities but restrict access to technology for student assessments. Niess et al. (2009) asserted that this model for developing TPACK needs to be tested and that the set of experiences for moving from one level to another needs to be
explored. They claimed that the set of experiences needed for progression within the TPACK levels may be different for different levels, teachers, and technologies.

Polly and Hannafin (2010) synthesis American Psychological Associations’ Learner-Centered Principle as quoted in Polly’s (2011) article claim that Learner-Centered Professional Development (LCPD) programs should;

- Focus on student learning outcomes
- Provide teachers with ownership of their professional development activities
- Address knowledge of both content and pedagogy
- Support reflections on teachers daily work,
- And include ongoing activities. (Polly, 2011, p. 84)

The purpose of Polly’s (2011) study was to examine participants use of technology-rich mathematical tasks in the classroom during their participation in a LCPD project, and how the enactment of these tasks revealed the teachers’ TPACK and how this aligned with the TPACK the project intended to foster. The purpose of Polly’s (2011) study can be interpreted as describing some experiences that could move teachers’ TPACK levels of incorporating technology. The goal of the professional development Polly (2011) implemented was to “improve student learning by supporting teachers’ enactment of specific standards-based pedagogies including rich mathematical tasks, using technology as a tool to support learning and posing high-level questions” (Polly, 2011, p. 84). This goal was to be achieved by

- Giving rich mathematical tasks to the teachers to work on with colleagues.
- Modeling: how to pose the task, how to use the technologies, and how to support student work through questioning rather than guiding the completion.
- Facilitating discussions of the teacher-participants approaches, the mathematical concepts embedded within the tasks, and how the task could be incorporated into their classroom.
Encouraging teachers to complete rich tasks with technology and discuss student learning while completing these tasks.

- Scaffolding teachers’ instruction by co-planning lessons with each teacher.
- Encouraging teachers to incorporate technology and mathematically rich tasks into their classrooms (Polly, 2011, pp. 85–86).

Polly (2011) found that the teachers in his LCPD project did incorporate technology into their classrooms. However, teachers rarely integrated technology into the lessons at a high level that provided opportunities for students: “to engage in an activity where the teacher uses technology and the activity involves solving problems and tasks with the assistance of the technology, … or to use technology to develop their mathematical knowledge and/or problem solving skills” (Polly, 2011, p. 88). On the occasions where teachers did incorporate technology at a high level, their lessons did not align with the pedagogies emphasized during the professional development.

Polly (2011) determined that for his participants there was not clear relationship between the planning support and the teachers’ implementations of the technology and mathematically rich tasks, but the co-planned lessons resulted in developing a TPACK orientation to those lessons. He proposed that a professional development program should: allow teachers to co-plan with more knowledgeable individuals such as coaches, be open-ended to account for differences in teacher knowledge, and provide support outside workshops.

**Decisions and Orientations**

Schoenfeld (2010b) claimed that “a teacher’s in-the-moment decision-making can be explained – indeed, modeled – as a function of the teacher’s knowledge [resources], goals, and orientations” (p. 84). He spent much of the last decade to investigate how the
interaction of the three components of resources, orientations, and goals produce teachers’ in-the-moment choices during teaching and developed a theory to model these interactions (Schoenfeld, 2010a). In the model, he defined orientations to be an amalgam of terms previously referred to in the literature as beliefs, dispositions, values, and preferences. A teacher’s resources include his or her knowledge as well as social and material resources (Schoenfeld, 2010b). Schoenfeld (2010b) described goals to be something people either consciously or unconsciously set out to achieve, and that people act towards attaining those goals by obtaining appropriate resources. Further, he claimed that teachers can be actively trying to attain a range of goals at different levels. He asserted that
decision making during teaching can be seen as the selection of goals consistent with the teacher’s resources and orientations… every sequence of actions [of the teacher] can be seen as consistent with a series of goal prioritizations that are grounded in the teacher’s beliefs and orientations, and the selection, once a goal has been given highest priority, of resources intended to help achieve that goal. (Schoenfeld, 2011, p. 460).

Applying this model to effectively incorporate CCT use into the classroom, teachers would need appropriate resources (access to the technology and TPCK for using it), goals for both mathematical and technological outcomes, and be oriented to using the technology to explore both mathematics and student thinking. Schoenfeld (2011) asserted that teachers develop their knowledge and perceptions about mathematics, mathematics teaching, and student thinking. These knowledge and perceptions experiences become part of their orientations that shape their teaching practices (Schoenfeld, 2011). Given the influence of a teacher’s orientation on shaping their instruction, a teacher’s conceptual or
calculational orientation (A. G. Thompson et al., 1994) to mathematics teaching could determine how CCT is used in the classroom.

A. G. Thompson et al. (1994) claimed that the middle school teachers in their study engaged with their students and the tasks differently because of fundamental differences in the teachers’ orientations to teaching. A. G. Thompson et al. (1994) offered two contrasting views to mathematics teaching: calculational, and conceptual. They characterize conceptual teachers as focusing students away from thoughtless application of procedures and toward a rich conception of ideas, situations, and relationships among ideas. Conceptual teachers focused on aspects of the situation that give meaning to the representations present and that in turn suggests the procedures that might be useful (A. G. Thompson et al., 1994). They claim that a conceptual teacher’s actions are driven by:

- An image system of ideas and ways of thinking that she intends students to develop;
- An image of how these ideas and ways of thinking can develop;
- Ideas about features of materials, activities, and exposition and the students’ engagement with them can orient the students’ attention in productive ways (a productive way of thinking generates a “method” that generalizes to other situations);
- An expectation and insistence that student should be intellectually engaged in tasks and activities (A. G. Thompson et al., 1994, p. 86).

A teacher with a calculational orientation to teaching is driven by a view that mathematics involves the application of calculations and procedures to produce numeric results and view mathematics learning as skills need for “doing” mathematics to get correct answers (A. G. Thompson et al., 1994). They described symptoms of a calculational orientation to teaching mathematics:

- A tendency to speak exclusively in the language of numbers and numerical operations.
A predisposition to cast solving a problem as producing a numerical result.
- An emphasis on identifying and performing procedures.
- A tendency to do calculations whenever occasion to calculate presents itself regardless of the overall context in which the occasion occurs.
- A tendency to disregard the context in which calculations might occur and how they might arise naturally from an understanding of the situation itself.
- An inclination to remediate students’ difficulties with calculational procedures independently of the context in which the difficulties manifest themselves.
- A tendency to treat problem solving as flat; that is nothing about problem solving is any more or less important than anything else, except that the answer is most important because getting the answer is the reason for solving the problem.
- A narrow view of mathematical patterns as limited to finding pattern in numeric sequences and in the sameness of operation across problems, as opposed to finding patterns in reasoning in the solution of the problems (A. G. Thompson et al., 1994, p. 87).

Although A. G. Thompson et al. (1994) described a calculational orientation to mathematics using numbers and numeric procedures, the descriptors of the calculational orientation can be modified to address algebra and symbolic manipulations. The algebraic version of the calculational orientation is the procedural orientation, described in more detail in Chapter 3. The conceptual orientation to teaching more closely resembles the TPACK orientation to teaching with technology than does the calculational orientation. Hence, teachers with a more conceptual orientation to teaching would likely incorporate technology more appropriately into their instruction.

Summary

Developing representational thinking requires intensive social co-construction of meaning of the representations, as well as practicing reasoning with and about representations (Pape & Tchoshanov, 2001). Understanding the concept of function, therefore, requires the understanding and flexibility to move between the different internal and external representations of functions (Moschovich, Schoenfeld, & Arcavi

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1993, Sfard 1992). Students must be given time to practice communicating and reasoning with these representations (Pape & Tchoshanov, 2001), and this practice must be intentional. The classroom has to support extensive social negotiation of the meanings of these representations (Cobb et al., 1993; Pape & Tchoshanov, 2001). Formative assessment and classroom discourse can provide a rich environment for students to engage in social negotiation (Black & Wiliam, 1998; McNair, 2000; Morrone et al., 2004). Connected classroom technology can aid formative assessment, discourse, and presentation of representations (Beatty & Gerace, 2009; Crossgrove & Curran, 2008; Hegedus & Kaput, 2004). Therefore, appropriate use of connected classroom technology should reasonably aid student representational thinking.
CHAPTER 3: METHODOLOGY

The purpose of this study is to create an empirically based theoretic model of change of the use of representations of functions in the classroom in the presence of connected classroom technology using data from the Classroom Connectivity in Promoting Mathematics and Science Achievement (CCMS) project.

CCMS project

The CCMS project is a randomized cross-over trial where the control group received the intervention sequentially and was generously funded through, Grant Number R305K050045 to The Ohio State University, by the Institute for Educational Sciences (IES) division of the U.S. Department of Education. The teacher participants were assigned to two cohorts by random selection. Cohort 1 treatment group received professional development on how to teach with connected classroom technology (CCT) during a week-long summer institute. Each year they were in the project they received follow-up professional development during an international technology convention (Irving et al., 2010). Cohort 2 teachers received similar professional development the summer before their second year of teaching in the project. Cohort 2 teachers used graphing calculators without the connected classroom technology with their students during the first year. The surveys and tests given to this cohort during the first year served as control data for comparison with treatment groups.
Participants of CCMS

Algebra 1 teachers from 28 states and 2 Canadian provinces and their students participated in this project. Initially, a total of 127 teachers (66 control, 61 treatment; 74% female) were randomly assigned to either control or treatment groups (Irving et al., 2010, p. 8). During the course of the project some teachers left the project or their data were unable to be collected.

Since Cohort 2 became a treatment group in years two through four of the project a cohort number dot year of project number was given to each treatment. For example treatment 1.1 is Cohort 1 during year 1 of the project, where as treatment 2.3 is Cohort 2 during year 3 of the project. The data from Cohort 2 year 1 served as a control group called control 2.1.

The student participants of the CCMS project were students of the teachers from both cohorts during year 1 to year 4 of the project. Each treatment and control group ranged from 271 students to 696 students. The total number of students participants enrolled each year of the project was typically over 1100 with a minimum of about 900. Students mostly identified themselves as white (60-75%) and 43-57% were female (Irving et al., 2010).

Participants for this Study

Since the purpose of the study was to create a model of growth in the use of representations of functions, only video-taped teachers who had instructed lessons using representations of functions for more than one year of the study were considered for investigation. Seven teachers were initially identified as suitable candidates for
participants for this study. After reviewing videotapes, teachers with similar instruction were catalogued. In order to better create a model for teacher growth relying on a diverse population, the number of participants was reduced to allow for more in-depth analysis of individual teachers. The teachers’ responses to the Teacher Instructional Practices and Beliefs Survey (TIBPS), described in more detail later, were studied to find the most diverse group of participants to be used. The candidates selected included Mr. L., Ms. A., and Ms. B. The participants’ school demographic data is displayed in Table 3.1.

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Location</th>
<th>Native Indian</th>
<th>Asian</th>
<th>Hispanic</th>
<th>African American</th>
<th>Caucasian</th>
<th>Free Lunch</th>
</tr>
</thead>
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<td>Urban Fringe of a Mid-size City</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>1</td>
<td>1</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>89</td>
<td>19</td>
</tr>
<tr>
<td>Ms. B</td>
<td>Rural Urban Fringe of a Large City</td>
<td>0</td>
<td>13</td>
<td>5</td>
<td>9</td>
<td>73</td>
<td>29</td>
</tr>
<tr>
<td>Mr. L</td>
<td>Urban Fringe of a Large City</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>7</td>
<td>89</td>
<td>8</td>
</tr>
</tbody>
</table>

Note: All numbers are percentages.
Table 3.1: Demographic data of participants’ schools.

**Data Sources**

Both quantitative and qualitative data were collected by the CCMS project. A subset of the project’s data sources were used in the current study. Teacher-level measures, teacher telephone interviews, teacher Post Observation Interviews, and video data from classroom observations collected for CCMS comprised the data set for this study. The student-level measures from the CCMS project were not used as the current study is focused on a developing a model for teacher growth.
Teacher-level Measures

The teacher participants completed three surveys, Teacher Instructional Practices and Beliefs Survey, Technology and Professional Development Use, and Demographic Information Form. Cohort 1 teachers participated in telephone interviews twice each year, while Cohort 2 teachers participated in the interview during years 2 through 4.

Demographic Information Form. All teachers during year one of the project completed the demographic survey to gather information about their gender, racial demographics, teaching experience and preparation.

Technology Use and Professional Development Survey. This survey was a 70-item Likert-type questionnaire that requested information about participants’ technology knowledge and skills, implementation of technology within their mathematics teaching, professional development activities, and level of mathematics expertise.

Teacher Instructional Practices and Beliefs Survey (TIPBS). The TIPBS was created to track teachers’ technology use, professional development outside the project and capture their practices and beliefs about mathematics teaching and learning. Ten subscales were used in the final version, school support for instructional innovation subscale (6 items, α=.79), technology availability (10 items, α=.54), NCTM Standards familiarity/implementation (3 items, α=.68), use of instructional technology (4 items, α=.86), strategy discussion (6 items, α=.85), explanations and justifications (5 items, α=.79), and data analysis (5 items, α=.90), the reform classroom discourse (4 items, α=.73), sense of efficacy (6 items, α=.80), beliefs about mathematics (6 items, α=.64). Of particular interest to this study are the technology use, strategy discussion, explanations
and justifications, data analysis, and reform classroom discourse subscales. The first four of these subscales are five point Likert-type scale including “never,” “once a month,” “once a week,” or “all or almost all mathematical lessons.” The reform classroom discourse subscale required teachers to indicate how strongly they identified with a teacher in one of two scenarios. One scenario was more teacher-directed and the other was focused more on discussion.

**Post Observation Interviews.** After one of the days of observation, researchers from the CCMS project interviewed the teachers about their use of CCT. Teachers were asked about: typically use of the CCT, changes in lessons due to CCT, the purpose of their CCT use, and gains in understanding of student thinking.

**Classroom Observations.** Fifty-five (Cohort 1 = 25, Cohort 2 = 30) of the 127 teachers in the CCMS project had videotaped classroom observations at least one of the four years of the project. Classroom observation videos are of two types, either two consecutive days of class or five consecutive days of class. The majority of classroom observations were of two consecutive days. One of the three participants chosen for this study had both two day and five day observations in one year on linear topics. Topics taught by the participants ranged from slopes of lines, properties of lines, systems of equations to finding best fit lines, and introduction to functions. Only the video data and verbatim transcripts of these three teachers will be used in this study.
<table>
<thead>
<tr>
<th>Years of CCT Use</th>
<th>Teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Mr. L.</td>
</tr>
<tr>
<td>1</td>
<td>Mr. L, Ms. A., Ms. B.</td>
</tr>
<tr>
<td>2</td>
<td>Ms. A.</td>
</tr>
<tr>
<td>3</td>
<td>Mr. L., Ms. B.</td>
</tr>
</tbody>
</table>

Table 3.2: Classroom observations data for participants.

Analysis

To create an empirically based theoretical model of growth of use and treatment of representations by teachers and students in CCT classrooms an in-depth qualitative analysis was conducted. Observational notes were taken during the first viewing of the video data. The videos, transcripts of both the classroom observations and POI’s, and observation notes were uploaded into data analysis software, Nvivo 9. The data were coded and categorized using a priori codes arising from the conceptual framework and a grounded theory approach (Creswell, 2005). Patterns of teacher growth were categorized by creating models for different levels of instruction with regards to representations, discourse, technology, and decisions. The literature (Moschkovich et al., 1993; Pape & Tchoshanov, 2001; Schoenfeld, 2010a; A. G. Thompson et al., 1994) points to many aspects influencing representations in the classroom such as treatment and usage of representations themselves, discourse surrounding representations, technology use, and in the moment decisions teachers make in response to learners’ ideas and feedback. Both a priori codes and grounded codes were used in each of the four main categories.

Individual case studies were created for each of the three teachers describing their classroom instruction with respect to the categories of representations, discourse,
technology and decisions for each year observed creating an outsider’s perspective on each teacher’s instruction.

Video data of the classroom observations were coded using a priori codes. Each of videos was coded multiple times at least once for each of the main categories of representations, discourse, decisions, and technology. Any patterns that emerged that seemed important to analysis were categorized to form codes, and the data was coded for the emergent themes. I created the case studies to capture growth or lack thereof of teachers’ instruction with respect to the four main categories. Another researcher and I watched the videos together of all the teachers for each year observed and discussed our observations until a consensus was reached of what was seen. A third researcher watched randomly selected portions of the videos and offered his analysis in places where disagreements existed among the coded categories. At a later time, a group of Ph. D. students watched selected portions of the videos and their analysis of the segments were discussed.

Classroom instruction was divided into segments by grouping clusters of similar activities, exercises, or discussion, such as teacher-led exercises demonstrating how to solve linear inequalities, or students submitting solutions to homework about linear inequalities to the CCT. The clusters were further divided into smaller subsegments, which contained a single exercise, single part of an activity, or topic of discussion. If a single part of an activity resulted in multiple discussions, each discussion was treated as a separate subsegment. The subsegments became the major unit of analysis for this study.
Coding and Ratings

Each of the main categories of representations, discourse, technology and decisions contained subcategories drawn from the literature and the emergent categories. Indicators were created for rating teacher classroom instruction low, medium, and high with regards to each of the four main categories. Initially, utterances were coded to generate the case studies. However, using utterances as the main unit of analysis seemed inadequate in capturing teacher growth as it was too fine tuned to detect the quality and nature of interactions globally. A more holistic approach of using the subsegments as the main unit of analysis was used to complete each of the cases and then the cross analysis. The teacher-level data were used to answer the main research question: what is the relationship between teacher use of CCT in algebra classrooms and teacher and student choice of representations of linear functions as manifested in the classroom discourse. Patterns within the codes, categories, and themes were noted and used in the model building process.

Representations. The category of representations was partially based on frameworks (Moschkovich et al., 1993; Pape & Tchoshanov, 2001) that captured the types of representations present during the instruction, the dual process/object nature of functions, and the treatment of representations. A coding of each subsegment for the source of representations that included a description of who or what created the representations present in the classroom, which seemed to influence the teachers instruction.

Types of representations present. Each subsegment was coded according to what type or types of representations were present. The three major representations of
functions, graphical, algebraic, tabular served as the basis for data coding. If more than one representation occurred within a subsegment, the subsegment was coded as a mixture of the types of representations present. For example, if a teacher asked students to sketch the graph of a specific equation, that subsegment would be coded as a mixture of Algebraic/Graphical. If students were asked to create a table of points for the equation of a line that were then displayed in a graph, then it would be coded as a combination of all three. Noting the specific combinations allowed for the codes to be mutually exclusive.

The literature (Moschkovich et al., 1993; Pape & Tchoshanov, 2001) offer that students need to experience and be able to make connections among multiple representations of similar and different functions. These constructs were used to create a rating for instruction with regards to the types of representations present in the classroom. Hence, classrooms with several different representations simultaneously present were rated more highly than classrooms with representations presented distinctly from one another. For each year a teacher’s instruction was rated as low, medium or high for the types of representations used in the classroom. A teacher was rated low if he or she used only used predominately only one representation during instruction even when another representation may have been beneficial to the students. A teacher was rated medium if two representations were simultaneously present during the majority of the time. A teacher was rated as high if all three representations were present simultaneously the majority of the instructional period or if different types of mixtures of two representations were present.
Process/Object nature. Moschkovich et al. (1993) argued that the dual process/object nature of functions is needed to nurture understanding of functions. Each type of representation within each subsegment was coded as to whether the representations of functions were treated as process, or object, or a combination of both. Moschkovich et al. (1993) argued that students needed not only to be able to switch among different representations of the same function but also to navigate the dual nature of functions. In light of this in this study, classrooms that treated functions as both process and object were rated more highly than classrooms that had predominantly a single treatment of functions. A teacher was rated low if he treated functions only as process a majority of subsegments. A teacher was rated medium if a representation of functions were treated as objects the majority of the time, or if they treated a representation of functions as process less than half of the time. A teacher was rated high if she treated the representation of function as a mix of process/object more than 30% of the subsegments containing the representation and treated function as solely process less than half the time.

Treatment of representations. Pape and Tchoshanov (2001) argued that how representations were treated in the classroom could influence student representational schemes. They argued that using representations only to create end products, that is creating representations for the sake of creating them, limits student understanding of representations. They further argued that to develop representational thinking among students, the students needed to interact with representations in at least three meaningful ways: 1) need to be reasoned about/with, 2) practiced upon, and 3) have explanations or
justifications about/with them and that all of the interactions should be present within a classroom for students to experience. Each subsegment was coded as to whether the representations present were treated as end products, if procedures were enacted upon them (practiced upon), as something to reason about/with, or explain/justify about/with. These codes were not mutually exclusive since representations could be treated in multiple ways, such as enacting procedures to make an end product, or enacting procedures to create something to think about.

Classrooms that encouraged students to interact with representations in the three meaningful ways were more highly rated than classrooms that encourage limited student interactions. A teacher was rated low if she treated representations as end products more than 50% of subsegments. A teacher was rated medium if he did not treat representations as end products for more than 50% of subsegments and treated representations as either something to reason with/about or something to explain/justify in more than 25% of subsegments but not both. A teacher was rated high if the subsegments where representations were treated as something to reason about/with and the subsegments where representations were treated as something to explain/justify were each in over 25% of subsegments.

Source of representations. The importance of the source of the representations present within a subsegment became apparent when creating the case studies. Typically different interactions occurred when representations were produced by different sources. Each subsegment was coded as to whom or what generated the representations present. They were coded as teacher, teacher using CCT, single student, or multiple students.
Typically the more student generated representations were used in the classroom the more the teacher focused on student thinking, treated representations as something to think about, and asked different types of questions. At times, students treated these student generated representations as something to think about even without teacher prompting. Therefore, the teacher’s instruction that used students as the source of representations in the classroom was rated higher than instruction where the teacher was the source. A teacher’s instruction was rated low if the teacher was the source of representations with or without the CCT over 60% of the subsegments. A teacher’s instruction was rated medium if the teacher was the source of representations in 40% to 60% of subsegments. A teacher’s instruction was rated high if the teacher was the source of representations less than 40% of the time or multiple students were the source of representations for more than 50% of subsegments.

**Overall Rating.** Teachers were rated low, medium, or high for four subcategories of usage and treatment of representations in the classroom; types of representations present in the classroom, source of representations, process/object treatment of functions, and treatment of representations. A composite of all the four subcategories was used to create indicators for an overall rating of representational usage and treatment in the classroom for each teacher as described in Table 3.3.
<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
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</table>
| High 2.5-3 | • The teacher used many different types of representations simultaneously, or different pairs of representations throughout instruction.  
• The teacher elicited and displayed representations from multiple students simultaneously.  
• The functions represented were treated from both the process and object perspectives.  
• Representations were treated both as something to reason about/with and something to explain/justify with/about. |
| Medium 1.5-2.5 | • The teacher used two different representations simultaneously for part but not most of instruction.  
• The teacher elicited and displayed representations from some students perhaps simultaneously but not for most of instruction.  
• The functions represented were treated primarily from the object perspective.  
• Representations were treated less frequently as end products; and were treated as something to reason about/with or something to explain/justify with/about but rarely as both. |
| Low 1-1.5 | • The teacher primarily used a single type of representation at a time, and the same representation throughout the day.  
• The teacher was the primary source of all representations present.  
• The functions represented were treated almost exclusively from the process perspective.  
• Representations were treated almost exclusively as something to enact procedures upon and as end products. |

Table 3.3: Model for rating overall representations.

Averaging the teachers’ ratings in each of the subcategories became their overall rating, where scores between 1.0 to 1.5 were considered low, 1.6 to 2.5 were considered medium and 2.5 to 3.0 were considered high. In this study the average rating corresponded with meeting at least three of the four indicators for the overall rating. However, a teacher could have an average rating of 2.0 by meeting only two of the criteria for medium if their rating was high in two subcategories and low in the other two.

**Discourse.** The discourse category contained three subcategories: average words per utterance, types of questions asked, and methods to elicit student discourse. These
categories were developed from the Thompson et al.’s (1994) framework or emerged from grounded methodology.

**Ratio of student to teacher talk.** The average words per utterance of the teacher’s and the students’ mathematical talk was computed, and the ratio of student to teacher talk was also calculated. Each subsegment was coded as to who spoke including the audience of the talk (teacher to student, student to teacher, or student to student). The number of words expressed by teachers and students were counted and an average number of words per utterance were calculated. The percentage for the ratio of student words per utterance to teacher words per utterance was computed as a means to compare the student and teacher talk. A high ratio of student to teacher talk indicated the potential for students to express their ideas, so teachers with higher ratio were rated more highly than teachers with a lower ratio. A teacher was rated low if the ratio of student talk to teacher talk was at or below 25%. A teacher was rated medium if the ratio was in between 25% and 45%. And a teacher was rated high if the ratio was at or above 45%.

**Types of questions.** Applying a modified version of the conceptual/calculational orientation to teaching (A. G. Thompson et al., 1994) as a guide to indentify different types of questions, each subsegment was coded as to the types of questions present within that subsegment, detailed later in Table 3.8. The questions asked within each subsegment were coded as procedural, conceptual, mixture of both, or transitional. Thompson et al. (1994) argued that conceptual questions encourage students to make deeper connections within mathematics. In this study, the more frequently conceptual questions were asked in a classroom the more highly the teacher’s instruction was rated. A teacher was rated
low if he asked only procedural questions for a majority (more than half) of the subsegments. A teacher was rated medium if she prominently asked transitional questions or a mixture of procedural and conceptual questions for over 50% of subsegments, but conceptual questions were also present in less than 50% of subsegments. A teacher was rated high if she or he asked conceptual questions in a majority of the subsegments.

**Method of eliciting student discourse.** The importance of the method that teachers used to initiate and maintain classroom discourse from students became apparent when coding the data including eliciting and evaluating simple answers or procedures from students; eliciting student ideas; or extending student thinking and focusing on student explanations. Each subsegment was coded as to how the discourse was elicited: Initiate Respond Evaluate (IRE) sequence that focused on correct answers or methods, IRE sequences that focused on exposing student ideas, or a sequence where a teacher would *elicit, extend, explain* or *clarify* students thinking or have students make *predictions* (EEECP).

Kazemi and Franke (2004) argued that teachers should be encouraged to elicit deeper explanations from students as it allows teachers to focus on the negotiation of shared meaning, build common ground and foster mathematical understanding. Therefore, the teachers that elicited student explanations more frequently were more highly rated. A teacher was rated low if he elicited discourse from students using an IRE sequence focused on answers or methods during a majority of subsegments. A teacher was rated medium if she used IRE sequences focused on thinking during a majority of the
subsegments. A teacher was rated high if he or she used EEECP during a majority of the subsegments.

**Overall discourse rating.** Teachers were rated low, medium, or high for each of three subcategories of discourse, ratio of student/teachers talk, types of questions asked, and method of eliciting student discourse. Indicators for an overall discourse rating were established based on the ratings of the subcategories as shown in Table 3.4. An average of the ratings of each subcategory was calculated for each teacher.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
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| High   | - The amount of student talk was a substantial portion of the amount of teacher talk, i.e. the ratio of student words uttered compared to the word uttered by the teacher was high.  
- The teacher asked conceptual questions for most of instruction.  
- The teacher’s method of eliciting student discourse not only revolved around student thinking, but elaborating and extending that thinking for most of instruction. |
| 2.5-3  |             |
| Medium | - The amount of students talk was a moderate portion of the amount of teacher talk; the ratio was medium.  
- The teacher asked questions that could be sometimes conceptual, but mostly moderately conceptual, or a sequence of questions that were mixed a mix of procedural or conceptual for most of instruction.  
- The teacher’s method of eliciting student discourse focused on exposing student thinking, but without exploring that thinking deeply. |
| 1.5-2.5|             |
| Low    | - The amount of student talk was a small portion of the amount of teacher talk, the ratio of student to teacher talk was low.  
- The teacher asked procedural questions for most of instruction.  
- The teacher’s method of eliciting student discourse focused on getting correct answers and methods in a systematic way for most of instruction. |
| 1-1.5  |             |

Table 3.4: Model for rating teachers overall classroom discourse.
An average rating of 1.0 to 1.5 corresponded with meeting most of the indicators for an overall rating of low. An average rating of 1.6 to 2.5 corresponded with meeting most of the indicators for medium. And an average rating of 2.6 to 3 corresponded with meeting two of the three indicators for high.

**Technology.** The technology category had two different subcategories: use of technology and treatment of technology. The treatment of technology was based on the model proposed by Goos et al. (2003). The use of technology coding emerged from grounded observations of teachers using the CCT.

**Technology use.** Four main uses of technology in the classroom emerged from the data, technology as: *information displayer, information gatherer, answer verifier,* and *discourse generator.* Each subsegment was coded as to how the CCT was used. These codes were not mutually exclusive as technology could be used in multiple ways in a single subsegment. Some subsegments could be coded as all four if a teacher collected and displayed student data to verify their answers that in turn spurred discourse about similarities/differences in student ideas. Such subsegments were counted as an instance for all the uses present.

Teachers who used the CCT more frequently to engage the students in doing mathematics, such as verifying answers and especially generating discourse were more highly rated. A teacher was rated low if he used the CCT almost exclusively as an *information displayer or information gatherer.* More specifically the rating was low, if a teacher did not use the CCT as either an *answer verifier or discourse generator* for more than 20% of the subsegments when the CCT was used. A teacher was rated medium if
she used the CCT as an *answer verifier* for more than 20% of subsegments with CCT use or as a *discourse generator* for more than 20% but less than 50% of subsegments with CCT use. A teacher was rated high if she or he used the CCT as a *discourse generator* for more than 50% of subsegments with CCT use.

*Treatment of technology.* Each subsegment was coded as to how the CCT was treated by the teacher: as a partner, master, or servant using Goos et al.’s (2003) framework. A teacher may treat technology as a master if his knowledge and usage are limited to a narrow range of operations; or to the extent the teacher calls on a student “expert” to use the technology. Given a teacher’s limited knowledge of technology, he or she may be reluctant to allow students to use technology to explore mathematics. A teacher treating technology as a servant implies using the technology as a computational tool to speed up certain processes, but ultimately the tasks remain unchanged and the technology is not used in creative ways to change the nature of the activities. Technology is used to mainly support the teachers’ preferred teaching methods when a teacher treats technology as partner he or she attempts to provide students access to new kinds of tasks or new ways of approaching tasks that may facilitate understanding, explore different perspectives, or mediate mathematical discussion.

Goos et al.’s model for treatment of technology provided a natural way to rank this subcategory. A teacher was rated low if he treated technology as a *partner* for less than 25% of subsegments with CCT use. A teacher was rated medium if she treated technology as a *partner* for 25% to 50% of subsegments with CCT use, and treated
technology as *servant* for more than 50%. A teacher was rated high if she or he treated technology a *partner* for more than 50% of subsegments with CCT use.

**Overall technology rating.** Teachers were rated low, medium, or high for two subcategories of technology, *technology use*, and *treatment of technology*. Indicators for an overall technology rating were based on the ratings of the subcategories as shown in Table 3.5.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
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</table>
| High   | • The teacher used the display of student generated data collected by the CCT to generate discourse for most of instruction with CCT use.  
• The teacher treated technology as a partner for most of instruction with CCT use. For example, a teacher treating technology as partner could provide access to new kinds of tasks or new ways of approaching tasks that may facilitate understanding, explore different perspectives, or mediate mathematical discussion. (Goos et al., 2003) |
| 2.5-3  |             |
| Medium | • The teacher used the display of student generated data collected by the CCT to frequently verify answers or generate discourse, but not for a majority of instruction with CCT use.  
• The teacher treated technology as a servant for most of instruction, but occasionally treated technology as partner. A teacher treating technology as a servant could use the technology as a computational tool to speed up certain processes, but ultimately the tasks remain unchanged and the technology is not used in creative ways to change the nature of the activities. Technology is used to mainly support the teachers’ preferred teaching methods (Goos et al., 2003). |
| 1.5-2.5|             |
| Low    | • The teacher used the CCT to collect and display student generated data without verifying answers or having discussions for most of the instruction with CCT use.  
• The teacher treated technology as a servant or treated technology as a master for most of instruction with CCT use and rarely treating it as a partner. A teacher may treat technology as a master if their knowledge and usage are limited to a narrow range of operations; to the extent the teacher calls on a student “expert” to use the technology. Given a teacher’s limited knowledge of technology, a teacher may be reluctant to let students use technology to explore mathematics (Goos et al., 2003). |
| 1-1.5  |             |

Table 3.5: Model for overall teacher’s technology use and treatment.
An arithmetic mean of the ratings in each subcategory was calculated for each teacher. An average rating of 1.0 to 1.5 corresponded with meeting most of the indicators for an overall rating of low. An average rating of 1.6 to 2.5 corresponded with meeting most of the indicators for medium. And an average rating of 2.6 to 3 corresponded with meeting two of the three indicators for high.

**Decisions.** Flow charts modeling the actions and apparent decisions for each teacher for each year were created based on classroom observations using Schoenfeld’s (2010a) framework by inferring the decisions a teacher made in the classroom as evidenced by his or her responsive actions to instances that prompt the need for a decision. The decisions flow charts for each teacher were created by compositing and generalizing from the decisions made in all the different situations observed. The flow charts provided not only a compact description of a teacher’s instruction, but also provided a prediction of how his or her instruction might unfold in other unobserved circumstances. Opportunities teachers missed to further develop the understanding of functions for their students were described and the decision flow charts were used to infer why the missed opportunities occurred.

**Orientation.** The orientation of each teacher was determined for each year of the data along a continuum from highly procedural to highly conceptual as evidenced in their instruction, such as how they treated functions, representations, the questions they asked, methods used to elicit discussion, and visible goals. Table 3.8 shows the indicators used the coding of the data.
<table>
<thead>
<tr>
<th>Orientation</th>
<th>Description</th>
</tr>
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</table>
| Highly Conceptual | • Tends to focus students away from thoughtless application of procedures and toward a rich conception of ideas.  
                   • Tends to focus on aspects of the situation that give meaning to the representations present that in turn suggests the procedures that might be useful.  
                   • Asks questions that move students to view representations in a nonprocedural context.  
                   • Has an expectation that students are intellectually engaged in tasks and activities.  
                   • When procedures are needed to be used, have students think about the products or the procedures themselves. |
| Conceptual      | • Meets all or most of the indicators for a Highly Conceptual teacher for most of instruction.  
                   • However, have occasional instances where procedure is supreme and little to no movement toward a rich conception of ideas. |
| Transitional    | • Tries to focus students away from pure procedures toward a rich conception of ideas, but the conception of ideas is lacking.  
                   • Gives some meaning to representations and procedures performed on them, but the meaning lacks depth.  
                   • Alternates between conceptual and procedural orientations, independent of context.  
                   • Reverts to procedural explanations without a hint of conceptual ideas when procedures are convenient or when conceptual explanations are difficult.  
                   • Frequently asks questions that can be easily answered with procedures or results of procedures, but also frequently asks questions that reveal the meaning of the mathematics. |
| Procedural      | • Meets all or most of the indicators for a Highly Procedural teacher for most of instruction.  
                   • However, has not completely abandoned meaning in mathematics as they tend to have occasional instances where representations are more than something to do or make, but something to think about and something that has meaning with connections to other representations. |
| Highly Procedural| • Tendency to speak in the language of symbols and symbolic manipulations.  
                   • Casts problem solving as performing the right procedures to produce solutions.  
                   • Emphasizes indentifying and performing procedures.  
                   • Tends to perform procedures whenever possible.  
                   • Tends to disregard the context of the problem in which procedures occur and how they might arise naturally from understanding the situation.  
                   • Tends to remediate student difficulties with procedures independently of the context in which the difficulties arose.  
                   • Treat problem solving as flat. Nothing is more important than anything else, except for the answer which is of supreme importance.  
                   • Has a narrow view of mathematical patterns as limited to single problems, rather than patterns across many types of problems.  
                   • Asks questions that can be easily answered with procedures or results of procedures. |

Table 3.6: Model for teacher’s orientation toward mathematics: a modified version of Thompson et al. (1994) calculational/conceptual orientation toward mathematics.
Teachers were rated according to where their orientations toward mathematics was on the procedural/conceptual spectrum: highly procedural with a rating of 1, mostly procedural with a hint of conceptual with a rating of 1.5, in transition between the ends of the spectrum with a rating of 2, conceptual with a hint of procedural with a rating of 2.5, and highly conceptual with a rating of 3.

Cross Analysis. The teachers for each year observed were rated in the four main categories of representations, discourse, technology, and decisions as previously described. A comprehensive case study for each of the participants was first constructed. A cross-comparison of the participants was completed for each year and across different years. The teachers’ instructional ratings were used to track the type of growth that occurred, if any. Patterns that emerged from the cross analysis were used to build an empirically based theoretical model of teacher growth using CCT so to create developmental trajectories for teachers based on their orientation to teaching.

Limitations

The data used in this study were obtained from the CCMS project and were not designed for the purpose of this study. The data collection for the CCMS project concluded before this study commenced, so the researcher did not have access to teachers to conduct interviews or to elicit member checks (Shenton, 2004). This certainly limited the exploration of why teachers had made certain decisions or modified their instructional practices. Additionally, the topics of lessons present in the video data were chosen by the teachers, so the videotapes had a different topic using representations of functions. Thus, comparisons of the same lesson by different teachers could not be made. The data
presented only a snapshot of the teachers’ use of the technology over only a few days. This limited the generalizations of the conclusions that can be drawn from the analysis. The video tapes did not focus much on group discourse or student work during the class; hence both the analysis and conclusions in this study focus on teacher and whole class discussion. Lastly, analysis of the data revealed, if only partially, the influence of teacher knowledge of content, content with technology, and pedagogy on the teacher’s instruction with CCT. However, the available data were insufficient to allow for an in depth exploration of this issue.

**Trustworthiness**

To ensure the trustworthiness of the analysis of this study several strategies commonly used to do so (Shenton, 2004) were employed; prolonged engagement, thick description, peer scrutiny of the research, triangulation, and connections to previous research. Another researcher and I watched videos for each of the original seven teachers initially chosen as candidates for each year of analysis for this study and discussed our observations until a consensus was reached of what was seen. We also discussed emergent patterns that may be important to analysis of teacher growth of their use of CCT and constructed codes. The videos were coded for each of the subcategories of the main categories of representations, discourse, decisions, and technology. After an initial coding, a third researcher watched randomly selected portions of the videos and offered his analysis in places where disagreements existed among the coded categories. The data were recoded for the subcategories with disagreements. This methodology and the fact that the teachers were observed over a period of a year or more provided prolonged
engagement with the data, persistent observation and also provided peer scrutiny of the research (Shenton, 2004). The case studies for each teacher were created for analysis as well as to provide a thick description (Shenton, 2004) of their instruction for each year observed. While the Post Observation Interviews (POI) were not able to provide triangulation for the codes used, they were able to offer confidence that the observations were typical of the teachers use of the CCT. Another way to provide triangulation is to use a “wide range of informants” (Shenton, 2004, p. 66). The method of participant selection was employed to provide a diverse group of participants from within the initial candidates. The diverse group of participants also provided triangulation for the codes of types of instructional practices that appeared in most of the teachers’ classrooms.

Chapter 8 details the connections to previous research.

Summary

The data used in this study were collected during the four years of the CCMS project. Qualitative analysis of videotaped classroom observation was conducted to create case studies for each teacher to determine the relationship of the use of connected classroom technology and student and teacher choice of representation. A detailed cross analysis of teacher’s instruction using technology was used to build an empirically based theoretical model of growth. The next three chapters present the case studies for Mr. L., Ms. A and Ms. B respectively. The chapter following these is the cross analysis. The study concludes with the discussion of the model, and implications for research and teaching.
CHAPTER 4: A CASE STUDY OF MR. L

This chapter is a report of the case study of Mr. L’s classroom observations. The case study is comprised of a section on year one, no years of CCT use, and year two, first year of CCT use, and year four, third year of CCT use in the CCMS project. For years one and two and four a description of an outsider’s perspective as observed from the classroom observations, a description of the insider’s perspective derived from Mr. L’s post observation interview (POI), and analysis are given. A summary of the themes and patterns across the years observed finishes the case study.

Introduction

Mr. A, a mechanical engineering undergraduate graduated in 1993, obtained his graduate degree in Curriculum and Instruction in 1996. At the beginning of the four year study, he had nine years of teaching experience with three years of experience teaching algebra.

The school in which Mr. L is employed is located the urban fringe of a large city with 89% of its students Caucasian, 7% African American, and 3% Asian. Eight percent of the students receive free and reduced lunches. The classroom observed was Algebra I part I, which is the lowest level math class offered at Mr. L’s school. Mr. L is part of a teaching team of different subjects who have very similar classroom procedures to help students transition from middle school to high school and to watch for students who are falling
behind in their classes. Mr. L’s class is “50% true freshman and 50% students who are still classified freshman cause they haven’t earned enough credits to be sophomores because they are repeating this and other classes multiple times.”

The measurement of teachers’ practices and beliefs about mathematics teaching and learning within the Teacher Instructional Practices and Beliefs (TIPBS) survey was used to differentiate participants and used for participant selection criteria. Table 4.1 shows Mr. L’s average scores of different constructs of the TIBPS survey. Interpretations of the scores are then discussed.

<table>
<thead>
<tr>
<th>TIBPS Constructs</th>
<th>Average Scores</th>
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<tbody>
<tr>
<td>Reform Classroom Practice</td>
<td>2.0</td>
</tr>
<tr>
<td>Tech Use Reform</td>
<td>1.4</td>
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<tr>
<td>Strategy Discussion</td>
<td>4.0</td>
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<td>1.2</td>
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<tr>
<td>Explanation/Justification</td>
<td>4.0</td>
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<td>2.5</td>
</tr>
<tr>
<td>Assessment Practice</td>
<td>2.3</td>
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<td>Teacher Efficacy</td>
<td>2.7</td>
</tr>
<tr>
<td>TB on Math</td>
<td>2.4</td>
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Table 4.1: Mr. L’s TIBPS average scores.

For questions within the constructs of Reform Classroom Practice, Tech Use Reform, Strategy Discussion, Data Analysis, Explanation/Justification, Reform Class Discourse, and Assessment Practice, responses were scored on a 5 points scale ranging from 1) never, 2) once a month, 3) once a week, 4) more than once a week, and 5) all or almost all lessons. For the Classroom Discourse construct, teachers were given descriptions of
two different scenarios of class discussions, where one was teacher led with simple questions and the other with more difficult questions arising from students. Then they were asked which scenario their instruction tended toward. The 5 point scale ranged from Definitely Ms. Hill's (simple questions and teacher led); Tend towards Ms. Hill; can’t decide; Tend towards Mr. Jones' (difficult questions from students); Definitely Mr. Jones'. The Teacher Efficacy and Teacher Beliefs on Math constructs were scored out of four points with a 1) strongly disagree, 2) disagree, 3) agree, and 4) strongly agree. Some of the Teacher Efficacy questions asked teachers to rate how prepared they were for certain situations on a four point scale.

Mr. L reported he almost never used technology in reform ways, by answering questions such as “using technology to develop conceptual understanding” or to “use technology to demonstrate mathematics principles.” The average score for Mr. L in the reform technology construct that asked how often do students engage in the following activities from the TIPBS was a 1.4 out of 5, with 1 being never and 5 being almost all lessons. A score of 1 was considered low and 5 was considered high, except for questions using a reverse scale where the opposite was true. Mr. L reported that he almost never used data analysis in the classroom and scored an average of a 1.17 out of 5. On average Mr. L reported that his students engaged in strategy discussion at least once a week with an average score of 4. All activities under this construct he scored a 4, except “talking about ways to solve mathematics problems, such as investigations” scored a 3 and “help students understand the relationships between the strategies they use and their mathematical understanding” scored a 5. In the explanations and justification construct,
Mr. L scored 4 or higher except “having students use multiple representations to explain a concept” which scored a 2 or “once a month.” Mr. L’s scores in the assessment practices construct were either a 1 or a 5 with an average of 2.3. Mr. L scored a 1 on the normal scale and a 5 on the reverse scale, which is to be consider to be low, for all but three items, a 5 on the normal scale “conduct a pre-assessment to determine what students already know,” “use assessments embedded in class activities to see if students are ‘getting it,’” and a 1 on the reverse scale “encourage students to compete with each other academically.” Mr. L reported he was more comfortable with a class discussion that tended toward Ms. Hill, thought students preferred class discussions that tended toward Ms. Hill, and that students gained more knowledge from class discussions that tended toward Ms. Hill. However, Mr. L believed that students gained more useful skills from class discussions that tended toward Mr. Jones. Mr. L reported he was moderately well prepared in mathematics instruction in all categories, except “using the internet in mathematics teaching for data acquisition” scored a poorly prepared and “some students are not going to make a lot of progress this year no matter what I do” scored “agree.”

When asked how often students engaged in reform classroom practices, Mr. L’s scores averaged a 2. Mr. L said the students in nearly every class period “reviewed homework/worksheet assignments,” “watched the teacher demonstrate a mathematical phenomenon,” and that he “introduced content through direct instruction.” Mr. L scored a 5 on these items that are reverse scaled. Mr. L scored a 1 on a reverse scale item “applying mathematics skills to solve routine problems.” For the normal scaled reform classroom practice items, Mr. L scores were 3 or lower.
Mr. L’s case study comprises a section on year one: no CCT, year two: one year of CCT use, and year four: three years of CCT use. For years one and two a description of an outsider’s perspective, a description of the insider’s perspective, and analysis are given. Since there was no post observation interview for Mr. L during year four, only a description of the outsider’s perspective and analysis are given. A summary of the themes and patterns across all three years finishes the case study.

<table>
<thead>
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<th>Data</th>
<th>Minutes</th>
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<tr>
<td>Post Observation Interview</td>
<td>24 minutes</td>
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<tr>
<td>Classroom Observation 5/17/2006</td>
<td>53 minutes</td>
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<tr>
<td>Classroom Observation 5/1/2007</td>
<td>54 minutes</td>
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<td>Post Observation Interview</td>
<td>12 Minutes</td>
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<td>Classroom Observation 5/2/2007</td>
<td>54 minutes</td>
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<tr>
<td>Classroom Observation 2/19/2009</td>
<td>44 minutes</td>
</tr>
</tbody>
</table>

Table 4.2: Mr. L’s case data.

**Year One: No CCT**

**Overview**

During the first year of the study, several prominent features of Mr. L’s teaching appeared to have derived from his procedural orientation to mathematics and his apparent major goal for students to obtain the correct answer. To aid students in coming up with the right answer, both the class structure and mathematics taught were segmented into manageable pieces. The segmentation of mathematical ideas seemed to limit the number of representations used on a given class day. In particular the graphical representation was the only representation used during the days observed and for exercise problems.
Even though the graphical representation could have depicted the object nature of functions more readily, the functions were treated by the teacher solely from the process perspective. Also, the classroom discourse seemed to be limited to sharing the correct answer or telling how to get the correct answer with no student to student interactions.

**Segmentation**

The structure of his class in the first year of data collection was highly regimented. Each day appeared segmented into five major components: 1) “Do Now,” 2) yesterday’s homework and review, 3) main lesson, 4) student practice, and 5) new homework. The class would begin with a “Do Now” activity, where students were given an exercise to complete. The teacher asked students for the correct answers to the “Do Now.” The “Do Now’s” did not always appear to be related to the lesson for that day. For instance, the topic of both observed days was “Given the graph of the line find the slope of the line,” the “Do Now” of the first day was to add various square root numbers and to simplify given expressions. The following day’s “Do Now” involved the Pythagorean Theorem that could have related to the lesson which involved creating triangles to find slope.

The second segment of each lesson was designated for yesterday’s material. This included going over and correcting homework from the previous day and a review of yesterday’s content. Mr. L typically read the answers with students checking their homework. Afterward he would ask students if they had wrong answers. If they did he would check their work by asking what they did and perhaps putting it up on the board. He reviewed content from previous days’ lessons that pertained to the main lesson of the
day. During the review, the teacher asked recollection type questions of the students, such as “what is the slope formula?”, and “what are the four types of slopes?”

Then the main lesson started, and Mr. L explained how to solve a particular type of exercise question and then carefully worked a few examples with the students. The explanations may have been a brief reminder of previous content and a short explanation of the connection to the new content or a lengthy explanation of how to do something new with little or no discussion of meaning. Afterward, the students were given ample time to practice exercises on their own. When it seemed most of the students were done with an exercise, Mr. L asked for the correct answer and whether anyone did not understand. If someone did not, he would ask students for their work to determine what they got wrong, as when he went over the homework. At the end of class he assigned homework for the next day.

Mr. L appeared to segment mathematics into manageable topics for a day’s lesson. “Given the graph of the line, find the slope of the line” was the topic for both days’ lessons. This topic was broken into two almost self-contained lessons each day. The first day focused on finding the slope given the graph by first using or finding points on the line and then using the slope formula. The second day’s lesson was on finding the slope using rise over run procedure. The topic of slope was split into segments working with only graphs, which was further into the different day’s lessons. During the first day, the focus was solely on how to find the slope using the slope formula in two situations, if points on the line were labeled and if they were not. Other than a brief review question on the second day, the previous day’s method of finding slope was not used. The exercises
chosen as part of the lesson were also segmented. The first few examples during the first
day had points labeled, then the class moved on to cases with no points labeled. Vignette
1 illustrates how he would segment the task for students in an attempt to get them to the
right answer.

Mr. L’s Vignette 1

1) T: Take a look up here; the first thing you ask yourself is the number underneath
the radical, the square root sign, a perfect square. Whitney, #1 is 162 a perfect
square?
2) S: Yes.
3) T: No, it's not. If the answer is no it's not, what do you ask yourself about 162?
4) S: Well it is a perfect square.
5) T: Did you figure out a perfect square factor of...
6) S: 9.
7) T: 9 is, but there's a greater number. How many times did you say 9?
8) S: 18.
9) T: 18; did you leave the 18 underneath there? Did you write in 9x18 and leave 18
underneath? 18's not a perfect square, but it has a perfect square. He's right that
9 is a perfect square factor of 162, but there's a greater perfect square factor of
162. Anybody get it?
10) S: 81?
11) T: 81. How many times does 81 divide into 162?
12) S: Twice.
13) T: Twice. I'll come back and tell you how you can still finish it your way. 128,
James, is that a perfect square?
14) S: No.
15) T: No, it's not. Nelson, what do ask yourself when 128 is not a perfect square?
16) S: What?
17) T: What do you ask yourself after you determine that 128 is not a perfect square?
18) S: Can you find any that go into it?
19) T: Yep. Did you?
20) S: No.
21) T: John, did you?
22) S: I didn't get that one.
23) S1: Yeah, it was like 64.
24) T: How many times will it divide in?
25) S: Twice.
26) T: Anybody, sq rt. of 81.
28) T: Sq rt. of 2.
29) S: Sq. rt. of 2.
30) T: Good. Sq rt. of 64.
31) S: 8.
32) T: Sq rt. of 2.
33) S: Sq rt. of 2.
34) T: 9 square root 2 and 8 square root 2 is...
35) S: 17 square root 2.

This sequence can be broken into 3 major segments: partially simplifying square root of 162 lines 1 to 12, partially simplifying square root of 128 (lines 13 to 25), and combining the two square roots and simplifying further (lines 26 to 35). During the first two segments, he broke the procedure into steps: what is the first thing to ask, and what is the second? Then he broke it down even more by asking how many times does this factor go into the original number and what is the square root of a particular number. He appeared to be modeling how to break down the task for the students so they could do it themselves.

**Representations**

The graphical representation of lines was the focus of both days. Although points were used extensively, Mr. L used these points in a graph, rather than in a table. This use had two effects: 1) focusing on the points of the graph emphasized the process nature of the graph, and 2) deemphasized the tabular representation. Therefore, for both days the only major representation presented was graphical treated from the process perspective. That singular focus appeared to be the result of segmentation of the lesson topics. On the first day observed, Mr. L said, “We've learned how to find the slope of a line given two points that lie on the line. We've talked about the formula for the slope of a line and we talked about the four possible slopes.” Mr. L stated that in previous days the class went
over the tabular view of lines with apparent focus on the one representation and found slope using only the formula.

In his classroom, the purpose of the graphical representation was not to discuss similarities and differences of lines with different slopes, nor to justify or refute the results of slope calculations given the shape of the line. The purpose of the presence of graphical representations in his classroom appeared to have students solve a specific set of exercises without making connections between the exercises.

The following is a typical example of how the graphical representation was used for solving exercises of finding slope with rise over run.

Mr. L’s Vignette 2

1) T: Another way to find the slope of a line which can be defined as the rise of a line over the run of a line, is to choose two points, form a right triangle, and then count how many blocks you have to rise to get from the first one to the second one, and put that number over how many blocks you have to run to the rt. to get from that block to the other one. Hopefully you can still see your blocks a little bit better than mine up here. I'm going to start at -4, 0 and count until I get up to that line. 1, 2, 3, 4. Stay with us, Brandon. 4 blocks. And then from that same spot, I counted up four, I'm right here. Everybody looking up here? I counted 4 blocks and I'm right here, and now I'm going to count how many I'm going to run to the rt. 1, 2, 3, 4, 5, 6, 7, 8. Should be 8.

2) S: It is 8.
3) T: My graph has shifted up here. The first point was -4, 0, the second point was 4, 4; to get from one the other one you should move 8 to the right. The slope is 4/8. It's a rise of 4 divided by a run of 8. We like to leave answers like that, 4/8?
4) SS: No.
5) T: Of course not; we simplify that. 4/8 is...
6) SS: 1/2.
7) T: 1/2.
8) S: That's a lot easier.
9) T: By easier you mean less work?
10) S: Yeah.
After this sequence, Mr. L moves on to the next graph. The graph of the line appeared to simply provide a platform for performing a procedure for finding the slope. No other connections between the $\frac{1}{2}$ and the graph of the line were given, nor did a comparison of different lines and their slopes occur during the lesson. Each exercise was solved in isolation from the others.

**Discourse**

The purpose of classroom discourse appeared to be to elicit the correct answer to the question or to show how to obtain the correct answer. The discourse in the Vignette 2 in the previous section showed students step-by-step how to obtain the correct slope without needing to identify points and without the need of the formula. Other than asking if the answer could be left as $\frac{4}{8}$ and the result of simplifying it, there was no discussion between teacher and students. This pattern of a long explanation with little student interaction occurred in the previous day’s lesson on how to find slope with points labeled on the graph, including a brief aside when a student asked if it mattered how the points get label as $(x_1,y_1)$ or $(x_2,y_2)$. In all the vignettes and throughout the two days of observations no student-to-student interaction occurred.

Discourse was to a specific student from teacher or to a student willing to answer or less often from student to teacher. Since the students and teacher took turns in speaking, an utterance is defined here as what was spoken during one turn. Mr. L’s utterances could be quite long, up to a maximum of 470 words, in between utterances of students. During that long utterance, he recapped what they learned the day before, explained that slope could be defined as rise over run and started explaining the process of finding rise over
run. Uttering 470 words was not typical. The next largest utterance was 260 words. Mr. L’s mathematical explanations could be 120 or more words. Student utterances were generally very short at 4 or less words and were answers to low cognitive load questions as evidenced in the vignettes. Out of over 300 student utterances over the two days only 6 were questions by the students to the teacher of a mathematical nature. The students asked the same question of “does it matter which points are chosen to find the slope of the line in three different instances”. Vignette 3 contains an example of one of those instances. Two of the remaining three questions were of similar nature: does it matter how you place the triangle, does it matter how you label your points. The last remaining student question asked how 9 was obtained for the y-coordinate.

As illustrated below, in all those instances the teacher answered the student question and/or gave an explanation. The first time a student asked if it mattered what points were chosen occurred several minutes after the teacher posed the same question. To convince the students, he had four different groups use four different pairs of points to find the slope. Below is an instance of student questions on the second day.

Mr. L’s Vignette 3

1) S: On these, could the answer no matter what way you do it, it still comes out the same?
2) T: Yes. Nelson said does it matter which two points you choose and the answer is no. For the first couple examples we're choosing the same ones so that everybody's seeing the numbers that we're getting. All right, slope can also be called rise over run. We need to talk about how much you have to rise to get from one point and then run to get from one point to another. I'm going to start right here at -2, 1 and I'm going to rise down. I'm going to rise down to get to this point. How many blocks do I have to go? How many blocks do you count to get from here down to that one?
4) T: 4, and we do go down. Someone said negative. The rise is down 4 so it's negative 4.
5) S: I have a question.
6) T: Hang on a second, let's finish this up. Then I can answer your question. We rose -4 to get down here and then we've got to count the blocks to get over. How many blocks to get from here to here, anyone?
7) SS: 6.
8) T: 6. Is it okay to leave the answer like that? -4, 6?
9) SS: No. -2/3.
10) T: -2/3. You have a question; I'm going to answer it. Are there any questions on what we just did to get -4, 6, it simplifies to -2/3. All right, Cory?
11) S: Does it matter which way you make the triangle?
12) T: Cory says does it matter which way you make the triangle? You have yours up here?
13) S: Yes.
14) T: It does not matter which way you make the triangle, but you need to remember that slope is rise over run, so you need to figure out how much you're going to rise to get from one point to the other and run to get from one point to the other. I'd prefer that you run to the rt., which means you'd do the triangle I did, but it's not wrong to do it that way.

Interesting discussions could have occurred as a result of student questions by asking “What is it about the line that causes the slope to be the same for all points chosen?”

However, the teacher answered the questions himself using results of procedures as explanations instead of involving students.

As stated earlier most student utterances were very short as a result of low cognitive load Initiate, Respond, Evaluate (IRE) sequences posed by Mr. L similar to that in the Vignette 1. Rather than opening up the questions of the sequence to all students he typically called on a specific student. He usually continued the IRE sequence until the student finished the task or subtask.

Mr. L’s response of “never” from the TIPBS survey to the question of how often do students use multiple representations to explain a concept was indicative of the quality of
representational discourse in the classroom. The quality of all representational discourse in this classroom was scored simple, where simple representational discourse can be characterized by an individual using one aspect of a single representation as a solution without support or multiple representations or aspects of one representation are present but not connected. Only the one aspect of finding slope was present in the discourse of the graphical representation. Not only were the individual conversations about this one aspect, but all conversations on both days pertaining to the graphical representation were only about this one aspect.

Orientation

Mr. L leaned heavily toward procedural, as defined by Thompson et al.’s (1994) spectrum of orientations to teaching from calculational to conceptual except replacing calculational with procedural. In Mr. L’s Vignette 2, he described to the students how to get rise over run of a line. To find rise over run, he said students need to

(step 1) choose two points, (step 2) form a right triangle, and (step 3) then count how many blocks you have to rise to get from the first one to the second one, and (step 4) put that number over how many blocks you have to run to the rt. to get from that block to the other one (parentheses are mine).

Mr. L explained how to find slope using a rise over run procedure step-by-step. He did not open up to the students a discussion of how they might find rise over run, nor was there a discussion of the reasonableness of the slope of $\frac{1}{2}$ given the line, nor how the slope is connected to the shape or orientation of the line. After the result of a slope of $\frac{1}{2}$ was obtained Mr. L moved on to the next exercise. The long step-by-step explanations occurred nine times over two days about half of which repeated the same procedure. In the first instance during the second day, Mr. L introduced the notion of rise over run.
Jot this down somewhere. It can be on the homework from last night, or another sheet of paper. Slope is also sometimes described as the rise of a line over the run of a line. Slope is defined as the rise over the run. When this is said what is meant is, how much the line moves up divided by how much the line moves left or right. I'm going to ask you to always run to the right. How much the line moves up divided by how much the line moves to the right. You can find the slope of a line by identifying two points on the line like you did yesterday, and counting how much you move up, divided by how much you move to the right to find the slope, and you don't need the slope formula to do that.

During the second day review, Mr. L briefly reminds the students that “slope can be described as how much a line is slanted, or the slant of a line.” During the above introduction of rise over run, no connection to steepness of a line and rise over run were addressed, no discussion of why rise over run would measure steepness occurred, no relationship between rise over run and the shape of the graph was given; only a process of how to find rise over run was described. The long explanations of instances of procedures illustrates why the procedural orientation was ascribed to Mr. L.

All the mathematical questions in the vignettes Mr. L posed to the students asked for the result of a procedure, such as: How many blocks did you move up? What is a greater square factor of 162? Can we leave 4/8 as is? Or a step of a procedure, such as, what do you ask yourself when 128 is not a perfect square? Two of the most conceptual questions that could have prompted discussions posed by the teacher occurred during the first day. He asked, how many points the line passes through, and which two points do we use to calculate slope? The student responses to the first question included: hard to tell, a million, and infinitely many probably. Mr. L validated the infinitely many points response and moved on to the second interesting question without discussing why there could be infinitely many points the line passes through. A student answered saying that
you could use any two points (on the line) you want to calculate slope. Mr. L identified this response as correct and had the students break into groups to calculate the slope of the line on the overhead projector. Mr. L used different pairs of points to show the students: “As Cory predicted, it doesn't matter which pair of points that you use; as long as the points do lie on the line you are going to be able to get the slope.” Mr. L seemed to consider that four equal results of slope calculations was sufficient proof for his students. Mr. L’s use of procedure on a finite number of instances to “prove” a general claim illustrates his procedural orientation to mathematics teaching.

In Mr. L’s Vignette 4, he revealed a glimpse into what he believed problem solving should be. In line one, he asked what the highly recommended problem solving procedure was. The problem solving procedure was not a general strategy for finding slope of a line given a graph, but was an algorithmic procedure to find slope of a line given a specific set of circumstances, i.e. two points are given.

Mr. L’s Vignette 4

1)  
   T: 0 slope. One other thing, who remembers the highly recommended problem solving procedure for how to calculate the slope of a line if I give you two points that lie on the line?

2)  
   S: First you write out the formula.

3)  
   T: Some point ___ would you like me to write the formula?

4)  
   S: Then you label.

5)  
   T: Label what?

6)  
   S: You label the coordinates.

7)  
   T: Uh, huh. Label them what?

8)  
   S: The first one would be X1 and Y1. The second one would be X2 and Y2.

9)  
   T: The what? We've got the formula...

10)  
    S: And then you take them and put them into formula form. You take Y2-Y1 and the X2-X1.
11) T: I'd like for you to label the coordinates for the points: X1, Y1. X2, Y2, write down the formula, plug it in and work it out like you did on the homework. It's not bad.

Decisions

From the classroom observation, by ascribing to him the procedural orientation to teaching mathematics and the goal of getting the students to the correct answers, the following decision flow chart was created modeling Mr. L’s instruction depicted in Figure 4.1. Rectangles represent actions in the classroom and diamonds represent decisions. One main feature of Mr. L’s chart is that once a student produced the right answer the teacher moved on to the next step or next exercise perhaps with added explanation. Another is for incorrect answers he began a cycle of breaking down the task by asking simpler questions of the student until the correct answer was produced by the student or he eventually told them the correct answer. The nature of the flow chart is recursive: it works for a larger task, its subtasks and when he reduced the load of the tasks.

The model works for the vignettes used. In Mr. L’s Vignette 1, he started posing the larger task of adding square root of 162 and square root of 128 and simplifying in line one by “take a look up here” and pointed to the question on the board. Also in line one, he reinforced the procedure of simplifying “the first thing you ask yourself is…,” and he
posed the subtask of working with the first term to a specific student. The outcome of this subtask, the student response of “yes” in line two, was undesirable. Mr. L then told her the correct answer of “no” and reduced the load of the task by asking “If the answer is no it’s not, what do you ask yourself…” in line three. The student response of 9 for the question of a perfect square factor of 162 was correct but ultimately was undesirable to Mr. L because there was a more efficient outcome. As a result of the undesirable but correct outcome, Mr. L used all three responses to undesirable outcomes in the model in line 9. He told the student how the response was correct, split the task further by mentioning the greater perfect square, and asked any other students had the answer. A student responded in line 10 with the desirable outcome of 81. Mr. L responded to the desirable outcome by validating the response and then posed the subtask of 162 divided
by 81 in line 11. Getting a desirable outcome, he validated that response then moved on to the subtask of simplifying square root of 128 by first reinforcing procedure of “what is first question we ask about square roots numbers?” in line 13 and “what is the second thing we ask?” in line 15. The response to “Did you?” was undesirable and since the task was sufficiently reduced he kept asking students until a desirable outcome was produced. The rest of the Vignette 1 was a series of posing subtasks and obtaining desirable outcomes until the main task was finished.

Mr. L’s Vignette 2 started in line one after posing the task of finding the slope of the line on the overhead projector using the rise over run method with a long reinforcing procedure segment showing students how to find the rise and run of a line. He produced desirable outcome himself of 4 for the rise and 8 for the run in line one. He reinforced the procedure again in line three and produced the desirable outcome of “the slope is 4/8” himself. Continuing in line three, he then posed the subtask of simplifying the slope expression. Lines four through seven could be modeled by posing subtasks and obtaining desirable outcomes.

Before Vignette 3, the larger task was to find the slope for question #3 on the worksheet. After posing subtasks, reinforcing procedure, and obtaining a few undesirable outcomes the question by the student was posed. In the model, if Mr. L was asked a question he answered it, and then moved on to the next item. In line two, Mr. L immediately answered the question, restated it in a slightly different way and answered it again. Moving on, he reinforced the procedure of finding rise over run “how much you have to rise to get from one point and then run to get from one point to another” in line
two. Also in line two, moving on again Mr. L posed the subtask of finding the rise. At least one desirable outcome of -4 was produced in line three. Mr. A validated this response in line four and after a brief interruption posed the subtask of finding the run in line six. The students produced another desirable outcome in line seven, which the teacher validated and then immediately posed the subtask of simplifying 4/6. In line ten, Mr. L allowed the student to ask his question. The student asked his questions; Mr. L repeated the question in line twelve and then answered the question with an explanation in line fourteen.

**Insider Perspective**

Mr. L’s post observation interview (POI) illuminated many of the classroom observations and his TIPBS responses. As noted above, Mr. L had a very procedural orientation to teaching at least Algebra I Part I. The procedural orientation is partially the result of choice, “…the Do Now assignments for the Algebra I class today was just a skill practice. Um part of that is because feel like the Algebra I part I students need the skill practice more than the advanced math students do.” Mr. L’s point for the lesson on the second day was to “try to give them the conceptual understanding of what it means . . . to understand the slope of a line by looking at the concrete way tomorrow.” Vignettes 2, 3 and when he offered lengthy step-by-step explanations that occurred during day two illustrated that he still had a procedural orientation to teaching. Since Mr. A claimed the second day was supposed to be more conceptual but appeared to be more procedural, he might not be able to switch teaching orientations as he suggested.
The segmentation of the lesson on slope was part of his goal. In his own words he stated that “yesterday’s lesson is where students were finding the line; slope of a line provided two points that lie on the line,” today’s lesson students “were provided a graph of the line and ah they had to identify two points on the line, the coordinates of those two points of a line, and then calculate the slope of the line,” and tomorrow’s lesson students will be “determining the slope of a line actually counting blocks by determining the rise over the run.” The order the lessons were sequenced seemed different from what most curricula would suggest. He decided to “to do the opposite of the last time I taught this course, which wasn’t last year, and that was to go from abstract to more concrete rather than concrete to more abstract,” because he found students relied too much on the rise over run procedure.” Mr. L’s above statement of his belief that the algebra students need more skills practice could also help explain why the lessons and tasks are segmented into more manageable pieces for the students.

Since Mr. L mentioned he had students work in groups student-to-student interactions may occur more frequently than during the two days observed. He wants his students to communicate in groups on their own; however, he admitted that he frequently has to tell them to do it. He did not push for group interaction the first day since he was running behind.

During the first year, CCT was not used in the classroom due to lack of availability. There were also no graphing calculators present due to the teacher’s lesson plan. Therefore, technology use was not addressed for the first year in the case study. However, the POI illuminated some of Mr. L’s beliefs about graphing technology. For
the construct of *using technology in reform ways* on the TIBPS, Mr. L scored a low value of 1.4 or almost never using it in that manner. When asked how he used graphing calculators in the past, Mr. L replied “primarily as another way to just replicate what we do on graph paper,” and that using graphing technology follows paper and pencil methods. However, he believed using graphing technology as a teaching tool first might be more useful, but he was uncomfortable doing that due to lack of training in how to use it in instruction.

Below is Mr. L’s description of how he approaches discussing homework in the classroom; this description appears consistent with his approach to discussing the “Do Now” activities and in class exercises. The description also illustrates Mr. L’s goal for his students to obtain the correct answer.

*If there are problems I think we need to discuss the procedure of or if they have multiple steps and I know the students are going to want to see them, I’ll ask them to put them on the board or they’ll volunteer to put them on the board. Some days we just go around the room and um give the answers. I will help a student come up with the correct answer if they give the incorrect answer. Some days I’ll just read off the correct answers. If we’ve been doing the same type of homework for several days, this is may be the third or fourth day, I’ll say get out the homework. Here are the answers, correct your papers. Which ones do we need to discuss. Um the earlier in um the sequence that I assign the homework, right after it’s new, a new concept the more we go over it.*

**Year 2: First Year of CCT Use**

**Overview**

Several of the prominent features of Mr. L’s instruction from year one, remained a part of his instruction in year two, such as segmentation of lessons and class time, working with a single representation, the use of the IRE discourse pattern, and his
orientation to mathematics. However, Mr. L appeared to have added a new goal for his students of having them indicate if they understood when asked. This new goal appeared to add multiple instances of Mr. L asking “You okay with that?” in the middle of a conversation seemingly with the expectation of a “no” response if they were not “OK.” Technology used during the first day seemed to be an add-on to the lesson with the purpose of reviewing material. Technology used during the second day was integrated into going over the homework, which helped to prompt a discussion that may not have occurred using last year’s homework review procedure.

**Segmentation**

The class structure was very similar to the structure of the prior year. The first day’s structure comprised the five segments as in year one with the addition of a brief review using CCT at the end. No new material was presented during the second day, so no main lesson segment existed. Instead the day was used to give students more time to practice and discuss linear inequalities. The “Do Now” activities appeared to either relate to the topic of the lesson, or were marginally connected. The first day’s “Do Now” was to solve linear equations, which was connected to solving linear inequalities. Finding the slope of a line given two points that lay on the line was the second day’s “Do Now,” which was not later connected to solving linear inequalities symbolically.

Both days’ topic was on how to solve linear inequalities and represent the solutions graphically on a number line. However, the exercises were given so that only addition and subtraction were used so that solving linear inequalities was the same as solving linear equations. The first day Mr. L mentioned this similarity adding, “Until we get to
the next section of this chapter, everything is going to be the same.” When stating answers for the homework during day two, a student reported he got the wrong answer. Mr. L duplicated the students work on the board discovering the error was an incorrect coefficient of the variable due to arithmetic, and Mr. L wrote the inequality with the correct coefficient. In order to keep the segmentation of the lessons, Mr. L had to start the problem over to produce the solution, because the student’s method of moving the variables to the right side of the inequality with the correct arithmetic naturally produced the need of dividing by a negative number. Mr. L said he would finish the problem using the student’s method another time and told the student that it was assumed the students would move the variables to the left side avoiding the negative. Despite the topic naturally arising from student work, Mr. L kept the segmentation of the lesson plan. Even though the exercise was “designed” to avoid the need to discuss the case of negative coefficients in inequalities, student natural work produced this need.

**Representations**

The presence of lines in the lesson arose from solving linear equations, solving linear inequalities and finding the slope of the line during the second day’s “Do Now.” The lines in the equations and inequalities were treated solely as something to be solved rather than as a thing. Lines were presented by the algebraic representation from the process perspective in every instance except for a “Do Now” problem. The notion of solving linear inequalities graphically was not mentioned nor hinted at in future lessons. The use of a single representation in the classroom was consistent with Mr. L’s TIBPS response of never having students use multiple representations for explanations.
Tabular and algebraic representations of the same line were present during the “Do Now” for finding slope, although the presence of these two representations was not intended by the teacher. A representation of the line other than the tabular during the “Do Now” would not have been discussed if a student had not asked after the correct slope was reported, “What if we kept going through it and figured out the M and the B.” Mr. L responded by saying that only M needed to be found and asked “Why were you finding the M and the B? What did you go on to do?” Mr. L and the student went through several IRE sequences to elicit the student’s procedure for finding the M and the B. The student finished with stating he found Y=MX+B. Vignette 5 shows how Mr. L tried to get students to connect Y=MX+B back to the original question of the “Do Now.”

Mr. L’s Vignette 5

1) T: okay, so what did Zack do even though the instructions didn’t ask for that? He found the slope and what else did he do?
2) S: Slope intercept?
3) T: Slope intercept what? What do you mean by that? What are you saying?
4) S: He found the slope intercept or whatever.
5) T: He found something called slope intercept. What was that thing that he found? The instructions said to find the slope of a line using these points. He went on further; what did he find that went through these points? He said he had an answer that was Y= MX+B. Craig is calling that slope intercept. What did he find?
6) S: The Y intercept?
7) T: He did find the Y intercept; that’s B. No one knows what Zack found? He found the equation of the line that passes through these points.

Line 1 is the second time Mr. L reminded the student that he did more than was necessary. The student that responded in lines 2) and 4) knew the name of what was found, but appeared not to know what kind of mathematical object was found. The
student response in line 6 indicated that some students were focusing on the pieces of what was found, rather than the whole. Two representations of lines were present in this instance. Mr. L tried to get students to connect the two or at least name the connection, but ultimately he decided to only mention the connection.

Multiple representations of lines were not present in the solution of the linear inequalities in class. However, the solutions of the inequalities were presented algebraically such as \( y < 0 \) or on a number line. The goal for the main lesson the first day was to show how to go from the algebraic solution to the number line solution. The algebraic representations of the solutions were acted on to produce the end product of the number line representation. The representations were not used to reason with, justify, or explain the solutions to the original questions.

**Discourse**

Several prominent features of Mr. L’s classroom discourse were remnants of the previous year: such as using IRE sequences, long teacher step-by-step explanations of procedure, and short student responses. The new features of the classroom discourse were Mr. L asking students if they understand in the middle of a problem rather than just at the end, Mr. L not directly answering a question and redirecting the question to a student, and asking if someone saw the pattern, and attempting to illustrate student thinking.

**Questioning.** When having a student enact a procedure upon a representation, the discourse in year two was very similar to year one. Vignette 6 from year two day two was a solution to a “Do Now” question. Mr. L broke the task of finding slope into steps. In line one he broke it into labeling points and plugging into the formula. Even though the
student responded with answers for both the numerator and denominator of the formula, Mr. L focused only on simplifying the numerator in line three. Line five was Mr. L asking for the subtask of simplifying the dominator and line seven was the subtask of simplifying the expression. This example illustrates a typical IRE sequence during year two, as well as the segmentation of tasks.

Mr. L Vignette 6

1) T: Well Nate said let’s call these X1, Y1, X2, Y2. And Nate says let’s use the formula Y2-Y1 over X2 –X1. Nathaniel, using Nate’s labels and Nate’s formula, tell me how to plug these in.
2) S: Subtract -2 from 12 and then 10 subtract 3.
3) T: Anybody, what’s 12 subtract -2?
4) S: 14.
5) T: And anybody what’s 10 minus 3?
6) S: 7.
7) T: Leave your answer like that? 14/7ths?
8) S: Nope.
9) T: What?
10) S: Slope equals 2.
11) T: The slope is 2.

At the beginning of the main lesson Mr. L had a 470 word long explanation of how to produce the number line solution from the symbolic one, while showing them on the board. Mr. A also had several 100+ word explanations throughout instruction. Student responses were usually under 5 words.

During the first year’s class times, all questions posed by the students to the teacher were answered immediately by the teacher, and students were not given much time to respond to “any questions on this problem” before moving on to the next problem. Instances of those kinds occurred during year two; however, they were not the only kind
of discourse. Vignette 7 illustrates asking students if they understand in the middle as well as not restating the correct response and moving on. Before the vignette during a long explanation that the number line solution for $d \leq 2$ will have a dot at $d=2$.

Mr. L Vignette 7

1) T: Over here I said that we were going to put a circle because it’s less than or equal to. Over here it’s just less than so I’m going to put an open circle. Go to the 4 on your number line and put it [inaudible]. The last one is also less than so I want my number line to point to the left side of the number line. So I’ll make an open circle because it’s less than and pointing to the left. So does anybody know the pattern yet for when to use an open circle and when to use a closed circle? Brandon, do you know?

2) S: Yes.

3) T: What is it?

4) S: When it’s greater than or equal to, you use a full dot and the circle when it’s less than or greater than.

5) T: So when it has what do you make it solid?

6) S: When there is a line under the sign.

7) T: Do you understand what he’s saying? Zack, do you understand what he’s saying? Kristin, you good?

8) S: Not really. (seems teacher did not hear this)

9) T: Dan, did you pay attention? Did you hear what he said? Huh? Say it one more time, Brandon.

10) S: When there’s a line under the greater than or less than you use a closed circle, then open when there is a greater than and less than.

11) T: You get that thing, what he said?

12) S: When there’s an open circle there’s a greater than...

13) T: You don’t know why we’re using the closed on that one and the open on this one from what he just said?

14) S: You use the closed for when you have a line under it.

15) T: Is that what he said?

16) SS: Yeah.

17) T: Richard, you understand?

18) S: Yeah.

19) T: Candy, how about you?
The correct answer was produced in line four. However, Mr. L did not validate the response, restate it and move on to the next problem. Rather, he had the student partially restate the answer, and completely restate it in line ten. In line thirteen, he asked another student for the answer. Mr. L never rephrased the answer in this instance. In lines seven, nine, seventeen and nineteen, Mr. L asked specific students if they understand what was said, rather than either moving on or asking a general “any questions” with little time to respond. Despite the change in flow of the discourse, the quality of the discourse did not improve. The connection of dots with less than or equal to and open circle with less than by the teacher was a mere statement. While this connection is by a matter of definition, Mr. L could have explained that the dot means “we want the point represented by the dot.” In other words, he could have explained the definitions of dot and open circle. The student with the correct answer responded with accurate mathematical phrasing. Mr. L accepted simple surface description in line fourteen, rather than push for mathematical terms.

**Choice of examples.** Connected to the quality of discourse, is the choice of examples used in the classroom. The choice of examples can hinder discourse, cause problems for students, or the opposite. On the first day of year two, to show the students how to produce the number line solution to the inequalities, Mr. L started with three examples containing only less than or less than or equal to. In Mr. L’s Vignette 8, he comments in class about his choice of examples, and in Vignette 9 is a consequence of the choice of examples.

Mr. L’s Vignette 8
1) T: … Questions about her choices for the number line? Alex, you understand the choices for the number line?

2) S: No, but if there’s a > symbol then the arrow will go the other way?

3) T: Yeah, I probably should have chose some examples where we would have something like that.

4) S: I got it.

5) T: You’re right.

Mr. L realized that he perhaps should have had an example with greater than. But he did not introduce such an example then, perhaps because the student asking the question in line two has the correct answer. A solution with greater than a number did not appear until after the main lesson and students started to put their solutions on the board.

Vignette 9 illustrates how the choice of examples can confuse students.

Mr. L’s Vignette 9

1) T: … Morgan has a question.

2) S: okay, on number 8, why does that go to the right, like the arrow?

3) T: Let me see what Brandon’s thinking was. Number 8, Brandon, she wants to know why the arrow goes to the right.

4) S: Because it’s greater than.

5) T: He says because it’s greater than. You okay with that?

6) S: I thought they always went to the left.

7) T: You thought they always went to the left?

8) S: Because it had...

9) T: She said she thinks they always go to the left. Why is that, probably?

10) S: Because we just did less than three times?

11) T: Yeah, on our three examples they all went to the left and I said that’s probably a bad idea and Kristin said, “No, I get it,” but she didn’t. She saw that was the pattern all the time. So left goes with less than and arrows to the right go with greater than. Anybody else have questions like that? Get back into Navnet...

Both the teacher and some students understood why the student responded the way she did in line six. She was trying to identify the pattern; however the examples presented did not illustrate different possibilities and led to an incorrect conclusion. Vignette 9 is
another example of Mr. L redirecting a student question to back to another student. Mr. L., in line five, appeared to think that student response in line four to Morgan’s question would be sufficient. Mr. L’s question at the end of line 5 and similar questions in Vignette 7 illustrated Mr. L’s new goal of asking students if they understand. Mr. L’s question in line nine was different than most of his questions, because he asked students to think about another student’s reasoning, which never occurred during the first year’s observations.

**Missed opportunities.** Not all of Mr. L’s choice of examples led to poor generalizations. In the exercises the students had to do, one had a non-integer solution. A student put his graphical number line solution, Figure 4.2, to this problem on the board and Mr. L asked them to compare their graphs to the ones on the board. Below is the resultant discussion.

![Figure 4.2: Student number line solution to a linear inequality.](image)

Mr. L’s Vignette 10

1) *T:* ...*All right, take a look up here and compare these graphs to yours. Number 5 we did yesterday. We got A is < -2. She got -3, -2, -1 on the number line. She has an open circle on the number line and the arrow points to the left. Problems with that?*

2) *S: Nope.*
3) T: Brandon’s got 3, 4, 5.6, 6 and 7 on his graph. He’s got an open circle on the 5.6 and the arrow going to the right. Okay with that?
4) S: No.
5) T: What’s wrong?
6) S: Ah, the decimal.
7) S1: The numbers are wrong.
8) T: What’s wrong?
9) S1: It’s supposed to be... wait a minute.
10) S: 5.4...
11) S1: 5.5 and 5.6 and 5.7 and 5.8.
12) T: okay, what about you Crystal, what were you going to say? that will work?
13) S: Yeah, I had 5.4, 5.5, 5.6.
14) T: What about you, Zack?
15) S: I had 5 and 6.
16) T: There’s no rule about how the number lines have to be set up. A couple things are okay. If you all notice that the solution had 10ths... the solution had 10ths in it, 5.6, so you all made your marking, your other tic marks being to the 10ths. 5.7 and 5.8 to the right and 5.5 and 5.4 to the left of it. Another thing he could have done here since he had all integers except for one is he could have made this one a 5. Where is 5.6 in relation to the numbers that he has up here if we change this? If we leave all integers where is 5.6 in there?
17) S: Would it be on the 6?
18) T: Is it on the 6?
19) S: Or like... / Close to the six.
20) T: Close to the 6?
21) S: A little bit past the middle.
22) T: A little bit past the middle of what?
23) S: 5 and 6.
24) T: 5 and 6. So we could put an open circle a little bit past 5, not quite 6; halfway between 5 and 6 and put the arrow to the right. This is probably a good idea down here, too, to make the other markings on your number line be to the 10ths in the solution. One of these would be fine. Brandon, you probably wouldn’t want to mix 10ths and integers, but there’s nothing really wrong with putting that there. Questions on 8? Now we have... (Moves on to next problem).

Students were fine with the number line representation for the first problem for A < -2, more than likely because it fit either the students’ number line representation or their internal one. Mr. L appeared to think that nothing was wrong with Brandon’s solution,
since there was a hint of surprise in his “what’s wrong?” in line five. Some students, however, believed that something was wrong. In fact, S1 from line seven seemed rather adamant that Brandon’s number line representation was incorrect. The combination of choosing a non-integer solution and the presentation of another student’s representation of that solution led to the controversy and discourse.

The excitement of the controversy could have been channeled into exploring the situation further, and be used to push the students’ reasoning. The students in lines ten, eleven and thirteen responded with what they thought the numbers should have been, rather than why they thought Brandon’s numbers were wrong. Mr. L missed an opportunity to ask, “Why couldn’t 3, 4, 5.6, 6, 7 work?” and have them test the numbers to see if they did work. Instead he explained that Brandon’s numbers were fine, saying that “There’s no rule about how the number lines have to be set up” in line sixteen.

During the third and fourth sentences in line 16, Mr. L drew the number line of the bottom of Figure 4.3. The rest of line sixteen of the vignette 10 to the beginning of line twenty-four, Mr. L modified Brandon’s number line to the one at the top of Figure 4.3 and stated at the end of line twenty four that one of these two number lines were more preferred.

![Figure 4.3: “Corrected” number line solution to an inequality.](image)
Casual comments. Taken from line eleven from Vignette 9, Mr. L said, “So left goes with less than and arrows to the right go with greater than.” This statement is not true in general but conditionally; the statement is true if the conditions of the variable is completely isolated, and the variable is on the left. During the second day’s review of homework, Mr. L had already gone over correct symbolic solutions and then went over the homework again asking for the graphical solutions. Below is an illustration of what happened in the classroom when the variable is isolated on the right.

Mr. L’s Vignette 11

1) T: Richard, number 3. The answer was 7 ≤H or we decided we could do it the same way as you had it, H ≥7, right? What kind of circle did you put on 7?
2) S: Closed.
3) T: And which way is the arrow?
4) S: Left.
5) T: Is that right or left?
6) S: Left.
7) T: To the left. You all okay with that, closed circle on 7, arrow going to the left? Alice, you okay with that? Ashley, you agree with that? Morgan, how about you?
8) S: Yes.
9) T: Closed circle on 7, arrow to the left? You okay with that, Amanda?
10) S: Number 3?
11) T: Number 3, closed circle on 7, arrow to the left?
12) S: Yes.
13) T: Everybody okay with that? Yes, Nate? ...Oh, let’s talk about that one then. The closed circle on 7 is right; why?
14) S: There’s a line underneath.
15) T: There’s a line underneath and it does need to be closed. Remember this is the one Ashley solved it this way. She likes her variable on the left hand side so she was trying to switch it around and when she switched it around she didn’t change the symbol with it. If 7 is <H, Richard, you told us earlier that H is bigger than 7. That H is ≥7. We can look at it like this. It is a closed circle on 7 but the arrow goes...
16) S: Right.
By following Mr. L’s casual comment from Vignette 9 and using the 7 less than or equal to H solution, students inferred the graph should go left. The casual comment did not hinder only a few students, but nearly the whole class. Rather than explaining why the arrow should go right using only the inequality with h on the right, he explained how to move the variable to the left to be able see which way the arrow goes in line fifteen.

**Technology Use**

CCT use during the first day was minimal, at the end of the lesson seemingly an add-on component rather than technology being integrated into the lesson. CCT was used during the last ten minutes of a fifty-five minute class session. The CCT use was meant as a review of the days lesson, but the questions were ambiguous and of low level. He used the CCT’s Quick Poll feature to ask the following questions, 1) True or false, linear inequalities look like linear equations; 2) true or false, linear inequalities is the same as solving linear equations; 3) he put a symbol on the board and asked for the name of it; 4) “Y is less than equal to 5. I’m going to make a solid dot on 5 with an arrow going to the right.” True or false; 5) and Do you understand how to solve and graph linear inequalities? For questions 1 and 2, the CCT displayed 11 students said yes and 2 said no. For question 3, there was a typo, 11 greater than’s, and 1 less than. Nine students said true and 5 said false for question 4. Everyone said they understood for the last question.

The first and second questions were vague and confused the students. Nearly all the students said that linear inequalities look like linear equations. Mr. L’s intended answer was that they do not look alike. Eleven students initially disagreed with his answer, but when he asked some who said false, they changed their answer from false to true. Mr. L
explained he intended the answer was that they did not look alike and commented “that you’re just saying true because you thought they looked close enough?” The question of are the procedures of solving linear inequalities and linear equations the same caused confusion due to lack of clarity. One student noticed “Just that we haven’t like divide yet.” Another student said getting the graphical solution is different, which Mr. L did not intend for them to use this in the comparison. Question 4 was the only one that addressed the issues that students had during class. Vignette 12 is the class discussion about question 4.

Mr. L’s Vignette 12

1) T: Y is less than equal to 5. I’m going to make a solid dot on 5 with an arrow going to the right. True or false?
2) S: Say that again?
3) T: To graph Y is less than equal to 5 I’m going to put a solid dot on 5 and make my arrow going to the right. One more person, send in your answer. [Repeats question] 9 said true and 4 said false. Can someone say why they said false? Why, Brandon?
4) S: Because when you go to the right it’s greater than?
5) T: okay, so I should do what?
6) S: Go to the left.
7) T: Do you all agree with that? Do you want to change to false?
8) S: No.
9) T: If you had true would you want to change? He’s right. If it says less than equal the arrow should go to the left and I said to the right. Maybe you didn’t pick up on that.
10) S: Oh.

Students were having trouble with the direction of the arrows given the symbolic solution on the first day continuing on to the second day (Vignette 11). The CCT displayed that nine of the thirteen students answered incorrectly. During line three, Mr. L asked a student who had the correct answer why they responded that way without
labeling that answer as correct. The student in line four responded with a reason sufficient for Mr. L and responded correctly to what needed to be done for the question in line six. Mr. L believing this explanation was sufficient asked the other students if they wanted to change their answer to false in line seven. Even after the student explanation, students did not want to change their answer. Even though CCT indicated several students had gotten the question wrong, and after an explanation for the correct answer was given, the classroom discourse revealed not all students were convinced by this explanation, Mr. L did not pursue the issue further. He ended the conversation with the simple statement of correctness in line nine. The first three and the last questions were of low cognitive load, uninteresting, and at times ambiguous. Question 4, although not higher cognitive load, addressed issues students had during the class and the CCT illuminated the need to further discuss the connections of the symbolic and number line solutions to linear inequalities. Unfortunately, this discussion never happened, and it became an issue during class the next day.

Mr. L used the CCT during the second day in the middle of the class time during the review of homework rather than the end and spent about ten minutes with it. He also used the feature, Screen Capture of the CCT during the second day. Students were asked to type on their calculator home screen using inequality symbols the solutions to the homework questions. Using the CCT, Mr. L took a snapshot of all the students’ calculator screens and projected them on to the board. Last year he would ask students for the answer, and if anyone got it wrong. With the CCT, he was able to see all the student responses and if they were correct. The homework question was to solve $16 \leq h + 9$. Mr.
L saw the student responses as, “7 <= H, I’ve got an H <= 7, I’ve got an H <= -7.”

Students were able to see not all of them had the same solution. Mr. L addressed the first two of the solutions present. The first solution of 7 <= h, was correct and he had a student who submitted that response explain how she did it. He did the same thing to a student who submitted h <= 7 and had her put her work on the board. Mr. L noticed the work on the board was correct, but the answer she submitted was incorrect. He then asked the students “What’s different about Amanda’s and Ashley’s responses up here?” Students replied that they were, switched or flipped. Mr. L asked Ashley why she submitted a response like that and she responded with “I always have the variable on the left.” The following is Mr. L’s explanation following Ashley’s response.

Mr. L’s Vignette 14

1) T: Like you always have the variable on the left hand side, which is okay to want to have the variable on the left hand side. If you did what Amanda did, which is what Ashley wrote up here you have it correct and Amanda just sent it in simply like she got on her paper. She sent that response in on her screen. Ashley says she likes to switch the variables on the left. That’s okay, but you’ve got to be careful how you send it in. If 7 is <= H, tell me, how does H compare to 7? Listen to the question again. If 7 is < H, how does H compare to 7? Richard? If 7 is < H, how does H compare to 7?

2) S: H is bigger.

3) T: It’s bigger than. So if Ashley wants to have the variable on the left, Brandon, what should she do when she decides to switch the H in the sentence?

4) S: She should put a different symbol?

5) T: She should put a different symbol. In this case what should she have put?

6) S: Greater than.

7) T: Greater than. So you did the work right, but you switched them. If you have what Amanda has then you’re right. If you had what Ashley had with the >= that would be right.
The displays of multiple solutions and the student response of wanting to put variable on the left indicated the need to discuss how to correctly “move” variables. Rather than having students figure and explain, Mr. L explained it himself in line one.

**Decisions**

Mr. L’s orientation appeared to be procedural during the second year. Most of the questions he asked are the result or a step of a procedure, such as in most questions in Vignettes 6, end of line one Vignette 7, several first questions from Vignette 11, and line 1 Vignette 12. The explanations he accepted are of procedural nature, such as line 4 Vignette 7, and line 4 Vignette 9. The explanations he gave are usually step-by-step procedures or explanations are the result of a procedure, such end of line eleven Vignette 9, line sixteen and 24 Vignette 10, line fifteen Vignette 12, and line one Vignette 14. The goals of Mr. L seemed to be for students to produce the correct answer and ask students if they understood it along the way.

The flow chart in Figure 4.4 models Mr. L’s decisions during year two of the study. The octagon shape represents a decision that is made possible or easier from the use of CCT and a rounded rectangle is an action made possible or simpler by using CCT.
Mr. L seemed to want to work with desirable outcomes first and then move on to see if students understood, or at least said they understood. If desirable and undesirable outcomes were present at the same time, Mr. L took the time to address the desirable one and at least one undesirable one. In few instances outside of using CCT were there
multiple outcomes present except for the instance in Vignette 10. This is why the corresponding action to multiple outcomes was made an octagon in Figure 4.4.

**Insider Perspective**

Mr. L explained that the point of the lesson was to get students to realize the similarities of linear equations and linear inequalities in that he was trying to “draw on their prior knowledge of how to solve linear equations and develop in the lesson.” This helps explain why he chose the questions at the end of the first day using CCT. If he really wanted to emphasize similarities between the two, it also explains why he stressed the segmentation even in the face of the differences appearing in the classroom.

During the first year and the second year, Mr. L stated that lessons did not differ much from what he planned. Although, Mr. L mentioned in the POI that he knew the three less than inequalities could be an issue, but figured since one student said, “No, that’s fine, I know exactly what you mean; I know that these are all to the left because it’s less than (Mr. L’s words)” that maybe it wasn’t an issue. Mr. L claimed that the CCT “did help me to understand a misconception that student had but it wasn’t early enough in the lesson that it made me change directions or anything that I was doing.”

Most of Mr. L’s CCT use revolved around Quick Poll because that was what he was comfortable with and because it “requires the least amount of additional preparation.” During the POI Mr. L explained that he initially used Quick Poll with open ended questions, but moved to true/false, yes/no, and multiple choice to avoid “silly” answers from students. Silly answers to Mr. L seemed to be ones where students drew attention to themselves and were usually meaningless. Therefore, the CCT use during the second day
was atypical since it used SC and may have been because of the presence of the CCMS classroom observations. The CCT use at the end of the first day was exactly how Mr. L said he uses the CCT.

**Year 4: Third Year of CCT Use**

**Overview**

Several features of Mr. L’s teaching changed after year two or during year four, such as segmenting the classroom differently, discussing multiple representations of lines at once, using technology during the whole class time, not asking for the result of a calculation, and having students do an activity rather than practice exercises. He continued his use of CCT for asking simple true or false questions from year two. No insider perspective is possible for year four, since no POI was conducted.

**Segmentation**

Every single class previously observed by Mr. L started with a “Do Now.” In the third year of CCT use as class began, the “Do Now” and the going over of yesterday’s homework were missing from his previous introduction. Instead Mr. L started with a statement of what they did the previous day and then asked questions using the CCT to see what the students remembered from the day before. After the questions, he started an activity for the students to send in equations of lines using the CCT to match pictures displayed on the Activity Center screen. The activity lasted about 30 minutes of a 43 minute class session. After the activity he asked students what their strategy was for the activity and concluded with simple true/false questions posed to the students. The
segmentation changed to review, activity, and wrap up, which was distinctly different from years one and two.

What the students needed to do to complete the exercises seemed to be a culmination of lessons that would have been taught by Mr. L in previous years. In years one and two, the lessons observed were of very specific sets of exercise such as find the slope of the line given the graph of the line with two points given and without two points given, or solve linear inequalities with only addition and subtraction and draw number line graphs. In year four, the students were not doing exercises since there was no single right answer. The lines they were to match in the picture were thick enough that different lines could fit inside so they could have argued which one was the best.

**Representations**

As mentioned earlier the goal of the activity was to have students send equations of lines in their calculator to match lines in a picture projected onto the board. Students had to do three of these pictures: the side of a house, a slanted book shelf, and a staircase. Using the Activity Center of the CCT, the picture was embedded into the XY plane with tick marks on each axis. The two different representations of lines were not presented by accident as happened in year two, but rather the point of the activity was to connect the two representations. Even with the graphical representation present, lines were treated from the process perspective since ultimately students enacted a process to find the lines. Treating lines as objects was only hinted at and was one of the largest missed opportunities of the activity.
A picture of the side of the house was centered so the vertex of the roof was on the Y-axis. The students were asked to match only the right side of the roof, which was a negatively sloped line. The students were able to resubmit lines as many times as needed. After giving the students a little bit of time to submit, Mr. L paused the activity to discuss lines that were out of place. Below is discussion that took place.

Mr. L’s Vignette 15

1) T: I am going to pause for just a second, I am not going to tell you the answer. I am still going to let you send in, just a minute. I am going to pause it. Let us talk about the ones we have up here so far. This happened yesterday. (Pointing to a horizontal line) Does anyone remember what the problem is?

2) S: The X.

3) T: They have forgotten the X. This person sent in the equation of a horizontal line. Maybe they had two numbers in there, but they had Y=2 + 4. But didn’t have an X behind the 2. If this one is yours make sure put an X in it. What about these three? (His hand covering three lines with two with positive slope one negative).

4) S: They all have the same Y-intercept.

5) T: They do all have the same Y-intercept, same B value. How about these two? (Pointing to the positive sloped lines).

6) S: [inaudible]

7) T: What do you mean by not that [inaudible]

8) S: It’s too big.

9) T: What’s too big?

10) S: The ...umm... the Y...umm the slope

11) T: okay. You said earlier they all had the same Y-intercept. What are you saying about the slope? That it is too large?

12) S: Either too large or too small.

13) T: Too large or too small. Yeah. ...Comment on this one. Kristin, what do you think about this two? (again to the two with positive slope)

14) S: They’re not on the right...

15) T: They’re not on the right... Do you mean right vs. left or right vs. wrong.

16) S: Umm. Both.

17) T: Both okay. They should be over here. (pointing to the right side of the XY plane). Can you tell me something about the line, something about the equation they must be sending in, if that’s the case?
18) S: they’re putting in a negative on the...
19) T: There putting in a negative what?
20) S: Slope.
21) T: There putting in a negative slope, not a positive?
22) S: yeah.
23) T: Is this one yours right here (pointing to the only negative sloped line)?
24) S: I am not sure.
25) T: you’re not sure. So are you putting in a positive slope?
26) S: Yeah.
27) T: do you all agree with her? That these two...
28) S: No, I [inaudible] negative and I still got that (pointing to screen).
29) T: One of these (positive slope) is yours? And you are putting in a positive slope?
30) S: Uh huh. Both of them positive.
31) T: okay.
32) S: [inaudible]
33) T: He is sure that one of these is his and he is putting in a positive slope.
34) S: The one that goes straight down, and I had 11+77x.
35) T: One of the questions I asked you earlier was true or false: a line that has a positive slope goes down and to the right. And the answer was....
36) S: [inaudible]
37) T: Huh? Positive slope down to the...
38) S: False.
39) T: it was false, so these right here (hand covering positive sloped lines) must have what kind slope?
40) S: Positive.
41) T: Positive because they are going which way?
43) T: Up to the...
44) S: Right.
45) T: So what do these people need to do?
46) S: Put one negative in the slope.
47) T: Put a negative value in the slope. This one (the horizontal line) needs to insert an X, and these people right here need to insert a negative slope and a negative coefficient on X. All right resume....

The representations of lines were used as both end products and something to reason about. When all the students’ lines reasonably fit the picture, Mr. L moved on to the next
question; the lines were treated as an end product. However, during the activity in several instances, if the lines were off in a distinctive way, Mr. L would pause the activity and have the students discuss what was wrong; treating the lines as something to reason about. Two of these instances occurred during the first picture and another occurring during the last one. Very little discourse occurred during the second activity. In line three and five of Vignette 15, Mr. L drew attention to the incorrect lines without labeling them as incorrect. Rather he asked them to talk about the ones that were present on the board and had them elaborate what they thought about the lines. Lines five to sixteen, the teacher and student discussed what was wrong with the lines visually with the student concluding that the two positively sloped lines were in the wrong place and needed to be on the right side of the graph. Mr. L moved beyond the graphical by pressing for the connection between the graphical and algebraic in line 17. The student in line 18 saw the connection between shape of the line and the negative slope. In instances like these, lines were not treated solely as an end product for students to produce, but something to discuss and think about.

In line 3 of Vignette 15, Mr. L classified one line as horizontal and focused students’ attention to a group of lines asking what they noticed about them. In line 5 of the vignette, he refined the students focus on the two positively sloped lines. The students noticed the slopes of the lines “were too big” (lines 8-10). Another student said that the two lines were also not in the right place (line 14). Mr. L hinted that the lines could be moved with his hand movement saying “They should be over here” (line 17). During this
Vignette, Mr. L hinted at the object nature of functions by focusing on properties of the lines and indicating the lines could be moved around the xy plane.

**Discourse**

The long teacher explanations, the IRE sequences for calculations, and the persistent asking of “Okay with that?” were not present in Mr. L’s classroom discourse during Year 4. Many of the newer features of the classroom discourse in Year 2 continued and expanded in Year 4, such as not immediately validating a student response as correct or incorrect, pressing for student explanation, and occasionally after a student gives a response asking if the other students agree. Although not pressed for by the teacher, a student interpreted another student’s comment.

Mr. L’s longest utterance was 140 words long rather than 400+ words of previous years. Mr. L’s longer utterances were around 50-80 words compared to 120-170 words. The majority of student comments were short. However, the maximum length of a student utterance changed from 17 words to 60 words.

**Questioning.** Perhaps due to the kind of lesson observed, Mr. L did not use calculational IRE sequences with the students. Vignette 15 contains small sets of IRE sequences that are part of a much larger sequence where the teacher asks a question line 5, a student responds line 6, the teacher ask for clarification line 7, the student clarifies line 8, another few cycles of clarification occurs lines 9 - 12, and the teacher evaluating in the beginning of line 13. Lines 13 – 17 are a short initiate, respond, clarification cycle, evaluate sequence. Another one of these sequences appeared in lines 17 to 27, except that
instead of evaluating he asks the students to evaluate, “Do you all agree with her.” Mr. L
not immediately telling the students the answer continued from his Year 2 instruction.

Mr. L asked a different style of questions during Year 4. In Vignette 15, he asked
students what they thought about the lines projected on the board. Starting in line 5,
perhaps even line 3, Mr. L wanted to steer the conversation toward a student explaining
the lines he pointed to needed a negative slope especially through his clarification
questions and more directly in line 17. The notion of negative slope comes from a student
in lines 18 and 20, but Mr. L did not get the student to say what needed to be done. His
asking the student to identify her line on the board appeared to slightly derail the
conversation. He attempted to get the conversation back on track by asking what other
students thought in line 27. However, this led to an unexpected result of line y = 11 + 77x
that goes “straight down.” Rather than exploring why that line appeared to go straight up
and down, he reset the conversation by asking a previous true/false question in line 35.
This ultimately led to a student saying the incorrect lines needed negative slopes in line
46, and he restated this in line 47. The connection for the students of the sign of the
coefficient of x and the direction of the line was not complete, since this mistake occurred
once more during the same activity and during the review questions at the end of the
lesson.

At the end of the activity for all three pictures, Mr. L asked a student perhaps the
most important question of the day. He essentially asked a student what her strategy was
for finding the best fit line. Below is the conversation that occurred.

Mr. L’s Vignette 16
1) **T**: Ashley, you seem to be pretty good at this at sending your line that’s correct with very few tries. What’s your secret?

2) **S**: I guess the first time and keep guessing until it works.

3) **T**: You say you guess; is it completely random?

4) **S**: Yes. I look at it and then I guess some numbers that at least work...

5) **T**: That’s not random. What numbers are you looking for?

6) **S**: If it’s sloping down, I will try like -1/1 and try that and put numbers in.

7) **S1**: I never do single numbers.

8) **T**: Never do what?

9) **S1**: Single numbers.

10) **T**: What do you mean by that?

11) **S1**: I always do fractions.

12) **T**: Fractions for...?

13) **S1**: The X. and this time I got like -1/1.2x +2.

14) **T**: Kristin. I said yesterday it seems like everybody starts typing right away, but you, you’re kinda studying the picture. What are you doing?

15) **S**: I look at where it crosses the y-axis...

16) **T**: okay.

17) **S**: …and what the slope is.

18) **T**: okay. Um Ashley you didn’t say anything about the y-axis do you look for that as well?

19) **S**: I do that last.

20) **T**: The y-intercept one?

21) **S**: Cause you’re supposed to that last.

22) **T**: Cause it is...

23) **S**: Cause of the MX+B.

24) **T**: Does that mean a person has to solve for that last?

25) **S**: No, but it is easier for me.

26) **T**: That’s fine.

The “What’s your secret?” question tied all three activities together and connected the algebraic and graphical representations of lines beyond the single connection of positive slope goes up to the right and negative slope goes down and to the left. This single connection appeared to be a major goal for the activity, since it was addressed in the beginning review in Vignette 19, in the middle of activities in Vignettes 15, 18, and

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during the review at the end of class in Vignette 17. Mr. L usually restated a student explanation of what needed to be done in a certain situation, such as line 47 in Vignette 15 or line 18 in Vignette 17. However, even though a few procedural strategies exist for finding a best fit line to match a picture, Mr. L did not explain or even mention them.

**Choice of examples.** The choice of examples for the pictures was better than his choice in Year 2. The choice of examples had more potential to better connect the algebraic and graphical representation and treating functions as objects than was realized in Mr. L’s instruction. This potential will be explored further in the missed opportunities section. The examples had both positive and negative slopes with different magnitudes as well as positive and negative y-intercepts. However, the combination of the choice of examples, a student’s assignment within those examples, and the lack of discussion at the end of each example led to incorrect generalizations. The first and third examples’ pictures to be matched had a negative slope and a positive y-intercept. The second picture had three parallel lines with positive and negative y-intercepts. At the end of class using the CCT to review, Mr. L asked, “Does y= 2x-4 slope up and to the right.” He asked any student who said true to say why they answered true and a student responded that they meant to say false. After the response, Mr. L started the conversation in Vignette 17.

Mr. L’s Vignette 17

1)  *T*: How can you determine from the equation how to decide if the line slopes up to the right or down to the left?

2)  *S*: okay, if it slopes, if it slopes down to the right... this one’s right here (making a gesture with his left hand) is going to be negative like this number over here is negative (making a clasping gesture with his left hand). And the non-X (making a gesture with his right hand) is going to be positive. And if it slopes up and to the
right this this X ones positive (left hand) and the other one (right hand) is negative.

3) T: Y’all agree? Answer in your calculator. Do you agree with what Devin just said?

4) S: I forgot what he said.

5) T: Can you repeat it again? You are sending in agree or disagree. I am going to clear anyone’s answers that have sent in.

6) S: okay were going to go zzzt. Okay when it goes negative slope, this number (gesture with left hand) over here is negative the X is negative and the B (right hand) will be positive. And when it goes it goes up to the right, positive slope, the X is going to be positive and the B is going to be negative.

7) S1: No.

8) T: Agree or disagree with Devin… One more person. Send your answer in. Do you agree or disagree? Oh, I guess its Devin. We assume, you will agree with yourself [laughter] right? Everyone agreed with you. Let me hear that explanation with you one more time.

9) S: okay. The number over the X, the negative slope, the X is going to be be ah [laughs] X is going to be negative and the B is going to be positive. And when its positive slope, B is going to be negative and X is going to be positive.

10) T: He’s half right. Which half of what he said is right? … The part you were saying about the coefficient of X is correct. The slope doesn’t affect what the B value, whether the B value is positive or negative.

11) S: Ah.

12) T: Can you repeat the part just about the coefficient of X.

13) S: Its positive slope when the coefficient of X is. . . I mean negative. Its negative slope when it’s positive.

14) T: Now you have mixed those up.

15) S: okay, now the other way around.

16) T: Well, let’s hear it.

17) S: Oh okay I got confused. Its positive slope X is positive. Negative slope when X is negative.

18) T: If the line has a positive slope, the coefficient of X will be positive. If the line has a negative slope the coefficient of X will be negative.

19) S: That means the B variable varies.

20) T: That it doesn’t have anything to do with the slope. Whether it is positive or negative has an effect on the graph, but it doesn’t affect the slope.
Devin was assigned to match the middle of the three lines that had a negative y-intercept, so his response in line 2, which was the longest student comment, made sense with this experience and the lack of discussion at the end of each activity. Consistent with his approach in Vignette 15, Mr. L asked the students if they agree or disagree rather than correcting him. Even though the student in line 7 did not agree with Devin, she did not send in a disagree response using the CCT. Despite other students having different experiences that contradicted Devin, he apparently convinced them he was correct as evidenced in line 8. Not having any correct answers to start from, he declared Devin to be half right in line 10. Although, Mr. L could have had a correct answer if he asked the student in line 7 to say why she said no but sent agree into the Quick Poll. Vignette 17 illustrates the influence of students to convince other students.

**Missed opportunities.** The combination of the choice of examples, the tasks the students performed, and different representations of lines coming from multiple students appeared to create more missed opportunities to connect the different representations and perspectives of lines in Year 4 than the previous years. Mr. L’s ability to take up these opportunities did not diminish over the years, rather the opportunities were far less likely to be present in previous years. Mr. L during Vignette 15 did take the opportunity to discuss some of the connections between the symbols and the graphs of lines. However, even with only the first picture more connections could have been made. The CCT had the ability to simultaneously display graphs and their equations. After most of the students were close to the picture, he could have clicked on the tab that displays both graphs and equations and asked students to compare. Not all the equations would look the
same, i.e. \( y = -1/1.2x +2 \) versus \( 11+77x \). The students could have been asked to discuss how to compare the different equations. After a standard form for the equation of the line was present, better comparisons among them could have been made. Since not all the lines fit perfectly on the picture, the impact of slight variations of the coefficient of \( x \) on the slope on the graph could have been explored. Unfortunately, when the students matched the pictures, Mr. L simply moved on to the next part of the lesson.

The edge of the house in the first picture looked like a thick upside down absolute value graph centered on the \( y \)-axis as shown in Figure 4.5. The students were asked to match the line segment on the right half of the plane. Near the end of the first picture activity, a student matched the picture well, but on the left side. Mr. L paused the activity and started the conversation in Vignette 18.

![Figure 4.5: Picture of rooftop for first activity during Mr. L’s third year of CCT use.](image)

Mr. L’s Vignette 18

1)  \( T: \) These right here are real close (hand waving over lines on the right). What about this line right here (pointing to line on the left).
2) S: It needs a negative
3) T: okay. So they need to send in what kind of equation sent in?
4) S: negative slope.
5) T: Negative slope, okay. Is this one yours? Everyone write down your...on their calculator hit graph. If yours looks like this then you’re the one that needs to change to a negative slope.
6) S: I am not sure what mine is, I sent mine in twice
7) T: So this one is yours?
8) S: Yeah I think so.
9) T: Go back to equations, put a negative coefficient for X. What do you have right now?
10) S: Um. I erased it

His apparent goal was to get this student to the correct answer, rather than to explore reflection of lines. Mr. L could have the students who fit the line on the right well enough, how could they move their line to match the line on the left without having to start over to emphasize the object nature of lines. To aid the discussion Mr. L could have displayed graphs and equations to compare the equations present. However at the end of the first activity, Mr. L asked a student who thought they had the best line what their equation was, had this student put the equation on the board, and had all the students send in that equation. When the students sent in that equation, Mr. L seemed content the correct answers had be found and moved on to the next activity without further discussion.

A similar missed opportunity occurred during the second activity. The picture of the second activity was a slanted book shelf with three shelves. The shelves were parallel to one another. As discussed above, he split the class into groups having each one match a different shelf. Not long after the students produced the picture in Figure 4.6, Mr. L asked some of the students individually what their equations were. He then cleared the
equations and picture to move on to activity three. Again using the graphs and equations display Mr. L could have asked students to compare and contrast the lines. After converting all slopes to decimals, the students could have noticed the lines had nearly the same slopes but different y-intercepts leading to or reinforcing the notion of parallel lines. He could have asked the students who had, for example, the bottom shelf matched how they could move the graph of the line to match the top shelf and what that move would do to the equation. This question emphasizes the object nature of lines, illustrates the impact of changing the b in the equation, and reinforces the notion of parallel lines. Discussing the lines in activity two could have prevented Devin’s incorrect generalization in Vignette 17, since he would have seen a contradiction to it. The shelf picture could have also been used to discuss perpendicular lines by having a group of students match the end of the shelves. However, the ends of the shelves do not have y-intercepts and could be more difficult to find.

Figure 4.6: Book case with parallel lines.

Perhaps the largest missed opportunity was Mr. L not pressing further and discussing more of Ashley’s secret, since it tied all the activities together and connected the
algebraic and graphical representations of lines further. Given the activities the students worked on, a major goal for the students should be to develop a good strategy for matching lines. However, Mr. L’s major goal for the students appeared to be for them to notice negative slopes go down to the right and positive slopes go up to the right, since this topic arose before, during, and after the activities. After line 6 in Vignette 16, Mr. L could have asked what numbers did she put in and how she got better numbers. Kristin in line 17 said she looked for what the slope was, but did not elaborate on how she did that nor did Mr. L press for the elaboration. He could have asked the students to develop a strategy for matching lines given what Ashley and Kristin said. Any one or all of these could have led to deeper connections between the representations of lines. Surprisingly, Mr. L did not give them a strategy, rather he moved on after line 26.

Casual comments. Mr. L’s question in line 1 of Vignette 19, “True or false: when you try to put a line up here on the screen the number that you add or subtract at the end of your equation is the place where the line is supposed to cross the y axis?” is an instance of a casual comment. The question was casual since in line 3, Mr. L treated the question as being true and it is only true if student submit lines in the form y= mx+b. If students submitted lines in an unsimplified point-slope form, such as y= -.77(x-2) +4.54, for the equation of the line then the number at the end, 4.54, is not the y-intercept. Since the lines Mr. L asked students to match all crossed the y-axis, all students chose the y=mx+b form probably due to ease of use. Furthermore, the casual comment did not have the immediate consequences as the one from Year 2 when students were confused about which arrow to use in number line solutions to inequalities.
Mr. L’s Vignette 19

1) T: When you try to put this line up on top of the picture here, remember that it's at the end of the equation is where the line goes across the y axis. True or false: when you try to put a line up here on the screen the number that you add or subtract at the end of your equation is the place where the line is supposed to cross the y axis?

2) S: Can you repeat that.

3) T: I’ll say it one more time: when you try to put the line up here, send the line up here to me, on top of the board, on the board on top of the photograph the number that you add or subtract to the end of the equation is where the line should cross the y axis. Think about that. Then send me true or false. I’ll say it one more time, and then I need two more people’s answers. Everyone’s just came in. They all said true.

Last one in those pictures. True or false: A line that goes up to the right has a negative slope, the line that goes up the right has a negative slope. Most of you said false. Would one of you that said false, tell me why you said false.

4) S: Cause it went up to the right which is positive.

5) T: Up the right which is positive, so what would be negative?

6) SS: Down to the right.

7) T: Down to the right. Down to the right is negative, couple people said true, do you understand that that one is positive. Answer on your calculator, yes or no do you have any more questions about how to send a line on top of this picture?

Several students’ comments were casual with mathematical language especially

Devin’s in Vignette 17. Even within line 2, Devin naturally attempted to move from a more casual position description to a more concise mathematical description, such as the non-x and the x one’s. Devin in line 6 continued to try to be more concise, by stating it’s the b and the x that are positive and negative. However, being more concise in this case made Devin’s comments technically incorrect, especially in line 17. Devin’s statement in line 17 lost all connection to the number in front of the x and technically referred directly to x itself. The statement also lost reference to multiple lines; it better described the graph of absolute value of x than the impact of the coefficient of x on the shape of the line. Mr.
L interpreted Devin’s statement in line 6, 9, and line 17 to mean the coefficient of x to be positive or negative, rather than the value of x itself. This interpretation appeared to be consistent with Devin’s gestural description in line 2, rather than what was said. Mr. L restated Devin’s comment using precise mathematical language but never pressed Devin to use it. No apparent consequences of this casual use of language were present during the rest of the observation.

Student interpretation. The first and only instance of a student interpreting another student’s comment occurred during Mr. L’s Vignette 20. During the third activity, three students had lines that matched the staircase well, while the other three students had the right direction but were far steeper than they needed to be. Mr. L interpreted the student, Devin’s, comment in line 2 to be referring to the lines’ slopes. This interpretation may not have been far off given Devin’s later treatment of “the X” in Vignette 17 and it was consistent with what Mr. L apparently thought was wrong about the lines. However, the student, Ashley, in line 4 had a different interpretation. She believed Devin was referring to moving the x-intercepts, which perhaps was her own view of what needed to be done. Keeping the y-intercept fixed and making the x-intercepts larger would in fact make the lines match the picture better. Ashley may have been acting on the graph itself rather than on the equation. However, the notion of moving the line is not pursued. Instead, Mr. L reset the conversation in line 5 like he did in line 35 in Vignette 15.

Mr. L’s Vignette 20

1) T: I’m going to pause this one more time. Everyone’s looking a little bit better. Everyone’s got a negative slope now. Ah. Just need to work on these three right here (pointing to the really steep lines). Here we need to closer to get closer to the other these right here to (pointing to the line better matching the staircase).
2) S: We gotta the X; we gotta do the X [laughs]
3) T: Large values of X, large but negative? They need to be...?
4) SI: No, he was talking about the x-intercepts.
5) T: Oh, how it’s forming [inaudible]. Well, the coefficient of the X happens to be a large negative number. The coefficient of X for these people right here is a large negative number. And needs to be...?
6) S: A small...
7) T: ...a small negative number.

Technology

The CCT was actively used during most of the lessons in Year 4 as it was used for review at the beginning and the end of the lesson and during every activity compared to the 10 minutes of active use each day during Year 2. Mr. L used CCT during the main part of the lesson in Year 4, instead of during review or review of homework in Year 2. The CCT made the activity possible or at least more convenient. The activity of picture matching would be different if students were given pictures on graph paper as students could draw the line then find the equation. The presence of the CCT let students interact directly with the equation of the line and the graph it produced.

Using the Quick Poll feature of the CCT to ask the dichotomous true or false, or agree or disagree questions used during Year 2 persisted in Year 4. At the beginning of class right before Vignette 19, Mr. L asked true or false, the slope intercept form of a line is \( y = mx + b \), and true or false, in \( y = mx + b \) the \( m \) is the \( y \)-intercept and the \( b \) is the slope. After Vignette 19, he asked using Quick Poll if students had any questions about sending in the equation of a line. An interesting difference of Mr. L’s use of Quick Poll between years 2 and 4 was he turned off the projector when asking the questions. Perhaps Mr. L noticed too many times that students when questioned about their response, joined the
majority like what may have happened in Year 2. For the first question and the third question at the beginning of Vignette 19 all students responded correctly. As evidenced in Vignette 19, Mr. L saw all students respond correctly and decided to move on without discussion. For the second and fourth questions, Mr. L said that most got it correct and a couple got it wrong. Since there are six students in the class, four students got it right and two got it wrong. Similar to how he proceeded using the Quick Poll in Year 2, Mr. L started by asking a student who got it right without identifying it as correct to explain their answer. Dissimilar to Year 2, Mr. L did not have a student who answered incorrectly to respond. In both instances, he asked them if the other student’s explanation was sufficient and moved on. Mr. L used the CCT as a data collection device to gather all students’ responses to his questions. The students’ response to question 4, true/false negative slope goes up to the right, indicated some students had an issue with that notion. Mr. L did not pursue discussing this notion during the beginning review and more students had issues with slope later in the day as evidenced in the first activity in Vignettes 15, and 18 and during the end review in Vignette 17.

Mr. L used two different features of the CCT in one class observed during Year 4. He used Activity Center during the main activity in addition to Quick Poll. The technology was used primarily as a display device projecting the picture for the activity and the graph of each student’s line. Since the technology was able to project all the students’ lines, Mr. L was able to pause the activity and call attention to lines that did not match. Mr. L used the multiple representations of the students displayed by the CCT to generate discourse. Some of the discourse generated was of low level as in Vignettes 18 and 20.
But the discourse generated in Vignette 15 was more involved and better connected the graphical and algebraic representations. The capability of the CCT to simultaneously display the graphs and equations of lines of all the students was a major source of missed opportunities not taken up by Mr. L.

Mr. L asked true/false questions at the end of class using the CCT. He asked if anyone had questions about how to look at a line and figure out the equation of the line. True or false, \( y = 2x - 4 \) slopes up to the right and true or false \( y = -3x \) has a negative slope. Although he did not directly state how many got the first and third questions correct, Mr. L moved on from these questions so perhaps all students got them correct. Not all the students got question 2 correct, since Mr. L began asking students who answered correctly why they chose that answer, which is his pattern if incorrect answers are present. Asking several students why they answered true and not getting a satisfactory answer with one student who changed her answer to being incorrect prompted Mr. L’s discussion in Vignette 17. Mr. L used the conflicting outcomes presented to him by the CCT and unsatisfactory explanations from the students to generate discourse about the sign of the coefficient of \( x \) and the direction of the line. If a more in-depth conversation occurred when this issue arose in the review at the beginning of class, the discussion at the end may not have been needed.

**Decisions**

Mr. L’s orientation appeared to remain procedural but less so. The type of true/false questions he asked at the beginning of class were simple recollection, \( y = mx + b \) is the slope intercept form, and \( m \) being the \( y \)-intercept and \( b \) being the slope is the exact
opposite of truth, reflected the procedural orientation. The procedural orientation is evidenced by Mr. L’s treatment of lines as processes and his acceptance of students treating lines in this manner. When a student treated a line as an object by suggesting moving the line’s x-intercepts to better match the graph, Mr. L did not pursue this line of thinking and reset the conversation to suit his goals in Vignette 20. Mr. L focused students’ attention away from the sole application of procedures towards a conception of ideas by pursuing the connections of representations of lines. This conception of ideas was not rich, nor completely coherent. In his third year of CCT use, not one question from Mr. L was to complete a procedure. Some of Mr. L’s questions were conceptual such as what do you notice about these lines, or what is your strategy for finding a matching line. Therefore, Mr. L was procedural but to a lesser extent.

One of Mr. L’s apparent major goals for his students to get correct answers remained year after year continuing in Year 4 as evidenced by moving on to the next activity without discussion when everyone matched the picture well enough. Mr. L appears to have replaced his goal of asking students if they get it along the way with having a student explain a desirable outcome when both desirable and undesirable outcomes are present. Figure 4.6 is the flow chart modeling Mr. L decisions during his third year of CCT use.
Figure 4.7: Decision flow chart depicting Mr. L’s instruction during third year of CCT use.

An anomaly to this decision flow chart was Vignette 16. While most of the vignette follows the left side of the flow chart, it does not end in the predicted manner. Ashley had a desirable outcome of finding the matching line without many tries. Mr. L started with her and asked some clarification questions, since not all outcomes were visible. This cycle was broken by a student interruption in line 7 and Mr. L began asking clarification questions with the new student. Reaching an undesirable outcome in line 13, Mr. L refocused the discussion by asking Kristin what she did in line 14. Kristin’s response in Line15 made Ashley’s discussion undesirable since it did not include anything about the
y-intercept. Mr. L refocused on Ashley again. If Mr. L found the discussion from line 18 to line 25 desirable, then he did not restate the outcome before moving on. If Mr. L found the outcome undesirable, then he did not reset or refocus the discussion with another question; he let it drop. Perhaps he did ultimately find the discussion undesirable, but because of Ashley’s frustrated tone in line 25 he ended the conversation.

**Summary**

**Segmentation**

The instructional segmentation was slow to change, but ultimately did change. From Year 1 to Year 2, the segmentation of class time was almost identical. The “Do Now,” homework and review, the main lesson, practice, and assign new homework was present during both years. Segmenting the lessons and the tasks within the lesson into manageable pieces was a consistent part of Mr. L’s years 1 and 2 instruction. Before or during Year 4, Mr. L’s segmentation changed. In Year 4, the “Do Now” disappeared, and the segmentation of the class observed became review, main activity, and review activity. The tasks Mr. L had students do in Year 4 would have combined several of his lessons from previous years. Mr. L did not segment the tasks for the students during the activity in Year 4; rather he discussed how the students’ solutions may be improved.

**Representations**

Similar to the segmentation aspect, Mr. L’s use of representations was slow to change. During Year 1 and 2 only a single representation of functions was present at any given time, except when a student did more than was asked. The representations were used either to enact a procedure upon them, (i.e., find the slope), or they were created as
an end product, that is, produce the number line. In the beginning years representations were not used to reason with, for justification, or explanation and connections among them may have been mentioned but not discussed. Year 4 is a year of change for Mr. L., in terms of representations. The activity given to the students requires the use of two representations of lines illustrating some of the connection among these representations. Although the pictures were acted upon by the students and the lines students produced were treated as end products, the lines were also used to reason with and the connections among the representations discussed. Although not required, the technology made using different representations simultaneously easier for teacher and students.

**Discourse**

Unlike segmentation and representation, Mr. L’s classroom discourse changed from Year 1 to Year 2 while keeping some elements as well. The IRE sequence, long explanations and short student responses noticed in Year 1 remained in Year 2. However, Mr. L., in Year 2, no longer directly answered student questions and began asking if students were “Okay with that?” during a task. Mr. L’s Year 2 discourse seemed to be in a state of flux by trying to hold on to how he did things before and incorporate what he learned in the summer professional development. The only remaining feature of the discourse from Year 1 in Year 4 was the use of IRE sequences, however even that was modified somewhat. Mr. L kept several features of his Year 2 discourse, such as not directly answering students’ questions, asking students to explain, and asking if students agree with another student’s comment. The persistent asking of “You okay with that” disappeared in Year 4. Mr. L also began to ask more conceptual questions like comparing
the lines on the board and what strategy did you use for finding a matching line. Despite the change in questioning from years 1 to Year 2 to Year 4, the quality of explanations Mr. L accepted as sufficient did not improve much. The discourse in Year 4 appeared to be aided by the use of technology; otherwise students would have had a difficult time discussing all the different lines produced by the members of the class.

**Technology**

Technology during the second year seemed to be an add-on to his typical first year instruction; it was used to briefly review the material at the end of the lesson on the first day and to review homework on the second day. The questions asked during the review were less clear and ill posed than the questions asked during the lesson and Mr. L did not use the responses gathered by the CCT to inform his instruction. Mr. L used the CCT in Year 2 day 2 to a lesser extent as a discourse generator to compare different student solutions. Mr. L’s use of the technology in Year 4 expanded greatly on this use of the CCT as a discourse generator to compare multiple representations produced by students. At the end of the lesson Year 4, the explanations of responses to a Quick Poll question prompted Mr. L to start a discussion. Thus, he seemingly used the technology to inform his instruction.

**Decisions**

Mr. L’s orientation to mathematics and its teaching remained largely procedural from year to year although it became slightly less procedural in Year 4. The major goals ascribed to Mr. L changed each year. Only one major goal was apparent during the first year, which was having students produce correct or desirable answers. While this was a
major goal in the other two years, other goals emerged as well. The goal of asking students if they understood during the task appeared during Year 2 and then was replaced with creating an occasion for students to explain their outcomes. The CCT became more integrated into Mr. L’s decision flow chart as some of the decisions and actions would have been hindered or impossible without the CCT.

**Answers to Research Questions**

**What kinds of representations are used in the classroom by teachers?**

Representations used by the teacher in the first year were graphical from the process perspective, except for the two “Do Now” questions. In all the exercises, students had to find the slope of a line given the graph of the line. No other representations were present to help students make connections. The “Do Now” was to add and simplify square roots from the algebraic representation. The exercises on the first day consisted of two graphs with points labeled and six graphs without points labeled. Three of these exercises were led by the teacher and seven were for student in class practice. Positive, negative, zero, integer and fractional slopes were displayed. For the following day’s homework, the problems contained undefined slopes in addition to the kinds of slopes seen in class. The following day’s in-class exercises consisted of finding the rise over run for two questions that the teacher led and 5 questions that students completed during class. All but irrational slopes were presented. The representations present for the entire slope finding exercises were graphical and were treated from a process perspective. The representations were presented on an overhead; none were produced by the students.
The dominant representational mode present during Year 2 was algebraic treated from a process perspective. All exercises were this combination of representation and perspective except for the two “Do Now” questions of the second day, which were tabular, treated from the process perspective. The purpose of the second day’s “Do Now” questions was to find the slope of a line given two points on that line. A student asked at the end of these “Do Now” questions what happens if we kept going to find the M and the B from the two points given. Mr. L initiated an IRE sequence with the student that led to the production of the algebraic representation. This was the only instance where different representations of functions were present for the same function, which was not intended by Mr. L. The first day’s lesson consisted of 6 linear inequalities that were already solved by the students for homework or solved that day in class. Mr. L showed them how to produce number line solutions for the 3 homework exercises and had the students create number lines for the other 3. Twelve linear inequality exercises were given as homework for the second day. For homework review on the second day, Mr. L read the correct answers for these problems, and corrected student mistakes. After the homework review, Mr. L asked that students produce number line solutions. Representations of the same solution produced by different people was uncommon the first day, and only occurred when he used the CCT the second day. The controversy over the numbers to be used in the number line was the only instance the first day that produced representations from multiple sources. The teacher used the CCT to capture the calculator screens to display all the students’ representations of the solutions to the
homework. Only one of the exercises prompted a discussion on how the solutions should look.

Of the six review questions used in class either at the beginning or at the end of class during Year 4 that involved representations of lines, two required some knowledge of both algebraic and graphical representations. The other four needed only strictly algebraic or strictly graphical knowledge. The three activity problems given to the students required graphical and algebraic representations to be present. Not only were two different representations of line needed to do the activity, representations offered by multiple students were also present due to the CCT. The graphs that the six students produced were continually being updated and displayed during all three activity problems. This enabled the teacher and the students to see the collection of one another’s graphs but not their corresponding equations. Three instances for students to see or hear another student’s equation occurred. The two equations that were spoken aloud were very brief and for the other instance Mr. L had a single student write her equation on the board. Since a lot of the students started plugging numbers into the equations to find the matching lines, most students seemed to treat lines as processes. Most of the discourse about the lines did not hint at the object nature of lines, except perhaps one student’s suggestion to move the x-intercepts of a line to better match the picture.

The number of different kinds and sources of representations of lines used in class increased from Year 1 to Year 4. The trajectory was from single representation/single source to single representation/potential multiple sources to two representations/multiple
sources. The perspective of the lines did not change however. The perspective held by the teacher and students was that of lines as process.

What is the quality of discourse about representations or use of representations by the teacher? The discourse surrounding the exercises during the first year was typically long explanations by Mr. L about how to do them or IRE sequences to get the correct answers from the students. The questions and answers were procedurally based and of low cognitive load. The only question that had potential to generate discussion among the students was quickly answered by a student, validated by Mr. L and then procedurally “proved.” The interaction was between teacher and a specific student or any students, rarely initiated by the students directed to the teacher, and never student-to-student. In an instance where a student asked a question, Mr. L would immediately and directly answer the question. Mr. L’s utterances could be quite long up to 470 words while student utterances were quite short usually under 5 words.

The texture of Mr. L’s classroom discourse changed from Year 1 to Year 2 while continuing some of its features as well. The IRE sequence, long explanations and short student responses persisted in Year 2. However in Year 2, he no longer directly answered student questions and began asking if students were “Okay with that?” during a task. Mr. L’s Year 2 discourse seemed to be in a state of flux by trying to hold on to how he did things before and incorporate what he had learned in the summer professional development. The cognitive load of the questions asked and the quality of student responses that were accepted as sufficient by Mr. L remained low during Year 2. These responses were mostly procedurally based. In the presence of representations from
multiple sources, discourse was prompted by students or by the teacher. The opportunity for interesting discourse was present in these situations but not fully realized by Mr. L.

The texture of Mr. L’s classroom discourse changed again from Year 2 to Year 4. The use of IRE sequences continued to persist from Year 1. These IRE sequences, however, were typically used to clarify student responses rather than to produce correct answers. Mr. L kept several features of his Year 2 discourse, such as not directly answering students’ questions, asking students to explain, and asking if students agree with another student’s comment. The persistent use of “You okay with that” disappeared in Year 4.

The questions asked using Quick Poll were dichotomous yes/no or true/false questions. These questions were not overtly procedural, but mentally demanded recollection of facts. Two instances of more conceptual questions occurred when he asked students to compare the lines present in the graph and to describe a strategy for finding a matching line. Mr. L’s purpose for the first instance was to get students to notice they needed negative slopes to match the roof, so he channeled the conversation in that direction. Student responses were short and of low cognitive load in this instance. In the second instance, Mr. L did not fully pursue the strategy that the students had offered and abandoned the question when a student became frustrated. In either case the quality of discourse was low. The discourse in Year 4 appeared to be aided by the use of technology; the students would not have been able to see all of the different lines produced by the students (without a time and effort) causing difficulty in discussing the comparisons of these lines.
The texture of Mr. L’s classroom discourse changed from year to year with the kinds of questions he asked, the interactions in the classroom, and the explanations he gave. However, the quality of discourse remained low each year, especially in terms of quality of students’ explanations accepted.

**For what purpose do teachers rely on multiple representations in classroom discourse?** For the two “Do Now” questions and the eight exercises of the first day of Year 1, the purpose of the presence of representations was to add and simplify symbols, or to find and/or label points, plug points into a formula, and calculate the slope. In other words, the students needed to perform a procedure on the representations. The second day was similar; the students needed to look at the graph of a line to find points, draw a triangle connecting the points, count how many blocks over and how many up is needed to get from one to the other, and calculate rise over run. Again the goal was procedural.

During one of the exercise problems, Mr. L asked which points should be chosen to find the slope. Briefly a student stated any two would work. Without giving other students a chance to reason about the representation, Mr. L identified the response as correct and had groups of students to enact the procedure of finding slope to “prove” the validity of the statement. The issue of which points to use cropped up multiple times over the two days of instruction and in each instance Mr. L told students the choice of points did not matter without providing them space for reasoning and discussion.

The students of Mr. L during the Year 2 on the first day had to solve or have solved the linear inequalities of the six exercises given. They performed procedures on the algebraic representation to produce a symbolic solution in each case. The purpose of the
algebraic representation present in the six exercises and the two “Do Now’s” appeared to reinforce the use of an appropriate procedure. The twelve homework exercises reviewed during the second day seemed to have a similar purpose. Enacting a procedure was not the only purpose apparent during Year 2. After stating or producing the correct symbolic solutions to the inequality questions, Mr. L created or had the students create a number line solution to them. The number line solutions were mostly treated as end products, since when they were produced Mr. L moved on except perhaps to clarify how the number lines were produced. The representations were briefly reasoned about during the instances with representations arising from multiple sources. The non-standard representation of the number line solution to \( y > 5.6 \) caused students to reflect on the representation. The students were given 24 seconds to reason about the representation before Mr. L told them that the initial representation offered by a peer was okay. Mr. L had not intended to use representations for promoting discussion in this instance, since he seemed surprised when other students raised issues with the solution and then he provided the answer himself. Reasoning about representations, at least briefly, seemed to be the purpose of the episode during the review of the homework problem that had a solution of \( h > 7 \) or \( 7 < h \). Mr. L saw the different correct and incorrect answers and pursued discussing them. However, the discussion lead to enacting procedures to show that \( h > 7 \) and \( 7 < h \) were correct and that \( h < 7 \) was not.

The purposes of the representations used in class during the fourth year appeared to be to produce answers treated as end products, and reason about representations. The goal of Mr. L for the students to produce end products was very apparent by the fact he ended
the current activity when all the students were fairly close to the picture each time. During the activities if the lines present were not close to the picture, Mr. L would pause the activity to have the students think about and discuss the lines. Mr. L asked what the students noticed about a few of the lines present during the first of these discussions leading them to the conclusion that to match the roof a negative slope was needed. The other instances of “pausing the activity” were much briefer and more direct. Mr. L attempted to have students reason about the connections between the algebraic and graphical representations of line by asking a few students what their strategy was for producing a matching line. But the students did not state, nor did Mr. L give a reasonable strategy.

The purposes of the presence of representations remained mostly steady over the years, with a slight increase of reasoning about representations. The trajectory for the purpose was to enact a procedure on and minimally reason about how to enact a procedure on, produce end products, and briefly reason about how to produce end products and moderately reason about them.

**What is the relationship between teacher use of CCT in algebra classrooms and the growth of teacher and student choice of representations of linear functions as manifested in the classroom discourse?** The use of CCT appeared to have increased the kinds and sources of representations present simultaneously and changed slightly the purpose of the representations. The texture of discourse changed over time, but only some of the changes appeared to directly stem from the use of CCT, such as having students look at all of the responses as discussed in the previous questions.
The segmentation of the lessons, tasks, and class structure appeared to be influenced by the use of the CCT. The segmentation was slow to change, but ultimately did change. From Year 1 to Year 2, the segmentation of class time was almost identical. The “Do Now,” homework and review, the main lesson, practice, and assign new homework was present during both years. Segmenting the lessons and the tasks within the lesson into manageable pieces was a consistent part of Mr. L’s years 1 and 2 instruction. Before or during Year 4, Mr. L’s segmentation changed. In Year 4, the “Do Now” disappeared, and the segmentation of the class observed became review, main activity, and review activity. The tasks Mr. L had students do in Year 4 would have combined several of his lessons from previous years. Mr. L did not segment the tasks for the students during the activity in Year 4; rather he discussed how the students’ solutions may be improved.

One of the major influences of the use of CCT was on Mr. L’s decision making. Mr. L’s procedural orientation to mathematics and its teaching remained intact from year to year. His questions and tasks were not explicitly procedural as in his first two years. But if Mr. L had truly become more conceptual in orientation, the multiple opportunities to discuss the connections of algebraic and graphical representations and to treat lines as objects would not have been missed by Mr. L. Despite his orientation not changing, he altered his goals. Only one major goal was apparent during the first year, which was having students produce correct or desirable answers. While this was a major goal in the other two years, other goals emerged as well. The goal of asking students if they understood during the task appeared during Year 2 and then was replaced with having on occasion students explain their outcomes.
The CCT became more integrated into Mr. L’s decision flow chart as some of the decisions and actions would have been hindered or impossible without the CCT. The rectangles in the flow chart represent actions of the teacher, while rounded rectangles represent actions enhanced by or made possible by the use of CCT. Diamonds represent decisions made by the teacher, while octagons are decisions enhanced or made possible by the use of CCT. The double sided arrows indicate cyclic actions and decisions.

The tasks in Mr. L’s Year 1 flow chart were to find the slope given the graph of a line exercises. Smaller steps of finding slope, such as finding points, setting up formula, simplifying expressions, or drawing triangles were the subtasks Mr. L would pose. Similar subtasks were posed for the “Do Now’s.” Mr. L asked few questions that were not tasks or part of IRE sequences during Year 1, i.e. “does it matter which points to use for finding slope?”

The reinforce procedure action is Mr. L’s long explanations of 140-470 words. When a student responded to a task, Mr. L appeared to decide if the outcome is desirable or undesirable. For a desirable outcome, Mr. L would restate the outcome, showed the outcome, or if the outcome produced a question he would immediately answer it. Afterwards he moved on to the next task or subtask. With an undesirable outcome he may have began an IRE sequence with the student to determine the source, he may have asked another student, or he may have produced the desirable outcome himself.
Figure 4.8: Mr. L’s Decision Flow Charts before CCT use (A), first year of CCT use (B) and third year of CCT use (bottom).
The tasks during Year 2 were the solving the linear inequalities and producing the number line solutions. The subtasks involved were the algebraic steps needed to solve the inequality, or drawing a line, picking points, and drawing a circle or dot and an arrow to produce the number line. Most of the non-task-posing questions were in IRE sequences or the Quick Poll questions. Mr. L during Year 2 seemed to only want to start with desirable outcomes. If no desirable outcomes were present, he would begin an IRE sequence, ask another student to try to produce the correct answer, or finally produce one himself, if needed. The presence of desirable and undesirable outcomes seemed to influence his decisions. If both were present, he would ask students to explain their outcomes eliciting at least one desirable and one undesirable. Multiple outcomes being present occurred more frequently in the presence of the CCT. The professional development for using CCT or perhaps the results of his polling questions appeared to have influenced Mr. L’s decisions by making him slightly aware that some students might not get it. This led to the decision where after this discussion or if no undesirable outcomes were present he asked students if they understand and he explained or had another student explain their work, especially if prompted by a student question. This action was cyclic and could occur many times during a lesson segment. After reaffirmation of a majority of his students on getting the right answer, Mr. L moved on to the next task, subtask or question.

The tasks posed by Mr. L during Year 4 involved matching equations of lines to lines in a picture. The questions not in clarification sequences posed by Mr. L were the Quick Poll true/false questions, questions about the students lines projected on the board, and
the strategy question. The presence of all desirable outcomes prompted Mr. L to move on to the next Quick Poll question, activity, or review questions at the end. This decision about all student outcomes was made much simpler by using CCT. Mr. L did not reveal the responses to the Quick Poll questions to the student and only he could see them. The students could not see the different outcome of the Quick Poll questions, but they could see all their lines during the activity. Whether or not the students could see the different outcomes seemed to influence how Mr. L proceeded. If the students could not see the different outcomes, he would begin by asking a student with a desirable answer how they arrived at the outcome, but without telling them their answer was right. While the students gave the explanation he may have asked clarifying questions along the way. After the student explanation of the desirable outcome he may have asked a student with an undesirable outcome to explain with Mr. L potentially asking clarification questions. If the clarification questions did not produce a desirable result, Mr. L appeared to at times reset or at least refocus the discussion by asking another question potentially with the CCT and start the previous action over. If the clarification questions produced a desirable outcome, Mr. L would move on. Mr. L proceeded similarly if students could see all outcomes. The main difference was that Mr. L would have students compare both desirable and undesirable outcomes simultaneously while still asking clarification questions along the way. In a similar tactic, if the discussion was not desirable Mr. L would reset or refocus the discussion and start over. If the discussion was desirable, he will move on after restating the outcome.
Year 1 has one major decision that appears to influence Mr. L’s actions, while Year 2 has three and Year 4 has three for each path. The decision in Year 1 repeats itself in the following years. Although Year 2 has three decisions the second decision was not exercised frequently due to the fact that the CCT was used for a brief time. Therefore, during Year 2 only two decisions were in effect most of the time. Regardless of the path, the decisions within the path were exercised throughout Mr. L’s instruction. The use of technology was integrated throughout the Year 4 decision chart. The decision about all desirable outcomes would be more difficult without the CCT as well as if the students being able to see outcomes.

Year 1, Year 2 and each path in Year 4 have five major actions. In Year 1, student participation occurred after the posing of task, subtask, question, and/or during the undesirable outcome cycle. The other actions are purely performed by Mr. L. Unlike Year 1, student participation in Year 2 occurred after the posing of the task, etc., the undesirable outcome cycle, the discussion of at least one desirable and undesirable outcome, and the student understanding cycle. For years 1 and 2 usually a single typically a specific student responded after posing the task, subtask, or question. The tasks and questions were posed in Year 4 to all students where they all were expected to respond. The questioning within a larger task or questions where students discuss outcomes resembled the previous year’s single student and teacher discourse.

While using CCT, Mr. L increased the number of major decisions he made during the lesson from one to two (three) to three decisions. The number of actions and reactions that students participated in increased over time, although the interactions were limited.
The technology became more integrated into Mr. L’s decision chart in terms of aiding both decisions and actions.
CHAPTER 5: A CASE STUDY OF MS. A.

This chapter is a report of the case study of Ms. A’s classroom observations. The case study is comprised of a section on year one (one year of CCT use), and year two (two years of CCT) use in the CCMS project. For years one and two a description of an outsider’s perspective as observed from the classroom observations, a description of the insider’s perspective derived from Ms. A’s post observation interview (POI), and analysis are given. A summary of the themes and patterns across the years observed finishes the case study.

Introduction

Ms. A graduated with a B.S. in education with a minor in mathematics in 1989. Ten years later she completed her M.S. in mathematics education in 1999 obtaining her licensure in mathematics for grades 6 to 9. At the beginning of the study in 2005, she had completed her course work for a Ph.D. Ms. A started the study with 3 years experience teaching algebra and a total of 16 years mathematics teaching experience.

The school that Ms. A teaches at is on the urban fringe of a mid-size city with about 650 students. The student population during Ms. A’s first year has the following composition of 89% Caucasian, 7% Hispanic, 2% Asian, 1% African American, and 1% Native American. Nineteen percent of the students are eligible for free and reduced lunch.
The measurement of teachers’ practices and beliefs about mathematics teaching and learning within the Teacher Instructional Practices and Beliefs (TIPBS) survey was used to differentiate participants and used for participant selection criteria. Table 5.1 shows Ms. A’s average scores of different constructs of the TIBPS survey. Interpretations of the scores are then discussed.

<table>
<thead>
<tr>
<th>TIBPS Constructs</th>
<th>Average Scores</th>
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<tr>
<td>Reform Classroom Practice</td>
<td>2.8</td>
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<tr>
<td>Tech Use Reform</td>
<td>1.6</td>
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<tr>
<td>Strategy Discussion</td>
<td>3.0</td>
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<tr>
<td>Data Analysis</td>
<td>2.3</td>
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<tr>
<td>Explanation/Justification</td>
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<td>Classroom Discourse</td>
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<td>3.4</td>
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<tr>
<td>Teacher Beliefs on Math</td>
<td>3.6</td>
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Table 5.1: Ms. A’s TIBPS average scores.

For questions within the constructs of Reform Classroom Practice, Tech Use Reform, Strategy Discussion, Data Analysis, Explanation/Justification, Reform Class Discourse, and Assessment Practice, responses were scored on a 5 points scale ranging from 1) never, 2) once a month, 3) once a week, 4) more than once a week, and 5) all or almost all lessons.

Ms. A’s average score indicated that she almost never used technology in reform ways, scoring an average 1.6. Ms. A reported she had students engage in reform classroom practices about once a week with a score of 2.82. On average Ms. A’s students
engaged in strategy discussion once a week with an average score of 3. This score of 3 resulted from averaging both small and large scores and does not describe the whole picture. The following two of the strategy discussion questions scored a 4 and 5 respectively: students engaged in talking about ways to solve mathematics problems, and describing and discussing strategies they use to learn mathematics. The other two questions scored a 1 and 2: Her students never compared and contrasted a variety of student responses to a single question and maybe once a month her students had to evaluate the validity of classmates’ responses. Ms. A’s students used data analysis techniques about once or twice a week with an average score of 2.33. The students of Ms. A engaged in explanation and justification more than once a week to nearly every day scoring an average of 4.4. The lowest item was having students use multiple representations in their explanations with a score of 3. Ms. A said she never poses questions that do not require student’s explanations. She reported that she does not evaluate student responses to her questions, except maybe once a month and that she daily encourages questions from her students scoring an average of 4.67 on reform classroom discourse practice.

The questions from the classroom discourse construct of the TIPBS are different from the previous ones. Rather than asking how often something occurs, teachers were given descriptions of two different scenarios of class discussions, where one was teacher led with simple questions and the other with more difficult questions arising from students. Ms. A believed students more preferred the teacher led classroom with simple questions. Despite her belief in this preference she rated herself most definitely toward the
classroom with questions arise from students. Ms. A usually uses assessment practices in reform ways about once a week with an average score of 2.89. Ms. A reported she was moderately prepared to well prepared in several circumstances in teaching mathematics that was used to determine teacher efficacy.

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<td>21 minutes</td>
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Table 5.2: Ms. A’s case data.

**Year One: First Year of CCT Use**

**Overview**

During the first year of the study, several features of Ms. A’s instruction emerged from the classroom observations. The three major representations of algebraic, graphical and tabular were used frequently both days. The purpose of the presence of representations was to help students understand the relationship between them. Discourse was frequently used to elicit and extend student thinking. Technology during the first day was used mostly for information gathering and display. However, during the second day, technology was used for generating discourse. Due to the length of some of the Vignettes, important phrases to the case study are underlined.
Segmentation

The first day was split into two large segments; each was roughly half the class time. The first segment was comprised of five activities to help students see the connection between the coefficient of \(x\) and the graph of the line. The activities were split into three parts; plotting points of the assigned line on graph paper, submitting points using CCT and checking points, and discussing observations. For the second large segment, the goal of the activities was to provide student insight into what the constant term does to the graph. A similar pattern of plotting points, submitting and checking points using the CCT and discussion was used during this second segment.

The second day was divided into three major segments. First, the day started with a review of the previous day (about 11 minutes) by having students graphing a line by first predicting what it would look like and the checking by creating a table. The two activities in this segment were split into two parts; predicting where the line should go and create a T-chart and sending points using the CCT to verify correctness. In the second segment, the following set of two activities had students go from the graph of the line to the equation for the line. The students quickly produced a way of finding the “\(b\)” term in the equation of a line during a discussion. Lastly in the third segment, they discussed how to find and practiced finding slope, since finding the equation of a line from its graph prompted the need to have a way of finding slope.

Representations

Graphical, algebraic, and tabular representations of lines permeated each activity of the first day of observation. Each activity had students create tables of points for the
equation of a line and then produce a graph of those points. The *embodied* representation, where the line is represented by a body part, made an appearance in class when Ms. A used her arms to show the slope of a line. The teacher used the embodied representation as shown in Figure 5.1 and asked students to make predictions of shapes of lines using their arms. Although the tabular representation was present in day two during the first two activities, the graphical and algebraic representations were the present during all of them.

Figure 5.1: Embodied representation in the classroom.

**Process/Object perspective.** The representations of lines were treated as both processes and objects. The tabular representation was entirely treated as a process since
students created the table by plugging in points and then used the points to plot a graph. The algebraic and graphical representations were treated as object and processes, but mostly as objects. At the start of the first day, Ms. A had students use the CCT to move points so they fit on the line $y= x$ by saying, “Find a place where your $y = your x$. Find a place where your $y = your x$. Your equation would be $y=x$. Y is X, so if your Y is 2 what’s your x?” By having students focus on the points generating the line and in particular asking what is the relationship between specific x’s and y’s, Ms. A was emphasizing the process nature of the graph. During the rest of the instruction, equations and graphs were treated as something that could be manipulated and as something that has properties, which can emphasize the object nature. Vignette 1 illustrates the teacher treating the lines as objects and students focusing on object like properties of lines.

Ms. A Vignette 1

1)  $T$: Um hum, because it’s a line. Ok, so on your $T$ charts, I want you to do one more. We’re not going to do it on here first; we’re going to do it second. I want you to now do $y=2x$. Find 4 to 5 points that would go on there and draw that on there and label it as well and then we’ll talk about that one. So the next one is $y=2x$. ...
2)  $T$: ok, so let’s check this one out and see what happened to it when I changed $y=x$ to $y=2x$. So the one that I have on there right now is the $y=x$; now let’s add the new one. Oh, hum, that’s the purple one. What do we notice about it? Casey?
3)  $S$: It’s steeper.
4)  $T$: It’s what?
5)  $S$: It’s steeper.
6)  $T$: It’s steeper than what?
7)  $S$: Just X.
8)  $T$: Ok, it’s steeper, and tell us what you mean by steeper.
9)  $S$: That it’s almost vertical.
10) $T$: It’s becoming more vertical? Yes, ma’am.
11) $S$: It has a higher slope? Is that what he said?
12) $T$: Ok. You can call it that, or becoming more steep. So in your mind picture what a line would look like that if I said $y=4x$. You don’t need to draw it, just picture it
in your mind and see if that would be the case. Let’s all another one. Everybody have it, in your head, I mean?

13) S: What?
14) T: You’re supposed to be picturing \( y = 4x \). I want you to guess where it’s going to show up and there we go. Is that what you thought?
15) SS: Um hum.
16) T: Huh.

In line 2, Ms. A asked “What happened to it (the graph of the line) when I changed \( y=x \) to \( y=2x \)” Ms. A did not ask students to compare the graph of one line to the other; rather she asked essentially what changed about the graph when something was done to the equation. A comparison question could have prompted students to compare the graphs as processes or as objects. But the question of what happened after an action was performed on the original equation reveals the object nature of function, since it is something that can be acted on. Viewing a graph of a line from the process perspective requires students to notice the individual points on the graph. Steepness is not a property of individual points, but of the whole line. Hence steepness is a property of an object and not a process. When the students noticed the change in steepness in lines 4 through 12, they were noticing the differences of properties of two lines. In line 13, Ms. A asked students to picture in their head what \( y=4x \) would look like. She is asking students to translate the algebraic external representation into an internal graphical representation. By having students represent internally what the line would look like, Ms. A had emphasized the object nature of the function.

In Vignette 2 line 19 a student said, “And since we already plotted the \( y=x \) and this is just the same thing except there’s a negative in there, so I knew just to flip it.” This student’s statement reveals she regarded \( Y=X \) and \( Y=-X \) as things. However, the
statement alone is not enough to infer that she is viewing the line from the object perspective, but her envisioning flipping one line to make it the other solidifies this inference of her using the object perspective.

**Treatment of Representations.** Representations were used to enact procedures, but these procedures were a means to an end, not an end in and of themselves. For example, students enacted procedures using the equations to create tables and plot points, but as exemplified in Vignette 1 they were ultimately used to make comparisons among representations of different lines.

In Vignette 1 students were asked to use representations to make predictions of representations of the graphs of other lines. Vignette 2 shows another student asked to make a prediction of representations. Underlined sentences within all Vignettes indicate important sentences that are central to the discussion. Not only were they asked to make predictions in line 1, but in line 2 Ms. A asked a student to justify their prediction with “How did you know which direction it was going to move from the y=-2x line.” After some clarification of Ms. A’s goals in lines 4 through 14, the student in line 17 explains that since the equation, Y = -X, has a negative multiplying the X that both X and Y cannot be positive making it impossible to be in the first quadrant. Even though the students response is correct in line 19, Ms. A presses for more connections among the different lines by asking “What if we only had y=-2x.” By restricting what representations students could use in their explanations, Ms. A was forcing them to reason about representations by focusing the students’ attention on the magnitude of the number as well as the sign. By restricting the focus of attention, Ms. A sparked the
discussion in Vignette 2 lines 22 through 56 that led to the student conclusion about a connection between the equation of a line and the graph of the same line in line 37 “S: For positives the lower the number the less the slope, but if it’s negative the lower the number the more slope it has.”

Vignette 2

1) T: …Picture what Y= neg. X is going to look like. Picture it in your head and on a scrap piece of graph paper draw it, kind of predict, and then I want you to chart it on your paper and label it Y= neg. X. Let’s look at what direction the line is going to go. Y= neg. X. Kind of make a guess and do a quick T chart and then on your regular graph paper go ahead and graph it. …

2) T: Let’s look up here, please. Let’s see if you’re right. There we go. Let me highlight it. This is the Y= neg. X line. How many of you had that correct? How did you know which direction it was going to move from the Y= neg. 2X line? Dylan?

3) S: It was a negative line.

4) T: Ok, but so was the other one.

5) S: [inaudible]

6) T: So we were at Y= neg. 2X, right?

7) S: Yeah.

8) T: So when you were trying to predict where Y= neg. X was going to move, how could you figure that out?

9) S: What do you mean?

10) T: Was it going to stay in the same place? So right now we’ve got Y= neg. 2X. And if I asked you to show me with your hand Y = neg. X, how did you know which way to move your arm or whether to keep it right where it was? You want to try, Dylan?

11) S: So like we had to solve for 2X this time so it was half way?

12) T: Since now it’s not neg. 2X what is really the coefficient in front of the X?

13) S: Neg. 1.

14) T: Neg. 1. So how did you know which way your line was going to move? That’s what I’m asking you. Was it going to go up, was it going to get steeper? More shallow? How did you know what was going to happen to it. Wendy?

15) S: Because I – we decided in class that the negative, it would turn that way because it couldn’t go into where there was a positive on the X and Y. So...

16) T: Say that again?

17) S: Since there’s a negative in the formula it can’t - the line can’t go thru a plane, as Sara called it, where there’s two positives because one of them has to be negative.

18) T: Ok, so what Sara was talking about a minute ago?
19) S: Yes. And since we already plotted the y=x and this is just the same thing except there’s a negative in there, so I knew just to flip it. There’s a negative.
20) T: Ok, so you looked more at your y=x line and said it’s just going to be inverse?
21) S: Yeah.
22) T: Ok. What if you didn’t have that line on your paper? What if the only thing that you had on your paper was $Y = -2X$? And I said show me with your arms what’s going to happen when I change that $-2$ to a $-1$ or just write it as $Y = -X$. How would you know without any other lines on the whole paper, or on the whole grid, how would you know what to do? You’re at $Y = -2X$ and I’m saying move your arm to $Y = -X$. We have no other lines to look at except one line. How are you going to know what to do with your arm? Jacob, what do you think?
23) S: If there’s a higher negative number it would move up because it would have to go further on the plane to like the right.
24) T: What do you mean by higher negative number?
25) S: Because if it’s just $Y = -X$ it’s like $Y = -1X$ and if it’s like $Y = -2X$ it’s a higher number so it would have to go to a different place on the plane. And if it was a lower negative number it would go down instead of up.
26) T: Ok. Dylan?
27) S: The smaller the number is the less the slope is going to be.
28) T: The smaller what number?
29) S: The number in front of the $X$.
30) T: Ok, so you’re thinking that if the number in front of the $X$ gets smaller what’s going to happen to your arm?
31) S: [inaudible]
32) T: You think it’s going this way? Stacey’s going... No? You don’t agree with him Stacey? What are you thinking?
33) S: A lower number would be like... Ok, neg. $1X$ and neg. $2X$; neg. $2X$ is the lower number because it’s further down negative so I think the lower it would be.
34) T: I think we need to define terms here. I think you all may be talking about the same thing, but that’s what threw me from what Jacob was saying, too, because he was calling it a higher number, a higher negative number when actually it’s a lower negative number but I understood what you meant because it’s going 2,3,4, but actually we’re going lower.
35) S: Yeah.
36) T: Is that what you were talking about, too, Dylan?
37) S: For positives the lower the number the less the slope, but if it’s negative the lower the number the more slope it has.
38) T: Is that what you’re saying, Stacey?
39) S: I don’t know.
40) T: So you think that $y=x$ is steeper than the $Y = -2X$. Why do you think that?
41) S: No, it will be steeper, but like the slope will be less because it’s a negative.
42) T: It will be steeper, but the slope will be less.
43) S: Since it’s negative.
44) T: Ok, so the line we’re talking about originally is this line with the dots.
45) S: Yeah.
46) T: That’s Y= neg. 2X, right? Ok. So show me with your arms, guys, if you’re looking at the screen. You’re going down. How did we know that Y= neg. X went like that instead of like that, because we’re that close. How would you know...? If I just said, “Draw the line,” and didn’t let you do any points... Show me where Y= neg. X is. How do you know which way to move the line? Stacey? What are you thinking?
47) S: Because the X line is 0X, and so the lower you go...
48) T: Are you talking about that coordinate right there?
49) S: The middle... No. That’s not right. Origin?
50) T: Are you talking about that coordinate right there?
51) S: No, yeah, the axis, the axon, it’s like there would be a positive 1 in there.
52) T: You mean there’s a 1 in front of the X?
53) S: Yeah, so the closer you get to 1 the closer it gets to the 1X.
54) T: Ok. The closer what gets to 1X?
55) S: The line. Like neg. 2X is farther away than neg. 1 so it’s...
56) T: Oh, I see what you’re saying. So you think the closer a number gets to one the flatter your line is going to be?
57) S: Yeah.

In Vignette 2 line 17 and 19, the student used knowledge of the appearance of the line y=x and the fact that for the line y=-x, the x’s and y’s cannot have the same sign to explain why the graph of y=-x looked like it did. In line 23, Jacob seemed to notice that increasing the magnitude of a negative slope, “higher negative number,” would cause the graph to rotate to the right. He was treating the equation and graph of the same line as objects and as something to reason about. After clarifying student comments in lines 24 through 36, Dylan explained the connection between the coefficient of x and the shape of the graph for both positive and negative numbers in line 37. After Dylan’s connection was made, Ms. A again asked her question, how to move the line y=-2x to get to the line y=-x, in line 46. Stacey tried to explain herself in lines 47 through 55 that led to an interesting conclusion that the closer the number got to 1 the flatter the line. This is an
example of students using graphical representation of a different line and the algebraic representation to explain what happened to the graphical representation of the line $y=-x$. In other words, the student was using different representations of the same object and different objects to explain the connections among them.

**Discourse**

The purpose of discourse in Ms. A’s classroom seemed to be to elicit student explanations to generate student understanding. The discourse was always from teacher to student or student to teacher. While students did not directly to speak to each other, in a few instances they would reference another student’s reasoning such as line 17 in Vignette 2. In the previous vignettes, not only did she ask students to make predictions but justify them as well. Ms. A pursued more than a single correct answer with a correct explanation as evidenced in Vignette 2 line 22. She pressed for different explanations under different constraints.

Ms. A’s utterances could be quite long; some ranging from 120 to 200 words. On average Ms. A spoke 26.5 words per utterance. The long utterances were typically explanation of a task instructing students what they needed to do or they were clarifying a question such as Vignette 2 lines 22 and 46. Students spoke on average 7.8 words per utterance.

The purpose of her discourse was almost always to extend and elaborate student thinking; it may be to get all students to a certain point so discussion about a common topic can occur. In Vignette 3 Mrs. A. gave students the task of moving their curser and essentially told them what to do without any discussion of mistakes. At the end of line 1
and line 4 she gave them big hints as to how to do the activity, rather than ask what the students think. The students had trouble reconciling the tabular and graphical representations of the line $y=x$ as evidenced in Vignette 3 line 5, since “it looks way different” as seen in Figure 5.2. Students appeared to have the expectation of a 45 degree angle for the line. The student did not explain what was different nor did Mrs. A. elicit elaboration or explanation; Mrs. A. decided to fix the problem of scale without discussion in line 6. To verify the correctness of student submitted points, Ms. A asked students to move their points on to the line she just graphed in Vignette 3 line 9. Ms. A asked the students who were not on the graph of the line to fix their points in lines 11 through 26.

Vignette 3

1) T: Almost everybody’s in. Ok, looks like everybody’s in. What I’m looking for is for you to have your graph paper out, a ruler of some sort and some notebook paper. I would clear your desk of everything else; that’s all you’ll need today. Alright, looks like we’re about ready. Today is a momentous day in your lives because it’s the one Casey’s been waiting for weeks, and that’s when you’re going to learn to crack the code on linear equations. After today you’ll know exactly where they go on the screen, you’ll know exactly what to draw, and you’ll know what all the numbers mean. So when you leave here your brain will be changed, won’t it? It will have a direction it didn’t have before. Before we start, what I’m going to have you do is I’m going to have you, just to warm up our brains a little bit, we’ll go to AC and I’m going to have you guys go to… Find a place where your $Y = your X$. Find a place where your $Y = your X$. Your equation would be $y=x$. Y is X, so if your Y is 2 what’s your X?

2) T: I’m a little confused with some of you guys over here, like what you’re thinking. This is over about 3 or 4 so what would you want to do? Go up about 3 or 4.

3) SS: But this is only over two. The one in yellow looks like...

4) T: A little hint: if you look on your calculator it tells you what coordinates you’re at.

5) SS: I know but like… [inaudible] It looks way different. I got it!

6) T: Ok, ok, that’s a good point, let me fix it. That will make it lighter.

7) SS: Whoa![Ms. A changed the viewing window so that the x and y had the same scale].
8) T: How’s that? Good point. Ok, I’ll give you another second. It was left over from my other class.
9) T: Ok, let’s see how we did. Are you on the line?
10) SS: Yeah.
11) T: If you’re not, see if you can get yourself onto the line.
12) S: Can you zoom out a little bit?
13) T: That’s as far as I’m going to zoom out for right now.
14) S: It says I’m (6,6) but I’m on the line.
15) T: Really; where are you?
16) S: I’m on neg. 5. I’m on X, the yellow one.
17) T: Right there, Casey?
18) SS: No. Down... Negative... oh, wait. There we go.
19) S1: What’s that?
20) T: Who’s this guy up here?
21) S1: Oh, that’s where I am.
22) T: Prudence.
23) S: Oh, that’s me?
24) [class laughs]
25) T: Yeah.
26) SS: [inaudible]. [Students speaking among themselves] Really? Yeah, I don’t know why it won’t... it won’t let me move, either. That’s what I had for a second. Prudence, you’re on the line now. You’re on the line.

Figure 5.2: Graph of student submitted points for the line y=x.
**Questioning.** Ms. A questioned students in a variety of manners to elicit and extend student thinking, such as asking them to predict what will happen, asking for observations, asking how do you know, and is there a different way to know the same thing. She also asked procedural questions aimed at eliciting student results rather than how the student obtained the result. The majority of Ms. A questions revolved around the former type.

In Vignette 1 line 2, Ms. A displayed both lines $y=x$ and $y=2x$ on the screen and asked “What do you notice about it (the line $y=2x$)” She was explicitly asking students to make observations about the line $y=2x$ and implicitly asking them to compare the two lines. In lines 3 through 7, Casey picked up on the implicit question and stated the new line is steeper than the old. Ms. A pressed further for more of Casey’s thinking by asking him what he meant by steeper in line 8. She seemed to be not satisfied with a correct answer, and continued to expose deeper student thinking by asking for observations and pressing for more meaning.

In Vignette 2 line 1, Ms. A asked the students to “Picture it (the line $y=-x$) in your head and on a scrap piece of paper draw it, kind of predict…” By asking the students to predict she was forcing the students to think about what they know and how this knowledge impacts the current situation. In previous tasks Ms. A did not ask students to predict until enough examples were present to make a prediction, but asked them to create graphs of different lines and to make observations about them as in Vignette 1 lines 1 through 11. During the prediction question in Vignette 1, Ms. A accepted the students’ “Uh huh” without justification. But during the same type of question about $y=-x$
in Vignette 2, she pressed for justification by asking “How did you know what direction it was going to move from the y=-2x line?” at the end of line 2 Vignette 2. Apparently not satisfied with the student response to this question she pressed again for students to justify their prediction in line 10. Wendy gave a reasonable justification of her prediction by using the line y=x and a reflection in lines 15 to 20. However even though Wendy’s response gave a meaningful way to make the prediction, Ms. A seemed to want students to make the prediction in another manner. She asked for a different way of predicting in line 22.

Ms. A’s push for different explanations led to an interesting discussion of what the number in front of the X really does to the graph of the line in lines 22 to 57. The discussion illustrates a pattern of questioning for Ms. A. She elicited students for more students’ thinking even in the presence of correct student thinking. Stacy’s conclusion at the end of Vignette 2, “you think the closer a number gets to one the flatter your line is going to be?” appeared to prompt Ms. A’s use of y=\(-1/2x\) as a counterexample to Stacy’s theory in Vignette 4 rather than immediately correct her. Ms. A began the counter example by having students test the theory by plotting “smart points” (x coordinates that produce integer y-values) for y=\(-1/2x\). In line 11 Ms. A was asking students to justify or refute Stacy’s theory. A student in line 12 believes that Stacy was almost correct, but needed to change her statement to “the closer it gets to zero” the flatter the line. Ms. A then asked what happened if the number was zero in line 15. The student response seemed to indicate to Ms. A that the line would be horizontal. The resulting discussion helped lead students to the connection between the part of the algebraic representation
and graphical representation of lines, more specifically the connection between the coefficient of \( x \) and the steepness of the line in Vignette 4 line 30.

Ms. A Vignette 4

1) \( T: \) Ok. Try this on your paper. Try \( Y= \text{neg. } 1/2 \times \). \( \text{Neg. } 1/2X. \) 2 or 3 points. Remember when I told you about choosing smart coordinates that don’t make you have to plot fractions? So choose some smart coordinates so your line will be easy for you to draw. \( Y= \text{neg. } 1/2X. \) Let’s test Stacey’s theory because now it’s going to be less than 1 and we’ll see what happens.

2) \( S: \) Ms. ____?

3) \( T: \) Yes.

4) \( S: \) I think when it goes like this, then if you do it times two or times 4, then you go up farther.

5) \( T: \) Ok.

6) \( S: \) But then if you just have [inaudible] \( X \) it’s just going to [inaudible].

7) \( T: \) Ok. We’ve got some interesting lines.

8) \( S: \) Do you want us to send it?

9) \( T: \) No, no, don’t send it this time. Right now we’re just plotting it on our papers. You guys are doing great. About 30 seconds. I have to remind myself when this class is over because I cut the last class short two or three minutes. All right. Let’s look up here. So the purple line is \( Y= \text{neg. } X. \) Is that correct? So now we’re going to put in \( Y= \text{neg. } 1/2X. \) Watch what happens on the screen. Kind of picture where you think it’s going to go. And here we go. Is that where you thought it was?

10) \( SS: \) Yeah.

11) \( T: \) So let’s talk about Stacey’s theory. She thought that the closer the number in front of the \( X \) got to 1 the flatter it was going to be. What do you think about that theory as far as what we know? So now we have – this is neg. \( 1/2 \) so we passed 1 and we’ve gone to the other side of 1. Yes.

12) \( S: \) I’d like to use the same concept as what she’s doing and I think this is what she was trying to say before. The \( X \) axis, it goes thru 0 on the \( Y \) axis, so the closer that the number gets to 0...

13) \( T: \) So you think she’s right, except instead of 1 it should be 0?

14) \( S: \) Yeah.

15) \( T: \) What happens if it got to 0?

16) \( S: \) It would be a straight line.

17) \( T: \) Ohhh.

18) \( S: \) Because \( X \) equals [inaudible].

19) \( T: \) Is that not what happened yesterday, Casey?

20) \( S: \) Yeah.

21) \( T: \) With the line that way. Yes, go ahead.

22) \( S: \) \( y=x \) is usually just a straight diagonal one. If we go more...
23) T: 45 degrees...
24) S...if you ever multiply by something you get it greater, it will tilt up like with 2X. And if you every divide it by 2 it will go down. A negative, it’s just a reverse 45 degree angle.
25) T: Do you have to divide to do that?
26) S: You multiply by ½.
27) T: Ok. I think I hear what you’re saying; it’s just slightly different from Wendy. Yes.
28) S: I think Casey’s right, but you can also subtract to get the line to go farther down, to be flat on the X axis, but you don’t only have to divide. Because you were saying the only way to get it to go down in is to divide and multiply to get higher.
29) T: Do you mean the number that’s in front of the X?
30) S: Yeah. If you added it (to zero) would still go higher and if you subtract it would go lower.
31) T: Ok. Other ideas? So on your notebook paper write your idea of how you would be able to tell me how the line is going to tilt. Just what you’re thinking right now. If I asked you to put a line up there and move it how would you know which way it was going to go?

Choice of examples. Mrs. A.’s choice of examples of representations of lines appeared to be a mix of examples chosen ahead of time and in response to student interaction. Some examples seemed to be planned contingencies and others seemed to be created in the midst of teaching. She chose a variety of good examples to reveal the connection between the coefficient of X and the steepness of the line using positive, negative and rational numbers. During the two days of observation she did not use non-integer y-intercepts.

During the second day when the class was discussing how to find the slope of a line, Ms. A asked if the order mattered when finding slope. Dylan said that it does matter which points we pick first. In Vignette 5, Ms. A planned for this student reaction and provided an example of points on the line where order doesn’t matter.
Ms. A Vignette 5

1) T... Now Dylan said... Listen up, this is really important. Nathan, Dylan said it matters which one we pick. So let's reverse it and see what happens. Let's call that one, one, and that one two. Ok, so now if I'm calling that one, one, and that one, two, now tell me – let's see, Sara, what is my Y sub 2? What is the Y in the second point?

2) S: Sorry – Y – I mean zero.

3) T: Very good. And let's see, Emma, what is my Y sub 1? The Y in the first point? Which is...? First point...

4) S: One.

5) T: Where's the Y?

6) S: One.

7) T: One. Ok, and now we're going to say X sub 2. Casey, which one is X sub 2?

8) S: Neg. 1?

9) T: Ok, he said it's the X in the second point and minus, and now I want to do X sub one. Emily, which one is X sub 1?

10) S: Zero.

11) T: The first point.

12) S: 1X.

13) T: Ok, so now we have zero take away one is?

14) SS: Neg. 1.

15) T: Neg. 1. Neg. 1 take away zero is...

16) SS: Neg. 1.

17) T: And neg. 1 divided by neg. 1 is?

18) SS: 1.

19) T: Oh, did it matter?

20) SS: No.

21) T: We'll have to try that again on some other problems and see if it matters. So it looks like you can pick whatever point you want and call it one and pick whatever point you want and call it two.

In line 1, Ms. A did not seem surprised by Dylan's response. She appeared to have planned for a student to respond in this manner and she quickly jumped into an example to counter the response. At the end of the vignette she acknowledged that one example wasn’t enough and alluded to trying more examples later.
On the first day Ms. A’s initial examples of coefficients of x lacked an important category, non-integer. Perhaps because of the lack of these examples, Stacy incorrectly concluded that the line gets flatter when the coefficient of x gets closer to 1. When Stacy proposed her theory, Mrs. A. appeared to have not anticipated this student response. Despite not having a contingency plan, Ms. A still was able to remedy the situation by using the example of $y=-1/2x$ and generating the discourse of Vignette 4.

**Missed opportunities.** There were a few instances where the CCT presented the opportunity to make connections between the graphical and algebraic representations of lines that Ms. A did not use. In most of these instances the CCT displayed students were making student error in systematic ways. The other major missed opportunity was not generated by the CCT, but rather Ms. A did not provide closure to an important discussion.

During the second major segment of the first day, Ms. A asked students to find points for the line $y=x+4$ and later submit two or three of them to the CCT. Figure 5.3 shows the points the CCT collected from the students. The dots above the solid line lie on $y=x+4$, except for one that is out of place. However, below the solid line lie 5 points that form a straight line. At least two students perhaps more submitted these incorrect points, assuming each student did submit 2 or 3 points and not 5. When Ms. A saw the points she claimed, “Oops, a little mistake there; probably just entered them wrong. About 15 more seconds.” While it may be true that students submitted the points incorrectly like the highest point at X=2, but the points below the solid line are systematically incorrect. The solid line looks to be in the middle of the two lines formed by the dots, meaning the 5
points below are on the line $y=x-4$. Perhaps the students misinterpreted the directions and thought they needed to find $y=x-4$. Although it may have been a more grievous error, such as “in order to find $y=x+4$ four needs to be subtracted from $X$ similar to solving equations.” However, Ms. A missed the opportunity to determine what caused the systematic errors. She could have asked the class, “what were the students who submitted the 5 points below thinking?” The CCT would have allowed her to determine how widespread the error was by enabling her to see how many students did make the error; since multiple people could have sent the point $(2,-2)$ and only one dot would be displayed.

![Figure 5.3: Student submitted points for $y=x+4$.](image)

A similar missed opportunity occurred during the next activity after the one discussed above. The task was the same as above but students needed to find points for $y=x-3$. Figure 5.4 shows the results of collecting the points the students found. The three solid lines in Figure 5.4 are $y=x+4$, $y=x$, and $y=x-3$ from top to bottom. To show the students
whether they are correct or not she has the CCT display the graph of the lines of the activities. The point in the bottom right corner may be one of the points entered incorrectly. However, the four other points not on the line $y=x-3$ look to be more systematic errors. The two incorrect points in the left half of the plane are slightly below and look parallel to the line $y=x+4$. These points more than likely lie on the line $y=x+3$ indicating the possibility of one or more students performing the opposite operation of what needs to be done, subtracting instead of adding and vice versa. The CCT would have allowed Ms. A to know if the same student(s) were making the same mistakes again, but she did not use that opportunity. Figure 5.4 also reveals another potential systematic error. The dots at (1,2), (2,1) and perhaps (3,0) look to be a reflection of the line $y=x-3$ about the x-axis. The points look to be on the line $Y=3-X$, maybe this was a simple mistake or perhaps the student thought they needed to remove x from 3 rather than 3 from x. Ms. A responded to the points the students submitted with,

*You guys are doing very well. About 20 seconds, it looks like. Here we go. $y=x-3$. Picture in your head… oh, look at that, all you brilliant people. Good job! Ok, on your paper without talking, we’ve now added another little thing, haven’t we? Either a – or + to the X. So write down what you think that does. What happened when we said $X+4$ or $X-3$ instead of just $y=x$. Which is our middle one? Our middle one is $y=x$. The top one is $y=x+4$, the bottom one is $y=x-3$. What happened when we added that to the end of the X? What happens to your line? Write down what you think, write down what you think, and then we’ll talk about it. About 30 seconds.*

Ms. A did not mention the potential systematic errors at all this time and moved on to wrap up the activity. Given that the same systematic error could have occurred twice in subsequent tasks, Ms. A missed an opportunity to illuminate the students’ thinking on such errors.
The other major missed opportunity not taken by Ms. A occurred at the end of Vignette 4, where she ended the discussion about the relationship between the coefficient of $x$ and the slope of the line. After the last line of Vignette 4, Ms. A moved onto the next group of activities dealing with the relationship of the constant term and the shape of the line. The missed opportunity was not generated by the CCT, but rather from the student discourse. The students were saying similar things that were close to the true relationship between the algebraic and graphical representations, but what they lacked was cohesion and clarity. Ms. A ended the group of activities with students writing in their own words the connection between the number in front of the $X$ and the slope of the line without revoicing or providing clarification about one of the student’s comments. She left the students with a non-concise, unclear description of this relationship with informal
terminology. Ms. A missed an opportunity to provide closure in recapping the main point of the first segment of activities with clarity.

**Technology Use**

Ms. A used the CCT for at least part of most of the activities during both days. A typical activity had a certain flow to it, such as students doing something on paper or in their heads, Ms. A collecting and displaying student data for the task, using the CCT to verify correctness, and perhaps discussing the results. Vignette 1 is an example of this pattern, Ms. A had students find points on the line $y=2x$, submit them, had the CCT plot the line, and discussed comparisons of the line $y=2x$ and $y=x$. At the end of Vignette 1 she had student do a mini-cycle of predicting in their head what $y=4x$ looks like and using the CCT to check their predictions. Three major uses for the CCT emerged from this pattern, using CCT as information gatherer/display, using the CCT as an answer checker, and using the CCT as a discourse generator.

The first two major ways Ms. A used the CCT went nearly hand-in-hand. On the first day and for the majority on the second day Ms. A used the CCT to collect student data and also used it to verify their answers by seeing if the points submitted by the students fell on the given line. The graphs of the lines Ms. A had the CCT display were used to generate discussion comparing lines rather than using the student data for discussion. Vignettes 1, 2, and 4 are examples of not using collected student data but using the verification lines for discourse. The resulting discussions in these vignettes could have been produced by using other technology such as the chalkboard.
Theoretically, using CCT as an information display/gatherer or answer checker could lead naturally to the use of CCT as a discourse generator by discussing student errors after the verification of information that seems out of place from the rest. However, on the first day Ms. A did not use the verification process to discuss incorrect points except perhaps through passing comments from Ms. A like “I bet someone just entered that in wrong.” Not discussing student errors especially when they were noticeably systematic errors became major missed opportunities for Ms. A. During the first activity of the second day, Ms. A used student data to generate discourse. She had students make predictions of what \( y = \frac{1}{4}x + 4 \) would look like, do a T chart and send in points. After the students submitted points, Ms. A had the CCT display the verification line shown in Figure 5.5. Vignette 6 is the resulting discussion.

Ms. A Vignette 6

1) \( T: \) Oh. All right, let’s look up here to see how closely we came. We have a few points all over the place and we’ll talk about it in just a minute. We have some people that are a little bit above it and a little bit below it. Let me take off “show student’s name” and we’ll look at those. Let’s look at some of these points here. (-8,6). Where might that have come from? Look at the equation that I gave you. Obviously that’s not correct, but what might that person have done that caused them to get a wrong point? Casey?

2) \( S: \) If you do positive 8 the Y is 6 so they might have accidentally put in the positive Y...

3) \( T: \) That’s a good point. So you say (8,6) is a point?

4) \( S: \) Yeah, (8,6) is a point. They probably just messed up a little.

5) \( T: \) Is there something they might have done in their equation where they actually did think it was -8? Is there any way that that might...Think about that.

6) \( S: \) Negative 8 won’t work.

7) \( T: \) Think about that. One second [interruption from computer technician]. Alright, Casey, you’re saying neg. 8 will work?

8) \( S: \) It will work; it’s not the 6.
9) T: Ok, so you think the 6 is wrong. Let’s look at another one, ladies. Let’s look at one more point. Let’s pick one that’s way off and figure out what this person did. So that one is a neg. 1, neg. 3. What might they have done or misunderstood to think that point would work? Neg. 1, neg. 3. They’re saying X should be neg. 1 and the Y should be neg. 3. Any ideas? What would you tell that person to try? It’s obviously wrong, isn’t it? So how can you be sure that it’s right? What could you do to say, oh, I know this point works? How can you, before you send it to me, be absolutely positive that it works? What could you do? Yes, sir.

10) S: Compare it to your line.

11) T: You could do that. You could say it doesn’t line up with my other points. There’s a math way you can also figure out if you’re right or not besides just looking which is a good strategy. What am I always telling you to do before you give me your tests?

12) SS: Check it.

13) T: How can we check these coordinates? [interruption from computer tech] I Ok, so neg. 3, neg. 1, how am I going to check that? You know what, I don’t want to see any wrong points and I want to make sure it’s right before I send it. How can you do that besides just eyeing it? Casey?

14) S: Use a neg. 3, neg. 1... or neg. 3, neg. 1.

15) T: Whatever it is, yeah. How could you check it? Yes ma’am.

16) S: Work backwards?

17) T: How would you do that?

18) S: You’d divide your X by ¼ so you’d multiply by 4 and then you’d add 4 to that and that would give you your Y?

19) T: All right, what else could you do? Stacy?

20) S: Like she said, but you get X by itself and then you can plug Y in and see if it works.


22) S: Can’t we just reverse the operations for the Y, like subtract 4 from neg. 3 and then divide by ¼?

23) T: Ok. There’s an easier way. If you can and said Ms. ____, what did you say it’s actually...? I forgot, did I say neg. 1, neg. 3?

24) SS: Yeah.

25) T: Ok, I say prove it. How can you prove it to me without a doubt that you are correct? Yes ma’am.

26) S: Plug it in first.

27) T: Plug it in! What does she mean by plug it in? Emma.
28) S: Do the equation with the number...
29) T: Try that right now; write down $Y = 1/4X + 4$ and plug in neg. 1 for $X$ and neg. 3 for $Y$ and see if your equation works. It was neg. 1, neg. 3, yes. Raise your hand and tell me if it worked then both sides of the equation are going to balance. Raise your hand and tell me if it worked or not. Aaron, did it work?
30) S: No.
31) T: How do you know it didn’t work?
32) S: [inaudible]
33) T: I’m sorry, what?
34) S: I plugged it in and it didn’t work.
35) T: What does your equation turn out to be?
36) S: Neg. 3 = 1.

In line 1 of Vignette 6, Ms. A recognized several student errors and felt the need to discuss them. Ms. A first picked a point to discuss that seemed to be a systematic error (-8,6) and asked the students “but what might that person have done that caused them to get a wrong point? Casey?” Casey noticed that (8,6) was a point on the line so maybe they messed up a little. While the student who submitted the points may have made a minus sign error, Ms. A appeared to want to direct the students’ attention to the systematic error by asking in line 5, “Is there something they might have done in their equation where they actually did think it was a -8?” Ms. A appeared to try to point out that one way for the point (-8,6) to occur is if the student thought the line was $y = -\frac{1}{4}x + 4$. The two points above the solid line in Figure 5.5 look to be reflections of the two points directly below over the line $y=3$, or reflections of the solid line about the y-axis. Unfortunately, Ms. A did not reveal the potential systematic error and ended discussion of the (-8,6) point in line 9 by looking at a point that was the largest outlier (-1,-3). During the course of line 9, Ms. A first asked for a discussion of “what might they have done or misunderstood to think that this point would work?” But then without
letting students answer this question she changed the discussion to “how can you be sure it’s right?” The student response in lines 16 and 18 revealed the potential cause of the systematic errors in the day before in for the major missed opportunities. In line 18, the student said you need to divide x by ¼, or multiply by 4, which is the opposite of what needs to be done to the x. This appears to be the same error that happened the day before: subtracting 4 instead of adding and adding 3 instead of subtracting. If Ms. A used the CCT as a discourse generator to talk about student errors the day before, the systematic error of “working backwards” could have been exposed and potentially eliminated during the second day. Unfortunately, Ms. A did not mention that the starting point matters when “working backwards.” Ms. A led rest of the discussion to finding an efficient way to verify points on the line. Even though Ms. A did not fully capitalize on the discussion generated in Vignette 6, she did illuminate the potential source of reoccurring student errors of working backwards from X more so than saying, “I bet someone just entered it in wrong” and then moving on to the next part of the activity.

The combination of three steps: using CCT for information display/gatherer, verification of answers and generating discourse about student errors appeared to have potential for illuminating student understanding and misconceptions even if not fully utilized. The third step seemed crucial to expose the error of working backwards from X, since during the first day when the first two uses of CCT in the sequence did not even hint at this error.
Figure 5.5: Points students submitted for line $y = \frac{1}{4}x + 4$.

**Decisions**

Ms. A’s goals appeared to be to elicit or extend student thinking in classroom discourse without immediately verifying the student’s response as correct. She consistently asked and expected students to answer questions like “what do you notice?” “what could this student have been thinking?,” “is there another way to think about this?” or “How could you figure that out?” Another of Ms. A’s goals seemed to be to have enough students participate, even when a correct statement arose from a student she would ask another student what they thought (see Vignette 2 line 22).

Ms. A’s orientation appeared to be farther on the conceptual end of the conceptual/procedural spectrum as evidenced in part by the classroom discourse. Ms. A may have had students perform procedural tasks such as creating T-charts, but procedural content was used to promote conceptual discussions or if discussions prompted the need for procedural content. The one exception to this appeared to be Ms. A’s focus on the
procedure of “plugging it in” to check the points students submitted in Vignette 6. The classroom discourse Ms. A promoted during the first day dealt with more conceptual matters, such as what is the relationship between the coefficient of x and the slope of the line, or the connection between the constant term and the shape of the line. The goals for the second day appeared to be to practice and reinforce the concepts of the day before. Thus, the discourse was focused more on procedures. Despite the focus being more procedural the second day, a very conceptual discussion on the meaning of infinite slope occurred. The procedure of finding the slope of a line given its graph did not appear in class until the activities led the students to the conclusion they needed a more precise way of finding slope giving the slope formula context and potentially reinforcing its meaning. Ms. A appeared to flexibly move from procedural to conceptual when needed. Ms. A’s treatment of representations of functions as both process and object revealed her conceptual orientation as both perspectives are needed to fully understand the notion of function.

Figure 5.6 presents a flow chart modeling Ms. A’s decisions during year 1 of the study. The rectangles represent actions taken by Ms. A and diamonds represent decisions. Rounded rectangles represent actions that seemed to need or were benefited by the use of CCT. Two loops appear in the flow chart. The purpose of the loop at the first decision appeared to be to create enough content for discourse. The purpose of the loop at the second decision appeared to be to elicit and extend student thinking. These two loops could occur many times in a single discussion of a single activity. The major feature of the decision flow chart is the *elicit, extend, explain or clarify* students thinking or have
students make *prediction* cycles (EEECP), which is the main method of Ms. A for reaching her goal.

The first segment of the first day can be modeled using the flow chart from above. The activity in Vignette 3 started the discussion of connection of the coefficient of x and the slope of the line. However, after Vignette 3 there were not enough cases to generate discussion by comparison. Vignette 1 was the result of Ms. A’s decision to generate more cases. In Vignette 1 line 3, Ms. A asked, “What do you notice” about the two lines displayed. Lines 3-11 were short elicit/extend/clarify cycles with at least two students. After a correct comparison and multiple students discussing the concept, Ms. A decided to have the students make a prediction for a similar case. At the end of Vignette 1 in line 12, Ms. A had students predict in their heads the outcome of the case of $y=4x$. Another category of important cases, negative coefficients, needed to be considered to establish the connection. Ms. A decided to go the “other way” and consider $Y=-2x$ in Vignette 7 after having students think about it, create a T-chart and send in points.
Ms. A Vignette 7

1) T: I bet they just entered it wrong. Ok, let’s check it out here, see if we’re right or not. There we go; almost everybody was right. I think somebody just entered it wrong. So what do we notice about the direction of our line? Yes, ma’am?
2) S: When you multiply X by negative it goes left to right?
3) T: Ok, so the line is going down instead of up.
4) S: Uh huh.
5) T: Ok, what else do we notice? Compare this on to the... This is Y= neg. 2X. Compare it to your Y= 2X line. What do you notice about that, Sara?
1) S: I guess you could call each one of the squares planes? It’s in the negative plane versus the positive one er...
2) T: Ok, all right. Casey?
3) S: It starts on the negative side and goes downward across the positive; all the positive ones which start at the negative 10 to positive?
4) T: Say that again?
5) S: The negative ones will go down from the positive – from pos. 1 down thru pos.
6) T: Ok, all right. Yes, ma’am.
7) S: The slopes are the same and the angles are the same, but one has a negative slope to it, so it goes the other way.
8) T: It’s going in the other direction, ok. In your head picture... This is neg. 2X. Picture what Y= neg. X is going to look like. Picture it in your head and on a scrap piece of graph paper draw it, kind of predict, and then I want you to chart it on your paper and label it Y= neg. X. Let’s look at what direction the line is going to go. Y= neg. X. Kind of make a guess and do a quick T chart and then on your regular graph paper go ahead and graph it.

At the beginning of Vignette 7 there were four cases displayed by the CCT, y=x, y=2x, y=4x and y=-2x. Ms. A appeared to have decided there were enough cases to consider and jumped right into discussion comparisons in line 1. Ms. A used several elicit/extend/clarify cycles in lines 2 though 9 by asking multiple students what they noticed and re-voicing what they said. Ms. A seemed to decide that more examples of negatives slope were needed and asked students to make a prediction of what the graph of y=-x looks like. With six cases being displayed by the CCT, Ms. A had students at least mentally verify their predictions and moved into discussion of the line y=-x at the beginning of Vignette 2. Ms. A tried to elicit a connection between y=-2x and y=-x by asking “How did you know which direction it was going to move from the line y=-2x.” However, students seemed to be focused on another case displayed such as y=x in line 19. Students made the connection that y=-x is the reflection of y=x, but not the connection that y=-x should go in the same direction as but less steep than y=-2x. Ms. A refocused the discussion to elicit this connection by telling students to only think about the two negatively sloped lines in Vignette 2 line 22.
Ms. A had to use several elicit/elaborate/clarify cycles in lines 24 to 33 in Vignette 22 to get to a common terminology in line 34 before moving back to the discussion of the connection between $Y=-2x$ and $y=-x$. The resulting cycles in lines 26 to 57 culminated into Stacy’s theory of “the closer a number gets to one the flatter your line is going to be?” Stacy’s theory appeared to prompt Ms. A to have students consider another category of cases, rational slope. Line 1 Vignette 4, Ms. A had students try $y=-1/2x$ to test Stacey’s theory. Vignette 4 is an example of several elicit/extend/clarify cycles of the last decision of the flow chart model about generalizing observations. At the end of Vignette 4, Ms. A had students write their observations about the connection between the coefficient of $x$ and the slope of the line.

**Insider Perspective**

Ms. A’s post observation interview illuminated many of the classroom observations. Ms. A’s observed goals of eliciting and extending student thinking without revealing the correct answers was verified by Ms. A’s statements of “I ask them what they are thinking because that is what I care about,” and “I very rarely if ever would say that an answer is good. I’d say that a question is good but I would never if somebody is answering that is good or that is right.”

Ms. A praised the CCT for being something the students thought was fun to use, but more importantly she appreciated the fact that “…now (with the CCT) the data gathering takes very little time but we spend most of our time conceptualizing, trying to pull things together and we had some great discussions in here,” which indicates she leaned more on the conceptual side of the procedural/conceptual spectrum.
Ms. A’s choice of examples and the progression of those examples were given careful consideration as she stated:

So, one way to think of that is to how we want the conversation to go, think about the questions, think about the mistakes they might make, and then the second was to how to sequence my example and the things I was going to have them to follow a logical progression from the slope to the Y-intercept and then of course the positives and negatives. So, I had to be very careful about the sequence of the equations.

During the interview, Ms. A claimed that she had not planned on using rational slope for the lesson, but the discourse led to needing that as an example.

The main use of the CCT in the classroom was observed to be information display/gatherer and answer checker, which was how Ms. A viewed the CCT “Ok, well, basically it [CCT]was as a tool to send data, to gather data from them… besides the obvious (ability) of letting them confirm their data.” Ms. A hinted at the use of CCT as a discourse generator since students can “Look there [at the CCT display] and see, I mean they can check immediately, ohh, wait a minute, that is what I have just said is not what I see it up there so they can quickly correct their thinking.”

The students in Ms. A’s classroom seemed free to speak their ideas without worrying if they were wrong. This freedom could be the result of Ms. A intentionally not correcting students as well as of the extensive time in the beginning of the year she said in the POI that she modeled (for) them a lot how to disagree with somebody without being personal about it because I don’t think most adults can even do that and so how that is Ok if I agree with you or I don’t agree with you as long as I express it an a polite way and how we reason for that and they don’t attack you personally.
Ms. A saw the errors during the missed opportunities, but perhaps did not see them as interesting errors, since she claimed that discussing the errors was “not the point [of the activity] because we practiced those [plugging in points into equations] a little.”

**Year Two: Second Year of CCT Use**

**Overview**

Several prominent features of Ms. A’s instruction either remained or expanded during year 2. Ms. A continued to use different representations for the same object, she used multiple representations from different students, and the notion of function as object was present. Discourse continued to elicit and extend student thinking. While discourse during the first year was exclusively student to teacher or teacher to student, student to student discourse emerged during the second year. The use of CCT as information display/gatherer and prediction checker continued from the first year, as well as an increased use of the CCT as a discourse generator from year 1 to year 2. Ms. A had the students work in groups with the CCT in year 2 rather than as individuals in year 1.

**Segmentation**

The class time of the first day of the second year was split into two multi-activity segments and a single activity segment. The first seven minutes of class was spent getting the class ready and logged into the CCT. During the first segment, Ms. A had the CCT display pictures of parabolic shaped objects such as a necklace or parabaloid beneath an XY plane. The students were given 5-8 minutes to determine the equation of the parabola that best matched the picture. Each student was in a group of about 4 students and only one equation per group could be sent to the CCT encouraging discourse within the group.
After the students sent in their equations to the CCT, Ms. A facilitated a whole class discussion usually lasting 2-4 minutes about improvements that could be made to a groups’ graph to better fit the picture. The three activities in the first segment lasted a total of 25 minutes. During the discourse in the first segment, Ms. A concluded that students were using a guess and check strategy to find the best fit parabola. Ms. A graphed \( y = x^2 - 7x + 12 \) without students seeing the equation and discussed with the whole class how they could determine the equation if the guess and check ability had been disabled for the second segment. The ten minute discussion revolved around a “math way” for knowing for certain that the equation would work before ever graphing it.

The third and final segment of the first day consisted of four smaller discussions totaling 14 minutes. Ms. A typed an equation into the CCT without displaying its graph and then had students predict its shape for the first two activities; for the third she had students discuss the graph of a parabola with one intercept, and for the fourth she had students discuss how to write an equation for a parabola that did not cross the x-axis.

The second day was split into three activities all similar to the first segment of the first day. Ms. A displayed a picture of the McDonalds’ arches under a XY plane for the first activity. The arches created four parabolic shapes and Ms. A had students find the best fit parabola in factored form for the bottom of the left arch. This activity was segmented into four parts, first students were given about 10 minutes to work in groups and find the best fit parabola. The next three parts of this activity were discussions of how to improve some of the student submitted parabolas. The first was discussing how to improve an equation that was not sent in the correct form. The second discussion dealt
with how to make the parabola narrower in the factored form. The third discussion was about how to make it taller and still keep the correct x-intercepts. The 33 minutes devoted to the first activity comprised 10 minutes of group work, 5 minutes for first discussion, 3 minutes for second discussion and 13 minutes for third discussion. The bottom of the right arch was the focus of the second activity with the same directions as the first. Students were given 5 minutes to work in groups to determine the best fit parabola and the whole class discussion comparing different student submitted equations lasted 6 minutes. Equations that were not close to the picture were not discussed as there were no major deviations. The third activity had pictures of streams of water from a pool fountain. The students had to submit the factored form of the best fit parabola to the topmost water stream. Students were limited to 3 minutes to find the parabola since it was near the end of class. No meaningful discussion took place due to the end of class; despite the success of the students finding a best-fit parabola for activity 2, students clearly were having difficulty with the third activity as several of their parabolas were not close to the picture of the stream.

**Representations**

The graphical and algebraic representations wove through all the activities of both days. Students had to go from the graphical representation to the algebraic representation for the first three activities of the first day and all the activities of the second day. The students used a cyclical process of going from graphical to algebraic and back again by essentially guessing an equation and graphing it then making adjustments to the equation until they got a decent fit to the picture. Therefore, both the graphical and algebraic
representations were not merely present but intimately connected. For the fourth activity of the first day students also had to go from the graphical to the algebraic, but they were no longer able to guess and check. Rather, Ms. A had them discuss a “math way” for them to know what the parameters were for the equation of parabola. The last four activities had students go in the reverse direction of the first four; they had to take an equation and predict what its graph would look like. Again the algebraic and graphical representations were more than merely being used in the same activity, but connections between the two representations were made. The tabular representation was not present for any activity either day.

During the activities of both days, students were exposed to different kinds of representations of parabolas but they also were exposed to different sources of representations. For the first three activities of the first day and the activities of the second day, students used a graphical representation of a parabola in a picture and sent the group’s algebraic equation to the CCT. After all the students submitted their equations, they could see the graphical representations of the other groups displayed on the screen in front of the class. During the discussion of the find the best fit parabola activities; Ms. A would draw the students’ attention to the display of the different equations submitted using the CCT (see Figure 5.7). The class discussed the connections between the different sources of the two different kinds of representations.
Process/Object perspective. The graphical and algebraic representations were treated from both the object and process perspectives. The students treated the graphical representation through the process perspective as inferred by how they discussed the representation. On multiple occasions students discussed how they could make the graph wider as in Vignette 8 line 4, or skewed to the left Vignette 12 lines 10 and 12. When finding a parabola that did not cross the x-axis a student added 1 to a parabola that had only one x-intercept. Another student noticed that the new graph was the “same thing, just moved up” as the old graph. On the second day when discussing how to fix a parabola, a student wanted to make the graph taller by “Grab it and pull it up” in Vignette 13 line 6. The students were treating the graph as something that could be moved, something that could be manipulated; they were treating the graph as something, an object. Ms. A seemed to expect students to discuss the object nature of the graph when asking questions about how to fix the parabolas. In Vignette 8 line 1 Ms. A asked “what would you suggest to this group that they need to do to make it a little bit more of the
same shape?” A student responded to make the A term smaller, but Ms. A not satisfied with just “the how” wanted to know “the why” in line 3 in Vignette 8.

Ms. A Vignette 8

1) T: Quiet please. Alright, let’s get rid of that one since it’s linear. So, let’s take a look at this one right here. Raise your hand. What would you suggest to this group that they need to do to make it a little bit more of the same shape? Hannah?
2) S: Make their A term smaller?
3) T: Why?
4) S: Because that will make it wider?
5) T: Ok, good. What else would we suggest to this group? Mason?
6) S: They added a B squared...
7) T: You mean the B term?
8) S: Yes, I mean the B term. They [inaudible].
9) T: Quiet please. Someone tell them why they didn’t need to do that.
10) S: Because the necklace is right in the middle so there’s no B needed.
11) T: No B needed, alright. So we said just to work on the A term and they may not have needed a B term. Let’s take a look at some of the other ones here.

When asked why by Ms. A the students focused on the object nature of the graph in lines 3 “because that will make it wider” and line 10 “because the necklace is right in the middle.”

By manipulating the graph through the algebraic equation, “Make their A (of the standard quadratic form) term smaller…because that would make it wider,” the object nature of the graph could extend to the algebraic equation. In Vignette 9, a student described a procedure of constantly manipulating the algebraic equation to get the best fit parabola through guessing and checking. The strategy of manipulating an algebraic equation by guessing and checking seemed to be common among all the student groups. Ms. A had been asking students to discuss how to fix parabolas that did not accurately
match the picture, and then she turned her attention to parabolas that were pretty close to
the picture in Vignette 9. In line 2, a student knew her group’s initial guess was off
because she noticed that the graph was too narrow to match the picture. The technology
enabled the student group to see the result of manipulating the algebraic equation on the
graph of the equation immediately, potentially reinforcing the object nature of the
algebraic representation.

Ms. A Vignette 9

1)  T: Ok, so that one is .25Xsq. minus 5. Let’s talk about that one for a minute.
Somebody that sent one of those equations – please stop talking over here. - How
did you determine that you should do a .25X? How did you determine that A
should be about .25?
2)  S: We did .5 first and it was too, like, narrow, and then we just guessed.
3)  T: You guessed and checked. Ok, alright, so how about the minus 5? How did you
determine that?
4)  S: It’s the Y intercept.
5)  T: Good, ok. Let’s try one more.

During the fourth activity of the first day, Ms. A drew the students’ attention to the
process nature of the graphical representation by having them focus on specific points of
the graph. The goal of the fourth activity was to force students to stop guessing and
checking and to determine the equation by focusing on the specifics of the graph and
using the connections between the algebraic and graphical representations. At the
beginning of this activity, she said

So, now I have an equation in there. There’s no picture. And also let’s suppose I
was sending you… I started the activity, but this time I didn’t allow you to plot it.
I noticed almost everybody in the classroom was putting in an equation, hitting
plot, seeing if it was right, going back and forth until it looked exactly perfect and
then sent it to me. But let’s suppose I disable the feature and don’t allow you to do
that anymore. How would you start figuring out what you’re going to do as far as
Ms. A started by focusing attention on the easier connections, such as what is the C term and is the A term positive or negative, big or small. Then Ms. A turned to finding the B term in the standard quadratic form, which led to focusing on the points of the graph rather than the shape of the graph emphasizing the process nature in the first 22 lines of Vignette 10. In line 3, Ms. A told students to look at the clues of the graph, specifically the x and y intercepts. The focus on the x-intercepts aided a student’s recollection that the b term can be obtained by adding the x-intercepts together and making it positive or negative depending if the parabola is to the left or right in line 12. In line 20 a student states how to determine if the b term should be positive or negative; however he did not know why his statement worked. To help the students know why the b term was negative rather than positive, Ms. A refocused the attention to the algebraic representation specifically the factored form and using the procedure of “setting things equal to zero” (line 23 Vignette 10). Ms. A in line 31 used Casey’s suggestion of looking at the factored form of the equation to aid the students’ connection between the x-intercepts and the b term. A student made a common error in factoring, and Ms. A used the error in line 33 to illustrate where the negatives come from. Ms. A ended the discussion by asking if students thought that adding the x-intercepts together would determine the b term of the parabola without answering the question. The statement of setting things equal to zero emphasized that the factored form is something to perform a
procedure on to find the relationship between specific x’s and y’s, that is the x-intercepts.

The procedure of finding x-intercepts using the factored form illustrates the process nature of algebraic representation by focusing on the points of the parabola rather than the whole shape.

Ms. A Vignette 10

1) T: Ok, I can tell you just from looking at this parabola what the B term is exactly. I can tell you exactly that it’s neg. 7.
2) S: Wow? Teach us how to do that.
3) T: [laugh] Well take a look at... Shh. There’s some clues on the screen that might help you figure out where could I have possibly gotten a 7?
   Quite please. Raise your hand when you want to talk, but first of all I want you to think a minute. Look at all the places that parabola is. Look at all the places it crosses both the X’s and the Y’s... Look at all the numbers and tell me where I could have ever said, just by looking at that, I know, the B term is neg. 7.
4) SS: There’s a trick. [inaudible]
5) T: There’s a trick, but there’s also an easy way to tell.
6) S: Because they cross...
7) T: Shh.
8) SS: [inaudible]
9) T: Shh. Quiet please.
10) S: I know we learned this.
11) T: Well, we briefly talked about it. Kyle.
12) S: Don’t you add the X intercepts together and then depending on which side it’s going to then you add the negative and positive because it goes the opposite way that you would have the...
13) T: Sort of. You’re right. What do you mean add them together?
14) S: Like since it’s 3 and 4, the X intercepts?
15) T: Well doesn’t 3 and 4 equal 7?
16) S: Yeah.
17) T: I thought I said it was neg. 7.
18) S: Since it’s going to the right...
19) T: I can’t hear him – sorry.
20) S: ...it’s going to the right and it’s already positive it has to be negative for it to go to the right.
21) T: Why?
22) S: It’s just the way it is.
23) T: You’re right, but do you know why? Do you remember me ever talking about zeros or setting things equal to zero?
24) SS: Yeah. I think I missed that.
25) T: Alright, Casey.
26) S: You have to set the X intercepts to zero and add those together to equal the number?
27) T: Ok, what do you mean set the X intercepts to zero?
28) S: You know that there are factored forms.
29) T: Okay.
30) S: And you need to set it zero equals X + 3 then minus 3 from both sides to get neg. 3 equals X.
31) T: Alright, so he’s saying the factored version of this… Let’s take a look at the regular version first, the expanded version. It’s X sq. minus 7X plus 12. We can see where the 12 came from – the 12 is the Y intercept. The X sq. is what makes it a parabola, and then minus 7, he’s saying, if we took a look at factoring this, when we factor it the two numbers when they multiply together have to equal C but when they’re added together have to equal B. So what if we did X plus 3 – oops, got to put my X in there, don’t I? [inaudible]
32) SS: [laugh]
33) T: Somebody said it’s X plus 3 times X plus 4. Let’s see. (displays equation using CCT) Oh, oh, what’s wrong with that?
34) S: It’s minus 4.
35) T: Why?
36) SS: Because you set it to zero. Isn’t it minus 3?
37) T: Yes, it is. Who said that?
38) S: Shelby.
39) T: Shelby, why is it?
40) S: Because you have to do... Because the opposite would be negative... Wait, hold on. Because... Because I mean like... Ok, I got it. Wait, Wait, wait. Ok, when you set it to zero so it would be the opposite so it would be negative...
41) T: Lauren, want to help her out?
42) S: Ok, the neg. 3 and the neg. 4 when you add them together you get neg. 7.
43) T: True. And I figured out the B term.
44) S: Are we trying to figure out the B term? Wait, what are doing?
45) T: Listen up, please. What someone said is because the graph went thru 3 and 4 that I should type in X plus 3 times X plus 4, but when I did that it moved the parabola onto the other side. So, why does X minus 3, when I type in X minus 4 it
just made the same exact parabola, so they obviously are equal, but why is that the case is what I'm asking you.

46) S: Because the opposite moved to zero, so like if you have 4 then to set it to there you have to subtract 4 so it’s neg. four and the three so you have to do negative 3.

47) T: So you’re saying set X minus 3 equal to zero, set X minus 4 equal to zero. When you solve them you would get a positive three and a positive four and that’s why they’re on this side of the graph. Jason?

48) S: Well it’s real obvious. All you have to know is basically the Y intercept – all the three intercepts, which is 12, and then you find the 3 and 4 – you just know – you make those negative and you add those together and you get neg. 7, and then you times those together you get neg. 12.

49) T: Positive 12? Do you think that would work for any equation?

50) SS: Yeah. Yes.

51) T: Any quadratic equation?

52) SS: Yes.

The factored from remained a prominent tool for the remaining activities on the first and second days. The four activities following the one in Vignette 10 continued to have a focus on specific points rather than on shape. The three activities of the second day emphasized shape and specific points emphasizing both the process and object perspectives of the graphical representation.

Treatment of representations. Representations in Ms. A’s classroom were used to enact procedures on, reason with, and make explanations. Students enacted the procedure of guessing and checking to find the best fit parabola to a picture. However, the purpose of the procedure was not to produce an end product and move on to the next picture, but to generate discussion about the representations. While the students performed the procedure of factoring the quadratic equation for activities 4 through 7 on the first day, finding the factored form was not the purpose of using the procedure. Rather the purpose
of finding the factored form was to illustrate the connection of the x-intercepts and the B term in activity 4 and to predict the position of the graph in activities five and six.

The discussions about how to fix the best fit parabolas made students reason about the connections between the algebraic and graphical representations. The students reasoned that the algebraic equation for a particular graph was not correct given the graph was not wide enough and was skewed to the right in Vignette 8. Vignette 9 is evidence that at least one group was not just randomly guessing and checking but used a reasoned trial and error approach since the students thought of how to make a better guess. During activities 5 and 6 of the first day Ms. A had students reason about the connections between the algebraic and graphical representations, since they had to use the algebraic equation to make predictions about the graph of the parabola. In Vignette 10, Ms. A used the factored form to facilitate with the students an explanation for the connection of the graph’s x-intercepts and the expanded form’s B term.

**Discourse**

Due to the intense chatter that occurred within the student groups during the best-fit parabola activities that made distinguishing student utterances difficult or impossible, only whole class discourse is analyzed. The purpose of the whole classroom discourse in Ms. A’s class seemed to be to elicit student explanations to aid student understanding as it seemed to be the year before. While several features of the classroom discourse remained unchanged from year 1 to year 2, some important aspects did change. One prominent change was that the discourse was not limited to teacher to student or student to teacher, but included student to student as well. Another change was that Ms. A
focused on student errors much more frequently. Ms. A during the year before did not bring attention to student errors during the first day and only during one activity on the second day. Another change was that student group discourse occurred during the activities, whereas the year before during the activities students had to work individually.

Ms. A’s longest utterance was 156 words long, the box quote after Vignette 9 down from 200 the year before. Seven utterances were above 75 words, which occurred during the explanation of an activity or an extended explanation or clarification of student thinking. Ms. A averaged 19.9 words per utterance down from 26.5 words per utterance the year before. The student average words per utterance rose from 7.8 the year before to 8.8 and the longest student utterance rose from 56 words to 95 words. During one of the student to student discourses, Casey had four utterances of 50+ words within a short time frame. Fifteen student utterances were over 30 words.

During the first activity of the second day, Casey saw in a previous case that the coefficient of x in one of the factors of the quadratic equation is similar to changing the A term in the expanded form, but he also noticed doing so changed the x-intercept. Casey proposed a method of adjusting the wideness of the graph in factored form while preserving the correct x-intercepts. He really proposed a method of using a factor of the quadratic equation to find x-intercepts. Shelby was confused by Casey’s statement in line 2 and with permission from Ms. A she asked him directly to clarify in line 14. Casey tried to use a few specific examples to explain to Shelby how he figured it out in line 15 and 19. Shelby was not satisfied and several students were confused by Casey’s method in Vignette 11 line 27. Sensing that seeing what Casey was doing might help students
understand, Ms. A had Casey go to the board. During this discussion Ms. A asked a few clarification questions to aid Casey’s explanation of his method and why he used this method.

Ms. A Vignette 11

1) T: I think we’re making it worse instead of better. Casey?

2) S: This is kind of off the point, but ah, I think that we should make it like -3 X and then put the minus 3 [inaudible]. If we do that, that will lead us to have the other X intercept because actually [inaudible].

3) S1: Why is it neg. 2?

4) S: Put the neg. 2 after the four.

5) SS: Yes. Plus 4, 2. Yeah, that’s the one to do.

6) T: That’s the one to do? Ok. What do you think is going to happen?

7) SS: It will be better. It will make it a little skinnier, but it will also make the X intercept neg. 1. Whoa! Oh shucks, we’re close.

8) T: That is better.

9) S: Can we move it down just a little bit.

10) T: How do I move it down just a little bit?

11) SS: Subtract... [inaudible]

12) T: Just a second – Casey? Casey, Shelby has a question for you.

13) S: I don’t understand why you changed -3x minus three – how did you get that?

14) S1: Oh, boy. When you put to the neg. 4 part it’s neg. 4X minus 1, so the X intercept was neg. ¼, so I figured that if it’s like 4 it would have one divided by four...

15) T: Where are you getting one divided by four?

16) S1: Factored it was -4X minus 1. That’s why it was the one divided by four.

17) T: Oh, ok.

18) S1: So I figured that’s the number for the one, so when I did that I used three instead.

19) S: I don’t get it.

20) T: Shelby’s shaking her head no. Try again.

21) SS: I get part of it, but not... Did you write it on paper, probably...? Like why is it a minus 3? What if you set it equal to zero? Like then you get to write...

22) T: Are you saying that you were thinking of it in expanded form first?

23) S: No. I was thinking of it factored form.
24) S1: I don’t get stuff in factored form, like how to change the quadratic if you have fractions.

25) T: That’s why we’re playing with it today, so we can figure that out. So Casey, what were you saying again?

26) S: Ok, I knew that pretty much whatever is minusing X is like what is suppose to be on like - I don’t know how to really explain it, but like the top of a fraction what would be the X intercept is divided by is what X is divided by, so I knew that this one is three, the neg. three minus of the X... And I figured out on top the fraction and I took the neg. 3 divided by X underneath it. That would equal one well neg. one. Are the other x-int.

27) SS: Wow. Wow. Huh? How’d you get that?

28) T: You want to show us on the board?

29) S: Please, please go up there.

30) T: That’s really amazing when you think about it. I’ve never, ever thought about it that way, but I can see what you’re getting at.

31) S: The teacher has been surpassed.

32) T: All eyes up on the board, please. Here we go.

33) SS: Write neatly. Like a girl; write like a girl.

34) S: [inaudible] this is pretty much the rest of it, so I figured out how to make it a little bigger.

35) T: So you’re saying the neg. 3 is telling you how big it is. The Y, ok.

36) S: And I saw from the neg. 2 that [inaudible] the four it would be a little thinner, so I made it that, and I wanted to make the other X intercept 1, neg. 1, so I thought that if I could put this there it would make it bigger. I put the 3 right there....

37) T: What made you think of doing that? Why are you doing that?

38) S: Because I noticed on the other one that we did, that was neg. 4X minus 1, and it said in that one that actually the intercept was ¼ and I was trying to figure out where we got the ¼ from, and so I figured that would be plus 1 in there and 4 in there and it would make it 1/4 so I figured that if I did make that 3 it would be 1/3.

39) SS: Cool. It really is. [inaudible] divide it by 3 to make it even. [inaudible]

40) T: Shh.

41) S: Why is the neg. 1?

42) S1: What?

43) S: Just at the right...

44) S1: The dividing part? The number that I think is being multiplied by X, the neg. 3, I put that on the bottom of the fraction, and then the number that’s being
minused from X would be on the top. I found out if there was four and 1 it equals \( \frac{1}{4} \).

45) S: So it equals one?
46) SI: Yup. That’s what I’m trying to do. And so...
47) SS: I don’t get it. Oh.
48) T: Casey, how can we test your theory? What did we do to this...? Let’s suppose that....

**Questioning.** Ms. A questioned students to elicit their understanding by asking in different manners, such as asking them how to make a parabola a better fit, asking them to predict what will happen, asking how a student determined a result, asking students to compare different student representations, asking “what made you think that,” “How would you start figuring out what you’re going to do” etc. Ms. A did ask procedural questions as in Vignette 10 lines 15 and 23, but these questions were part of a bigger discussion leading to the connection between the x-intercepts and the B term.

Ms. A started by asking the students to compare the highlighted equations to the unhighlighted equations in Figure 5.7 (page 231) during the second activity of the second day. She clarified the question further in line 1 Vignette 11 by stating that the students in the unhighlighted group did almost the same thing, and the two highlighted equations were similar. Taylor responded with a correct observation that the two highlighted equations' A terms were too small. In a similar fashion to the year before, she did not respond if the statement was correct, rather she pressed for more differences in line 5. Ms. A drew the students’ attention to the B term and proceeded to ask them what they thought it did to the graph starting in line 9. Ms. A continued to question multiple students about what made them think which graph belonged to the equation with the .2x.
The students responded with several different ways of describing the graph, its skewed (line 10), not really centered (line 13), the one of the right (line 15), more uneven line 23, and scooted over (line 28).

Ms. A Vignette 12

1) T: Alright, all eyes up here. Much better this time. Let’s talk about what you think… Quiet please. Let’s talk about what you think might be the difference between this group of equations and these two equations? What do you think might be the difference between those two? Because these guys almost did exactly the same thing, and I think these guys equations are going to be very similar as well. Taylor, what are you thinking?
2) S: I think their A value was too small.
3) T: Too small?
4) S: Yeah.
5) T: You think that’s the only difference?
6) SS: Yeah. There should be a B. Maybe a C.
7) T: Let’s take a look at those two equations and see. So let’s see, it was .15X sq. minus 8; .2Xsq minus .2X?
8) S: We can’t see them.
9) T: Oh, sorry. There you go. So the second one had a minus .2X in it. They’re both about the same place with their C values and their A values. What do you think…? I’m not going to tell you which one it is… But what do you think adding in the minus .2X did to the parabola?
10) S: Made it skewed.
11) T: What do you mean, Mason?
12) S: Ah, by adding – they added B, just like last time. It’s the same problem. By adding B it would make it move in the direction...
13) SS: It’s not really centered.
14) T: So which of these parabolas do you think it is?
15) S: Probably the one on the right.
16) T: This one or this one?
17) SS: No, that one. Yeah.
18) T: You think it’s that one?
19) S: Ah, the other one.
20) T: That’s that one. What made you think it was that one?
21) S: Because… I [inaudible].
22) T: Sure you were. What was in your thinking that made you think it was that one?
23) S: Because it seems like that one, it seems like it was even... Yeah, I guess now
that I look at it, it looks a lot more uneven. So I just didn’t look at it long enough.
24) SS: [laugh]
25) T: So you were looking for un-evenness? In what way.
26) S: Yeah. In the other one you can see that it’s actually scooted over more than
this one.
27) T: This one?
28) S: Yeah. See, it’s more scooted over.
29) T: Ok, so what was throwing you was this one was just farther to the right, but it’s
still... Ok, I see what you’re saying. What are you thinking, Casey?
30) S: That one that looked like it was to the right had the B value, but they didn’t
have the smaller A value looked wider, and it looks like it was [inaudible].
31) T: So I think it’s a combination of what you and Mason were saying. Ok, good.
Let’s get rid of these two guys...

Casey’s method of finding x-intercepts prompted Ms. A to have the students check it. She
tried different values for the parameters for a factor of the quadratic and had students use
the method to predict the x-intercept and then verify the result with the graph.

**Choice of examples.** Ms. A’s choice of examples seemed to be either selected ahead
of time like activities 1 and 2 on the first and second days, chosen in class from a
predetermined list like activity 3 on both days, or generated on the spot like activities 5-8
on the first day. The purpose of the examples seemed to be to help students make
connections between the graph of a parabola and the equation of a parabola by either
going from the graph to the equation or vice versa.

The examples of the parabolas used in the activities on the first day were limiting and
seemed to add to the difficulty students faced the second day. While the examples were
limiting, they did include important cases. The first three activities included non-integer
coefficients of the squared term as well as non-integer y-intercepts. Ms. A also made sure
that students saw the graphs and related equations of parabolas that had different ways of
crossing the x-axis including crosses once, twice, or not at all. Beyond this, the variety of examples was limited. All of the parabolas opened up. The pictures that the students had to find best fit parabolas for were all centered on the y-axis, so the students were not able to find a B term by guessing and checking. After the first three activities, the rest of the examples on the first day all had an A term of one. Three activities had a non-zero B term, but only during the fourth activity did the class discuss the B term. One of the purposes of the fourth activity was to discuss how to find the B term, but the method discussed can only work with an A term of one. The first four activities students went from graph to the equation of a parabola. Students knew what the B term did as evidenced by their claims of it skewing the graph, making it uneven, or scooting the graph over. But they did not have a chance to go from the graph of a non-centered parabola to its equation. Activities 5-7 changed the focus to the factored form for the equation of the parabola rather than the standard form and as mentioned above the coefficients of the x’s in the linear factor were always one. Activities 5 and 6 went from algebra to graph, which the students seemed to have discussed in previous lessons. So students were not able to see a combination of non-centered and a wider or skinnier parabola. Ultimately, students were not able to see an upside down non-centered parabola with a non-one “a” term on the first day.

The examples of the second day remedied the limitations of the first day with the first activity. The students had to find the best fit parabola to an upside down non-centered skinny McDonalds’ arch. As evidenced by a student’s claim of not getting stuff in factored form (line 25 Vignette 11), the lack of examples from the first day appeared to
cause students trouble on the second day, since they never discussed how to combine the factored form and making the graph stretch or shrink. The lack of discussing this combination was remedied during the 23 minute classroom discourse during activity one, including the student to student discourse about a student generated method of stretching or shrinking the parabola while preserving the x-intercepts in Vignette 11. The students seemed at least initially to get the combination after the 23 minute discussion as evidenced by all the student groups’ graphs fitting the next arch very well as depicted in Figure 5.8.

![Image of graphs](image)

**Figure 5.8: Students’ best fit parabolas for the second activity of the second day.**

However, the initial understanding of the combination seemed to be lost during the third activity. The picture on the left in Figure 5.9 shows the parabola to be fitted, the uppermost one, and the picture on the right shows the best fit parabolas submitted by the students. The student submitted parabolas were “all over the place.”
Missed opportunities. A major source of missed opportunities that occurred the year before disappeared during the second year largely due to Ms. A focused attention on student error. However, other sources of missed opportunities cropped up during the second year, such as not making connections from different activities, limiting choice of examples, and not discussing one of the forms of the equation of parabolas fully.

For activities 1 and 2 for the first day students had to find best fit parabolas that matched the shape of the top and bottom of a necklace. The top and bottom had very similar shapes being nearly parallel. However, during the discussion of the second activity neither Ms. A nor the students mentioned the parallelness. Ms. A did not have students compare the better fitting equations students found from activity 1 with the better fitting equations from activity two, which did not give students the opportunity to notice that the equations are similar except that y-intercept was different by about 3. Ms.
A missed an opportunity to show students an efficient method of finding the equation of a parabola if you know the equation of a similar one as well as not making the connection that parabolas with the same “a” term all have the same shape just moved about the xy plane. A similar missed opportunity occurred during the second day after the second activity. The picture students had to find the best fit parabola for the inner right arch of the McDonalds’ sign, which is a reflection about the y-axis of the picture the students had to find in the first activity. However, no mention that the two pictures are reflections of one another occurred. Also, no comparisons of the better fitting equations of activity 1 to the better fitting equations of activity 2 were made preventing students from having the opportunity of seeing that the equations were exactly the same except that the constant terms of the linear factors had opposite signs. Ms. A missed an opportunity to show students that changing the sign of the constant terms in the factored form reflect a parabola over the y-axis further emphasizing the object nature of the quadratic equation.

While Ms. A’s choice of examples did include interesting ones like parallel parabolas and reflections, the choice of examples of the first day were limiting as discussed above. Beyond just limiting examples, the combination of choice of example along with choice of task seemed to cause problems during the second day. After discussing how to find a non-zero B term in activity 4, students were not given the opportunity to find the equation in the standard form given the graph of a non-centered parabola. In the beginning of Vignette 13, the class discussed which parabolas better fit the picture. One student noticed that one of the parabolas was close to the picture but needed to be taller by grabbing it and pulling it up, lines 6 and 8. After the students declared another parabola to
be pretty close, Ms. B stated that it had the wrong x-intercepts and asked the students who submitted it to explain how she got her equation. Perhaps because of the incomplete choice of examples the day before, Sarah made an incorrect connection in Vignette 13 line 12. She seemed to have made the connection that the opposite of the constant term of a linear factor of a quadratic equation is an x-intercept as it happened correctly in all cases in the previous day. However, this connection is only true when the coefficient of x is one. Sarah knew that the coefficient of at least one factor should not be one because the graph needed to be taller. Sarah may have tried to explain that the smaller the number the wider it is in line 24. Ms. A’s choice of examples from the first day potentially hindered students by allowing them to make generalizations without sufficient cases.

Ms. A Vignette 13

1) T: Ok, so it looks like the B term does have an effect on that. Ok, let’s take those two off... Ok, let’s take a look at some of these other ones.
2) S: We never figured out how to make it wide.
3) T: Ok, there we go. Let’s go to one that looks like it’s really close.
4) S: The first one is close.
5) T: What about that one?
6) SS: Yeah. The first one – what about the first one? I don’t know how to make it taller. Neither do I.
7) T: What do you mean taller?
8) S: Like go upward. Grab it and pull it up.
9) T: Ok, that looks like probably the closest... Well, no...
10) SS: I think that other one. That’s good. That was the closest.
11) T: The problem with this is that it doesn’t have the X intercept correct. Let’s take a look at this one because it’s in factored form. We’ve got – do you mind if I show whose it is?
12) S: No.
13) T: Ok, Sarah. So Sarah, we’ve got a negative 2X minus one. Where did you get the minus one from?
14) S: The X intercept.
15) T: Ok, so you said that it went thru negative what?
16) S: Two.
17) T: Ok, why do you think it’s not going thru neg. one here?
18) S: Because the X intercept [inaudible]
19) T: Which was what?
20) SS: Over two. X is positive. Why is the second one positive and the first one negative? Because [inaudible]. I’m confused.
21) T: What are you confused about?
22) S: I don’t even get why I did that.
23) T: You don’t know why you did that? Ok, Casey, can you help?
24) S: You have to make one neg. X because if you had both neg. X then it would be positive X. So you can only have one X negative.
25) T: Does that make sense? Ok. So what made you think of the idea of putting a neg. 2 in front of the X because normally we have X plus something, X minus something. So what made you think of putting the number in front of it?
26) S: Because the smaller...
27) T: Ok, but what made you think that would make it smaller?
28) S: Because it just does.
29) T: Because it just does? Ok... What does putting that neg. 2 there – how would that affect your equation when I ask you to change it to expanded form?
30) S: You have to double it.
31) T: What do you mean, double it?
32) S: You have to double everything that you – when you multiply the thing thru instead of [inaudible] 1 it would be 2.
33) T: Ok, so what do you think we would have normally multiplied that X term by would be twice as thick is what you saying?
34) S: Yeah.
35) T: Ok, good. Alright.

Ms. A did not have a discussion about how to make a parabola wider or thinner using the factored form on the first day. Several students were able to figure out that changing the coefficient of x in one of the linear terms of the quadratic equation can make the graph taller. However, this method of making the graph taller changed one of the x-intercepts if not compensated for. Casey determined a method for compensating for the change in the x-intercept as he explained in Vignette 11. Ms. A did take the opportunity
to connect Casey’s method to a more standard method in Vignette 14 as she led the
students through seeing that setting the linear factors equal to zero returns the same
results of Casey’s method. Ms. A pointed out in lines 15 and 17 that using a general
method from the day before that the x-intercept was opposite the constant term of the
factor by saying that x-4 was a factor and 4 was an x-intercept and that -3x+3 was a
factor but -3 is not an x-intercept. Ms. A had students determine why that was not the
case in lines 17-24, by looking at setting the factors equal to zero and solving for x. Ms.
B connected what was discussed to Casey’s method in line 32 and Casey affirms that
what was discussed was what he did earlier in line 38. However, Ms. A did not have the
students completely factor -3x+3 to get -3(x-1), which reveals that the x-1 the students
seemed to expect to be present really is. By not factoring completely, Ms. A lost an
opportunity to show the connection that r is an x-intercept of a parabola if and only if, x-r
is a linear factor of the parabola. Factoring this quadratic equation could have led to the
more efficient general factored form of y=a(x-b)(x-c) rather than a more cumbersome
y=(x-b)(ax-d) that is not completely factored. In the efficient general form, each
parameter changes the graph in a specific manner, the A term makes it taller, shorter,
and/or reflects the graph about the x-axis, and the b and c terms change the x-intercepts.
In the not completely factored form, the “a” term does everything the “a” term did in the
factored form but also changes one of the x-intercepts. The efficient general form was not
discussed when the topic arose for the second time in Vignette 14 nor was it discussed
when the topic originated in Vignette 13, nor when Casey explained his method in
Vignette 11. The discussion of the efficient general form could have made finding the
best fit parabolas easier for the students and more importantly it could have illustrated the connections between the factored algebraic representation and the graphical representation better than the cumbersome form.

Ms. A Vignette 14

1) T: So far, look up here. It’s kind of the same thing Casey was saying, just a different way to look at it. So we’ve got neg. 3x plus 3 and we have the X minus 4. Let’s look at a simpler version first. Let’s suppose we have what we’ve been doing in the past. X plus 2 times X minus 4. What are we setting these equal to?

2) SS: Zero.

3) T: Zero. And we solve and we get neg. 2. X minus 4 equals zero, so X equals...

4) S: Four.

5) T: Four. So we just looked at those and we said oh, we’re just going to say the opposite. So four and neg. 2 is where it crosses the X axis, is that correct? AJ?

6) S: Yes.

7) T: Yes, ok. So, we come over here and we go X minus 4 is equal to...


9) T: Well what do we always say it’s equal to?

10) SS: Zero.

11) T: Zero, so X equals what?

12) SS: Four.

13) T: And did it cross the X axis at 4?

14) SS: Yeah.

15) T: So we got used to saying, well, you can just look at this and we can say it’s just the opposite so it crosses at 4. So I followed that thru and said I’m going to look at this and it’s plus 3 I’m going to take it the opposite which is what?

16) SS: Negative 3.

17) T: Neg. 3. It doesn’t cross the X axis at neg. 3. Why? Yes, ma’am

18) S: Because you have a negative three before you add?

19) T: What difference does that make?

20) S1: Divide it by neg. 3.

21) S: Because you have to divide it by neg. 3 to make X alone.

22) T: So you’re saying I have to take this one and set it equal to...

23) S: Zero.

24) T: Zero as well. So I get neg. 3x and I have to take away 3 from both sides, is that correct?

25) SS: Yeah.

26) T: Ok. So those go away and I get neg. 3x equals neg. 3, but X still is [inaudible].

27) S: Divide by neg. 3.

28) T: Divide by neg. 3 and we get what? X equals...

29) SS: One.
30) T: One. Is it going thru one?
31) SS: Yes.
32) T: Ok, is that kind of what Casey was trying to tell you?
33) SS: Yes.
34) T: Yeah. Does that make more sense now?
35) SS: Yes.
36) T: I’m getting yes’s but not...
37) S: Zombie answers.
38) S1: That’s exactly what I was doing.
39) T: That’s exactly what you were doing, I just had never heard it explained that way before.

The purpose of activity 4 on the first day as stated by Ms. A was to make the parabola perfect without testing it ahead of time. The class discussed ways to determine the A term, B term and C term of the standard form of the quadratic equation. The students quickly remembered that the C term is the y-intercept. Vignette 10 is part of the class discussion about determining the B term. They determine that the B term was negative the sum of the x-intercepts, which only works if the A term is one. Ms. A even left the students thinking this method worked for any quadratic equation in Vignette 10 lines 51 and 52. However, Ms. A never discussed a way to make the A term perfect. Ms. A spent a short amount of time discussing the A term during activity 4 in Vignette 15. A method of pinpointing the A term was never discussed, which is perhaps the reason the student in line 8 does not want to guess and why the students line 9 had widely varying answers. Ms. A left the students with 3, 4 and 1.5 as being possible A terms when she changed the discussion to the B term in line 10. Ms. A did not make the connection that by going to the left or right one from the x-coordinate of the vertex the y-value of the graph will go up or down A units from the y-coordinate of the vertex. Nor did she explain how to plug
in points from the graph in the equation to determine the A term, like the y-intercept into
the factored form and showing that A is one.

\[ y = a(x - 4)(x - 3) \]
\[ 12 = a(0 - 4)(0 - 3) \]
\[ 12 = 12a \]

Even though the above method of finding A is procedural it could have illuminated the
connection between the graph of a parabola and its factored form. Either method of
finding the “a” term seemed would have helped students during the activities of the
second day.

Ms. A Vignette 15

1) T: Let’s start with what we know. Is it pointing up or pointing down?
2) SS: Up.
3) T: Ok, so we know that A is positive. We know it’s a pretty big A, or a pretty small
   A?
4) SS: Small A. Big A
5) T: What do we mean by big, AJ? I’m going to wait just a second because I don’t
   have everyone’s attention. I’m sorry, go ahead.
6) S: The bigger the A the skinnier it is, and it’s pretty thin.
7) T: So what would you guess that is?
8) S: I don’t want to guess.
9) SS: 3. 4. I guess 1.5.
10) T: Alright, and what about, is there a B term?

Technology Use

CCT use was an integral part of Ms. A’s instruction during the second year. Ms. A
only used the CCT Activity Center feature during both days of instruction. The tasks and
discussions during the first three activities of both days would have been difficult to
perform without the CCT. While Ms. A used the CCT during activities 4-8, the resulting
discussions and tasks could have been accomplished by other means, such as a graphing
calculator connected to an overhead projector. Ms. A continued to use the CCT as an information display/gatherer and answer checker from year 1 to year 2. However, Ms. A increased her use of the CCT as a discourse generator from year 1 to year 2.

Perhaps Ms. A realized the potential for the combination of using the CCT as an information display/gatherer and answer verifier and finally as a discourse generator, since that was how she used the CCT for the first three activities of both days. These activities all followed a similar pattern, Ms. A used the CCT to display the object that needed to be fitted, the students used the display of the CCT to verify their best fit parabolas, Ms. A gathered the students best fit parabolas and displayed them with the CCT, she then used this display to verify student answers and to generate discourse surrounding student errors or comparisons of student groups. Vignette 8 is an example of how Ms. A used student errors displayed by the CCT to generate discourse. Vignette 9 is a short example and Vignette 12 is a longer example of Ms. A using the CCT to generate discourse about comparing student groups. These discussions were aided by the fact that the CCT was able to display both the equations and the graphs of student groups. Vignettes 11 and 13 are parts of discussions generated by Ms. A questioning the class about student errors displayed by the CCT. Use of the CCT made these discussions possible/more feasible than without it by being able to collect equations of parabolas from all student groups and displaying them for the whole class to see in a short time frame.

When Ms. A was not collecting data from students, she used the CCT to display graphs or equations for students to discuss in activities 4-8 day 1. She also used the CCT
display to verify predictions of students during these activities as well as to verify
Casey’s method of finding x-intercepts.

Decisions

Ms. A’s goal of eliciting and extending student thinking remained in year 2 as
evidenced by her generating student discourse about comparison, student errors, and why
things are true. However, this goal appeared to be slightly modified in year 2 to focus
more attention on student thinking about other student’s work or mistakes, since that was
a frequent method for Ms. A to generate discourse. Ms. A’s goal of wanting multiple
students to participate remained in year 2, as evidenced by her asking another student to
help explain in line 41 Vignette 10, allowing Casey to take several minutes to explain to
another student his method Vignette 11, or asking multiple students what they thought
about a non-centered parabola in Vignette 12.

Ms. A’s orientation appeared to remain further on the conceptual end of the
procedural/conceptual spectrum as evidenced by her treatment of functions as both
process and object and her asking students to make comparisons, make predictions,
explain what they think, and explain how to fix something and why it needed to be fixed.
The purpose of the first three activities of the first day appeared to be to promote the
conceptual connections between the algebraic equation of a parabola and its graph. Ms. A
had a flexibility moving from conceptual to procedural topics, such as using procedures
to determine equations in graphs to promote precise connections between the graph of a
parabola and its equation in activity 4 (Vignette 10). The goal activities of the second day
appeared to be to make new connections or reinforce the connections of the day before
between the algebraic and the graphical representations of parabolas. During these activities when discussing how to fix a group’s parabola, Ms. A and the students relied on procedural methods of adjusting it, such as Casey’s method Vignette 11 and setting linear factors equal to zero in Vignette 14.

Figure 5.10 presents a flow chart modeling Ms. A decisions during year 2 of the study. The rectangles represent actions taken by Ms. A and diamonds represent decisions. The octagon shape represents a decision that is made possible or easier from the use of CCT and a rounded rectangle is an action made possible or simpler by using CCT.

Two main branches occurred in Ms. A’s flow chart depending on the type of activity she wished to use. If the activity required group work like the first three activities of both days, Ms. A followed the left branch. If the activity required more whole class discussion like activities 4-8 day one, Ms. A went down the right path in the flow chart. Another way to describe the two branches is that CCT use was integral to the left branch and the CCT was only somewhat beneficial or not needed in the right branch. Both branches contain elicit, extend, explain or clarify students thinking elements or have students make prediction cycles (EEEC) within larger cycles of exploring student outcomes or exploring aspects of the task.
Figure 5.10: Decision flow chart depicting Ms. A’s instruction during second year of CCT use.

Vignettes 8 and 9 are short examples of the left branch of the flow chart. After she collected the parabolas from the students she asked about the parabolas, she used a short, 2 to 4 minute, EEECP cycle to determine what the students were thinking before moving to the next activity. During activity 1 on the second day, Ms. A had students work for 10 minutes on finding the best fit parabola for a McDonalds’ arch starting down the left branch. One of the students sent parabolas was upside down and Ms. A asked students how to fix it starting the “anything interesting” cycle in the left branch. After a few
EEECP cycles Ms. A focused on another interesting difference among student outcomes that occurred when a group of students sent a parabola that did not match the picture and was not in the factored form. This parabola prompted Ms. A to ask students to think about the equation and how to fix the graph. Two theories emerged as to how to fix it and Ms. A had students test these theories using multiple EEECP cycles. After the parabola was fixed by the students’ directions, Ms. A began another “anything interesting” cycle by asking students what they thought about another parabola in lines 3, 5 and 9 Vignette 13. In line 9, Ms. A decided to focus on a parabola in factored form with an incorrect x-intercept and began asking students how they found their linear factor of -2X+1. Ms. A then used several EEECP cycles to determine what the group was trying to do and resolved confusion by drawing on students outside the group for help during the rest of the Vignette 13. Ms. A then drew the class’ attention to a parabola that was nearly perfect and asked students how to make it even better starting a third anything interesting cycle. After several EEECP cycles to elicit what students thought about fixing the parabola, Ms. A realized the student suggestions were making the parabola worse in line 1 Vignette 11. The result of Casey’s suggested fix in line 2 was described by another student as “Whoa! Oh shucks, we’re close” line 7. Ms. A heard that Shelby and more than likely others did not understand how Casey knew his fix would work and let Casey take over the EEECP cycles for explaining his thinking with guidance from Ms. A throughout the rest of Vignette 11. At the end of the Vignette 11, Shelby stated she still didn’t understand and Ms. A decided to switch gears by focusing on the prediction part of the EEECP cycle having students make predictions using Casey’s method and potentially increase their
understanding. After the students tested a few predictions using Casey’s method, a student asked if his method worked for the second linear factor prompting a short EEECP cycle focused on prediction. After the students were able to make predictions using Casey’s method for both linear factors Ms. A appeared confident that the students could start the next task ending activity 1 on the second day.

Ms. A followed the right branch of the flow chart during activities 4-8 on the first day as they focused mostly on whole class discussion rather than generating student work for discourse. Ms. A began activity 4 with a long statement of objectives to make it clear to the students that their strategy of guess and check needed to change to a way to make the graph perfect ahead of time starting down the right branch. She then asked the students what they would try so that they knew it was perfect before sending it in. A student replied with “put it on paper,” which was similar to guessing and checking but without the technology. After asking another student what they would do and getting essentially an “I don’t know” response, Ms. A decided that the students needed a new perspective and drew that students’ attention to what they did know. For example, students would know if the graph is pointing up or down in line 1 Vignette 15. Ms. A used a few EEECP cycles to elicit student guesses to the value of the “a” term. The decision of Ms. A to move on to the next important aspect of the task at the end of Vignette 15 is one of the anomalies of the flow chart. Ms. A did not give or have students determine a way to pinpoint the A term as discussed in the Missed Opportunities section. After Ms. A started on the aspect of finding the B term, she used EEECP cycles to elicit what students thought the B term would be. The students knew the B term should be negative since the
parabola was on the right side of the plane. However, they could only guess at its value. Before resolving the issue of the B term, Ms. A quickly resolved the important aspect of finding the C term. Ms. A knew the students needed a new perspective for finding the B term from the random student guesses, and gave them a new perspective first by telling them what the B term was and second by drawing the students’ attention to the process perspective of the graphical representation lines 1 and 3 Vignette 10. Students were able to determine that the -7 for the B term came from the x-intercepts but were unable to explain why in lines 12 through 22. Ms. A determined the students needed another change in perspective to aid their understanding. In line 23, Ms. A drew the students’ attention to setting things equal to zero. Ms. A then was able to elicit finding the factored form from the students in lines 26 to 30. Ms. A used the graph of a student prediction of \((x+3)(x+4)\) as the factored form of \(x^2 - 7x + 12\) to generate more discourse about the correct connection between the x-intercepts and the B term. Perhaps since the focus of the rest of the activities was on the factored form of the parabola, Ms. A did not discuss that the connection between the x-intercepts and the B term only work for an A term of one and started activity 5.

**Insider Perspective**

Ms. A’s post observation interview revealed insight into her decisions as well as illuminated many classroom observations. Ms. A’s goal of eliciting and extending student thinking appeared again in the second year with her focus on letting students see the big picture and enabling students to “make a conceptual leap.” The observation that Ms. A had pre-planned examples, examples from a list, and examples generated on the spot was
verified by Ms. A’s comment of “I just have a variety of examples that are there for me as I see where I need to go, but then I also can deviate from those as needed if I need to as well.” The increase in difficulty in finding best fit parabolas from day 1 to day 2 was intentional on the first day she “wanted them to concentrate on the A term, whether it was negative or positive and concentrate on the C term.” The activities on the second day she “on purpose made it pretty hard today because I wanted them to think about it… they were really struggling with it which is exactly what I wanted them to do and it led to a really great discussions.” Ms. A stated the purpose of the activities was

*for them to be able to look at a quadratic function and be able to pretty much in their mind picture where it would fall on the coordinate plane; where would it intercept on the Y axis, about what size and shape is it, about where its location is, so they could just by looking at the equation do that. I want them to be able to take... look at a parabola and then use some of the data they gather from it to be able to come up with a function or an equation. I want them to be able to work both ways. I want them to be able to take and to do that either in factored form or in standard form.*

In other words, Ms. A wanted students to establish deep multidirectional connections between the algebraic and graphical representations of quadratic functions.

Ms. A’s view of the CCT changed from “Ok, well, basically it (CCT) was as a tool to send data, to gather data from them… besides the obvious (ability) of letting the students confirm their data” (parentheses added) in year 1 to an instruction organization tool that allows the teacher to enable conceptual leaps.

*I would say that with the Navigator I’m able to organize instruction. As I watch them I figure out pretty quickly what they don’t know and what their misconceptions are and so I’ll organize my instruction to pull those out so they can see them and change them and that allows them to make much bigger conceptual leaps because if I don’t know what their misconceptions are and they*
The CCT allowed Ms. A to know when most students understood the content and what potential misconception they had, which in turn allowed her to alter her instruction to address those issues. When Ms. A described how she taught the same lesson without technology, she stated that it would take many days to complete enough examples for students to make generalizations and hopefully conceptual leaps, but with the CCT she “could do that several times during a lesson and before maybe once, but sometimes it took me several days.”

Summary

A summary of the themes that emerged within each of the four main categories of representations, discourse, technology, and decisions will be discussed. Following the summary of the categories, a discussion of answers to the research questions for Mr. A. will be presented.

Segmentation

The manner in which Ms. A segmented her lessons did not change much from year 1 to year 2. Ms. A seemed to segment her lessons by structuring sequences of activities with similar goals to ultimately allow students to make connections between the algebraic and graphical representations, or in Ms. A’s words “make big conceptual leaps.”

Representations

Perhaps due to time constraints and the nature of the activities, Ms. A did not use the tabular representation once during the days observed during the second year. However, the tabular was an integral part of the activities of the first year. While the process
perspective was present during most of the activities of the first year when students were finding points for the tabular representation of a line, it was only the focus of one brief discussion. During the second year, the process perspective played a more prominent role in the discourse when Ms. A focused attention on specific points of the graph of the parabola and finding the x-intercepts of the factored form in multiple activities. The object perspective of the representations of functions remained an important manner for both Ms. A and the students to discuss manipulating both the graphs and equations from year 1 to year 2. Ms. A’s treatment of representations as something to reason with/about, use for explanations, and never as only an end product continued from year 1 to year 2.

**Discourse**

The purpose of Ms. A’s classroom discourse of eliciting and extending student understanding appeared to not change over the years. One change in the discourse was in its focus; Ms. A paid more attention to student errors during the second year. A major change in the discourse during year 2 was the change from only teacher to student, student to teacher discourse to discourse that also included student to student interactions. The manner in which Ms. A chose her examples for the activities of a combination pre-planned, from a list, and generated on the spot remained the same from year 1 to year 2. The combination of tasks of an activity and choice of examples in that activity became more limited from year 1 and 2. In year 1, Ms. A chose a wide variety of examples of important cases to aid students to make connections between the graphical and algebraic representations. In year 2 the choice of examples as a whole over both days had most of the important cases, but the combination of the examples with the tasks prevented
students from being exposed to all important cases such as finding a non-zero B term with a non-one A term. The major category of missed opportunities of not discussing systematic student errors from year 1 did not occur during year 2 because of Ms. A’s specific attention to using student errors to generate discourse. Ms. A explicitly had students discuss multiple activities, however during the second year only implicit connections between different activities were made. The use of EEECP cycles was a prominent feature of Ms. A’s discourse during both years.

Technology Use

The CCT was used throughout Ms. A’s instruction during both years. Ms. A continued to use the CCT as an information display/gatherer and answer verifier both years. However, Ms. A used the CCT more frequently as a discourse generator in combination with the other two uses of CCT in year 2. Ms. A’s view of the CCT changed significantly from year 1 to year 2 from as a tool for displaying information to an instruction organizer that aids students in making conceptual leaps. The evidence of change in Ms. A’s view of the CCT was most prominent in Ms. A’s decisions as discussed below.

Decisions

Ms. A’s goal of eliciting and extending student thinking and having multiple students contribute to the discourse was apparent during both years. Ms. A’s orientation of more on the conceptual end of the procedural/conceptual spectrum appeared to have not changed from year 1 to year 2. Using CCT became an integral part of the apparent
decision making process for Ms. A in year 2. However, Ms. A frequently used CCT in year 1, most of the actions and decisions of Ms. A did not require its use.

**Answers to Research Questions**

**What kinds of representations are used in the classroom by teachers?** Ms. A and her students used the graphical, algebraic and tabular representations where the tabular was treated from the process perspective and the other two were treated almost exclusively from the object perspective. For the first day activities of the first year the students used the tabular representation to find points of an equation of a line, and then they submitted those points to the teacher using the CCT. The points were displayed as well as the graph of the line passing through those points. Students used three different representations of the same line for the activities on the first day first year. During the second day first year the students had to predict the shape of a graph of a line given its equation. Then students had to go in the reverse direction from the graph to the equation. The points were generated by the students, the graphs were displayed using the CCT and the equations were given as part of the task by Ms. A

Ms. A and her students used the graphical and algebraic representations of lines where both were treated from both the process and object perspectives, but mostly from the object perspective. During the first three activities of both days students were given a picture that they had to find the equation of a parabola that fit the picture. They sent the equations to the teacher using the CCT, where the graphs of the equations were displayed and later discussed. The graphs and equations were generally treated as objects to be manipulated Not only did different representations of the same parabola occur, but the
different representations came from multiple sources. The remaining activities of the first day students were either given a graph or an equation of a parabola or had to determine how to find its equation or predict the shape of its graph. During the discussion of these activities, the representations were treated from the process perspective. Different representations of the same parabola were present but only from one source from the CCT display.

The number of kinds of representations decreased from three year one to two year 2. Both years had multiple representations of the same line/parabola in each activity. The trajectory of representation use depended on the representation. The tabular representation went from multiple sources treated from the process perspective to no occurrence. The graphical representation went from single source treated from the object perspective to multiple sources treated from both perspectives. The algebraic representation followed the same trajectory as the graphical representation.

**What is the quality of discourse about representations or use of representations by the teacher?** The discourse surrounding the activities during year 1 typically involved Ms. A explaining the task and then having students discuss the task or the results of the task. Ms. A frequently used questions to elicit, extend, explain, or classify student thinking or have students make predictions or EEECP to aid students in making connections between the different representations of lines. The questions could be conceptual or procedural depending on the needs of the task or student. The apparent point of many activities was to promote discourse to expose student thinking.
The discourse surrounding the activities of year 2 was very similar to the discourse of year 1. Ms. A would typically explain a task and have students discuss the task or the result of the task. However, during year two Ms. A focused more discourse on student generated outcomes of the task versus a teacher displayed outcome of the task. The EEECP cycles were a prominent part of her discourse. However, she allowed a student to take over an EEECP sequence in year 2, but not in year 1. Her questions could elicit either conceptual or procedural responses depending on the flow of the discourse.

Important aspects of Ms. A’s discourse changed like the focus on student generated outcomes and a shift in control from strictly the teacher to mostly teacher and infrequently to student. The EEECP cycles and the types of questions asked remained the same as well as the quality of discourse remained largely the same.

**For what purpose do teachers rely on multiple representations in classroom discourse?** The purpose of the representations occurring in the activities of the first day first year seemed to be to allow students to create graphs of line, and then discuss connections between the graph of the line and the equation of that line. The questions Ms. A asked had students reason about representations and to explain with them. While the students enacted procedures on representations the goal was to never produce a representations’ end product, but rather produce something to be discussed. The second day was similar; the students had to make predictions about or reason about representations during the first segment of class. During the second segment of the second day students had to reason about how to find the values for the m and the b in the \( y=mx+b \) form. During the last segment the students performed procedures on the
representations, the slope formula, and discussed important aspects of the formula like does the order of points used matter. The purpose of the presence of representations during the first three activities of the first day second year was to “let students focus on the connection between the A term and the shape of the parabola and the connection between the C term and the y-intercept.” The students were given a variety of examples to help establish that connection by reasoning about student generated representations. During the fourth activity, a graph of a parabola was given so that the students could make more precise connections between the algebraic representation’s standard and the graphical through discourse and potentially enacting procedures on the representations. The purpose of representations in the remaining activities of the first day second year appeared to be similar to the fourth activity except for the factored form. These remaining activities excluded an important case, the non-one A term. But the purpose of the activities of the second day year 2 was to remedy the situation allowing students to make connections to a more general factored form and more general graphs of parabolas. Ms. A claimed that the ultimate goal of the presence of representations on both days was for students to be able to “look at a quadratic function and be able to pretty much in the mind picture where it would fall on the coordinate plane.”

The purpose of representations of both years appeared to be to allow students to make deep connections between different representations of lines/parabolas by performing procedures on them, reasoning about them and to explain results. The trajectory for the purpose of the presence of representations remained consistently high year 1 to year 2.
What is the relationship between teacher use of CCT in algebra classrooms and the growth of teacher and student choice of representations of linear functions as manifested in the classroom discourse? Ms. A’s change in view of the CCT as a display tool to an instruction organizer appeared to influence different aspects of her teaching. While Ms. A’s goal of eliciting and extending students thinking or having multiple students participate did not change from year 1 to year 2, the different focus of her CCT use enabled different methods of attaining that goal. Her increased focus on discussing student generated representations appeared to be influenced by this change in view. This focus allowed for different kinds of connections to be made by being able to explore more student thinking.

The use of CCT by Ms. A increased the amount and sources of representations. During the POI of the second year, Ms. A described how she previously and without CCT taught a similar lesson to the one observed that year. She explained that the activity was laborious for the students since she could only put one example on the board at a time. The students would have to spend a lot of time graphing a “bunch” of equations that have something in common. Eventually after seeing enough families they could try to determine generalities. Students were only able to see one graph produced by the teacher on the board at any given time without the CCT, but with it students were able to view several graphs and their related equations from different sources such as the teacher and different students.
Figure 5.11: Ms. A’s Decisions Flow Charts during First and Second Years.
The major influence of extended CCT use on Ms. A was the change in her decision making. She appeared to have created separate but similar ways of engaging students in discourse depending on if CCT use was central to generating that discourse. She focused on interesting similarities and differences of students generated representations if the CCT was central. Figure 5.11 shows Ms. A’s flow charts for both years.

The rectangles in the flow chart represent actions of the teacher, while rounded rectangles represent actions enhanced by or made possible by the use of CCT. Diamonds represent decisions made by the teacher, while octagons are decisions enhanced or made possible by the use of CCT. The double sided arrows indicate cyclic actions and decisions.

The decision flow chart for year one has one major decision and two minor ones. The minor decisions are to determine if enough is present to discuss and to determine if students need to submit work for the activity. The major decision in the year 1 flow chart deals with students understanding and participation which generates Ms. A’s EEECP cycles. While the major decision of year 1 remained in year 2, two other major decisions emerged during year 2. The major decision of determining if there was any interesting student generated outcomes to discuss seemed to be the result of the shift in view of the CCT. The second major decision of determining if other important aspects of the task need to be discussed was not entirely new in year 2, since other important aspects of a task were generally separate but related activities. For example, the important aspect of negative slope was addressed by a new activity year 1, whereas determining the A, B, and the C were different aspects of the same activity in year 2. CCT use was not central to
any of the decisions in the year 1 flow chart, as the CCT seemed to only influence Ms. A’s decisions once during Vignette 6. CCT use seemed central to her ability to decide if any student generated outcomes were interesting, since otherwise it would be laborious for her to see them all at once especially on the same screen.

Ms. A collected data from students frequently in year 1 and usually verified student data with a graph. Usually nothing more was done with the students’ data, as the resulting discussion revolved around the graph of the line Ms. A displayed. The exception was Vignette 6, when Ms. A asked how a student could get some of the wrong points that were being displayed. The CCT use was central to the decisions and actions of the first three activities of both days of year 2. The action of having students discuss different student generated outcomes would not have been possible without them being able to view these different outcomes, which would have been difficult and laborious without the CCT.

While using CCT over time, Ms. A increased the amount and sources of representations present, increased the methods of attaining her goal of eliciting and extending student thinking, and increased the number of decisions made. The actions she performed and the actions she had the students perform increasingly were benefitted by the CCT. The next chapter presents the case of Ms. B
CHAPTER 6: CASE STUDY OF MS. B

This chapter is a report of the case study of Ms. B’s classroom observations. The case study is comprised of a section on year one, one year of CCT use, and year three, three years of CCT use in the CCMS project. For years one and three a description of an outsider’s perspective as observed from the classroom observations, a description of the insider’s perspective derived from Ms. A’s post observation interview (POI), and analysis are given. A summary of the themes and patterns across the years observed finishes the case study.

Introduction

Ms. B graduated with a major in math, a minor in education in 1970. She earned 32 credits towards masters in mathematics education but never graduated. She has licensure is in 7-12 grade mathematics. She taught 14 years full time and 7 years part time. She had no response for how many years of algebra teaching experience she had.

The school at which Ms. B taught is located in rural area of North Carolina with about 526 students. The student population was composed of no Native Americans, 13% Asian, 5% Hispanic, 9% African American, and 73% Caucasian. Twenty nine percent of the students receive free or reduced price lunches.

The measurement of teachers’ practices and beliefs about mathematics teaching and learning within the Teacher Instructional Practices and Beliefs (TIPBS) survey was used
to differentiate participants and used for participant selection criteria. Table 6.1 shows Ms. B’s average scores of different constructs of the TIBPS survey and following is an interpretation of the scores.

<table>
<thead>
<tr>
<th>TIBPS Constructs</th>
<th>Average Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reform Classroom Practice</td>
<td>3.2</td>
</tr>
<tr>
<td>Tech Use Reform</td>
<td>3.0</td>
</tr>
<tr>
<td>Strategy Discussion</td>
<td>4.2</td>
</tr>
<tr>
<td>Data Analysis</td>
<td>3.0</td>
</tr>
<tr>
<td>Explanation/Justification</td>
<td>4.6</td>
</tr>
<tr>
<td>Ref Class Discourse</td>
<td>3.3</td>
</tr>
<tr>
<td>Classroom Discourse</td>
<td>4.0</td>
</tr>
<tr>
<td>Assessment Practice</td>
<td>3.1</td>
</tr>
<tr>
<td>Teacher Efficacy</td>
<td>2.9</td>
</tr>
<tr>
<td>TB on Math</td>
<td>3.1</td>
</tr>
</tbody>
</table>

Table 6.1: Ms. B’s TIBPS average scores.

For questions within the constructs of Reform Classroom Practice, Tech Use Reform, Strategy Discussion, Data Analysis, Explanation/Justification, Reform Class Discourse, and Assessment Practice, responses were scored on a 5 points scale ranging from 1) never, 2) once a month, 3) once a week, 4) more than once a week, and 5) all or almost all lessons. For the Classroom Discourse construct, teachers were given descriptions of two different scenarios of class discussions, where one was teacher led with simple questions and the other with more difficult questions arising from students. Then they were asked which scenario their instruction tended toward. The Teacher Efficacy and Teacher Beliefs on Math constructs were scored out of four points with choices of 1) strongly disagree, 2) disagree, 3) agree, and 4) strongly agree. Some of the Teacher
Efficacy questions asked teachers to rate how prepared they were for certain situations on a four point scale.

Ms. B claimed using *reform classroom practices* such as working in groups, using activities with concrete materials, or using concepts to solve applications about once a week on average. She also claimed to use technology in reform ways such as using technology to demonstrate mathematical principles, or to develop conceptual understanding about once a week on average. She reported that her classrooms engaged in *strategic discussions* activities such as encouraging whole class discussions about student incorrect errors, or comparing or contrasting a variety of student responses more than once a week on average. Ms. B stated that she used *data analysis* in class to aid student understanding about once a week on average. She alleged that she required students to *explain or justify* their answers, mathematical behaviors and models almost every day. When determining if her *classroom discourse* was closer to the traditional classroom or the reform classroom, she claimed her discourse tended towards the reform classroom discourse. Ms. B reported that she used *assessment practices* to inform her instruction about once a week on average. She claimed that she was on average moderately prepared for reform practices to assess her *teacher efficacy*. She reported that she on average agreed that mathematics is a way of thinking rather than a collection of facts.

Table 6.2 shows the type and length of the video and audio data for each year observed for Ms. B.
<table>
<thead>
<tr>
<th>Data</th>
<th>Minutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom Observation 5/8/2006</td>
<td>75 minutes</td>
</tr>
<tr>
<td>Classroom Observation 5/9/2006</td>
<td>83 minutes</td>
</tr>
<tr>
<td>Classroom Observation 5/10/2006</td>
<td>81 minutes</td>
</tr>
<tr>
<td>Classroom Observation 5/11/2006</td>
<td>83 minutes</td>
</tr>
<tr>
<td>Post Observation Interview</td>
<td>12 minutes</td>
</tr>
<tr>
<td>Classroom Observation 5/12/2006</td>
<td>82 minutes</td>
</tr>
<tr>
<td>Post Observation Interview</td>
<td>15 minutes</td>
</tr>
<tr>
<td>Classroom Observation 12/11/2007</td>
<td>55 minutes</td>
</tr>
<tr>
<td>Classroom Observation 12/12/2007</td>
<td>52 minutes</td>
</tr>
<tr>
<td>Classroom Observation 12/13/2007</td>
<td>50 minutes</td>
</tr>
<tr>
<td>Post Observation Interview</td>
<td>17 minutes</td>
</tr>
</tbody>
</table>

Table 6.2: Ms. B’s case data.

**Year One: First Year of CCT Use**

**Overview**

During the first year of the study, several features of Ms. B’s instruction emerged from the classroom observations. Two of the major representations, graphical and algebraic, of functions were used each of the five days of instruction. Representations were treated almost exclusively from the process perspective with occasional hints of the object nature. The purpose of representations appeared to be to perform tasks on them or to create representational end products. The purpose of discourse appeared to be to mainly get the correct answer or correct solution from the students. A few important exceptions occurred, such as when she asked students about their understanding on the fifth day. Ms. B willingly let students take charge of the discourse on occasion, even if only briefly. Ms. B used technology frequently during the five days mostly for information gathering and display. However, during the second day, technology was used briefly for generating discourse. Ms. B appeared to have a mostly procedural orientation
to mathematics. However, if students did not seem to “get it,” she seemed willing to alter her plan for teaching. This alteration of her planned instruction did not occur in that moment but rather in subsequent days. Ms. B seemed to have limited mathematical knowledge that impacted what happened in the classroom and the decisions she made. However, even if she didn’t appear to fully understand them, she was willing to try suggestions from students.

Segmentation

Day one. Each day of year one lasted for about 80 minutes. The first day of year one, May 8, 2006, was segmented into three major components. The class started with what Ms. B called a class opener (CO), she went over the previous day’s quiz, and then she started a new activity. For the CO, Ms. B gave her students a worksheet that they completed in about 15 minutes. Ms. B spent about 6 minutes asking the students for answers to the CO. This day’s CO appeared to cover an array of topics, percent increase, probability, and scientific notation. The last topic led to a brief explanation of the base-ten number system. Then she spent about 6 minutes discussing what issues students had with the CO. After the CO, Ms. B spent about 30 minutes going over the 12 question quiz from a previous day, presumably the Friday before. She took 8 minutes for the first question of how to graph the line for the equation \(-7x+7y=49\). She showed the class three different methods to do so, finding the x and y – intercepts, getting the equation into slope-intercept form, and picking points on the line and plotting those. She spent about 2 to 3 minutes on the rest of the problems from the quiz. The quiz questions ranged from graphing a line given its equation in multiple forms, how to find the slopes of lines given
points, and finding equations of lines given points. The questions about finding slope
given two points lead to a brief discussion of the difference between zero slope and
undefined slope. The final segment of the first day of her first year was comprised of Ms.
B giving a demonstration of how to use a calculator program and then letting students
play with it. The program was designed to let students practice finding the slope and y-
intercepts of a line given only its graph. The program would randomly create a line and
graph it. Then it asked the user to enter a value for the slope of the line and the y-
intercept. After which, the program graphed the line the user inputted. Students were
supposed to compare the graphs and the program asked the user to try again or start a
new problem.

**Day two.** The second day of year one included four major segments. The first
segment was another CO that took about 20 minutes. However, Ms. B never went over
the content of this CO. The next 21 minute segment was the main activity for the day.
Ms. B used the Activity Center (AC) to display a picture of a stair rail against the
backdrop of a dune in a desert. The picture contained multiple objects with straight edges
and Ms. B had students submit equations to best fit these edges. The first edge she had
the students find a best fit line for was the stair rail. Ms. B let students work for about 4
minutes, before discussing for 6 minutes the lines that did not match. Another straight
edge in the picture was the posts supporting the rail. She asked students to submit their
best fit lines, but admitted that they might not be able to. After discovering the students
could not submit the lines she asked the class what the equation should be. She repeated
the question for the other vertical post taking about 3 minutes for both posts. Since the
multiple students’ lines did not match the picture during the first activity, she wanted to try other slanted lines. Ms. B had them find the slanted hill in the picture the stair rail was on and then the shadow of the stair rail. The class took about 5 minutes to find the best fit lines and discuss the lines that did not match for each of these straight objects. After the main activity, Ms. B gave them a quiz that the class finished in 25 minutes. During the last segment of day two year one, Ms. B provided motivation for why finding the equations of lines is important. She used about 7 minutes to explain that a major part of science is making sense of data and that finding equations to best fit the data can allow for extrapolation. Ms. B spent the last 7 minutes of class setting up an example of data that was not quite linear, cost of postage given the year, and how to graph it. The class ended before Ms. B could explain or have the class find a method to find a best fit line.

**Day three.** The third day of year one was segmented into six pieces. Ms. B started again with a CO that lasted 23 minutes. As in the second day, Ms. B did not go over the CO. The majority of the class did not do well on the quiz given the day before. Ms. B called out the students by name who got a C+ or higher on the quiz. She had these students group together and she gave them material to move on to the next topic in the class. Addressing the rest of the class, Ms. B stated she needed to re-teach the chapter. Ms. B used the next 8 minute segment of the second day to re-teach finding the slope of the line given two points. Using the overhead, she plotted two points on an XY plane and demonstrated the two methods for finding slope, the graphical way and the algebra way. She repeated using both methods for another set of points. Ms. B gave the students 30 minutes to work on problems 1-15 to practice finding slope of a line that they had to
finish as homework, which was the third segment of the day. The rest of the third day was split into 3 smaller segments. Ms. B started to re-teach a lesson on how to graph a line given a point and given a slope, but a student said she knew how to do this kind of problem. Ms. B let Chelsea briefly take over the class to explain how to graph a specific line as an example. The short segments containing Chelsea’s explanation lasted 2.5 minutes. Afterwards, Ms. B asked the students if they needed more examples and a student asked about the difference between zero slope and undefined slope. Ms. B took the next 5 minutes to explain the difference using calculations as well as visual aids.

During the last 8 minute segment of the class, Ms. B had students work on the next set of questions, 16-27, on the same sheet she gave them earlier and finish for homework.

**Day four.** The fourth day of year one was segmented into 5 major pieces. Ms. B started the class with another 21 minute CO that was not discussed. The second major segment was split into two smaller pieces, one 4 minutes and the other 9 minutes. Ms. B went over the class work/homework the students worked on the day before. She read the slopes that answered questions 1-15 and stated errors would be discussed later. Ms. B used an overhead to plot the graphs that answered question 16-27 from the worksheet. After going over the homework from the previous day, Ms. B took the next 15 minutes to re-teach the next lesson in the chapter, which was the third major segment. The topic of this lesson was finding an equation of a line given two points. First Ms. B used the points (5,7) and (2,3) and split the class in half, where one half needed to find the point slope form of the equation given (5,7) and the other half used the other point. Afterwards, Ms. B had students take her through the steps to get the equation into slope intercept form.
Ms. B changed the points to (-3,4) and (8,2) and repeated the splitting the class in half to find the point slope equation to later find the slope intercept form. During the fourth major segment, Ms. B gave the students 11 minutes to practice finding an equation of the line given two points by doing problems from the book. The last 12 minute segment of the fourth day was comprised of Ms. B using the CCT to check the work the students recently finished. She plotted the two points given in the first problem using the CCT and asked the students to submit the equations fitting these points that they recently found. As the students submitted their equations, Ms. B sounded discouraged since many of them were incorrect. She did not discuss the student errors but rather tried to move on to the next problem. However, she had trouble plotting another set of points and had a student help her with the CCT. Trying to do another point took too long, so Ms. B went back to the first problem and worked out how to find the answer. She ended the class with assigning homework.

**Day five.** Filling out the student survey for the CCMS project took the place of the CO on the last day of the classroom observations during year one, which took about 30 minutes. Ms. B used the CCT to check student homework for finding equations of line given two points in a similar manner as the previous day. She had some minor difficulties with the CCT, but she was able to overcome them. For each of the problems, Ms. B did not discuss student errors despite many being present and systematic. Ms. B going over the homework lasted about 13 minutes, which comprised the second major segment. Ms. B stated she was not confident that the students were “getting it.” Then Ms. B took about 3 minutes to ask students what they did not understand. Students replied with forgetting
minus signs, and the difficulty of the fractions/decimals. Ms. B asked the students if they needed to review it a little more and has Chelsea work through her answers to find where the errors occurred since Chelsea stated she got half of them wrong. After about 5 minutes of Chelsea leading the class, Ms. B decided to move past her re-teaching of the chapter. The questions about student understanding and Chelsea leading the class comprised the third major segment.

She discussed what the small group that was split off from the main class at the beginning of Wednesday’s class had been doing since then. The small group had been working on finding solutions to systems of equations. She discussed that the solution to a system would occur when the graphs intersected. Using a problem that the small group had done, Ms. B had the whole class find the solution to the system of \( y=4x-8 \) and \( y=2x+2 \) graphically. After discussing some of the student solutions, she had the students find the solution to another system of equations. These two problems and the discussion of their answers composed the fourth major segment lasting 13 minutes. During the last 13 minutes of class, Ms. B used a “real world” example of when solutions to systems of equations might be important. Using a problem from the book, Ms. B had students find the equations that modeled the cost for renting two different studios to record music for \( x \) hours. Ms. B originally was not going to graph the equations of \( y=100+50x \) and \( y=50+75x \) by hand until students suggested using a scale. Ms. B had difficulty plotting the graphs since she went by 25’s on both the x and y axis. Some student suggested using a different scale like 10’s, one student mentioned using a different scale for just the x’s, while Casey mentioned that they could divide everything by 10 and graph \( y=10+5x \) and
Ms. B thought this idea was “so cool,” until she divided the equation by 10 and stumbled when she saw $1/10y$. Ms. B then let the student who suggested the division by 10 to take over the class and show the other student what he did using a calculator connected to an overhead. Ms. B used Casey’s graph display to discuss the solution of which studio would be the best choice.

**Representations**

The graphical and algebraic representations were present during all segments that focused on lines. The content of the CO’s were never made explicit to anyone not participating in the class, so the presence of representation of the CO’s will not be discussed. The activities and lessons observed in class had students graphing equations, finding slope given points, graphing lines given slope and a point, finding equations given two points, or finding the intersection of two lines. While the students used points frequently, they used the points as part of a graph rather than on a table. Hence the tabular representation was not used frequently. However, the tabular representation was used during the last segment of day 2 when Ms. B discussed the line that would best fit data. Ms. B used embodied representations three times during the five days: twice to show students the difference between zero slope and undefined slope after students said they did not understand the difference and once to show students how two lines might intersect using her arms. The first time she used the tilting of a piece of paper to provide a physical representation of slope, and the second time she used the cover of a book titled at different angles.
On days 2, 4, and 5 students encountered graphical and algebraic representations from multiple sources when Ms. B collected best fit lines to either pictures or points from the students. However, only during the activity on the second day were these multiple sources of representations explicitly discussed by Ms. B. On days 4, and 5 Ms. B used the CCT to check and display the student’s homework response, but never talked about student errors. Students were able to diagnose some of the errors they saw in the CCT display even without being prompted from Ms. B. The right picture of Figure 6.1 shows the lines that students submitted going through the points on the picture on the left. A student noticed an error the student who submitted the horizontal line, which was “somebody forgot the X.” Ms. B appeared to not have noticed the student’s comment and only mentioned Jay got his wrong before moving on to the next one.

![Figure 6.1](image.png)

Figure 6.1: Student submissions of lines (right) that go through the points on the (left).

During other segments when the CCT was not used to collect student equations, students encountered a single source of representations. During each non-CO segment either Ms. B was the source of the algebraic and graphical representations, or in a few instances a single student was the source. Only one of the latter instances allowed student
to be the true source of the representations, when Casey displayed his calculator to the class. In the other instances, Ms. B interpreted representations discussed by Chelsea on days 3 and 5 and displayed them on the overhead.

**Process/Object perspective.** The representations of lines were treated primarily as processes. Ms. B hinted at the object nature of lines during day 1 when she used embodied representations using a piece of paper to represent a line to illustrate the difference between zero slope and no slope. She tilted the piece of paper differently to illustrate the different slopes. During the third day when asked again what is the difference between zero slope and no slope, Ms. B used a book cover to represent a line opening the cover to different angles to show how the slope of the line changes the shape of the line as shown in Figure 6.1 and illustrated in Vignette 1. Line 3 student asked what the difference between the different slopes was. Ms. B attempted to explain procedurally in lines 6, 8, 12. However, until line 20 Ms. B focused on the calculations of finding slope and using a calculator to justify the no slope (error). At line 20 Ms. B switched her focus to the conceptual by using her movement of the book cover to illustrate the impact of changing the slope on the graph of the line. These two instances were the exception to her using the process perspective.

Ms. B Vignette 1

1) S: Ms. B?
2) T: Yes.
3) S: What’s the difference between undefined and 0?
4) T: Good question; who can answer that? One at a time.
5) S: [students talk over one another]
6) T: If you typed this into your calculator you would get error. That means there’s no definition, no definition – root word – for that number. That’s why it is
undefined. When you type this into your calculator you get 0, so 0 is the number. Are you asking me what they look like, how they’re different? Ok. Slope.

7) S: 0.
8) T: Let’s do it by the formula first. What’s the formula?
9) S: 0X + 2
10) T: We’re finding slope. That’s what we’ve been doing all this period. The formula you’ve been writing.
11) SS: (talk over each other)
12) T: And I’m just writing these so I don’t make a mistake because I tend to do this, too. These are my Y’s. 2 - 2. Negative 2 - 2. I get 0/-4. If I type that into my calculator I get 0.
13) S: It’s 3, minus 2, positive 5.
14) T: I’m sorry, that was a 3.
15) S: -5.
16) T: -5, you’re right.
17) S: It’s not negative.
18) T: It’s not negative?
19) S: It’s 3 - -2.
20) T: Oh, thank you. So the answer is 0 because the slope is 0. There’s no slope to that line; it’s just straight, ok? Jay. No slope. The second I change the Y’s it starts becoming a slope. I keep changing it, I keep changing it, this keeps getting higher. There’s still a change here, right? But when I get to here and the line looks like this, I have a lot of change this way but I have nothing this way. There’s no change. Y’s over 0. This is undefined when it looks like this. This is 0. That help?

Figure 6.2: Ms. B tilting book cover to represent changing slope.

As noted above, Ms. B typically treated representations as processes. Vignette 2 is an example of her typical treatment. The class was going over a quiz question asking them to graph the line -7x + 7y = 49 in Vignette 2. Ms. B., in line 6, drew the students’ attention
to the form of the equation of the line. In line 8, Ms. B treated the standard form as a process since there was “only one thing we really did with it.” A student recalled what the one process they did with the standard form was in line 13 and Ms. B began this process by asking for the y intercept in line 14. Ms. B’s question at the end of line 14 had students focus on how to from a specific x to a specific y or the process nature of the algebraic representation of functions. While Ms. B was talking in line 20, she was plotting points on an x-y-plane on an overhead. She had Seth count over several points to find the x- and y- intercepts focusing on the specific points rather than the whole line. By focusing the students’ attention to the specific points on the graph of the line Ms. B was treating the graphical representation as a process as well. Ms. B could have treated the equation and graph of the line as an object in line 30 by focusing on the shape of the whole line, but she continued to focus on the points of the line using her “way” to graph it. This method seemed to be to pick any two points on the line and plot them to draw the line going through them, but Ms. B’s third method ultimately ended up being identical to the intercept method in line 36.

Ms. B Vignette 2

1) T: Oh, right. But we’re going to retake it. Ok, let’s take a look at #1, which many of you did not get.
2) -7x+7y=-49
3) S: -7/8+7Y=-[inaudible].
4) T: Equals what?
5) S: -49.
6) T: Tyler, what form is this in?
7) S: Standard.
8) T: What do you remember about the standard form and what do we do with it when it was [inaudible]?.. There was only one thing we really did with it; what was that?
9) S: Changed it to Y.
T: Oh, we could do that. That's all we did in the section.
S: We divided the Y and the X's by the answer?
S: Oh, I remember this. You have to substitute the X and the Y for 0.
T: Right. They said when it was in standard form the easiest thing to do is to find the X intercept and the Y intercept. So if X = 0, then the Y intercept is what?
S: Negative.
T: And if the Y is = 0...
S: A positive.
T: That's one thing that you can do, and then how are you going to match that with your graphs? How are you going to match it?
S: Take those points and then graph the line.
T: You're going to look at your X, Seth, and count over 1, 2, 3, 4, 5, 6, 7, right? Then you have Y, count over 1, 2, 3, 4, 5, 6, 7. Which is the answer?
S: [inaudible].
T: That's not the only way to do it. So Y = what? Because I heard somebody say that. How many of you put this into slope intercept form? You're going to what to both sides, Chris?
S: Add 7X.
T: Add 7X to both sides. Then what?
S: Divide by 7.
T: So what kind of a line are you looking for?
S: The Y intercept is -7?
T: And... S: And a slope of one.
T: Right. So doing it this way you're going to be looking for a Y intercept of -7, but you have to look at the slope. Doing it the other way you don't have to worry about the slope. That's the second way to do it. Is there another way? There's a way I teach my [inaudible] to do it. Keep this equation. I pick out a point, let's say A, on that line. This is the long way, but if you don't understand everything else, you can still get it right. So if you pick a point on line A... Pick a point on line A.
S: 4, 2
T: 4 for X?
S: Yeah.
T: 4 for Y. 7x4 is -28+2x7 is 14. Does that go to -49? Is that on 49?
S: No, there is no...
T: How about using the intercepts.
S: 7 and -7.
T: So, 7 for X. So the point is then 0, right? Or it's 0, -7. When you replace, -7x7 is -49+0 is -9. You can always work backwards, take points, and put them back into the equation. Let's do #13 since we're already here.
The slope of a line, one of the major properties of lines that can reveal the object
nature, was treated as a process by Ms. B and her class. When Ms. B went over the quiz
during the first day, the students treated slope as something to find by plugging in points
or by counting on a graph rather than as a characteristic of a line. Mathew in Vignette 3
line 2, described how he attempted to find the slope between the points (-5,2) and (5,-1)
by doing a calculation. Ms. B had him work through his calculation, which let her find
the source of his error in lines 3-7. She asked the class what his error was and two
students responded with similar answers that he made a mistake by mixing the x’s and
y’s in lines 11 and 12. They treated slope as something to be plugged into. Another
student mentioned in lines 14 and 16 that Matthew could have just counted which is a lot
easier according to her. This student saw slope as something to be counted to be found or
as a process. Neither the teacher nor the students said that the slope should be small and
the line should be pretty flat since the change in the y-coordinates is small (-3) compared
to the change in the x-coordinates (10). Matthew’s slope of -1/2 was pointing in the right
direction and flat but not nearly as flat as it needed to be. The student and teacher
comments did not connect to the meaning of slope, which could have illuminated the
object nature of lines.

Ms. B Vignette 3

1) T: All right, and #10? Matthew, what did you get 10 right?
2) S: I got it wrong. I got like 5-2 over -1+-5, and I got 3/-6. I don’t know if I got
   that wrong or right.
3) T: Tell me how you did it again. You used the points negative 5 and 2 and one like
   that?
4) S: No, I did 5, -2. Yeah, I just didn’t write it down?
5) T: Did you do the formula? How did you find your slope?
6) S: I did 5-2 over -1 minus 5.
7) T: It’s -1 minus -5. What did he do wrong?
8) S: It was supposed to be -5 instead of positive 5.
9) S1: That’s mean it should have been a positive 2.
10) T: He did these two points and he’s finding the slope for #10.
11) S: He did y1-x1 and y2-x2.
12) S1: He subtracted in the same – instead of using the different, from the different categories, he used the same...
13) T: Is that what you were saying? He used an X from here, a Y from here, a Y from here and an X from here. The formula is... Is you had your notes. Y2-Y1 over X2-X1. So Y2 is -1. Y1 is 2. You should have -1 minus 2 in the numerator. In the denominator you should have 5-2.
14) S: He could have just counted.
15) T: He could have just counted, you’re right.
16) S: It’s a lot easier.
17) T: It is a lot easier when there’s a graph to look at. You’re right.

The students appeared to have treated slope in the manner that Ms. B taught them as evidenced by her re-teaching slope on Wednesday, which is illustrated by Vignette 4. In line 1, Ms. B stated there are two ways to find slope by graphing it or “the algebra way.” She gave the students a precise way of using the graph to find the slope by “starting with the lowest point” and counting over to the second point. Ms. B did not connect how to find slope with its meaning of the steepness of the line. She treated the graphical representation of slope as a process. To Ms. B the algebra way for finding slope was using the formula lines 5 and 7. In line 11, she told students to put the coordinates above one another and showed them that they could subtract columns to avoid making mistakes. In lines 13-21 Ms. B has the class address whether -2/5 is the same as 2/-5, but she did not address the meaning of slope.

Ms. B Vignette 4
1) T: There’s two ways to find the slope; there’s graphing it, and there’s the algebra way, right? So graphing it we’re going to put points on the graph. When we’re graphing I’m going to always ask start at the lowest point, so in this case it’s this
point right here. And go up - and I want to see you mark this on your graph – up 2 and then over 1, and then the 2 there, 1,2,3,4,5.

2) S: Negative 5?
3) T: Negative 5, you are right Katie. So slope is...
4) SS: 2...
5) T: 2 over -5 because we’re always talking about rise over run. The algebra way is what?
6) SS: 6-4 and -3-2.
7) T: I want the formula first.
8) SS: Y-Y1 ...
9) T: Y-Y1 you want to say?
10) S: Yes.
11) T: Ok, Y-Y1 over X-X1. Now maybe I’m thinking, if when you write your points, when you copy them off your paper, if you write them like this maybe it will help you avoid making mistakes. Y is 4, so we’re doing 4-6/2- -3. 4-6 is -2, 2- -3 is...
12) S: -5.
13) T: Are those the same? Are those two the same?
14) S: Yes.
15) T: Why?
16) S: ______
17) T: No, are they the same number is what I’m asking.
18) S: Yeah, because
19) T: Because what?
20) S: It doesn’t matter where the negative is.
21) T: Right, it doesn’t matter where the negative is. So for each slope problem that we do I want a graph, I want this marked out with 2, or whatever numbers go here, and I want the formula and the work. For every single one I’m going to give you today. Shall I do another one?
22) SS: No. Yeah.

**Treatment of representations.** Representations in Ms. B’s classroom were used to enact procedures, and become end products. In Vignette 2 line 8, Ms. B seemed to indicate that the only purpose of the standard form for the equation of a line was to do something with it, or treating it as something to enact procedures. A student wanted to change the standard form into something else to get the answer in line 9. When Ms. B was counting over to plot points, Ms. B was enacting a procedure on the graph of the line rather than treating it as something to reason about, or to use in explanations.
Ms. B treated slope as something to enact procedures to find an end product. In Vignettes 3 and 4, Ms. B had students enact the procedures of “the algebra way” or counting on a graph to find the slope to either find the slope or determine an error in the procedure. Once the correct end product, slope, was found, Ms. B moved onto another part of the lesson. Ms. B treated more than just slope as something to enact procedures to create an end product, she frequently treated representations in general in this manner. In Vignette 2 line 8, Ms. B reminded students that there was only one procedure they used on the standard form to graph the line. In lines 14-22, Ms. B helped the students work through the procedure as well as a procedure to create a graph. Once the graph was produced, Ms. B moved on to finding the graph using a different procedure of putting the line in slope-intercept form line 22. In lines 23-30, Ms. B led the students through this procedure and creating the graph. In Vignettes 2 and 3, Ms. B had the class enact procedures on the algebraic standard form to inform a procedure for creating a graph of a line without discussing or reasoning about lines after they were created. During the last segment of the first day when Ms. B described the graphing program to the students, Ms. B moved on to another graph when the correct line was produced without discussing it. During the second major segment of the last day, after students created the graphs in Figure 6.1, Ms. B treated these graphs as end products by moving on to the next one without discussing the errors present. During the segments where Ms. B went over homework or quizzes on days 1, 4 and 5, she treated the representations in a similar manner to either Vignette 2 or the manner described for the graphs in Figure 6.1. She would treat the representations that occurred during these segments as end products as
Ms. B would either state the answers or lead the students through finding the answers. Except during part of Vignette 1, Ms. B had herself or her students finding slope or producing graphs of lines from equations. The majority of Vignette 1 (lines 1-19), Ms. B tried to initially explain the difference between zero slope and no slope using the formula for slope and the calculator. Hence, Ms. B had the class treat representations as something to enact procedures to create end products the majority of the time.

Ms. B had a few occasions where she had the class treat representations as more than end products. Line 20 of Vignette 1 is an example of such an occasion. In this line, Ms. B used embodied representations to have students think about the connection between the shape of a line and its slope by illustrating the shape with a book cover.

*The second I change the Y’s it starts becoming a slope. I keep changing it, I keep changing it, this keeps getting higher. There’s still a change here, right? But when I get to here and the line looks like this, I have a lot of change this way but I have nothing this way. There’s no change. Y’s over 0. This is undefined when it looks like this. This is 0. That help?*

During the main activity of the second day, Ms. B took the time to discuss the incorrect graphs sent by students to match lines on the picture in Figure 6.3. Ms. B focused the students’ attention on the line that deviated the most from the picture of the stair rail, the positively sloped line, they were supposed to find at the beginning of Vignette 5. Ms. B asked students the connections between the equation of the line and its graph by asking what the blanks meant in lines 3-8. During line 11, Ms. B used the CCT to display the equations of the lines submitted by the students and circled the y-intercepts (right) Figure 6.3. Ms. B used the fact that stair rail crossed the y-axis at negative two to have Becky think about her equation in line 11. She briefly had the students using
representations as something to think about. In lines 17-27, Ms. B had the student who had positive slope describe how he found the slope treating the representation as something to enact procedures on to produce the correct answer.

Ms. B Vignette 5

1) T: Ok, let’s look at this one first. Did you write Y=1X+0?
2) S: Yeah.
3) T: And we were graphing this, right? That’s what you were supposed to be doing? What does… What goes here in the equation? [Writing y= _x+ _ on the board and pointing to the first blank]
4) S: In that one?
5) T: In any equations.
6) S: Slope.
7) T: And what goes here?
8) S: Y-intercept.
9) T: Ok. Here’s your line. Can you see this here? What’s the Y intercept?
10) S: -2?
12) S: [inaudible]
13) T: I can’t hear you.
14) S: [inaudible]
15) T: What bar? This bar? We’re trying to graph this equation right here. So we’re looking at where it crosses the Y axis. It crosses at -2. Chris, can you point that out to her? Ok, Darien.
16) S: I had that on my paper.
17) T: Oh, ok. Now talk to me about the slope. What points did you use to find the slope, Darien?
18) S: I used 0, -2 and…
19) T: I’m sorry, did you use this one?
20) S: Yeah.
21) T: Ok, and what other one?
22) S: 2,-4.
23) T: Ok, that’s good. Those are good points to use. Now what’s the slope?
24) S: uh…1
25) T: Is it positive or negative?
26) S: Negative.
27) T: Where are you? You put in a positive. So you put in a positive 1 instead of a negative and you didn’t put the right intercept.
During another part of the main activity on the second day, Ms. B explicitly had Adam think about what was wrong with his graph without telling him the error first in line 1 Vignette 6. Adam could see his line compared to the picture and the other students and reasoned that his slope was wrong (line 4). Ms. B disagreed as to which error was more severe pointing out something else was wrong, and in line 5, she had Adam think about what error she was more concerned about. Adam reasoned about his graph and concluded his slope was wrong. After interaction with Ms. B he concluded his y-intercept was also wrong. In other words, Ms. B had Adam treat his representation as something to reason about rather than something to simply produce. After Ms. B had a student discuss his incorrect line that was the furthest below the stair rail in (line 9), in line 11 a student mentioned that the two horizontal lines were wrong and why he thought they were. Ms. B asked for clarification in line 12 and asked Matthew what he did wrong in line 14. Andy appeared to clarify what Mathew meant by “forgot to put in the rest of the equation” by stating specifically they left out the x. Ms. B explicitly had specific students reason about
their representations and what errors they had. It seemed to be in part because of the display of both the graphs and the equations that a student took it upon himself to reason that the two horizontal lines were wrong because they only had part of the equation present. Hence the class and Ms. B were treating representations as more than end products but as something to reason about.

Ms. B Vignette 6

1) *T:* So you know how to do that? Adam, what did you do wrong?
2) *S:* _____
3) *T:* What?
4) *S:* I didn’t get the right slope.
5) *T:* I’m not too worried about your slope because it looks, you know… this one doesn’t match exact. I’m more worried about something else. What am I worried about?
6) *S:* Y-intercept?
7) *T:* The Y intercept?
8) *T:* The Y intercept… Becky, we talked about. What’s this?
9) *S:* Change the y-int. I put in an extra 2. [the y-intercept was -22]
10) *T:* Otherwise we’re all set.
11) *S:* There’s two straight lines of people that were supposed to put in slope and only had the Y intercept.
12) *T:* Why?
13) *S:* _____
14) *T:* Matthew?
15) *S:* I forgot to put in the rest of the equation.
16) *T:* Andy?
17) *S:* I think they forgot to put in an x.
18) *T:* What?
19) *S:* _____
20) *T:* So any questions on this? Let me clear this and send me this one. I don’t know if you can do this, but let’s see because I haven’t done this before. See what you get when I send this.

After Ms. B had the all the students regroup during day five so that she could teach a lesson about simultaneous linear equations, she picked a specific problem that the “advanced” group had been working on. At the beginning of the problem she did
not have students starting a procedure, but rather asked students to compare the two lines in a specific manner, parallel, in Vignette 7 line 1. She had the students explain why the two lines were not parallel in lines 3-8. Then she had the students compare the lines in another manner, perpendicular, and had the students explain why in lines 9-15. Ms. B had students think about the two lines together, reason about them, and to a limited extent justify their responses. Ms. B found that Adam’s intersection was (-3,4) and Jonathan stated he didn’t think that was right in line 15. Ms. B found that several students did not have the same answer in lines 15 – 17. After Ms. B used embodied representations to illustrate how the lines could meet high up or down low as depicted in Figure 6.4, the students began to reason about which of the two possibilities was true in line 23. Rather than letting the students continue to reason about the intersection meeting high up or down low, Ms. B returned to her usual treatment of representations as something to enact procedures upon and began plugging into the equations to check her answers in line 24. During the instance captured in Vignette 7, the students appeared to try to reason about the representations, while Ms. B appeared to want to use procedures as explanations.

Ms. B Vignette 7

1) T: What I’m asking you to do is to graph these on this paper and then I’m going to start class. The equations that I’m writing down on the overhead. First of all, Tyler – I’m picking on Tyler today, I don’t know why. Are these two lines parallel?
2) S: No.
3) T: No? Why?
4) S: Because not the same...
5) T: They’re not the same what?
6) S: slope
7) T: So one of the slopes is...
8) SS: 4,2.
9) T: Right. One slope is 4, one is two, so these two lines are not parallel. Are they perpendicular?
10) SS: No.
11) T: Why not?
12) S: Because one is not negative?
13) T: Y is not negative and when you multiply them together they don’t equal...-1.
14) SS: -1.1.
15) T: -1, right? 4x2 is 8, not -1. So they’re not parallel, they’re not perpendicular; therefore they’re just two lines that intersect. So I’m asking you to just graph those two lines. Adam has them intersecting at -3,4. Is that right? Jonathan, you don’t think that’s right? What do you have?
16) S: I graphed and got -1,8
17) T: You got -1,8? Alex has something else.
18) S: They’re saying “How would you graph it?”
19) T: Whoa... Are those just two...
20) S: lines.
21) S1: They are not parallel or perpendicular.
22) T: Right, but the problem is because of the slope. It’s [the intersection] meeting either high up [large y-values] or down low [small y-values].
23) SS: Yeah. It’s high up. It’s going to be like low.
24) T: We’re doing this on paper. ...um... Ok. Let me just check the two solutions that I got. If this is the point, you can see when two lines intersect the point, namely this point or this point – Jay, please look up – this point belongs to both lines. It’s sitting on this line and it’s sitting on that line. And so that would mean that -3,4 would make this equation true and it would make that equation true. 4x-3, Chelsea?
25) S: I got the wrong thing.
26) T: You’re missing the explanation; that’s why I called your attention to it. 4x-3 is -12. -12+8 is -20. It does not equal 4. That point isn’t even on that line. Maybe it’s on this line. 2x-3 is -6, -6+10 is 4. So this point is on that line, but it’s not on that one. -1,8. Kaylin, it really would be more helpful if you’re looking up here while I’m pointing so you know what I’m doing with the 1. -1 here. 4x-1 is -4. -4 minus 8 is 0. This point is not even on that line. -1x2 is -2. -2+10 is 8. So this point is also on that line. You’ve each given me two points that are on the second line, but neither of those are on the first line. Can you give me another one that comes out somewhere in the 1,2,3,5 range from your homework yesterday?
During the last segment of the fifth day, Ms. B had the class determine which recording studio would be better to rent from. The numbers involved in the equations that modeled the rent were large, $y=100+50x$ and $y=50+75x$, so Ms. B originally had planned to have the students use their calculators to graph the lines and find their intersection. But some students mentioned that a scale could be used to graph it, and she decided to try a scale. After Ms. B ran into trouble with her scale, she let a student take over the class and show his solution depicted in Figure 6.5. Ms. B guided the students thinking by comparing the lines at certain $x$-values, “So after 2 hours: 3, 4, 5 hours, you’re going to spend less money choosing this one [pointing to the less steep line] than you are choosing this one.” Later she drew the students’ attention what was happening before the two lines intersected, “Before the two hours you going to spend less money choosing this one [pointing to the steeper line] than you are this one. And the only time you’re spending the same amount is at 2 hours.” Ms. B did not use PJ’s representations as an
end product and then move on to the next problem as she had done frequently during the observed
days, but rather she used his representations to have a guided discussion comparing the
lines.

Figure 6.5: PJ displayed graphs modeling the rent for both studios on the overhead.

**Discourse**

Due to the inability to clearly hear students while viewing video tapes only whole class discourse is analyzed. The purpose of most classroom discourse appeared to be to elicit correct answers, correct methods of finding solutions, or how to fix errors from the students.

The discourse was almost exclusively teacher to student or student to teacher. One student, Mathew, would infrequently remark on other students’ comments to no one in particular, such as after one student said she got a “hundred” he exclaimed, “we don’t care,” or after PJ figured out an interesting method to graph lines another student said “he’s so smart!” Mathew exclaimed, “but not smart enough.” In three instances, Ms. B let a student briefly take over the class and these students would talk to the class.
Ms. B spoke 10,026 words over the five observed days with 471 utterances during her 90 minute classes. During most days she spoke close to her average of 2005 words per day and 94 utterances per day. The first and third days were exceptions, on the first day she spoke 2881 words in 140 utterances and 1211 words in 45 utterances on day three. The lack of words during the third day resulted from large portions of class time devoted to in class student work. On average Ms. B spoke 21.29 words per utterance. Some of her utterances could be quite long over 100 words with the longest being 189 words. She averaged 3.6 such utterances per day, with six of them occurring during day five. Ms. B used the long utterances either as introduction to a topic, instructing the students of the task they were to perform, or as an explanation using procedures.

Ms. B’s students spoke 2995 words in 529 utterances over the five days. The students averaged 599 words per day in an average of 105.8 utterances per day. On day three, students only uttered 199 words, whereas on day five they uttered 1010 words. On average the students spoke 5.66 words per utterance. Most of the students utterances’ were short, but 15 utterances were over 20 words with 8 of these utterances occurring during day five. The longest student utterance was 51 words.

**Questioning.** The purpose of the most of the questions Ms. B asked appeared to be to elicit answers or procedures from the students by typically using Initiate, Respond, Evaluate (IRE) sequences. In Vignette 1 after a student asked what was the difference between zero slope and undefined slope, Ms. B tried to use the calculator as an explanation in line 6, “*If you typed this into your calculator you would get error. That means there’s no definition, no definition – root word – for that number. That’s why it is*
undefined.” Ms. B appeared to notice the student was not content with this explanation and proceeded to discuss how the two slopes would look. She began an IRE sequence in line 8, “Let’s do it by the formula first. What’s the formula?” The students and Ms. B appeared to switch roles after she worked out the slope in line 12, “Negative 2-2. I get 0/-4” since the student evaluated her response in line 13 “It’s 3, minus -2, positive 5.” Another student said the result was -5 in line 15 and Ms. B evaluated this response as correct in line 16. But a student evaluated her evaluation by saying “It’s not negative.” While the discussion in Vignette 1 did not contain a typical IRE sequence, the kinds of questions present that could be answered with results of procedures or short statements were typical of IRE sequences. Beginning in line 4 of Vignette 2, Ms. B asked “Equals what?” starting an IRE sequence. She continued this sequence by asking questions such as what form the line was in, what can be done with this form, what were the x- and y-intercepts, and what was the answer in even numbered lines from 6 to 22. In line 22 Vignette 2 Ms. B began another IRE sequence to demonstrate by using student input how to graph the line by finding the slope intercept form of the line ending in Vignette 2 line 30. At the end of line 30, Ms. B asked the students to pick a point on the line beginning another IRE sequence to demonstrate another method for finding the graph, which ultimately ended up being the same as the first method in Vignette 2. Vignettes 3, 4, 5 and 6 are examples that contain many IRE sequences Ms. B used to determine what a student did to get either the wrong slope or the wrong equation of a line matching the stair rail, or to demonstrate with student input how to find slope. While Vignette 7 has a long IRE sequence in lines 1-16, an interesting discussion could have occurred after Ms.
B said in line 22 “Right, but the problem is because of the slope; it’s meeting either high up or down low”. The students appeared to want to think about it in this manner in line 23, but Ms. B began using procedure as explanation in line 24 starting another IRE sequence.

Ms. B mentioned on multiple occasions on multiple days that she was not comfortable that the students knew the material. After asking students to submit their lines from the homework that went through two points and seeing that many of the students were still getting them wrong, Ms. B asked the students to talk to her about what they learned. Vignette 8 contains the resulting discussion that broke away from her typical IRE discussions. During this discussion Ms. B elicited from students why they thought they were getting answers wrong, rather than what were the right answers in lines 1, 5, 10, 16, 20 and 22. The students’ responses ranged from only finding part of the question line 4, mixing up negative signs line 7, trouble calculating decimals line 13, and using fractions line 25. None of the students mentioned that finding slope was difficult, other than perhaps working with decimals or fractions. Most of students were getting the correct slope as evidenced by their lines submitted to and displayed by the CCT in Figures 6 and 7. The students appeared to have trouble finding the y-intercept when going from the point-slope form to the slope-intercept form since they were creating several parallel lines. The students seemed to determine that the arithmetic for finding the y-intercept was their problem as a student said in line 13, “I forgot how to add and subtract the decimals. They messed me up.”
Ms. B Vignette 8

1) Guys, I don’t have a good feeling about last night’s homework. And I need for you to talk to me about it.
2) SS: [students talk over one another]
3) T: One at a time. Yes.
4) S: What I did was I thought this meant find the slope so I didn’t do the whole problem.
5) T: Chris?
6) S: I have no idea what I was thinking. I did the things on the ...I don’t know what I got them wrong...
7) S1: Most of them, I got the negative signs mixed up.
8) T: Are you using a calculator at home?
9) S: Yes. If there’s a negative ten and a negative 5 sometimes I forget there’s a negative there.
10) T: Gary? Looking at this I feel like I have to re-teach it. But what I’m hearing is, no you don’t, I understand it.
11) SS: I understand it. I understand it.
12) T: Wait a minute. Hands up.
13) S: I forgot how to add and subtract the decimals. They messed me up.
14) T: All right. Were you trying to add fractions by hand at home, or were you trying to add them on the calculator?
15) S: Calculator.
16) T: Right...um...You need a common denominator, right? Did you remember that? Anybody else? Did you get them all right.
17) S:
18) T: Ming, talk to me about...
19) S: I got 3 wrong.
20) T: But talk to me about your understanding of this.
21) S: It was hard to...
22) T: What makes it hard?
23) S: ______
24) T: What?
25) S1: When I was asking him to find the right answers, he’s say a number without fractions.
26) T: He never had any fractions. You’re right. Even if you don’t reduce them they’re right. Anybody over here? So yes, do we need to go over this anymore? Yes? Raise your hands yes.
27) SS: Yeah. Yes, because I got half of them right. Yes. We need to review it and keep going over it.
28) T: Yes, Chelsea.
29) S: I got like half of them right and half of them wrong.
**Choice of examples.** Ms. B’s choice of examples of representations of lines appeared to be mostly chosen ahead of time with several chosen from the book. The segments in which Ms. B went over previous days homework all contained representations from the book. The questions on the quizzes were similar to the homework questions, so the book appeared to influence the choice of representations on the quizzes as evidenced during the segments where she went over the quizzes. During the segments on days 3 and 4, Ms. B re-taught the main lessons of the chapter. Ms. B had representations of lines or points on a line ready to go when she taught these lessons that appeared to be influenced by the book. Ms. B’s representations used during the main segments of day 2, the picture with the stairs and the cost of postage data, seemed to be chosen ahead of time without the appearance of being influenced by the book. Ms. B had semi-planned representations present during the segment on the first day where the students were to work with the lines randomly generated by the calculator. Ms. B chose to introduce the program, and the program generated the representations present.

Not all of Ms. B representations were chosen ahead of time. Ms. B’s use of unplanned representations appeared to have occurred in response to a student question or a student suggestion. Ms. B’s use of embodied representations was prompted by students’ questions, such as the difference between zero and undefined slope. Ms. B did not shy away from using student’s suggestions. Ms. B had planned to use the calculator to display the graph as evidenced by her saying, “*So for an equation like this, who would want to graph it on graph paper?*” line 7, until a student suggested a scale could be used in line 10. Ms. B liked the idea of using a scale in line 14 and works with the students’
suggestion by placing the scale on the x,y-axis on the overhead. But she ran into trouble trying to plot a slope of 50 using a scale of 25’s for both the x- and the y-axis and she exclaimed “Oh yuck” line 18. A few students offered suggestions lines 20, 23, and 25 and one student appeared to have predicted that the scale used would cause problems (line 21). PJ offered the suggestion that they could divide everything by ten and graph y=5+7.5x and y=10+5x instead and multiply the answer by ten in lines 25, 27, 29, 31, and 40. Ms. B thought the idea was “cool” line 32 and used his suggestion by writing out his steps on the overhead. Ms. B said she thought the method was going to work until she saw that y was being divided by ten in line 39. Ms. B let PJ take over the class and gave him a cord so all the students could see his calculator screen on the overhead. The events of Vignette 9 led up to the events described for Figure 6.5. During the events of this Vignette and Figure 6.5 Ms. B used several impromptu representations that were all suggestions made by the students. Ms. B appeared willing even pleased to use student generated/suggested representations, despite not knowing ahead of time if the representations will work.

Ms. B Vignette 9

1) T: Look at #39. Why do we even worry about systems of equations? Two equations? Suppose you and your friends form a band. You want to record a demo. Studio A rents for $100 plus $50 an hour. Studio B rents for $50 plus $75 an hour. Solve the system by graphing. Before we could solve, we need an equation. What is the equation if you are renting and you are spending $100 plus $50 an hour, what would be an equation that would represent the cost of renting that studio? What? And that would equal the cost, or Y. How about the other one?

2) SS: (talk over each other)

3) T: Do I hear 50?

4) SS: Yes.

5) T: Ok. At some point both of those costs will be the same because these are lines and they’re in Y=form. Y=50, the slope is 50, 100, 75, slope here, Y intercept at 50.
6) S: At 75 it’s almost a straight line.
7) T: No... Pretty close. So for an equation like this who would want to graph it on graph paper?
8) S: Nobody.
9) T: Nobody, you’re right. So this would be a good one to graph on your calculator. You got the equation down, because I’m going to erase it. Have you got it written.
10) S: Ms. B______, you could always use a scale.
11) S1: You could go by 25.
12) T: You could go by 25’s?
13) S: If you have a scale.
14) T: I like that idea, let’s do it.
15) S: Ms. B______, you don’t remember anything I said.
16) T: Ok, let’s go by 25’s. 25, 50, 75, 100. 25, 50, 75... We’re going to run into trouble. Yeah, I think you’re right.
17) S: I say go by 5 and 10’s.
18) T: So the Y intercept on the first one is here at 100. The slope is 50, which means 50 over one... Oh, yuck.
19) S: It’s going to be about that big.
20) S1: I say go by 5’s.
21) S2: I knew this was going to happen.
22) T: Let’s go back to the calculator.
23) S: 25’s.
24) T: Yeah, but you’re going to go up 50, which isn’t bad, but I’ll be going over 1.
25) SS: You haven’t changed the X axis. You know you can simplify a little. You can make it 10 and then 5 and then 5 and then 7.5.
26) T: Do you think so?
27) S: Yes, I do. And for the Y’s move the decimal.
28) T: What did you say?
29) S: I said you could simplify it to make it 10+5X=
30) T: So what are you doing?
31) S: Dividing by 10.
32) T: He’s dividing – watch this, this is so... This is so cool.
33) S: He’s just too smart.
34) S1: He’s not smart enough.
35) T: We have... Follow his thinking. Is this going to work, though, PJ?
36) S: Whenever you get your answer...
37) S1: Multiply it by 10.
38) S: Multiply it by 10. I bet it will work.
39) T: We need to have it... I was thinking gee, that might be good until I went to do this and divide both sides by 10 and got 10+5X=1/10Y. But the thing is we want it in Y= form.
40) S: What I’m saying is just do it like regular but just put in your calculator. 10+5X, and then Y = that and then Y=5+ 7.5X and when you get your final answer times it by 10.
41) S1: That thing you kept back when you did the last thing.
42) T: One at a time.
43) S: Like here’s what I’m saying; I’m going to do it on the calculator. And I’m going to do the large scale on the calculator.
44) T: Hold up, don’t do it before I give you the cord.

The representations present during the class time spent “going over” quizzes and homework had a wide selection. Students were able to see positive slopes, negative slopes, integer slopes, fractional slopes, zero slope and no slope as well as positive and negative y-intercepts and integer and fractional y-intercepts. Ms. B’s use of the CCT exposed students to sets of parallel and perpendicular lines when students submitted incorrect lines to match points or the picture. However, the only times where multiple lines or slopes were compared to one another occurred during the embodied representations episodes.

**Missed opportunities.** Ms. B had several missed opportunities to reinforce the object nature of lines or make connections among different/multiple representation to varying degrees. Whenever Ms. B used the CCT to elicit and display equations of lines of the students, Ms. B missed an opportunity to make connections among the correct and incorrect lines. Ms. B not connecting the multitude of representations of lines present in the homework, class work, or quizzes was a major missed opportunity. Another missed opportunity occurred when Ms. B failed to capture the enthusiasm after she said the lines should meet high up (large y-values) or down low (small or negative y-values).

As mentioned above, Ms. B used lines with many different kinds of slopes. However, only twice during the episodes where Ms. B used embodied representations to illustrate zero and undefined slope did Ms. B make connections between the value of slope and its
impact on the shape of the line. As discussed above the choice of these representations appeared to have been unplanned. During Ms. B’s apparent planned lessons, she did not discuss the connections of the shape of the graph of line, its slope and its equation. Rather Ms. B seemed intent on enabling her students to be able to find the slope of a line correctly. In Vignette 5, Ms. B used an IRE sequence to determine how a student found their slope lines 17-27. But she did not mention that the height of the stair rail had a shape similar to a decreasing line and would have to have a negative slope. When Ms. B re-taught the lesson on how to find slope, captured in Vignette 4, she never made the connection between slope and its impact on the shape of the line. Again she seemed intent on students being able to find it correctly. During Vignette 3 lines 1-13, Ms. B wanted the student to be able to plug into the formula correctly to find the slope, but did not state that line should be fairly flat compared to the slope he found. Ms. B briefly discussed that the coefficient of x and the slope of the line in Vignette 5 lines 3-6. Rather than discussing how the slope determined the steepness of the line, Ms. B turned the discussion to a procedure for finding slope lines 17-27. Ms. B discussed on several occasions how to find the slope, but almost never discussed the meaning of slope and its impact on the graph of the line except in a few unplanned instances.

When Ms. B used the CCT to gather and display equations from students, she mostly stated that a certain group of students were correct and the rest were wrong before moving onto the next problem. However, during the main activity segment of day two, Ms. B took time to discuss student errors as partially captured in Vignette 5 and 6. The graphs of lines submitted by the students are shown in Figure 6.3. In Figure 6.3, students
submitted equations that were positively sloped, horizontal, negatively sloped as well as
groups of parallel lines. During Vignette 5, Ms. B displayed the equations that the
students submitted and compared the constant term in each noting that many were -2.
However, she did not go further in her comparison of the lines to ask students what they
noticed about the coefficient of $x$ and the slopes of lines present. She could have
mentioned that the wrong slope in Vignette 5 was the right magnitude but wrong
direction and that it was a reflection over the $y$-axis of the correct line illustrating the
object nature of the line.

At the end of day four, Ms. B used the CCT to collect equations students found that
went through two points given in the homework. The equations the students submitted
are shown in Figure 6.6. Almost all of the lines were parallel to one another except two
that appear to be perpendicular to one another, so nearly all of the students found the
correct slope. The negatively sloped line appeared to be a reflection of a correct line, so it
had the right magnitude wrong direction. The other line not parallel to the others
appeared to be perpendicular to the negatively sloped line, so it probably had the right
numbers for the slope in the wrong place. In other words the student probably did change
in $x$ over change in $y$. However, as evidenced in Vignette 10, none of the above was
discussed or perhaps even noticed by Ms. B Although the student in line 7 figured out his
mistake was putting in a negative. Ms. B appeared to be disappointed by the fact many of
the students’ lines did not go through the points as her “Oh, guys…” in line 4 sounded
disheartened. Perhaps because of her disappointment, Ms. B did not seem to realize that
most of the students were getting at least half of their equations correct. Instead of
discussing student errors like she did on day 2, she showed the class who got it correct and moved onto the next homework problem. The lines present in Figure 6.6 could have been used to illustrate that parallel lines are the same except for the constant term, changing the constant term shifts the graph up or down, that putting a negative in front of the coefficient of x causes the line to reflect over a horizontal line and that flipping the numerator and denominator in the slope causes a reflection over a line with a slope of 1.

Ms. B Vignette 10

1) T: Get into the activity center, please.
2) S: What are we going to be doing?
3) T: We’re going to check a couple of these to make sure we’re on the right track.
4) T: I did it! Ok. This is the first one, so you’re going to send me your equation. It should run through those two points if you were right. You have to send it in slope intercept form. Oh, guys...
5) SS: Oh, oh. Oh, oh.
6) T: Nicole’s is right, and everybody else... Oh, no, I’m sorry.
7) S: I put a negative on accident.
8) S1: All those people are right.
9) T: All those people.
10) SS: I can’t believe that, I got it right. I messed up. I don’t know what I did.
11) T: All right, I want the next one.

Figure 6.6: Student submissions (right) to lines going through the two points (left).
Whenever Ms. B used the CCT to gather equations students found for homework during the next day, similar missed opportunities as discussed in the preceding paragraph occurred as she mostly moved on after collection. Both Figure 6.7 and 8 shows some of the lines that students submitted and similar errors of using the wrong sign, forgetting the x, or finding reciprocal slope occurred. Students appeared to be finding the slope correctly, or have a minor calculation error for many of the problems. The lines in the right picture in Figure 6.8 contained more errors, but two pairs of lines are parallel and these pairs have equal but opposite signed slope. One or more students forgot to put an x in the equation, the horizontal line. The two unaccounted lines appear to have reciprocal slopes. Since most of the time students were submitting a series of parallel lines seen in the CCT, they appeared to have more trouble finding the y-intercepts than slope as the students nearly say later during the episode in Vignette 8. However, Ms. B did not appear to notice the systematic errors the students were making when using the CCT or later when she asked them what was difficult about finding the equations of line in Vignette 8. Although Ms. B did not take time to discuss errors, a few students did take the opportunity to think about what was wrong with their or others equations, such as “somebody forgot the x” for Figure 6.1 or Vignette 10 line 7.
During Vignette 7 line 22 and shown in Figure 6.4, Ms. B used embodied representations to demonstrate that the two non-parallel and non-perpendicular lines would either meet high up or down low. The students seemed excited about thinking about the intersection of the lines in this manner and had differing opinions in line 23. Using the y-intercepts of both lines and the meaning of slope, the students could have justified that the two lines should have met “high up” by saying something like “the one
with the steeper slope starts below the flatter line, so eventually the steeper line will catch up to the flatter line by moving to the right.” However, Ms. B did not give students the opportunity to reason about the lines in this manner by quickly exclaiming, “We’re doing this on paper…” line 24. She then proceeded to use procedures to check the intersection points students found earlier.

Technology Use

Ms. B used technology as an information display/gatherer, drill practice aid, a mathematical authority, and as a limited discourse generator. Ms. B integrated the CCT into her main activity on day two lasting 21 minutes, but the CCT appeared to be an add-on to her instruction on days four and five lasting for less than 15 minutes each day. On day one Ms. B demonstrated and had students use a calculator program that randomly generated lines and asked for the slope and y-intercept of this line. The program acted as a drill practice aid, since it could repeatedly ask students the same type of questions.

When Ms. B used the CCT, she used it to gather and display equations from the students that either went through two points on days four and five, or that best matched straight objects in a picture (day two). On days four and five, the purpose of collecting and displaying the students’ equations appeared to be for her to merely check the students’ answers as she moved on after pointing out who had correct answers as evidenced in Vignette 10 and the missed opportunities described above. On these days, Ms. B used the CCT as an information display/gatherer.

On the day two, Ms. B had students find the best fit lines to match pictures. She collected and displayed these lines using the CCT, so she used the CCT as an information
displayer/gatherer. Unlike what she did on days four and five, Ms. B took time to discuss student errors. Ms. B used display of the graphs of students’ lines to pick a line that did not match the picture and ask the student who sent the line what was wrong. She had used the CCT to generate discourse about some of the student errors that occurred. In Vignette 5, the first instance Ms. B used the CCT to discuss student errors, lines 3-8, Ms. B asked students what the coefficients of x and the constant term were using an IRE sequence when she asked what fills in the blanks in \( y = \_ x + \_ \). Ms. B controlled the discussion by first asking where the rail crossed the y-axis in line 9 and then comparing the y-intercepts. Ms. B described the y-intercepts she saw, “Yeah, -2. Most of you knew that. I see -2 in a lot of places here. I see -22, Gary. I'm seeing -2. Becky why -6?” line 11, rather than letting students compare/contrast the y-intercepts in the equations. In lines 17-27, she had the student who had the wrong equation in line 1 discuss how he found his slope and how to fix it. Later during the main activity of day two in Vignette 6 line 1, Ms. B loosened control of the discussion slightly by asking what Adam did do wrong, rather than leading the student through his calculations. Adam realized that he had the wrong slope by comparing his line to the stair rail. Ms. B said she was not too worried about the slope since the slopes were similar, but she was more concerned about something else in line 5. Adam asked if it was the y-intercept and his answer was confirmed by Ms. B in lines 6 and 7. Ms. B clicked on other lines that did not match the stair rail. Becky’s had been discussed during Vignette 5 before she clicked on the line that was far below the others in line 7. The student who submitted the line knew exactly what error he made since he put in an extra 2 in line 8. Perhaps because the discussion focused more around
what errors were present than how to specifically fix them, Andy noted his observation in line 10 that “There’s two straight lines of people that were supposed to put in slope and only had the Y intercept.” Andy appeared to notice two of the lines were horizontal and that the likely error was forgetting the x term. Ms. B’s appeared confused about Andy’s statement as her “Why?” in line 11 was higher pitched and lasted longer than her other “whys.” Ms. B asked for clarification in lines 13 and 15 before moving on to the next task. During the main activity on day two, Ms. B took time to explicitly discuss student errors. Even though she largely controlled the discussion, the presence of representations from multiple sources appeared to have enabled some discourse to be generated. As discussed in the missed opportunity even though Ms. B did not intend to generate discourse about the problems displayed during days four and five, students discussed errors anyway.

Ms. B appeared to defer her mathematical authority to the calculator as she frequently relied on it for justifications rather than giving mathematical explanations. During Vignette 9, Ms. B after asking if anyone would want to graph the equations with large numbers, stated, “you’re right. So this would be a good one to graph on your calculator.” Later, after the students mentioned that a scale could be used to graph the lines and Ms. B did not make the scale work in line 18, she said wanted to go back to the calculator (line 22). In Vignette 1 when she was asked to explain the difference between zero slope and undefined slope, Ms. B tried to explain using the calculator, “If you typed this [4/0] into your calculator you would get error. That means there’s no definition, no definition – root word – for that number.” Ms. B’s initial explanation tried to use the calculator to
justify why 4/0 was no slope and 0/-4 was zero slope. The student did not seem convinced of this argument and Ms. B later used embodied representations to explain.

During the discussion of what was difficult about graphing lines in Vignette 8, both times when students said they were having trouble with minus signs or adding decimals, Ms. B first observed reaction was to ask if they were using calculators at home. The student, who mixed up the negatives in line 7, seemed to say in line 13 that the calculator would not help much since he sometimes forgot there was a negative. The student who had trouble adding decimals in line 13 said he was using a calculator at home. Ms. B seemed surprised that the student was using a calculator and appeared to be temporarily stumped as to how to help the student in line 16 “Right…um…” Afterwards, Ms. B stated that he needed a common denominator and quickly asked if anyone else had problems in the remainder of line 16. Ms. B appeared to think that students using calculators would have solved student issues with computation errors.

**Decisions**

Ms. B’s goals appeared to be to wanting students to “get it”, which to her “getting it” seemed to be able to find correct answers; have the students at least see the correct answers; she appeared to want to use student suggestions even if they might not work out; and address student questions. During the segments where she went over homework or quizzes she would use IRE sequences to elicit the correct answers and/or the correct methods for finding those answers as evidenced in the first 19 lines of Vignette1, and all of Vignettes 2, 3, 5. When Ms. B re-taught how to find slope of a line in Vignette 4, she demonstrated using student input how to find slope with two different methods. Ms. B
mentioned multiple times that she was not comfortable that the students were getting it and on day five she asked them what was causing the problems in Vignette 8, which revealed her concern for her students “getting it.” During the last segment of day five, Ms. B used many suggestions from the students such as using a scale to graph the equations or dividing the equations by ten to make graphing them easier in Vignette 9. Ms. B said the students’ ideas were cool or that she liked them and appeared happy to try to use them. Ms. B appeared willing to answer questions posed by the students as evidence by Vignette 1.

Ms. B’s orientation appeared to be highly procedural. Ms. B’s explanations used results of procedures such as the beginning of Vignette 1 where she plugged into the calculator to find slope, the end of Vignette 7 where she plugged in points to find the intersection, during Vignette 4 where she demonstrated step by step with student input how to find slope, during Vignette 2 where she demonstrated three procedures to graph the equation of a line in standard form, and when Ms. B tried to explain the student suggestion of dividing by ten in Vignette 9. Only when she used embodied representations were her explanations conceptual by demonstrating the impact of slope on the shape of the line, but these instances appeared unplanned. However, Ms. B never appeared to explicitly discuss the meaning of slope. Most of the questions Ms. B asked the students were in IRE sequences: either what was the result of a certain procedure or what was the procedure used. The exceptions were in Vignette 8 where she asked students what they thought was difficult and in Vignette 10 where she let a student decide what was wrong rather than leading him through fixing it. Ms. B’s treatment of
representations as almost exclusively as process as well as her treatment of representations as end products or something to enact procedures upon revealed her procedural orientation.

Ms. B appeared to have limited mathematical knowledge as evidenced in part by her deferment of authority to the calculator. In Vignette 9, Ms. B did not seem to think of using a scale to graph the lines with large numbers before the students suggested it and this part of the lesson appeared planned ahead of time. Ms. B had trouble reconciling the scale and the graph later in this Vignette. The thought of using different scales for the x- and y-axis seemed to elude her even when a student made this suggestion in line 25. Later in Vignette 9, when PJ suggested they could divide the equations by ten, Ms. B appeared to get stuck when she saw the y/10 claiming, “But the thing is we want it in $Y = \text{form}$.” She seemed to not make the connection that the second equation after dividing it by ten would also have a y/10 and could be used to find the intersection. Ms. B repeatedly did not address the series of parallel lines students submitted to the CCT during the going over homework segments on days four and five. During these segments Ms. B focused only on her identifying correct lines before quickly moving on. Ms. B appeared to focus on the fact that most of the students were wrong and it disheartened her in Vignette 10. Ms. B did not seem to realize the fact that most of the lines were parallel meant most of the students were getting most of the problem correct and that the students were having trouble finding the y-intercept and perhaps would have been less discouraged. However, Ms. B did not address the issue of the difficulty of finding the y-intercept in the slope-intercept form of a line when given two points on the line that were
not intercepts, until after the students mentioned that this was their difficulty in Vignette 8. When she did address the issue, she tried to defer authority to the calculator. In Vignette 11 line 1, Ms. B asked students to try to find the equation of the line that would match the post holding the stair rail and she was unsure if the students could since the line was vertical. In line 10 Ms. B made the suggestion that a student could try \( y=\frac{3}{0}x \). However, \( \frac{3}{0} \) is undefined so \( \frac{3}{0} \times X \) is undefined meaning the calculator has nothing to graph. Ms. B’s inferred lack of knowledge of the interesting connections of the representations of the lines submitted by the students appeared to have prevented Ms. B from discussing the connections and the underlying student difficulty.

Ms. B Vignette 11

1) T: So any questions on this? Let me clear this and send me this one. I don’t know if you can do this, but let’s see because I haven’t done this before. See what you get when I send this.
2) S: Is \( y\times \) undefined.
3) T: [points to graph] \( y=x \) is right there.
4) S: That’s \( 1\times \)?
5) S1: Ms. B I don’t have an X.
6) T: You don’t have an X? Excuse me, I can’t even talk. Change in Y; what’s the change in Y for those 2 points?
7) S: 3.
8) T: What’s the change in X?
9) S: 0.
10) T: I wonder, I don’t know, if you can put in \( y=\frac{3}{0}X \).
11) S: I’ll send it. I just sent it.
12) T: Does that work?
13) S: It’s not popping up. I think it’s the straight lines at the top. Go over to my name on the list.
14) T: What do you want to be able to put in?
15) S: \( X\times \).
16) T: \( X= \) what?
17) S: 4.5.
18) T: \( X= 4 \frac{1}{2}? \) Ok, I can put it in.
19) S: How can you do that?
20) T: I don’t know, it’s giving me this option over here. \( X= \) what, PJ?
21) S: 4.5.
22) T: Ok. How about the other one. What’s the other one?
23) S: =-5. X=-5.
24) T: X=-5.

Below is a flow chart modeling Ms B.’s decisions in the classroom during year one of the CCMS study. The rectangles represent actions taken by Ms. B and diamonds represent decisions. The octagon shape represents a decision that is made possible or easier from the use of CCT and a rounded rectangle is an action made possible or simpler by using CCT. Ms. B was willing to alter her plans of instruction if students seemed to not “get it.” However this alteration was not in the moment or in the same day, but between days. Nor did the instruction seem like it would have been much different than the first time she taught it.

Ms. B’s instruction followed the flow chart the majority of the time around the second cycle of students needing correct answers/procedures and using IRE sequences to elicit or demonstrate them in the flow chart as evidenced by Vignettes 2-7. This cycle appeared to have broken down when many corrections are needed and Ms. B moved on rather than discuss correct answers/procedure as what happened when student submitted parallel lines in Vignette 10. Vignette 1 is an example of Ms. B addressing student questions, and Vignette 9 is an example of Ms. B incorporating multiple student suggestions.
Insider Perspective

The insider perspective stems from analysis of Ms. B’s Post Observation Interview on 5/11/2006. Ms. B recognized that students were having difficulty writing equations of line and she “felt that I really I needed to take them back to the basic finding the slope then writing the equations.” However, she never claimed to know what that difficulty was. Ms. B acted on this feeling and decided to re-teach her lesson on slope and finding

Figure 6.9: Decision flow chart depicting Ms. B’s instruction during first year of CCT use.
equations on day three. Although Ms. B’s instruction re-teaching the lesson on slope appeared that it would have been similar to the first time she taught it, Ms. B did change how she taught slope on day three:

*The first time that I learned it I practically did. I mean one day we did the graph and then another day we did the formula. And, we moved on and I was, this time I thought if I have done the graph and the formula together, they are going to be at least self-checking if they’re making any mistake on one, they are going to catch it and check it again.*

Ms. B taught the two methods of finding slope on different days, on day three she decided to essentially mesh the two lessons together so the students could reflect on their answers. However, Ms. B did not state that she altered how she explained each method, nor did Ms. B seem to alter her explanation of each method in the classroom.

When asked “What decisions did you make while planning this lesson,” Ms. B responded that she thought the students needed to review by more practice doing it over and over again.

*I mean my thoughts were the kids needed to review, they needed more practice it. Sometimes I feel, I think what I have done over the year and I think I don’t give them enough practice. I give them summing class and then we do some stuff with navigator and we do a lot with the calculator but yet the basic practice of writing these equations over and over again, those steps, I just I am thinking they need more of them.*

Ms. B does not mention that perhaps focusing on the meaning of slope by making comparisons of the impact of the slope on the graph would be beneficial for the students. Rather she believed that the basic practice of writing of these equations over and over would help her students.

Ms. B only saw wrong answers during Vignette 10 on day four as she said, “But it is also very discouraging when you see so many wrong answers like I did today,” and she
did not indicate that she saw mostly correct answers with minor errors. Ms. B probably felt the same way on day five when she encountered similar sets of “wrong” answers.

**Year 3: Third Year of CCT Use**

**Overview**

Several features of Ms. B’s instruction observed in year one changed in some manner by year three, such as incorporating technology into most of her lessons, using technology more frequently as a discourse generator, asking students to make comparisons of mathematical objects, treating functions as objects and processes, and treating representations as something to reason about. While these changes were apparent, the amount of change had varying degrees. Ms. B retained several aspects of her first year’s instruction such as, using IRE sequences to elicit correct answers/procedures from students, she continued using different kinds of representations from multiple sources, and continued to lead the discourse. While a major purpose for discourse in Ms. B’s class was to elicit correct answers from students, Ms. B also used discourse to elicit and aid some student thinking. Ms. B appeared to be less procedurally oriented in year three.

**Segmentation**

**Day one.** Each day of year three had about 55 minutes of class time, where the students had a break for lunch in the middle. The first day of year 3 was divided into four major segments; she went over homework from the day before, introduced a new lesson about functions, had students predict rules of functions, and display graphs of functions if rules were given. During the first segment, Ms. B took 6 minutes to elicit correct answers
to part of the homework. For the next 9 minutes to go over the remaining homework, Ms. B gave a student a touchpad with a stylus wirelessly connected to the computer with an overhead. Ms. B had the students with the touchpad show their answers to the homework questions by using the touchpad to sketch distance/time graphs or speed/time graphs. During the second major segment, Ms. B introduced the notion of function using a function machine applet on her computer. To use the applet Ms. B could drag numbers into the function machine, the animated machine moved as if it were performing an operation, and then it spit out a number as the result of the input. The applet had you input 1, 2, 3, and 4 into the machine and had blanks for inputs 5, 6, and 7 for guessing. Ms. B had students predict what the outputs would be for the inputs with blanks. She used the applet for four different functions. The first was linear; the second was constant, which she thought was an error so she skipped it; the third was the sum of the first n natural numbers; and the fourth was an exponential function. She gathered and displayed the points to students found for the functions using the CCT. After the break, she took about 5 minutes to have students predict the rule for each function without revealing what the rules were if they could not figure it out. During the last major segment of day one, Ms. B had students for about 10 minutes plot points of functions given by a rule and displayed the points with the CCT.

**Day two.** The second day was divided into 3 major segments. Ms. B began day two with a CO (class opener) that asked students to graph the height of the water being poured into two different pitchers. One of the pitchers was cylindrical, the other more hour glass shaped. Ms. B gave the students about 6 minutes to work on the problem. As
the students worked on the problem, Ms. B walked around the room observing and aiding students. Ms. B used her touchpad to sketch the graphs she saw her students produce and display them. For 8 minutes, Ms. B and the class discussed the graphs for the two different pitchers. For the next major segment, Ms. B had the students get out their homework from the previous day and submit answers to the CCT. For the homework, students needed to find coordinates of points on different functions. For the first question, students had to send their points that were on the graph of \( f(x) = -3x \). The students submitted several points that did not lie on the line. Ms. B picked wrong points and asked students how to fix the points. After correcting a few points, Ms. B concluded it would take too long to fix them all. So she cleared the mistakes and had one student send in points. After the single student send in her points, Ms. B graphed the line \( y = -3x \) using the CCT. Ms. B had the class determine the domain and range of the function. Ms. B had a different single student submit points for and asked similar questions for each of the following functions \( f(x) = |x|, \ x^2, \ \frac{1}{2}x + 1, \text{ and } 2x-3 \). A few differences in questions occurred for the different functions: Ms. B asked the students what was it about the quadratic that made its range nonnegative. During the discussion of the last line, a student appeared to recognize a pattern that the domain is always everything and asked Ms. B if that was always the case. Ms. B said she had to think about it. Then a student claimed he had an example of a function without a domain of everything. The student suggested a vertical line would not have a domain of everything, and Ms. B said his example was good. Ms. B took about 20 minutes to go over the homework. After lunch during the last major segment of the day, Ms. B assigned each student a different integer number,
ranging from about -10 to 10. For the first function discussed, \( y=x+1 \), Ms. B had the students find the correct output to the function using their assigned number as the input. After the students found the points, Ms. B collected and displayed them with the CCT. She then graphed the line \( y=x+1 \) using the CCT. Ms. B used this example to discuss the range of the line has values in between the integer values the students found. Ms. B had the student do the same thing for the function \( y=2x \). After correcting a student error, Ms. B asked the students to compare the two lines. She appeared to try to elicit from the students the notion of slope. A student suggested that the \( y=2x \) line was closer to the \( y=x+1 \) line. After doing the same task for the function \( y=-3x-2 \), Ms. B asked them to compare the three lines displayed and elicited the notion of steepness of the lines from the students. The last function Ms. B looked at on day two was \( y = x^2 + 1 \). After collecting and correcting student points, Ms. B had the students compare this function with the previous three. Students determined that it was “crooked, Not crooked, but curvy.” The third segment of day two lasted about 17 minutes.

**Day three.** Day three was divided into three major segments. Ms B. began the day by going over the homework from the day before. The homework asked students to find and plot points for different functions. Ms. B rather had the students use their assigned point for the homework, instead of multiple points. Ms. B collected points from the students like she did the day before. Two students misunderstood and submitted all the points they found for the entire homework set instead of one point at a time. After correcting this misunderstanding, Ms. B asked the students to correct their points if they were wrong. The homework problems included two quadratic, two linear, and one absolute value of a
linear function. After two or more functions were displayed using the CCT, Ms. B asked the students to compare the graphs. She asked the students what about the function made it either a V or a U shape versus a straight line. For the second quadratic before collecting points, she had students predict its shape. Ms. B spent 25 minutes going over the homework. For the next 7 minutes, Ms. B asked the students what conclusions about graphing they learned. A student concluded that a negative in front of the x causes the line to slope down. Another student mentioned that an absolute value has a V shape and it would be positive. Ms. B acknowledged the V shape as correct and questioned whether the graph would always be on the positive side. After some disagreement from the students, one student suggested putting a negative outside the absolute value bars. After the suggestion, Ms. B asked students to send in their points to the function \( y = -2|x| \) using their assigned points to test the students prediction. After Ms. B corrected points and graphed the function, the class saw that the graph was on or below the x-axis. After lunch, the third major segment consisted of three smaller segments. Ms. B asked students to look at functions in the textbook numbered 26-37 and predict without plotting or graphing whether their graphs would be a line, a V, or a U. Ms. B gave the students about 5 minutes to make their predictions and then Ms. B used the CCT to collect and display student responses. For the second part of the third segment, Ms. B asked students to send which functions they thought were lines using the number from the textbook. She saw a lot of students said problem 26 was a line. She asked for the equation for this problem and asked if the class agreed if it was a line. Only Jacob said problem 27 was a line, and Ms. B asked him what the equation was. After hearing the equation, Ms. B asked why the
rest of the students did not say it was a line. A few students remarked that it was not a line because it had an exponent. A few students said number 28, an absolute value function, was a line. A lot of students said that number 29, \( y = x + \frac{1}{2} \), was a line, but fewer students said that \( y = 7 - 5x \) was a line. Curious, Ms. B asked why there were fewer students for number 30. Ms. B., after several questions, elicited from a student why she thought the equation was not a line; she said it was not a line because of the negative in it. Ms. B went through the rest of the textbook problems in a similar manner. Ms. B took 7 minutes for the second part of the third segment. For the last part of the third segment, Ms. B took 2 minutes summarize what kind of equations have V shapes, U shapes or are lines.

**Representations**

Ms. B used the tabular, graphical, and algebraic representations in her classroom. She more frequently focused on the algebraic and the graphical. Except for the homework or CO that involved time graphs, all segments of instruction had at least two kinds of representations of the same object. Sources of representations varied, some were produced by Ms. B., some by the students, and some by several students at once using the CCT. Ms. B used the same representation, graphical, of two different objects for students to compare/contrast.

During the homework of the first segment of the first day Ms. B asked students about the graphical representation such as what did the flat parts of a distance/time graph represent, or when was the speed increasing. For the second part of first segment, Ms. B asked students to display their generated distance/time or speed/time graphs for different situations using the touchpad. Ms. B focused on the tabular representation during second
segment of day one using the function-machine applet. Ms. B also asked the students to find points for the different functions the applet created that she later collected and displayed using the CCT. During the second segment of day one, the tabular and graphical representations were present for the same object. After the break, Ms. B tried to elicit the algebraic representations for the applet functions by asking students to determine the rule for the function. The students quickly found the exponential function. Students had trouble finding the rule for the linear function, since they were finding rules for each individual input/output relationship rather than one that would work for all of them. A student recognized the pattern for the sum of the first n natural numbers, but could not find one rule for the sequence. Ms. B did not “want to give away answers” during this segment, and did not elicit or give them. During the last segment, Ms. B incorporated three major representations into one lesson. She gave the students the algebraic representation, the rule, and asked them to find points on this function creating a tabular representation of this function on the display. Then she plotted the points to make a graphical representation of the function.

In Ms. B’s class opener, only the graphical representation of the height of water in a pitcher at a given time was present. The second segment incorporated three major representations; students created a table of points for an algebraic equation that Ms. B graphed using CCT. The graphical/tabular representation came from multiple sources at once, since each student submitted their table of points to the CCT. However, given the number of errors present, Ms. B decided that one student at a time to submit points may be more efficient. Originally during segment two, representations had multiple sources
for same function but changed to a different source for each function. After the break, Ms. B focused on the algebraic and graphical representations. She assigned each student a number they were to use as an input. She gave the students an algebraic function the students were to plug in their number in to find the corresponding output. Ms. B then collected and displayed the graph of these points without looking at the table of points generated. After the points were plotted, Ms. B graphed the function connecting most of the points. The algebraic representation and a graphical representation came from one source, Ms B., but the plotted points came from multiple sources, the students. Ms. B left the graphs of each function displayed after initially discussing it, so she had the graphical representation for different mathematical objects. The same representation of different objects became the focus of discourse when she had student compare and contrast the functions.

Ms. B used representations similarly during the first segment of day three as she did the last segment of day two. Ms. B focused on the graphical and algebraic representation using single points from students using their assigned inputs to generate graphs. However, Ms. B shifted focus from what was different/similar about the functions to what about the functions were making them different. During the second segment of day three, Ms. B had students summarize the connections they made earlier between graphical and algebraic representations of the types of functions they observed. For the last segment of day three, Ms. B asked students to make predictions about the shapes of the graphs of algebraic equations using the connections they established. While the
graphical representation was not displayed for any of the functions, Ms. B asked students to picture in their head roughly what each graph should look like.

**Process/Object perspective.** Ms. B introduced the notion of function as a function machine illustrating the process nature and continued to emphasize the process nature of functions when she treated functions as a rule and when she elicited the points coming from the input/output calculations students performed. However, Ms. B had students compare/contrast shapes of graphs and algebraic equations to eventually see them as U’s, V’s and lines or as objects. Ms. B did not discuss the dual nature of functions, but she treated functions from both perspectives.

When Ms. B introduced the concept of function, she used a function-machine applet as shown in Figure 6.10 that animated the process nature of function. Ms. B first explained that “a number is put into the machine, something happens to that number and it spits out a number, alright?” in Vignette 12 line 3. In other words, a function performed some sort of procedure on a number to get another number. Further in line 3, Ms. B emphasized the input/output relationship between the ordered pairs of numbers the machine related rather than stating a function could be thought of as a set of ordered pairs. At the end of Vignette 12, Ms. B began using the function-machine applet to generate several points for a function.

Ms. B Vignette 12

1) **T:** We’re going to talk about functions today. Oh. oh, I have to find two more of these. I might have two more. We’re going to talk about functions today and a function is like a machine. And there’s a machine.

2) **SS:** Hey./ Cool.

3) **T:** Alright, and what happens is the machine... [inaudible aside to student] What happens is a number is put into the machine, something happens to that number
and it spits out a number; alright? The number that goes into the machine is called input and if you look at the sheet that I set up for you do you see input? It’s also the X value if we’re talking about ordered pairs. Another name for these numbers that you’re putting in is domain, and another name is independent variable. All those names for the same thing, and that’s why I wrote them above there. So what I’m going to do is I’m going to drag one into the machine and when I drag it in… out comes 1. Go ahead and put that down on your sheet.

Figure 6.10: Function machine applet.

Ms. B treated functions as process during several different segments, such as when she tried to elicit the rule for the functions produced by the function applet, when she asked students to find points of a function and graphed them, when she corrected errors of the student submitted points, and during her explanations of the shapes of graphs. Vignette 13 is an example of Ms. B’s treatment of function as process during her elicitation of the rule for the first applet function, y=2-x, during the third segment of day one. Before Vignette 13, the students had seen the points plotted for the function. But
Vignette 13 occurred after lunch and Ms. B re-created the table of points that ended at 
x=7 in Figure 6.11 for the function but she did not plot the points again. In line 1, Ms. B 
appeared to want students to think about extrapolations so they could determine “If I put 
in 100 what would happen…?” By focusing on specific numbers, 10 and 100, and the 
notion of putting the number in, Ms. B emphasized the process nature of functions. Since 
Ms. B did not redisplay the plot of the points but only the table of values, Ms. B further 
emphasized the process nature by not letting students reason with the graph. The graph 
could have let students see that globally the function moved down and to the right. Ms. B 
explicitly stated she wanted students to think of the procedure the function was doing to 
the inputs in line 5. Dorothy treated the function as the process of “adding -2” in line 6. 
Ms. B did not ask the question of adding -2 to what. Rather in line 7 perhaps because Ms. 
B wanted a single rule for the function, Ms. B treated Dorothy’s comment as adding -2 to 
the inputs. However, Dorothy could have meant adding -2 to the previous line to get the 
next. The student in line 8 did not see a single process for the whole function, but rather a 
different yet predicable rule for each input. She appeared to think that the function is 
subtracting increasing even numbers, 2, 4, 6. This student treated the function not only as 
a process, but as several different processes instead of as a whole single process. In line 
9, Ms. B tried to get the students to see a whole process, or a single rule that works for all 
the numbers by giving them a few examples of procedures. Students may have better 
seen the single rule, if functions had been treated as a thing or an object. But the students 
did not think of any. At that time, Ms. B did not want to give away answers and moved
on to the next applet function. During the extrapolation task, functions were treated entirely as processes by Ms. B and the students.

Ms. B Vignette 13

1) T: -5. What do you think is happening there? Like if I asked you what 10 was, or what 100 – If I put 100 in what would happen, can you Figure 6. out what would happen?
2) S: Wouldn’t it be neg. [inaudible]?
3) T: I don’t know, what are you thinking?
4) SS: I was thinking that when you put in a certain number… / [inaudible] / Yeah.
5) T: What do you think the machine is doing to the number when you put it in? Dorothy, what do you think?
6) D: Adding -2?
7) T: Ok, so by adding -2 to this one it would be -1. If I add a -2 to this it would be 0, and if I add a -2 to that I don’t think I get -1… What do you think?
8) S: I think it’s subtract 0 for the first one and you get 1, and then you subtract 2 for the second one, and subtract 4 for the third one and 6 for the fourth.
9) T: Ok, but, the machine only has one rule, and you’re giving me like a different rule for everything. The machine takes a number that comes in, does something to the number – whatever it does to 2 it does the same thing to 5, like if it doubles it or if it cuts it in half or whatever. No thoughts?
10) S: No.
11) T: That’s a hard one, huh?
12) S: Yeah.
13) T: Ok, let’s look at the second one first. I don’t want to give away any answers and maybe you can get extra credit. Let’s look at the next one.

Figure 6.11: Re-creation of table for first applet function.
Ms. B typically treated functions as process when correcting points submitted by students to the CCT. Vignette 14 is an example representative of how Ms. B corrected Brandon’s point for the curve $y=|{-2x}|$ during the first segment day three. Brandon had been assigned the number 7 the day before. Ms B. used the CCT list of names attached to the points each student submitted to see that Brandon’s point was wrong. Ms. B told Brandon the meaning of absolute value in line 1. She asked him for his assigned number, but changed her mind and asked other students to figure out his number. Ms. B focused on the procedure of finding the output from Brandon’s input in lines 5 – 9. Ms. B broke the function down into smaller pieces and asked Brandon what one step of the procedure for finding the output for his input in line 5. In line 7, Ms. B asked Brandon to calculate the next step of the procedure for finding the absolute value of $-2$ times 7. By asking students to do the calculations to fix errors, Ms. B had students working through the process of relating outputs with specific inputs. She was drawing students’ attention to the process nature of functions. During the instances where students’ errors occurred when they submitted points for functions, Ms. B treated the errors in a similar fashion by having them work through their calculations. Ms. B did not discuss how to correct the errors by looking at the larger pattern as shown by the graph. Ms. B treated the functions as pieces, rather than as a whole.

Ms B. Vignette 14

1) T: Brandon. What does absolute value mean? Absolute value means distance from 0, ok? So what was your number... No, somebody else tell me what Brandon’s number was. Who knows what number I gave him?

2) SS: (7, 7)/ 7.
3) T: 7.
4) S: Oh. [laughter]
5) T: So -2 times 7 is what?
6) S: -14.
7) T: -14. Now Brandon, absolute value means how far is -14 from 0?
8) B: 14.
9) T: Right, so he was fine except that he had...
10) S: Negative.
11) T: Yeah, he had a negative, so he’ll jump right into place.

Ms. B did not always treat functions as process. During the segments when she asked the class to compare and contrast different functions, she appeared to want students to notice the properties of the whole function to do the comparisons. During these instances, she drew students’ attention to the shape of the whole graph treating the function as an object. During the last segment of the last day, Ms. B asked students to make predictions about the shape of the graph of an equation without plotting or graphing beforehand. Ms. B appeared to want students to visualize the functions as being different shaped objects such as lines, V,’s and U’s. The discussion during Vignette 15 was the first time Ms. B asked students to compare two different functions. Ms. B started the discussion by asking students to look at the two sets of points, although when she collected the second set of points the first set disappeared leaving only the graph of the line displayed. The students compared the functions y=x+1 and y=2x. She asked students to “just say something” about what they noticed about the two sets of points in Vignette 15 line 1. The students began noticing process-oriented comparisons, or students were focused on the points of the lines rather than the lines as a whole such as going through the origin line 2, and intersecting at (1,2) lines 4-7. By not moving on after having a few correct observations and continuing to ask what they notice, Ms. B pressed the students to think more about
the graphs of the functions. Beginning in line 8, students began seeing more object-oriented comparisons or characteristics of whole function rather than its pieces. The student in line 8 noticed that both functions had a positive trend, which is a property of all of the points of the function rather than a property of a single point. In line 12, a student stated both functions were straight lines. He seemed to notice that the functions were *things*, lines, and had a global property, straightness. In line 13, Ms. B reaffirmed the importance of the classification of straight lines, since they might not always have them and then pressed for more comparisons. In line, 15 Ms. B summarized the student observations. Students appeared to think Ms. B wanted a specific observation, since after her “come on, something else” in line 15 the student in line 18 questioned what Ms. B was looking for. After getting a few repeat observations and a clarification about the task in lines 16-19, Ms. B pressed again for more observations. Shaniqua seemed to notice that the majority of green line, $y=2x$, was closer to the $y$-axis than the red line, $y=x+1$. She appeared to focus on the lines as wholes rather than as points since both lines touch the $y$-axis at specific points, so perhaps she focused more on the object nature in her observation or did not notice the specific points that contradicted her actual statement of “the green line is like closer to the Y.” Perhaps because a student stated an observation that was closely related to slope or perhaps she did not notice the contradictions in Shaniqua’s statement, Ms. B claimed she was correct and wanted students to make the same observation but using different words in line 21. Another student noticed the angle the green line made with the $y$-axis was smaller than the angle formed with the red line and the $y$-axis. Ms. B did not immediately claim this was correct; rather, she went up to
the board to measure the angles to verify this statement in the first part of line 23. Even though a student brought up the notion of slope, Ms. B did not pursue this notion and moved on to the next problem.

Ms. B Vignette 15

1) T: Alright. Look at your second set of points compared to the first, and I thought I had changed the color on that so it would actually... I thought I changed it. There we go. Look at your second set of points compared to the first. Talk to me about that. Just say something; make a statement about it, but raise your hand to do it. Go ahead.
2) S: The green one’s going through the origin.
3) T: Oh, interesting. Yes. The green one went right through the origin, the other one did not. Good observation. Something else, Jacob?
4) J: They meet at (1, 1). They intersect.
5) T: They intersect at (1, 1).
6) SS: (1, 2)
7) T: (1, 2), you’re right, it is, (1, 2). So let me just talk about that for one second. Look, that means that the first one which was X + 1, right? 1 + 1 is 2, and the second one is 2 times 1 and 2 times 1 is 2, and that’s why because they both make that equation true. Something else? Anything else you want to say?
8) S: There’s a positive trend.
9) T: There’s a pos. trend, awesome. Anything else? [several second pause] Something else, Dorothy?
10) D: They’re all real numbers?
11) T: Yeah, they do, they’re all real numbers, um hum.
12) S: It’s a straight line.
13) T: It is a straight line. Anything else? And that’s important because pretty soon we might not see a straight line. Anything else? Shaniqua, you want to say something else?
14) Sh: I don’t know, I mean I know [inaudible]
15) T: [inaudible]? You can’t see... We talked about... this one goes through the origin, this one actually goes through the point (0, 1), right? That they meet here, they’re a straight line... come on, something else. Jacob?
16) J: They all increase.
17) T: They all increase, and I think somebody said positive.
18) S: Are you looking for Y is...
19) T: I’m looking for observations about those two lines; anything you want to say about them. Shaniqua?
20) Sh: The green line is like closer in to the Y.
21) T: Yeah, it is, it’s closer in to the Y. How can you say the same thing? How else can we talk about that?
The next function after the two were discussed in Vignette 15 was \( y = 3x - 2 \). After collecting and correcting student points like Ms. B did in Vignette 14, she asked students to make observations about comparison of the three lines like she did for the previous functions. Ms. B began eliciting observations about the three functions by first clarifying she wanted them to say something about the line and that there was no right or wrong answers in Vignette 16 line 1. In line 2, Carla repeated one of the observations from the previous discussion that it was another straight line. Another student noticed it had a positive slope. Ms. B appeared to treat this observation as a repeat of the observation of “positive trend,” since she stated it was “another positive slope.” Ms. B wanted students to make the observation that the new line was steeper than the other two lines as evidenced in line 14. Even though the student mentioned the word “slope” that as a concept is highly dependent on the notion of steepness Ms. B did not pursue the meaning of this word to reach the one she intended. Rather she waited until an informal definition of steepness, “Instead of like one more down it goes like up.” was given in line 7 to pursue the notion of steepness. In line 8, Ms. B asked for another way to say “more up.” In line 10, she gave students a visual aid as a hint to the word she was looking for. Again she did not pursue the word “slope” in line 11. After another hint, a student said the right buzzword in line 13. The students cheer when they got the right word in line 15. Ms. B asked the students to make the comparisons between the lines using the notion of
steepness in line 16. Ms. B pursued a characteristic of the whole line as a measure of comparison, steepness, which emphasized the object nature of the lines. The students jumped right into observations about the whole function in line 2 and continued to make similar observations throughout Vignette 16, instead of observations about single points. So the students appeared to be treating functions as objects. Ms. B attempted to see if students could make the connections between what about the algebraic function was making the lines steeper or not in line 18. She pursued this question and the question about what make the graph cross the y-intercept at a point in lines 18-28, but apparently she only intended to introduce the idea in line 28.

Ms. B Vignette 16

1) T: Oh, ok. Let me get so I can see all your numbers. Ok, alright, so talk to me about this one compared to the other two. Say something about that line. Carla, just say something. There’s no right or wrong, I’m just asking you to make an observation.
2) C: It’s a straight line.
3) S: It’s a positive slope.
4) T: It’s another positive slope, good.
5) C: It’s a straight line.
6) T: Yeah, it’s a straight line.
7) S: Instead of like one more down it goes like up.
8) T: Oh, give it another name; more up.
9) SS: Vertical / Closeness
10) T: Think about hills.
11) SS: Slope? / Height?
12) T: Think about skateboard ramps.
13) SS: Vertical / Height? / Steep?
14) T: Steep, yeah.
15) SS: [clap]
16) T: Is it steeper? Is the green line steeper than the yellow one and steeper than the orange one?
17) SS: Yes.
18) T: Yeah, so we’re going to start to figure out what is it that makes it steep; what is it that’s making that one steeper than the other.
19) SS: The formula / The points? / The numbers.
20) T: You’re all using the same points, guys, and you’re all using the same X’s.
21) S: The function.
22) T: Yeah, what about the function, though?
23) SS: It changes/ It’s different.
24) T: It is different, and also you were talking about where the line – what did you
tell me before? This one went through the origin? What does this one go through?
25) SS: The middle./ 1... / -2?
26) T: Yeah, it goes through -2 and that one goes through 1; what is it that’s making
it do that?
27) S: The function.
28) T: I’m only putting these questions in your head; you don’t have to answer them
right this minute.

Figure 6.12: CCT displaying functions y=x+1, y=2x and y=3x-2.

**Treatment of representations.** Representations were used to enact procedures upon,
such as plugging a point into an equation to either plot the points or to make corrections
to these plots. Representations of functions did not appear to be created for the sole
purpose of creation as an end product, but rather representations were used as something
to reason about, something to discuss, and something to make (brief) explanations with.
Even though the discussions ended with unanswered questions or unpursued concepts,
students were asked to eventually think about the representations they created. Some of the applet generated functions were not discussed immediately, but students had to think about them when Ms. B wanted them to find the rule for these functions.

While the purpose of the homework from day two appeared to be create the end product of the graph of a given function, Ms. B did not have students create and display the functions during segment two day two to simply create them and move on to the next function. Rather during Vignette 15, Ms. B continually asked what observations students noticed about the different functions. In Vignette 15, Ms. B did not direct the observations until the end when a student came close to the notion of steepness. In essence, not only did she have students think about the representations but by not directing the observations she implicitly had students think about what differences/similarities were important enough to mention. Ms. B wanted students to notice that the lines had different steepness but she appeared to not want to give out the answers. Ms. B guided the observations students gave in Vignette 16 by offering hints using connections to “real world” objects that had the property of steepness she wanted them to see. In this instance Ms. B had the students think about the representations of lines in a certain manner.

During the first segment day one, Ms. B asked students to use her touchpad to display the graphs they made for homework. After Ms. B helped the student with the setup, he created a height versus time graph to model the elevation of a person hiking in the mountains. He drew a graph where the person went up the mountain, had lunch, and came back down shown in Figure 6.13. Ms. B in Vignette 17 line 1 drew the students
attention to the near vertical line segments and what they would mean in the problem context. A group of students suggested the hiker fell in line 2, 4 and 6. Ms. B did not accept this explanation claiming (in line 7) there needs to be some slant, since even when falling would take some time to change heights. Ms. B did not have the student create the graph to display it to the class and move on; in other words she did not use the representation as an end product. But after the students created the representation, she used it to discuss the meaning of the representation in context of the hiker. She used representations as something to reason about.

Ms. B Vignette 17

1)  T: Whoa, look at that section... Yeah, look at the section right here. What does this section... actually, these two sections, but this one is almost straight. What would that say?
2)  SS: [inaudible]/He like fell off the mountain and he had to recover. [laughter]
3)  T: He went from this height to this height in how much time?
4)  SS: He was rappelling./No time.
5)  T: No time. Yes. Alright?
6)  S: He fell.
7)  T: If he fell like this, you’re going from that height to that height in no time, which is pretty impossible. So there needs to be a little bit of slant there, Jacob. Your speed as you travel from home to school – thank you – do you have your graphs done?
8)  S: Yeah.
Discourse

The purpose of discourse in Ms. B’s classroom appeared to be to elicit correct answers/procedures from students, but also to elicit and aid some student thinking. The discourse in Vignette 14 was an example of the former and the discourse in Vignette 15 was an example the latter. The discourse was always teacher to student or student to teacher. Only a few instances occurred where a student spoke without initially being prompted by Ms. B

Ms. B spoke 10271 words over the three observed days in a total of 671 utterances. Ms. B averaged 3423.7 words per day and 223.7 utterances per day. Ms. B did not vary much in her words spoken per day or the number of utterances. Ms. B spoke an average of 15.3 words per utterance. Ms. B spoke 4 utterances that were over 100 words with the largest being 160 words. None of these utterances occurred during day three. Ms. B
spoke 8 utterances between 70-100 words with most of them spoken during days one and two.

Ms. B’s students spoke a total of 3557 words in 658 utterances over the three days. They averaged 1185.7 words per day in an average 219.3 utterances per day. The students did not vary much in how many words/utterances were spoken per day. The students spoke an average of 5.64 words per utterance. Twelve of the student utterances were over 20 words with the largest being 54 words.

**Questioning.** Ms. B used IRE sequences to elicit correct answers/procedures from students such as leading the student through the calculation of absolute value of -2 times 7 in Vignette 14. However, she also used open ended questions to elicit and aid student thinking such as when she asked students to make observations about the graphs in Vignette 15 and 16. She also asked students what parts of the graph meant as in Vignette 17 and she asked them to make predictions about patterns of inputs as in Vignette 13 and about shapes of graphs during the last segment of day three.

During the segment where Ms. B asked students to predict the shape of the graph based on its equation, Ms. B occasionally asked students to justify briefly the correct prediction. In Vignette 18 line 1, Ms. B noticed one student said the equation $y = x^2 - 4x + 4$ would form a straight line and asked why the rest of the students did not say it was a line. Initially, a group of students essentially said it’s wrong to think it is a line. Ms. B did not accept the “Cuz he is wrong” argument and pressed for a reason why the student was wrong. Even though she pressed for a reason why the student was wrong, she appeared to want a simple explanation that the equation had an exponent as
evidenced in lines 7 and 9. A student in line 6 began to offer an explanation as to why the exponent would cause the function to not be a line. His argument, while it did not address all the subtleties of the function such as the function could be zero or negative somewhere, appeared to be that the function could not be a line because the square made the function all positive. Either the student or Ms. B could have additionally stated that lines eventually become all negative after or before the x-intercept further justifying his argument. But Ms. B appeared to want to focus on the simpler explanation of “it has an exponent” in line 7. Ms. B’s explanation of “it gets bigger faster” was a casual comment that did not address important aspects of the problem such as addressing faster than what and that lines have a constant rate of change. Her statement is untrue in certain cases such as numbers that are close to zero as squaring them makes them smaller and parabolas have slow growth near its vertex. Even though Ms. B pressed for explanations, the students’ and her explanations were not deep.

Ms. B Vignette 18

1) T: Y = X sq. minus 4X + 4, and you think that’s going to form a straight line. You seem to be the only one. [laughter] So why did the rest of you not put that?
2) SS: Cuz he is wrong.
3) T: Give me a reason why he’s wrong.
4) S: There’s an exponent.
5) T: Because there’s an exponent in the equation.
6) S: We don’t know what X is... it could be 12 or.../ It doesn’t matter, neg. or pos. it’s still going to be a pos. number because the neg. times a neg. means it’s going to be a pos.
7) T: But what makes it not a straight line? It’s the fact that...
8) S: It’s an exponent.
T: It’s an exponent, and so when you square something it gets bigger faster, alright?
Vignette 19 is an example of an instance where Ms. B did ask for more than a few word explanations. During the CO of the second day, Ms. B asked students to sketch graphs of water being poured into a curved pitcher at a constant rate after first doing the same thing for a cylindrical pitcher. Ms. B sketched the graph she saw most students produced, which was a straight line that eventually became flat. Ms. B noticed when she went around the room while the students were graphing, many had sketched the same graph for both. Before Vignette 19, Ms. B asked the class if they should have gotten the same graph again. Shaniqua knew the graph should not be a straight line and that something happened at the neck of the pitcher. She claimed in line 2 that the graph was a line that became horizontal for a time before it continued to fill up to the top. When Ms. B asked her to state what she said again, Shaniqua had difficulty expressing herself in line 4. Given Ms. B’s statement in line 5 about Shaniqua knowing the flat section should not be there, perhaps Shaniqua changed her mind in line 4 during the inaudible part. Ms. B explained that the height should never be flat until the pitcher filled because water was constantly being poured in lines 7 and 9. Jacob recognized that the water would fill more quickly in the thin part of the pitcher in line 13. As Ms. B and Jacob were discussing in lines 9-19, Ms. B sketched the graph she thought Jacob was describing. Ms. B created a piecewise graph with two line segments with the second being steeper than the first as seen in Figure 6.14. Jacob confirmed Ms. B’s interpretation of his graph in line 19. Another group of students said that since the pitcher was curved the graph should curve in line 20. During the explanations of why the graph should be curved and why the change in steepness in lines 21 and 23, Ms. B sketched the curved graph over top of the
straight lines. Ms. B did not ask any student to perform a procedure, but to think about
the relationship between the context of the problem and the representation of it as a
graph. The students and Ms. B gave more in depth explanations than the few word
explanations in Vignette 18.

Ms. B Vignette 19

1) T: Ok. Somebody else? I mean she’s saying that what she had was this; just
another straight line. Shaniqua?
2) Sh: I got the line was going up and it stopped and then kept on going.
3) T: Say that again? Hold on; let me see it.
4) Sh: Because it was kind of being... I guess because... I think I just [inaudible].
5) T: So you know it shouldn’t be straight; you know you shouldn’t have that section
right there, is that what you’re saying?
6) Sh: Yes.
7) T: Ok, because it’s not going to stay the same height in the pitcher if you keep
pouring into it, right?
8) Sh: Um hum.
9) T: I mean if you’re pouring in, the height is going to change. Alright, Jacob?
10) J: [inaudible] but my height kept constant – not constant, but like what Dorothy
had, like that, the first one. Jake was...
11) S: Dorothy?
12) T: That was Leslie; that’s ok.
13) J: Yeah. Sorry, yeah, and then it really shoots up because the thinner it is the
more water the water takes. [laughter]
14) T: So you’re saying when it reached here...
15) J: It went higher.
16) T: Because they said the thinner the thing is, it went higher faster.
17) J: Yeah, yeah.
18) T: So you wanted it to go –woosh – like that.
19) J: Yeah.
20) SS: Because the curve; it’s supposed to curve./ Yeah.
21) T: It’s supposed to curve and it curves slowly at first because this is curving and
this is changing. As this goes up... picture water in this and you’re looking at it
after a second. When it gets to here... not water, juice... it takes more juice to fill
that area so the height changes not so fast. Everyone with me?
22) SS: yeah./ Yes.
23) T: And then right in here its almost like the height changes because it’s almost
straight right in there, and then, Jacob’s right, that was key there. When that gets
narrow if you keep pouring at the same rate it’s going to fill faster in that section
because there’s not as much space to fill, so you’re right. A nice curve up there would be good. Good job.

Figure 6.14: Ms. B’s graph of height of juice in a pitcher after being poured at a constant rate.

Choice of Examples. Ms. B’s choice of examples appeared to be mostly chosen ahead of time. The representations present came from the students’ homework during the second segment day two and first segment day three, from their workbooks during the third segment day two, or from their textbooks segment three day three. Ms. B chose the problems from these texts ahead of time. However, Ms. B did not choose the representations generated by the students during the first segments of days one and two, nor did she choose the representations the function applet generated.

By the time the students were asked to compare observations about different functions during day two, students had seen positively and negatively sloped lines with integer and fractional coefficients and positive and negative y-intercepts, quadratic
functions, absolute value functions, and an exponential function. Although students had seen graphs of piece-wise functions and functions that were not defined everywhere such as the graphs for the height of the water, Ms. B never made the connection that these graphs were graphs of functions. Also missing from the list of examples were important cases such as upside down parabolas and absolute value functions. The choice of examples seemed to have led a student to make an incorrect generalization that the domain of functions is always everything since that was the case in all the examples he had seen.

Another example of an unplanned representation occurred during day three when a student introduced the example of the upside down V after being asked if absolute value functions were always positive. Ms. B asked students to summarize what they learned about graphs of functions after the second segment of day two. Before Vignette 20, a student said she noticed that a negative in front of the x caused the line to slope downward. In line 1, Ms. B asked for something else they learned. A group of students made the observation that absolute value functions are like a V and they will always be positive. Given the examples they had seen, this was a reasonable conclusion. Ms. B incorrectly affirmed the statement about the V, but in line 3 pressed the students to think about the second statement about always being positive. To make her affirmation of the V statement correct, she would have needed to preface the statement with the absolute value of linear functions shaped like V’s. The absolute value of a quadratic could be shaped like a curvy and pointed W. A student, in line 8, mentioned that a negative in front of the absolute value bars should cause the values to all be negative. Ms. B asked
for a specific example in line 9, and the student gave her \( y = -2|x| \). The class took about 34 lines of dialog explaining how student can use their calculators to find the outputs to their assigned inputs, since students were having trouble in the segment before. In line 12, Ms. B started to evaluate the list of points submitted by the students. After making a few corrections to points in lines 14-19, in line 23 Ms. B explained why this function should be opened downward since the absolute value of a number will always be positive and when multiplied by a negative will always be negative. In line 24, a student noticed that the graph looked like a Christmas tree. In Vignette 20, a gap in Ms. B’s choice of examples of functions was exposed. But Ms. B appeared to have recognized this gap and decided to fill it by having the students think about how an absolute value function could be negative and graphing it.

Ms. B Vignette 20

1) ...Ok, somebody else?
2) SS: If you have absolute value bars are the equations – it will always be on the positive side. It will be like a V.
3) T: Right. It will be a V, and you think it will be on the positive side all the time?
4) SS: Yes./ yeah./ No./ No, it would be [inaudible]
5) T: It might be like this?
6) SS: yeah.
7) T: What did you say?
8) S: When there’s a neg. outside the absolute value bars.
9) T: What if there’s a neg. outside the absolute value... What if it’s \( Y = \ldots \)
10) S: -2.
11) T: ...-2, absolute value of \( X \). Alright, store... let’s practice....
12) T: -2, a-b-s-x. that’s all I have to do. So let’s just look at your list; you got it?
13) SS: Oh, no./ No.
14) T: If it’s down that means every single one of these should be...
15) SS: Neg.
Missed opportunities. Ms. B had several missed opportunities to reinforce the object nature of functions, the connections among the representations of function, or the meaning of function. The kinds of missed opportunities Ms. B had varied. The following are examples of the missed opportunities. When Ms. B corrected student submitted points, she focused solely on the process nature of functions. To save time when collecting points, Ms. B limited the number of students who could submit. When Ms. B introduced the notion of function, she left out a very important part of the definition; one and only one output for a given input. Using her machine analogy, she could have said the machine would always do the same thing to the same number. Ms. B connected the representations of many of the functions present, but she failed to connect some of the more interesting ones. Ms. B did not notice increased student enthusiasm and thus failed to capitalize on it. Ms. B did not pursue a line of inquiry the students seemed interested in pursuing.
During Vignette 16 line 18, Ms. B asked the students “what is it that’s making that one [green line] steeper than the other [yellow line].” Several students offered suggestions in line 19 such as the formula, the points, and the numbers. Ms. B focused on the numbers/points suggestion by stating “You’re all using the same points, guys, and you’re all using the same X’s” in line 20. After establishing in line 22 that the function was causing the one line to be steeper than the other, Ms. B pressed the students again for what about the function was causing the difference. Apparently unsatisfied with the responses of “it changes/ it’s different,” Ms. B tried to draw the students’ attention to some of the observations they made earlier in line 24. Ms. B knew she would address the question of what caused the one line to be steeper later, so she decided to move on to the next problem in line 28. Ms. B had the students looking at the graphs of the function during this instance. If Ms. B asked students to compare and contrast the algebraic formulas for the functions and the shapes of their graphs, the students could have noticed the difference in the coefficients of x and the constants. And perhaps they could have noticed that the constants related to the y-intercepts and the bigger the coefficient of x the steeper the line.

The graphs students created as part of the homework before day one and during the CO were interesting examples of piece-wise functions not defined for every x. However, Ms. B never mentioned that these graphs were graphs of functions. The graph in Figure 6.13 revealed why functions having only one output for a given input is important since a person cannot have two different heights above ground at the same time. However, Ms. B failed to mention this important aspect of function in her introduction and could have
used this graph as an illustration as to why it is important. If Ms. B had made the connection that the different graphs were functions, the student during day two may have not incorrectly concluded that functions are defined for all x.

During the second major segment of day two, Ms. B asked students to submit points they found on the graph of $y=-3x$. The students found these points for homework. In Vignette 21 lines 1-7, Ms. B gave students instructions and asked for a reminder as to what function the students were supposed to model with points. At the end of line 7, Ms. B could see the points students submitted in Figure 6.15. Several students sent correct points, but several more did not. Many of the points submitted appeared to lie on the graph of $y=3x$, while others seem to fit a line with a slope of positive or negative 4.

Instead of looking at patterns formed by these points, Ms. B began correcting individual points and asked students how to fix them in lines 7-13. After making a correction, Ms. B began to look at other points in line 15. She noted that there were “some” mistakes and the students corrected her with “some?” in lines 17-19. At the end of line 19, Ms. B corrected several points by focusing on the process nature of function by performing the calculations. Ms. B came to the conclusion in line 23 that students might be forgetting how to multiply negative numbers. Instead of focusing only on the calculations, Ms. B could have asked students to look at the pattern of the correct points to determine which points did not follow the pattern. Ms. B had an opportunity to illustrate the object nature of functions by looking at reflections and observing patterns of points, instead Ms. B focused on the process nature of function by focusing on the calculations of finding outputs from inputs. After Vignette 21, Ms. B decided that too many students made errors
and it would take too long to correct them all. So Ms. B cleared the student submitted points and asked a single student to submit. Ms. B’s method of recalculating each incorrect point would have taken a while to complete. However if Ms. B focused on the pattern formed by the points, correcting points could have been quicker. Ms B. eliminated multiple sources for representations of the same function by only allowing one student to submit points, which in and of itself limited opportunities to make connections and to reason about representations.

Ms. B Vignette 21

1) T: It is, because I’m thinking. I have to get a list ready. You’ve got points that you’re going to send me, right? You’re going to send me a list. Alright, and I want to make sure that I plot it once it gets here, so the X… so that you get to see the same thing I see. Oops, not down there… that one… alright. Can you refresh my memory; what’s number 1? What’s the equation?
2) Ss1: -2… Ss2: Y= -3x. Ss3: -3x.
3) T: Y = -3x. Ok. I’m ready. Are you ready?
5) T: Here you go.
6) S: So at L1 you put the first number and then L2 you put the other number?
7) T: Yes. I see one I think is wrong. That one right there. Cherise, did you mistype something?
8) C: Yes. It’s supposed to be -1.
9) T: It’s supposed to be -1?
10) C: Yeah.
11) T: Ok.
12) SS: Negative. -1.
13) T: Oh. Be careful when you’re typing so that you type what you want to type. Alright, do you want to fuss with the graph? Oh!, Oh!
14) SS: Slow… it, it…
15) T: Yeah… Alright, let’s look at these. What did you say the equation was?
16) S: Y = -3x.
17) T: So we’ve got some mistakes here.
18) SS: Yup. Some?
19) T: Yeah. [laughter] We have many mistakes. Ummm, let’s go here. -2 times -3 is 6. -3 times -1 is 3. -3 times 0 is 0. -3 times 1 is...
20) SS: -3.
21) T: So that one’s not right. How about the next one?
22) SS: -6.
23) T: Alright, are we forgetting how to multiply negative times a positive?
24) SS: Maybe. No. I didn’t.

Figure 6.15: Points students submitted for the equation y = -3x.

During another task in the second segment of day two, Ms. B asked students to submit points to the curve $y = x^2 + 1$. For all the previous functions during this segment, Ms. B needed only the standard graphing window. When students started sending in points to the quadratic curve, she realized she would need to zoom out since some y-values were quite large (Vignette 22 line 1). Ms. B pressed a zoom button on the CCT that automatically creates a graphing window so that all the data points can be viewed. The TI-84 graphing calculators have this feature as well; however, when the zoom feature is used the graph screen is cleared and everything is redrawn. The CCT instead dynamically changed the viewing window and the graphs in it. The students saw the parabola and the lines in the left picture Figure 6.16 collapse into the graphs in the far right picture. The students saw a dynamic transformation of functions and they seemed impressed by it in line 2 “Whoa! [laughter].” Ms. B seemed to not notice what happened.
and did not pick up on the student enthusiasm. Instead Ms. B began to correct points by again focusing on recalculating specific points in lines 3-5. After correcting a few more points she began asking students to compare the new function to the two previous lines. If Ms. B noticed the thin parabola transform on the screen into the wide parabola, she could have used the students’ enthusiasm to illustrate the effect of changing scales on the graphs of functions which could emphasize the object nature of function. Since the projector’s display window of the graphs did not change, the students witnessed the transformation of one group of functions into another group. Ms. B could have used the dynamic nature of the zooming windows to introduce and illustrate transformations of functions. Again Ms. B seemed not to notice and began focusing on the process nature of the function.

Ms. B Vignette 22

1) T: My goodness, I got numbers up to 145; I can’t see those, I’ll have to change this.
2) SS: Whoa! [laughter] You need a negative...
3) T: Ok, we’ve got a problem here, right? Let’s look at...
4) S: Johnny’s.
5) T: ...Johnny’s. What is -9 sq. ...

Figure 6.16: Altering viewing window dynamically changed the appearance of the parabola.
Technology Use

Ms. B used technology throughout her instruction during year three in some form or another. Ms. B used either the CCT or a touchpad and Smartboard™ software technologies every day observed. Ms. B no longer used the overhead projector during the observed days. Ms. B frequently used CCT as information displayer/gatherer, answer checker, and a discourse generator. Ms. B used the CCT to collect from students and graph the points generated from the function applet on day one. Ms. B during the last segment of day one, the second and third segments of day two, and first and second segments day three asked students to find points for different equations of functions. During these segments, Ms. B would then collect and display the student generated points. Ms. B would usually correct points that did not fit the graph and after a few functions were graphed she may have students discuss comparisons of the graphs. During the last segment of day three, Ms. B (instead of collecting points from students) collected predictions from students about the shapes of the graph of given equations of functions.

Although the CCT was not instrumental for comparing the graphs in Vignettes 15 and 16, Ms. B used the display of the graphical representation of different functions to generate discourse. In both Vignettes she started the discourse by asking what did they notice about the graphs displayed leaving it open to the students to come up with their own observations. During these Vignettes Ms. B elicited several observations. During Vignette 15, Ms. B did not guide the student observations and let them say what they wanted to. The students’ observations during Vignette 2 included both process nature and
object nature observations about the graphs. Ms. B during Vignette 16 gave hints to guide the students to the notion of steepness. Ms. B used technology other than CCT to generate discourse. The touchpad connected to the computer was portable to allow students to use it. Ms. B used the touchpad and Smartboard™ software to collect and display information gathered from students as in Vignette 17 and 19. The student used the touchpad to sketch his height/time graph and to display it to the class. Ms. B took time to discuss the graph and its meaning rather than display correct answers and move on as she did her first year. Ms. B also used the collection and display of points to generate procedural discourse around the corrections of points as in Vignette 14, 21, and 22.

Ms. B used the collection of student predictions about the shape of graphs of functions to generate discourse. During the last segment of day three, Ms. B asked students to submit the problems that contained functions that would have a graph of a line. Ms. B used the CCT display of the student predictions to ask a student who predicted problem X was a line why they thought it was a line. Ultimately, Ms. B consecutively discussed each problem because at least one student predicted each problem was a line. However because of the CCT, Ms. B was able to determine how many students were making incorrect predictions. Typically only a few students made the incorrect predictions. The CCT also allowed Ms. B to see when students were not making correct predictions in Vignette 23. In line 1, Ms. B noted that many students predicted problem 30 was a line and asked for the equation. Because of the CCT display of the results, Ms. B noticed in line 3 that more students said problem 29 was a line versus
problem 30. Had Ms. B simply asked the class if problem 30 was a line without the CCT collection, she may have missed this discrepancy. Ms. B seemed curious as to why some students were thinking \( y = 7 - 5x \) was not a line. After several lines of dialog, the student finally understood in line 16 that Ms. B wanted to know why she thought one problem was a line while this one was not a line. In line 16, the student thought that since problem 30 had a negative in the equation that it would not be a line. Ms. B convinced the student that lines could have negative slopes in lines 17-21.

Ms. B Vignette 23

1) T: Ok, 30. Lots of 30’s. What’s the equation for 30?
2) S: It is \( Y = 7 \) minus \( 5x \)?
3) T: I don’t have as many for 30 as I did for 29. Why?
4) S: There’s more 29’s than 30’s.
5) T: Why did some of you think 30 wasn’t? Anybody?
6) S: I think it is.
7) T: You think it is?
8) SS: I think it is./ I don’t.
9) T: You don’t think it is?
10) S: I think it’s [inaudible]
11) T: Ok, but I’m asking you what do you think.
12) S: I don’t think it is.
13) T: Why?
14) S: I didn’t put that number in.
15) T: Why did you pick one number and not another? That’s what I’m asking you.
16) S: Because one had a neg. in it and the others had pos.
17) T: Yeah, it could be a neg. [sloped] line, right?
18) S: Yeah.
19) T: Yeah, vs. a pos. [sloped] line, but it’s still a line.
20) S: Yeah.
21) T: Ok, so I think 30’s a line.
After Ms. B collected points from students that fit graphs of equations, Ms. B would use the CCT to graph the equation to leave a semi permanent record of the function and to verify the correctness of the points. The class knew if the points were correct if they fell on the graph displayed. Ms. B also used the collection and display of points, and the graph to verify the prediction that $y=-2|x|$ was beneath the y-axis in Vignette 20.

Ms. B also used technology as a calculation tool. Ms. B wanted to help students who were making calculational errors when sending points that fit functions in Vignette 24 line 1. Ms. B explained to students that they could use the calculator memory to help them in line 3. Ms. B demonstrated how they could use the calculator memory and variables to help them calculate function outputs in lines 7 and 9. The students thought this ability of the calculator was “cool” lines 8 and 10. Ms. B showed them they could quickly calculate another output of the same function by storing 10 for x and doing the same thing in line 13. Prompted by a student question, Ms. B explained with student help that they could use more variables than just x for storing numbers in lines 16-28. Ms. B instructed students how they could use their calculators as a computational aid as well as an answer checker.

Ms. B Vignette 24

1) T: Alright, I’m going to show you a trick so you don’t get these wrong. I don’t want you to get them wrong when you’re doing this. Alright, how do I go back on...
2) SS: [inaudible]
3) T: Alright... you can store a value in your calculator...
4) S: What’s that?
5) T: What’s store?
6) S: No, never mind.
7) T: ...a value in your calculator. So if your $X$ was $-3$, let’s say, for instance. You could store it, which is s-t-o, right here, as $X$. So now your calculator thinks $X$ is $-3$.
8) S: Oh, that’s cool.
9) T: So what I’m trying to do... $2X \text{ sq} + X \text{ minus} 2$, all I have to type in is $2X \text{ sq} + X \text{ minus} 2$, hit enter, and get 13.
10) SS: Oh, that’s cool./ Sweet.
11) T: And if I want to change that value...
12) S: You do the same thing?
13) T: ...I just... let’s say I want it to be $-10$. Then I just hit $-10$, store it as $X$. So until you change it... right now, before you store anything, I think your calculator thinks $X$ is 0. But you can make it think it’s anything at all and it will store that variable and then...
14) S: Can you change variables?
15) T: What?
16) S: Can you change the variable?
17) T: Um... yeah, I think it’s more work.
18) S: I think you’re right.
19) T: I haven’t really tried it. I think you can... I think if I can do $-3$, store, and because you’ve got all these letters here, I could probably store it as $A$.
20) S: You hit alpha.
21) T: Oh, I clicked alpha, you’re right. Second $A$...
22) S: You clicked [inaudible]
23) T: That is right. What am I supposed to hit?
24) SS: Alpha. / A
25) T: $A$. Enter. But then, see, Tyler, every time you’re going to have to go 2, alpha, $A$, sq., plus say 5$A$, whatever. It’s just harder... $+6$, and then it will give you... yeah, you can do it, but it’s easier to use $X$ because you don’t have to do that alpha.
26) S: Can you store a lot more?
27) T: Well you could store a value for every single letter of the alphabet, I guess.
28) S: What’s on the other letters? You have $X$, $K$ and [inaudible]?
29) T: Ummm, very higher math things which...
30) S: Oh.
31) T: ...I’m not real sure of. Higher, higher math. Variables... can’t help you; I’d have to research it. Alright, so let’s make a conclusion about graphs here. What did you learn?
Decisions

Ms. B’s goals appeared to be to elicit and aid student thinking without always revealing correct answers; address student questions; use student suggestions or student generated representations; and to want students to get the right answers. Ms. B elicited student thinking in: Vignette 13 as she asked students to predict what would happen with numbers outside their data; Vignette 15 as she asked students to make comparisons about the two different lines graphed; Vignette 16 as she asked students to compare three different lines while steering them toward the notion of steepness; Vignette 19 as she asked students to think about the shape of a graph of the height of water versus time; Vignette 20 as she asked students if absolute value graphs would always be positive; and Vignette 23 as she asked students to explain why they predicted an equation was not a line. During the third segment of day one when Ms. B had students try to find the rule for the applet generated functions and during the events of Vignette 16, Ms. B did not reveal the answers to the questions she posed. Rather she let students think about them and indicated they would revisit the questions. Ms. B elicited correct answers from students using IRE sequences in Vignettes 14, 21, and 22 when she had students correct points they submitted. During the first segments of days one and two when the class went over different kinds of time graphs, Ms. B used representations created by students instead of her own. During segments with collecting points, Ms. B elicited representations from all the students. When a student suggested that $y=-2|x|$ would not be positive in Vignette 20, Ms. B asked the students to verify his suggestion by finding points on its graph.
Ms. B’s orientation appeared to switch back and forth between procedural and more conceptual. Ms. B appeared to have a procedural orientation in several instances such as in her introduction of functions: Vignette 12; her asking students to extrapolate beyond the data by focusing on finding “the rule” Vignette 13; her correcting student submitted points Vignettes 14, 21, and 22; her explanation of why a quadratic was not a line in Vignette 18; and her demonstration of how the calculator can help students perform computations in Vignette 24. Ms. B’s orientation appeared more conceptual in other instances, such as her asking students to compare and contrast the graphs of lines in Vignettes 15 and 16; her asking students to think about what a vertical line in a distance time graph would mean in Vignette 17; her asking students to reason about the shape of a graph of the height of water in a pitcher versus time in Vignette 19; and her asking students to make predictions about the shapes of graphs of different equations in Vignette 23. Although Ms. B appeared to be able to switch orientations, the switching did not appear to be fluid. Ms. B’s orientation appeared to be related to the task she performed such as when she was correcting points she went into procedural mode to correct them, or when she had students compare lines she went into a more conceptual mode. Within a given task she did not appear to switch orientations such as correcting student’s submitted points. Not switching orientations within a task appeared to add to the missed opportunities present in Ms. B instruction such as in Vignettes 21 and 22. So her orientation for a given task was not always appropriate and she seemed to be in a state of transition between the conceptual and procedural orientations.
While the gaps in Ms. B’s knowledge were less apparent during year three, Ms. B appeared to have some gaps remain. When the applet function outputted 1 three times in a row during the second segment of day one, Ms. B thought the applet made a mistake and skipped the function in Vignette 25. Ms. B did not appear to realize the function machine did not make a mistake but rather generated a constant function. Not only did Ms. B encounter the constant function in class, but also when she practiced the lesson earlier. Both times she thought the function applet created an incorrect function in line 3.

Ms. B Vignette 25

1) T...Alright, let’s try another one. Ready? Second one. 1 in, 1 out. 2 in, oh, oh, it looks like the same one. Hold up, let me just do one more and let’s see.
2) SS: Yeah./ Yes.
3) T: Ok, let me start another one. That happened to me last night when I was practicing...

During the second segment of day two when the class went over domain and range of different functions, a student appeared to notice a pattern that the domains of functions are everything (Vignette 26 line 10). When questions about whether the domains of function were everything, Ms. B did not have an example ready and had to think about it in line 12. Ms. B did not have an example ready even though she had students use several functions that were not defined for all x, such as the time graphs, only the points generated by the applet function, or only the points submitted by students. Perhaps Ms. B did not view those objects as functions. A student suggested that a “straight down” line would be a function that did not have a domain of everything in line 17. Ms. B seemed excited when realized the student described a vertical line in lines 17-20 and drew one using the touchpad. Ms. B saw that the line would only be defined for one x-value in
lines 26-28. Ms. B thought the suggestion was “cool,” and “very good” in line 28. However, the student was wrong. Vertical lines are not functions of x, and do not have a domain. Ms. B did not appear to realize the student suggested something that did not work because it simply was not a function. Although Ms. B did recognize that vertical lines were a problem in time graphs as evidenced by her response to the mountain climb graph in Vignette 17, perhaps Ms. B did not view these time graphs as functions. When introducing the concept of functions, Ms. B did not address a defining characteristic of them (only one output for a given input). This could explain why the students thought vertical lines could have a domain, but Ms. B’s acceptance of this answer indicated that she did not know or had temporarily forgotten this important characteristic.

Ms. B Vignette 26

1) T: Let me type it in: \(1/2X + 1\)?
2) SS: Um hum.
3) T: Ok, what’s the domain of that function? Domain, X. What are we using for X’s?
4) SS: -6 and [inaudible]/ Everything.
5) T: Everything.
6) S: Oh.
7) T: Everything, right? I want you to look at this X. I’m using it to get that answer, I’m using it to get that answer, I’m using it to get all those answers, so the domain is everything. How about the range?
8) SS: Everything, too?/ Everything.
9) T: Everything again, you got it.
10) S: Do you always use X as everything?
11) SS: No./ Maybe.
12) T: Let me think about that. Maybe there are some times I can’t use it.
13) S: I know one thing you can’t.
14) T: You think there’s one that you can’t?
15) S: Yes.
16) T: What is it?
17) S: Whenever there’s a line straight down with the X on it, or a line like horizontal would use the Y.
18) T: He’s asking about X’s, not Y’s.
19) S: It’s like the X, if there’s a line going straight down, then?
20) T: Oh, I know what you’re talking about! Omigosh! Do you know what he’s talking about?
21) SS: Yes./ [inaudible]
22) T: [inaudible] be that long?
23) S: Yes.
24) T: Are you talking about when the line goes like this?
25) SS: Yes./ Whoa./ You see the vertical going straight down.
26) T: Oh, gosh, is that good, yes! Let’s just say this is Y and this is X – you’re right, because then all you’re using for X’s is...
27) S: That one number.
28) T: That one number like X = 5 and you’re not using anything else, right? Cool, Armando. Very good. Who wants to do the last one?
29) SS: Me. / Me.

Below is a flow chart modeling Ms B.’s decisions in the classroom during year three of the CCMS study. The rectangles represent actions taken by Ms. B and diamonds represent decisions. The octagon shape represents a decision that is made possible or easier from the use of CCT and a rounded rectangle is an action made possible or simpler by using CCT. Although Ms. B appeared to be able to switch from procedural mode to a more conceptual mode, she seemed locked into a single mode in a cycle in the flow chart.

The majority of Ms. B’s instruction flowed around the incorrect answer cycles, and the important aspect cycles. The incorrect answers present typically followed IRE sequences to either elicit correct answers/procedures or IRE sequences to elicit brief explanations of predictions. During the important aspect cycles, Ms. B would focus on more conceptual questions such as the meaning of certain parts of the graph, to ask students to make predictions about shape, ask students to predict the next numbers from the pattern of the function applet, or to have them generalize the impact of the equation of
the function and the shape of its graph. The comparison cycle also had a substantial presence in Ms. B’s instruction. Students made only one suggestion and a few questions.

Insider Perspective

The insider perspective stems from Ms. B Post Observation Interview during the lunch break on 12/13/2007. Ms. B clearly stated her goals for the students as she “wanted
them to discover things about the connection between the equation and the graph without me telling them,” and she wanted “them to see that each different equation gave you a different graph and I wanted them to figure out why that was.” Ms. B’s perceived goal of eliciting and aiding student thinking aligned with her stated goal of discovering connections with her facilitating. Treating representations as both objects and processes and as something to reason about appeared to have aided her stated goal. Further, asking different kinds of questions and using technology to generate discourse appeared to have aided her new stated goal.

Ms. B felt the CCT encouraged student learning since it enabled all the students to be involved: “each one of them has the responsibility for contributing something, they are all there and involved.” Ms. B felt the CCT aided her instruction since she “Can see that they really understand.” She also felt the CCT helped her let her students discover “if there is a square in it, it will always be a U; if absolute values [it is] going to be a V, and I think they are discovering that and they seem, some of them seem to understand why.” The CCT enabled Ms. B to see how many students struggled with negative inputs and she altered her instruction to show them how to store values in the calculator to simplify computations.

Ms. B’s view of technology changed from a master that had to be served every day in predetermined ways to a partner that she relied upon during appropriate times. At first Ms. B was apprehensive about using the technology because of the planning, troubleshooting and the feeling that she needed to use it every day. Her view changed when she became more comfortable with the technology since she could “almost fly by
the seat of my pants, not that I do…” After using the CCT, Ms. B realized that she did not have to use it every day and that it was not always appropriate to use it.

I guess I really rely on it now. I rely on it for screen shot; I rely on it for student feedback through quick poll. Definitely on graphing. I mean it’s just awesome for graphing. I just rely on it a lot. Um where as at first I didn’t, you know, and at first I felt… um not uncomfortable. I’m not sure what the word is, apprehensive about planning the lessons it was a major effort to plan the lessons, now that I’m more comfortable with the technology itself I can almost fly by the seat of my pants, not that I do… But you know what I’m saying, I can change mid-stream; I can also alter something if I get, I may get, I may get more relaxed or I may get an idea in the middle of class and be able to pick it up and go with it. Whereas my first year this is the plan there are the steps this is the way I need to do it… Although I guess, you know I’m thinking at first I thought I have to use this every day, I have to find a use for it every day and I’m now I’m feeling like there are times I don’t need to use it and there are times when it’s really, really important to use it.

Summary

Segmentation

Ms. B started class by either going over homework or with a CO (class opener) in year three as she did in year one. Ms. B during year one segmented class time so that each task was singularly focused such as finding the graph of a line given a point and a slope, or finding slope of a line given two points. However during year three Ms. B had segments that had multiple foci such as segment two of day two where she had students submitting points, correcting those points, comparing those points to other sets of points, and then generalizing observations.

Representations

Except for a few segments, Ms. B used multiple representations of the same function in year three as she did in year one. Ms. B continued to use graphical, tabular, and algebraic representations of function from year one to year three. Ms. B did not however
use embodied representations during year three. Ms. B continued to use representations from multiple sources such as herself, technology, or single and multiple students from year one to year three. However, most of the representations in year three were generated by a student. Ms. B and her students treated representations of function almost exclusively from the process perspective during year one. But in year three Ms. B and her students also treated functions as a process as well as objects. Another major change in Ms. B’s use of representations was how representations were treated. Ms. B treated representations as something to enact procedures on, or as an end product in year one. While Ms. B and her students still enacted procedures on representations, their production was not the end goal for Ms. B. Rather, Ms. B used representations as something to reason about, to discuss, and to make brief explanations with. Ms. B drew students’ attention to connections among different representations of the same function during year three, even if she did not fully explain them at the time. Ms. B during year one did not attempt to make connections between the different representations of functions, except during the use of embodied representations.

**Discourse**

The purpose for Ms. B is classroom discourse expanded from year one to year three. In year one, the purpose of discourse appeared to be to elicit correct answers/procedures. The purpose of discourse in year three also included year one’s purpose, but expanded to elicit and aid student thinking. While Ms. B had 80 minute classes during year one, due to CO’s, quizzes, etc. only during 50 minutes involved people talking. So the amount of minutes that people spoke each year was about the same. In the three days in year three,
Ms. B spoke as about as many words as she did in year one. Ms. B spoke about 200 more utterances in three days during year three than she did during the five days during year one. Ms. B’s average words per utterance dropped from 21.29 to 15.3. In year three, Ms. B’s students spoke more often per day with about 600 more words per day on average and about 100 more utterances per day on average. However, the students’ words per utterance remained about the same. Ms. B continued to use IRE sequences to correct answer/procedures from students from year one to year three, but in year three Ms. B also began asking students open ended questions to elicit and aid student thinking or having them make predictions. Ms. B choice of examples of representations appeared to continue to be mostly chosen ahead of time, and she seemed willing to use student generated/suggested representations. Ms. B continued to rely on texts for her source of representations. However, the textbook in year three appeared to treat mathematics less procedurally than the book in year one. Ms. B continued to not fully utilize the opportunities generated by the CCT when discussing representations generated by multiple students. However, in year three Ms. B did discuss representations that did not match the pattern, rather than moving on to the next problem without discussion as she did in year one. Ms. B partially remedied one of her missed opportunities from year one in year three. During her first year Ms. B never established connections between the different representations of the same or different functions, which was one of major missed opportunities. However, one of the major components of Ms. B’s instruction during year three was to compare representations of different functions and to introduce
the connections between the representations of the same function. Ms. B did not fully capitalize on the latter component even with increased student interest.

**Technology Use**

Ms. B appeared to have discontinued her use of technology as a drill and practice aid in year three. While Ms. B continued to use the calculator as a mathematical authority, the extent that she used technology in this manner diminished from year one to year three. Ms. B continued to use technology as an information displayer/gatherer in year three. However, Ms. B used technology in this manner in combination with technology as a discourse generator more frequently during year three than year one. Ms. B appeared more comfortable with technology in year three and she seemed more able to troubleshoot on her own. Ms. B used some kind of computer technology during all segments of her instruction either with the CCT or the Smartboard™ software.

**Decisions**

Ms. B’s apparent goals expanded from year one to year three. While her goals of wanting students to “get it”, using student suggestions and addressing student questions continued from year one to year three, Ms. B goals expanded in year three to also include elicit and aid student thinking without always revealing correct answers. Ms. B’s mathematical knowledge still appeared to have gaps in year three; however these gaps were less apparent. Ms. B’s orientation to mathematics appeared to change from highly procedural to being able to switch from procedural to more conceptual. However, the fluidity of the switching appeared limited and not always appropriate for a given task. Ms. B’s instruction during year one revolved around the correct answer/procedure cycle
in her decision flow chart. Ms. B’s instruction during year three had a similar cycle, but also revolved around more conceptual cycles such as the comparison cycle and the important aspects cycle.

**Answers to Research Questions**

**What kinds of representations are used in the classroom by teachers?** Ms. B and her students used graphical, tabular, and algebraic representations of function during both years and Ms. B also used embodied representations, using physical objects as representations, during year one. The students were asked to find slopes of lines, graph lines given a point and slope, graph lines given two points, find the equation of a line given two points, or to submit equations of lines that fit pictures or points to the CCT. All of these tasks require at least two representations of functions. With the exception of the embodied representation, all other representations were treated from the process perspective during year one.

During year three, representations were treated from both perspectives depending on the task. For example if Ms. B needed to make corrections she treated functions as process, or if she wanted students to make comparisons she focused more on the object nature of functions. Within a given task she did not switch perspectives. When students were asked to compare lines, they first noticed process nature observations such as where the lines intersected or that one went through the origin. After making a few observations student began to notice properties of the entire function such as classifying it as a line, or straight, or closer to the y-axis. In other words the students were treating representations of functions as objects as well or at least noticing their object nature.
The graphical, algebraic representations changed from multiple sources treated from the process perspective to multiple sources treated from both perspectives. The tabular representations changed from being generated by Ms. B to being generated by the students. Ms. B did not use the embodied representation during year three.

**What is the quality of discourse about representations or use of representations by the teacher?** The discourse surrounding the activities of year one typically involved IRE sequences where Ms. B elicited correct answers/procedures from students. The questions were highly procedural. The point of the activities seemed to be to enable the students to do the given tasks.

The discourse surrounding the activities of year three included aspects of the discourse from year one, but moved to include other aspects such as eliciting and aiding student thinking. While part of the goal of year three’s activities seemed to be to enable students to do the tasks, part of the goal seemed to be to help students to understand functions. Ms. B’s questions could be quite procedural during year three; however Ms. B also asked more conceptual questions. In spite of being able to ask different kinds of questions, Ms. B did not mix the two kinds in a given task.

**For what purpose do teachers rely on multiple representations in classroom discourse?** During year one, Ms. B treated representations as something to enact procedures on or as something to produce as an end product. The tasks Ms. B gave the students typically had students use the representations to find slope, the equations of the line, or graph the line. Once the graph, slope, or equation was found, Ms. B moved on to
the next problem. Representations were not used as something to think about or reason with.

During year three, Ms. B used representations to find outputs with given inputs, rules of patterns, or their graphs. But the production of the representations was not an end as it was during year one. Ms. B used representations to ask students to reason about and with them. She had students make predictions about the shapes of the graphs of functions given their equation and to briefly justify their reasons. Ms. B asked students to make observations about the graphs of different functions. Ms. B asked students to reason about the shape of different time graphs. Ms. B also asked students to generalize their observations about the shapes of functions given equations.

**What is the relationship between teacher use of CCT in algebra classrooms and the growth of teacher and student choice of representations of linear functions as manifested in the classroom discourse?** In year one, Ms. B’s goal for the students appeared to be for them to do the tasks correctly. Ms. B retained this goal for her students, but she also adopted a goal of eliciting and aiding student thinking as evidenced by her statement from her POI of wanting students to discover the connections among the different representations of graphing without her telling them. The new goal appeared to be a major source of the changes in her instruction.

The tasks in year one revolved around “doing” the math such as find the slope; graph the line; find the equation; or find the intersection, which aligns with her apparent year one goal. While the tasks of year three had components of doing mathematics correctly, the tasks had a component of thinking about mathematics as well. Ms. B wanted the
students to reason about the context and the time graphs, to make comparisons between graphs to discover for themselves the impact of the equation on its graph, and to make predictions using those connections.

The kinds of representations present in Ms. B’s classroom remained mostly unchanged from year one to year three as she continued to use the graphical, algebraic, and tabular representations from single or multiple sources. Only the embodied representation disappeared. However, her treatment of representations changed to include both the object and process perspectives, and to treat representations as something to reason about. Representations were used to “do stuff” to or to make in year one, which aligned with her apparent goal of students doing math correctly. Ms. B continued to use representations to “do stuff” to in year three as the year one goal did not disappear. However, representations were also used as something to think about their meaning and connections between them, which aligned with her new goal of students discovering connections on their own. Treating functions as solely processes would make seeing connections among the different representations of the same function difficult since the focus would not be on large scale properties such as slope or shape. Ms. B’s treatment of function as objects at times aided her new goal since students could view representations of functions as objects with properties that were connected to different objects with other properties.

Ms. B’s new goal appeared to influence her choice of questions in classroom discourse. She continued to use IRE sequences to elicit correct answers/procedures from her students, again as her goal from year one did not disappear. However, to achieve her
new goal she could not continue with the same type of discourse since it focused on doing mathematics rather than making connections within mathematics. Ms. B seemed to realize that her questions could not remain the same and began asking students to think about the meaning of graphs given context, to think about connections among representations, to make comparisons among different representations of functions, and to make predictions about functions.

The opportunities that Ms. B did not capitalize on using could have further achieved her new goal for her students in year three. Treating functions as objects when Ms. B corrected points could have allowed her students to explore reflections illustrating the impact of a negative sign for the coefficient of x or to reason about patterns. Ms. B introduced but did not pursue the specific impact of the equation of the line on the steepness of the line, which aligned perfectly with her new goal. Ms. B appeared to want to pursue the specific connections: perhaps she did not have the pedagogical content knowledge as to how in that moment to enable students to discover the connections own their own. Ms. B’s apparent lack of mathematical knowledge seemed to be an influence as to why Ms. B missed some opportunities during year one as she saw students lines as wrong rather than slightly wrong. Ms. B’s content knowledge appeared to influence some of the missed opportunities during year three especially those concerning the meaning of function.

Ms. B’s use of technology in year three appeared to change as a result of her new goal. She used the touchpad technology to let more students display their own representations to allow the class to discuss their meanings. Ms. B used the CCT more
frequently and more proficiently as a discourse generator. In year one, Ms. B’s use of the CCT as a discourse generator revolved around correcting points using computations. This use of the CCT occurred during year three. However Ms. B also used the CCT display to ask students to make comparisons among graphs, and given equations to discuss predictions about shapes of their graphs.

The use of technology increased Ms. B’s comfort level with technology as she used some sort of computer technology all the time during year three. Ms. B no longer used blackboard, whiteboard, or overhead transparencies during year three. Ms. B’s continued use of technology increased her proficiency in troubleshooting as she had problems with the technology in year three but she was able to correct it on her own. In year one, on multiple occasions a student had to help her with the technology.

To accommodate Ms. B’s new goal, her decisions had to change. Ms. B could no longer only pursue correct answers/procedures. Ms. B appeared to pursue her new goal by adding two major decisions in her flow chart by pursuing student comparisons among multiple representations and exploring important aspects of single representations. Ms. B’s use of the CCT became more embedded in her flow chart in year three. Ms. B used the CCT to decide if correct answers or predictions were present as well as to ask specific students to make corrections or explain their predictions. Figure 6.18 shows Ms. B’s decision flow charts for year one and year three side by side.

Ms. B., during her third year, appeared to be able to switch her treatment of, perhaps her orientation toward mathematics between procedural or conceptual. Not all of her questions or tasks in year three had students perform procedures; some tasks and
questions asked students to reason about meaning of representations, make observations about different representations, or to make predictions based on the observations. While Ms. B treated mathematics as only procedural in year one and could somewhat switch in year three, Ms. B did not switch mid task.

Figure 6.18: Ms. B’s Decision Flow Charts First Year (left) and Third Year (right).

Ms. B’s new goal of having students discover connections/meaning on their own without her telling them appeared to have influenced many of the changes that occurred in her instruction from year one to year three. Ms. B appeared to either not have this goal during year one or perhaps this goal was dormant since she did not know how to achieve it. Ms. B’s participation in the CCMS project and continued use of CCT in her classroom appeared to influence the new goal or let the goal surface.
CHAPTER 7: CROSS ANALYSIS

This chapter is a cross analysis of the three teachers each presented as a case study in the previous chapters. The report is comprised of an analysis of all the teachers using the four major categories of representations, discourse, technology, and decisions along with their subcategories. The analysis compares and contrasts growth or respective lack thereof of the participants within these categories over time.

Unless otherwise indicated all charts show percentage of subsegments where the code was observed. Each class day observed for each teacher was divided into segments based on similar themes of the lesson, activities, or exercises. The segments were further divided into subsegments by individual exercises, activities, examples, and other relevant instructional pieces. Each subsegment was coded for each of the subcategories of the four major ones. While each subsegment could only be coded one way for most of the subcategories, some subsegments could receive multiple codes within a single subcategory such as treatment of representations. Since a representation could be treated as something to enact procedures upon to later be treated as something to reason about within the same activity, a representation in a subsegment could be coded as both. Since Mr. L did not have CCT during year 1 of the CCMS study and the other two did, Mr.
L.’s year one data are not considered for this final comparison. For the cross analysis each teacher’s first year of CCT use will be compared as well as a subsequent year that is their second or third year of CCT use. Refer to Table 7.1 for the number of subsegments observed for each teacher during each year.

<table>
<thead>
<tr>
<th></th>
<th>Mr. L</th>
<th>Ms. A</th>
<th>Ms. B</th>
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<tbody>
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<td>14</td>
<td>31</td>
</tr>
<tr>
<td>Beyond First Year</td>
<td>13</td>
<td>15</td>
<td>28</td>
</tr>
</tbody>
</table>

Table 7.1: Number of subsegments during each teacher’s instruction for each year.

**Representations**

Table 7.2 displays the indicators for different levels of classroom instruction with regards to representations. Teachers were rated low, medium, or high for four subcategories of usage and treatment of representations in the classroom: types of representations present in the classroom, source of representations, process/object treatment of functions, and treatment of representations. A composite of all the four subcategories was used to create indicators for an overall rating of representational usage and treatment in the classroom for each teacher as described in Table 7.2. Averages of the teachers’ ratings in each of the subcategories became their overall rating, where scores between 1.0 to 1.5 were considered low, 1.6 to 2.5 were considered medium and 2.5 to 3.0 were considered high. In this study the average rating corresponded with meeting at least three of the four indicators for the overall rating. However, a teacher could obtain an
average rating of 2.0 by only meeting two of the criteria for medium if their rating was high in two subcategories and low in the other two.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
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| High 2.5-3 | - The teacher used many different types of representations simultaneously, or different pairs of representations throughout instruction.  
- The teacher elicited and displayed representations from multiple students simultaneously.  
- The functions represented were treated from both the process and object perspectives.  
- Representations were treated both as something to reason about/with and something to explain/justify with/about. |
| Medium 1.5-2.5 | - The teacher used two different representations simultaneously for part but not most of instruction.  
- The teacher elicited and displayed representations from some students perhaps simultaneously but not for most of instruction.  
- The functions represented were treated primarily from the object perspective.  
- Representations were treated less frequently as end products; and were treated as something to reason about/with or something to explain/justify with/about but rarely as both. |
| Low 1-1.5 | - The teacher primarily used a single type of representation at a time, and the same representation throughout the day.  
- The teacher was the primary source of all representations present.  
- The functions represented were treated almost exclusively from the process perspective.  
- Representations were treated almost exclusively as something to enact procedures upon and as end products. |

Table 7.2: Indicators for overall representations rating.

**Types of Representations**

Each subsegment was coded for the type of representation present. To indicate when different representations were used for the same function, the subsegment was coded as a
mix rather than one of each of the individual representations. For each year a teacher’s instruction was rated as low, medium or high for the types of representations used in the classroom. A teacher was rated low if the teacher used only single representations at a time and used predominately only one representation during instruction. A teacher was rated medium if two representation were simultaneously present the majority of the time. A teacher was rated as high if all three representations were present simultaneously the majority of the time or if different mixtures of two representations were present.

Mr. L. Figure 7.1 shows the types of representations in Mr. L’s classroom during his first and third years using CCT. Mr. L decreased his sole use of algebraic representations from 88% of all subsegments in his first year to 53% in his third year as shown in Figure 7.1. He decreased his use of purely tabular representations from 8% of subsegments his first year to none his third year. Mr. L increased his use of mixed representations from 4% of subsegments in his first year to 47% the subsegments in his third year. The presence of a mixture of representations appeared to have occurred by accident in his first year. When Mr. L asked students to find only the slope of a graphed line, a student went further and found the equation of the line instead. The overall composition of the type of representations in his classroom in his first year was a mix of purely algebraic, purely tabular, and the accidental mixture, although the algebraic representations dominated the class time. The overall composition changed in his third year from purely singular representations present to a combination of singular representations and a mix of two representations present at the same time. The composition in his third year was split
almost evenly between a single representation present and a mix of two representations present at a time. Given that Mr. L only used a single representation at a time during his first year except for the accidental mixture, he was rated as low for the types of representations present during that year. In his third year, Mr. L spent 47% of subsegments with two representations of lines at the same time, so since he was close to the 50% cut off he was rated as medium in his third year.

Figure 7.1: Mr. L’s types of representations used in the classroom.

Ms. A. Figure 7.2 shows the types of representations present in Ms. A’s classroom during her first and second years of CCT use. Ms. A decreased her use of purely algebraic representations from a small portion in her first year to none in her second year. She did not use purely graphical representations during both years. She decreased her use of all three main representations in a single activity during year one from 80% of
subsegments to none during her second year. She increased her use of both algebraic and graphical representations from less than 20% of subsegments in her first year to all the subsegments in her second year. The overall composition of Ms. A’s types of representations present during her first year was a mix of mostly using all three during the same subsegment, a mix of algebraic and graphical, and purely algebraic. The composition of the types of representations changed to algebraic and graphical representations used at the same time. She did not use CCT for only the subsegments with the purely algebraic representation during her first year, while she used CCT for all subsegments during her second year. However, Ms. A’s use of multiple representations at the same time did not change from her first year to her second year. The majority of class time during her first year she used all three representations. Thus, Ms. A was rated high for her first year. She used only algebraic and graphical representations during her second year, so she was rated medium for her second year.

![Graph showing the types of representations used in the classroom.](image)

Figure 7.2: Ms. A’s types of representations used in the classroom.
Ms. B. Figure 7.3 shows the types of representations present in Ms. B’s classroom during her first and third year of CCT use. Ms. B did not use purely algebraic representations in either year. Ms. B increased her use of purely graphical representation from none in her first year to about 20% of subsegments during her third year. Ms. B rarely used only the tabular representation during her first year and not at all in her third year. Ms. B decreased her use of algebraic and graphical mix from 80% her first year to about 10% her third year. However, Ms. B increased her use of the main three representations from 0% in her first year to approximately 60% in her third year. She also focused on the graphical and tabular representations for approximately 20% of the subsegments in her third year, which did not occur during her first year. The composition of the type of representations during her first year was almost exclusively algebraic and graphical mixed. The composition of the types of representations present in Ms. B’s classroom changed to a majority of a mixture of the three major representations, two different mixtures of two representations, and purely graphical. For approximately a third of subsegments with algebraic and graphical mix, Ms. B used the CCT. During her third year with CCT, Ms. B used the CCT during the subsegments with all three representations present and with the graphical and tabular. For the instances with purely graphical representations and when both graphical and algebraic representations were used, she did not use CCT. Ms. B used two representations the majority of the time for her first year, so she was rated medium. During her third year, Ms. B used all three
representations simultaneously the majority of the time as well as a variety of different representational modes, so she was rated high.

Figure 7.3: Ms. B’s types of representations used in the classroom.

Comparison. Figure 7.4 shows the types of representations present in each of the teachers’ classrooms during their first year. During the first year of CCT use, all the teachers used one representational mode the majority of the time, such as purely algebraic for Mr. L., using all three representations for Ms. A and algebraic and graphical for Ms. B. However, Mr. L used a single representation predominantly at a time and the instance of mixed representations appeared to be the result of an accident, whereas both Ms. A and Ms. B used multiple representations simultaneously. Mr. L and Ms. B did not spend much time using the tabular representation, but the tabular representation was used frequently by Ms. A. Although both Mr. L and Ms. B had the three major representations present during their first year, Ms. A was the only one to use all three simultaneously and
to mix the tabular representation with any of the others. While each teacher seemingly used preferred type(s) of representations for over 75% of all subsegments in the first year, Ms. A was the only teacher to switch from her preferred type of representations present in the classroom to different types for more than 20% of the time. The ratings for each teacher were different with Mr. L being low, Ms. A being high, and Ms. B in between.

![Figure 7.4: Representations present in teachers’ classrooms during first year of CCT use.](image)

Figure 7.4 shows the types of representations present in each teacher’s classroom beyond their first year of CCT use. Ms. A switched from being the teacher using some variety of different mixtures of representations present to the only teacher using only one mixture. Mr. L switched from using a single representation for nearly all instruction to an almost equal split between using a single representation and a mixture of two. Ms. B also
switched from one using a mixture of two representations to using a variety of different mixtures of representations with the mixture of all three major representations being the most used. All three teachers used multiple representations simultaneously beyond their first year of CCT use. However, Mr. L was the only teacher to devote a significant amount of class time to a single representation. Ms. B was the only teacher to use all three representations at all and she used all three simultaneously for a majority of the subsegments. Ms. B and Ms. A shifted positions in the ranking, Ms. B was rated high and Ms. A was rated medium. Mr. L moved up from low to medium. The continued use of the CCT appeared to increase the types of representations present and the amount of representations occurring simultaneously for teachers who started low to medium, whereas teachers who started high either remained the same or decreased.

Figure 7.5: Representations present in teachers’ classrooms beyond first year of CCT use.
Sources of Representations

Each subsegment was coded for the source of the representations present. The charts below depict the percentage of subsegments with the coded source. A teacher was rated low if the teacher was the source of representations with or without the CCT over 60% of the subsegments. A teacher was rated medium if the teacher was the source of representations 40% to 60% of subsegments. A teacher was rated high if the combined percentage of subsegments with either single students or multiple students as the sources of representations totaled 60% or more.

Mr. L. Mr. L was the source of representations for over 80% of all subsegments during his first year of CCT use as shown in Figure 7.6. Mr. L decreased as the source of representations to approximately 50% of the subsegments during his third year of CCT use. Students were the source of the representations present for approximately 10% of the subsegments during his first year. However, students became the source of representations for approximately 40% of the subsegments during his third year. Multiple sources of representations for the same object was only present for less than 5% of the subsegments during the first year and again for more than 40% during the third. The composition of sources representations changed from almost exclusively Mr. L and some students to an almost equal split between Mr. L and multiple students. The CCT was used during the subsegments with multiple students as sources of representations both years and it appeared the CCT made it possible for multiple students to be sources. Since Mr. L
was the primary source of representations during the first year, he was rated low. Mr. L increased his students as sources of representations during his third year and was rated medium.

![Figure 7.6: Sources of representations in Mr. L’s classroom.]

**Ms. A.** Ms. A was the source of the representations less than 25% of the time as shown in Figure 7.7 during the first year either creating the representations herself or using the *CCT*. Ms. A was the source of representations in approximately 35% of subsegments during her second year of *CCT* use. Ms. A created the representations using *CCT* in approximately 16% of the subsegments during her first year of *CCT* use. She increased using *CCT* as a source of representations to 35% in her second year. She created representations without the *CCT* less than 8% in her first year to 0% in her second year. Approximately 75% of the subsegments had *multiple students* as sources of the
representations present. *Multiple students* as sources of representations dropped to approximately 60% of subsegments during her second year. The composition of sources of representations changed from mostly multiple students and Ms. A with or without CCT to a little less *multiple students* and a little more Ms. A with the CCT. All instances with *multiple students* as sources of representations occurred with the use of the CCT, and it seemed the CCT made the multiple student sources possible. Since the majority of the time *multiple students* were sources of representations both years Ms. A was rated high both years.

![Bar chart showing sources of representations in Ms. A's classroom.](image)

**Figure 7.7:** Sources of representations in Ms. A’s classroom.

**Ms. B.** Ms. B was the source of representations with or without the CCT in about 61% of subsegments during her first year of CCT use as shown in Figure 7.8. The percentage of subsegments with Ms. B as the sole source dropped to below 15% during her third year of CCT use. Ms. B using CCT as the single source of a representations
decreased from about 6% of subsegments to 0% her third year. A *single student* as the source of representations increased from about 6% of subsegments her first year to 14% her third year. *Multiple students* as the sources of representations increased from about 32% of subsegments during her first year to 71% during her third year. Students were the source of representations about 85% of subsegments during her third year, which was about an increase of 45% of subsegments from her first year. The composition of the sources of change from a majority of representations produced by Ms. B and some from the students to a large majority of representations produced by multiple students, some produced by *single students*, and some produced solely by Ms. B The subsegments with *multiple students* as sources occurred while Ms. B used the CCT. Ms. B was rated low for her first year and high during her third year for the sources of representations.

![Bar chart showing sources of representations in Ms. B's classroom](image)

Figure 7.8: Sources of representations in Ms. B’s classroom.
Comparison. Both Mr. L and Ms. B were the primary source of representations during their first year of CCT use, whereas Ms. A was rarely the sole source of representations as depicted in Figure 7.9. While Ms. B had a substantially larger number of subsegments with students as sources for representations than Mr. L., Ms. A had a substantially larger number of subsegments with multiple students as sources than both Mr. L and Ms. A. Both Mr. L and Ms. A had operated in similar manner for more than 80% of subsegments, teacher as source and multiple students as sources respectively. However, Ms. B had an almost 60/40 split between two manners of operation, teacher as source and multiple students as sources. Both Ms. B and Mr. L were rated as low due to their role as the primary source of representations. And Ms. A was rated as high due to the large amount of student generated representations present simultaneously.

Figure 7.9: Sources of representations in teachers’ classrooms first year of CCT use (left) and beyond first year (right).
Only Mr. L continued to be the sole source of representations for a majority of subsegments beyond the first year as shown in Figure 7.9. Ms. B and Ms. A had *multiple students* as sources of representations for more than the majority of the subsegments, whereas Mr. L had *multiple students* as source approximately 50% of the time. All three teachers had different modes of operation, the sources of representations in Mr. L’s classroom was 50% of subsegments from *multiple students* and the rest from himself, Ms. A had mostly *multiple students* as source and herself using CCT as a source, and Ms. B had *multiple students* as sources and some *single students* as sources. For both teachers rated low during their first year of CCT use, a large increase in the amount of subsegments with *students as sources* occurred after using the CCT more. Although Ms. A had a decrease in the amount of subsegments with *multiple students* as sources, students remained the source of representations for over half of the time. Mr. L changed from low to medium, Ms. A stayed the same at high, and Ms. B changed from low to high. The continued use of CCT appeared to increase the amount of students as sources of representations for teachers rated low during their first year of CCT use, whereas teachers rated high appeared to remain about the same.

**Process/Object Treatment of Functions**

Each subsegment was coded to show how each representation of functions was treated, as processes, as an object, or a mix. A teacher that treats the concept of function only as a process implicitly, perhaps even explicitly, emphasizes the notion that mathematics is a set of skills to be learned, such as the skills needed to find the y-
intercept, or calculate slope, or read points from a graph. Some teachers emphasize only the process nature of functions (Knuth, 2000). A teacher that treats functions only as an object draws students’ attention to properties of the object, how these properties can be acted upon, and connections among these properties and properties of other objects. However, Moschovich et al. (1993) argued that both the process and object natures of function required to understand functions as each are useful in different situations. Hence, a teacher’s rating for this subcategory increased with the additional focus on the object nature and the mixture of both perspectives. Specifically:

1. A teacher was rated low if they treated functions as process the majority of the time.
2. A teacher was rated medium if a representation of functions was treated as objects the majority of this time, or if they treated a representation of functions as process less than half of the time.
3. A teacher was rated high if they treated the representation of function as a mix of process/object more than 30% of the subsegments containing the representation and treated function solely as process less than half the time.

Mr. L. Mr. L treated the algebraic representations of functions as process during his first year and during more than a majority (about 77%) of subsegments of his third year for both as shown in Figure 7.10. During his third year, by focusing on properties of lines and indicating that the lines could be moved around the xy plane, he hinted at the object nature of functions during the remaining 23% of subsegments. Although he treated functions as objects during his third year, his treatment did not change enough to alter his rating. Hence, he was rated low for both years.
Figure 7.10: Mr. L’s process/object treatment of algebraic (top) and graphical (bottom) representations of functions.

**Ms. A.** In Ms. A’s case, she nearly doubled the percentage of subsegments that she treated representations as *process* from 7% in her first year of CCT use to 14% in her second year of CCT use as shown in Figure 7.12. She also reduced the amount of subsegments in which she treated functions solely as *objects* from 92% her first year to 50% her second year. However, her treatment of representations as a mix of
process/object increased by 35% from 0% her first year to her second year. During her second year, she had a variety of modes that were more evenly divided among the subsegments. Her rating increased from medium her first year to high her second.

Ms. A did not treat the graphical representation as a process during her first year of CCT use. Her treatment of the graphical representation as process increased to 14% her second year as shown in Figure 7.11. Ms. A treated the graphical representations of functions solely as objects in 92% in subsegments her first year, which dropped to 42% of subsegments her second year. Ms. A increased her treatment of functions as both process and objects from 0% her first year to 42% her second. During her first year, Ms. A only treated the graphical representations of functions as objects. She did not treat them in any other manner. This single treatment of functions changed her second year as she had a variety of manners to treat function, mostly as objects or a mix of process and objects. Ms. A increased her rating from medium her first year to high her second.

![Figure 7.11: Ms. A’s process/object treatment of algebraic representations.](image-url)
Ms. A did not use the tabular representation her second year at all, so no comparisons of how she treated this representations could be made. However, during her first year, Ms. A treated the tabular representations of functions solely as process for all subsegments with the tabular representation.

**Ms. B.** Ms. B decreased her treatment of the algebraic representations as functions as purely process from 100% of subsegments during her first year with CCT to 28% her third year with CCT shown in of Figure 7.13. Ms. B did not treat the algebraic representation as an object at all during her first year. Ms. B increased her treatment of functions as an object and as a mix of both by 21% and 18% of subsegments respectively. Ms. B had one mode of treatment of the algebraic representation during her first year. Ms. B had a fairly even divide between the three modes during her third year, as seen in Figure 7.13. Ms. B was rated low her first year and medium her second year.
Ms. B decreased her treatment of the graphical representation of functions as a *process* from 93% of subsegments during her first year of CCT use to 53% during her third year of CCT use as shown in Figure 7.14. Unlike the algebraic representation, Ms. B treated the graphical representation of functions as an *object* during 6% of subsegments her first year. The treatments of the graphical representation as an *object* increased to 21% during her third year. Ms. B did not treat the graphical representation as a mixture of *process* and *objects* ever during her first year, but she did treat them as a mix for 18% of subsegments during her third year. Ms. B had one mode of treatment of the graphical representation of function during her first year. Ms. B had a variety of modes of treatment during her third year that was mostly functions as *process*, but then a fairly even split between *object* and a mixture after that. Ms. B was rated low during her first year and medium during her third year.

![Graphical representation of functions](image)

Figure 7.13: Ms. B’s process/object treatment of algebraic representations.
Ms. B used the tabular representation only once during her first year, so comparisons between her treatments of this representation are not possible. However, during her third year Ms. B treated the tabular representation of functions as purely process.

Comparison. Both Ms. B and Mr. L treated the algebraic representation as process 100% of subsegments containing that representation during their first years as shown in Figure 7.15. Ms. A was the only teacher to treat the algebraic representation differently during her first year and she was the only teacher to treat the algebraic representation of function as objects at all during their first years. The teachers consistently treated the algebraic representation in the same manner during their first year. Each teacher had their apparent preferred treatment of the representation: Mr. A’s and Ms. B’s preferred treatment was that of function as process, and Ms. A was that of function as object.

Mr. L slightly changed his preferred treatment of the algebraic representation to hint at the object nature of functions. Ms. A and Ms. B changed from a single apparent
preferred treatment of functions to a variety of treatments of functions. Ms. A changed from almost entirely object treatment of functions to a split between mostly an object treatment and a mixture of the two. Ms. B changed from entirely process oriented to include the object treatment approximately half of the subsegments as shown in Figure 7.15. Mr. L did not change his rating, Ms. A increased her rating from medium to high, and Ms. B increased her rating from low to medium.

![Figure 7.15: Process/object treatment of algebraic representation first year of CCT use (left) and beyond first year of CCT use (right).](image)

Mr. L did not use the graphical representation during his first year of CCT use. Both Ms. A and Ms. B had one major manner in treating the graphical representation during their first year; Ms. B treated the representation almost entirely as process and Ms. A as object. Ms. B was the only teacher to have a variety, albeit small, of treatments of the graphical representation.
Mr. L did not use the graphical representation during his first year, but treated it mostly as *process* and some as *object* during his third year as shown in Figure 7.16. Mr. L did not use the graphical representation during his first year, so his treatment of the graphical representation changed from not using it to treating the graphical representation of functions as *process* and *object*. Mr. L was the only teacher to rarely treat the graphical representation as *object*. Both Ms. A and Ms. B changed their treatment of the graphical representation from predominantly one mode to a variety of modes. Ms. B increased her treatment of the graphical representation as *objects* substantially, and Ms. A decreased her treatment of functions as *objects* to include as mixture of *process* and object. Ms. A increased from medium to high and Ms. B increased from low to medium.

![Figure 7.16: Treatment of representation during first year of CCT use (left) and beyond first year (right).](image)

None of the teachers used the tabular representation for more than one year for any substantial length of time. Mr. L rarely used the tabular representation at all during the
years observed. Ms. A used the tabular representation during her first year but not the second. Ms. B rarely used the tabular representation during her first year and used it frequently during her third year. Interestingly, all teachers treated the tabular representation from the process perspective perhaps indicating a difficulty to treating this representation as an object.

**Treatment of Representations**

Each subsegment was coded as to how the representations were treated by the teacher and the students as **enact procedures**, **end products**, **reason with/about**, **explanation/justify with/about**. Since the representations in a subsegment could be treated in multiple ways the codes are not mutually exclusive, except the treatment of representations as **end products** prevented the treatment of representations as either something to **reason about/with** and/or **explanation/justification** and vice versa. **Enact procedures** on the representation was the most common treatment to share with another treatment in a subsegment, since a teacher might have students do something to/with the representation and then either treat it as an end product and move on or discuss what was done. A teacher was rated low if they treated representations as **end products** more than 50% of subsegments. A teacher was rated medium if they did not treat representations as **end products** for more than 50% of subsegments and treated representations as either something to **reason with/about** or something to **explain/justify** in more than 25% of subsegments but not both. A teacher was rated high if the minimum percentage of subsegments was 25% or above for each of the subsegments where representations were
treated as some to *reason about/with*, and the subsegments where representations were treated as something to *explain/justify*.

**Mr. L.** Mr. L substantially decreased his treatment of representations as *end products* from 88% of subsegments his first year of CCT use to 23% his third year as shown in Figure 7.17. Mr. L decreased his treatment of representations as something to *enact procedures* upon from 88% his first year to 53% his third. Not only did Mr. L decrease his treatment of representations as *end products*, he increased his treatment of representations as something to *reason about/with* from nothing his first year to 30% of subsegments his second year. Mr. L did not nor did his students use representations to explain or justify mathematical thinking.

![Figure 7.17: Mr. L’s treatment of representations.](image)

Mr. L primarily had students enact a procedure upon a representation to create an *end product* of/with the representation during his first year of CCT use. There were a few
subsegments where he treated representations one way but not the other. In his third year, he treated representations in three ways, *end products*, *enact procedures*, and something to *reason about/with*. He treated representations mostly as something to *enact procedures* upon, and had an almost equal split between treating representations as *end products* and something to *reason about/with*. Mr. L continued the combined treatment of enacting procedures upon the representations to create an end product from his first year, but in his third year he also combined the treatment of *enacting procedures* to create something to *reason about/with*. Mr. L. used CCT for the majority of subsegments where Mr. L asked students to *reason about/with* representations. Mr. L was rated low his first year and medium his second.

![Figure 7.18: Mr. L’s treatment of representations by year.](image)

**Ms. A.** Ms. A never treated representations as *end products* in either year as shown in Figure 7.19. Her treatment of representations as something to *enact procedures* upon
remained about the same both years at about 75% of subsegments. Her treatment of representations as something to *reason about/with* increased from 62% of subsegments in her first year to 86% her second year. Her treatment of representations as something to *explain/justify* increased from 54% to 71%. Ms. A did not treat representations in a substantially different manner in her second year. During both years, Ms. A typically asked students to *enact procedures* on/with the representations and to then *reason about/with* them and/or *explain/justify* with them.

![Bar chart](image.png)

**Figure 7.19:** Ms. A’s treatment of representations.

Figure 7.20 illustrates that the composition of her treatment of representations did not change much from her first year to her second. For her second year her treatment of representation became more evenly spread between her three ways of treating representations. Ms. A was rated high both years.
Ms. A’s treatment of representations by year.

Ms. B. Ms. B treated representations as something to *enact procedures* upon to produce *end products* for over 90% of subsegments during her first year of CCT use as shown in Figure 7.21. Ms. B treated representations as something to *enact procedures* upon to then reason about for less than 10% of subsegments. While Ms. B treatment of representations as something to *enact procedures* upon decreased slightly from 100% of subsegments in her first year down to 92% of subsegments in her third year, her treatment of representations as *end products* disappeared her third year. Also her treatment of representations as something to *reason about/with* increased dramatically from less than 10% in her first year to 79% of subsegments her third. Ms. B rarely (4%) treated representations as something to *explain/justify* during her third year, which is a minor change from 0% in her first year. The composition of her treatment of functions changed from an almost equal split between *end products* and *enact procedures* to an almost equal
split between *enact procedures* and reasoning as shown in Figure 7.22. Ms. B was rated low her first year and medium her third year.

![Figure 7.21: Ms. B’s treatment of representations.](image1)

![Figure 7.22: Ms. B’s treatment of representations by year.](image2)

**Comparison.** Mr. L was the only teacher to treat representations in only two different ways as shown in Figure 7.23, although Ms. B was very similar in her treatment of representations except for a few subsegments where students needed to reason about the
representations. Ms. A was different from the other two participants as she never treated representations as *end products* and she was the only one to spend substantial amount of time treating them as something to *reason about/with* and as something to *explain/justify* about/with.

Ms. A was the only teacher to not change much in her treatment of representations as her treatment of representations changed to be more evenly divided among the three ways as shown in Figure 7.23. Mr. L and Ms. B both changed their treatment of representations, although Ms. B changed more than Mr. L since she dropped her treatment of representations as *end products* and substantially increased her treatment of representations as something to *reason about/with*. Mr. L did increase his treatment of representations in a similar manner but the shift was not as visible than that of Ms. B’s Ms. B treated representations in the three manners that Ms. A had, but Ms. B’s treatment of representations as something to *explain/justify* about/with occurred far less frequently than in Ms. A’s class.

![Figure 7.23: Teacher treatment representations during first year of CCT use.](image)
Each teacher was rated in four subcategories under representations. The average of these values was found for each teacher for each year by using 1 as low, 2 as medium and 3 as high. Figure 7.25 captures the change in the average rating for each teacher. Ms. A had zero slope between her average ratings from her first year of CCT use to her second as she remained slightly less than high both years. Both Mr. L and Ms. B have positive slope for their change in their average rating from their first year to their third. Mr. L’s slope of increase of his average representations rating was .375 per year as he changed from a low average his first year a medium-low average his second. Ms. B changed the most as her slope of increase in her average rating was .625 per year as she increased from slightly above low to medium-high. Mr. L who was highly procedural and rigid in his teaching of mathematics was slow to change the types, sources, and treatment of
representations in his classroom. Ms. A who had shown the tendency to be conceptual in her orientation and tried to elicit student understanding did not alter her instruction much with regards to her treatment and use of representations in the classroom. However, Ms. B who was highly procedural during her first year, but less rigid and willing to use student input changed her instruction of representations the most by using CCT in her classroom.

Figure 7.25: Change in average rating of *representations* of each teacher overtime.

Although Ms. A did not receive an overall growth from her first year to her second, her instruction did change. Figure 7.26 shows each teacher’s ratings for each of the subcategories of representations for the years observed. Ms. A decreased in one category, *types of representations*, from high to medium and increased in another, *process/object treatment*, from medium to high. Hence, her overall average rating stayed the same. Mr. L grew in nearly all subcategories of representations from low to medium,
except for the process/object treatment of functions where he remained low both years. Ms. B grew from low to medium in two categories, treatment of representations and process/object treatment of representations, and from medium to high in types of representations present in the classroom. Ms. B was the only teacher to increase by two ranks in a category as she went from low to high in the sources of representations present in the classroom. All teachers remained the same or grew in the three different categories of treatment of representations, the source of representations present, process/object treatment. The only category with a decrease was types of representations present.

Figure 7.26: Ratings of teacher by subcategories of representations during first year (left) and beyond first year (right). Teachers are represented with different colors: Ms. A is blue, Ms. B is red, and Mr. L is green. The categories are types of representations present, sources of representations, treatment of representations and process/object treatment of functions.
Discourse

Teachers were rated low, medium, or high for three subcategories of discourse, ratio of student/teachers talk, types of questions asked, and method of eliciting student discourse. Indicators for an overall discourse rating were based on the ratings of the subcategories as shown in Table 7.2. An average of the ratings of each subcategory was calculated for each teacher. An average rating of 1.0 to 1.5 corresponded with meeting most of the indicators for an overall rating of low. An average rating of 1.6 to 2.5 corresponded with meeting most of the indicators for medium. And an average rating of 2.6 to 3 corresponded with meeting two of the three indicators for high.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
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| High   | • The amount of student talk was a substantial portion of the amount of teacher talk, i.e. the ratio of student words uttered compared to the word uttered by the teacher was high.  
  • The teacher asked conceptual questions for most of instruction.  
  • The teacher’s method of eliciting student discourse not only revolved around student thinking, but elaborating and extending that thinking for most of instruction. |
| Medium | • The amount of students talk was a moderate portion of the amount of teacher talk; the ratio was medium.  
  • The teacher asked questions that could be sometimes conceptual, but mostly moderately conceptual, or sequence of questions that were mixed a mix of procedural or conceptual for most of instruction.  
  • The teacher’s method of eliciting student discourse focused on exposing student thinking, but without exploring that thinking deeply. |
| Low    | • The amount of student talk was a small portion of the amount of teacher talk, the ratio of student to teacher talk was low.  
  • The teacher asked procedural questions for most of instruction.  
  • The teacher’s method of eliciting student discourse focused on getting correct answers and methods in a systematic way for most of instruction. |

Table 7.3: Model for rating teachers overall classroom discourse.
Words per Utterance

Words of teachers and students were counted and an average number of words per utterance were calculated. The percentage for the ratio of student words per utterance to teacher words per utterance was computed as a means to compare student and teacher talk. A teacher was rated low if the ratio of student talk to teacher talk was at or below 25%. A teacher was rated medium if the ratio was in between 25% and 45%. And a teacher was rated high if the ratio was at or above 45%.

Mr. L. Mr. L decreased his average number of words per utterance by less than one from 23.3 his first year to 22.5 in his third year of CCT use as shown in Figure 7.27. His students became more talkative during his third year as the average number of words per utterance increase by almost two from 4.6 to 6.5. The ratio of words uttered by students to the words uttered by Mr. L in his first year was 19.6%; he spoke more than five times more words than his students. He was rated low in his first year. The ratio in his third year was about 29%, so he spoke a little under 3.5 times as much as his students so he was rated medium.

Figure 7.27: Words per utterance in Mr. L’s classroom.
Ms. A. Ms. A decreased her average number of words per utterance by over five words from 26.5 her first year to 19.9 her second as shown in Figure 7.28. The students in her second year class increased their average word by utterance by one above the average for her first year students, from 7.8 first year to 8.8 during her second year. The ratio of words uttered by the students to the words uttered by Ms. A was about 29% in her first year, so she was rated medium. The ratio increased to 44% in her second year, so she spoke a little more than twice as much as her students. Since she was less than one percent from the cut off for high, she was rated high her second year.

![Figure 7.28: Words per utterance in Ms. A’s classroom.](image)

Ms. B. Ms. B decreased her average number of words spoken per utterance from 21.3 her first year to 15.3 her second. Her students spoke about the same amount of words per utterance on average at about 5.6. Ms. B uttered about 4 times as many words as her
students her first year with a ratio of 27%. During her third year she spoke a little less 3
times as many words as her students with a ratio of 37%. Since both ratios were in
between 25% and 45%, she was rated medium both years.

Figure 7.29: Words per utterance in Ms. B’s classroom.

**Comparison.** All the teachers were similar in the average number of words per
utterance during their first year of CCT use as shown in Figure 7.30. Ms. A’s students
were the most talkative during the first year of CCT use, whereas Mr. L’s students were
the least. Ms. B’s students’ average number of words per utterance matched closely to
Mr. L’s students. Ms. A’s and Ms. B’s ratios of words spoken were very similar during
their first year, whereas Mr. L’s students spoke the least compared to the teacher.

All teachers decreased their average number of words per utterance from their first
year to their second or more years of CCT use as indicated in Figure 7.30. Ms. A and Ms.
B decreased their average by about the same amount of 6 words per utterance, whereas
Mr. L decreased his average by less than one word. All the students’ average number of words per utterance increased or stayed the same. Ms. B’s students’ average stayed the same. Both Mr. L’s and Ms. A’s students’ average increased, but Mr. A’s student average increased the most. The rank for the most talkative students changed. Ms. A’s students remained the most talkative, but Mr. L’s students switched with Ms. B’s students as the second most talkative. All teachers’ ratio of student words uttered to teachers’ words uttered increased from their first year as shown in Figure 7.31. However, Ms. A’s ratio increased the most from 29% to 44%. Mr. L increased the second from 19% to 29%. While Ms. B increased her ratio by about 10%, since she was near the bottom of the medium cut off, this increase was not enough to change her rating.

Figure 7.30: Words per utterance in teachers’ classroom during first year of CCT use (left) and beyond first year of CCT use (right).
Figure 7.31: Ratio of student to teacher talk over time.

**Types of Question**

The questions in each subsegment were coded as to whether they were highly *procedural*, highly *conceptual*, a *mix* of procedural and conceptual questions or questions that are procedural but had conceptual components (*in between*). A teacher was rated low if they asked only procedural questions for a majority (more than half) of subsegments. A teacher was rated medium if they asked only procedural questions less than a majority of the subsegments, but asked highly conceptual questions less than 35% of subsegments. A teacher was rated high if they asked conceptual questions a majority of the subsegments.

**Mr. L.** Mr. L asked only procedural questions during his first year of CCT use. However, he substantially decreased his use of procedural questions by over 50% of subsegments to 46% during his third year of CCT use as shown in Figure 7.32. Mr. L increased the number of questions with conceptual components from none at all his first year to a majority of subsegments his third year. Mr. L asked only one type of question in his first year as shown in the Figure 7.32. However, in his third year, Mr. L had two types
of question that he asked in about an equal amount of the subsegments. Mr. L was rated low his first year and medium his third year.

![Figure 7.32: Mr. L’s Types of questions asked during whole class discussion.](image)

**Ms. A.** Ms. A decreased her asking of procedural questions from 23% of subsegments her first year to 14% of subsegments her second. She increased her asking of questions with conceptual components and mixing of procedural and conceptual questions by about 20% (from 15% her first year to 35% her second). Ms. A decreased her asking of conceptual questions from 62% of subsegments to 43% of subsegments. Ms. A asked three different types of question her first and second years, however the composition changed from mostly conceptual questions to an almost even split between conceptual questions and mixing the two. Ms. A was rated high her first year and medium her second.
Ms. B. Ms. B asked only procedural questions during her first year as shown in Figure 7.34. She decreased her asking of procedural questions by over 60% of subsegments to 39% during her third year. She increased her use of both procedural and conceptual questions in the same subsegment by 32% from none at all her first year. Ms. B did not ask conceptual question her first year, but she increased her asking of those questions to 25% her third year. Ms. B’s types of questions changed from only one type to a variety of types shown in the bottom of Figure 7.34. During her third year she had an almost even divide between the three types of questions. Ms. B was rated low her first year and medium her third year.
Comparison. All teachers asked procedural questions in some of the subsegments. However, both Mr. L and Ms. B only asked procedural question during their first year. Ms. A was the only teacher to ask a mix of conceptual and procedural questions. Furthermore, Ms. A was the only teacher to ask conceptual questions a majority of the time.

All teachers changed their types of questions asked during their second or third year of CCT use. Ms. A was the only teacher to decrease her asking of conceptual questions. Mr. L was the only teacher to not ask highly conceptual questions, whereas both Ms. A and Ms. B did. All teachers increased their asking of questions with conceptual components or mixing the two types of questions. Both Mr. L and Ms. B increased their rating from their first year low to medium their third year, while Ms. A decreased her rating from high to medium.
Methods of Eliciting Student Discourse

Each subsegment was coded as to how the teacher elicited discourse from the students. Three main methods were observed: Initiate Respond Evaluate (IRE) with a focus on eliciting students’ answers or methods for solutions (IRE A/M), an IRE style of
questioning but with a focus on students’ ideas or thinking (IRE T), or the teacher could elicit, extend, explain or clarify students thinking or have students make predictions (EECP). A teacher was rated low if they elicited discourse from students using an IRE sequence focused on answers or methods a majority of subsegments. A teacher was rated medium if they used IRE sequences focused on thinking a majority of the subsegments. A teacher was rated high if they used EEECP for the majority of the subsegments.

**Mr. L.** Mr. L elicited discourse from students using only IRE sequences focused on answers or methods his first year of CCT use as depicted in Figure 7.37. Mr. L decreased his use of this method by about 30% from his first year to his third. He increased his focus on student thinking in his IRE sequences by 30% from his first year to his third. Mr. L had only one method his first year and had two methods his second year. Although the majority of subsegments during his third year, he used the method he exclusively used his first year. Mr. L was rated low both years.

![Figure 7.37: Mr. L’s methods of eliciting discourse from students.](image)

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**Ms. A.** Ms. A infrequently used IRE sequences focused on eliciting answers or method from students both years with 15% and 7% of subsegments respectively as shown in Figure 7.38. The rest of the subsegments Ms. A elicited, extended, explained, clarified or had students make predictions for 85% of subsegments her first year and 93% her second year. Ms. A did not change the composition of the different kinds of methods of eliciting student discourse much at all from her first year to her second. She was rated high both years.

![Figure 7.38: Ms. A’s methods of eliciting discourse from students.](image)

**Ms. B.** Ms. B used only IRE sequences focused on eliciting answers and methods from student her first year. During her third year, she cut her reliance on this method by almost half to 43% of subsegments. For the remaining 57% subsegments during her third year she continued to use IRE sequences, but the focus was on student ideas and thinking.
versus answers up from 0% her first year. Ms. B changed the composition of her methods of eliciting student discourse from a single method her first year to an almost even divide between two different methods as shown in the bottom of Figure 7.39. Ms. B was rated low her first year. Since Ms. B was very close to the cut off for the medium rating, she was rated medium her third year.

![Figure 7.39: Ms. B’s methods of eliciting discourse from students.](image)

**Comparison.** All teachers used IRE sequences focused on eliciting answers or methods from students during their first year of CCT use as shown in Figure 7.40. However, this method was barely used by Ms. A and exclusively used by both Mr. L and Ms. B. Ms. A was the only teacher to elicit student discourse with a focus on student thinking during their first year of CCT use. In fact, Ms. A spent a large majority of time with this focus.
All teachers continued to use IRE sequences focused on eliciting answers or methods from students during their second or third year of CCT use as shown in Figure 7.40. However, this method decreased for all teachers. Mr. L continued to use this method for a large majority of subsegments and Ms. B continued to use this method for a near majority of subsegments. All teachers elicited student discourse with a focus on student thinking during their second or third years of CCT use. However, the teachers elicited student thinking for different amounts of class time: Mr. L elicited some of the time, Ms. B almost half of the time, and Ms. A a large majority of the time. Even though both Mr. L and Ms. B elicited student discourse with a focus on student ideas or thinking, they did not explore student thinking like Ms. A did. Mr. L remained low both years, Ms. A remained high both years and Ms. B increased from low to medium.

Figure 7.40: Teachers’ methods of eliciting student discourse during first year of CCT use (left) and beyond first year (right).
Overall Discourse Rating

The ratings for methods of eliciting discourse and types of questions asked were averaged for each year the teachers were observed. The graph in Figure 7.41 displays the change in the average rating for each teacher. Both Mr. L and Ms. B increased their average discourse rating from their first year to their third at the same rate of .33 per year. Ms. A’s overall discourse rating did not change over time. Despite the lack of growth, she was still rated higher than either Ms. B or Mr. L. Since Ms. A was more conceptual to begin with, she did not alter her instruction much from her first year to her second with the types of questions asked and methods of eliciting discourse. Ms. B who was highly procedural her first year changed her instruction the most to include more conceptual questions and methods of eliciting discourse from students indicating she may have become more conceptual. Mr. L who was also highly procedural his first year changed his discourse patterns, but he did not alter his patterns much indicating he probably remained procedural. He continued to ask mostly procedural questions and used mostly IRE sequences focused on eliciting methods or answers.

Both Ms. B and Mr. L remained unchanged in one category, but grew in the other two as shown in Figure 7.42. However, the category without growth was different for these teachers. Mr. L did change his method of eliciting discourse was not enough to reach the next rating level; and Ms. B did not increase the ratio of her students words uttered compared to her own enough to go to the next rating. As to what happened in the representations categories, Ms. B’s overall rating did not change but her instruction did.
She increased in one area but decreased in another. She increased her ratio of student to teacher talk, but decreased the amount of conceptual questions asked in class.

Figure 7.41: Change in teachers’ average discourse rating.

Figure 7.42: Teachers’ rating by subcategories of discourse first year (left) and beyond (right). The teachers are represented by color: Ms. A is blue, Ms. B. is red, and Mr. L is green. The categories are the subcategories of discourse.

Technology Use

Table 7.3 shows the indicators created for each level of technology use and treatment in the classroom. Teachers were rated low, medium, or high for two subcategories of technology, technology use, and treatment of technology. Indicators for an overall
technology rating were based on the ratings of the subcategories as shown in Table 7.3. An average of the ratings of each subcategory was calculated for each teacher. An average rating of 1.0 to 1.5 corresponded with meeting most of the indicators for an overall rating of low. An average rating of 1.6 to 2.5 corresponded with meeting most of the indicators for medium. And an average rating of 2.6 to 3 corresponded with meeting two of the three indicators for high.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>2.5-3</td>
</tr>
<tr>
<td></td>
<td>- The teacher used the display of student generated data collected by the CCT to generate discourse for most of instruction with CCT use.</td>
</tr>
<tr>
<td></td>
<td>- The teacher treated technology as a partner for most of instruction with CCT use. For example, a teacher treating technology as partner could provide access to new kinds of tasks or new ways of approaching tasks that may facilitate understanding, explore different perspectives, or mediate mathematical discussion. (Goos et al., 2003)</td>
</tr>
<tr>
<td>Medium</td>
<td>1.5-2.5</td>
</tr>
<tr>
<td></td>
<td>- The teacher used the display of student generated data collected by the CCT to frequently verify answers or generate discourse, but not for a majority of instruction with CCT use.</td>
</tr>
<tr>
<td></td>
<td>- The teacher treated technology as a servant for most of instruction, but occasionally treated technology as partner. A teacher treating technology as a servant could use the technology as a computational tool to speed up certain processes, but ultimately the tasks remain unchanged and the technology is not used in creative ways to change the nature of the activities. Technology is used to mainly support the teachers’ preferred teaching methods (Goos et al., 2003).</td>
</tr>
<tr>
<td>Low</td>
<td>1-1.5</td>
</tr>
<tr>
<td></td>
<td>- The teacher used the CCT to collect and display student generated data without verifying answers or having discussions for most of the instruction with CCT use.</td>
</tr>
<tr>
<td></td>
<td>- The teacher treated technology as a servant or treated technology as a master for most of instruction with CCT use and rarely treated it as a partner. A teacher may treat technology as a master if their knowledge and usage are limited to a narrow range of operations; to the extent the teacher calls on a student “expert” to use the technology. Given a teacher’s limited knowledge of technology, a teacher may be reluctant to let students use technology to explore mathematics (Goos et al., 2003).</td>
</tr>
</tbody>
</table>

Table 7.4: Model for overall teacher’s technology use and treatment.
CCT Use

How the CCT was used in the classroom was coded for each subsegment. The CCT could be used in multiple ways for each subsegment, such as both an information displayer and information gatherer. Therefore, the codes for the subsegments are not mutually exclusive. A teacher was rated low if they used the CCT almost exclusively as an information displayer or information gatherer when the CCT was used. More specifically they were rated low, if a teacher did not use the CCT as either an answer verifier or discourse generator for more than 20% of the subsegment when the CCT was used. A teacher was rated medium if they used the CCT and an answer verifier for more than 20% of subsegments with CCT use or as a discourse generator for more than 20% but less than 50% of subsegments with CCT use. A teacher was rated high if they used the CCT as a discourse generator for more than 50% of subsegments with CCT use.

Mr. L. Mr. L used the CCT exclusively simultaneously as both as an information displayer and gatherer his first year of CCT use. However, he used only the CCT for 32% of subsegments his first year as shown in the left of Figure 7.43. Mr. L increased his use of CCT in general from 32% to 100% his third year. He increased his use of CCT as an information displayer as a percentage of total subsegments by about 30% from his first year to his third. But as a percentage of subsegments with CCT use, he decreased the use of CCT in this manner by 40% to 60% of subsegments with CCT use. He increased his use of CCT as an information gatherer from 32% of total subsegments to 100% in his third year. However, he continued to use the CCT in this way in 100% of subsegments
with CCT use both years. Mr. L never used CCT as an *answer verifier* or *discourse generator* his first year, but he increased his use of CCT as a *discourse generator* by 32% of subsegments with CCT use. Mr. L appeared to have used the information gathered by the CCT that was displayed to the students to generate discourse. He used CCT in only two ways his first year and three ways his third year as shown in the right of Figure 7.43. He was rated low his first year and medium his third year.

![Graph](image)

**Figure 7.43:** Mr. L’s technology use in the classroom by year and stacked to 100%.

**Ms. A.** Ms. A used CCT for approximately 93% of subsegments her first year and 100% her second year. Ms. A increased her use of CCT as *information displayer* by about 25% from 76% to 100% of total subsegments as shown in the Figure 7.44. However, she continued to use the CCT in this manner for all of the subsegments with CCT use. Ms. A decreased her use of the CCT as an *information gatherer* from 100% of
subsegments with CCT use her first year to 43% her second year. She also decreased her use of CCT as an *answer verifier* from 90% of subsegments with CCT use to 43% of subsegments. However, she increased her use of CCT as a *discourse generator* from 30% of subsegments with CCT use her first year to 57% her second. Ms. A frequently used the technology to display information gathered from the students to verify answers her first year. She occasionally used the CCT in this manner to also generate discourse her first year. Ms. A frequently used the CCT to display information gathered from the students to generate discourse her second year. Ms. A was rated medium her first year and high her second year.

![Figure 7.4](image-url)

Figure 7.44: Ms. A’s technology use in the classroom by year and stacked to 100%.

**Ms. B.** She used CCT for about 41% of total subsegments her first year. She used CCT for 69% of total subsegments her third year. Of the subsegments she used CCT, she
used it as an *information display* for 100% of those subsegments for both years. She used the CCT as an *information gatherer* for about 32% of total subsegments and 77% of subsegments with CCT use. She increased her use as an *information gatherer* both in terms of total subsegments and subsegments with CCT use to 56% and 81% respectively.

The biggest changes in Ms. B.’s use of CCT were her use of CCT as an *answer verifier* and *discourse generator*. Ms. B never used the CCT as an *answer verifier* her first year and only 6% of total subsegments and 15% of subsegments with CCT as a *discourse generator* her first year as shown in Figure 7.45. Ms. B used the CCT to verify answers about 35% of total subsegments or 51% of subsegments with CCT use her third year. She increased her use of CCT as a *discourse generator* from 15% of subsegments with CCT use her first year to 41% her third year. When Ms. B used CCT as a *discourse generator* her first year, she used the CCT to display information gathered from the students. She used this method of generating discourse with the addition of verifying answers more frequently her third year. Ms. B predominately used the CCT in two ways her first year. She changed to using the CCT in a variety of manners her third year. Ms. B was rated low her first year and medium her second.
Comparison. Ms. A used the CCT for the majority of the classroom in a variety of manners, whereas both Mr. L and Ms. B used the CCT infrequently and predominantly in only two manners. Both Ms. B and Mr. L used the CCT almost exclusively as something to display collected information and not much else. Whereas Ms. B used the CCT to do something with the collected information such as verify its correctness or discuss it. During their first year, all teachers always used the CCT to display information when using the technology. All teachers almost always used the CCT to gather information from students when using the technology to display information. Usually some of the information displayed was the information gathered from the students. All teachers also used the CCT to display other information not gathered from students as well.
The disparity in the amount of technology use disappeared as they continued to use the technology. The most common use of the technology after continued use was still as an information displayer, followed by information gatherer as in their first year. Using the CCT as an information gatherer always prompted the use of the CCT as an information displayer for all teachers the first year and the teachers rarely used the CCT as an information displayer without gathering student information. This pattern changed after continued use of the CCT. Both Ms. A and Ms. B continued to always use the CCT to display gathered student information, but both used the CCT to display information not gathered from the students. Mr. L reversed the trend and used the CCT as an information gatherer more frequently than as an information displayer. In other words, he would gather information from the students and sometimes not display any of the collected information to them. Ms. B frequently used the combination of displaying gathered
student information. However, Ms. A used the CCT to display information not collected from students on multiple occasions.

All teachers increased their use of the technology beyond just as an information displayer and gatherer. Both Mr. L and Ms. B increased their use of the technology to do something with the information they collected and displayed to the students. All teachers substantially increased their use of the CCT as a discourse generator and Ms. B increased her use of CCT as an answer verifier substantially as well. For both Ms. A and Ms. B the most common method of generating discourse was discussing the verification of student answers gathered and displayed by the CCT. In other words, the most common way of generating discourse was to use the CCT in the four main methods sequentially. For Ms. B this was the only combination of uses of CCT that led to generating discourse. Ms. A could use information displayed by the CCT other than information gathered from students to generate discourse, although this occurred infrequently. Mr. L used generated discourse by having students talk about the information displayed that was gathered from the students. Mr. L was similar to the other two teachers in how he generated discourse, except that he skipped the answer verification. All teachers increased their rating by one for technology use.
Treatment of CCT

Each subsegment for each teacher was coded as to how they treated the technology, either as a *master*, *servant* or *partner*. A teacher was rated low if they treated technology as a *partner* for less than 25% of subsegments with CCT use. A teacher was rated medium if they treated technology as a *partner* for 25% to 50% of subsegments with CCT use, and treated technology as *servant* for more than 50%. A teacher was rated high if they treated technology a *partner* for more than 50% of subsegments with CCT use.

**Mr. L.** Mr. L treated technology as an add-on his first year. He used it for a small portion of class time, 32% of subsegments. He used the technology once per day for about 10 minutes his first year; the first time to ask students questions using Quick Poll (QP), and the second time he asked students to type solutions to their homework in their calculator to display them using Screen Capture. Both of these uses were typical of his
normal class instruction, and so when he used the technology he treated it as a servant 100% of the subsegments with CCT use as shown in Figure 7.48.

During his third year, Mr. A. used technology to ask simple multiple choice questions before and after the main activity. Since he did not use the QP to help students to see mathematics in new ways during these questions, he treated the technology as a servant extending his usual teaching approach. However, the main activity of finding best fit lines asked students to explore mathematics and the connections between two different representations as lines. During these subsegments he treated technology as a partner, but during the subsegments of the main activity where he treated representations as end products he treated technology as servant. Mr. L typically used exercises to be performed by the students or himself as his main method of instruction his first year, but the main lesson his third year was an activity. Mr. L seemed to allow the incorporation of technology to change his mode of instruction. Mr. L dropped his treatment of technology as a servant from 100% to about 70% of subsegments with CCT use. His treatment of technology as partner increased from 0% to 30%. He began to use the technology in his their year to allow students to see different perspectives in mathematics. Since he treated the CCT as a servant, as an add-on, when he used it, he was rated low his first year. Since he began to treat technology as a partner for a large portion of class time but not most of class time he was rated medium his third year.
Ms. A. Ms. A appeared to be comfortable with the CCT her first year of use. Ms. A was able to elicit discourse from the students, but she did not seem to be aware of the potential for the CCT to help facilitate the discourse and allow students to explore mathematics in different ways. Her lesson could have been accomplished with a graphing calculator overhead display or even plots by hand. She primarily treated the CCT as a *servant* to add efficiency to her usual way of teaching for most of the subsegments, 76%. During the subsegments where she used the display of student errors made available by the CCT to generate discourse, Ms. A treated the CCT as a way to aid understanding or as a *partner*. Ms. A treated the CCT as a *partner* for 24% of subsegments her first year.
Over the course of the next year, Ms. A seemed to have realized at least some of the potential of the CCT to engage students in new ways as she used the CCT to help facilitate discourse. Ms. A used the display of all student groups’ generated representations to generate discourse about the connections of the shape of the parabola and its equation. During these instances she treated technology as a partner. She treated the CCT as partner for 64% of subsegments. During the subsegments where Ms. A wanted students to find a “math way” to determine the b-term of the quadratic, she used the CCT primarily as a display device as she did before. During these instances she treated the CCT as a servant to her teaching as normal. She treated the CCT as a servant for 36% of subsegments. Ms. A decreased her treatment of the CCT as a servant from 76% down to 35% and increased her treatment of CCT as partner from 24% to 64%. Since she was close to the cut off for medium she was rated medium her first year, and
she was rated high her second year. Technology treatment was one of the few areas of growth for Ms. A.

Ms. B. Ms. B appeared to be intimidated with the technology most of the time during her first year. She asked a student to help her when she got stuck using the CCT, which happened on several instances her first year. Ms. B appeared to be overwhelmed at times by all the “wrong” answers the CCT displayed during the missed opportunities and seemed unsure as to how to use the information displayed by the CCT. During these instances she treated the CCT as *master* for 29% of all subsegments or 69% of subsegments with CCT use. When Ms. B demonstrated the calculator program that asked students to guess the equation of the line graph, she seemed to use the technology as a tool to further her own teaching style. Here she treated technology as *servant* for 6% of total subsegments or 15% of subsegments with CCT. On the second day when Ms. B asked her students to find the best fit lines that were not vertical, she asked students to discuss errors displayed by the CCT of all the student submitted lines. Ms. B used the CCT to facilitate at least some student discourse and so she treated the CCT briefly as a *partner* for 6% of the total and 15% with CCT use.
Two years later, Ms. B appeared much more comfortable with technology as she used some sort of technology throughout her instruction mostly with ease. While there were times the technology presented issues, she was able to troubleshoot them herself. Technology no longer seemed to be the master of Ms. B. When she used the function applet the first day, asked students to use the sketchpad to draw curves, corrected student submitted representations using only procedures, she treated the CCT as a servant. When she used the CCT to collect student generated representations to discuss connections among them or collecting student predictions about representations, she treated the CCT as a partner in facilitating discourse or let students see mathematics in new ways. She treated technology as a servant for 35% of all subsegments and 40% with CCT. Ms. B treated CCT as partner for 53% of all subsegments and 60% with CCT. As shown in Figure 7.50 Ms. B decreased her treatment of technology as master from 69% down to
0%. She increased her treatment of technology as *servant* from 15% to 40%, while she increased her treatment of technology as *partner* from 15% to 60% from her first year to her third. Since she treated the CCT primarily as *master* in her first year she was rated low, and since she treated technology as *partner* for more than half of the time with CCT use she was rated high. This jump from low to high was the biggest increase out of all the teachers for any category. Even though the discussion Ms. B used the CCT to help facilitate did not lead to a rich conception of ideas, these discussions could lead to such a conception and Ms. B seemed to be aware of the technology’s potential for aiding student understanding.

**Comparison.** Figure 7.51 shows each teacher’s treatment of technology during their first year. Both Mr. L and Ms. A appeared to use technology primarily as a *servant* to continue instruction as normal. Ms. A used the technology to make her lesson more efficient to help the class get to the conceptualizing quicker, but without the aid of the CCT. Mr. L seemed to use the technology the first day because he had to as part of the CCMS study. The second day he used it for a more efficient manner to check student work, but treated the corrections as he normally did without the CCT. However, Ms. B seemed to have trouble with the technology which limited what she could do with the technology. Both Ms. B and Ms. A treated the technology as *partner* for some of the time during their first year. However, both missed multiple opportunities to treat the CCT as *partner* to facilitate discourse.
None of the teachers treated the CCT as master beyond the first year of CCT use as shown in Figure 7.51. Both Ms. A and Mr. L decreased their treatment of CCT as servant to increase their treatment of the technology as partner. Ms. A decreased her treatment of the CCT as servant more than Mr. L., perhaps as she realized the potential for the CCT to facilitate discourse and allow students to see mathematics from different perspectives. While Mr. L began treating the CCT as a partner to facilitate discourse, he used the technology to continue instruction as normal more than 60% of the time. Ms. B appeared to embrace technology during her third year and no longer seemed intimidated by it. She treated the CCT in a similar manner as Ms. A did. The treatment of CCT was the category with the most growth for all the teachers. After spending time using the technology, the teachers appeared to be able to use the CCT to help facilitate mathematical discourse or to change their method of instruction. Each teacher had the
most growth in the technology category as a whole and all the teachers increased by one rating level or more for the two subcategories as shown in Figure 7.51. Both Mr. L and Ms. A increased their average rating by one level; Mr. L from an average of 1.0 or low to 2.0 or medium, and Ms. A increased from 2.0 or medium to 3.0 or high. Ms. B had the largest increase from 1.0 or low to 2.5. Ms. A had the highest rate of change of 1 average rating per year; Ms. B had a rate of change of .75, and Mr. L had a rate of change of .5, which were the highest rates of change for any of the major categories.

When teachers used the CCT, the major change to their instruction appeared to be how the technology was used and how it was treated. Ms. A only had overall growth in the technology category, whereas for the others she had no overall growth. Mr. L grew by one whole rank in this category, but in the other categories he grew overall for only part of a rank. Ms. B grew by one whole rank before, but for technology she almost grew by two ranks.

**Overall Technology Rating**

As shown in Figure 7.52 both Mr. L and Ms. B used and treated technology at similar levels during their first year. Ms. A was one level higher in each category her first year than both Mr. L and Ms. B Both Ms. A and Mr. L increased their rating in both categories by one level, whereas Ms. B tied Mr. L in use and tied Ms. A in treatment.
Figure 7.52: Change in teachers’ average technology rating.

Figure 7.53: Teachers’ ratings for technology by category first year (left) and beyond (right).
In the Moment Decisions

A decision flow chart was created for each year for each of the teachers using Schoenfeld’s (2010a) framework and the apparent decisions the teachers made in the classroom. The rectangles represent actions taken by the teacher and diamonds represent apparent decisions the teachers made. The octagon shape represents a decision that is made possible or easier from the use of CCT and a rounded rectangle is an action made possible or simpler by using CCT. A rectangle depicts an action the teacher made, and the text inside describes that action. A rectangle with rounded edges depicts an action aided by or made possible by the use of CCT. A diamond represents an apparent decision made by the teacher, and an octagon represents an apparent decision aided by or made possible by the use of CCT. Figure 7.54 is the legend for the decision flow charts. Other than the dashed lines or borders, the items in the legend represent changes over time within flow charts created for each teacher. The first of the items on the left represent new or modified actions and decisions in the flow chart that were not present during the first year, where the color of the glowing border of the first two items indicate where the teacher changed her or his discourse patterns, or their treatment and/or usage of representations or technology. The faded box represents decisions or actions that no longer appeared in the subsequent decision flow chart for a teacher.
Mr. L. Mr. L’s decision tree his first year shown in Figure 7.5 contained three decisions, two of which contained the major cycles of his instruction. The first major cycle revolved around the presence of outcomes Mr. L desired students to make. If a desired outcome of a task or question was not present, Mr. L would try to produce one from the students by rephrasing the question, asking someone else, or provide it himself. With the CCT Mr. L could see all outcomes of a task or question. If he could see undesired outcomes, he would explain or have students explain at least one desired and an undesired outcome. However, this decision was only made when the CCT was used, which occurred for less than 32% of his instruction in his first. The second major cycle of Mr. L’s instruction revolved around Mr. L’s desire to have at least an acknowledgement
of understanding of the material by the students, whether or not the students truly understood. Before he moved on to the next subsegment he would typically asked multiple students if they “got it.” Most of the time students answered yes, but if one did not he would explain or have a student explain again. Mr. L was the only teacher to appear to have a decision influenced by the CCT during the first year.

Figure 7.5: Decision flow chart depicting Mr. L’s instruction during first year.

The dashed border within the action associated with the third decision in Figure 7.55 indicates that a missed opportunity occurred within this action. Mr. L had few missed opportunities to deepen connections among representations in his first year, not because
he was adept as picking up the opportunities when they arose but rather he had control of
the discourse so that those opportunities never occurred. However, when the missed
opportunities did happen, they happened within the second major cycle of “are you okay
with that?” Students might acknowledge understanding when they do not understand. In
one particular instance a student put on the board a number line solution to a linear
inequality that had integers and decimal numbers. Mr. L saw the solution as correct but
the other students did not. He asked what was wrong with the solution and they stated the
combination of the two types of numbers were incorrect. After putting up a solution more
desired by the students, he stated that it is okay to mix numbers like that and moved on to
the next exercise.

The appearance of three new actions, one modified action and three new decisions
occurred in Mr. L’s third year flow chart from his first year as indicated in Figure 7.56
with the glowing borders and shading. The second major cycle of his first year
instruction, the asking of “are you ok with that,” disappeared from the flow chart for his
third year as indicated by the faded shapes in Figure 7.56. The new actions account for
most of the shifts in Mr. L’s instruction from his first year to his third year. Mr. L’s focus
appeared to remain on undesired/desired outcomes, but he treated the pursuit of this focus
differently if students could see the outcomes or not. This became his first new decision,
the second in the flow chart, while his first decision was similar to his first year decision
of moving on if all desired outcomes were present. If students could see all the different
outcomes of the task or question, the action he would take would be to ask student for the
different outcomes while asking clarification questions and without initially indicating which outcomes were desirable. This action and the attached cycle in the flow chart occurred during the activities where students found the best fit lines and discussed the lines present. The purple glowing border of this action in Figure 7.56 indicates that Mr. L shifted his instructional practices in relation to the three categories of representations, discourse and technology. By having students compare and contrast the different lines, Mr. L was having students reason about representations, a shift from
only considering representations as doing something to. Students could see different representations produced by different sources, so he shifted from only himself as the source of producing representations to multiple students as sources. Mr. L asked students to send equations of the best fit line to be graphed by the CCT; so he changed from only one representation at a time to two simultaneous representations. Since he asked students to compare and contrast different lines, he was asking different types of questions that were more conceptual than producing the right answers and he used IRE sequences to focus on their thinking rather than their answers.

During the purple glowing action, he treated the CCT as a partner since he used it so that students could do an open ended activity rather than closed ended exercises and as a discourse generator. Hence shifts in the three major categories occurred within this action; in fact the majority of the growth detected occurred within the cycle associated with this action. The new decision within the cycle on the right after the purple glowing action in Figure 7.56 is a modified decision from the first year; however the focus of the decision was different in that not correct answers were desired but rather correct conclusions from discussion. After asking students to compare and contrast the different lines present, if students had not reached the desired conclusions he would reset or refocus the discussion. This “reset or refocus discussion” action with the blue border also resulted in shifts within Mr. L’s instruction. For example, he might, after asking students to compare all the lines displayed, ask students to compare a certain group of lines with similar properties such as a positive slope. Within the “reset or refocus” action, he asked
more conceptual questions and asked students to think about the lines present. Hence, the blue bordered action resulted in the shift in those two categories. Different from his first year, Mr. L did not display the results of the Quick Poll questions. Asking questions with Quick Poll at the beginning and end of class accounted for the majority of instances, when Mr. L answered “No” to the decision of students seeing all the outcomes and he went down the cycle on the left within the flow chart. The major action within this cycle, the one with the small blue border, was very similar to his action from his first year of “discussing a desired and undesired outcome.” However, this action was modified to not always discussing undesired outcomes, using clarification questions rather than questions to lead them through the process, and occasionally focusing on students thinking rather than correct answers. Within this left cycle most of his instruction remained the same as it was during his first year. Even though the borders of the actions within this cycle indicate shifts occurred, his instruction remained the same as his first year during most of the instances when he entered this cycle. For example, he might focus on student thinking within the left cycle, but he mostly focused on correct answers and methods. During this cycle, he mostly focused on a single representation at a time produced by himself while asking procedural questions that treated representations as something to do rather than as something to think about. He used the CCT to collect data, but not to generate discourse, and he treated the CCT as a servant. The instance where shifts occurred was when Mr. L asked a student what her secret was for finding the best fit lines quickly. In this instance he treated representations as something to think about, he asked conceptual questions,
and focused on student thinking. The different instructional practices within the two major cycles of Mr. L’s decision flow chart explain why there was growth in about half of his instruction while about half remained the same as the previous year. Most of his growth occurred within the cycle focused on comparing and contrasting representations, whereas most of his lack of growth occurred within the cycle focused on desired outcomes. Incorporating the new action of using the CCT to discuss the displayed representations produced by all the students seemed to have spurred a lot of Mr. L’s growth with regards to his treatment and usage of representations, treatment and usage of technology, and discourse.

Mr. L had more missed opportunities occurring within his third year, seemingly because he allowed more students to produce representations, had more than one type of representation present, and asked questions that were not looking for results of procedures. One major source of missed opportunities occurred within the action that occurred when only desired outcomes were present. After all the students produced lines the fit the picture relatively well, Mr. L would stop the students, have a few state their equation for their lines, and move onto the next task by clearing all the lines without discussing them further. Other missed opportunities occurred when Mr. L decided to move on from a discussion even though his desired outcomes of the discussion were not reached. For example when he asked a student what her secret was for finding the best fit line, he tried to elicit her method, and she said she only picked numbers at random. He tried to push further, but she would not elaborate. A few other students did mention what
they did. But they nor did Mr. L explain an efficient method for finding a best fit line. The missed opportunities occurred as a result of both Mr. L’s actions and decisions. Sometimes the opportunities were missed because his actions prevented them from being taken up, such as moving on to the next activity with only stating the right answers, or resetting the discussion when the students could have been coaxed to the desired outcome on their own. While other opportunities were missed because Mr. L decided to move on when his desired outcome of a discussion had not been reached, such as the student’s secret for finding best fit lines.

Ms. A. Ms. A’s major focus appeared to be about eliciting student ideas through discourse during her first year, and her decisions reflected that focus. In her first two decisions of the flow chart (Figure 7.57), she determined whether the students had enough content to discuss. Once Ms. A was satisfied with the amount of material to discuss, she asked the students to compare/contrast graphs, what they noticed, and consolidate their ideas. The major cycle of her instruction began after the discourse started and she seemed to focus on student understanding and participation by eliciting, elaborating, extending, or clarifying student thinking or asking students to make predictions.

The dashed line connecting the last decision to the “Start new Subsegment” action in Figure 7.57 indicates some instructional opportunities to make connections among different representations of functions that Ms. A did not take up when student errors were displayed by the CCT indicating potential misunderstanding of the activity. When
students submitted points that fell on the line \( y = x + 4 \) and several points were systematically wrong indicating the potential for student misunderstanding, Ms. A decided to not pursue the misunderstanding and dismissed the incorrect points as a typing error. A similar missed opportunity occurred when Ms. A decided to switch from a focus on slope to the \( y \)-intercept even though the discussion about slope revealed that some students were not stating the relationship between slope and the coefficient of \( x \) clearly. Ms. A’s missed opportunities resulted when she seemed to ignore evidence of students’ lack of understanding.

Figure 7.57: Decision flow chart depicting Ms. A’s instruction during first year.
Ms. A’s decision flow chart in her second year depicted in Figure 7.58 differed from the flow chart for her first year, since it has two branches, more decisions and actions, and the CCT was central to some of the decisions and actions that occurred. The major cycle of her first year, the EEECP cycle, appears in each branch of her flow chart. The actions of collecting student data and asking students about their thinking were present in her second year’s flow chart. Two new actions and three new decisions appeared in her second year flow chart. The glowing blue border around the rectangle labeled “Draw attention to new perspective or method” in the branch moving to the right indicates that Ms. A shifted her instruction within this action. Ms. A used the CCT during the right branch, but it did not seem to influence the decisions and actions. In particular, Ms. A shifted her treatment of representations and discourse, but not her treatment of technology. When she asked students to figure out a way to determine the value of the “b” term in a quadratic equation from its graph, she drew students’ attention to the process and object nature of parabolas. This was a shift from her previous year’s treatment of functions as prominently as an object. While focusing on functions as process, Ms. A also shifted to asking the students more procedural questions in class. Hence, the blue glowing action resulted in a shift in her discourse. Although during her first year, Ms. A controlled the flow of discourse by asking questions rather than by making statements. However, during her second year she still controlled the flow of discourse with questions, but she also redirected the students when she believed they
needed a new direction. This new decision seemed to lead to the new blue glowing action that resulted in a shift in her instruction.

Ms. A went through the left branch of the flow chart (Figure 7.59) when she focused on student generated representations. During her first year, she seemed to wait for the final outcome of different activities to be completed in class to begin a discussion. However during her second year, she tried to use student generated data for discourse.
during or after each activity. Therefore, the red glowing action became the major action of the left branch. Then she asked students to compare/contrast the representations produced by different groups of students, how they might fix certain representations, and about their interpretations of the representations produced. Ms. A did not treat or use representations differently, nor did she change her discourse patterns within this new action. Hence, she changed both her treatment and her usage of technology; she increased her usage of the CCT to generate discourse. The new action appeared to be the result of a new decision about explaining similarities and differences in student outcomes. In her second year she considered all of the student outcomes collected by the CCT rather than the final representation of an outcome of an activity. She seemed intent on revolving the discussion around student produced representations that she characterized as “interesting differences and similarities among student outcomes.” Since the majority of Ms. A’s actions and decisions in her second year were similar to those in her first year, her instruction did not change much.

The missed opportunities occurred during her second year at times when she would normally continue or begin her EEECP cycles, but rather she decided to move on as depicted by the dashed lines in Figure 7.59. For example within the left branch, when student errors created parallel parabolas, Ms. A could have illustrated deeper connections between algebra and graphs. But Ms. A did not use this opportunity. An example for a missed opportunity for the right branch: Ms. B tasked students to find the “b” term of a
quadratic equation given its graph, but for only the special case of one for the coefficient of the $x$ squared term in the quadratic function.

**Ms. B.** Ms. B seemed to focus on correct answers or methods as her decisions and actions led to eliciting them from students as indicated in the decision flow chart for Ms. B in Figure 7.59. Ms. B was willing to let some of the control of the class go to the students at times, and use their suggestions even if she did not know if they worked. Even though Ms. B appeared to make four decisions during a subsegment, three of those decisions were infrequent as students did not always ask questions, or make suggestions. The major cycle for Ms. B focused on eliciting correct answers or methods from the students or simply stating correct answers or methods herself.

Figure: 7.59: Decision flow chart depicting Ms. B’s instruction during first year.
Some of Ms. B’s missed opportunities occurred during the action where many errors were encountered that seemed to end the flow chart abruptly. When she collected student lines that fit through two points, the CCT displayed that many students did not submit the correct line. So she knew students needed correct answers or methods for finding the answers, but she would highlight the correct answers and then move on to the next question. Ms. B appeared to be overwhelmed by the many incorrect answers disrupting the decision flow chart in Figure 7.60 and she skipped to a new problem. She did this every time she collected lines to fit through two points.

Ms. B appeared to have no decisions or actions concerning making connections across activities, exercises, or tasks exploring the meaning of different representations. She created several different lines with many different slopes, but exploring connections among these representations she illustrated did not seem to be a part of her instruction. She had several slope finding tasks, but none connecting the slopes back to its graph, its equation or what slope means.

Four new decisions, three new actions and one modified action appeared in the decision flow chart in Ms. B’s third year as shown in Figure 7.60. In the presence of incorrect answers that were not the result of asking students to make predictions, Ms. B would perform nearly the same action as she did the year before by using IRE sequences to elicit correct answers or procedures from students. However, she did change her treatment and use of the CCT. She treated the technology as a servant rather than as a master (Goos et al., 2003), and she used the CCT to generate discourse albeit not highly
conceptual. If students made incorrect predictions about the shapes of graphs of equations, she asked for student thinking and pushed for explanations resulting in one of the new action/decision pairs. Ms. B asked students to make predictions about the shapes of graphs of different functions and collected their predictions using the CCT, which was something she had not done during her first year. While performing the new action related to predictions, she would ask students to explain their thinking. By having students make predictions about the graphs, she was treating representations as something to think about and reason with rather than as something to do or make. She also asked

Figure 7.60: Decision flow chart depicting Ms. B’s instruction during third year.
students to explain their thinking about the representations, although the explanations accepted by Ms. B were limited in depth. Since she asked students to make predictions about the shape of the functions, she was revealing the object nature of the functions. Hence, Ms. B shifted in her treatment of representations. Ms. B shifted from her focus on correct answers and methods in her first year to student thinking and asking more conceptual questions in her third year by asking students to make predictions, by focusing on student thinking, and by pressing for at least minor explanations. In particular, Ms. B shifted her instructional practice pertaining to regarding her treatment of discourse with incorrect predictions. Her use of the CCT enabled her to decide which predictions to pursue during discourse.

One of the new decisions in the third year flow chart for Ms. B., “Are there multiple functions present,” created a left and a right branch within the flow chart in Figure 7.60. The majority of her instruction in her third year occurred within these branches. Within both branches a new decision/action paired cycle arose. If only one function was present, her decisions focused on important aspects of displayed functions that should be discussed. If important aspects were present, a new action occurred where she did one of three things 1) ask students to predict a pattern or rule for the function, 2) ask them about the meaning of the shape of the graph within context of the problem, 3) have them predict the shape of the graph of the function. Generally, these actions engaged students in reasoning about/with representations of functions. In other words, she treated functions as
more than something to do but something to think about, which was a shift in her
treatment of representations from her first year.

Ms. B traversed the left branch in Figure 7.60 stemming from the diamond labeled
“Are there multiple functions present?” decision when more than one function was
displayed to the class. The left branch contains another new decision/action cycle. The
functions were typically gathered by first asking students to create a table given the
equation of the functions. Then she collected and displayed the points the students found
using the CCT. Lastly she displayed the graph that fit the points. Within this action the
three main representations were present rather than only the graphical and algebraic
present her first year. While she used IRE sequences to elicit discourse, her focus within
the upper left action was on what students thought rather than correct answers. She asked
more conceptual questions than answers to procedures since she elicited similarities,
differences, or observations about the functions from the students. Therefore, Ms. B
shifted the types of questions and methods of eliciting discourse within this action.
Within this action, she used technology to collect and display student data as she did her
first year. However, Ms. B also used the technology to display multiple functions
generated by the students to elicit observations and discourse from the students. In other
words she used the technology as a discourse generator. Also by using the CCT as a
discourse generator, Ms. B treated the technology more like a partner than a servant or as
a master as she did during her first year. Again Ms. B shifted in the three categories of
representations, discourse, and technology of her instruction within the upper left action.
The left and right branches within Figure 7.60 accounted for most of the growth of Ms. B within the three categories.

Ms. B’s missed opportunities occurred within both actions and decisions in her third year. Ms. B had similar missed opportunities as in her first year, but she handled them differently in her third. She treated functions as a process and occasionally had only one student do the work when correcting student submitted data in her third year, but she did not simply move on as she did her first year. Here, Ms. B missed opportunities to treat functions as objects and use errors in student representations to make more connections among representations. These kinds of missed opportunities occurred within the upper right action in Figure 7.60 with the dashed border. Other missed opportunities occurred when she made decisions to move on in the both the left and right branches in Figure 7.60, such as moving on when more important aspects of the functions present could have been discussed or moving on when more observations about multiple functions could have been made.

**Comparison.** All teachers’ instruction could be depicted with streamlined decisions flow charts in their first year as shown in Figure 7.61, no branches appeared and only small and large cycles existed. The teachers’ movement through the charts was a spiral. At each decision the teacher would cycle, if needed, before moving on to the next level of the chart and continue with the spiral motion. The CCT did not seem to influence most of the decisions across all the teachers in their first year as only Mr. L had such a decision that was influenced by it. Each teacher had an action aided by or made possible by using
CCT; however these actions were not a major part of their instruction. Each teacher had one major decision/action paired cycle that seemed pivotal to their instruction. The focus of these cycles was different for each of the teachers. Both Mr. L’s and Ms. B’s major cycles focused on correcting student answers, whereas Ms. A’s major cycles focused on eliciting student thinking. Both Mr. L’s and Ms. B’s missed opportunities occurred in their treatment of functions. Ms. A’s missed opportunities occurred within her decisions to move on when more could have been discussed.

The use of CCT appeared to have complicated the teachers’ decisions as their instruction could no longer be depicted with streamlined flow charts and the CCT became more embedded and more central to their instruction in the flow chart both in actions and decisions as shown in Figure 7.61 continued. For Mr. L and Ms. B the branch that focused on CCT use contained mostly new actions and/or decisions. Mr. L’s growth mostly occurred within these new actions/decisions and a portion of Ms. B’s growth occurred within these same actions/decisions. Ms. A had a branch focused on CCT use as well, but this branch was composed of old and new decisions. Within this branch, Ms. A grew the most in technology usage and treatment.

Growth occurred within each of the new actions in their decision flow charts for all the teachers as depicted in Figure 7.61 continued. In other words, if a new action appeared in a teacher’s flow chart, then some type of growth occurred within that action. For Ms. A., growth occurred only within actions new to her instruction Ms. B had the most new actions with growth in the three main categories and a modified action with
growth in one category. Mr. L had one new action with growth in three categories, one new action and one modified action with growth in two categories, and one new action with growth in one category. Ms. A had one new action with growth in two categories, and one new action with growth in one category. Teachers’ overall growth appeared to be related to the number of new/modified actions and the categories of growth within these actions.

Figure 7.6: Continued multipart figure: Teachers’ first year decision flow charts, Mr. L (green), Ms. A (blue), and Ms. B (pink). Teachers beyond first year decision flow charts continued on next page with the same colors.
Figure 7.61 continued
Mr. L’s instruction expanded as depicted in the decision flow charts from his first year to his third. He remained focused on desired outcomes; however the desired outcomes could include student ideas rather than just answers or methods as shown in the top of Figure 7.61 continued. Mr. L appeared to augment his instruction from his first year by adding decisions or actions rather than completely alter his practice. The CCT had become more incorporated into his instruction and was central to two new decisions and two new actions. The first decision influenced by the CCT in his third year was a remnant of a major decision from the previous year, which was if only desired outcomes were present he would move on to the next subsegment. The next decision caused the flow chart to branch as he acted differently depending on the answer of the decision. If students could see undesired and/or desired outcomes, he would go through the right branch in his flow chart by asking the students what they noticed about the outcomes presented by the CCT. Usually he appeared to have a desired point in mind when he had students compare the outcomes, such as the roof sloped down so the equations of the lines matching the picture needed to have a negative number in front of the x. Mr. L appeared out of his element when having these kinds of discussions, since if the students did not acknowledge his desired outcome with questions he would reset the discussion with another question that he thought may produce his desired point.

If Mr. L hid the results of student outcomes, his instruction followed the pattern depicted in the left branch of the flow chart. In this branch, his prime focus was on establishing desired outcomes as he began asking students who had desired outcomes to
explain how they arrived at their outcome. However, unlike the right branch he may or may not discuss undesired outcomes and how students may arrive at them. He may accept a student’s explanation of the desired outcome, which typically were answers and methods, as sufficient to move on to the next subsegment. Mr. L’s slight change in orientation appeared in the flow chart as he began in the right branch to ask different kinds of questions and focus some on student thinking rather than purely answers and methods of the left branch.

Like Mr. L., Ms. A’s decision flow chart branched depending on how she wanted to use the CCT. If she wanted collect and display student representations using the CCT, her instruction would follow the pattern depicted in the left branch. If she wanted to use the CCT as an information display of her own representations, her instruction flowed as depicted in the right branch of Figure 5.58. Like Mr. L., Ms. A’s flow chart appeared to be an augmentation of her first year. For example, after collecting student generated representations of best fit parabolas during the first three activities on both days her second year, if there were interesting similarities/differences present Ms. A would ask students about what they noticed about the representations displayed as she did the previous year. Her focus in this branch remained about student understanding and participation as it had her first year. She continued using her EEECP cycles as her main method of instruction within this branch.

Ms. A went down the right branch during the last five subsegments of the first day her second year, when she wanted students to understand how to find the b-term of the
parabola and make predictions about the shapes of equations. After describing the task and displaying important information, she began in a similar manner as depicted in the left branch and asked students how they thought the task could be accomplished or what their predictions about what the equations of the parabolas should look like. Since these tasks had a more specific goal in mind, like predicting shape or figuring out a way to determine the b-term, Ms. A needed to focus students’ attention on a new perspective, such as looking at the x-intercepts rather than the whole graph, to try to lead the students to the goal. She also had the EEECP cycles depicted in the right branch as a significant part of her instruction.

Like the other two teachers, Ms. B’s flow chart for her third year appeared to be an augmentation of her first year’s instruction. The major parts that remained the same as the first year were the IRE cycles she used to correct wrong answers and incorporating student suggestions into her lesson. Despite her use of IRE sequences to correct student generated representations as evident in her first year, she did not stop there. Rather, she used the corrected student generated representations as something to discuss in class later. New to the flow chart was Ms. B’s new focus on student thinking, student predictions and treating representations as something to think and talk about rather than just generate. The flow chart revealed that Ms. B’s orientation was in a transitional state between the conceptual and procedural as she began to focus on more conceptual ideas but remained procedural for corrections. For each teacher the CCT became more
embedded in their decisions flow charts, as revealed the change in orientation or lack thereof for the teachers.

**Orientation**

The orientation of each teacher was determined each year on a continuum from highly procedural to highly conceptual as inferred by their actions in the classroom, such as how they treated functions, representations, the questions they asked, methods used to elicit discussion, apparent goals, etc. Teachers were rated according to where their orientation toward mathematics was on the procedural/conceptual spectrum: highly procedural with a rating of 1, mostly procedural with a hint of conceptual with a rating of 1.5, in transition between the ends of the spectrum with a rating of 2, conceptual with a hint of procedural with a rating of 2.5, and highly conceptual with a rating of 3. The indicators for each of these ratings are described in Table 7.4.
<table>
<thead>
<tr>
<th>Orientation</th>
<th>Description</th>
</tr>
</thead>
</table>
| Highly Conceptual  | - Tends to focus students away from thoughtless application of procedures and toward a rich conception of ideas.  
                        - Tends to focus on aspects of the situation that give meaning to the representations present that in turn suggests the procedures that might be useful.  
                        - Asks questions that move students to view representations in a nonprocedural context.  
                        - Has an expectation that students are intellectually engaged in tasks and activities.  
                        - When procedures are needed to be used, have students think about the products or the procedures themselves. |
| Conceptual         | - Meets all or most of the indicators for a Highly Conceptual teacher for most of instruction.  
                        - However, have occasional instances where procedure is the primary focus and little to no movement toward a rich conception of ideas. |
| Transitional       | - Tries to focus students away from pure procedures toward a rich conception of ideas, but the conception of ideas is lacking.  
                        - Gives some meaning to representations and procedures performed on them, but the meaning lacks depth.  
                        - Alternates between conceptual and procedural orientations, independent of context.  
                        - Reverts to procedural explanations without a hint of conceptual ideas when procedures are convenient or when conceptual explanations are difficult.  
                        - Frequently asks questions that can be easily answered with procedures or results of procedures, but also frequently asks questions that reveal the meaning of the mathematics. |
| Procedural         | - Meets all or most of the indicators for a Highly Procedural teacher for most of instruction.  
                        - However, has not completely abandoned meaning in mathematics as they tend to have occasional instances where representations are more than something to do or make, but something to think about and something that has meaning with connections to other representations. |
| Highly Procedural  | - Tendency to speak in the language of symbols and symbolic manipulations.  
                        - Casts problem solving as performing the right procedures.  
                        - Emphasizes indentifying and performing procedures.  
                        - Tends to perform procedures whenever possible.  
                        - Tends to disregard the context of the problem in which procedures occur and how they might arise naturally from understanding the situation.  
                        - Tends to remediate student difficulties with procedures independently of the context in which the difficulties arose.  
                        - Treat problem solving as flat. Nothing is more important than anything else, except for the answer which is of supreme importance.  
                        - Has a narrow view of mathematical patterns as limited to single problems, rather than patterns across many types of problems.  
                        - Asks questions that can be easily answered with procedures. |

Table 7.5: Model for teacher’s orientation toward mathematics: a modified version of Thompson et al. (1994) calculational/conceptual orientation toward mathematics.
Mr. L

Mr. L’s orientation appeared to be procedural during the first year of CCT use. Most of the questions he asked were for the result of a procedure. The explanations he gave were usually step-by-step procedures or explanations are the result of a procedure. He treated functions exclusively as processes and not objects. He treated representations as something to do and then as something to create. Once the representations were created, he moved on to something else. Mr. L was rated highly procedural his first year of CCT use.

Mr. L’s orientation appeared to remain procedural but to a lesser degree during his third year of CCT use. The dichotomous true/false questions he asked at the beginning of class reflected his procedural orientation to teaching mathematics. The procedural orientation is evidenced by Mr. L’s treatment of lines as processes and his acceptance of students treating lines in this manner. When a student treated a line as an object by suggesting moving the line’s x-intercepts to better match the picture, Mr. L did not pursue this line of thinking and did reset the conversation to suit his goals. Mr. L focused students’ attention away from the sole application of procedures towards a conception of ideas by pursuing the connections of representations of lines. This conception of ideas was not rich, nor completely coherent. Not one question from Mr. L was to complete a procedure in his third year. Therefore, he was procedural but to a lesser extent.
Ms. A

Ms. A’s orientation appeared to be farther on the conceptual end of the conceptual/procedural spectrum as evidenced in part by the classroom discourse. Ms. A may have asked students to perform procedural tasks such as creating T-charts, but content was rarely about procedure unless the students had recently learned a new procedure like the slope formula. The one exception to this appeared to be Ms. A’s focus on the procedure of “plugging it in” to check the points students submitted. The classroom discourse Ms. A promoted during the first day dealt with some conceptual issues, such as what is the relationship between the coefficient of x and the slope of the line, or the connection between the constant term and the shape of the line. Her goals for the second day appeared to be to practice and reinforce the concepts of the day before. As such, the discourse was focused more on the procedures like convincing students that order does not matter using the slope formula. She also guided the students towards realizing an efficient procedure for checking points on a line. Despite the focus being more procedural the second day of her first year, a very conceptual discussion on the meaning of infinite slope occurred. The procedure of finding the slope of a line given its graph did not appear in class until the activities led the students to the conclusion they needed a more precise way of finding slope. Ms. A appeared to flexibly move from procedural to conceptual when needed. Ms. A’s treatment of representations of functions as both process and object revealed her conceptual orientation as both perspectives are needed to fully understand the notion of function.
Ms. A’s orientation remained further on the conceptual end of the procedural/conceptual spectrum during her second year. Ms. A showed flexibility in moving from conceptual to procedural topics. For instance, while the purpose as stated by Ms. A for activity 4 was to promote precise connections between the graph of a parabola and its equation, she spent a large portion of this discussion on a procedural method of finding the b term and setting factors equal to zero and solving for x. The goal activities of the second day appeared to be to make new connections or reinforce the connections of the day before between the algebraic and the graphical representations of parabolas. During these activities when discussing how to fix a group’s parabola, Ms. A and the students relied on procedural methods of adjusting it, such as Casey’s method and setting linear factors equal to zero.

Ms. B

Ms. B’s exhibited a highly procedural orientation toward mathematics during her first year. The explanations she offered concerned the results of procedures. Only when she used embodied representations did her explanations became conceptual by demonstrating the impact of slope on the shape of the line; but these instances appeared unplanned. However, Ms. B never explicitly discussed the meaning of slope. Using IRE sequences, most of her questions asked the students for the result of a certain procedure or for the procedure needed. Ms. B treated representations almost exclusively as process and her treatment of representations as end products, or something to enact procedures upon, revealed her procedural orientation.
Ms. B’s orientation vacillated between procedural and more conceptual during her third year of CCT use. She exhibited a procedural orientation in several instances such as in her introduction of functions; when asking students to extrapolate beyond the data by focusing on finding “the rule”; and in correcting student submitted points. Moreover, her explanation of why a quadratic was not a line; and her demonstration of how the calculator can help students perform computation were also procedural. Ms. B’s orientation appeared more conceptual in other instances such as her asking students to compare and contrasts the graphs of lines: her asking students to think about what a vertical line in a distance time graph would mean; her asking students to reason about the shape of a graph of the height of water in a pitcher versus time and her asking students to make predictions about the shapes of graphs of different equations. Although Ms. B seemed to be able to switch orientations, the transition was not fluid and closely limited to the task she performed. Within a given task she did not appear to switch orientations such as correcting student submitted points. Not switching orientations within a task appeared to add to the missed opportunities present in Ms. B’s instruction.

**Comparison**

Ms. A’s orientation toward mathematics remained highly conceptual as shown in Figure 7.62. She was the only teacher whose orientation did not change. Both Mr. L and Ms. B changed from highly procedural orientations toward mathematics to a less procedural orientation to different degrees. Mr. L largely remained procedural, but at times focused on his students’ ideas. Mr. L changed from highly procedural to
procedural. Ms. B appeared to want to push her students toward a rich conception of ideas. However, she did not appear able to always do so, since she frequently relied on the procedural. And if she did push the students to a conception of ideas it was not always rich. Ms. B changed from highly procedural to being in a transition between the two ends of the continuum, and she changed the most of the three teachers.

Figure 7.62: Change in teachers’ orientation toward mathematics over time.

**Missed Opportunities**

Each teacher had multiple missed opportunities to establish deeper connections among the different representations present in the classroom. Mr. L had the fewest his first year, not because he was adept at using these opportunities when they occurred but rather he controlled the events of the classroom to the extent that the opportunities did not present themselves. The teachers shared a similar kind of missed opportunity involving
parallel lines, but they missed the opportunity due to different reasons. Some of the missed opportunities occurred because the teacher thought the errors were minor, the students achieved the goal of activity and no discussion was needed, or the teacher was overwhelmed by the amount of errors.

Using parallel lines to further connect the graphical and algebraic representations was the most common missed opportunity for each teacher. Figure 7.63a shows near parallel lines that nearly fit different shelves of a slanted book case from Mr. L’s class during his third year. Mr. L asked different students to find lines that fit each of the shelves. After a few minutes the students produced the lines in the left picture. He asked a couple of students what their equations were before moving on to the next activity. To further the connections between the graphical and algebraic representations, he could have displayed the equations and graphs of the students submitted lines and asked for similarities and differences. However, Mr. L still focused largely on correct answers.

Figure 7.63: Missed opportunites from Mr. L (a), Ms. A (b), and Ms. B (c).
Once the desired outcome was made available by the students, he stated the desired outcome and moved on to the next task as he had done in previous years. This missed opportunity was built into his flow chart, since reaching a desired outcome concluded his instructional cycle.

Figure 7.63b shows points that form parallel lines in Ms. A’s class in her first year. The students were asked to find points that fit on the line $y=x+4$. At least two students submitted points that appear to fit the line $y=x-4$, rather than the line that was asked. When Ms. A saw the points she claimed, “Oops, a little mistake there; probably just entered them wrong...” While it may by possible that a few points were entered incorrectly, the likelihood that several points were entered incorrectly to form a line parallel to the desired line seemed low. More likely, the students found the points incorrectly and submitted these incorrect points correctly. Perhaps the students thought, “in order to find $y=x+4$ four needs to be subtracted from x similar to solving equations.” Promoting student understanding was one of Ms. A’s major decisions in her flow chart her first year; however Ms. A dismissed the systematic errors as a simple mistake. A break-down of her flow chart occurred here as she moved on rather than beginning her typical EEECP cycle.

Figure 7.63c shows parallel lines, reflected lines, and perpendicular lines from Ms. B’s class her first year. Ms. B asked the class to submit lines that connected the two points displayed on the CCT after giving them similar tasks to do in class. The lines
displayed in the picture are the results of the student submissions. Many of the lines submitted were incorrect. Normally, when incorrect answers were present she used IRE sequences to correct them. However, Ms. B appeared overwhelmed by all the incorrect lines as she stated the correct answers and proceeded with the next task. So there was a break-down in her flow chart. Moreover, the lines submitted were systematically and only partially incorrect as most had the right slope but the wrong y-intercept. The lines that had the wrong slope were reflections or perpendicular to the correct line. That is their slopes were each a common mistake away from being correct. The break-down in Ms. B’s flow chart seemed to occur because her limited mathematical knowledge prevented her from seeing the partially correct lines rather than all wrong lines.

**Overall Growth**

Figure 7.64 shows each teacher’s average rating for the four main categories for their first year (left) and beyond first year (right). All of the teachers grew in one or more of the major categories. Ms. A was the only teacher to grow in one category while the overall growth in the other categories remained the same. The category that all teachers had growth, in fact had the most growth, was the technology category. Each teacher grew by one or more overall ranking. When a teacher used CCT over multiple years, the area that changed the most was how teachers treated and used the technology as she or he began to use it in more sophisticated ways to elicit and generate discourse. The representations category had the second highest overall growth. Ms. A’s practice remained unchanged but Mr. L grew by almost a full rank in representations overall and
Ms. B grew by a little more than a full rank. While the teachers treated and used technology in more sophisticated ways, they also seemed to do the same when dealing with representations. The orientation toward mathematics had the second to last most growth. Ms. A’s orientations remained unchanged; Mr. L grew slightly from being highly procedural to procedural; while Ms. B grew from highly procedural to being in a state of transition. The teachers, who did not already try to move students toward a rich conception of ideas, began to do so more to some degree as they continued to use the CCT. The discourse category had the least amount of growth. Again Ms. B’s overall discourse rating remained unchanged and both Mr. L’s and Ms. B’s rating increased by less than a full rank. Ms. A’s discourse already focused on student thinking in her questions and her method of engaging students in discourse. However, Both Mr. L and Ms. B began to focus more on conceptual questions and focusing on student thinking rather than just answers. Perhaps changing how a teacher engages his or her students is more difficult than changing how one treats and uses technology and representations.

The teachers’ average rating for each of the four major categories were then averaged to assign their respective overall rating. Figure 7.65 shows the overall rating at the beginning and end of the observations. Ms. A started the highest and remained the highest overall, but she grew the least. Mr. L started the lowest, although not much lower than Ms. B and remained the lowest. However, he grew more than Ms. A; Ms. B started near the bottom and grew the most of the three teachers.
Figure 7.64: Teachers’ ratings by major categories first year (left) and beyond first year (right). The teachers are represented by color: Mr. L is green, Ms. B is red, and Ms. A is blue.

Figure 7.65: Change in teachers average rating for the four major categories.
Change in Other Categories Compared to Orientation

A teacher’s orientation toward mathematics influenced their ratings in the other three categories of representations, discourse, and technology. A teacher who exhibited a conceptual orientation toward mathematics typically had high ratings in the three categories. A teacher with a procedural orientation typically was rated low in the other categories. A teacher whose orientation was in a state of transition had ratings ranging from low to high with most being medium.

**Mr. L.** Figure 7.66 shows the change in Mr. L’s average rating for each of the four major categories. During his first year, Mr. L was highly procedural and he was rated low in the other three categories. In fact, his procedural orientation seemed to limit how he treated and used representations as something to do and to make, using a single representation at a time produced by him and treated as a process. His orientation seemed to limit his discourse to revolving around correct answers and procedures. Being procedurally oriented, Mr. L did not use the CCT his first year to allow students to see mathematics from different perspectives, but rather to proceed as usual. While he remained largely focused on answers and procedures, Mr. L’s orientation changed over time to include some conceptual ideas for at least a small portion of instruction, The shift in focus from all procedural to mostly procedural and some conceptual seemed to allow Mr. L to treat representations as something to think about and not just something to do or make. To treat representations as something to think about, Mr. L focused more of the
classroom discourse on student thinking by asking questions beyond finding answers and methods. Mr. L’s shift in orientation seemed to allow the growth in discourse. He used technology as a *discourse generator* to connect representations and as a way to give students a new perspective on mathematics in his third year. Had he remained highly procedural, he more than likely would have used the technology to reach his main goal of producing desired outcomes. Mr. L’s shift in orientation seemed to increase the levels of the other categories.

Figure 7.66: Mr. L’s change in each of the average ratings of each of the major categories.

**Ms. A.** Figure 7.67 shows change in the average ratings in each of the major categories for Ms. A. She had a highly conceptual orientation toward mathematics her first and second years, so she did not change orientations. Her use and treatment of representations, and her classroom discourse remained unchanged. Since Ms. A exhibited a conceptual orientation toward mathematics, she seemed to attempt to provide, with a
relatively high degree of success, a rich representational and a rich discourse environment for students. The only category that changed was technology in which she used and treated the CCT as an aid to facilitate discourse in her second year. The categories that were marked high (above 2.5 average rating) were also the categories in which she had no overall growth. The category with overall growth started at a medium rating. Ms. A’s rating for overall technology use and treatment seemed to eventually match the rating of her orientation.

![Chart showing Ms. A’s change in each of the average ratings of each of the major categories.](image)

**Figure 7.67**: Ms. A’s change in each of the average ratings of each of the major categories.

**Ms. B.** Figure 7.68 shows changes in Ms. B’s average rating in each of the four major categories. Like Mr. L., Ms. B had a highly procedural orientation toward mathematics and was also rated low in all the other major categories. As with Mr. L., her orientation during her first year seemed to limit her treatment and use of representations;
reduce her discourse to answering dichotomous question and eliciting procedures from students. It also limited her technology use. Ms. B experienced the greatest change in orientation and in each of the other three major categories of representations, discourse and technology. Her overall scores changed the most in every category. As she shifted more toward a conceptual orientation toward teaching, she seemed to try to use technology to provide a richer representational and generating a discourse environment for her students. Again as with Mr. L., Ms. B’s change in orientation seemed to increase her level of use in all of the other categories.

![Figure 7.68: Ms. B’s change in the average ratings of each of the major categories.](image)

Each teacher’s level of use in the three categories of representations, discourse, and technology seemed to converge to the level of their orientation over time, since the conceptual teacher’s ratings either remained high or became high, and the procedural
teachers’ levels increased as they shifted their orientations. The teachers’ orientations seemed to act as an attractor for the other categories, meaning the orientation seemed to push or pull the levels of the other categories up or down to about the level of their orientation. If the teacher had a high level orientation, any category that was not close already would be pulled up. If the teacher had a low-level orientation, the other categories were pulled down. If the teacher had a medium-level orientation, the other categories were pushed or pulled to the middle.

Research Questions

**What kinds of representations are used in the classroom by teachers?** Each of the three main representations appeared across all teachers’ instruction their first year, however the algebraic representation was the most visible even when presented in combination as a unit. The use of graphical representation appeared the second most frequently with the tabular appearing the least. The representations could appear alone, with one other representation, or all three at once. The treatment of the representations of functions and treatment of representation in general appeared to be linked to the orientation of the teacher. In a highly procedural teacher’s classroom, the representations of functions were treated as process and representations were treated as something to manipulate. In a conceptually based classroom, representations of functions were treated as objects and representations were treated as something to reason about or explain with. The orientation of the teacher also influenced the source of representations. In a highly
procedural classroom, the teacher generated most of the representations. However, in a conceptual classroom, the students generated most of the representations.

While not all three representations appeared in each of the teacher’s classrooms beyond their first year, they were more likely to appear in combination with other representations. The algebraic/graphical combination of representations appeared the most frequently across all teachers. The next most frequent combination of representations was all three at once, then the graphical/tabular combination. Only the algebraic and graphical representations appeared alone in any of the teachers’ instruction. Representations of functions were treated solely as process in the procedural teacher’s classroom and used as end products. However, representations were also treated as something to reason about as well in the procedural classroom. Representations of functions were treated as a process alone, object alone, or process/object mix in both the conceptually oriented teacher’s classroom and the teacher in transition’s classroom. The latter two treatments occurred more frequently in the conceptual teacher’s classroom. Both the conceptual teacher and the teacher in transition treated representations as something to reason about/with and to explain/justify about/with and again the latter treatment occurred more frequently in the conceptual teacher’s classroom.

**What is the quality of discourse about representations or use of representations by the teacher?** The questions asked to elicit discourse from students ranged from procedural questions to conceptual questions in the teachers’ first year. The conceptual questions were only asked by the conceptually oriented teacher, whereas only procedural
questions were asked by the procedurally oriented teachers. All discourse was from teacher to student or student to teacher in their first year. Within the procedurally oriented teachers’ classrooms, the teachers used only IRE sequences focused on answers or methods of finding answers as the only method of eliciting discourse from students. While the conceptual teacher used similar types of IRE sequences to elicit discourse, she elicited student thinking, extended their ideas, made explanations, clarified student statements, or expected students to make predictions.

The amount of purely procedural questions that the teachers asked decreased over the years. These types of questions were mostly replaced by questions that were in transition between conceptual and procedural or a mix of the two for all three teachers. However, a large portion of the procedural teacher’s and the teacher in transition’s questions were procedural. The teacher in transition had a portion of the procedural questions replaced by conceptual ones, whereas the procedural teacher asked no conceptual questions. The procedurally oriented teacher and the teacher in transition most frequently used IRE sequences on eliciting answers or procedures from students to elicit discourse from students. However, both began using the IRE sequences to focus on student thinking as a method of eliciting discourse beyond their first year. The conceptual teacher remained mostly the same with regards to methods of eliciting discourse and types of questions asked.

**For what purpose do teachers rely on multiple representations in classroom discourse?** The purpose of representations in procedurally oriented teachers’ classroom
during their first year was to perform procedures to create end products. In other words representations were present in the classroom to allow students to do something or to make something. Representations were not used to explain or to argue. In the conceptually oriented teacher’s classroom, representations were treated as something to think about and as something to use in explanations. Despite this, only the conceptually oriented teacher pressed more frequently for explanations. The procedurally oriented teachers’ purpose for representations in the classroom continued to be something to do or something to make. However, the purpose of representations in these teachers’ classroom also appeared to be to something to think about.

**What is the relationship between teacher use of CCT in algebra classrooms and the growth of teacher choice of representations of linear functions as manifested in the classroom discourse?** Growth in teachers’ instruction with respect to each of the four categories of representations, discourse, technology, and decisions occurred for most teachers, but one teacher experienced no growth in some categories. By using CCT in the classroom, teachers grew the most in the category technology itself. Teachers began to see and use CCT more as a partner rather than as a servant and began to use it in more sophisticated ways rather than doing the same things more efficiently. They began to use displays of student generated representations to spur student discourse. Teachers quickly became more comfortable and proficient with the technology.

Growth within the *representations* and *discourse* categories occurred, but not as much as *technology*. While using CCT, teachers increased or remained the same in the types
and combinations of different representations of functions present in the classroom. Teachers that started with fewer combinations of representations simultaneously present increased the amount of combinations present. In classrooms where representations were treated as something to do or something to make, representations eventually also became something to think about. Representations increasingly were generated by all or most of the students in the classroom rather than being created by the teacher with the continued use of CCT in the classroom. Some teachers treated representations of functions as objects or as a mixture of process and object more frequently. However, not all teachers shifted their treatment of the dual nature of representations of functions of process/object.

Discourse in teachers’ classrooms remained the same or grew with the continued use of CCT, but this category had the least growth overall the teachers. Several elements of discourse from teachers previous years remained in subsequent years. However, new elements of discourse arose in subsequent year, especially when teachers focused on multiple student generated representations. During this focus, teachers asked more conceptual questions, and focused more on student thinking with their methods of eliciting discourse.

The category that seemed to be the most influenced by continued CCT use was the teachers’ decisions. The decision flow charts became more complicated with the continued CCT use by branching in multiple directions rather than the more streamlined flows of previous years. The CCT became more embedded within the decision flow charts for all teachers. Teachers devoted part of their decision flow charts to focusing on
student ideas present when students generated representations that were displayed by the CCT. Only one teacher seemed to use the CCT within their decision flow chart their first year. While core cycles from teachers in flow charts from previous years remained, CCT use became more central to new decision/action cycles that appeared in subsequent years. Nearly all growth for the other three categories of representations, discourse, and technology, occurred within these new actions. The new actions seemed increasingly focused on student ideas, making connections among representations, or asking more conceptual questions. The shift in focus to student thinking seemed to change some of the teachers’ orientation toward mathematics as they could not ignore differences in student ideas. The teachers who were not already conceptual in orientation became more conceptual with continued CCT use.

After presenting a case study characterizing each teacher’s instruction during the observations in previous chapters, this chapter has focused on a cross-case comparison of the three teachers’ orientations to teaching mathematics, their use and treatment of representations, use and treatment of technology, and questions used and methods classroom discourse. The next chapter will summarize the study and present conclusions and recommendations.
CHAPTER 8: SUMMARY, CONCLUSIONS, DISCUSSION, AND IMPLICATIONS

This chapter is dedicated to a discussion of results of the cases studies and the subsequent cross analysis of these individual cases. An overview of the problem statement, research questions, and methodology are presented. Findings and their implications for research, professional development, and pre- and in-service teacher education will be discussed. A theoretical model for CCT use and teacher growth will also be offered.

Problem Statement

Several features of Connected Classroom Technology (CCT), in particular the TI-Navigator, have the potential to elicit and display ideas from all students in the classroom exposing differences in their thinking. Quick Poll (QP), Activity Center (AC), Screen Capture (SC), and Learn Check (LC) all can invite all students to participate in activities and practice using representations. Through Activity Center and Screen Capture, students can see their own representations as well as peers’ representations allowing for reflection on and negotiation of the representations. Activity Center has the capability to move among three major representations, algebraic, graphical, and tabular of the same object. With several students participating in sharing ideas, alternative representations are likely
to occur. Pursuing explanations and justifications of particular representations may or may not be reasonable can aid students to see representations as tools for exploring mathematics.

The purpose of this study was to develop a model for tracing teacher growth in usage and treatment of representations, discourse, and technology with continued use of CCT in the classroom. The following question guided the data analysis:

What is the relationship between teacher use of CCT in algebra classrooms and the growth of teacher choice of representations of functions as manifested in the classroom discourse?

a. What kinds of representations are used in the classroom by teachers?
b. What is the quality of discourse about representations or use of representations by the teacher?
c. For what purpose do teachers rely on multiple representations in classroom discourse?
d. What changes occurred within teachers’ in-the-moment decisions over time with continued use of CCT?

Methodology

Data from the Classroom Connectivity in Promoting Mathematics and Science Achievement (CCMS) project were used to in this study to create an empirically based theoretic model of teachers’ over time growth in their treatment and usage of representations in the presence of CCT. The CCMS project was a randomized cross-over trial where the control group received the intervention sequentially. The teacher participants were assigned to two cohorts by random selection. Cohort 1 treatment group received training on how to use connected classroom technology (CCT) during a week
long summer institute. Cohort 2 teachers received similar training the summer before their second year of the project. During each year that they were in the study they received follow-up professional development during an international technology convention (Irving et al., 2010). Cohort 2 teachers used graphing calculators without the connected classroom technology with their students during the first year. The surveys and tests given to Cohort 2 students during the first year served as control data for comparison with treatment groups.

**Participants**

Among the 127 original teacher participants of the CCMS study, seven were initially identified as candidates for this study. The initial participants were chosen according to whether videotaped lessons on the representations of functions for multiple years existed in the projects’ data bank. To allow for more in-depth analysis the participant selection process was further refined. The teachers’ responses to the Teacher Instructional Practices and Beliefs Survey (TIBPS) survey were used to find the most diverse group of participants to generate the model of teacher growth. Those participants are Mr. L., Ms. A., and Ms. B
<table>
<thead>
<tr>
<th>Participant</th>
<th>Location</th>
<th>Native Indian</th>
<th>Asian</th>
<th>Hispanic</th>
<th>African American</th>
<th>Caucasian</th>
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<td>77</td>
<td>1</td>
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<td>89</td>
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<td>5</td>
<td>9</td>
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<td>0</td>
<td>7</td>
<td>89</td>
<td>8</td>
</tr>
</tbody>
</table>

Note: All numbers are percentages.

Table 8.1: Demographic data of participants’ schools.

Data Sources

Quantitative and qualitative data were gathered by the CCMS project. This study used several of the data sources from the project. Teacher-level measures, teacher Post Observation Interviews (POI), and video data from classroom observations comprised the data set for this study.

The Teacher Instructional Practices and beliefs (TIPBS) was established to be able to track teachers’ technology use, professional development outside the project and measure their practices and beliefs about mathematics teaching and learning. Of particular interest to this study were the technology use, strategy discussion, explanations and justifications, data analysis, and reform classroom discourse subscales.

Fifty-five (Cohort 1 = 25, Cohort 2 = 30) of the 127 teachers in the CCMS study had videotaped classroom observations at in least one of the four years of the study. Videos observed either two or five consecutive days of instruction. Topics ranged from slopes of lines, properties of lines, systems of equations/inequalities, finding best fit lines or
quadratics, to the introduction of functions. Only the video data and verbatim transcripts of these three teachers were used in this study.

<table>
<thead>
<tr>
<th>Years of CCT Use</th>
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<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>Mr. L, Ms. A., Ms. B.</td>
</tr>
<tr>
<td>2</td>
<td>Ms. A.</td>
</tr>
<tr>
<td>3</td>
<td>Mr. L., Ms. B.</td>
</tr>
</tbody>
</table>

Table 8.2: Classroom observation data of participants.

**Data Analysis**

Field notes were taken during the first viewing of the video data and Post Observation Interview (POI) with each teacher. The videos, transcripts of both the classroom observations and POI’s, and observers’ notes were uploaded into data analysis software. The data were coded and categorized using a priori codes arising from the conceptual framework and a grounded theory approach (Creswell, 2005). Emergent patterns within the codes, categories and themes were used to create a model of teacher growth while using CCT. The literature (Moschkovich et al., 1993; Pape & Tchoshanov, 2001; Schoenfeld, 2010a; A. G. Thompson et al., 1994) points to many characteristics influencing representations in the classroom such as treatment and usage of representations themselves, discourse surrounding representations, technology use, and in the moment decisions teachers make in the classroom. Both a priori codes and grounded codes were used within these four main categories.
Individual case studies were created for each of the three teachers describing their classroom instruction for each year observed with respect to the categories of representations, discourse, technology and decisions, creating an outsiders perspective on the teacher’s instruction. The POI’s were used in the cases studies to provide an insider’s perspective on the teacher’s own instruction.

**Coding.** Video data of the classroom observations were coded using a priori codes and grounded methodology to create the case studies and to capture growth, or lack thereof of teachers’ instruction with respect to the four main categories. Classroom instruction was segmented by grouping clusters of similar activities, exercises, or discussion, such as teacher-led exercises demonstrating how to solve linear inequalities, or students submitting solutions to homework about linear inequalities to the CCT. The clusters were further divided into smaller subsegments, which contain a single exercise, single part of an activity, or topic of discussion. If a single part of an activity resulted in multiple discussions, each discussion became a different subsegment. The subsegments became the major unit of analysis for this study.

**Representations.** The category of representations had three a priori subcategories and one grounded category. The types of representations present, dual process/object nature of functions, and treatment of representations are important for students to learn about/with representations based on frameworks from (Moschkovich et al., 1993; Pape & Tchoshanov, 2001). A priori codes were created based on the practices associated with these three subcategories including: who or what created the representations present in
the classroom seemed to influence the teachers’ instruction and the source of representations became a grounded subcategory of representations.

**Types of representations present.** Each subsegment was coded according to what type or types of representations were present. The three major representations of functions: graphical, algebraic, tabular were the basis of the coding. If more than one representation occurred within a subsegment, the subsegment was coded with the specific combination of the types present. For example, if a teacher asked students to sketch the graph of a specific equation, that subsegment would be coded as an Algebraic/Graphical mixture. If students were asked to create a table of points for the equation of a line that were then displayed in a graph, and then it was be coded as a combination of all three. By coding the specific combinations, each subsegment could be coded uniquely.

**Process/Object nature.** Moschkovich et al. (1993) argued that the dual process/object nature of functions is needed to truly understand functions. Each type of representation within each subsegment was coded as to whether the representations of functions were treated as a process, an object, or a combination of both.

**Treatment of representations.** Pape and Tchoshanov (2001) argued that how representations were treated in the classroom could influence students’ representational development. They posited that only using representations to create only end products limits student understanding of representations. They further argued that to develop representational thinking within students, representations need to be reasoned about/with, practiced upon, and have explanations or justifications about/with them. Each
subsegment was coded as to whether the representations present were treated as end products, if procedures were enacted upon them, as something to reason about/with, or explain/justify about/with. These codes were not mutually exclusive since, representations could be treated in multiple ways, such as enacting procedures to make an end product, or enacting procedures to create something to think about.

**Source of representations.** The importance of the source of the representations present within a subsegment became apparent when creating the case studies. Typically different interactions occurred when representations were produced by different sources. Each subsegment was coded as to whom or what generated the representations present. They were coded as teacher, teacher using CCT, single student, or multiple students.

**Discourse.** The average words per utterance of the teachers and students were calculated, and the ratio of student to teacher talk was also calculated. Each subsegment was coded as to who spoke to whom, teacher to student, student to teacher, or student to student. Using (A. G. Thompson et al., 1994) as a guide, types of questions asked were coded as procedural, conceptual, or in-between or mixed. The importance of the method teachers used to elicit discourse from student became apparent when the data were coded. Each subsegment was coded as to how the discourse was elicited; Initiate Respond Evaluate (IRE) sequence that focused on correct answers or methods, IRE sequences that focused on exposing student ideas, or sequence where a teacher would elicit, extend, explain or clarify students thinking or have students make predictions (EEECP).
Technology

Technology Use. Each subsegment was coded as to how the CCT was used. Four main treatments emerged from the data. Technology as information display, as information gatherer, as answer verifier, and discourse generator were the four treatments. These codes were not mutually exclusive as technology could be used in multiple ways in a single subsegment. Some subsegments could be coded as all four if a teacher collected and displayed student data to verify their answers that in turn spurred discourse about student ideas.

Treatment of Technology. Each subsegment was coded as to how the CCT was treated by the teacher; as a partner, master, or servant using Goos et al.’s (2003) framework. A teacher may treat technology as a master if their knowledge and usage are limited to a narrow range of operations and to the extent the teacher calls on a student “expert” to use the technology. Given a teacher’s limited knowledge of technology, a teacher may be reluctant to let students use technology to explore mathematics. A teacher treating technology as a servant could use the technology as a computational tool to speed up certain processes, but ultimately the tasks remain unchanged and the technology is not used in creative ways to change the nature of the activities. Technology is used to mainly support the teachers’ preferred teaching methods. A teacher treating technology as partner could provide access to new kinds of tasks or new ways of approaching tasks that may facilitate understanding, explore different perspectives, or mediate mathematical discussion.
**Decisions.** The orientation of each teacher was determined each year on a continuum from highly procedural to highly conceptual as inferred by her or his actions in the classroom, such as how she or he treated functions, representations, the questions she or he asked, methods used to elicit discussion, apparent goals, etc. Table 8.3 shows the indicators for each of the different orientations to teaching.

Flow charts modeling the actions and apparent decisions for each teacher for each year were created based on classroom observations using Schoenfeld’s (2010a) framework by inferring the decisions a teacher made in the classroom as evidenced by his or her responsive actions to instances that prompt the need for a decision. The decisions flow charts for each teacher were created by compositing and generalizing from the decisions and actions made in all the different situations observed in the videos. The flow charts provided not only a compact description of a teacher’s instruction, but also a prediction of how his or her instruction might unfold in other unobserved circumstances. Comparing first year and beyond first year decision flow chart for each teacher allowed for an examination of where growth occurred within the teacher’s instruction. Each action for the second or third year of each teacher was coded as to whether a shift each of the categories of *representations, discourse, or technology* occurred.
<table>
<thead>
<tr>
<th>Orientation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Highly Conceptual</td>
<td>• Tends to focus students away from thoughtless application of procedures and toward a rich conception of ideas.</td>
</tr>
<tr>
<td></td>
<td>• Tends to focus on aspects of the situation that give meaning to the representations present that in turn suggests the procedures that might be useful.</td>
</tr>
<tr>
<td></td>
<td>• Asks questions that move students to view representations in a nonprocedural context.</td>
</tr>
<tr>
<td></td>
<td>• Has an expectation that students are intellectually engaged in tasks and activities.</td>
</tr>
<tr>
<td></td>
<td>• When procedures are needed to be used, have students think about the products or the procedures themselves.</td>
</tr>
<tr>
<td>Conceptual</td>
<td>• Meets all or most of the indicators for a Highly Conceptual teacher for most of instruction.</td>
</tr>
<tr>
<td></td>
<td>• However, have occasional instances where procedure is supreme and little to no movement toward a rich conception of ideas.</td>
</tr>
<tr>
<td>Transitional</td>
<td>• Tries to focus students away from pure procedures toward a rich conception of ideas, but the conception of ideas is lacking.</td>
</tr>
<tr>
<td></td>
<td>• Gives some meaning to representations and procedures performed on them, but the meaning lacks depth.</td>
</tr>
<tr>
<td></td>
<td>• Alternates between conceptual and procedural orientations, independent of context.</td>
</tr>
<tr>
<td></td>
<td>• Reverts to procedural explanations without a hint of conceptual ideas when procedures are convenient or when conceptual explanations are difficult.</td>
</tr>
<tr>
<td></td>
<td>• Frequently asks questions that can be easily answered with procedures or results of procedures, but also frequently asks questions that reveal the meaning of the mathematics.</td>
</tr>
<tr>
<td>Procedural</td>
<td>• Meets all or most of the indicators for a Highly Procedural teacher for most of instruction.</td>
</tr>
<tr>
<td></td>
<td>• However, has not completely abandoned meaning in mathematics as they tend to have occasional instances where representations are more than something to do or make, but something to think about and something that has meaning with connections to other representations.</td>
</tr>
<tr>
<td>Highly Procedural</td>
<td>• Tendency to speak in the language of symbols and symbolic manipulations.</td>
</tr>
<tr>
<td></td>
<td>• Casts problem solving as performing the right procedures for solutions.</td>
</tr>
<tr>
<td></td>
<td>• Emphasizes indentifying and performing procedures.</td>
</tr>
<tr>
<td></td>
<td>• Tends to perform procedures whenever possible.</td>
</tr>
<tr>
<td></td>
<td>• Tends to disregard the context of the problem in which procedures occur and how they might arise naturally from understanding the situation.</td>
</tr>
<tr>
<td></td>
<td>• Tends to remediate student difficulties with procedures independently of the context in which the difficulties arose.</td>
</tr>
<tr>
<td></td>
<td>• Treat problem solving as flat. Nothing is more important than anything else, except for the answer which is of supreme importance.</td>
</tr>
<tr>
<td></td>
<td>• Has a narrow view of mathematical patterns as limited to single problems, rather than patterns across many types of problems.</td>
</tr>
<tr>
<td></td>
<td>• Asks questions that can be easily answered with procedures or their results.</td>
</tr>
</tbody>
</table>

Table 8.3: Model for teacher’s orientation toward mathematics: a modified version of Thompson et al. (1994) calculational/conceptual orientation toward mathematics.
Close inspection of teachers’ actions and decisions as depicted in the flow charts allowed for isolating events that distinguished instructional opportunities present in the sessions that teachers had missed. These missed opportunities appeared to be the result of pursuing students ideas or taking advantage of a surprise unanticipated display. Lastly, these missed opportunities directly influenced the teachers’ movement within the flow charts.

**Cross Analysis**

Following the completion of the individual case studies, a cross case analysis comparing and contrasting patterns, and identifying emerging themes across the cases was conducted. For the a priori codes, the literature (Goos et al., 2003; Moschkovich et al., 1993; Pape & Tchoşhanov, 2001; A. G. Thompson et al., 1994) was used to create a model to rank a teacher’s instruction as low, medium, high along each of targeted indicators (subcategories of the main categories) determined by percentages of total subsegments that codes within the subcategory occurred. The ratings for the subcategories were then averaged to determine overall rating for the main category. The ratings were used to track growth over time for each teacher. The decision flow charts were modified to identify not only changes in the decisions/actions of the teachers but also shifts to their instruction with regards to *representations, discourse, and technology*. Patterns among teacher ratings were used to create a model for teacher growth using CCT over time.
Findings

Findings with regards to some of the subcategories of the four main themes of representations, discourse, technology, and decisions are discussed in this section. The discussion starts with the decisions category because of the observed importance of teachers’ decisions in the classroom and the influence of teacher’s orientation on other aspects of classroom instruction.

Decisions

**Decisions/Actions within the classroom.** Teachers’ actions are depicted in each of their decision flow charts. The actions within a teacher’s second or third year flow chart were coded according to whether over time they: remained unchanged (UN), were altered (AL), or were completely new (CN). Instructional shifts within these actions were investigated and the findings of these shifts are discussed in this section.

Continued use of CCT seemed to produce substantial changes to the teachers’ instruction as depicted in their decision flow charts. Teachers’ instructional actions changed to allow for making student ideas more visible by using CCT. Even though the more recent decision flow charts were different, core decision/action pair(s), called core cycles, from their first year appeared in the new flow charts. In fact, the teachers seemed to have embedded core cycles from their first year into their new decision flow charts, sometimes in multiple places. Hence, teachers’ main method of instruction seemed slow to change. However, within a year, other changes to a teacher’s decisions/actions could occur. Even though slow to change, slight alterations in core cycles could produce growth.
in the teacher’s instruction. These alterations typically included a shift from representations as something to do/make to something to think about. Consequently, changes in discourse from a focus on answers to a discussion of ideas occurred. The relationships between growth and decision flow charts are discussed more fully below.

Teacher actions that were slightly modified (AL) from one year to the next could and in most instances produced growth in instruction with regards to at least one of the three targeted categories. Clearly, actions that remained unchanged (UN) produced no growth. In the three cases observed, any AL action typically produced growth in at most two of the categories. For example, the action depicted in Figure 8.1 where Ms. B started “Using IRE sequences to elicit correct answers from students” only had as shift in one category (technology). However, growth in all three categories occurred within two of the teachers’ new actions. Mr. L had one new action with growth in all three, and Ms. B had growth in all three categories in all of her new actions as depicted by the purple borders in Figure 8.1. Ms. A did not have growth in all three categories for any action as she was ranked highly in each of the main categories during her first year. Most growth for all the teachers occurred within their new decision/action pairs. The more a new decision/action pair focused on electing ideas/representations from multiple students and contrasting similarities and differences among those ideas/representations then growth occurred in more categories within that new action. Most of the new decision/action pairs with growth in all three categories focused on using the CCT to elicit and display student ideas/representation and discuss connections among them. In fact, the changes to a
teacher’s decision flow chart seemed to occur to accommodate for the differences in student ideas/representations that the CCT made visible in which the teacher could not ignore.

Figure 8.1: Decision flow chart depicting Ms. B’s instruction during her third year.

Breakdowns within a decision flow chart (indicated by dashed arrows as in Figure 8.1.) depicted opportunities that the teachers missed during interactions. These included making connections among ideas presented, or to further explore the concept of functions with students. These breakdowns typically indicated that the teacher had opted not to
continue a discussion, and instead moved on pursuing a new issue. For example, a teacher that decides to discuss important aspects of functions when they are present in the classroom may move on to a new issue even when important aspects remain. Other missed opportunities included instances of resetting/refocusing discussion could cause a missed opportunity if the discussion was refocused too quickly before students had time to investigate connections for themselves, and correcting student submitted data with procedures rather than rather than unpacking similarities in errors, or eliciting explanations for why errors had occurred.

Orientation. A teacher’s initial orientation toward mathematics teaching seemed to dictate the teacher’s ratings in growth in how representations where treated and used, how discourse evolved in the classroom, and how technology was treated and used in their first year. Teacher with visible attributes of a highly procedural orientation was also rated to low to medium-low within the subcategories of the other three main categories of representations, discourse, and technology. A teacher starting with a conceptual orientation toward mathematics seemed to initially have mostly high or medium ratings in the other subcategories. Over time, a teacher’s orientation rating seemed to attract the other ratings to it. For example, if a teacher remained conceptual over time categories that were high remained at that level, and categories that were not rated high were rated higher. The more the teacher’s apparent orientation toward mathematics changed from procedural to conceptual the more overall growth occurred. Indeed a smaller shift toward a conceptual orientation toward mathematics appeared to produce smaller growth. For
example, Mr. L’s orientation influenced some growth in the other main categories as depicted in Figure 8.2.

![Figure 8.2: Mr. L’s change in the mean ratings for each main category.](chart)

**Representations**

**Usage of representations.** The CCT appeared to encourage and enable teachers to elicit representations from multiple students as over time all teachers began to use this ability of the CCT. Teachers who did not begin to use the CCT to elicit representations began using the CCT in that manner over time. These teachers continued to do so even more frequently than before. All teachers increased the number of students who generated representations in the classroom. Most teachers had either a large shift to using multiple students as sources for representations or continued to use multiple students as a
source for generating representations during a large majority of instruction as depicted in Figure 8.3.

![Figure 8.3: Source of representations during first year (left) and beyond first year (right).](image)

Collecting representations from learners using the CCT typically involved students submitting equations or points that were instantaneously displayed to the entire class. The focus on eliciting representations from multiple students increased the presence of different types of representations as well. A teacher who typically used only one representation at a time would have to use at least two representations simultaneously to use the CCT in this manner. The CCT’s capability to collect and display student representations seemed to encourage teachers to ask a larger number of students to generate and share representations which could in turn increase the number of types of
representations present. As shown in Figure 8.4., most teachers shifted toward simultaneous diverse representations in the classroom.

![Figure 8.4: Representations present in teachers’ classroom during first year (left) and beyond first year (right).](image)

**Treatment of representations.** Treating representations as something to do or make was the main method that teachers who exhibited procedural orientation toward mathematics used to engage students with representations. While teachers who exhibited a conceptual orientation toward mathematics treated representations as something to do, they tended to also ask students to engage with representations as something to think about and explain with. This practice became more commonplace in instruction as, the more procedural teachers moved toward a conceptual orientation to teaching over time. The treatment of representations as something to think about occurred more frequently
when teachers elicited student generated representations from the class and asked them to consider similarities/differences among the responses. Most teachers shifted away from treating representations as end products and all teachers shifted to treating representations as something to reason about/with more frequently as shown in Figure 8.5. However, most teachers rarely, if at all, treated representations as something to make explanations/justifications about/with.

![Figure 8.5: Teachers treatment of representations first year (left) and beyond first year (right).](image)

The teachers, even those with a conceptual orientation, never treated the tabular representation of a function as an object, but always as a process. This might be because the tabular representation lends itself more naturally toward the process natures since it shows individual relationships between the variables. The treatment of the other
representations of functions seemed to change slowly for the teachers. The teacher who exhibited the most procedural orientation over time did not change from his process treatment of the representations of functions. The more conceptual orientation a teacher exhibited the more they treated the non-tabular representations of functions as objects, or as a mixture of both a process and an object.

**Technology**

The CCT appeared to have a learning curve as none of the teachers were highly rated in the subcategories of the treatment and use of technology in their first year of use. However, the teacher who exhibited the most conceptual orientation toward mathematics was rated higher than the procedurally orientated teachers. The teachers who did not appear to be intimidated by the technology seemed to use it as a simple add-on tool or to make their usual instruction more efficient. In other words, they were treating it as a servant (Goos et al., 2003). The teacher who seemed intimidated by the technology treated it mostly as a master (Goos et al., 2003). The teachers prominently used the CCT to display and/or collect information in their first year. Even though they collected information from students using the CCT, they rarely focused on student ideas that were made visible. Instead, teachers in their first year, tended to focus on correct answers or representations they generated themselves. All the teachers shifted, in different degrees, their focus toward using the student ideas to encourage classroom discourse as depicted in Figure 8.6. As stated earlier this shift in focus, altered how teachers treated and used representations, complicated the decisions the teachers made within the classroom.
Discourse

The patterns within teachers’ classroom discourse changed more slowly when compared to shifts in the other categories of representations, technology, and decisions as shown in Figure 8.7. Each of the major category ratings were averaged across all teachers for each year, the changes in ratings are depicted in Figure 8.7. The teacher who exhibited a conceptual orientation to teaching did not shift in her discourse, as her methods and questions continued to focused on student thinking and asking students to engage in explanations. While the teachers who started with a highly procedural orientation continued to use procedural questions with a focus on correct answers, they began to ask more conceptual questions to elicit student thinking to varying degrees. But the teachers’ new focus on eliciting student ideas centered around what they were
thinking rather than asking them to explain their ideas. Only rarely did these teachers ask students for explanations, and when they did, the explanations validated by the teachers were simple. The shift in the types of questions asked and the focus on student thinking typically occurred when teachers asked students to compare/contrast different representations of functions or asked them to make predictions about the functions.

Figure 8.7: Over time change in averages in major categories across all teachers.

**Conclusion: Model of Teacher Growth**

A teacher’s orientation to teaching mathematics determined their treatment and use of *discourse, representations, and technology* in the classroom. Specifically, a teacher with a procedural orientation is predicted to have low ratings in representations, discourse, and technology; a teacher in transition is expected to have medium ratings; and a conceptual teacher is expected to have high ratings in all targeted categories. In this work, changes in a teacher’s orientation to teaching influenced changes in her classroom instruction. The
model of teacher growth presented in Figure 8.8 was based on the participants’ case studies, the cross analysis of these cases, and conjectures using the definitions of the targeted indicators of the main categories. The teachers’ initial orientation determines their initial instruction with regards to representations, discourse, and technology as shown in Figure 8.8. As teachers use the CCT, they naturally become more comfortable and efficient in using it. Several of the capabilities of CCT allow teachers to view and display input from most or all of the students. Almost certainly student input, including student errors, alternate solutions, or alternate ways of representing the same solution, will be displayed to the class. Teachers may not initially focus on these differences, but over time they are inclined to do so. This, in turn, encourages the teacher to adjust their decision making process to accommodate for exploring student ideas as shown in Figure 8.8. If the teacher already focused on eliciting student thinking before using the technology, they will eventually find ways to use the CCT to better explore student ideas. If the teacher initially focused on eliciting correct answers from students, they will have to change their focus from answers and methods to what students thought to begin focusing on student thinking. In other words, they will have to change their orientation to teaching. The level of change in a teacher’s orientation was closely linked to the amount and quality of the accommodations they made to focus on student ideas.
The process of teacher growth depicted in Figure 8.8 is expected to continue as a finite cycle where teachers start over with their current orientation to teaching. Changes in teachers’ practice using technology could encourage them to find more and better ways to explore student thinking, which in turn could cause a greater shift in their orientation. Several indicators for high ratings of instruction include student explanations or justifications. In other words, to grow beyond the medium levels, the teacher needs to focus on *why* students are thinking they way they do and to press for explanations.
Teacher’s already at high levels of instruction could continue to grow by using solutions to problems to generalize abstract connections.

The inability to ignore the differences in student ideas that were made present by the CCT altered the teachers’ decision making mechanisms within the classroom to account for exploring the differences. The teacher's implementation of the new/ altered decision making mechanisms seemed to be influenced by his or her previous willingness to endorse some of the control of the classroom to the students. The level of implementation of decisions to elicit student thinking determined the change in the teacher’s orientation. Further growth is conjectured to occur when teachers begin to focus not only on student thinking, but encouraging students to explain and justify their thinking.

**Expected Trajectory of Growth for a Teacher with a Procedural Orientation**

Figure 8.9 shows a trajectory of development of a teacher who initially has either a highly procedural or procedural orientation to teaching mathematics. The red boxes represent predicted ratings within each category for most teachers with this orientation. The lines indicate less probable instructional ratings that could occur. These teachers are predicted, initially, to have low instructional ratings in the three categories of discourse, representations, and technology as evidenced in the case studies and as predicted by the indicators of the ratings. The model predicts that continued use of CCT would encourage a change in the teacher’s orientation. The anticipated trajectory for the teacher’s orientation is predicted to range between procedural and in transition as depicted by the box in Figure 8.9. The lines in the orientation category in this figure indicate that the
highly procedural teachers could remain so, but it is predicted that there would be some shift in orientation as a highly procedural and highly controlling teacher had instructional growth. The technology category (usage and treatment) was the quickest to grow across all teachers in the current study and it’s the rating for a procedural teacher is predicted to increase the most among the three main categories. CCT use encourages the presence of multiple representations and this feature causes a small increase to the representations rating. The representations category grew the second most and is predicted to grow about as fast as the technology category. In this study, teachers’ practice with regards to classroom discourse was the slowest to change for them whose initial orientations were procedural. Hence, the anticipated trajectory predicts the discourse rating to have the least growth and widest variability. A predictor of the level of teacher growth is how they embrace eliciting student thinking.

Figure 8.9: Expected trajectory of a teacher’s instruction that began with a procedural orientation to teaching.
Expected Trajectory of Growth for a Teacher with a Conceptual Orientation

Figure 8.10 shows the instructional ratings for a teacher who has a highly conceptual or conceptual orientation toward teaching. These teachers are predicted to have high initial ratings among the main categories. Technology is an exception to this prediction as the conceptual teacher was not initially rated high and needed time to acclimate to its use. Since discourse was the slowest growing instructional category and higher discourse ratings require eliciting student explanations, it is predicted to have slightly lower ratings than representations. All of the teachers’ orientations
remained the same or shifted toward being more conceptual, so teachers are predicted to continue to reside within the conceptual to high conceptual range over time. The technology rating is predicted to quickly receive a high rating as this category changed the most across all teachers. The representations category is predicted to be initially high and to continue to remain at that level. The discourse category is predicted to become high if it was not initially high.

**Expected Trajectory of Growth for a Teacher with a Transitional Orientation**

Figure 8.11 shows the anticipated trajectory of a teacher who starts with an orientation that is in a state of transition between conceptual and procedural. None of the participants in this study started with such an orientation, so the trajectory is based on observations of the teacher whose orientations became transitional, speculating that the teachers will grow over time depending on their accommodations of exploring student thinking.

![Figure 8.11: Expected trajectory of a teacher’s instruction that began with a transitional orientation to teaching.](image)
As with the other orientations, the initial instructional ratings are predicted to be close to the orientation rating with discourse being slightly lower and technology being lower still for the same reasons for the lower ratings with the conceptual teacher. After the continued use of CCT, the teacher’s orientation is predicted to increase from transitional to ranging between transitional and conceptual. The representations and technology categories are predicted to increase the most with a larger shift towards a more conceptual orientation and a small increase is predicted for the discourse category. Since the teacher participants in this study rarely elicited explanations from students, growth is predicted to level off until teachers begin to navigate though explanations from students.

**Individual Models Depending on Initial Orientation**

Figures 8.12 to 8.14 provide more detail of the teacher growth model corresponding to the particular orientations they initially exhibited. Figure 8.12 depicts a procedurally oriented teacher’s growth while using CCT. Note that the amount of growth of a teacher is dependent on which of the two paths he or she may take; embracing exploring student thinking, or minimally addressing student thinking in his or her instruction. The former path is projected to lead to more growth than the latter.

Figure 8.13 shows the conjecture that a teacher with a transitional orientation to teaching will continue to have medium ratings if he remains focused on student ideas and not exploring student explanations or justifications of these ideas. However, if the teacher moves toward eliciting deep student explanations, he is conjectured to grow.
A teacher with a conceptual orientation to teaching is conjectured to have high initial ratings in all categories except the technology category since learning how to use the tool may require time as shown in Figure 8.14. This category of teacher is not projected change substantially except the technology category. The high rating levels for each category were based on literature and observations, however even higher ratings could be created, such as abstracting connections among representations of functions, using particular solutions to problems to generalize mathematical concepts, or to treat technology as an extension of self (Goos et al., 2003). A conceptual teacher who already has high ratings is not predicted to grow unless high level generalization or abstraction occurs, essentially if the teacher provides an environment to foster students becoming mathematicians.
Figure 8.12: Model of predicted growth for teachers with an initial procedural orientation to teaching.
Figure 8.13: Model of predicted growth for teachers with an initial transitional orientation to teaching.
Figure 8.14: Model of predicted growth for teachers with an initial conceptual orientation to teaching.
Discussion and Implications for Teaching and Research

Implications for Teaching

The orientation of the teacher toward mathematics teaching determined, for the most part, her instructional practices with regards to representations, discourse, technology and the in the moment decisions in the classroom. A change in a teacher’s orientation, moving from procedural to being in a state of transition caused also a shift in their focus from seeking answers/methods to eliciting and explaining student thinking. Sowder (2007) argued that professional development efforts should focus on increasing the teachers’ attention to student ideas and their knowledge of student thinking as a means to facilitate professional growth. Sowder (2007, p. 173) claimed that the build up of three types of knowledge should guide professional development programs, two of which include: Knowledge-for-practice, knowledge acquired from formal instruction; knowledge-in-practice, knowledge acquired from teachers consciously reflecting on their own classroom practice that encourages transformation of perspectives. Prominent in Sowder’s (2007) advocacy is a need to focus teachers on making sense of student mathematical thinking. This study confirms her assessment and further concludes that CCT use can provide an avenue for professional development to aid teachers to learn how to engage all or most of the students in learning.

The teachers not being able to ignore differences in student thinking present by the use of CCT seemed to spur knowledge-in-practice to encourage a transformation of orientation. In other words, continued use of CCT seemed to naturally encourage
opportunities for knowledge-in-practice. The CCT’s capability to collect and display responses from all or most of the students in the classroom increased the potential variability among student ideas to be revealed. Also if the teacher focuses on student exploration of ideas, she can potentially enhance the discussion of those ideas. A common instructional method that teachers in this study used to focus on student ideas was asking students to compare and contrast the different ideas that they had generated themselves. However, the teachers were not always persistent in pursuing discussions in depth. Encouraging a similar stance among teachers who primarily stress right answers or procedures might provide a vehicle for their growth and development, which is consistent with Sowder’s (2007) argument that teachers need to be provided the opportunity to explore different student approaches to the same problem.

The typology and elaboration of the construct of missed opportunities described in this study provides an illustration of how the use of technology can create space for exploring student thinking whilst helping students to make mathematical connections among their ideas. The exploration of the missed opportunities similar to these presented in this study could be used as a learning tool in professional development to model how teachers could make these opportunities richer by exploring differences in student thinking, pursuing explanations and hopefully generalizations whilst engaging teachers thusly in mathematical sense making.

Kazemi and Franke (2004) observed that teachers found it difficult to elicit explanations from students. However after working with a facilitator and other teachers
for a year during which they were exposed to effective models of questioning, the teachers shifted in their approach and tended to engage with student ideas. The participants in this study who after shifting away from their highly procedural orientation to mathematics teaching either only asked students “what” they were thinking rather than “why” they were thinking that way, or if they did ask “why” they accepted simple responses as adequate. However, a continued focus only on what students are thinking versus why they are thinking limited the teacher’s ascent to high levels of representations, discourse and technology. As conjectured by the model, to reach the high levels of performance in these categories, the teacher must press for deeper explanations from the students.

Mishra and Koehler (2006) developed a framework to describe the types of knowledges teachers need in order to be able to teach rich mathematics with technology (technology, content, and pedagogy) and the connections, interactions, affordances, and constraints among these knowledges. They claimed that in order to foster a rich learning environment with the use of technology, researchers need to examine the combinations of the knowledge domains of technology, content, and pedagogy. Mishra and Koehler claimed that technological pedagogical content knowledge (TPCK) is an emergent form of knowledge that is more than the sum of its three components and defined TPCK to be:

the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students’ prior knowledge and theories of epistemology; and knowledge of how technologies can be used to
build on existing knowledge and to develop new epistemologies or strengthen old ones. (2006, p. 1029)

Bowers and Stephen (2011) interpreted the notion of teachers using technology in “appropriate and responsible ways” to mean using technology to explore mathematical relationships. Bowers’ and Stephen’s (2011) vision of Mishra’s and Koehler’s (2006) TPACK (TPCK became later known as TPACK) framework to be “to help teachers develop a technological habit of mind oriented toward using advanced computation and communication tools to help students explore and understand the underlying concepts and their relation to the larger world outside of school” (2011, p. 286). Bowers and Stephens (2011) concluded that TPACK is not comprised of a list of specific knowledge pieces but rather a stance that views “technology as a critical tool for a identifying mathematical relationships” (2011, p. 290) and the goal of the development of pre- and in-service teachers is to develop such as stance. This perception of teaching approaches a natural subset of a conceptual orientation to teacher mathematics in general. An implication for using CCT use in the classroom is that teachers receive guidance on how to use this tool to elicit student ideas and to encourage students to explain and justify these same ideas so to enhance learners’ mathematical thinking.

This study provides evidence that Thompson et al.’s (1994) construct of conceptual orientation to teaching and a TPACK stance toward effective practice are interconnected. A teacher who treats technology as a partner (Goos et al., 2003) exhibits tenets of a TPACK as well. The fact that the conceptually oriented teacher increased her instruction
with regards to technology to high levels can be interpreted as that such an orientation encouraged the development of a TPACK stance over time. Growth (Figure 8.8) of a procedural teacher is conjectured to occur if they to interact with different student ideas made visible by the CCT, which encourages a shift their orientation to teaching in general. Thus, an implication for professional development is to model the use of CCT for exploring differences in student ideas and to make the strategy specific to the teachers to foster a TPACK orientation in the teachers.

Bowers and Stephens (2011) and Goos and Benson (2002) found that teachers who were simply given access to, and training with technology did not allow them to develop the type of stance TPACK envisioned as effective. Contrary to their finding, this study provided evidence that the CCT exposed teachers over an extended period of time to many different student ideas which in turn led to the emergence of a more conceptual orientation to teaching. The results do however, confirm the conclusions of Bowers and Stephens (2011) and Zbiek and Hollebrands (2008) that for teachers to use technology effectively they need to become aware of how to create activities so that using technology to elicit student thinking, and encourage student conjectures and explanations become the classroom norm.

The goal of the Learner Centered Professional Development (LCPD) project that Polly (2011) implemented was to “improve student learning by supporting teachers’ enactment of specific standards-based pedagogies including rich mathematical tasks,
using technology as a tool to support learning and posing high-level questions” (2011, p. 84). This goal was to be achieved by:

- Giving rich mathematical tasks to the teachers to work on with colleagues.
- Modeling: how to pose the task, how to use the technologies, and how to support student work through questioning rather than guiding the completion.
- Facilitating discussions of the teacher-participants approaches, the mathematical concepts embedded within the tasks, and how the task could be incorporated into their classroom.
- Encouraging teachers to complete rich tasks with technology and discuss student learning while completing these tasks.
- Scaffolding teachers’ instruction by co-planning lessons with each teacher.
- Encouraging teachers to incorporate technology and mathematically rich tasks into their classrooms (Polly, 2011, pp. 85–86).

Polly (2011) determined that in the case of his participants the relationship between the planning support and the teachers’ implementations of the technology and mathematically rich tasks was not clearly established, but the co-planned lessons resulted in developing a TPACK orientation to those lessons. He proposed that a professional development program may need to: allow teachers to co-plan with more knowledgeable individuals such as coaches, be open-ended to account for differences in teacher knowledge, and provide support outside workshops. The practice of exploring student ideas within teachers’ regular instruction could help foster a more mature approach to the use of technology as an instructional tool, compatible with TPACK’s proposed model. Considering the pivotal role that teachers’ particular orientation to teaching may play on how lessons are implemented, I argue that encouraging the use of CCT in lessons may not be sufficient in motivating change. But systematic efforts must be made to also
confront traditional practices pointing out the types of learning students can exhibit in class with close attention paid to their ideas and explanations of those ideas.

Using CCT, the professional developers could collect teacher-participant responses to rich mathematical tasks to model how to use the technology, model how to facilitate generative discourse surrounding the responses. Teacher educators can also help teachers unpack the mathematical concepts within the tasks as well as providing the teachers with direction on how these tasks might be incorporated into their classrooms. In such a setting, the use of CCT seems to be an effective tool for making teachers’ ideas visible and providing a model of how to incorporate their ideas into their classroom instruction.

**Summary of Implications for Professional Development and Teaching**

- CCT use can provide an avenue aid teachers to learn how to engage all or most of the students in learning.
- The exploration of the missed opportunities could be used as a learning tool in professional development.
- Model predicts that to reach the high levels of performance in the main categories, the teacher must press for deeper explanations from the students.
- Professional development could model the use of CCT for exploring differences in student ideas and to make the strategy specific to the teachers to foster a TPACK orientation in the teachers.
- Using CCT, the professional developers could collect teacher-participant responses to rich mathematical tasks to model how to use the technology, model how to facilitate generative discourse surrounding the responses.

**Implications for Further Research**

Thompson et al. (1994) described a teacher’s orientation to teaching mathematics as an either/or position: either the teacher has a procedural orientation, or has a conceptual
This study provided evidence that teachers could actually lie on a continuum between procedural and conceptual. The study also provided a model for determining a teacher’s position along this continuum. In the current work, the focus of analysis concerned teachers’ practice in the context of one specific topic. The indicators developed and used for cataloging teachers’ orientations were closely linked to how this topic was treated by teachers. Future inquiry is needed to determine whether similar phenomena may occur when considering other mathematical concepts.

Zbiek, Heid, Blume, and Dick (2007) conducted a synthesis of research on technology and mathematics teaching. They propose that the notion of pedagogical fidelity of a technology for teachers (defined below) to be fruitful for future research with regards to technology. Zbiek et al. (2007) expand on Dick’s (2007) notion of pedagogical fidelity to be the extent that a teacher believes a tool provides students the ability to act mathematically in ways that are consistent with the teacher’s notion mathematical learning in their practice. “A teacher’s readiness to use a particular form of technology and the nature of how a teacher’s use of the technology unfolds center around how the teacher’s practice and the nature of that technology align” (Zbiek et al., 2007). In other words, the incorporation of technology use into a teacher’s practice was influenced by the technology’s pedagogical fidelity. Zbiek et al. (2007) argue that successful incorporation of technology into the teacher’s classroom practice can be facilitated or impeded by degree of pedagogical fidelity of that technology. In other words, the teacher’s use of technology is influenced by how well this teacher believes the technology can assist in
their teaching practice. This study problematizes this claim in the sense that this relationship did not appear to be static or trivial as continued use of the technology influenced teacher goals and orientation. A need exists for a more careful inspection of how beliefs, knowledge and technology use might be linked.

Pape et al. (2011) expresses a need for future research to examine teachers trajectory though Niess et al.’s (2009) stages of incorporating technology into the classroom, which are

1. **Recognizing** (knowledge), where teachers are able to use the technology and recognize the alignment of the technology with mathematics content yet do not integrate the technology in teaching and learning of mathematics.
2. **Accepting** (persuasion), where teachers form a favorable or unfavorable attitude toward teaching and learning mathematics with an appropriate technology.
3. **Adapting** (decision), where teachers engage in activities that lead to a choice to adopt or reject teaching and learning mathematics with an appropriate technology.
4. **Exploring** (implementation), where teachers actively integrate teaching and learning of mathematics with an appropriate technology.
5. **Advancing** (confirmation), where teachers evaluate the results of the decision to integrate teaching and learning mathematics with an appropriate technology (Niess et al., 2009).

While evidence for a complete explanation of how and why a teacher progresses through the stages, this study offers insight into what may occur in a teacher’s trajectory through the stages with regards to CCT and eliciting student ideas. In Figure 8.13 and 8.14, the “increase comfort level and exploring student ideas” can be expounded upon by applying Niess et al.’s (2009) model to teacher’s incorporation of CCT use into their instruction. As a result of the application, the levels for using CCT to explore student thinking/representations become:
1. **Recognizing** that CCT has the capability to allow for exploration of student thinking/representations but do not incorporate this capability into their instruction.
2. **Accepting** the CCT’s capability to expose student thinking/representations with either a favorable or unfavorable view.
3. **Adapting** activities that mimic activities in the teachers already established repertoire to use the CCT use to make student thinking visible.
4. **Exploring** the curriculum for areas to incorporate the capability of CCT to make student thinking visible.
5. **Advancing:** Evaluating the incorporation of CCT into classroom instruction. Teacher at this level incorporate technology into more of their instruction, and even challenge the curriculum looking for ways to change classroom tasks as a result of how CCT use can be used to explore student thinking.

The teachers who exhibited a procedural orientation initially seemed to be at the recognizing level or below during their first year since they did not address the differences in student ideas made visible by CCT. The teacher with the conceptual orientation to mathematics teaching seemed to be at least at the accepting level, since she at times explored differences in student representations. The teachers seemed to move beyond accepting level to at least the adapting level overtime, since they used activities to explore student thinking/representations made visible by the CCT. Since the activities of matching pictures were similar to ones model during CCMS’ professional development, the ability to distinguish as to whether the teachers were adapting by mimicking these activities or incorporating these activities through exploration lacks sufficient evidence.

The case study for Mr. L revealed less growth in his instruction when considered as a standalone case. Mr. L’s case study used conventional qualitative methods of coding teacher/student utterances. However, in the cross case analysis, using the careful
segmentation of classroom instruction into clusters of similar activities, exercises, or discussions as a basis for coding analysis revealed more growth on the part of each teacher than previously thought. The multidimensional analysis of the different categories allowed for tracing growth along different areas. This study provides evidence that teachers did not grow at the same rate for each of the four categories. Future research may take into consideration as to whether conventional methods of coding utterances and focusing on one aspect of promoting growth are appropriate and if a more holistic multidimensional approach is warranted.

Schoenfeld (2011) called for research to explore how teachers develop in-the-moment decision making mechanisms. This study provides some insight on the issue in that teachers who were unable to ignore differences in student ideas altered their instructional decisions to account for those differences. However, a complete explanation of how/why these decisions were altered is lacking. Echoing Shoenfelds’ call, I propose the need for careful study of the relationship between technology usage and how and why decisions are made and might be changed in the classroom.

**Novel Contributions and Scholarly Insights**

Thompson et al. (1994) has described how a teacher’s conceptual or calculational orientation to mathematics teaching influences other aspects of their instruction. No link between this orientational framework and technology use has previously been considered. This research suggests that relationship between teacher orientation and technology use may be reciprocal and in need of further investigation. Indeed, I posit that understanding
the relationship/interactions between these two elements might provide venues for development of fruitful opportunities for design of learning experiences for teachers.

Appropriate use of technology has been theorized based on "an all or none" foundation in the past; that is, documentation of how teachers use or fail to use technology to advance children's mathematical learning (National Center for Education Statistics, 2000). Such a perspective has been grounded in data drawn on snapshot instruction considering one or two instructional variables. In this work, four key areas of teacher practice became the basis for analyzing teacher growth. This study provides evidence that teacher growth is gradual and multifaceted; focusing on one or two elements of practice as indicators of "action" is insufficient in theorizing about their decision making (Lampert, 2001). Given this and the above point, professional development may consider giving teachers differentiated instruction with regards to their orientation to teaching. For instance, modeling how to engage in what students are thinking may be more beneficial to procedurally oriented teachers than modeling eliciting explanations.

The construct of missed opportunities presented in this study should be unpacked to explore the range and types of these opportunities to increase the repertoire of teacher educators in engaging teachers in critical reflection of their practice, mathematics, and student thinking.

In this work, I focused on four key variables for capturing teachers' practice. Expanding the number of variables impacting teachers' instructional decision
making when analyzing and interpreting practice allows for the creation of a more sensitive model for teacher development. This study hints that teacher knowledge, and tasks used in the classroom may be fruitful variables to include in a model of teacher growth.

**Limitations and Ideas for Future Research**

The data used for analysis in this study were collected for the CCMS project in years before this research was proposed. The existing data were collected to answer questions specific to CCMS objectives. Also, the researcher did not have access to the participants to collect other data that became pertinent to this investigation, such as interviews with the teachers, and classroom observations over multiple topics. The constructs of *discourse, technology* and *orientations* were generalized so that even in the presence of only snap-shots of a teacher’s instruction, conjectures could be made about their growth in other topics. However, the construct of representations had more specific detail to the topics covered in class. Future research is needed to address these gaps in order to further expand on the teacher growth model present in this study.

The findings of this study hinted that teacher mathematical knowledge might have an influence on the teacher’s ability to use the technology in ways to enhance student conceptual exploration of mathematical topics, but the lack of teacher knowledge data prevented the exploration of this relationship. Future research using this model of teacher growth could benefit from an instrument that could gauge teacher knowledge with regards to content, pedagogy, and technology (TPACK). An avenue to pursue future
research could include incorporating teacher knowledge into the model of teacher growth while using CCT. Despite limited data, this study provides a detailed description and analysis of how teachers’ decisions influenced classroom instruction over time.

The analysis of the decision flow charts accounted only for whether there were changes in the major categories of representations, discourse, and technology, rather than what specifically changed and by how much. The results indicated generally that the actions that seemed to spur teacher growth focused on either eliciting student thinking or student explanations. Given the found importance of teachers’ decisions/actions within the classroom to influence teacher growth, a future study could modify the models for rating a teacher’s instruction. The new model should account for growth in specific individual teacher actions for the three main categories of representations, discourse, and technology to better catalog the types of actions that lead to high ratings of these categories and how to encourage the implementation of these actions.

Determining whether having a conceptual orientation to teaching and high levels of instruction with regards to discourse, representations, and technology influences student understanding of representations is an area left unexplored in the current study. While the CCMS study has data on student general algebra achievement, it did not have data designed to determine student understanding of representations of functions. Future research might consider if teacher’s ratings in representations, discourse, technology, and orientation influence student mathematical achievement.
REFERENCES


