TQFTS FROM QUASI-HOPF ALGEBRAS AND GROUP COCYCLES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By
Jennifer George, MS
Graduate Program in Mathematics

The Ohio State University
2013

Dissertation Committee:
Thomas Kerler, Advisor
Henri Moscovici
Sergei Chmutov
ABSTRACT

In three dimensions, a topological quantum field theory, or TQFT, is a functor from the category of 3-dimensional framed cobordisms to the category of vector spaces. Two well-known TQFTs are the Hennings TQFT and the Dijkgraaf-Witten TQFT. The Hennings TQFT is built from a link invariant, by applying elements of a Hopf algebra in a systematic way to tangle diagrams. The Dijkgraaf-Witten TQFT is built by counting principal bundles on a 3-manifold which have been weighted by a 3-cocycle. We prove that the Hennings TQFT applied on the double of the group algebra is equivalent to the Dijkgraaf-Witten TQFT applied on a trivial cocycle. In order to extend this result to the more general case of a non-trivial cocycle, we discuss the notion of a quasi-Hopf algebra, which is an almost-cocommutative Hopf algebra. We then extend the definition of the Hennings TQFT so that instead of applying elements of a Hopf algebra to the tangle, we instead apply elements of a quasi-Hopf algebra. The specific quasi-Hopf algebra in which we are interested is the twisted double of the group algebra, where the twisting occurs via a 3-cocycle. Finally, we conjecture that the Hennings TQFT applied on the twisted double of the group algebra is equivalent to the Dijkgraaf-Witten TQFT applied on the same cocycle.
For Jay and Kathy, who inspire me to never stop learning.
ACKNOWLEDGMENTS

My most heartfelt thanks to the following people, who have made a significant impact on my life during this process.

To my advisor, Dr. Kerler, for giving me this project as well as the patient help and encouragement I’ve needed to see it through. For encouraging me to get back on an airplane and go to another conference, no matter how much trouble I had with the previous trip.

To my parents, brother, and sister-in-law, for all of your love, encouragement, and support. For giving me a place to escape to when I needed to get away for the weekend, and ears to hear all my stories. For reminding me that while I’m not physically present, I’m always present in your hearts.

To my George family, for your love, hugs, prayers, and great questions about what getting a PhD is like. Most of all, for your pride in me.

To Tim, for sharing the worst of my bad moods and the best of my successes. For loving me constantly and sincerely, and for always trying to understand what I’m going through. You’ve been a huge part of my solid ground for the past two and a half years, and I couldn’t be more thankful for you. I love you.

To Kayre, for your prayers and your food. For our shared love of learning, no matter the topic. It’s been such a blessing to know that every other Thursday night I can relax with a great meal and even better conversation. I’m honored to have
shared this journey with you, and I know your prayers have seen me through. You truly inspire me to always be my best.

To Isabel, for always answering my questions, no matter how many times I ask them. For eating pumpkin-flavored food experiments. For mid-day coffee breaks and half-muffins (with half-calories!). For being one of the most thoughtful and generous people I know, and allowing me to be a part of your thoughtfulness and generosity. Seeing your journey, always a couple steps ahead of mine, has always encouraged me. I’m so grateful for your friendship.

To Stewie, for many cups of tea and late-night conversations. Our paths in life will take us in very different directions, but I know that any time we get together, it’ll be just like we never left.

To Bruce, for the use of your couch during my candidacy exam, and appropriate Mario Kart breaks.

To Carol and Matt, for making me feel like a part of your family. For corny jokes and too much candy. For hugs, advice, support, and use of that awesome massage chair. Oh, and thanks for those pages of complex analysis you found the other day.

To Emily and Rebecca, for making home a place I love to be.

To Laura, for making every activity into an adventure.

To Cindy, Denise, and Roman, for quietly doing everything you do to keep the department running. For listening to my complaints, fixing my problems, and always being willing to take time out of your busy days for me.

To Marilyn, Dawn, and Ann, for being the rest of the glue that holds this department together. It’s my firm belief that we have the best support staff ever.

To my Miami Professors: Dr. Davis, Dr. Burke, Dr. Holmes, Dr. Schaefer, Dr. Ortiz, and others, for starting me down this road.

Finally, to Jesus Christ, who has made every step possible.
VITA

1984 ............................. Born in Ashland, OH

2006 ............................. BS *Summa Cum Laude* in Mathematics and Statistics, Miami University

2012 ............................. MS in Mathematics, The Ohio State University

2006-Present ...................... Graduate Teaching Associate, The Ohio State University

FIELDS OF STUDY

Major Field: Mathematics

Specialization: Low-Dimensional Topology
TABLE OF CONTENTS

Abstract ......................................................... ii
Dedication ....................................................... iii
Acknowledgments ............................................... iv
Vita ................................................................. vi
List of Figures .................................................. ix

CHAPTER PAGE

1 Introduction .................................................. 1
2 Hopf Algebras ............................................... 6
   2.1 Definitions ............................................... 6
   2.2 Integrals ................................................ 11
   2.3 Example: Double of the Group Algebra ................. 13
3 Quasi-Hopf Algebras ......................................... 17
   3.1 Definitions ............................................... 18
   3.2 Special Elements in Quasi-Hopf Algebras ............... 22
   3.3 Integrals and Cointegrals ................................ 26
   3.4 Example: The Dijkgraaf-Pasquier-Roche Algebra ....... 32
4 Tangle Categories ............................................ 47
   4.1 The Category $Tgl$ of Admissible Tangles ............... 47
   4.2 The Categories FLAT and $DTgl$ ......................... 51
   4.3 The Category $Tgl(1)$ .................................... 54
   4.4 A Coproduct Map on Strands in $Tgl(1)$ ................. 63
5 Cobordism Categories and Topological Quantum Field Theories .... 72
   5.1 TQFTs and the Category $Cob_3^*$ ........................ 72
   5.2 Example: The Hennings TQFT ............................ 79
5.3 Example: Dijkgraaf-Witten TQFT ........................................ 83

6 A Hennings TQFT Construction for Quasi-Hopf Algebras .......... 92
  6.1 Introduction ............................................................... 92
  6.2 Rules for Computation .................................................. 94
  6.3 Proof of Theorem 6.3.24 .................................................. 106
  6.4 Example: Computations on $D^\omega[G]$ ................................. 134

7 Equivalence of TQFTs .......................................................... 145
  7.1 Case: $\omega = 1$ or $D[G]$ ............................................... 145
  7.2 Further Work ............................................................... 161

Bibliography ........................................................................... 163
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>49</td>
</tr>
<tr>
<td>4.2</td>
<td>50</td>
</tr>
<tr>
<td>4.3</td>
<td>51</td>
</tr>
<tr>
<td>4.4</td>
<td>52</td>
</tr>
<tr>
<td>4.5</td>
<td>53</td>
</tr>
<tr>
<td>4.6</td>
<td>53</td>
</tr>
<tr>
<td>4.7</td>
<td>56</td>
</tr>
<tr>
<td>4.8</td>
<td>56</td>
</tr>
<tr>
<td>4.9</td>
<td>57</td>
</tr>
<tr>
<td>4.10</td>
<td>58</td>
</tr>
<tr>
<td>4.11</td>
<td>59</td>
</tr>
<tr>
<td>4.12</td>
<td>60</td>
</tr>
<tr>
<td>4.13</td>
<td>61</td>
</tr>
<tr>
<td>4.14</td>
<td>64</td>
</tr>
<tr>
<td>4.15</td>
<td>65</td>
</tr>
<tr>
<td>4.16</td>
<td>65</td>
</tr>
<tr>
<td>4.17</td>
<td>67</td>
</tr>
<tr>
<td>4.18</td>
<td>68</td>
</tr>
<tr>
<td>4.19</td>
<td>69</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------</td>
</tr>
<tr>
<td>6.18</td>
<td>Move IV on $Tgl^0$</td>
</tr>
<tr>
<td>6.19</td>
<td>Equivalence of Move IV Tangles</td>
</tr>
<tr>
<td>6.20</td>
<td>Beaded Tangles Corresponding to Move IV</td>
</tr>
<tr>
<td>6.21</td>
<td>An Isotopy on Tangles</td>
</tr>
<tr>
<td>6.22</td>
<td>Equivalent Tangles</td>
</tr>
<tr>
<td>6.23</td>
<td>Applying the Map $\tilde{\nu}_H$</td>
</tr>
<tr>
<td>6.24</td>
<td>Set Up for the Fenn-Rourke Move</td>
</tr>
<tr>
<td>6.25</td>
<td>The Quasi-Hennings Procedure Applied to the Tangle in Figure 6.24</td>
</tr>
<tr>
<td>6.26</td>
<td>The Fenn-Rourke Move</td>
</tr>
<tr>
<td>6.27</td>
<td>The Tangle of Figure 6.28 Gives an Integral</td>
</tr>
<tr>
<td>6.28</td>
<td>A Tangle for the Integral</td>
</tr>
<tr>
<td>6.29</td>
<td>A Tangle for the Modified 1st Kirby Move</td>
</tr>
<tr>
<td>6.30</td>
<td>The $\sigma$ Move</td>
</tr>
<tr>
<td>6.31</td>
<td>The Map $\tilde{\nu}_H$ Applied to the $\sigma$-Move</td>
</tr>
<tr>
<td>6.32</td>
<td>Framed 1st Reidemeister Move</td>
</tr>
<tr>
<td>6.33</td>
<td>Move II Beads</td>
</tr>
<tr>
<td>7.1</td>
<td>Genus One Punctured Torus, $\Sigma_1^*$</td>
</tr>
<tr>
<td>7.2</td>
<td>Generators of the Fundamental Group</td>
</tr>
<tr>
<td>7.3</td>
<td>Intersection of $A_i$ and $b_i$</td>
</tr>
<tr>
<td>7.4</td>
<td>Hennings Procedure Applied to $B_j$</td>
</tr>
<tr>
<td>7.5</td>
<td>Intersection of $B_j$ and $a_j$</td>
</tr>
<tr>
<td>7.6</td>
<td>Hennings Procedure applied to $C_k$</td>
</tr>
<tr>
<td>7.7</td>
<td>Intersections of $C_k$ with Generators</td>
</tr>
<tr>
<td>7.8</td>
<td>Deforming the Twist $C_k$</td>
</tr>
<tr>
<td>7.9</td>
<td>Tangles for the Special Cobordisms</td>
</tr>
<tr>
<td>7.10</td>
<td>Hennings Procedure applied to $H_n^+$</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

Functorial invariants of 3-manifolds, more commonly referred to as Topological Quantum Field Theories, or TQFTs, have opened up a new view in low-dimensional topology and spurred much research since their axiomatic formulation by Atiyah in [2]. Numerous non-trivial constructions began to emerge around 1990, including those of Dijkgraaf-Witten, Reshetikhin-Turaev, Turaev-Viro, Witten, and others. For some examples, see [28, 6, 10, 19].

Here we will define a TQFT to be a functor

\[ \nu : Cob_3 \to Vect, \]  

(1.0.1)

where \( Cob_3 \) denotes the category of 2-framed cobordisms between standard surfaces, up to homeomorphism. For technical reasons, we will instead consider a related category \( Cob_3^\bullet \) of relative cobordisms between standard surfaces with one boundary component. See [20, 4]. Viewing a TQFT as a functor between categories allows us to decompose spaces into smaller pieces, making invariants more computable. This view also allows us to use the structure of the mapping class group to further simplify our approach, as well as define representations of the mapping class group.

There are typically two types of rigorous constructions of TQFTs. The first are geometric, mostly using the ideas of counting flat connections in appropriate bundles. The Dijkgraaf-Witten TQFT [6] is an example of this first type. The second type
utilizes combinatorial methods based on diagrammatic presentations of cobordisms, where calculations are done in Hopf algebras. An example of the second type is given by the Hennings TQFT [19].

Both types of TQFTs are useful: the first can usually be related directly to topological and geometric properties of the underlying manifold, while the second are explicitly and algorithmically computable. Finding TQFTs of the second type that are equivalent to those of the first type is a powerful tool, as it combines geometric content with computability. This thesis is motivated by one such equivalence, using the examples cited.

The Hennings TQFT is the more combinatorial TQFT, and it is constructed from a link invariant given by Hennings in [14]. We have a functor

$$\nu_H : \text{Cob}_3^* \to \text{Vect},$$

(1.0.2)
given by first sending \(\text{Cob}_3^*\), the category of relative cobordisms between standard surfaces with one boundary component, to an isomorphic category of admissible, framed tangles. The algorithm then applies elements of a Hopf algebra to these tangles, and uses these elements and their positions on the tangle to give an element of \(\text{Vect}\).

The Dijkgraaf-Witten TQFT, our geometric TQFT, is defined in [6] using an invariant of 3-manifolds \(M\). Let \(G\) be a finite group and \(\omega\) a 3-cocycle

$$\omega : G \times G \times G \to U(1).$$

(1.0.3)

If \(M\) is a closed 3-manifold, we essentially count principal \(G\)-bundles over \(M\) up to equivalence. The 3-cocycle gives us a map from \([M, BG]\) to the complex numbers \(\mathbb{C}\) by

$$[f] \mapsto \langle f^*(\omega), [M] \rangle,$$

(1.0.4)
where \([M]\) denotes the fundamental class of \(M\). The invariant is defined by summing these images over all classes \([f]\). In the case of manifolds with boundary, we adjust this definition using a triangulation of \(M\) to compensate for the fact that the fundamental class of \(M\) is not well-defined. Dijkgraaf and Witten give an explicit definition of the TQFT. In [10], Freed and Quinn expand on the definition of this TQFT more from the standpoint of algebraic topology. These original definitions were made for standard surfaces, but we instead use standard surfaces with one boundary component. This allows us to fix a basepoint on each standard surface, a basepoint on the manifold, and a path in the boundary of the manifold connecting the basepoints of each standard surface. Fixing basepoints means we need not consider the adjoint action on the fundamental group, and thus our calculations are simplified.

Using the same data, namely a finite group \(G\) and a 3-cocycle \(\omega\), Dijkgraaf, Pasquier and Roche define in [5] a so-called quasi-Hopf algebra denoted \(D^{\omega}[G]\). Quasi-Hopf algebras are Hopf algebras in which the coassociativity condition has been weakened. It has been implied in the literature that these two constructions using the same data, namely the Dijkgraaf-Witten TQFT and the Dijkgraaf-Pasquier-Roche algebra, should be related. The most likely candidate for this relation is the Hennings TQFT, but the Hennings TQFT has thus far only been defined for ordinary Hopf algebras.

It has also been suggested in the literature that in the case of a finite group \(G\) and trivial cocycle (that is, \(\omega(g,h,k) = 1\) for all \((g,h,k) \in G^3\)) that we have an equivalence of TQFTs. Explicit constructions have thus far been missing, but will be provided in this thesis.

**Theorem 1.0.1. (Theorem 7.1.3).** Let \(G\) be a finite group. The Hennings TQFT applied with the Hopf algebra \(D[G]\) is equivalent to the Dijkgraaf-Witten TQFT applied with the same group \(G\) and trivial 3-cocycle.

We then conjecture that this result should hold in a more general setting.
Conjecture 1.0.2. (Conjecture 7.2.1) Let $G$ be a finite group and $\omega : G \times G \times G \to U(1)$ a 3-cocycle. The Hennings TQFT applied with the quasi-Hopf algebra $D^\omega[G]$ is equivalent to the Dijkgraaf-Witten TQFT applied with the same group $G$ and 3-cocycle $\omega$.

In order to formulate this conjecture, however, we first need to generalize the known Hennings algorithm for ordinary Hopf algebras to the quasi-Hopf case. We will do this in generality, and then discuss the specific case of the quasi-Hopf algebra $D^\omega[G]$.

Theorem 1.0.3. (Theorem 6.3.24) Given a unimodular quasi-Hopf algebra $H$ satisfying suitable non-degeneracy conditions, there exists a well-defined Hennings TQFT $\nu_H : \text{Cob}_3 \to \text{Vect}$, defined as a composition of a functor from the category of admissible tangles to a category of flat tangles decorated with elements of the quasi-Hopf algebra $H$, followed by a functor from the category of such decorated, flat tangles to $\text{Vect}$.

The plan for extending the definition of the Hennings TQFT is to loosely parallel the work of Kauffman and Radford in [17], and the extension to a TQFT in [19]. In particular, we must define an appropriate category of unoriented admissible bracketed tangles, and discuss how the functor to flat, decorated tangles is defined. All of our constructions will reduce to those in [19] in the case that the cocycle is trivial, and will reduce to the case of [17] in the case that the cocycle is trivial and we work with links. Some pieces of this puzzle were discussed in 1992 by Altschuler and Coste in [1], but they do not give a full definition of the TQFT or detailed proofs. We will also need more recent work of Hausser and Nill and others, who have been working to define integrals and cointegrals for quasi-Hopf algebras, as well as to generalize various
results on integrals and cointegrals in the ordinary Hopf case to the quasi-Hopf case. See, for example, [13].

We will prove and state these results in essentially the opposite order just given. The outline for this paper is as follows. In Chapter 2, we make the relevant definitions for Hopf algebras, including definitions and properties of integrals and cointegrals, and present the double of the group algebra $D[G]$ as our main example. Chapter 3 is then a parallel to Chapter 2, where we describe quasi-Hopf algebras, their properties, their integrals and cointegrals, and the explicit example $D^\omega[G]$. Chapter 4 discusses tangle categories, which we will need to formulate the Hennings algorithm. Chapter 5 presents the relevant definitions for TQFTs and the specific examples of the Hennings TQFT for ordinary Hopf algebras as well as the Dijkgraaf-Witten TQFT for the case of a trivial cocycle, or $D[G]$. In Chapter 6, we extend the definition of the Hennings TQFT to quasi-Hopf algebras and prove that this TQFT is well-defined. Finally, in Chapter 7 we give a direct proof that in the case of ordinary Hopf algebras, the Hennings TQFT on the double of the group algebra $D[G]$, is equivalent to the TQFT of Dijkgraaf and Witten with trivial cocycle, and conjecture that this equivalence extends to the quasi-Hopf case and non-trivial cocycle.
CHAPTER 2
HOPF ALGEBRAS

Throughout, let $G$ denote a finite group and $k$ a field with characteristic zero. Let $	au : A \otimes B \to B \otimes A$ be the “flip” map; that is, $\tau(a, b) = (b, a)$.

2.1 Definitions

In this section, we review some of the basic definitions for Hopf algebras, as well as relevant properties. The reader is referred to [16] for more details. We use $\text{id}_H : H \to H$ to refer to the identity map on a space $H$; when the context is clear we omit the subscript.

**Definition 2.1.1.** A Hopf algebra is a tuple $H = (H, \mu, \eta, \Delta, \epsilon, S)$ defined by the following.

- $H$ is a finite-dimensional vector space over $k$ \hfill (2.1.1)
- $\mu : H \otimes H \to H$ is the multiplication map \hfill (2.1.2)
- $\eta : k \to H$ is the unit map \hfill (2.1.3)
- $\Delta : H \to H \otimes$ is the coproduct \hfill (2.1.4)
- $\epsilon : H \to k$ is the counit \hfill (2.1.5)
- $S : H \to H$ is an antiautomorphism called the antipode \hfill (2.1.6)
The maps in (2.1.1) to (2.1.6) must satisfy the following properties. Let \( x, y, z \in H \) be arbitrary, and let \( 1 \in k \) be the unit in our field.

**Multiplication is associative:** \( \mu(\mu(x, y), z) = \mu(x, \mu(y, z)) \) (2.1.7)

**The unit is a multiplicative unit:** \( \mu(x, \eta(1)) = x = \mu(\eta(1), x) \) (2.1.8)

**Comultiplication is coassociative:** \( (\Delta \otimes \text{id})(\Delta(x)) = (\text{id} \otimes \Delta)(\Delta(x)) \) (2.1.9)

**The counit property:** \( (\epsilon \otimes \text{id})(\Delta(x)) = x = (\text{id} \otimes \epsilon)(\Delta(x)) \) (2.1.10)

\[ \Delta(\mu(x, y)) = (\mu \otimes \mu)(\text{id} \otimes \tau \otimes \text{id})(\Delta \otimes \Delta)(x, y) \] (2.1.11)

\[ \epsilon(\mu(x, y)) = \epsilon(x)\epsilon(y) \] (2.1.12)

\[ \Delta(\eta(1)) = \eta(1) \otimes \eta(1) \] (2.1.13)

\[ \epsilon(\eta(1)) = 1 \] (2.1.14)

\[ \mu(S \otimes \text{id})(\Delta(x)) = \epsilon(x) \cdot 1 = \mu(\text{id} \otimes S)(\Delta(x)) \] (2.1.15)

**Remark 2.1.2.** Hopf algebras are often presented with diagrammatic notation, see [16], for example.

Let \( H \) be a Hopf algebra and \( h \in H \). We use the following notation, due to Sweedler. See [16].

\[ \Delta(h) = \sum_{(h)} h' \otimes h'' \]

The primes refer to the factors of \( \Delta(h) \): the first factor is \( h' \) and the second is \( h'' \). The \( (h) \) underneath the summation denotes that we sum over all of the summands of \( \Delta(h) \) without needing to know how many summands there are. In general, we will suppress the summation index.

This notation can be used with any number of applications of the coproduct. For instance,

\[ (\Delta \otimes \text{id})(\Delta(h)) = \sum h' \otimes h'' \otimes h''' \] (2.1.16)
Remark 2.1.3. The notation in equation (2.1.16) is only valid when the coproduct is coassociative. That is, we have

\[(\Delta \otimes \text{id})(\Delta(h)) = \sum (h')' \otimes (h')'' \otimes h''\]

and

\[(\text{id} \otimes \Delta)(\Delta(h)) = \sum h' \otimes (h'')' \otimes (h'')'',\]

but by coassociativity of the comultiplication these two expressions must be equal. Hence we may use the notation in (2.1.16).

Remark 2.1.4. We will simplify the notation by using \(1_H\) to refer to \(\eta(1)\). Thus, we have \(1 \in k\) and \(1_H \in H\). Typically the context will be clear, and we will write \(1\) for either \(1\) or \(1_H\).

Notation 2.1.5. Let \(A = \sum a_i \otimes b_i \in H \otimes H\). We use subscripts on \(A\) to denote using the two factors of \(A\) as factors of a larger tensor. In the other factors, we use the identity in \(H\).

For example, in \(H^{\otimes 3}\),

\[A_{13} = \sum a_i \otimes 1 \otimes b_i.\] (2.1.17)

Also, let \(\Delta^{\text{op}} : H \to H \otimes H\) be \(\tau \circ \Delta\) for \(\tau\) the flip map.

Definition 2.1.6. A Hopf algebra is called quasitriangular if there is an invertible element \(R \in H \otimes H\), denoted \(R = \sum s_i \otimes t_i\), satisfying the following properties.

\[\Delta^{\text{op}}(h) = R \Delta(h) R^{-1}\] (2.1.18)

\[(\Delta \otimes \text{id})(R) = R_{13} R_{23}\] (2.1.19)

\[(\text{id} \otimes \Delta)(R) = R_{15} R_{12}\] (2.1.20)

This element is called the universal \(R\)-matrix.
Remark 2.1.7. Note that the $R$-matrix also satisfies the Yang-Baxter equation, given by:

$$R_{13}R_{12}R_{23} = R_{23}R_{13}R_{12};$$  \hfill (2.1.21)

using Notation 2.1.5.

Quasitriangularity is a desirable condition for several reasons. First, it produces solutions to the Yang-Baxter equation. Second, the $R$-matrix gives elements in the Hopf algebra with certain properties that will simplify our calculations. However, Hopf algebras are generally not quasitriangular. Drinfeld [7] gives us a construction to remedy this issue. Given a Hopf algebra $H$, we construct the quantum double of $H$, denoted $D(H)$, which is always quasitriangular. Let $A = (H^{op})^* = (H^*, \Delta^*, \epsilon, (\mu^{op})^*, \eta, (S^{-1})^*)$ where $H^*$ denotes the dual vector space, and the asterisk on a map means to compose with the flip map $\tau$. Note that $A$ is a Hopf algebra, satisfying all properties in Definition 2.1.1.

Definition 2.1.8. [7], [16] The double of $H$ is the bicrossed product $D(H) = A \bowtie H$.

Specifically, the underlying vector space is given by $A \otimes H$, and the maps are given by the following. Let $f, g \in A$ and $x, y \in H$.

**Multiplication:** $(f \otimes x)(g \otimes y) = \sum_{(a)} fg(S^{-1}(a^{m'})\ldots a') \otimes a''b$,

where $g(S^{-1}(a^{m'})\ldots a')$ denotes the map $z \mapsto g(S^{-1}(a^{m'})za')$ \hfill (2.1.22)

**Unit:** $1_A \otimes 1_H$ \hfill (2.1.23)

**Comultiplication:** $\Delta(f \otimes x) = \sum_{(f)(x)} (f' \otimes x') \otimes (f'' \otimes x'')$ \hfill (2.1.24)

**Counit:** $\epsilon(f \otimes x) = \epsilon(x)f(1)$ \hfill (2.1.25)

**Antipode:** $S(f \otimes x) = \sum_{(f)(x)} f''(S^{-1}(a^{m''}S(a^{m'''}))) \otimes f'(a''S^{-1}(a'))S(a'')$ \hfill (2.1.26)
Since both $H$ and $A = (H^{op})^*$ can be thought of as Hopf subalgebras of $D(H)$, we also have a “straightening formula” [16] for the double. Essentially, the straightening formula tells us which bicrossed product we choose for the double. We multiply an element of $H$ by an element in $A$ and resolve this multiplication using the definition. The formula is as follows. Let $f \otimes 1_H$ be an element of $A$ in $D(H)$, and $1_A \otimes h$ be an element of $H$ in $D(H)$. Then $hf := (1_A \otimes h)(f \otimes 1_H)$ simplifies to

$$hf = \sum_{(h)} f(S^{-1}(h''')h'h'') \in D[H].$$

(2.1.27)

**Remark 2.1.9.** The straightening formula allows us to simplify the multiplication by passing all elements of $A$ through any element of $H$. We collect all elements of $A$ at the front of the product, all elements of $H$ at the back of the product, and then use regular multiplication in $A$ and $H$ on each of the respective groups of elements.

**Proposition 2.1.10.** [7], [16] The double $D(H)$ of any finite-dimensional Hopf algebra $H$ is quasitriangular.

**Proof.** By computation. Take the $R$-matrix to be

$$R = \sum_i (e^i \otimes 1_H) \otimes (1_A \otimes e_i)$$

(2.1.28)

where $\{e_i\}$ is a basis for the vector space underlying $H$, and $\{e^i\}$ is its dual basis, which is hence a basis of the underlying vector space for $A$. In [16], this is Theorem IX.4.4.

As we mentioned, quasitriangular Hopf algebras always contain certain elements.

**Notation 2.1.11.** Let $u \in H$ be given by

$$u = \sum S(t_i)s_i,$$

(2.1.29)

where $R = \sum s_i \otimes t_i$. 

10
Lemma 2.1.12. [16] The element $u$ satisfies the property

$$S^2(h) = uhu^{-1} \quad (2.1.30)$$

for every $h \in H$.

Definition 2.1.13. A quasitriangular Hopf algebra is called a ribbon Hopf algebra if there exists a grouplike element $G \in H$ so that $v := G^{-1}u$ is central in $H$ and $G^2 = S(u)^{-1}u$.

We have that “grouplike” means that $\Delta(G) = G \otimes G$.

Lemma 2.1.14. [16] The element $v$ has the following properties.

$$v^2 = uS(u) \quad (2.1.31)$$

$$S(v) = v \quad (2.1.32)$$

$$\epsilon(v) = 1 \quad (2.1.33)$$

2.2 Integrals

We illustrate another type of special element in Hopf algebras, originally considered by Sweedler in [27].

Definition 2.2.1. A nonzero element $\Lambda \in H$ is called a left integral if for any $h \in H$,

$$h\Lambda = \epsilon(h)\Lambda. \quad (2.2.1)$$

A nonzero element $\tilde{\Lambda} \in H$ is called a right integral if for any $h \in H$,

$$\tilde{\Lambda}h = \epsilon(h)\tilde{\Lambda}. \quad (2.2.2)$$

Sweedler showed that the space of non-zero left integrals is always nonempty.

Since our antipode $S$ is invertible, we have the following.
Proposition 2.2.2. If \( \Lambda \) is a non-zero left integral, then \( S(\Lambda) \) is a nonzero right integral.

Proof. Suppose that \( \Lambda \neq 0 \) is a left integral. Consider \( S(\Lambda)h \) for any \( h \in H \).

\[
S(\Lambda)h = S(S^{-1}(h)\Lambda) \tag{2.2.3}
\]
\[
= S(\epsilon(S^{-1}(h))\Lambda) \tag{2.2.4}
\]
\[
= \epsilon(h)S(\Lambda) \tag{2.2.5}
\]

Hence \( S(\Lambda) \) is a right integral.

Thus, the spaces of nonzero left and right integrals are both nonempty.

Definition 2.2.3. A Hopf algebra in which every left integral is also a right integral is called unimodular.

By Proposition 2.2.2, we can see that in the case that \( H \) is unimodular, every integral is invariant under the antipode.

Theorem 2.2.4. [21] If \( H \) is a finite-dimensional Hopf algebra over a principal ideal domain, the space of left integrals over \( H \) is one-dimensional.

We may also consider the dual Hopf algebra \( H^* \) and its integrals. Since \( H \) is finite-dimensional, \( H^* \) is also finite-dimensional, and so similar results hold.

Definition 2.2.5. A left (resp. right) cointegral on \( H \) is a left (resp. right) integral on \( H^* \). In particular, a left cointegral \( \lambda \) satisfies

\[
(1 \otimes \lambda)(\Delta(h)) = 1 \cdot \lambda(h) \tag{2.2.6}
\]

for every \( h \in H \).

Proposition 2.2.6. Suppose \( \lambda \) is a left cointegral on \( H \). Then \( \lambda \circ S^{-1} \) is a right cointegral on \( H \).
Proof. Suppose \( \lambda \) is a left cointegral. Recall that for any \( h \in H \), we have

\[
\sum S^{-1}(h') \otimes h'' = \sum (S^{-1}(h))^\prime \otimes S(S^{-1}((h'))) \tag{2.2.7}
\]

We then have the following.

\[
(\lambda \circ S^{-1})(\Delta(h)) = \sum \lambda((S^{-1}(h)))' h'' \tag{2.2.8}
\]

\[
= \sum \lambda((S^{-1}(h))''(S^{-1}(h))') \tag{2.2.9}
\]

\[
= \sum S(\lambda((S^{-1}(h))''(S^{-1}(h))')) \tag{2.2.10}
\]

\[
= \sum S(1 \cdot \lambda(S^{-1}(h))) \tag{2.2.11}
\]

\[
= 1 \cdot \lambda \circ S^{-1}(h) \tag{2.2.12}
\]

Using the result for integrals on Hopf algebras, we also know that the space of left (resp. right) cointegrals is one-dimensional.

Finally, a result on integrals which is necessary to define our invariants.

Lemma 2.2.7. \([17, 26]\) For a unimodular finite-dimensional ribbon Hopf algebra \( H \) over a field \( k \), there is a right cointegral \( \lambda \) satisfying the following properties for all \( x, y \in H \).

1. \( \lambda \) is unique up to scalar multiplication

2. \( \lambda(xy) = \lambda(S^2(y)x) \)

3. \( \lambda(S(x)) = \lambda(G^2x) \) for the special grouplike element \( G \).

2.3 Example: Double of the Group Algebra

Our main example of a Hopf algebra will be the double of the group algebra. First we define the group algebra, denoted \( k[G] \). Suppose that \( g, h \in G \) are group elements and \( e \in G \) is the unit in the group.
Lemma 2.3.1. [16] The group algebra $k[G]$ is a Hopf algebra with the following properties.

- Multiplication is multiplication in the group: $\mu(g, h) = gh$.
- The unit is the unit in the group: $\eta(1) = e$.
- The coproduct makes every group element grouplike: $\Delta(g) = g \otimes g$.
- The counit: $\epsilon(g) = 1$.
- The antipode: $S(g) = g^{-1}$.

This Hopf algebra is not quasitriangular, so we form its double $D(k[G])$, or simply $D[G]$. The Hopf algebra $k[G]$ has the set $\{g \in G\}$ as its basis, so its dual basis is given by \{\(\delta_g(x) : g \in G\)\} where $\delta_g(x) = 1$ if $x = g$ and is zero otherwise. More often, we will use the notation $\delta_{g,x} = \delta_g(x)$.

Notation 2.3.2. We introduce the following notation as in [5] to describe elements of $D[G]$:

$$\delta_g \otimes h := g \downarrow_h.$$

Our basis for the underlying vector space of $D[G]$ is then all elements in the above form for all pairs $(g, h) \in G \times G$. We need only define our Hopf algebra maps on basis elements.

Lemma 2.3.3. [16] With the above notation, $D[G]$ is a Hopf algebra. The calculations of Definition 2.1.8 for $D[G]$ simplify as follows.

- Multiplication: $g \downarrow_h \cdot x \downarrow_y = \delta_{g,hx^{-1}} \cdot y \downarrow_h \cdot y$
- Unit: $1_A \otimes e$, where $1_A = \sum \delta_g$; hence $1_A \otimes e = \sum g \downarrow_e$
• **Comultiplication:** \( \Delta(g \downarrow_h) = \sum_{uv=g}^v \downarrow_h \otimes u \downarrow_h \)

• **Counit:** \( \epsilon(g \downarrow_h) = \delta_{g,e} \)

• **Antipode:** \( S(g \downarrow_h) = (h^{-1}g^{-1}h) \downarrow_{h^{-1}} \)

**Remark 2.3.4.** Since we are using \( A = (H^{op})^* \), our comultiplication involves the flip map.

**Lemma 2.3.5.** [16] The Hopf algebra \( D[G] \) is quasitriangular, with the \( R \)-matrix, computed from (2.1.28) given by

\[
R = \sum_{g,h \in G}^h \downarrow_g \otimes g \downarrow_e .
\] (2.3.1)

Finally, we calculate the straightening formula using the multiplication map in Lemma 2.3.3.

\[
1_A \downarrow_h \cdot g \downarrow_e = \text{hgh}^{-1} \downarrow_h
\] (2.3.2)

**Lemma 2.3.6.** In \( D[G] \), the square of the antipode is the identity map.

**Proof.** Let \( x \downarrow_y \) be an arbitrary basis element. We have the following.

\[
S^2(x \downarrow_y) = S(S(x \downarrow_y))
\] (2.3.3)

\[
= S(y^{-1}x^{-1}y \downarrow_{y^{-1}}) = (g^{-1}g) \downarrow_y = x \downarrow_y
\] (2.3.4)

\( \Box \)

Finally, we give the special elements of the double outlined in Section 2.1.

**Lemma 2.3.7.** The special element \( u \) is given by the formula

\[
u = \sum_{g \in G}^g \downarrow_{g^{-1}} \]
(2.3.5)

and this element is central in \( D[G] \).
Proof. We calculate the formula for $u$ using the definition of $R$ given in (2.3.1) and the formula in (2.1.29). The following calculation shows that this element $u$ is central in $D[G]$. As usual, let $x \rhd y$ refer to an arbitrary basis element.

\[
x \rhd y \cdot u = x \rhd y \sum_{g \in G} g \rhd y g^{-1} = \sum_{g \in G} x \rhd y g \rhd y^{-1} = \sum_{g \in G} \delta_{x, yyy^{-1}} x \rhd y_g^{-1} = x \rhd y^{-1}x^{-1} = \sum_{g \in G} \delta_{y, x^{-1}g^{-1}y_g} g \rhd y = \sum_{g \in G} g \rhd y_g^{-1} = u \cdot x \rhd y
\]

This is the desired result. $\square$

Lemma 2.3.7 then gives that the special element $v$ is equal to $u$, and the special grouplike element $G$ is the identity in $H$. The reader may then verify the properties of $v = u$ given in (2.1.31), (2.1.32), and (2.1.33).
CHAPTER 3
QUASI-HOPF ALGEBRAS

Quasi-Hopf algebras, originally defined by Drinfeld in [8], are a natural extension of Hopf algebras. We therefore present the basic definitions and theorems for quasi-Hopf algebras as a parallel to our definitions in the case of ordinary Hopf algebras.

The major difference between Hopf algebras and quasi-Hopf algebras is that the comultiplication map \( \Delta \) is not coassociative in a quasi-Hopf algebra. That is,

\[
\text{id} \otimes \Delta(\Delta(x)) \neq (\Delta \otimes \text{id})(\Delta(x)).
\]

We do not eliminate coassociativity entirely, however, simply weaken it in a way motivated by representation theory. Let \( H \) be a Hopf algebra. Let \( V_j \) denote a vector space and \( \pi_j : H \to \text{End}(V_j) \) an action. We denote the image of \( h \) under \( \pi_j \) as \( \pi(h, -) \in \text{End}(V_j) \). Suppose we have two representations \( (V_1, \pi_1) \) and \( (V_2, \pi_2) \) of \( H \). We define a tensor product of these two representations, \( (V_1 \otimes V_2, \pi_1 \otimes \pi_2) \) by \( \pi_1 \otimes \pi_2 \). The image of \( h \in H \) under \( \pi_1 \otimes \pi_2 \) is given by

\[
(\pi_1 \otimes \pi_2)(h, v_1 \otimes v_2) = \sum_{(h)} \pi_1(h', v_1) \otimes \pi_2(h'', v_2) \tag{3.0.1}
\]

for any \( v_j \in V_j \) where \( \sum h' \otimes h'' = \Delta(h) \) as usual. If the coproduct is coassociative, then this tensor product of representations is strictly associative. That is, we have

\[
(\pi_1 \otimes \pi_2) \otimes \pi_3 = \pi_1 \otimes (\pi_2 \otimes \pi_3). \tag{3.0.2}
\]
If instead of strict associativity, we only require the two tensor representations in (3.0.2) to be equivalent, then we introduce some element $\Phi \in H \otimes^3$ so that the intertwiner between the two representations is

$$\Phi_{123} = (\pi_1 \otimes \pi_2 \otimes \pi_3)(\Phi) : (V_1 \otimes V_2) \otimes V_3 \rightarrow V_1 \otimes (V_2 \otimes V_3).$$

(3.0.3)

We call the resulting space $H$ a quasi-Hopf algebra. The element $\Phi$ will be called the “coassociator” or “reassociator” and satisfies the following property on the quasi-Hopf algebra $H$, which is equivalent to the property in (3.0.3).

$$(\text{id} \otimes \Delta)(\Delta(x)) = \Phi \cdot (\Delta \otimes \text{id})(\Delta(x)) \cdot \Phi^{-1} \quad (3.0.4)$$

**Notation 3.0.8.** We will denote $\Phi = \sum X_i \otimes Y_i \otimes Z_i$ and $\Phi^{-1} = \sum \bar{X}_i \otimes \bar{Y}_i \otimes \bar{Z}_i$.

### 3.1 Definitions

We present here the basics, following [1]. The reader is also referred to [8] for more details. Let $H$ be a finite-dimensional vector space over our field $k$, as in Chapter 2. Suppose further that this $H$ is an associative algebra with multiplication $\mu$ and unit map $\eta$ where multiplication is associative and the unit is a multiplicative unit, as in Definition 2.1.1.

**Definition 3.1.1.** The space $H$ is a quasi-bialgebra if we have algebra homomorphisms $\Delta : H \rightarrow H \otimes H$ and $\epsilon : H \rightarrow k$ and an invertible element $\Phi \in H \otimes^3$ so that the following properties are satisfied, where $h \in H$ is arbitrary.

$$(\text{id} \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \text{id})(\Delta(h))\Phi^{-1} \quad (3.1.1)$$

$$(\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\Phi)$$

$$= (1 \otimes \Phi) \cdot (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \cdot (\Phi \otimes 1) \quad (3.1.2)$$

$$(\epsilon \otimes \text{id})(\Delta(h)) = h = (\text{id} \otimes \epsilon)(\Delta(h)) \quad (3.1.3)$$

$$(\text{id} \otimes \epsilon \otimes \text{id})(\Phi) = 1 \otimes 1 \quad (3.1.4)$$

18
Note as in [1] that the second and fourth properties together imply that
\[(\epsilon \otimes \text{id} \otimes \text{id})(\Phi) = 1 = (\text{id} \otimes \text{id} \otimes \epsilon)(\Phi). \tag{3.1.5}\]

**Remark 3.1.2.** As discussed in the introduction to this chapter, these properties are essentially the same as for ordinary Hopf algebras, except we remove the coassociativity of $\Delta$ and add properties of the element $\Phi$.

An equivalent definition of a quasi-bialgebra is as follows, see [16].

**Definition 3.1.3.** Let $H = (H, \Delta, \epsilon)$ be an algebra with comultiplication $\Delta$ and counit $\epsilon$. It is a quasi-bialgebra if the category $H$-mod equipped with the tensor product of $\text{Vect}(k)$ is a tensor category.

A proof that these two definitions are equivalent is found in [16]. The main idea of the proof is that we must produce an associativity constraint, as well as left and right unit constraints. We may take the left and right unit constraints to be the identity, and for the associativity constraint, we have the following. For $H$ modules $U, V, W$ and arbitrary elements $u \in U, v \in V, w \in W$,
\[a_{U,V,W}((u \otimes v) \otimes w) = \Phi(u \otimes (v \otimes w)). \tag{3.1.6}\]

Furthermore, a tensor category must satisfy the Pentagon Axiom, which in this case is equivalent to condition (3.1.2) in Definition 3.1.1. The Pentagon Axiom is also expressed by the following commutative diagram, where $T, U, V$ and $W$ are $H$-modules.

\[
\begin{align*}
&T \otimes (U \otimes (V \otimes W)) \\
\rightarrow & \\
(T \otimes (U \otimes V)) \otimes W \\
\rightarrow & \\
((T \otimes U) \otimes V) \otimes W \\
\rightarrow & \\
(T \otimes U) \otimes (V \otimes W)
\end{align*}
\]

(3.1.7)
This axiom on modules is given in Definition 3.1.1 by (3.1.2). Specifically, we may take
\[ a_{\mathcal{U},\mathcal{V},\mathcal{W}}(u \otimes v \otimes w) = \sum X_i u \otimes Y_i v \otimes Z_i w \]  

(3.1.8) for \( \Phi = \sum X_i \otimes Y_i \otimes Z_i \). The conditions of Definition 3.1.1 are those that would make \( H\text{-mod} \) a tensor category, hence we use only Definition 3.1.1.

**Definition 3.1.4.** A quasi-bialgebra \( H \) is called a quasi-Hopf algebra if there exists an antiautomorphism \( S : H \to H \), called the antipode, and elements \( \alpha, \beta \in H \) so that the following hold for every \( h \in H \). Recall that \( \Phi = \sum X_i \otimes Y_i \otimes Z_i \in H^{\otimes 3} \).

\[
\sum S(h')\alpha h'' = \epsilon(h)\alpha \\
\sum h' \beta S(h'') = \epsilon(h)\beta \\
\sum X_i \beta S(Y_i) \alpha Z_i = 1 \\
\sum S(\bar{X}_i) \alpha \bar{Y}_i \beta S(\bar{Z}_i) = 1
\]

(3.1.9) (3.1.10) (3.1.11) (3.1.12)

**Remark 3.1.5.** As noted after Definition 2.1.1, we emphasize the fact that Sweedler’s notation must be used with extreme care. In the case of quasi-Hopf algebras, we have that
\[
\sum (x')' \otimes (x')'' \otimes x'' \neq \sum x' \otimes (x'')' \otimes (x'')''
\]

For the calculations in this paper, the above notation will suffice. That is, we retain the parenthesis when taking a second coproduct. This notation will become extremely cumbersome in the case of calculations involving more than two applications of the coproduct to a single element; in that case the is referred to the notation in [13] and [3], which deals with this problem more elegantly.

**Lemma 3.1.6.** [8] Properties 3.1.9 and 3.1.11 imply that
\[ \epsilon \circ S = \epsilon \]  

(3.1.13)
and

\[ \epsilon(\alpha\beta) = \epsilon(\alpha)\epsilon(\beta) = 1. \tag{3.1.14} \]

**Remark 3.1.7.** By rescaling if necessary, (3.1.14) implies that we may assume \( \epsilon(\alpha) = \epsilon(\beta) = 1. \)

**Notation 3.1.8.** Let \( \sigma \) denote a permutation of \( \{1, 2, 3\} \). If we denote \( \Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \), we take

\[ \Phi_{\sigma(1)\sigma(2)\sigma(3)} = \sum \Phi^{-1}_{\sigma(1)} \otimes \Phi^{-1}_{\sigma(2)} \otimes \Phi^{-1}_{\sigma(3)}. \tag{3.1.15} \]

**Definition 3.1.9.** A quasi-Hopf algebra \( H \) is called quasitriangular if there exists an invertible element \( R \in H \otimes H \) so that the following properties hold for all \( h \in H \).

\[ \Delta^{\text{op}}(h) = R\Delta(h)R^{-1} \tag{3.1.16} \]

\[ (\Delta \otimes \text{id})(R) = \Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi \tag{3.1.17} \]

\[ (\text{id} \otimes \Delta)(R) = R^{-1}_{231}\Phi_{213}\Phi_{12}\Phi_{13}^{-1} \tag{3.1.18} \]

**Remark 3.1.10.** Similarly to the case of Hopf algebras, the definition of a quasi-Hopf algebra implies that the quasi-Yang Baxter equation is satisfied. That is,

\[ R_{12}\Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi = \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{123}^{-1}. \tag{3.1.19} \]

**Remark 3.1.11.** We may make an equivalent definition corresponding to Definition 3.1.3 in terms of \( H \)-mod: a quasi-bialgebra is a quasi-Hopf algebra if \( H \)-mod is braided. In this case, we need a commutativity constraint, given by

\[ c_{U,V}(u \otimes v) = \tau_{U,V}(R(u \otimes v)), \tag{3.1.20} \]

where \( U \) and \( V \) are \( H \)-modules with \( u \in U \) and \( v \in V \), and \( R \) is the \( R \)-matrix. See [16], Proposition XII.1.4. The map \( \tau \) is the usual flip map.
A commutativity constraint in a braided tensor category must satisfy the Hexagon Axiom, which is given by the following commutative diagrams. Again, \( U, V, \) and \( W \) are \( H \)-modules and \( c \) refers to the commutativity constraint. Note that the subscripts on \( c \) and \( a \) have been suppressed.

\[
\begin{array}{c}
\xymatrix{ (U \otimes V) \otimes W \ar[r]^a & U \otimes (V \otimes W) \\
W \otimes (U \otimes V) \ar[u]^{a^{-1}} \ar[d]_{1 \otimes c} & U \otimes (W \otimes V) \ar[d]^{a^{-1}} \ar[u]_a \\
(W \otimes U) \otimes V \ar[r]_{c \otimes 1} & (U \otimes W) \otimes V }
\end{array}
\quad (3.1.21)
\]

\[
\begin{array}{c}
\xymatrix{ U \otimes (V \otimes W) \ar[r]^{a^{-1}} & (U \otimes V) \otimes W \\
(V \otimes W) \otimes U \ar[u]^{a^{-1}} \ar[d]_{c \otimes 1} & (V \otimes U) \otimes W \ar[u]_a \ar[d]^{a} \\
V \otimes (W \otimes U) \ar[r]_{1 \otimes c} & V \otimes (U \otimes W) }
\end{array}
\quad (3.1.22)
\]

These commutative diagrams are equivalent to \((3.1.17)\) and \((3.1.18)\) of Definition 3.1.9, respectively.

### 3.2 Special Elements in Quasi-Hopf Algebras

As in the case of quasitriangular Hopf algebras, there are some special elements of quasitriangular quasi-Hopf algebras of which we should take note. Many of these elements are nicely outlined in [1]. We again define an invertible element \( u \) so that \( S^2(x) = uxu^{-1} \) for any \( x \in H \). This element \( u \) is given by

\[
u = \sum_{i,j} S(\tilde{Y}_i \beta S(\tilde{Z}_i))S(t_j)\alpha s_j \bar{X}_i.
\quad (3.2.1)
\]
Lemma 3.2.1. [1] The element $u$ has the following properties.

\[ S^2(u) = u \]  
\[ uS(u) = S(u)u \text{ is central.} \]  
\[ \sum S(t_i)\alpha a_i = S(\alpha)u = S(u)u \sum S(\tilde{s}_i)\alpha \tilde{t}_i \]  

To define the special elements $G$ and $v$, we need some additional elements. First, we define two elements $\gamma$ and $\delta$ which are related to the reassociator $\Phi$.

Let

\[ (\Phi \otimes 1)(\Delta \otimes \text{id} \otimes \text{id})(\Phi^{-1}) = \sum_i A_i \otimes B_i \otimes C_i \otimes D_i. \]  

Then define

\[ \gamma = \sum_i S(B_i)\alpha C_i \otimes S(A_i)\alpha D_i. \]  

Let

\[ (\Delta \otimes \text{id} \otimes \text{id})(\Phi)(\Phi^{-1} \otimes 1) = \sum K_i \otimes L_i \otimes M_i \otimes N_i. \]  

Then define

\[ \delta = \sum K_i \beta S(N_i) \otimes L_i \beta S(M_i). \]  

Next, we define an element $f$ which helps simplify some later algebra. Let

\[ f = \sum_i (S \otimes S)(\Delta^{\text{op}}(\bar{X}_i)) \cdot \gamma \cdot \Delta(\bar{Y}_i \beta S(\bar{Z}_i)). \]  

These elements enjoy the following relations.

Lemma 3.2.2. [1] The elements $\gamma$, $\delta$, and $f$ satisfy the following, for any $h \in H$.

\[ f\Delta(h)f^{-1} = (S \otimes S)(\Delta^{\text{op}}(S^{-1}(h))) \]  
\[ \gamma = f\Delta(\alpha) \]  
\[ \delta = \Delta(\beta)f^{-1} \]
Definition 3.2.3. [1] We call a quasitriangular quasi-Hopf algebra a ribbon quasi-Hopf algebra if there exists an invertible element $v \in H$ so that the following properties are satisfied.

\begin{align*}
v^2 &= uS(u) \quad (3.2.13) \\
S(v) &= v \quad (3.2.14) \\
\epsilon(v) &= 1 \quad (3.2.15) \\
\Delta(uv^{-1}) &= f^{-1}((S \otimes S)(f_{21}))(uv^{-1} \otimes uv^{-1}) \quad (3.2.16)
\end{align*}

Here, we let $f_{21} = \tau(f)$ for the flip map $\tau$.

Remark 3.2.4. Unlike the ordinary Hopf algebra case, for quasi-Hopf algebras it is more convenient to first define the element $v$, then denote the element $uv^{-1}$ by $G$. An important distinction here is that the element $G$ is not grouplike for quasi-Hopf algebras, but instead satisfies condition (3.2.16) above. However, because of the properties of the elements $u$ and $v$, many of the original properties of the element $G$ still hold.

Lemma 3.2.5. The following properties hold for the special element $G = uv^{-1}$, and for all $x \in H$.

\begin{align*}
S(G) &= G^{-1} \quad (3.2.17) \\
S^2(x) &= GxG^{-1} \quad (3.2.18) \\
uG &= Gu \quad (3.2.19) \\
S(u) &= G^{-1}uG^{-1} \quad (3.2.20)
\end{align*}

Proof. For property (3.2.17), we know that $S(v) = v$, and so we have the following.

\begin{align*}
S(G) &= S(uv^{-1}) = S(v^{-1})S(u) = v^{-1}S(u) = v^{-1}u^{-1}v^2 = vu^{-1} = G^{-1} \quad (3.2.21)
\end{align*}
In the fourth equality, note that we have used \( S(u) = v^2u^{-1} \) from \( v^2 = S(u)u \). For property (3.2.18), we know that \( v \) is central in \( H \). We have the following.

\[
S^2(x) = u xu^{-1} = u xv^{-1} v^{-1} = uv^{-1} xv u^{-1} = G x G^{-1}
\]  

(3.2.22)

For property (3.2.19), we know that \( S^2(u) = u \), hence the following hold by property (3.2.18).

\[
S^2(u) = G u G^{-1}
\]  

(3.2.23)

\[
u = G u G^{-1}
\]  

(3.2.24)

\[
u G = G u
\]  

(3.2.25)

Finally, then, the proof for property (3.2.20) is as follows.

\[
G^{-1} u G^{-1} = vu^{-1} uv u^{-1}
\]  

(3.2.26)

\[
= v^2 u^{-1}
\]  

(3.2.27)

\[
= u S(u) u^{-1}
\]  

(3.2.28)

\[
= S^3(u) = S(u)
\]  

(3.2.29)

\[
\square
\]

We continue with additional elements unique to quasi-Hopf algebras.

We define the element \( \mathcal{M} \), called the monodromy element, as follows.

\[
\mathcal{M} = \mathcal{R}^{op} \mathcal{R} = \sum t_j s_i \otimes s_j t_i
\]  

(3.2.30)

**Definition 3.2.6.** [4] We say that \( \mathcal{M} \) is nondegenerate if

\[
\mathcal{M} : H^* \to H : l \mapsto (\text{id} \otimes l)(\mathcal{M})
\]  

(3.2.31)

is an isomorphism.
Finally, we have four more special elements in $H \otimes H$, which will help simplify later calculations.

\begin{align*}
p_L &= \sum Y_i S^{-1}(X_i \beta) \otimes Z_i \quad (3.2.32) \\
p_R &= \sum \bar{X}_i \otimes \bar{Y}_i \beta S(\bar{Z}_i) \quad (3.2.33) \\
q_L &= \sum S(\bar{X}_i) \alpha \bar{Y}_i \otimes \bar{Z}_i \quad (3.2.34) \\
q_R &= \sum X_i \otimes S^{-1}(\alpha Z_i) Y_i \quad (3.2.35)
\end{align*}

We will sometimes use the shorthand notation $p_L = \bar{p}$, $q_L = \bar{q}$. We will use further shorthand notation when the individual terms are needed. That is, we use the following.

\begin{align*}
p_R &= \sum p^1 \otimes p^2 \quad (3.2.36) \\
p_L &= \sum \bar{p}^1 \otimes \bar{p}^2 \quad (3.2.37) \\
q_R &= \sum q^1 \otimes q^2 \quad (3.2.38) \\
q_L &= \sum \bar{q}^1 \otimes \bar{q}^2 \quad (3.2.39)
\end{align*}

### 3.3 Integrals and Cointegrals

The existence of cointegrals in quasi-Hopf algebras is an interesting discussion in itself, and the interested reader is referred to [13]. Finding integrals is not difficult, as the only complication arises from the non-coassociativity of the coproduct. Finding cointegrals, on the other hand, is much more difficult. The dual $H^*$ of a quasi-Hopf algebra is no longer a Hopf algebra, as the multiplication is not associative. With some careful manipulation, however, we can indeed find cointegrals in quasi-Hopf algebras, and these integrals and cointegrals behave in the way we expect from the ordinary Hopf algebra case.
Definition 3.3.1. Let $H$ be a quasi-Hopf algebra. An element $\Lambda \in H$ is called a left integral if

$$h\Lambda = \epsilon(h)\Lambda$$  \hspace{1cm} (3.3.1)

for all $h \in H$. The element $\tilde{\Lambda}$ is called a right integral if

$$\tilde{\Lambda}h = \epsilon(h)\tilde{\Lambda}$$  \hspace{1cm} (3.3.2)

for all $h \in H$.

Lemma 3.3.2. If $\Lambda$ is a left integral in a quasi-Hopf algebra $H$, then $S(\Lambda)$ is a right integral in $H$.

Proof. Identical to the ordinary Hopf case.

Definition 3.3.3. A quasi-Hopf algebra $H$ is called unimodular if every left integral is also a right integral.

Lemma 3.3.4. In a unimodular quasi-Hopf algebra $H$, the integral satisfies the following properties.

$$\Lambda$$ is a central element in $H$  \hspace{1cm} (3.3.3)

$$S(\Lambda) = \Lambda$$  \hspace{1cm} (3.3.4)

Proof. Identical to the ordinary Hopf case.

Theorem 3.3.5. [13], Theorem 4.3. The space of left integrals on a finite-dimensional quasi-Hopf algebra $H$ is one-dimensional.

We would again like to dualize this construction to produce cointegrals on quasi-Hopf algebras. Even though $H^*$ is not a quasi-Hopf algebra, we can make $H^*$ into a
left quasi-Hopf $H$-bimodule, see [3]. In order to define a left cointegral, we need the following two elements of $H \otimes H$.

\[ U = \sum (f^{-1})^1 S(q^2) \otimes (f^{-1})^2 S(q^1) \]  
\[ V = \sum S^{-1}(f^2 p^2) \otimes S^{-1}(f^1 p^1) \]  

(3.3.5)  
(3.3.6)

Here $f \in H^\otimes 2$ is the element given in (3.2.9) and $p_R = p^1 \otimes p^2$, $q_R = q^1 \otimes q^2$ are given in (3.2.33) and (3.2.35), respectively. We also have $\Delta(h) = \sum h' \otimes h''$ as usual.

**Definition 3.3.6.** [3] An element $\lambda \in H^*$ is a left cointegral of a unimodular finite-dimensional quasi-Hopf algebra $H$ if and only if

\[ \sum_{(h)} \lambda(V^2 h'' U^2) V^1 h' U^1 = \lambda(h) \]  

(3.3.7)

for every $h \in H$, where $U = \sum U^1 \otimes U^2$ and $V = \sum V^1 \otimes V^2$.

**Remark 3.3.7.** This definition, given on page 7 of [3], has been simplified for the unimodular case.

**Lemma 3.3.8.** [13], Theorem 4.3. The space of left cointegrals on a finite-dimensional quasi-Hopf algebra $H$ is one-dimensional.

**Definition 3.3.9.** A right cointegral on a finite-dimensional quasi-Hopf algebra $H$ is a left cointegral on $H^{\cop}$. The quasi-Hopf algebra $H^{\cop}$ is given by taking opposite comultiplication on the quasi-Hopf algebra $H$.

**Remark 3.3.10.** The definition of a right cointegral can also be given by the following. An element $\lambda \in H^*$ is a right cointegral of a unimodular finite-dimensional quasi-Hopf algebra $H$ if and only if

\[ (\lambda \otimes \text{id})(S \otimes S)(\tilde{p}_{21}) f \Delta(h)(S^{-1} \otimes S^{-1})(f_{21}^{-1})(S^{-1} \otimes S^{-1})(\tilde{q}_{21})) = \lambda(h) \]  

(3.3.8)

for every $h \in H$, where $\tilde{p} = p_L$ and $\tilde{q} = q_L$. 

28
Lemma 3.3.11. [3] If $H$ is a finite-dimensional unimodular quasi-Hopf algebra with left cointegral $\lambda$, then $\lambda \circ S$ is a right cointegral on $H$.

Proof. This follows from Corollary 4.4 in [3].

We will generally be concerned with right cointegrals, and so we would like to tailor the definition to our needs. Towards this end, we make two important assumptions. First, since $\epsilon(\alpha)\epsilon(\beta) = 1$, we may assume without loss of generality that $\epsilon(\alpha) = \epsilon(\beta) = 1$ by rescaling $\alpha$ and $\beta$ if necessary. We also assume that $\mathcal{M}$ is nondegenerate in the sense of Definition 3.2.6.

Lemma 3.3.12. [3] Let $\lambda \in H^*$ be nonzero, and let $\Lambda$ be any left integral in $H$. Then $\lambda$ is a right cointegral on $H$ if and only if one of the equivalent relations below is satisfied.

\begin{align*}
(\lambda \otimes \text{id})(q_L\Delta(\Lambda)p_L) &= \epsilon(\beta)\lambda(\Lambda) \cdot 1 \quad (3.3.9) \\
(\lambda \otimes \text{id})(\Delta(\Lambda)p_L) &= \epsilon(\beta)\lambda(\Lambda)S^{-1}(\beta) \quad (3.3.10) \\
(\lambda(h\Lambda^p))\Lambda''\tilde{p}^2 &= \epsilon(\beta)\lambda(\Lambda)S^{-1}(h\beta) \quad \text{for all } h \in H \quad (3.3.11)
\end{align*}

where $p_L = \sum \tilde{p}^1 \otimes \tilde{p}^2$.

Proof. This is Corollary 3.10 in [3] when applied to a unimodular quasi-Hopf algebra. In this cited result, the authors applied the maps of $H^{\text{cop}}$ to a previous result for left cointegrals, and mistakenly took the $\beta$ element to be $S^{-1}(\alpha)$. In reality, in the quasi-Hopf algebra given by $H^{\text{cop}}$, the $\beta$ element is given by $S^{-1}(\beta)$, and hence the characterization to which we are referring is correct when $\alpha$ is replaced by $\beta$.

We now use Lemma 3.3.12 to give our tailored definition.

Lemma 3.3.13. Let $\lambda \in H^*$ be a nonzero map. Then $\lambda$ is a right cointegral on $H$ if and only if it satisfies the following property for any $h \in H$.

\begin{align*}
(\lambda \otimes \text{id})(q_L\Delta(h)p_L) &= 1 \cdot \lambda(\alpha h S^{-1}(\beta)) \quad (3.3.12)
\end{align*}
Proof. First, suppose \( \lambda \) is a nonzero map satisfying (3.3.12). Let \( \Lambda \) be a left integral on \( H \). Since \( H \) is unimodular, \( \Lambda \) is also a right integral. We have the following.

\[
(\lambda \otimes \text{id})(q_L \Delta(\Lambda)p_L) = 1 \cdot \lambda(\alpha \Lambda S^{-1}(\beta))
\]
(3.3.13)
\[
= 1 \cdot \lambda(\epsilon(\alpha)\Lambda S^{-1}(\beta))
\]
(3.3.14)
\[
= \epsilon(\alpha)\lambda(\Lambda S^{-1}(\beta))
\]
(3.3.15)
\[
= \epsilon(\alpha)\epsilon(S^{-1}(\beta))\lambda(\Lambda)
\]
(3.3.16)
\[
= 1 \cdot \epsilon(\beta)\lambda(\Lambda)
\]
(3.3.17)

Note that the last line follows from the previous line as we have assumed \( \epsilon(\alpha) = 1 \) and we know that \( \epsilon \circ S = \epsilon \) from properties of a quasi-Hopf algebra.

By (3.3.9) in Lemma 3.3.12, \( \lambda \) is a right cointegral.

Next, let \( \lambda \) be a right cointegral. We show that (3.3.12) is satisfied. We substitute \( \alpha h S^{-1}(\beta) \) in the definition of \( \lambda \) given in (3.3.8). This gives us the following equality.

\[
(\lambda \otimes \text{id})((S \otimes S)(\tilde{p}_{21})f\Delta(\alpha h S^{-1}(\beta))(S^{-1} \otimes S^{-1})(f_{21}^{-1})(S^{-1} \otimes S^{-1})(\tilde{q}_{21}))
\]
(3.3.18)
\[
= \lambda(\alpha h S^{-1}(\beta))
\]

Now, we know that \( f\Delta(\alpha) = \gamma \), and we can see that

\[
(S \otimes S)(\tilde{p}_{21} = \tilde{q} = q_L).
\]
(3.3.19)

Then we can simplify (3.3.18) to the following.

\[
(\lambda \otimes \text{id})(q_L \Delta(h)\Delta(S^{-1}(\beta))(S^{-1} \otimes S^{-1})(f_{21}^{-1})(S^{-1} \otimes S^{-1})(\tilde{q}_{21}))
\]
(3.3.20)
\[
= \lambda(\alpha h S^{-1}(\beta))
\]

Next, we use (3.2.12) to see that \( \Delta(S^{-1}(\beta))(S^{-1} \otimes S^{-1})(f_{21}^{-1}) = (S^{-1} \otimes S^{-1})(\delta_{21}) \), and then

\[
(S^{-1} \otimes S^{-1})(\delta_{21})(S^{-1} \otimes S^{-1})(\tilde{q}_{21}) = \tilde{p} = p_L.
\]
(3.3.21)
Finally, then, we simplify (3.3.20) into the desired result.

\[
(\lambda \otimes \text{id})(q_L \Delta(h)p_L) = \lambda(\alpha h S^{-1}(\beta)) \tag{3.3.22}
\]

**Notation 3.3.14.** We use \( \lambda \) to denote a right cointegral on \( H \).

Next, we turn to properties of the right cointegral, and highlight our assumptions about the relationships between \( \lambda, \Lambda \), and the element \( v \) which are possible by rescaling the elements \( \lambda \) and \( \Lambda \). Recall that the spaces of integrals and cointegrals are both one-dimensional, so rescaling produces another integral or cointegral. We may assume the following.

\[
\lambda(\alpha v S^{-1}(\beta))\lambda(\alpha v^{-1} S^{-1}(\beta)) = 1 \tag{3.3.23}
\]

\[
\lambda(\Lambda) = 1 \tag{3.3.24}
\]

Since we have assumed \( H \) is unimodular, Lemma 5.1 in [13] implies that for all right cointegrals \( \lambda \) and for all \( a, b \in H \), we have

\[
\lambda(ab) = \lambda(S^2(b)a). \tag{3.3.25}
\]

Furthermore, we may define a map \( \text{tr} : H \to \mathbf{k} \) by

\[
\text{tr}(h) = \lambda(h G^{-1}) \tag{3.3.26}
\]

for all \( h \in H \). Since \( H \) is unimodular, Hausser and Nill also show in [13] in the proof of Proposition 5.6 that this map is a nondegenerate trace. Using the properties of the special element \( G \) and (3.3.25), which corresponds to the trace property

\[
\text{tr}(ab) = \text{tr}(ba), \tag{3.3.27}
\]
we can show that
\[ \lambda(S(h)) = \lambda(G^2h) \] (3.3.28)
for all \( h \in H \). Then (3.3.28) corresponds to
\[ \text{tr}(S(a)) = \text{tr}(a). \] (3.3.29)
on the trace map defined in (3.3.26).

### 3.4 Example: The Dijkgraaf-Pasquier-Roche Algebra

In [5], Dijkgraaf, Pasquier and Roche presented an example of a non-trivial quasi-Hopf algebra associated to any finite group \( G \). This algebra is referred to as the Dijkgraaf-Pasquier-Roche Algebra, or as the twisted Drinfeld double of the group algebra, and is denoted \( D^\omega[G] \). The original construction involves modifying the usual quantum double \( D[G] \) by introducing a phase into the multiplication and comultiplication. The results of [5] describe why this modification gives rise to a quasi-Hopf algebra, and that studying its representations is interesting.

Various attempts to understand this quasi-Hopf algebra have been made. In [30], Willerton showed that the twisted Drinfeld double could be thought of as a twisted groupoid algebra in a manner similar to the way that the non-twisted Drinfeld double can be thought of as a groupoid algebra. This perspective is useful when considering Chern-Simons theory.

Majid in [23], and Hausser and Nill in [13, 12], developed a double construction for quasi-Hopf algebras. Majid considers the double of a quasi-Hopf algebra as constructed from the automorphisms of the forgetful functor. Hausser and Nill present more explicit formulae and use the bicrossed product. Specifically, for a quasi-Hopf algebra \( \mathcal{A} \), we have \( D(\mathcal{A}) = \mathcal{A}^* \bowtie \mathcal{A} \). They then prove that the Dijkgraaf-Pasquier-Roche algebra is the double of the quasi-Hopf algebra \( \mathcal{A} = \text{Fun}(G) = (k[G])^* \) of
functions on $G$. This $A$ has its standard coproduct, counit and antipode, and coassociator given by

$$\Phi = \sum_{x,y,z \in G} \omega(x,y,z)^x \otimes y \otimes z$$

(3.4.1)

for $\omega : G \times G \times G \to U(1)$ a normalized 3-cocycle. This means that

$$\omega(x,y,z) = 1$$

(3.4.2)

whenever at least one of $x, y, z = e \in G$, and

$$\omega(x,y,z)\omega(tx,y,z)^{-1}\omega(t,xy,z)\omega(t,x,yz)^{-1}\omega(t,x,y) = 1$$

(3.4.3)

for every $t, x, y, z \in G$. We refer to this cocycle as the "twist".

**Remark 3.4.1.** If we have $\omega(x,y,z) = 1$ for all triples $(x,y,z) \in G^3$, then $\Phi = 1 \otimes 1 \otimes 1$, so that the resulting quasi-Hopf algebra is in fact an ordinary Hopf algebra, and will be called "non-twisted". In this case, we say that $\omega$ is trivial.

This construction of the Dijkgraaf-Pasquier-Roche algebra as the double of the quasi-Hopf algebra of functions on the group algebra gives us a uniqueness for $D^\omega[G]$: given a particular finite group $G$ and a 3-cocycle $\omega$, $D^\omega[G]$ is the only quasitriangular quasi-Hopf algebra constructed in this fashion. The construction also gives us that the original quasi-Hopf algebra $A$ is a sub-quasi-Hopf algebra of the double. The dual space $A^*$ is not a quasi-Hopf algebra, however, though we still have a map $A^* \to D(A)$ sending $x \mapsto x \otimes 1_A$.

**Remark 3.4.2.** We would like for the quasi-Hopf algebra $D^\omega[G]$ to reduce to the ordinary Hopf algebra $D[G]$ in the case that $\omega$ is trivial. For this reason, instead of working with the original Dijkgraaf-Pasquier-Roche algebra, we will work with its opposite. See [11]. The opposite structure includes the opposite coproduct, defined by composition with the flip map. The opposite coproduct is necessary for the desired reduction.
As in the original paper of Dijkgraaf, Pasquier, and Roche, we present the definitions of the various maps. Suppose we are given a 3-cocycle \( \omega \) as above, satisfying (3.4.3) and (3.4.2).

As in Chapter 2, the underlying vector space for \( D^\omega[G] \) is \( A \otimes H \) for \( H = k[G] \) and \( A = (H^\text{op})^* \). We then define the operations on basis elements of the form \( g h \).

We will need the following coefficients, constructed from products of the cocycle \( \omega \).

\[
\theta(g, h, k) = \omega(g, h, k)\omega(h, k, (hk)^{-1}g(hk))\omega(h, h^{-1}gh, k)^{-1}
\]

(3.4.4)

\[
\gamma(g, h, k) = \omega(h, k, g)\omega(g, g^{-1}hg, g^{-1}kg)\omega(h, g, g^{-1}kg)^{-1}
\]

(3.4.5)

Note that if any one of \( g, h, k \) is the identity \( e \in G \), then \( \theta(g, h, k) = 1 = \gamma(g, h, k) \).

**Remark 3.4.3.** The context should clarify whether we mean the coefficient \( \gamma(g, h, k) \) or the element \( \gamma \in (D^\omega[G])^{\otimes 2} \) defined in (3.2.6).

To shorten the notation, we use the following.

\[
\omega_g = \omega(g, g^{-1}, g)
\]

(3.4.6)

**Lemma 3.4.4.** The following relation holds.

\[
\omega_g^{-1} = \omega_{g^{-1}}
\]

(3.4.7)

**Proof.** By the cocycle condition (3.4.3), we have the following.

\[
\omega(g, g^{-1}, g)\omega(g^{-1}g, g^{-1}, g)^{-1}\omega(g^{-1}, gg^{-1}, g)\omega(g^{-1}, g, g^{-1}g)^{-1}\omega(g^{-1}, g, g^{-1}) = 1
\]

(3.4.8)

Using the notation defined in (3.4.6) and the fact that if any of \( x, y, z = e \) then \( \omega(x, y, z) = 1 \), (3.4.8) simplifies to the following.

\[
\omega_g \cdot 1 \cdot 1 \cdot \omega_{g^{-1}} = 1 \in k
\]

(3.4.9)

Hence, \( \omega_g^{-1} = \omega_{g^{-1}} \), as desired. \( \square \)
We now define the coassociator, though we take the opposite of the coassociator given in the introduction to this chapter.

**Definition 3.4.5.** [5] Let $\omega$ be the given 3-cocycle. The element $\Phi \in D^\omega[G]^\otimes 3$ and its inverse are given by the following.

\[
\Phi = \sum_{g,h,k \in G} \omega(k, h, g) g \downarrow_e \otimes h \downarrow_e \otimes k \downarrow_e \quad (3.4.10)
\]

\[
\Phi^{-1} = \sum_{g,h,k \in G} \omega(k, h, g)^{-1} g \downarrow_e \otimes h \downarrow_e \otimes k \downarrow_e \quad (3.4.11)
\]

**Lemma 3.4.6.** Together with the coassociator given in Definition 3.4.5, the following maps make $D^\omega[G]$ into a quasi-Hopf algebra. Let $A$ denote $(k[G])^*$, the space of functions on $G$, and $1_A$ denote the identity element in $A$, given by $\sum_{g \in G} \delta_g$.

**Multiplication:** $g \downarrow_h x \downarrow_y = \theta(g, h, y) \delta_{g, h, hx^{-1}} g \downarrow_{hy}$

**Unit:** $1_A \otimes e = \sum_{g \in G} g \downarrow_e$

**Comultiplication:** $\Delta(g \downarrow_h) = \sum_{uv=g} \gamma(h, u, v) v \downarrow_h \otimes u \downarrow_h$

**Counit:** $\epsilon(g \downarrow_h) = \delta_{g,e}$

**Antipode:** $S(g \downarrow_h) = \theta(g, h, h^{-1})^{-1} \gamma(h, g^{-1}, g)^{-1}(h^{-1}g^{-1}h) \downarrow_{h^{-1}}$

\[
\alpha = 1 \quad (3.4.17)
\]

\[
\beta = \sum_{g \in G} \omega^{-1} g \downarrow_e \quad (3.4.18)
\]

**Proof.** This is the opposite structure applied to the Dijkgraaf-Pasquier-Roche algebra given in [5]. For convenience and familiarity with the notation, we highlight some aspects of the proof. Let $g \downarrow_h$, $x \downarrow_y$ and $w \downarrow_z$ be arbitrary basis elements in $D^\omega[G]$.

In order to show that multiplication is associative, we must have the following.

\[
(g \downarrow_h x \downarrow_y) w \downarrow_z = g \downarrow_h (x \downarrow_y w \downarrow_z) \quad (3.4.19)
\]
Using the definition of multiplication inside the parenthesis on either side of the equality in (3.4.19), we find the following.

$$\theta(g, h, y)\delta_{g,hxh^{-1}} g \frac{g}{h} \frac{g}{y} = \theta(x, y, z)\delta_{x,gyy^{-1}} g \frac{x}{h} \frac{y}{z}$$  \hspace{1cm} (3.4.20)

Applying the definition of multiplication on (3.4.20), we see the following.

$$\theta(g, h, y)\theta(g, hy, z)\delta_{g,hxh^{-1}} \delta_{g,hyw(hy)^{-1}} g \frac{g}{hyz} = \theta(x, y, z)\theta(g, hy, z)\delta_{x,gyy^{-1}} \delta_{g,hyw(hy)^{-1}} g \frac{g}{hyz}$$  \hspace{1cm} (3.4.21)

Because of the $\delta$-coefficients, this is only nonzero on the following element.

$$\theta(g, h, y)\theta(g, hy, z) g \frac{g}{hyz} = \theta(h^{-1}gh, y, z)\theta(g, hy, z) g \frac{g}{hyz}$$  \hspace{1cm} (3.4.22)

That is, we must have

$$\theta(g, h, y)\theta(g, hy, z) = \theta(h^{-1}gh, y, z).$$  \hspace{1cm} (3.4.23)

We may verify (3.4.23) using the cocycle condition (3.4.3) and the definition of $\theta$.

Similarly, in order to show that the coproduct is quasi-coassociative (that is, to verify equation (3.0.4)), we need the relation

$$\gamma(x, g, h)\gamma(x, gh, k)\omega(x^{-1}gx, x^{-1}hx, x^{-1}kx) = \gamma(x, h, k)\gamma(x, g, hk)\omega(g, h, k).$$  \hspace{1cm} (3.4.24)

As in the case of the associativity of multiplication, this relation is verified using the cocycle condition (3.4.3) and the definition of the coefficient $\gamma$. Furthermore, we need the comultiplication map to be an algebra morphism. This requires the formula

$$(\mu \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)(g \frac{g}{h} \frac{x}{y}) = \Delta(\mu(g \frac{g}{h} \frac{x}{y})),$$  \hspace{1cm} (3.4.25)

which results in a condition on the maps $\gamma$ and $\theta$ in the form of

$$\theta(g, x, y)\theta(h, x, y)\gamma(x, g, h)\gamma(y, x^{-1}gx, x^{-1}hx) = \theta(gh, x, y)\gamma(xy, g, h).$$  \hspace{1cm} (3.4.26)
As with the other equations of this type that we have encountered, this is verified by using the cocycle condition (3.4.3) as well as the definitions of \( \theta \) and \( \gamma \) in (3.4.4) and (3.4.5).

**Remark 3.4.7.** The element \( \theta \) gives us the bicrossing from the double construction, as discussed in [12].

At this juncture, it is useful to note several additional relations. First, it is convenient to have an explicit formula for the inverse of the antipode.

**Lemma 3.4.8.** The inverse of the antipode map is given by the following. Let \( g \in D^w[G] \) be an arbitrary basis element.

\[
S^{-1}(g \downarrow_h) = \theta(g^{-1}, h, h^{-1})^{-1} \gamma(h, g, g^{-1})^{-1} (h^{-1}g^{-1}h) \downarrow_{h^{-1}}.
\] (3.4.27)

**Proof.** By computation. We verify that \( S \circ S^{-1}(g \downarrow_h) = g \downarrow_h = S^{-1} \circ S(g \downarrow_h). \)

We have some additional relations amongst the maps \( \omega, \theta, \) and \( \gamma \).

**Lemma 3.4.9.** The following relations hold.

\[
\gamma(y, x, x^{-1})\omega_{y^{-1}x^{-1}}\gamma(y, x^{-1}, x)^{-1}\omega_{x}^{-1} = 1
\] (3.4.28)

\[
\theta(ghg^{-1}, g, g^{-1}) = \theta(h, g^{-1}, g)
\] (3.4.29)

**Proof.** By computation. Recall the definition of \( \omega_x \) in (3.4.6). Equation (3.4.28) is derived from (3.4.23), (3.4.24), (3.4.26) and the cocycle relation. Equation (3.4.29) is verified using the definition of the element \( \theta \).

**Lemma 3.4.10.** The square of the antipode is given by the following.

\[
S^2(g \downarrow_h) = \omega g \omega_{h^{-1}g^{-1}h} g \downarrow_h.
\] (3.4.30)

**Proof.** By computation using (3.4.28).
The most frequent use of (3.4.29) is to cancel factors of \( \theta(g^{-1}, h, h^{-1})^{-1} \) which occur with an application of the antipode to the element \( \sigma_h \). Some additional properties of the antipode are as follows.

**Lemma 3.4.11.** We have \( S(\beta) = \beta^{-1} \).

**Proof.** By computation.

\[
S(\beta) = S\left( \sum g^{-1}_e \sigma_g \right) = \sum g^{-1}_e S(\sigma_g) = \sum g^{-1}_e g^{-1} \sigma_g = \sum g^{-1}_e \omega_g \sigma_e = \beta - 1
\]

Note that in the final step we have simply re-indexed the sum over \( g^{-1} \). We claim that \( \sum g\sigma_e = \beta^{-1} \). Multiplying on the right we see the following.

\[
\beta \cdot \sum g \sigma_e = \sum g^{-1} \omega_h \sigma_h \sigma_g = \sum g^{-1} \omega_h \delta_{h,g} \sigma_g = \sum g^{-1} \omega_g \sigma_g = \sum g \sigma_e = 1
\]

Multiplying on the left we also have the desired result.

\[
\sum g \sigma_e \cdot \beta = \sum g^{-1} \omega_h \sigma_g = \sum g^{-1} \omega_h \delta_{h,g} \sigma_g = \sum g^{-1} \omega_g \sigma_e = \sum g \sigma_e = 1
\]

Hence, \( S(\beta) = \beta^{-1} \), as desired.

**Lemma 3.4.12.** The following relation holds for every \( h \in D^e[G] \).

\[
S^2(h) = \beta^{-1} h \beta
\]
Proof. We have from (3.2.18) in Lemma 3.2.5 that

\[ S^2(h) = GhG^{-1} \quad (3.4.42) \]

for the special element \( G \) with \( G = uv^{-1} \). In this case, we define the element \( v = \beta u \) (see Lemma 3.4.16) so that \( G = uv^{-1} = uu^{-1}\beta^{-1} = \beta^{-1} \). The direct consequence of this is that (3.4.42) becomes

\[ S^2(h) = \beta^{-1}h\beta, \quad (3.4.43) \]

as desired. \( \square \)

We now turn to the quasitriangularity of \( D^\omega[G] \). Recall that

\[ \sum_g g\downarrow_h = 1_A \otimes h \in D^\omega[G]. \quad (3.4.44) \]

**Lemma 3.4.13.** The quasi-Hopf algebra \( D^\omega[G] \) is a quasitriangular, with

\[ R = \sum_{g,h \in G} g\downarrow_h \otimes h\downarrow_e \quad (3.4.45) \]

\[ R^{-1} = \sum_{g,h \in G} \theta(h^{-1}gh, h^{-1}, h)^{-1} g\downarrow_h \otimes h^{-1}\downarrow_e \quad (3.4.46) \]

**Proof.** The definition of the \( R \)-matrix is given in [5]. The three conditions of Definition 3.1.9 must be satisfied. First, we show that (3.1.16) holds. Let \( x \downarrow_y \) be an arbitrary basis element of \( D^\omega[G] \). We show that \( \Delta^{op}(x) = R\Delta(x)R^{-1} \).

\[ R\Delta(x \downarrow_y)R^{-1} = \sum_{uv=x} \gamma(y, u, v)\theta(n^{-1}mn, n^{-1}, n)^{-1} \]

\[ g\downarrow_h \otimes h\downarrow_{ue} \quad (3.4.47) \]

\[ = \sum_{uv=x} \gamma(y, u, v)\theta(n^{-1}mn, n^{-1}, n)^{-1}\theta(g, h, y)\theta(h, e, y)\delta_{h,hev^{-1}}\delta_{h, u} \]

\[ g\downarrow_{hy} \otimes h\downarrow_{n^{-1}e} \quad (3.4.48) \]

39
\[
= \sum_{uv=x} \gamma(y, u, v)\theta(n^{-1}mn, n^{-1}, n)^{-1}\theta(uvu^{-1}, u, y)
\]

\[
\theta(u, y, e)\delta_{uvu^{-1}, uyyn^{-1}y^{-1}y} \otimes u \quad (3.4.49)
\]

\[
= \sum_{uv=x} \gamma(y, u, v)\theta(n^{-1}mn, n^{-1}, n)^{-1}\theta(uvu^{-1}, u, y)\theta(uvu^{-1}uy, n)
\]

\[
\theta(u, y, e)\delta_{uvu^{-1}, uyyn^{-1}y^{-1}y} \otimes u \quad (3.4.50)
\]

\[
= \sum_{uv=x} \gamma(y, u, v)\theta(yvu^{-1}y, y^{-1}uy, y^{-1}u^{-1}y)^{-1}\theta(uvu^{-1}, u, y)
\]

\[
\theta(uvu^{-1}, uy, y^{-1}u^{-1}y) \otimes u \quad (3.4.51)
\]

We make a substitution to clarify. Let \( k = uvu^{-1} \), so that \( v = u^{-1}ku \). Then \( ku = x \), so that (3.4.51) becomes the following.

\[
= \sum_{ku=x} \gamma(y, u, u^{-1}ku)\theta(y^{-1}ky, y^{-1}uy, y^{-1}u^{-1}y)^{-1}\theta(k, u, y)
\]

\[
\theta(k, uy, y^{-1}u^{-1}y) k \otimes u \quad (3.4.52)
\]

We would like to see that (3.4.52) is equal to \( \Delta^\text{op}(x \underline{\quad/y}) \). The two tensor factors of (3.4.52) are the two tensor factors of \( \Delta^\text{op}(x \underline{\quad/y}) \), as \( ku = x \). We need to see that

\[
\gamma(y, u, u^{-1}ku)\theta(y^{-1}ky, y^{-1}uy, y^{-1}u^{-1}y)^{-1}\theta(k, u, y)\theta(k, uy, y^{-1}u^{-1}y) = \gamma(y, k, u).
\]

(3.4.53)

Toward that end, (3.4.23) may be rewritten as

\[
\theta(k, y, y^{-1}uy)\theta(k, uy, y^{-1}u^{-1}y) = \theta(k, y, e)\theta(y^{-1}ky, y^{-1}uy, y^{-1}u^{-1}y).
\]

(3.4.54)

Then, (3.4.53) becomes

\[
\gamma(y, u, u^{-1}ku)\theta(k, u, y)\theta(k, y, y^{-1}uy)^{-1} = \gamma(y, k, u).
\]

(3.4.55)

The relation (3.4.55) holds by the definitions of \( \theta \) and \( \gamma \), as all terms cancel. Thus, (3.1.16) holds as desired.
Next, we verify (3.1.17), and first consider the left-hand side of the equation.

\[
(\Delta \otimes \text{id})(R) = (\Delta \otimes \text{id})(\sum g | h \otimes h | e) \\
= \sum \Delta(g | h) \otimes h | e \\
= \sum_{g,h \in G, uv = g} \gamma(h, u, v) | v \otimes u | h \otimes h | e
\]  

(3.4.56)  
(3.4.57)  
(3.4.58)

On the other hand, the right-hand side of the equation is as follows.

\[
\Phi_{312} R_{13} \Phi_{132} \Phi = \sum \omega(c, b, a) \omega(z, y, x)^{-1} \omega(t, s, r)^{b | h \otimes h | e} \\
\times c | z | k | e \otimes a | y | l | e \\
= \sum \omega(c, b, a) \omega(z, y, x)^{-1} \omega(t, s, r) \delta_{b, g} \delta_{x, z} \delta_{c, k} \delta_{k, l} \delta_{a, h} \delta_{y, l}
\]

(3.4.59)  
(3.4.60)

\[
= \sum (3.4.58) \text{ and (3.4.64) are the same element in } D^\omega[G], \text{ as we sum over all elements of } G \text{ for each index, so (3.1.17) is verified.}
\]
Finally, we verify (3.1.18), again by starting with the left-hand side of the equation.

\[(\text{id} \otimes \Delta)(\mathcal{R}) = (\text{id} \otimes \Delta)(\sum g \mathcal{L}_h \otimes h \mathcal{L}_e) \quad \text{(3.4.65)}\]

\[= \sum g \mathcal{L}_e \otimes \Delta(h \mathcal{L}_e) \quad \text{(3.4.66)}\]

\[= \sum_{uv=h} g \mathcal{L}_e \otimes v \mathcal{L}_e \otimes u \mathcal{L}_e \quad \text{(3.4.67)}\]

Note that the coefficient is \(\gamma(e, u, v) = 1\). On the other hand, the right-hand side of the equation is as follows.

\[\Phi^{-1}_{211} \mathcal{R}_{13} \Phi_{213} \mathcal{R}_{12} \Phi^{-1} = \sum \omega(c, b, a)^{-1} \omega(z, y, x) \omega(t, s, r)^{-1} c \mathcal{L}_h y \mathcal{L}_e k \mathcal{L}_l r \mathcal{L}_e \quad \text{(3.4.68)}\]

\[= \sum \omega(c, b, a)^{-1} \omega(z, y, x) \omega(t, s, r)^{-1} \delta_{c,g} \delta_{y,k} \delta_{a,x} \delta_{l,s} \delta_{b,h} \delta_{z,t} \quad \text{(3.4.69)}\]

\[= \sum \omega(g, h, a)^{-1} \omega(z, k, a)^{-1} \omega(z, l, r)^{-1} g \mathcal{L}_h k \mathcal{L}_l r \mathcal{L}_e \quad \text{(3.4.70)}\]

\[= \sum \omega(g, h, l)^{-1} \omega(h, l)^{-1} \omega(h, l^{-1} h^{-1}) \omega(h^{-1} gh, l) \theta(g, h, l) \quad \text{(3.4.71)}\]

\[= \sum_{g, h, l \in G} \theta(g, h, l)^{-1} \theta(g, h, l) g \mathcal{L}_{hl} \otimes l \mathcal{L}_e \otimes h \mathcal{L}_e \quad \text{(3.4.72)}\]

Now (3.4.67) and (3.4.73) are the same element of \(D^\omega[G]\), so (3.1.18) is verified. Thus the definition of \(\mathcal{R}\) given in (3.4.45) does in fact give rise to a universal \(\mathcal{R}\)-matrix for the quasi-Hopf algebra \(D^\omega[G]\). We may verify the definition of \(\mathcal{R}^{-1}\) by multiplication. \(\square\)
With a specific definition for the $R$-matrix, we may now calculate our special elements.

**Lemma 3.4.14.** The element $u$ is given by

$$u = \sum_{g \in G} g \frac{1}{g^{-1}}. \quad (3.4.74)$$

*Proof.* By computation, using the specific definition of $R$ in (3.4.45) and the first equality in (3.2.4). Recall that $\alpha = 1$, making this the most convenient formula to use. We could alternatively calculate $u$ using the general formula (3.2.1). \qed

**Lemma 3.4.15.** The element $f$ is equal to the element $\gamma \in (D^\omega[G])^\otimes 2$. Specifically,

$$f = \sum_{g,h} \omega(g^{-1}, g, h) \omega(h^{-1}, g^{-1}, gh)^{-1} g \frac{1}{e} \otimes h \frac{1}{e}. \quad (3.4.75)$$

*Proof.* The equation $f = \gamma$ is true since we know $\gamma = f \Delta(\alpha)$ and $\alpha = 1$. See [1]. We then calculate this element for $f$ using (3.2.6). \qed

**Lemma 3.4.16.** We take $G = \beta^{-1}$, so that $v = G^{-1}u$ is given by

$$v = \sum_{g \in G} \omega^{-1}\frac{1}{g} g \frac{1}{g^{-1}}. \quad (3.4.76)$$

*Proof.* The statement of this result is in [1]. We may also directly check that the properties of Definition 3.2.3 are satisfied. \qed

We then have that

$$v^{-1} = \sum_{g \in G} g \frac{1}{g}. \quad (3.4.77)$$
Lemma 3.4.17. Our other special elements are as follows.

\[
M = \sum_{x,y} x \downarrow \otimes \frac{xyx^{-1}}{x} \quad (3.4.78)
\]

\[
p_L = \sum_{x,y} \omega_x \omega(y, x, x^{-1}) \frac{x}{e} \otimes \frac{y}{e} \quad (3.4.79)
\]

\[
p_R = \sum_{x,y} \omega_y \omega(y^{-1}, y, x)^{-1} \frac{x}{e} \otimes \frac{y}{e} \quad (3.4.80)
\]

\[
q_L = \sum_{x,y} \omega(y, x, x^{-1})^{-1} \frac{x}{e} \otimes \frac{y}{e} \quad (3.4.81)
\]

\[
q_R = \sum_{x,y} \omega(y^{-1}, y, x) \frac{x}{e} \otimes \frac{y}{e} \quad (3.4.82)
\]

The element \(M\) is non-degenerate.

Proof. By computation on previous formulae. To see that \(M\) is non-degenerate, consider the map given by the following.

\[
\overline{M} : D^\omega[G]^* \to D^\omega[G] \quad (3.4.83)
\]

\[
l \mapsto (\text{id} \otimes l)(M) \quad (3.4.84)
\]

\[
l \mapsto \sum_{x,y} x \downarrow \frac{xyx^{-1}}{x} \quad (3.4.85)
\]

Now \(x \downarrow \frac{y}{y}\) is clearly a basis element in \(D^\omega[G]\). Hence \(\overline{M}\) is an injection. Since \(G\) is a group and \(k\) is a field, \(\overline{M}\) is also an isomorphism.

We take the following as our definition of the integral and cointegral.

Lemma 3.4.18. The element

\[
\Lambda = \sum_y e \downarrow \frac{y}{y} \quad (3.4.86)
\]

is a left and right integral in \(D^\omega[G]\). The map

\[
\lambda(x \downarrow \frac{y}{y}) = \omega_x^{-1} \delta_y, e \quad (3.4.87)
\]
is a right cointegral in $D^\omega[G]$. The trace map corresponding to this integral is

$$\text{tr}(\underline{x}_y) = \lambda(G \underline{x}_y) = \delta_{y,e}.$$ \hfill (3.4.88)

**Proof.** To verify that (3.4.86) gives both a left and right integral in $D^\omega[G]$, let $g \underline{\_}_h$ be an arbitrary basis element. We have the following.

$$\Lambda g \underline{\_}_h = \sum_y e \underline{\_}_y g \underline{\_}_h \quad (3.4.89)$$

$$= \sum_y \theta(e, y, h) \delta_{e, yg^{-1}} e \underline{\_}_y \quad (3.4.90)$$

$$= \sum_y \delta_{g, e} e \underline{\_}_y \quad (3.4.91)$$

$$= \epsilon(g \underline{\_}_h) \Lambda \quad (3.4.92)$$

Note that in the last equality, we have used the fact that summing over all $y \in G$ is equivalent to summing over all $yh \in G$ for fixed $h \in G$. Thus, $\Lambda$ is a right integral.

To see that $\Lambda$ is also a left integral, consider the following.

$$g \underline{\_}_h \Lambda = \sum_y g \underline{\_}_y e \underline{\_}_y \quad (3.4.93)$$

$$= \sum_y \theta(g, h, y) \delta_{g, e} g \underline{\_}_h \quad (3.4.94)$$

$$= \sum_y \theta(g, h, y) \delta_{g, e} g \underline{\_}_h \quad (3.4.95)$$

$$= \epsilon(g \underline{\_}_h) \Lambda \quad (3.4.96)$$

Next, we must verify that (3.4.87) defines a right cointegral. By Lemma 3.3.13, we must verify that

$$(\lambda \otimes \text{id})(q^L \Delta(h)p^L) = 1 \cdot \lambda(\alpha h S^{-1}(\beta)) \quad (3.4.97)$$

for any $h \in D^\omega[G]$. We have previously calculated that

$$p^L = \sum_{a,b} \omega_a \omega(b, a, a^{-1}) a \underline{\_}_e \otimes b \underline{\_}_e \quad (3.4.98)$$
and
\[
q_L = \sum_{x,y} \omega(y, x, x^{-1})^{-1} x \downarrow_e \otimes y \downarrow_e.
\] (3.4.99)

Again let \( g \downarrow_h \) be an arbitrary basis element in \( D^\omega[G] \). We first consider the left-hand side of the equality.

\[
(\lambda \otimes 1)(q_L \Delta(g \downarrow_h) p_L)
\] (3.4.100)
\[
= (\lambda \otimes 1)(\sum_{uv=g} \omega(y, x, x^{-1})^{-1} \omega_h \omega(b, a, a^{-1}) \gamma(h, u, v)

\quad x \downarrow_e \otimes y \downarrow_e \otimes u \downarrow_h \otimes b \downarrow_e).
\] (3.4.101)
\[
= (\lambda \otimes 1)(\sum_{uv=g} \omega(u, v, v^{-1})^{-1} \omega_{h^{-1}vh} \omega(h^{-1}uh, h^{-1}vh, h^{-1}v^{-1}h) \gamma(h, u, v)

\quad u \downarrow_h \otimes u \downarrow_h).
\] (3.4.102)
\[
= \sum_{uv=g} \omega(u, v, v^{-1})^{-1} \omega_{h^{-1}vh} \omega(h^{-1}uh, h^{-1}vh, h^{-1}v^{-1}h) \gamma(h, u, v)

\quad \omega_v^{-1} \delta_{h,e} u \downarrow_h.
\] (3.4.103)
\[
= \sum_{u} u \downarrow_e = 1
\] (3.4.104)

Now consider the right-hand side of the equality.

\[
1 \cdot \lambda(\alpha(g \downarrow_h) S^{-1}(\beta)) = \lambda(g \downarrow_h \beta^{-1})
\] (3.4.105)
\[
= 1 \cdot \omega_k \lambda(g \downarrow_h k \downarrow_e)
\] (3.4.106)
\[
= 1 \cdot \omega_{h^{-1}gh} \lambda(g \downarrow_h)
\] (3.4.107)
\[
= 1 \cdot \omega_{h^{-1}gh} \omega_g^{-1} \delta_{h,e} = 1
\] (3.4.108)

Hence, the two sides are equal. \( \square \)
CHAPTER 4
TANGLE CATEGORIES

As discussed in the introduction, the TQFT in which we are primarily interested is the Hennings TQFT, which we shall define shortly. In essence, the Hennings TQFT generalizes the Hennings invariant on knots and links, which can be described in terms of a functor on tangle categories.

We define various tangle categories, as well as their relevant properties, for use in the following sections.

4.1 The Category $\mathcal{Tgl}$ of Admissible Tangles

**Definition 4.1.1.** Let $n$ and $m$ be integers. A framed tangle $T : m \rightarrow n$ is a framed embedding of a union of circles and intervals in $\mathbb{R}^2 \times [0, 1]$ with $m$ strands attached to the top of the diagram and $n$ strands attached to the bottom of the diagram. The set of tangles is generated by the maps $\cap : 0 \rightarrow 2$, $\cup : 2 \rightarrow 0$, $c : 2 \rightarrow 2$ and $c^{-1} : 2 \rightarrow 2$, as well as a straight strand labeled $\text{id} : 1 \rightarrow 1$. These generators are pictured in Figure 4.1.

**Remark 4.1.2.** Instead of considering embeddings in $\mathbb{R}^2 \times [0, 1]$, we may take a generic immersion of a union of circles and intervals in $\mathbb{R} \times [0, 1]$ with over- and under-crossing information at double points. This second definition is mapped to the
first by pushing strands off each other at double points using the over- or under-crossing information and the blackboard framing.

**Definition 4.1.3.** The usual tangle category will be denoted $TGL$. Its objects are given by integers, and its morphisms in $\text{Hom}(m,n)$ are given by equivalence classes of framed tangles with $m$ top end points and $n$ bottom end points, as defined in Definition 4.1.1. The equivalence classes are generated by isotopies in the plane, as well as Moves I, II, III, IV, and R, pictured in Figure 4.2. The tensor product is given by juxtaposition, and the composition is given by stacking, with $T \circ S$ given by stacking $S$ on top of $T$.

**Remark 4.1.4.** Contrary to the usual convention, we read our tangle diagrams from the top to the bottom. This author feels that while composition is easier to understand with bottom-up diagrams, the top-down variety are easier to understand at a glance.

We are interested not in the full category $TGL$, but in a subcategory $Tgl$ of admissible framed tangles. To define these admissible tangles, we first quotient by extra equivalence relations.

**Definition 4.1.5.** We say that two equivalent tangles $T$ and $T'$ are admissibly equivalent if they are related by a sequence of the following additional moves.

1. The $\sigma$ Move, pictured in Figure 4.2

2. The Modified First Kirby Move, or cancellation of an isolated pair of circles, one with $+1$ framing and the other with $-1$

3. The Second Kirby Move, namely slides of any component over internal components.
Definition 4.1.6. The category $T_{gl}$ of admissible tangles is defined as follows. The objects of $T_{gl}$ are even integers. The object $2n$ will be written as a sequence

$$1^-, 1^+, ..., n^-, n^+.$$ (4.1.1)

The morphisms in $\text{Hom}(2m, 2n)$ of $T_{gl}$ are equivalence classes of tangles with $2m$ top end points and $2n$ bottom end points, where the end points $(j^-, j^+)$ are either connected by a single component of the tangle, or are connected via two components to the pair $(i^-, i^+)$ at the opposite end of the tangle. There may also be closed components of the tangle, isomorphic to $S^1$. The equivalence of admissible tangles is given in Definition 4.1.5.

The tensor product of morphisms is given by juxtaposition, and the composition of morphisms is given by stacking, with $T \circ S$ given by stacking $S$ on top of $T$.

![Figure 4.1: Generators for $T_{gl}$](image)

Remark 4.1.7. In Figure 4.2 and similar diagrams, there is no difference between tangles with smooth lines, as in Move IV and those with angular lines as in Move III.

Moves I, II, III, and IV are named in [17], where Move I is called "=". For more information on the $\sigma$-move, see [18].

49
Figure 4.2: Equivalence of Tangles
4.2 The Categories FLAT and DTgl

**Definition 4.2.1.** Let \( n \) and \( m \) be integers. A flat tangle \( T : m \rightarrow n \) is a generic immersion of a union of circles and intervals in \( \mathbb{R} \times [0,1] \) with \( m \) strands attached to the top of the diagram and \( n \) strands attached to the bottom of the diagram. Here, double points have no over- or under-crossing information.

The set of flat tangles is generated by the maps \( \cap : 0 \rightarrow 2 \), \( \cup : 2 \rightarrow 0 \), and \( \times : 2 \rightarrow 2 \), as well as a straight strand labeled \( \text{id} : 1 \rightarrow 1 \). These generators are pictured in Figure 4.3.

**Definition 4.2.2.** We say that two flat tangles \( T \) and \( T' \) are equivalent if they are related by a sequence of the moves pictured in Figure 4.4.

![Figure 4.3: Generators for FLAT](image)

**Definition 4.2.3.** The category FLAT of flat tangles is defined as follows. The objects of FLAT are integers.

The morphisms of FLAT are equivalence classes of flat tangles, as defined in Definitions 4.2.1 and 4.2.2.

The tensor product of morphisms is given by juxtaposition, and the composition of morphisms is given by stacking, where \( T \circ S \) means we stack \( S \) on top of \( T \).
Definition 4.2.4. Let $n$ and $m$ be integers. A decorated, flat tangle $T : m \to n$ is a generic immersion of a union of circles and intervals in $\mathbb{R} \times [0, 1]$ which are decorated with beads, with $m$ strands attached to the top of the diagram and $n$ strands attached to the bottom of the diagram. The beads will be decorated with elements of a Hopf algebra $H$. Again, double points have no information about over- or under-crossing.

The set of decorated, flat tangles is generated by the maps $\cup, \cap, \times$, a straight strand labeled $id$, and straight strand decorated with a bead. Any bead labeled by $1 \in H$ may be omitted. These generators are pictured in Figure 4.5.

Definition 4.2.5. Two decorated, flat tangles $T$ and $T'$ are equivalent if they are related by a sequence of the moves pictured in Figure 4.4 and Figure 4.6.
Definition 4.2.6. The category $DTgl$ of decorated, flat tangles is defined as follows.

The objects of $DTgl$ are integers.

The morphisms of $DTgl$ are formal sums of equivalence classes of decorated flat tangles, as defined in Definitions 4.2.4 and 4.2.5.

The tensor product of morphisms is given by juxtaposition, and the composition of morphisms is given by stacking, where $T \circ S$ means we stack $S$ on top of $T$.

![Figure 4.5: Generators for $DTgl$](image)

![Figure 4.6: Equivalent Beaded Tangles in $DTgl$](image)
In order to clarify the situation in which the elements \( h \in H \) are expressed as sums, the following remark may be useful.

**Remark 4.2.7.** An equivalent definition of \( DTgl \) is given in [4] as follows. Let \( H \) be a Hopf algebra. Let the objects of \( DTgl \) be integers. The morphisms of \( DTgl \) are given by equivalence classes of pairs \( (D,a) \), where \( D \) is a tangle in \( FLAT \) with \( N \) ordered markings and \( a \) is an element in \( H^{\otimes N} \). If

\[
a = \sum_{\nu} a_1^{\nu} \otimes ... \otimes a_N^{\nu}
\]  

we write \((D,a)\) as a formal linear combination, also with summation index \( \nu \), of the same tangle in \( FLAT \) marked with the same \( N \) ordered markings, where the label at the \( j \)-th marking of the \( \nu \)-th diagram is \( a_j^{\nu} \).

The relations between such tangles are still those pictured in Figure 4.4 and 4.6.

### 4.3 The Category \( Tgl^{(0)} \)

The categories \( Tgl \) and \( DTgl \) are examples of strict braided tensor categories. We can also view quasitriangular Hopf algebras as strict braided tensor categories, and we apply the elements of such a Hopf algebra to the strands of \( DTgl \). Quasi-Hopf algebras, on the other hand, are braided tensor categories, but they are not strict. See Section 3.1 for details. We would thus like to define a tangle category \( Tgl^{(0)} \) which is a braided tensor category, but not strict. We will then apply the elements of a quasi-Hopf algebra to the strands of \( DTgl \). We will build \( Tgl^{(0)} \) piece-by-piece to suit our needs.

First, we define a category of bracketed tangles, \( TGL^{(0)} \). As in the case of \( TGL \) and \( Tgl \), we will then quotient by additional relations to reduce to a subcategory of admissible bracketed tangles.
Definition 4.3.1. Let $n$ be an integer. By $\hat{n}$, we mean a sequence of $n$ elements which have been bracketed in pairs, as if to multiply. See Definition 4.3.2 for a specific example.

Definition 4.3.2. Let $n$ be an integer. By $\overline{2n}$, we mean a sequence of $2n$ elements which have been bracketed in adjacent pairs, and then the adjacent pairs are bracketed from the left. For example, labeling the $2n$ elements as $1^-, 1^+, ..., n^-, n^+$, we have the following.

$$\overline{2n} = (......((1^-, 1^+), (2^-, 2^+)), ..., (n^-, n^+)) \quad (4.3.1)$$

Definition 4.3.3. Let $\hat{n}$ and $\hat{m}$ be bracketed integers, as in Definition 4.3.1. A bracketed (framed) tangle $T: \hat{m} \to \hat{n}$ is a framed embedding of a union of circles and intervals in $\mathbb{R}^2 \times [0,1]$ which are decorated with parenthesis indicating a bracketing, and with $m$ bracketed strands attached to the top of the diagram and $n$ bracketed strands attached to the bottom of the diagram. At the source and target objects, the strands must be bracketed to match the bracketing on the objects.

The set of bracketed tangles is generated by the maps $\cap: \hat{0} \to \hat{2}$, $\cup: \hat{2} \to \hat{0}$, $c: \hat{2} \to \hat{2}$, $c^{-1}: \hat{2} \to \hat{2}$, a pair of straight strands labeled $\text{id}$, and a change of parenthesis $a: (1,2), 3 \to 1, (2,3)$ on three strands. The generator $a$ is pictured in Figure 4.7. The other generators are as depicted in Figure 4.1, except we add necessary parenthesis.

Remark 4.3.4. The bracketing on $\hat{2}$ must be $(1^-, 1^+)$. Hence, the generators $\cup$, $\cap$, $c$, and $c^{-1}$ must have matching brackets on the strands.

Remark 4.3.5. The category $TGL^{(1)}$ will have bracketed integers as objects and equivalence classes of bracketed tangles as morphisms. We proceed with understanding the equivalence relation.
We must have bracketed versions of the moves in $TGL$, given by Moves I - IV and Move R in Figure 4.2. Furthermore, the conditions of a braided tensor category must be satisfied. See [16]. We begin with the Pentagon and Hexagon Axioms.

A braided tensor category must have an associativity constraint $a$, here given by extending the generator $a$ to three arbitrary objects. We begin with the change of parenthesis on three individual strands in the diagram. To extend $a$ to a map $a : (\hat{l} \otimes \hat{n}) \otimes \hat{m} \rightarrow \hat{l} \otimes (\hat{n} \otimes \hat{m})$ we collapse each object into a single strand in the diagram. See Figure 4.8. After we have collapsed each object into a single strand, we change the parenthesis as depicted in Figure 4.7. Since the associativity constraint $a$ is not trivial, we have that the category cannot be a strict braided tensor category.

$$\left(\begin{array}{c}
\hat{n} \\
\hat{m}
\end{array}\right) \otimes (\hat{n} \otimes \hat{m}) = \left(\begin{array}{c}
\hat{n} \\
\hat{m}
\end{array}\right)$$

Figure 4.8: Collapsing Two Strands into One
The Pentagon Axiom, given for quasi-Hopf algebras in (3.1.7), is repeated in the present case in (4.3.2). Let $\hat{l}$, $\hat{m}$, $\hat{n}$, and $\hat{r}$ be four arbitrary objects in $TGL^{(i)}$.

\[
(\hat{l} \otimes \hat{m}) \otimes (\hat{n} \otimes \hat{r}) \quad (4.3.2)
\]

(\(\hat{l} \otimes (\hat{m} \otimes \hat{n})\) ) \otimes \hat{r} \longrightarrow \hat{l} \otimes (\hat{m} \otimes (\hat{n} \otimes \hat{r}))

On tangles, the diagram in (4.3.2) is equivalent to the equality of bracketed tangles given in Figure 4.9. To satisfy the Pentagon Axiom, we assume that the two tangles depicted in Figure 4.9 are equal, for any source objects $\hat{l}$, $\hat{m}$, $\hat{n}$ and $\hat{r}$.

Remark 4.3.6. In order to apply the map $a$ to objects of the form $\hat{m} \otimes \hat{n}$, we again combine strands as in Figure 4.8.

Remark 4.3.7. Because the Pentagon Axiom is satisfied, MacLane's coherence theorem [22, 31] gives that on a collection of straight strands with fixed bracketing at the top and bottom, any internal change of parenthesis is equivalent to any other internal change of parenthesis.
We may then omit intermediate changes of bracketing on our tangle diagrams, if desired.

Next, the Hexagon Axiom must be satisfied. We use the generator $c$ to define a commutativity constraint, which we also call $c$, by collapsing strands as we did for the change of parenthesis generator and the associativity constraint $a$. The commutative diagrams for this axiom for quasi-Hopf algebras are given in (3.1.21) and (3.1.22), and the corresponding equality of bracketed tangles is depicted in Figure 4.10, where equality (a) corresponds to (3.1.21) and equality (b) corresponds to (3.1.22).

Figure 4.10: Hexagon Axioms

Again, the Hexagon Axiom is necessary for a braided tensor category, so we assume the equality of the tangles in Figure 4.10. If we consider our definition to be a set of tangles modulo a set of relations, we add the Pentagon and Hexagon axioms to our set of relations.

The identity object is $\hat{0}$ and we take the left and right unit constraints to be trivial. This gives us automatically that the Triangle Axiom is satisfied.
There are two more commutative diagrams necessary for a braided tensor category. Let \( \hat{n}, \hat{n}', \hat{m}, \hat{m}', \) and \( \hat{r}, \hat{r}' \) be objects in the category \( TGL^{(i)} \). The first diagram is given in (4.3.3). Let \( f : \hat{n} \to \hat{n}' \) and \( g : \hat{m} \to \hat{m}' \).

\[
\begin{array}{ccc}
\hat{n} \otimes \hat{m} & \xrightarrow{c_{\hat{n}, \hat{m}}} & \hat{m} \otimes \hat{n} \\
\downarrow f \otimes g & & \downarrow g \otimes f \\
\hat{n}' \otimes \hat{m}' & \xrightarrow{c_{\hat{n}', \hat{m}'}} & \hat{m}' \otimes \hat{n}'
\end{array}
\]  

(4.3.3)

On tangles, (4.3.3) corresponds to the equality in Figure 4.11.

Hence, we assume that we can push morphisms through crossings. For the bottom half of the square in (4.3.3), we have used that the tensor product of morphisms is given by juxtaposition. See Figure 4.12.

Our second square is given in (4.3.4). Let \( f : \hat{n} \to \hat{n}' \), \( g : \hat{m} \to \hat{m}' \), and \( h : \hat{r} \to \hat{r}' \).

\[
\begin{array}{ccc}
(\hat{n} \otimes \hat{m}) \otimes \hat{r} & \xrightarrow{a_{\hat{n}, \hat{m}, \hat{r}}} & \hat{n} \otimes (\hat{m} \otimes \hat{r}) \\
\downarrow (f \otimes g) \otimes h & & \downarrow f \otimes (g \otimes h) \\
(\hat{n}' \otimes \hat{m}') \otimes \hat{r}' & \xrightarrow{a_{\hat{n}', \hat{m}', \hat{r}'}} & \hat{n}' \otimes (\hat{m}' \otimes \hat{r}')
\end{array}
\]  

(4.3.4)
On tangles, (4.3.4) corresponds to the equality in Figure 4.13. Recall again the relation in Figure 4.12.

Hence, we will assume that morphisms can pass through a change of parenthesis, provided that the morphism can be written in the form \( f \otimes g \otimes h \) on the three objects. That is, we cannot exchange a crossing and a change of parenthesis if the exchange would result in a crossing which does not have the two strands grouped together.

With the bracketed versions of the moves in \( TGL \), these are all the assumptions necessary for our category \( TGL^0 \). That is, we make the following definition.

**Definition 4.3.8.** Two bracketed tangles are equivalent if they are related by a sequence of the following moves.

1. Pentagon and Hexagon Axioms

2. Figure 4.11 and Figure 4.13

3. Isotopies

4. Bracketed Moves I, II, III, IV on tangles

\[
\begin{array}{c}
\text{Figure 4.12: Tensor Product of Morphisms}
\end{array}
\]
5. The framed 1st Reidemeister Move, or Move $R$

**Definition 4.3.9.** The category $TGL^{(1)}$ of bracketed tangles is given as follows. The objects of $TGL^{(1)}$ are bracketed integers.

The morphisms in $\text{Hom}(\hat{m}, \hat{n})$ of $TGL^{(1)}$ are equivalence classes of framed bracketed tangles with $m$ top end points and $n$ bottom end points, with the equivalence given in Definition 4.3.8.

The tensor product of morphism is given by juxtaposition with parenthesis, that is $S \otimes T$ has parenthesis around all of $S$ and parenthesis around all of $T$. Composition of morphisms is given by stacking, where $T \circ S$ means we stack $S$ on top of $T$. In this case, the target bracketing on $S$ must match the source bracketing on $T$.

**Definition 4.3.10.** Two equivalent bracketed tangles $T$ and $T'$ are admissibly equivalent if they are related by a sequence of the following additional moves.

6. 2-handle slides or the 2nd Kirby Move
7. The Modified 1st Kirby Move, or removal of an isolated Hopf link in which one component has zero framing

8. The $\sigma$-move

**Definition 4.3.11.** The category $Tgl^{(1)}$ of admissible, bracketed tangles is defined as follows. The objects of $Tgl^{(1)}$ are even bracketed integers $\overline{2n}$ with fixed bracketing from the left as discussed in Definition 4.3.2.

The morphisms in $\text{Hom}(\overline{2m}, \overline{2n})$ of $Tgl^{(1)}$ are equivalence classes of bracketed, admissible tangles with $2m$ top end points and $2n$ bottom end points, where the end points $(j^-, j^+)$ are either connected by a single component of the tangle, or are connected via two components to the pair $(i^-, i^+)$ at the opposite end of the tangle. There may also be closed components of the tangle, isomorphic to $S^1$. The equivalence of admissible tangles is given in Definition 4.3.10.

The tensor product of morphisms is given by juxtaposition with parenthesis, and the composition of morphisms is given by stacking, with $T \circ S$ given by stacking $S$ on top of $T$. See Definition 4.3.9.

We will generally omit the objects from our tangle diagrams, and assume that each strand is a single strand unless otherwise indicated.

**Lemma 4.3.12.** We have an injective functor $Tgl \hookrightarrow Tgl^{(1)}$.

**Proof.** On objects, the functor is given by

$$2n \mapsto \overline{2n}. \quad (4.3.5)$$

By the Coherence Theorem, since $\overline{2n}$ corresponds to a fixed bracketing from the left, this map is injective.

On morphisms, the functor is given by applying appropriate parenthesis to the generators, and then changing the parenthesis on the straight strands to match the
bracketing from the left on $\overline{2n}$. This is well-defined by the Pentagon Axiom, and injective because two bracketed tangles with the same non-bracketed generators differ only by a change of parenthesis.

4.4 A Coproduct Map on Strands in $Tgl(l)$

In [17], Kauffman and Radford describe a coproduct on the category $DTgl$ given by doubling of strands and taking the Hopf algebra coproduct on beaded elements. We would like to redefine this coproduct for $DTgl$ with elements in a quasi-Hopf algebra and then define a coproduct on $Tgl(l^0)$.

Since a general coproduct map $\Delta$ is evaluated on one factor at a time, we will not make a formal definition of a coproduct functor for either of these categories. We instead define a coproduct map on one component of a tangle at a time. We will refer to “open” components as those isomorphic to $[0, 1]$ and to “closed” components as those isomorphic to $S^1$.

First, we modify the coproduct on $DTgl$ given in [17] to take care of beaded extrema as well as the coproduct in the quasi-Hopf algebra. Let $T$ be a tangle in $DTgl$ and $S$ a component in $T$. Define $\Delta_S(T)$ as follows. If $S$ is an open component of $T$, we define the coproduct according to Figure 4.14. Let $f = \sum f^1 \otimes f^2 \in H \otimes H$ be the special element defined in (3.2.9), and $\Delta(h)$ the usual coproduct on $H$. The selected component $S$ is one of those pictured on the left side of the diagram, hence we label the component with $S$.

Remark 4.4.1. We have explicitly labeled the formal sum of tangles for the topmost strand in Figure 4.14. That is, in terms of our more formal notation, for the top pair of tangles in Figure 4.14 we have $\Delta((S, h)) = (S', \Delta(h))$ where $S'$ is the doubled component. The other two images of $\Delta$ are also formal sums over all factors of $f$ or $f^{-1}$, though we have suppressed the summation.
To find the image $\Delta_S(T)$ for $S$ a closed strand of $T$, we decompose $S$ into compositions of the open components in Figure 4.14 and then map these compositions as depicted in the figure.

Using the relations on $f$, $\gamma$, and $\delta$ given in Lemma 3.2.2, it is not difficult to see that given a tangle $T$ and a component $S$, this map is well-defined. We must check that the relations in Figure 4.4 and Figure 4.6 are satisfied.

One consequence of the definition is depicted in Figure 4.15. We will need these images for later calculations. Recall the definitions of $\delta = \sum \delta^1 \otimes \delta^2 \in H \otimes H$ and $\gamma = \sum \gamma^1 \otimes \gamma^2 \in H \otimes H$ given in (3.2.8) and (3.2.6), respectively.

Remark 4.4.2. This definition reduces to that in [17] for ordinary Hopf algebras, since for ordinary Hopf algebras, we have $f = \gamma = \delta = 1 \otimes 1$ and $\alpha = \beta = 1$.  

64
Next, we consider the case of $TGL^0$, and consequently the subcategory $Tgl^0$. For a framed tangle $T \in TGL^0$ and a component $S$ of $T$, we use $\Delta_S(T)$ to denote the tangle with one additional component $S'$ obtained by pushing a copy of $S$ off itself along the framing of $S$. If $S$ is a closed component and $p$ and $q$ are the labels of the endpoints of $S$ at (for example) the top of the tangle, the source object of the resulting tangle $\Delta_S(T)$ would have $p$ and $q$ replaced by either $(p, p')$ or $(p', p)$ and either $(q, q')$ or $(q', q)$, respectively. See Figure 4.16.

Figure 4.16: Applying $\Delta_S(T)$ on an Example
Remark 4.4.3. In our example, the source object must now be rebracketed so that if $T : \hat{m} \to \hat{n}$, we have $\Delta_S(T) : m + 1 \to \hat{n}$. Also, the tangle $\Delta_S(T)$ should be bracketed so that the individual crossings $c$ can be evaluated. Finally, note that pushing off the strand $S$ means we must take care that the proper order of $p$ and $p'$ or $q$ and $q'$ is observed.

On closed components, we use the same strategy of defining $\Delta_S(T)$ by adding $S'$, a push-off of the strand $S$, but we need not worry about the objects.

Definition 4.4.4. The coproduct $\Delta_S$ on various strands $S$ is given in Figure 4.17 for the generators of $TGL(\cdot)$ except the change of parenthesis $a$. For the generator $a$, we define the coproduct by doubling the required strand. In the figure, we assume that the strand $S$ is not connected to the strand not labeled $S$.

Remark 4.4.5. Defining the coproduct on components means we have different available choices for the strand $S$ on some of the generators. For the generator $c$, we have depicted two different choices for $S$ in Figure 4.17, and could have drawn a third where $S$ is both strands of the crossing, hence both are doubled. Similarly, we have seven different choices for the component $S$ in the generator $a$.

Remark 4.4.6. This $\Delta_S$ is well-defined on equivalence classes in $TGL(\cdot)$. For $Tgl(\cdot)$, $\Delta_S$ is well-defined with respect to the Second Kirby Move, but is only almost well-defined with respect to the Modified First Kirby Move. In this second case, doubling a component may introduce additional isolated zero-framed unknots.

In general, we will not be concerned with the objects on our tangles. We are instead concerned with the relationship between the coproduct on $DTgl$ and the coproduct on $Tgl(\cdot)$. This relationship will be discussed in Chapter 6.

Recall our discussion in the previous section about combining two or more tangle strands into a single strand. See Figure 4.8. We may also apply the coproduct in this
\[ \Delta_S \left( \left\| \right\| \right) = \left\| \left\| \right\| \right\| \]
\[ \Delta_S \left( \frac{S}{\left\| \right\|} \right) = \left( \frac{\left\| \right\|}{\left\| \right\|} \right) \]
\[ \Delta_S \left( \frac{S}{\left\| \right\|} \right) = \left( \frac{\left\| \right\|}{\left\| \right\|} \right) \]
\[ \Delta_S \left( \frac{S}{\left\| \right\|} \right) = \left( \frac{\left\| \right\|}{\left\| \right\|} \right) \]
\[ \Delta_S \left( \frac{S}{\left\| \right\|} \right) = \left( \frac{\left\| \right\|}{\left\| \right\|} \right) \]
\[ \Delta_S \left( \left\| \right\| \right) = \left( \frac{\left\| \right\|}{\left\| \right\|} \right) \]

Figure 4.17: The Coproduct \( \Delta_S \) on Generators of \( Tgl^0 \) for Indicated Strands \( S \)
situation. Applying the coproduct on such a combined strand would mean applying the coproduct on each of the individual strands which have been combined. The result may be depicted by doubling the combined strand, with the understanding that the picture refers to the doubling of the original strands, before they were combined into one. Because of the potential for confusion, we will only use this process when straight, uncrossed strands are combined.

The Hexagon Axiom allows us to interchange thinking of two crossing strands either together or separately. For instance, $\Delta_S(c)$ for $S$ the over-crossing strand is pictured on the left-hand side of (a) in Figure 4.10. The left-hand side of this figure represents two strands which together cross the third strand. The right-hand side depicts two strands individually crossing a third strand. These two notions are thus interchangeable. Similarly, tangle (b) in Figure 4.10 depicts $\Delta_S(c^{-1})$ for $S$ the under-crossing strand.

For our work in Chapter 6, we will want to generalize to an $n$-fold coproduct by induction. First, some notation on tangles. We will use two strands decorated with $(n)$ to denote that the diagram consists of $n$ parallel strands. Since these strands must be bracketed, we will use a subscript $R$ to denote that the objects have been bracketed from the right. Consider the example in Figure 4.18.

$$
\left( \begin{array}{c}
|\\
\end{array} \right)_R = \left( \begin{array}{c}
|\\
\end{array} \right) \left( \begin{array}{c}
|\\
\end{array} \right) \left( \begin{array}{c}
|\\
\end{array} \right) \left( \begin{array}{c}
|\\
\end{array} \right) \left( \begin{array}{c}
|\\
\end{array} \right)
$$

Figure 4.18: Four StrandsBracketed from the Right
\[ \Delta_R^{(n)} \left( \begin{array}{c} \vline \\ \vline \end{array} \right) = \begin{array}{c} (n) \\ R \end{array} \]

\[ \Delta_R^{(n)} \left( \begin{array}{c} S \\ \times \end{array} \right) = \begin{array}{c} (n) \\ R \end{array} \]

\[ \Delta_R^{(n)} \left( \begin{array}{c} S \\ \times \end{array} \right) = \begin{array}{c} (n) \\ R \end{array} \]

\[ \Delta_R^{(n)} \left( \begin{array}{c} \vline \\ \vline \end{array} \right) = \begin{array}{c} (n) \\ (n) \\ R \\ R \end{array} \]

\[ \Delta_R^{(n)} \left( \begin{array}{c} \vline \\ \vline \end{array} \right) = \begin{array}{c} (n) \\ (n) \\ R \\ R \end{array} \]

Figure 4.19: The Coproduct \( \Delta_R^{(n)} \) on the Generators of \( T_{gl(1)} \)
Notation 4.4.7. Let $\Delta^{(n)}_R$ denote an $n$-fold coproduct on a component $S$ of a tangle $T$, where the bracketing is concentrated on the right-hand side. That is, the coproduct pushes off $n - 1$ copies of $S$ according to the framing, and the bracketing on the $n$ strands (the $n - 1$ copies plus the original $S$) is from the right. For an example of bracketing from the right, see Figure 4.18. This $n$-fold coproduct is defined inductively by

$$\Delta^{(n)}_R = (\text{id} \otimes ... \otimes \text{id} \otimes \Delta)(\Delta^{(n-1)}_R).$$

Due to notational complexity, we have suppressed the subscript $S$ indicating which component is selected. This component will be clear from the context. Figure 4.19 depicts the $n$-fold coproduct on various components of the generators of $T gl^{(0)}$.

Remark 4.4.8. In Figure 4.19, we have not marked all the parenthesis for the generators $\cup$ and $\cap$. The important parenthesis are marked; the rest of the diagram may be parenthesized in any appropriate fashion.

We may also generalize the Hexagon Axiom by induction. For the first step, take $\Delta^{(3)}_R(c^{-1})$ with $S$ the under-crossing strand to get the left-most tangle in Figure 4.20. Peel off the left-most strand of the three under-crossing strands by considering the two right-most strands to be a single strand, and use the Hexagon Axiom. This means we can evaluate the top-most crossing separately from the other two crossings. See the middle tangle in Figure 4.20. Finally, use the Hexagon Axiom on the two right-most strands to evaluate the crossings separately. We then have the right-most tangle in Figure 4.20. This generalizes the Hexagon Axiom, depicted in Figure 4.10, to three strands.

The induction procedure for a crossing on $n$ strands, using the Hexagon Axiom $n - 1$ times, is now obvious, see Figure 4.21. In this figure, we have taken $S$ to be the over-crossing strand.
Remark 4.4.9. We have discussed only one type of crossing, as this will be the only type we will need. The inductive argument on the other type of crossing, corresponding to part (a) of Figure 4.10, can be computed in a similar fashion.
CHAPTER 5
COBORDISM CATEGORIES AND TOPOLOGICAL
QUANTUM FIELD THEORIES

Our work is concerned with 3-manifolds, hence throughout this chapter, we make our definitions only for dimension 3. It is worthwhile to note, however, that these definitions were originally made for an arbitrary dimension. Since we make this dimension restriction, some of the original definitions are simplified. For instance, we may work with diffeomorphisms instead of homeomorphisms, as these are equivalent in dimension 3.

5.1 TQFTs and the Category $Cob^3$

We begin by defining a category of 3-dimensional cobordisms. For more details, see [20, 4]. Let $\Sigma_n^\bullet$ denote a connected, compact, oriented genus $n$ surface with one boundary component. The boundary component should be isomorphic to the circle $S^1$, and we will think of this as a punctured standard surface. We associate to each surface $\Sigma_n^\bullet$ an orientation-preserving homeomorphism $\partial \Sigma_n^\bullet \to S^1$. Let $-\Sigma$ denote $\Sigma$ with opposite orientation.
Notation 5.1.1. Let $C_{mn}$ denote the stratified surface obtained by sewing a cylinder $S^1 \times [0,1]$ between $-\Sigma^*_m$ and $\Sigma^*_n$ using the two boundary homomorphisms. That is,

$$C_{mn} = (-\Sigma^*_m) \coprod_{\partial \Sigma^*_m \sim (S^1 \times 0)} (S^1 \times [0,1]) \coprod_{\partial \Sigma^*_n \sim (S^1 \times 1)} (\Sigma^*_n).$$

(5.1.1)

Definition 5.1.2. A cobordism between surfaces $\Sigma^*_m$ and $\Sigma^*_n$ is a compact oriented 3-manifold $M$ with corners together with a strata- and orientation-preserving homeomorphism $\xi : C_{mn} \to \partial M$. See Figure 5.1.

![Figure 5.1: A Cobordism $M : \Sigma^*_m \to \Sigma^*_n$](image)

Notation 5.1.3. We will use the notation $M : \Sigma^*_m \to \Sigma^*_n$ to denote a cobordism $M$ between $-\Sigma^*_m$ and $\Sigma^*_n$.

Definition 5.1.4. We say that two cobordisms $M$ and $M'$ are equivalent if $m = m'$, $n = n'$, and there is a homeomorphism $\eta : M \to M'$ so that $\eta \circ \xi = \xi'$.
Definition 5.1.5. The category of framed cobordisms \( \text{Cob}_3^\bullet \) in dimension 2 + 1 is defined as follows. The objects are the standard surfaces \( \Sigma_n^\bullet \), with one surface for each \( n \in \mathbb{N} \). The morphisms are the equivalence classes of cobordisms \([M, \xi]\) as in Definition 5.1.4, together with a 2-framing of \( M \), or equivalently the signature of a 4-manifold bounding a standard closure of \( M \).

Composition of morphisms is defined by gluing cobordisms along a common boundary component. The tensor product is given by disjoint union.

Remark 5.1.6. We may define a related category \( \text{Cob}_3^\emptyset \) of cobordisms whose objects are standard surfaces \( \Sigma_n^\bullet = \Sigma_n^\bullet \cup D^2 \), where we have glued a disc in the boundary of \( \Sigma_n^\bullet \). The morphisms in this category are then the same as the morphisms in \( \text{Cob}_3^\bullet \).

The two categories are related by a fill functor \( \mathcal{F}_\partial : \text{Cob}_3^\bullet \to \text{Cob}_3^\emptyset \).

Theorem 5.1.7. [18] The category \( Tgl \) of admissible tangles in Definition 4.1.6 is isomorphic to the category \( \text{Cob}_3^\bullet \).

We summarize the construction of the isomorphism \( \mathcal{F} : Tgl \to \text{Cob}_3^\bullet \).

Let \( T : 2n \to 2m \) be an admissible tangle in \( Tgl \). Add \( m \) 1-handles \( D^2 \times [0, 1] \) at the \( 2m \) points at the bottom of the tangle as follows. Write \( 2m = (1^-, 1^+, \ldots, m^-, m^+) \) as usual. The two endpoints of each 1-handle are attached at the pair \((i^-, i^+)\). For any arcs connecting \( j^- \) to \( j^+ \) at the bottom of the tangle, we continue the 1-handles through these components of the link. This creates a copy of \( \Sigma_m^\bullet \) along the bottom of the tangle.

Next, we drill out holes along the components attached to the top of the tangle, creating a copy of \( \Sigma_n^\bullet \) along the top of \( T \). What remains of \( T \) after this procedure is a framed link in the interior of the tangle, surrounding the drilled-out pieces. We do surgery along this link to obtain the manifold \( M \in \text{Cob}_3^\bullet \), which is the image of \( T \) under the functor \( \mathcal{F} \).
In order to simplify our later calculations, we would like to relate cobordisms to
the mapping class group, and then use the structure of the mapping class group to
further reduce our calculations.

First, we relate coordisms with the mapping class group. Let $\psi : \Sigma_n \to \Sigma_n$ be a
homeomorphism. Define a particular cobordism

$$I_{\psi} = [\Sigma_n \times [0, 1], \xi],$$

where $\xi : C_{nn} \to \partial M$ is the map which is the identity on the bottom copy of $\Sigma_n$ and
the map $\psi$ on the top copy of $\Sigma_n$.

The homotopy class of this cobordism depends only on the isotopy class $\{\psi\}$ of
$\psi$, and in Proposition 1.4.2 of [20] it is shown that the map

$$\Gamma_n \to \text{Aut}(\Sigma_n) : \{\psi\} \mapsto [I_{\psi}].$$

is an isomorphism, where $\Gamma_n$ denotes the mapping class group of $\Sigma_n$ and $\text{Aut}(\Sigma_n)$ is
the group of invertible cobordisms on $\Sigma_n$.

Now, when a cobordism has the same standard surface at both boundary compo-
nents, we can then use the fact that the mapping class group is generated by Dehn
Twists to simplify our calculations.

Most cobordisms do not have the same surface at both boundary components,
however. In order to use the above simplification, then, we define two more special
cobordisms. For the first special cobordism, let $H_n^+$ be the manifold obtained by
adding a 1-handle to $\Sigma_n \times [0, 1]$ at two discs in $\Sigma_n \times 1$, and let $\xi_n^+ : C_{n(n+1)} \to \partial(H_n^+)$
be the identity on both standard surfaces. Define

$$H_n^+ = [H_n^+, \xi_n^+]$$

(5.1.4)
to be the cobordism which essentially adds another hole to $\Sigma_n$.

Remark 5.1.8. We add the 1-handle in a way that $\Sigma_n \to \Sigma_{n+1}$ is an inclusion.
For the second special cobordism, define $H_n^-$ to be the mainfold obtained by glueing a 2-handle into the surface $\Sigma^{n+1} \times [0,1]$ along a curve $b_{n+1}$ in the last torus added when we build $\Sigma^{n+1}$ from $\Sigma^n$ which has geometric intersection number one with the meridian of the 1-handle added in $H_n^+$. Define $\xi_n^- : C(n+1)_n \to \partial(H_n^-)$ to be the identity on both standard surfaces. Then define a cobordism

$$H_n^- = [H_n^-, \xi_n^-]$$

(5.1.5)

which essentially deletes the last hole from $\Sigma^{n+1}$.

**Remark 5.1.9.** *By their definitions, these two cobordisms have the property that $H_n^- \circ H_n^+$ is equivalent to the identity.*

Using basic Morse theory, we can then decompose any cobordism $M : \Sigma^m \rightarrow \Sigma^m$ into a Heegard Splitting

$$M \simeq H_n^- \circ H_{n+1}^- \circ \ldots \circ H_{N-1}^- \circ I_{\psi} \circ H_N^+ \circ \ldots \circ H_m^+$$

(5.1.6)

for some $\psi \in \text{Homeo}^+(\Sigma_N)$ and for some integer $N$.

As an example of the isomorphism in Theorem 5.1.7 and the structure of $Cob^*_3$, we present the following correspondence. The Dehn Twists which generate the mapping class group are given by the twists in Figure 5.2. See [19]. The corresponding admissible tangles in $Tgl$ under this isomorphism are given in Figure 5.3.

**Remark 5.1.10.** *In Figure 5.3, we have labeled the $i$-th pair of strands as $i^-, i^+$ as we discussed for the objects in $Tgl$.*

We are now ready to define TQFTs. Following Atiyah in [2] and its summary in [19], we consider topological quantum field theories, or TQFTs, from the perspective of functors.

Let $Vect$ be the category of vector spaces over our field $k$. 76
Figure 5.2: Dehn Twists for $\Gamma(\Sigma^*_n)$
**Definition 5.1.11.** A topological quantum field theory, or TQFT, is a functor

\[ \nu : \text{Cob}^* \rightarrow \text{Vect.} \]  

By our remarks on the structure of cobordisms, a TQFT is completely determined by the images of \( H_n^+ \), \( H_n^- \) and \( I_\psi \), as all cobordisms can be constructed from these families. This means that if two TQFTs coincide on the images of those cobordisms induced by Dehn twists, as well as the cobordisms \( H_n^+ \) and \( H_n^- \), then the two TQFTs are equivalent.
5.2 Example: The Hennings TQFT

For a treatment of the knot invariant version of the Hennings algorithm, see Hennings' original paper [14] or its reformulation by Kauffman and Radford in [17]. We expand the definition slightly in order to define a TQFT as in [20]. The TQFT functor is defined as a composition of functors as follows.

\[
\nu_H : \text{Cob}_3^* \cong Tgl \xrightarrow{\Psi} DTgl \xrightarrow{\hat{\Psi}} \text{Vect}
\]

We discussed the isomorphism between \( \text{Cob}_3^* \) and \( Tgl \) in the previous section, so we must define the functors \( \Psi : Tgl \to DTgl \) and \( \hat{\Psi} : DTgl \to \text{Vect} \). We begin with the functor \( \Psi \).

Let \( T : 2m \to 2n \in Tgl \). We map \( T \in Tgl \) to a decorated flat tangle \( \hat{T} \in DTgl \) by systematically replacing the generators of \( Tgl \) by elements of \( DTgl \). See Figure 4.1 for the generators of \( Tgl \). The generators id, \( \cap \), and \( \cup \) in \( Tgl \) are replaced by the generators id, \( \cap \), and \( \cup \), respectively, in \( DTgl \). The generators \( c \) and \( c^{-1} \) are replaced as in Figure 5.4.

![Figure 5.4: Replacement of Crossings](image)

The functor \( \Psi \) is then defined on the integer objects as \( \Psi : 2m \to 2m \) and on tangles as \( \Psi(T) = \hat{T} \).

**Lemma 5.2.1.** [19, 4] The functor \( \Psi \) factors through Moves I - IV and Move R.
Remark 5.2.2. We have abused the notation by calling $\Psi$ a functor. It is not well-defined on $Tgl$, but rather on $TGL$. The full functor $\hat{\Psi} \circ \Psi$ is well-defined on $Tgl$.

Remark 5.2.3. We could have defined $\Psi$ on the full tangle category $TGL$, but as we are concerned with cobordisms, we will only need the definition on the subcategory $Tgl$.

We next define the functor $\hat{\Psi}$, which is only defined on the image $\Psi(Tgl)$, hence we must have tangles in $DTgl$ whose source and target objects are both even numbers, as well as restrictions on the types of components. See Definition 4.1.6. Let $T : 2m \to 2n$ be such a decorated tangle. Using the equivalence relations in the category $DTgl$, depicted in Figure 4.6, find an equivalent tangle $T'$ in which all of the beads on each component are collected into a single bead, and that bead and component is arranged in one of the following options. For the image $\Psi(Tgl)$, these are all the possibilities that can occur. See [19].

1. On the right-hand side of a closed component

2. On the left-hand side of a maximum

3. On the right-hand side of a minimum

4. One bead on each of two parallel strands.

See Figure 5.5 for tangle diagrams of these possibilities. This assignment is unique up to factors of $S^2$ applied to elements of the bead on the circular component when combining these beads. We have also indicated our next step in Figure 5.5: we assign to each component of the resulting tangle $T'$ a map on the Hopf algebra $H$.

The map $f_T : H^{\otimes m} \to H^{\otimes n}$ corresponding to the tangle $T'$ is given by the following. For each pair of strands at either the top or bottom of the tangle, we have one factor of the underlying Hopf algebra $H$, and so we need one variable $x_i$. 

80
The maps in the following list each correspond to a single factor of $H$. We have:

- (1) : $k \to k$ sends $1 \mapsto \lambda(a_i) \cdot 1$
- (2) : $k \to H$ sends $1 \mapsto b_j$
- (3) : $H \to k$ sends $x_k \mapsto \lambda(S(x_k)c_k)$
- (4) : $H \to H$ sends $x_n \mapsto p_nx_nS(q_n)$.

For example, the tangle given in figure 5.6 would give rise to the map $H \otimes H \to H$ given by $(x_1, x_2) \mapsto \lambda(d)\lambda(S(x_1)a)\lambda(S(x_2)b)c$. 

---

81
The map $f_T$ is then the tensor product of the relevant maps. We define $\hat{\Psi}$ on objects to send the integer $2n$ to $H^\otimes n$. On morphisms, $\hat{\Psi}(T) = f_T$.

**Lemma 5.2.4.** [19, 4, 17] The functor $\hat{\Psi}$ is well-defined on the image $\Psi(Tgl)$.

**Proof.** Since the equivalent tangle $T'$ in standard position is unique up to $S^2$ applied to elements of the bead on the circular component, we need only check that

$$\lambda(S^2(b)a) = \lambda(ab). \quad (5.2.2)$$

This is one of the assumptions required for the cointegral $\lambda$. See Lemma 2.2.7.

**Remark 5.2.5.** We may, using the Hennings map, associate tangles to particular elements in a Hopf algebra. Once we have defined our Hennings map on quasi-Hopf algebras, we may then associate tangles to elements in a quasi-Hopf algebra. We discuss the basics of this association here.

In the category $Tgl$, the positive crossing $c$ corresponds to the $R$-matrix, as these strands are replaced by a flat crossing decorated with the two elements of the $R$-matrix. We may, in essence, think of the element $R \in H \otimes H$ and the positive crossing in $Tgl$ interchangeably. In the same way, we may think of the negative crossing and the inverse of $R$ interchangeably.

The association is more interesting in the case of the Hennings algorithm for quasi-Hopf algebras, and so we will discuss this association further after making the extended definition of the Hennings algorithm.

**Remark 5.2.6.** We recover the original Hennings algorithm by evaluating our invariant on links instead of tangles. See [17]. When all of the components are closed, the algorithm produces a number in the field $k$ which we denote $TR(L)$ for a link $L$. 

82
We then normalize as follows. Let \( c(L) \) denote the number of components of \( L \), and \( \sigma(L) \) the signature of the linking matrix of the components of \( L \). We have

\[
INV(L) = [\lambda(v)\lambda(v^{-1})]^{-c(L)/2}[\lambda(v)\lambda(v^{-1})]^{-\sigma(L)/2}TR(L).
\] (5.2.3)

### 5.3 Example: Dijkgraaf-Witten TQFT

The Dijkgraaf and Witten TQFT was originally defined in [6], then expanded and clarified in [10] by Freed and Quinn. The TQFT was also considered from a more combinatorial perspective by Wakui in [29]. Let \( G \) be a finite group and \( \omega \in H^3(G, U(1)) \) be a 3-cocycle. Here, we consider only the case that the cocycle \( \omega \) is trivial, or \( \omega = 1 \).

**Remark 5.3.1.** All of the above cited papers deal with cobordisms \( M : \Sigma_1 \to \Sigma_2 \). We adjust the definitions for \( M : \Sigma^*_1 \to \Sigma^*_2 \), which gives a simpler structure when we fix a basepoint in the boundary of our cobordism.

Let \( M : \Sigma^*_1 \to \Sigma^*_2 \) be a cobordism. Recall that the boundary of \( M \) is homeomorphic to

\[
C_{12} = (-\Sigma^*_1) \coprod_{\partial \Sigma^*_1 \sim (S^1 \times 0)} (S^1 \times [0, 1]) \coprod_{\partial \Sigma^*_2 \sim (S^1 \times 1)} (\Sigma^*_2)
\] (5.3.1)

via a map \( \xi \). Fix a basepoint \( p_j \) on \( \Sigma^*_j \), and a path \( P \) connecting \( p_1 \) to \( p_2 \) with \( P \) on the part of \( \partial M \) isomorphic to \( S^1 \times [0, 1] \).

**Notation 5.3.2.** We use \( C(X) \) to denote \( C(X, k) \), the space of continuous functions from \( X \) to \( k \).

**Theorem 5.3.3.** [6, 29, 10] The Dijkgraaf-Witten TQFT is a well-defined functor \( \nu_T : Cob_3 \to Vect \).

**Remark 5.3.4.** The functor will be defined shortly. We adjust the original definition for closed surfaces \( \Sigma \) to punctured surfaces \( \Sigma^* \). The proof that the original TQFT is
well-defined still applies with careful attention to the fixed basepoints and adjusting spaces and coefficients.

To define this functor, we first assign the vector space

$$V(\Sigma^\bullet) := C(\text{Hom}(\pi_1(\Sigma^\bullet), G))$$

(5.3.2)

to each surface $$\Sigma^\bullet$$. To each cobordism $$M : \Sigma^\bullet_1 \to \Sigma^\bullet_2$$, we assign a map

$$V(M) : V(\Sigma^\bullet_1) \to V(\Sigma^\bullet_2)$$

(5.3.3)

as follows. Let $$\xi_j : \Sigma^\bullet_j \to M$$ be the restriction of the map $$\xi : C_{12} \to \partial M$$ to the surfaces $$\Sigma^\bullet_j$$.

**Notation 5.3.5.** Let $$a \in \text{Hom}(\pi_1(\Sigma^\bullet_1, p_1), G)$$ and $$b \in \text{Hom}(\pi_1(\Sigma^\bullet_2, p_2), G)$$. Note that we choose the basepoint $$p_1$$ in $$\Sigma^\bullet_1$$ as the basepoint for $$M$$. Denote

$$\text{Hom}(\pi_1(M, p_1), G)_{ab} = \{ \gamma \in \text{Hom}(\pi_1(M, p_1), G) | \gamma \circ \xi_{1*} = a \text{ and } \gamma \circ \xi_{2*} = b \}.$$ 

(5.3.4)

To make this definition for $$b \in \text{Hom}(\pi_1(\Sigma_2, p_2), G)$$, we consider $$\gamma \in \pi_1(\Sigma^\bullet_2, p_2)$$ as an element of $$\pi_1(M, p_1)$$ by using the designated path $$P$$ from $$p_1$$ to $$p_2$$. That is, we may think of $$\gamma$$ as $$P^{-1} \gamma P \in \pi_1(M, p_1)$$. Then define

$$n_{ab} = \#\text{Hom}(\pi_1(M, p_1), G)_{ab}.$$ 

(5.3.5)

With these clarifications, we omit reference to the basepoints, though they remain fixed.

Given $$a \in \text{Hom}(\pi_1(\Sigma_1), G)$$ and $$b \in \text{Hom}(\pi_1(\Sigma_2), G)$$, the procedure outlined for closed surfaces in Dijkgraaf and Witten’s paper details a number

$$Z(M, a, b) = \frac{1}{|G|} \sum_{\gamma \in \text{Hom}(\pi_1(M), G)_{ab}} W(\gamma)$$

(5.3.6)

where $$\text{Hom}(\pi_1(M), G)_{ab}$$ refers to the same space with closed surfaces instead of punctured ones. In the case of $$\omega = 1$$, this number is equal to $$n_{ab}.$$
Remark 5.3.6. We use the same definition for punctured surfaces, but omit the coefficient $\frac{1}{|G|}$. In the original definition, this coefficient comes from the adjoint action of $G$, since the basepoint is not fixed.

We then define $V(M)$ by

$$V(M)(f)(b) = \sum_{a \in \text{Hom}(\pi_1(\Sigma_1^•), G)} f(a)n_{ab} \quad (5.3.7)$$

for $f \in V(\Sigma_1^•)$ and $b \in \text{Hom}(\pi_1(\Sigma_2^•), G)$.

Theorem 5.3.7. The functor $\nu_T : \text{Cob}_3 \to \text{Vect}$ given by

$$\nu_T(\Sigma^•) = C(\text{Hom}(\pi_1(\Sigma^•), G)) \quad (5.3.8)$$

and

$$\nu_T(M : \Sigma_1^• \to \Sigma_2^•)(f)(b) = \sum_{a \in \text{Hom}(\pi_1(\Sigma_1^•), G)} f(a)n_{ab} \quad (5.3.9)$$

is well-defined.

Proof. Throughout, assume the basepoints previously mentioned to be fixed.

First, we claim that $\nu_T(\Sigma^• \times [0, 1])$ is the identity map on $\nu_T(\Sigma^•)$. Suppose that $\gamma \in \text{Hom}(\pi_1(\Sigma^• \times [0, 1]), G)_{ab}$ for $a, b \in \text{Hom}(\pi_1(\Sigma^•), G)$. The maps $\xi_1$ and $\xi_2$ mapping the surface $\Sigma^•$ to $\Sigma^• \times [0, 1]$ are both the identity, so we have the following.

$$\gamma|_{\Sigma^•} \circ \xi_1 = a \quad (5.3.10)$$

$$\gamma|_{\Sigma^•} = a \quad (5.3.11)$$

$$\gamma|_{\Sigma^•} \circ \xi_2 = b \quad (5.3.12)$$

$$\gamma|_{\Sigma^•} = b \quad (5.3.13)$$
Hence \( n_{ab} = 1 \) if \( a = b \) and is zero otherwise. That is,

\[
\nu_T(\Sigma^\bullet \times [0, 1])(f)(b) = \sum_a f(a)n_{ab} \tag{5.3.14}
\]

\[
= \sum_a f(a)\delta_{a,b} \tag{5.3.15}
\]

\[
= f(b) \tag{5.3.16}
\]

so \( \nu_T(\Sigma^\bullet \times [0, 1]) \) is the identity map, as desired.

Second, we claim that if \( M : \Sigma_3^\bullet \to \Sigma_2^\bullet \) and \( N : \Sigma_2^\bullet \to \Sigma_1^\bullet \), then \( \nu_T(N \circ M) = \nu_T(N) \circ \nu_T(M) \). Let \( f \in C(\text{Hom}(\pi_1(\Sigma_3^\bullet), G)) \) and \( b \in \text{Hom}(\pi_1(\Sigma_1^\bullet), G) \).

\[
\nu_T(N \circ M)(f)(b) = \sum_{a \in \text{Hom}(\pi_1(\Sigma_3^\bullet), G)} f(a)n_{ab} \tag{5.3.17}
\]

\[
= \sum_{c \in \text{Hom}(\pi_1(\Sigma_2^\bullet), G), a \in \text{Hom}(\pi_1(\Sigma_1^\bullet), G)} (f(a)n_{ac})n_{cb} \tag{5.3.18}
\]

To see that (5.3.17) is equal to (5.3.19), we must show that

\[
n_{ab} = \sum_{c \in \text{Hom}(\pi_1(\Sigma_2^\bullet), G)} n_{ac}n_{cb}. \tag{5.3.20}
\]

The Seifert-Van Kampen Universal Property of Push-Outs gives us a unique map \( \tau \) in the following diagram. Let \( \xi_j : \Sigma_j^\bullet \to M \) and \( \xi'_i : \Sigma_i^\bullet \to N \). Let \( \tau' \in \text{Hom}(\pi_1(N), G) \) and \( \tau'' \in \text{Hom}(\pi_1(M), G) \).

\[
\begin{array}{c}
\pi_1(\Sigma_2^\bullet) \\
\downarrow \xi_{2*} \\
\pi_1(M) \\
\downarrow \pi_1(\Sigma_1^\bullet) \\
\downarrow \xi_{2*} \\
\pi_1(N) \\
\downarrow \tau' \\
\pi_1(N \circ M) \\
\downarrow \tau'' \\
G
\end{array}
\tag{5.3.21}
\]
That is, \( \tau \) is equivalently given by maps \( \tau' \) and \( \tau'' \) so that

\[
\tau' \circ \xi_2^* = \tau'' \circ \xi_2^*
\]

in \( \text{Hom}(\pi_1(\Sigma^*_2), G) \). Equivalently,

\[
\text{Hom}(\pi_1(N \circ M), G)_{ab} = \bigcup_{c \in \text{Hom}(\pi_1(\Sigma^*_1), G)} \text{Hom}(\pi_1(N), G)_{ac} \times \text{Hom}(\pi_1(M), G)_{cb},
\]

so that

\[
n_{ab} = \sum_{c \in \text{Hom}(\pi_1(\Sigma^*_2), G)} n_{ac}n_{cb},
\]

as desired. \( \square \)

**Remark 5.3.8.** This proof shows that the Dijkgraaf-Witten TQFT is well-defined for the specific case of a trivial cocycle.

Previously, we have decomposed cobordisms into a composition of basic cobordisms \( H^+_n \) which adds a hole, \( H^-_n \) which deletes a hole, and \( I_\psi \) corresponding to Dehn Twists in the mapping class group. These building blocks of cobordisms are easy to study from this perspective, as we need only consider what happens to the generators of the fundamental group of \( \Sigma^*_1 \) under the cobordism \( M : \Sigma^*_1 \rightarrow \Sigma^*_2 \).

We first consider the case of cobordisms \( I_\psi \), corresponding to the elements of the mapping class group.

**Lemma 5.3.9.** The map

\[
\nu_T(I_\psi) : C(\text{Hom}(\pi_1(\Sigma^*_n), G)) \rightarrow C(\text{Hom}(\pi_1(\Sigma^*_n), G))
\]

is given by

\[
\nu_T(I_\psi) : \phi \mapsto \phi_\psi,
\]

as desired.
where the map $\phi_\psi$ is defined by

$$\phi_\psi(b) = \phi(b \circ \psi_*),$$  \hspace{1cm} (5.3.27)

with $b \in \text{Hom}(\pi_1(\Sigma_n^\bullet), G)$.

**Proof.** Consider $n_{ab} = \#\text{Hom}(\pi_1(M), G)_{ab}$. Suppose that $\gamma \in \text{Hom}(\pi_1(M), G)_{ab}$ for $M = I_\psi$. By definition of $M$, we have $\Sigma_1^\bullet = \Sigma_n^\bullet = \Sigma_2^\bullet$ and $\xi_1 = \psi$ and $\xi_2 = \text{id}$. Now on $\gamma' = \gamma|_{\Sigma_n^\bullet}$ we must have

$$\gamma' \circ \xi_{1*} = \gamma' \circ \psi_* \hspace{1cm} (5.3.28)$$

$$\gamma' \circ \xi_{2*} = \gamma' \circ \text{id} = \gamma' \hspace{1cm} (5.3.29)$$

so that $\gamma' \circ \xi_{1*} = a$ means $\gamma' \circ \psi_* = a$ and $\gamma' \circ \xi_{2*} = b$ means that $\gamma' = b$. But since $\gamma' = b$, we see

$$b \circ \psi_* = a. \hspace{1cm} (5.3.30)$$

This means that $n_{ab} = 1$ when $a = b(\psi_*)$ and is zero otherwise, so we have the following.

$$\nu_T(I_\psi)(\phi)(b) = \sum_{a \in \text{Hom}(\pi_1(\Sigma_n^\bullet), G)} \phi(a)n_{a\gamma} \hspace{1cm} (5.3.31)$$

$$= \sum \delta_{a,b(\psi_*)}\phi(a)n_{ab} \hspace{1cm} (5.3.32)$$

$$= \phi(b(\psi_*)) \hspace{1cm} (5.3.33)$$

This is the desired result. \hfill \Box

Next, we turn to our special cobordisms, starting with $H_n^+$. This cobordism is given by adding a 1-handle to one of the boundary surfaces, as described in Section 5.1 in (5.1.4). We make a similar definition to the previous case.
Lemma 5.3.10. The map \( \nu_T(H^+_n) \) is given by the following.

\[
\nu_T(H^+_n) : C(\text{Hom}(\pi_1(\Sigma^*_n), G)) \rightarrow C(\text{Hom}(\pi_1(\Sigma^*_{n+1}), G))
\]

(5.3.34)

\[
\phi \mapsto \phi_+,
\]

(5.3.35)

Here \( \phi_+ \) is defined as follows. Let \( \gamma \in \text{Hom}(\pi_1(\Sigma^*_n), G) \). Let \( \psi^+_* \) be the map describing the images of the generators of the fundamental group of \( \Sigma^*_n \) under the cobordism \( H^+_n \).

Then

\[
\phi_+(\gamma) = \phi(\gamma \circ \psi^+_*).
\]

(5.3.36)

Proof. Let \( \gamma \in \text{Hom}(\pi_1(M), G) \) with \( \gamma_1 \) the restriction to \( \Sigma^*_n \) and \( \gamma_2 \) the restriction to \( \Sigma^*_{n+1} \). Let \( a_i, b_i \) for \( 1 \leq i \leq n \) be the generators for \( \Sigma^*_n \), with the \( a_i \) the longitude and \( b_i \) the latitude. See Figure 5.7.
The maps $\xi_1$ and $\xi_2$ are both the identity on $\Sigma_n^*$ and $\Sigma_{n+1}^*$, respectively. That is, if $\gamma \in \text{Hom}(\pi_1(M), G)_{ab}$, we must have

$$\gamma_1 \circ \xi_1^* = \gamma_1 = a \quad (5.3.37)$$
$$\gamma_2 \circ \xi_2^* = \gamma_2 = b \quad (5.3.38)$$

This means that $n_{ab} = 0$ if $a$ and $b$ do not agree on the first $n$ generators. For the generators $b_{n+1}$ and $a_{n+1}$, we have the following. Adding a 1-handle to the $n$-th hole, we cut out a copy of $S^0 \times D^2$ and glue in $I \times S^1$. This creates a generator $b_{n+1}$ along the interval $I$. That is, the image of $b_{n+1}$ under $\gamma_2$ is free, hence we sum over all possibilities for the image of this generator in $G$. Since we glue along $S^0$, we only require cutting along this new $b_{n+1}$ in order to flatten the handle, so we must map the generator $a_{n+1}$ to be the identity. That is, $n_{ab} = 1$ if the images of $a$ and $b$ agree on the first $n$ generators, and $b$ sends $a_{n+1}$ to the identity. It is zero otherwise. This describes the desired map.

Finally, for $H_n^-$, we essentially make the same definition.

**Lemma 5.3.11.** The map $\nu_T(H_n^-)$ is given by the following.

$$\nu_T(H_n^-) : C(\text{Hom}(\pi_1(\Sigma_n^*), G)) \rightarrow C(\text{Hom}(\pi_1(\Sigma_{n+1}^*), G))$$

$$\phi \mapsto \phi_- \quad (5.3.39)$$

Here $\phi_-$ is defined as follows. Let $\gamma \in \text{Hom}(\pi_1(\Sigma_{n+1}^*), G)$. Let $\psi^-_\ast$ be the map describing the images of the generators of the fundamental group of $\Sigma_{n+1}^*$ under the cobordism $H_n^-$. Then

$$\phi_-(\gamma) = \phi(\gamma \circ \psi^-_\ast) \quad (5.3.41)$$

**Proof.** Let the generators of $\Sigma_n^*$ again be those pictured in Figure 5.7. Let $\gamma \in \text{Hom}(\pi_1(M), G)$ with $\gamma_1$ the restriction to $\Sigma_{n+1}^*$ and $\gamma_2$ the restriction to $\Sigma_n^*$. The
maps $\xi_1$ and $\xi_2$ are both the identity on $\Sigma_{n+1}^\bullet$ and $\Sigma_n^\bullet$, respectively. That is, if $\gamma \in \text{Hom}(\pi_1(M), G)_{ab}$, we must have

$$\gamma_1 \circ \xi_1* = \gamma_1 = a \quad (5.3.42)$$
$$\gamma_2 \circ \xi_2* = \gamma_2 = b \quad (5.3.43)$$

Again, this means that $n_{ab} = 0$ if $a$ and $b$ do not agree on the first $n$ generators. For the generators $b_{n+1}$ and $a_{n+1}$, we have the following from the definition of the space $H^-$. We cut out a cylinder along the generator $b_{n+1}$ and glue in two discs in its place. That is, we trivialize the generator $b_{n+1}$, hence must send it to the identity in $G$. We sum over all possibilities for the image of the generator $a_{n+1}$ in $G$ as the image of this generator is not restricted by the map $b$. To summarize, we have $n_{ab} = 1$ if the images of the first $n$ generators of $\pi_1(\Sigma_{n+1}^\bullet)$ agree and $a$ sends $b_{n+1}$ to the identity. It is zero otherwise. This describes the desired map. \qed
6.1 Introduction

Hennings’ original definition is presented in [14] as an invariant of oriented links giving way to a 3-manifold invariant. In [17], Kauffman and Radford simplified computations considerably by showing that the invariant could be defined for unoriented links, and presented an algorithm for computing the invariant in this case. As the method of Kauffman and Radford is easier to visualize and understand, our extension is formulated along the same lines.

Since we are interested in TQFTs, we will instead define this invariant on tangles as in [19]. We will also directly extend the case of ordinary Hopf algebras, so that if we set the coassociator $\Phi$ to be the identity on $H^{\otimes 3}$, we will recover the previous invariants.

First, we review the necessary assumptions. Let $H$ be a quasitriangular ribbon
quasi-Hopf algebra. We assume the following, where $\lambda$ is a right cointegral on $H$.

$H$ is unimodular

$\alpha, \beta$ are both invertible

$\epsilon(\alpha) = \epsilon(\beta) = 1 \quad (6.1.1)$

$\lambda(\alpha v S^{-1}(\beta)) \lambda(\alpha v^{-1} S^{-1}(\beta)) = 1$

$\lambda(\Lambda) = 1$

$\mathcal{M}$ is nondegenerate in the sense of Definition 3.2.6

**Remark 6.1.1.** These assumptions correspond directly to those necessary to define the Hennings algorithm in the ordinary Hopf case.

**Remark 6.1.2.** Recall that the assumption of unimodularity gives us the desired properties on integrals and cointegrals outlined in Section 3.3. In particular, we have a trace map $\text{tr}$.

In fact, as in [17], a trace map with the properties

$\text{tr}(xy) = \text{tr}(yx) \quad (6.1.2)$

and

$\text{tr}(S(x)) = \text{tr}(x) \quad (6.1.3)$

is all that is required to make the definition for the link invariant. Under our assumptions, we have such a trace map if and only if we have a cointegral with the desired properties. Our work will mostly involve the cointegral.

**Remark 6.1.3.** In the ordinary Hopf algebra case, the double construction produces a quasi-triangular Hopf algebra satisfying the necessary conditions. A double construction exists to produce quasi-triangular quasi-Hopf algebras, but is exceptionally
complicated, so this author will not treat the subject here. Since we assume that $H$ is
unimodular, the double construction for quasi-Hopf algebras will produce a quasi-Hopf
algebra satisfying the necessary conditions.

6.2 Rules for Computation

Suppose we are given a quasitriangular ribbon quasi-Hopf algebra $H$ satisfying the
assumptions in (6.1.1).

The Hennings TQFT is given by the following diagram of functors.

$$\tilde{\nu}_H : \text{Cob}_3 \cong T gl \xrightarrow{\Psi} DT gl \xrightarrow{\hat{\Psi}} \text{Vect}$$

(6.2.1)

The isomorphism $\text{Cob}_3 \cong T gl$ is given in Theorem 5.1.7. The injection $T gl \hookrightarrow T gl^{(1)}$ is
given in Lemma 4.3.12. We have two steps left to define. First, we define a functor $\Psi$
from $T gl^{(1)}$, our category of admissible bracketed tangles, to the category of decorated
flat tangles, $DT gl$. We then map from $DT gl$ to a map on the quasi-Hopf algebra $H$
via a functor $\hat{\Psi}$. The entire Hennings TQFT is the composition of these functors,
and will be denoted $\tilde{\nu}_H$.

First, we define $\Psi : T gl^{(1)} \to DT gl$. Let $T : \overline{2m} \to \overline{2n}$ be a morphism in $T gl^{(1)}$. We
map $T$ to a decorated flat tangle $\hat{T} \in DT gl$ by systematically replacing the generators
of $T gl^{(1)}$ by elements of $DT gl$.

First, we replace the crossing generators $c$ and $c^{-1}$ as in Figure 6.1, where
$$R = \sum s_i \otimes t_i.$$

Remark 6.2.1. Recall that for the crossing generators, no change of parenthesis are
needed.

Next, we replace the generators $\cup$ and $\cap$ with beaded versions of these generators.
See Figure 6.2. This bead must always be the closest bead to the maximum or
minimum.
Finally, we replace the generator $a$ changing the parenthesis by appropriate factors of $\Phi$. If we must change parenthesis on more than three individual strands, we can consider strands grouped together to be the coproduct $\Delta_S$ on one of the strands. This consideration allows us to collapse the change of parenthesis into three strands only. For instance, the left-hand side of Figure 6.3 depicts applying beads to the generator $a$. On the right-hand side of Figure 6.3, we consider the first two strands to be the coproduct of a single strand, hence we must apply $(\Delta \otimes \text{id} \otimes \text{id})(\Phi)$ to the four strands.

\[
\begin{align*}
\left(\left( \begin{array}{c} X_i \\ Y_i \\ Z_i \end{array} \right) \right) & \quad \rightarrow \quad \left( \begin{array}{c} X'_i \\ Y'_i \\ Z'_i \end{array} \right) \\
\left( \begin{array}{c}
\left( \begin{array}{c} X_i \\ Y_i \\ Z_i \end{array} \right)
\end{array} \right) & \quad \rightarrow \quad \left( \begin{array}{c}
\left( \begin{array}{c} X'_i \\ Y'_i \\ Z'_i \end{array} \right)
\end{array} \right)
\end{align*}
\]
Note that we have suppressed the summation symbol in Figure 6.3; this will be the case for the rest of our diagrams.

This replacement procedure produces a tangle $\hat{T} \in DTgl$. The functor $\Psi$ is then defined on the integer objects as $\Psi : 2n \to 2n$ and on bracketed tangles as $\Psi(T) = \hat{T}$.

**Remark 6.2.2.** Proving that $\hat{\Psi} \circ \Psi$ is well-defined is the aim of Section 6.3, and is the content of Theorem 6.3.24. As in the non-twisted case, $\Psi$ will factor through Moves I - IV and Move R, hence is only well-defined on $TGL(1)$. The remaining moves in $Tgl(1)$ will factor through the composition.

**Remark 6.2.3.** We could have defined $\Psi$ on the full tangle category $TGL(1)$, but as we are concerned with cobordisms, we need only define the functor on the subcategory $Tgl(1)$.

We next define the functor $\hat{\Psi}$, which is only defined on the image $\Psi(Tgl(1))$, hence we must have tangles in $DTgl$ whose source and target objects are both even numbers, as well as restrictions on the types of components. See Definition 4.3.11. Let $T : 2m \to 2n$ be such a decorated tangle. Using the equivalence relations in the category $DTgl$, presented in Figure 4.6, find an equivalent tangle $T'$ in which on each component, all of the beads have been collected into a single bead, and each component is arranged in one of the options depicted in Figure 6.4. For the image $\Psi(Tgl(1))$, these are all the possibilities that can occur.

**Remark 6.2.4.** To replace a twist by the element $G$ or $G^{-1}$, we must have no other beads in the twist.

Finally, we define $f_T \in H$-mod as we did in the case of the original Hennings algorithm. See Section 5.2. We present this map again for the reader’s convenience in Figure 6.4. Recall that each pair of strands denotes one factor of the resulting map, and for a genus $n$ surface, we have $2n$ total strands.
The maps corresponding to each of the types of tangles in Figure 6.4 are given in the following list.

- (1) : \( k \to k \) sends \( a_i \mapsto \lambda(a_i) = \text{tr}(a_i G^{-1}) \)
- (2) : \( k \to H \) sends \( 1 \mapsto b_j \)
- (3) : \( H \to k \) sends \( x_k \mapsto \lambda(S(x_k)c_k) \)
- (4) : \( H \to H \) sends \( x_n \mapsto p_n x_n S(q_n) \)

The functor \( \hat{\Psi} \) sends objects \( 2m \) to \( H^\otimes m \) and morphisms \( \hat{\Psi}(T) = f_T \) as for ordinary Hopf algebras, but now the beads are elements of a quasi-Hopf algebra.

**Remark 6.2.5.** The functor \( \hat{\Psi} \) is still well-defined on the image \( \Psi(Tgl^{1}) \). These images have the same underlying linear combinations of flat tangles, but the beads are now in the quasi-Hopf algebra \( H \). The relationships amongst the special elements in \( H \) ensure that the relations on tangles are preserved. We are also assuming that we have a cointegral \( \lambda \) so that

\[
\lambda(ab) = \lambda(S^2(b)a) \tag{6.2.2}
\]

as in the ordinary Hopf algebra case. See (3.3.25).
We have now defined the functor $\tilde{\nu}_H : \text{Cob}_3^\bullet \to H\text{-mod}$, and we would like to show that this in fact gives rise to a TQFT. In general, our procedure will be to check that the functor $\Psi$ is well-defined, or that we map equivalent tangles in $Tgl^{(i)}$ to equivalent tangles in $DTgl$. Recall from Lemma 4.3.8 that the equivalence of tangles is generated by the following.

Isotopies (6.2.3)

Moves I, II, III, IV on tangles (6.2.4)

The framed 1st Reidemeister Move, or Move R (6.2.5)

Rebracketing, or the Pentagon and Hexagon Axioms (6.2.6)

2-handle slides or the 2nd Kirby Move (6.2.7)

The Modified 1st Kirby Move, or removal of an isolated Hopf link in which one component has zero framing (6.2.8)

The $\sigma$ Move (6.2.9)

We will verify that the functor is well-defined in the next section.

As a first example, we have the following.

**Lemma 6.2.6.** A single-strand twist maps to an untangled strand decorated with the special element $\nu$ under the functor $\Psi : Tgl^{(i)} \to DTgl$.

**Proof.** For the replacement of individual parts of the diagram, see Figure 6.5. In the figure, recall that we sum over all indices $i, j, k$. We gather the beads together at the top of the diagram, using the rule that to move clockwise through a maximum or a minimum we apply the antipode to the bead. We then combine the beads using our multiplication rule. This results in a single bead at the top of the diagram whose element is

$$\sum S^2(\bar{Z}_k \bar{s}_j Y_i) S(\bar{X}_k X_i \beta) \alpha \bar{Y}_k \bar{t}_j Z_i. \quad (6.2.10)$$
Figure 6.5: A Single-Strand Twist

Figure 6.6: Diagram Corresponding to $u$
We would like to show that this element is the same as our special element \( u \) defined in (3.2.1). The algebraic calculations to show that the two elements are the same would be cumbersome and unenlightening, so we work instead using the diagrams. This method of producing equivalent expressions for \( u \) will be valid once we have proven that tangles related by Moves I - IV and R in \( Tgl^{(l)} \) give the same beaded diagrams in \( DTgl \).

We first write a diagram corresponding to the element \( u \) as described in (3.2.1). This results in the diagram on the left-hand side of Figure 6.6. Mapping backwards to \( Tgl^{(l)} \) we will get the diagram with parenthesis given on the right-hand side of Figure 6.6.

The tangle on the left-hand side of Figure 6.5 is equivalent to the tangle on the right-hand side of Figure 6.6 when we use Move IV to move the strand at the top of the tangle in Figure 6.5 past the maximum. Hence, the two diagrams are equivalent, and the element in (6.2.10) is the same as the element in (3.2.1).

We have collected our beads into a single bead at the top of the figure, but the diagram still contains a twist. In \( DTgl \), this twist may be replaced by the element \( G^{-1} \). We then combine the two beads into one final bead, giving the desired result. See Figure 6.7.

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Figure 6.7: Gathering the Beads}
\end{array}
\end{array}
\end{align*}
\]
Remark 6.2.7. Using Move R, this also means that a negative single-strand twist, or
the inverse of the single-strand twist in Figure 6.5, must map to an untwisted strand
decorated with the inverse of the ribbon element $v^{-1}$.

Remark 6.2.8. The result in Lemma 6.2.6 is identical to the ordinary Hopf case.

At this point, we may continue some of our previous discussions. First, we provide
the continuation of Remark 5.2.5.

Remark 6.2.9. We may, using the extended Hennings map, associate tangles to par-
ticular elements in a quasi-Hopf algebra $H$. We discuss the basics of the association
here.

The association is between tangles decorated under the first part of our map
with quasi-Hopf algebra elements and the quasi-Hopf algebra elements themselves.
For example, we associate the reassociator $\Phi$ with the tangle given in Figure 4.7 in
which the parenthesis change. We associate the $R$-matrix with the positive crossing
$c \in Tgl^{(i)}$.

Similarly, the tangles in Figure 6.8 will correspond to the elements $p_L, p_R, q_L,$ and
$q_R$ by the beads which will be attached to the tangles under the first part of our
Hennings algorithm.

Finally, we see that relations in the quasi-Hopf algebra have tangle equivalences.
For instance, the Pentagon Axiom, depicted in Figure 4.9, corresponds exactly to
the Pentagon Axiom (3.1.2) in the algebraic definition of a quasi-Hopf algebra. The
Hexagon Axiom behaves similarly.

Second, we return to the coproduct on components. Suppose $T \in TGL^{(i)}$ and $S$
is a component of $T$. By an abuse of notation, use $\Psi : TGL^{(i)} \to DTgl$ as a functor
on the full category $TGL^{(i)}$, defined in the same way we defined $\Psi : Tgl^{(i)} \to DTgl$. 

101
Denote $\Psi(T) = \bar{T}$ and $\bar{S}$ the respective component in $\bar{T}$. Then we would like to show that $\Delta_S(T) = \Psi(\Delta_S(\bar{T}))$. In fact, we will only be concerned with some specific examples. The relevant definitions are found in Section 4.4.

As a first example, consider the diagram in Figure 6.9. On the right-going arrows, we have applied the functor $\Psi : TGL \to DTgl$. On the downward arrows, we have applied our coproduct on the single-strand generator $\cap$. This square of maps
commutes when we consider the appropriate change of parenthesis on the diagram in the lower left corner.

The generator $\cup$ is a similar second example, and we have already discussed the case of the generators $c$ and $c^{-1}$, which is equivalent to the Hexagon Axiom. This same procedure gives us insight into the image of the generator $a$ under the functor $\Psi$, as we have defined $\Psi(a)$ so that a square similar to Figure 6.9 will commute. Applying $\Delta_S$ to Move I in $TGL^0$ is an interesting exercise.

As an example we will need shortly, we consider the single-strand twist, decorated by $v$ under the functor $\Psi$. Recall the properties of $\delta$, $\gamma$, and $f$ in Lemma 3.2.2. Consider Figure 6.10.

![Figure 6.10: The Strand Coproduct Applied to a Single-Strand Twist](image-url)
Remark 6.2.10. The bracketing on the left-hand side of Figure 6.10 is implied. Also, we have used large beads decorating two strands at once. The label on the bead is an element in $H^\otimes 2$ which denotes the two beads that would decorate the two strands.

Lemma 6.2.11. The tangle on the left-hand side of Figure 6.10 maps under $\Psi$ to a tangle in $DT_{gl}$ consisting of a formal sum of two straight strands labeled with the factors of $\Delta(v)$.

Proof. We work in $H \otimes H$. Gathering the beads at the bottom of the diagram, we have the following.

\[
\sum \Delta(X_k)\Delta(\bar{s}_j)\Delta(\bar{Y}_i)\delta(S \otimes S)(\Delta^\text{op}(\bar{Z}_i))(S \otimes S)(\Delta^\text{op}(Z_k))(S \otimes S)(\gamma_{21})
\]

\[
(S^2 \otimes S^2)(\Delta(Y_k))(S^2 \otimes S^2)(\Delta(\bar{t}_j))(S^2 \otimes S^2)(\Delta(\bar{X}_i))
\]

\[
(6.2.11)
\]

\[
= \sum \Delta(X_k\bar{s}_j\bar{Y}_i)\delta(S \otimes S)(\Delta^\text{op}(\bar{Z}_i))(S \otimes S)(\Delta^\text{op}(Z_k))(S \otimes S)(\gamma_{21})
\]

\[
(S^2 \otimes S^2)(\Delta(Y_k\bar{t}_j\bar{X}_i))
\]

\[
(6.2.12)
\]

Note that when moving the beads across a maximum or minimum we apply the antiautomorphism $S$ to each factor, hence we must switch the order in which the factors occur. Also, $\gamma_{21}$ refers to $\tau(\gamma)$. We use that for all $h \in H$,

\[
f\Delta(h)f^{-1} = (S \otimes S)(\Delta^\text{op}(S^{-1}(h)))
\]

\[
(6.2.13)
\]

and apply this to the middle terms in (6.2.12).

\[
= \sum \Delta(X_k\bar{s}_j\bar{Y}_i)\delta f\Delta(S(Z_i)f^{-1}f\Delta(S(Z_k))f^{-1}(S \otimes S)(\gamma_{21})
\]

\[
(S^2 \otimes S^2)(\Delta(Y_k\bar{t}_j\bar{X}_i))
\]

\[
(6.2.14)
\]

\[
= \sum \Delta(X_k\bar{s}_j\bar{Y}_i)\delta f\Delta(S(\bar{Z}_i)S(Z_k))f^{-1}(S \otimes S)(\gamma_{21})
\]

\[
(S^2 \otimes S^2)(\Delta(Y_k\bar{t}_j\bar{X}_i))
\]

\[
(6.2.15)
\]

In (6.2.15), we use $\delta f = \Delta(\beta)$ and simplify. Also, we simplify

\[
f^{-1}(S \otimes S)(\gamma_{21}).
\]

\[
(6.2.16)
\]
We know that $f^{-1} \gamma = \Delta(\alpha)$, so

$$\gamma_{21} = f_{21} \Delta^{\text{op}}(\alpha)$$  \hspace{1cm} (6.2.17)

$$(S \otimes S)(\gamma_{21}) = (S \otimes S)(f_{21} \Delta^{\text{op}}(\alpha))$$  \hspace{1cm} (6.2.18)

$$= (S \otimes S)(\Delta^{\text{op}}(\alpha))(S \otimes S)(f_{21})$$  \hspace{1cm} (6.2.19)

$$= f\Delta(S(\alpha))f^{-1}(S \otimes S)(f_{21})$$  \hspace{1cm} (6.2.20)

$$= f\Delta(S(\alpha))\Delta(G)(G^{-1} \otimes G^{-1})$$  \hspace{1cm} (6.2.21)

Note that in (6.2.21), we have used the property that $\Delta(G) = f^{-1}(S \otimes S)(f_{21})(G \otimes G)$ from (3.2.16). We also substitute (6.2.21) into (6.2.15).

$$= \sum \Delta(X_k \bar{s}_j \bar{Y}_i)\Delta(\beta)\Delta(S(Z_i)S(Z_k))f^{-1}f\Delta(S(\alpha))\Delta(G)(G^{-1} \otimes G^{-1})$$  \hspace{1cm} (S^2 \otimes S^2)(\Delta(Y_{k \bar{t}_j \bar{X}_i}))  \hspace{1cm} (6.2.22)

$$= \sum \Delta(X_k \bar{s}_j \bar{Y}_i)\beta S(\bar{Z}_i)S(Z_k)S(\alpha)G(G^{-1} \otimes G^{-1})$$  \hspace{1cm} (S^2 \otimes S^2)(\Delta(Y_{k \bar{t}_j \bar{X}_i}))  \hspace{1cm} (6.2.23)

Finally, we would like to simplify $(S^2 \otimes S^2)(\Delta(h))$ for any $h \in H$.

$$f\Delta(\alpha)f^{-1} = (S \otimes S)(\Delta^{\text{op}}(S^{-1}(h)))$$  \hspace{1cm} (6.2.24)

$$(S \otimes S)(f\Delta(\alpha)f^{-1}) = (S^2 \otimes S^2)(\Delta^{\text{op}}(S^{-1}(h)))$$  \hspace{1cm} (6.2.25)

Switching the two factors and rearranging (6.2.25), we have the following.

$$(S^2 \otimes S^2)(\Delta(h)) = (S \otimes S)(f_{21}\Delta^{\text{op}}(S(h))f_{21}^{-1})$$  \hspace{1cm} (6.2.26)

$$= (S \otimes S)(f_{21}^{-1})(S \otimes S)(\Delta^{\text{op}}(S(h)))(S \otimes S)(f_{21})$$  \hspace{1cm} (6.2.27)

$$= (S \otimes S)(f_{21}^{-1})f\Delta(S^2(h))f^{-1}(S \otimes S)(f_{21})$$  \hspace{1cm} (6.2.28)

$$= (S \otimes S)(f_{21}^{-1})f\Delta(S^2(h))\Delta(G)(G^{-1} \otimes G^{-1})$$  \hspace{1cm} (6.2.29)

$$= (G \otimes G)\Delta(G^{-1})\Delta(S^2(h))\Delta(G)(G^{-1} \otimes G^{-1})$$  \hspace{1cm} (6.2.30)

$$= (G \otimes G)\Delta(G^{-1}S^2(h)G)(G^{-1} \otimes G^{-1})$$  \hspace{1cm} (6.2.31)
We have made heavy use of the definition of $\Delta(G)$, here. Finally, recall that $S^2(h) = GhG^{-1}$, from (3.2.18). We further simplify (6.2.31) to find our intermediate result.

\[
(S^2 \otimes S^2)(\Delta(h)) = (G \otimes G)(\Delta(h))(G^{-1} \otimes G^{-1}) \tag{6.2.32}
\]

Finally, we substitute (6.2.32) into (6.2.23).

\[
= \sum \Delta(X_k \bar{s_j} \bar{Y_i} \beta S(\bar{Z_i})S(\bar{Z_k})S(\alpha)G)(G^{-1} \otimes G^{-1})
\]

\[
(G \otimes G)(\Delta(Y_k \bar{t_j} \bar{X_i}))(G^{-1} \otimes G^{-1}) \tag{6.2.33}
\]

\[
= \sum \Delta(X_k \bar{s_j} \bar{Y_i} \beta S(\bar{Z_i})S(\bar{Z_k})S(\alpha)GY_k \bar{t_j} \bar{X_i}))(G^{-1} \otimes G^{-1}) \tag{6.2.34}
\]

\[
= \sum \Delta(X_k \bar{s_j} \bar{Y_i} \beta S(\bar{Z_i})S(\bar{Z_k})S(\alpha)S^2(Y_k \bar{t_j} \bar{X_i})G)(G^{-1} \otimes G^{-1}) \tag{6.2.35}
\]

\[
= \sum \Delta(X_k \bar{s_j} \bar{Y_i} \beta S(\bar{Z_i})S(\bar{Z_k})S(\alpha)S^2(GY_k \bar{t_j} \bar{X_i}))G)(G^{-1} \otimes G^{-1}) \tag{6.2.36}
\]

We have used again that $S^2(h) = GhG^{-1}$, so that $Gh = S^2(h)G$. Finally, we have that the quantity inside the coproduct here is the element $S(u)$ for our $u$ discussed earlier, though in a different equivalent form. Then (6.2.36) simplifies to the following.

\[
= \Delta(S(u)G)(G^{-1} \otimes G^{-1}) \tag{6.2.37}
\]

\[
= \Delta(v)(G^{-1} \otimes G^{-1}) \tag{6.2.38}
\]

When we untwist the two strands in $DTgl$, each strand is multiplied by a factor of $G$ on the right, so the final element on the straightened strand is $\Delta(v)$, as desired. □

### 6.3 Proof of Theorem 6.3.24

We now turn to the verification that our functor $\tilde{\nu}_H$ is well-defined. Our goal is to prove a series of lemmas checking that the various types of equivalent tangles in (6.2.3) through (6.2.9) give rise to equivalent elements in $H$-mod. We look mostly at the functor $\Psi : Tgl \to DTgl$, as we have already seen that $\hat{\Psi}$ is well-defined.
Lemma 6.3.1. The functor $\hat{\nu}_H$ factors through Moves I and II, depicted in Figure 6.11.

Proof. We begin with Move I, also called a cancellation. See Figure 6.12. The beaded diagram for the left-hand side of Figure 6.11a corresponds to the equation

$$\sum X_i \beta S(Y_i) \alpha Z_i$$  \hspace{1cm} (6.3.1)

when collected at the bottom of the diagram. Similarly, the right-hand side of Figure 6.11a corresponds to the single bead

$$\sum \bar{Z}_i S^{-1}(\beta) S^{-1}(Y_i) S^{-1}(\alpha) \bar{X}_i = S^{-1}(\sum S(\bar{X}_i) \alpha Y_i \beta S(Z_i))$$  \hspace{1cm} (6.3.2)

when all of the beads are collected at the bottom of the diagram. In $DTgl$, we use Move I to cancel a maximum followed directly by a minimum, or a “wiggle”. This cancellation gives a straight strand with a single bead, labeled with (6.3.1).

In a quasi-Hopf algebra, the sum in (6.3.1) is equal to $1 \in H$, see (3.1.11). For the right-hand side of Figure 6.11a, a similar process gives (6.3.2), which is the inverse of the antipode applied to the element in (3.1.12), also equal to $1 \in H$. Since $S^{-1}(1) = 1$, we have the desired result. This means that both both the diagrams on left-hand side and the right-hand side are equivalent to a straight line with a single bead equal to the identity, or simply just a straight line. Thus, the functor factors through Move I.
For Move II, we note that no change in parenthesis is needed, since we have isolated the move inside its own set of parenthesis. This means that the proof for this case is precisely the same as the case in the regular Hopf case - the beads applied to this move by the functor $\Psi$ are precisely those associated to $R$ and $R^{-1}$. Since $RR^{-1} = 1 = R^{-1}R$, the functor factors through Move II.

\[\]

Lemma 6.3.2. The functor $\tilde{\nu}_H$ factors through Move III, depicted in Figure 6.13.

Figure 6.12: The Functor $\tilde{\nu}_H$ Applied to Move I

Figure 6.13: Move III
Proof. Note that we must ensure that the parenthesis at the top and bottom of each diagram are the same, so that the interior of one side of the diagram may be replaced by the other. This move corresponds to the quasi-Yang Baxter equation,

$$\Phi_{321} R_{23} \Phi^{-1}_{231} R_{13} \Phi_{213} R_{12} = R_{12} \Phi_{312} R_{13} \Phi^{-1}_{132} R_{23} \Phi_{123}. \quad (6.3.3)$$

Consider Figure 6.14, where the elements $a_i$ and $b_j$ are given by the following.

![Diagram](image)

Figure 6.14: Proof for Move III
\begin{align*}
  a_1 &= Z_k \tilde{Z}_j s_m Y_i s_n \quad (6.3.4) \\
  a_2 &= Y_k s_r \tilde{X}_j X_i t_n \quad (6.3.5) \\
  a_3 &= X_k t_r \tilde{Y}_j t_m Z_i \quad (6.3.6) \\
  b_1 &= s_r Y_k s_m \tilde{X}_j X_i \quad (6.3.7) \\
  b_2 &= t_r Z_k \tilde{Z}_j s_n Y_i \quad (6.3.8) \\
  b_3 &= X_k t_m \tilde{Y}_j t_n Z_i \quad (6.3.9)
\end{align*}

We have
\begin{equation*}
  a_1 \otimes a_2 \otimes a_3 = \Phi_{321} \mathcal{R}_{23} \Phi_{231}^{-1} \mathcal{R}_{13} \Phi_{213} \mathcal{R}_{12} \quad (6.3.10)
\end{equation*}
and
\begin{equation*}
  b_1 \otimes b_2 \otimes b_3 = \mathcal{R}_{12} \Phi_{312} \mathcal{R}_{13} \Phi_{132}^{-1} \mathcal{R}_{23} \Phi_{123}. \quad (6.3.11)
\end{equation*}

Now under our functor \( \Psi \) from \( T_{gl}(^0) \) to \( DT_{gl} \), the equality of the two tangles in Figure 6.13 corresponds to the equality
\begin{equation*}
  a_1 \otimes a_2 \otimes a_3 = b_1 \otimes b_2 \otimes b_3, \quad (6.3.12)
\end{equation*}
or the quasi-Yang-Baxter equation. We know that the Quasi-Yang Baxter equation holds in quasi-Hopf algebras, so these beads represent the same element in \( H \) and thus the same element in \( DT_{gl} \). Hence the full functor \( \nu_H \) also factors through Move III.

In order to prove equivalence under Move IV, we make an alternate definition for the generator \( c^{-1} \) of \( T_{gl}(^0) \). We must prove that the tangle in Figure 6.15 is the inverse of the tangle for \( c \) in \( T_{gl}(^0) \).

**Remark 6.3.3.** Since \((\epsilon \otimes 1 \otimes 1)(\Phi^{-1}) = 1 \otimes 1 = (1 \otimes 1 \otimes \epsilon)(\Phi)\), the first and last changes of parenthesis in Figure 6.15 are unnecessary.
Lemma 6.3.4. The tangle in Figure 6.15, which we label $d$, may be substituted for the negative crossing $c^{-1}$ in $T_{gl}^l$.

Proof. We use the Hexagon identity in $T_{gl}^l$, see Figure 4.10. We first draw in Figure 6.16 the tangle corresponding to $cd$. 

Figure 6.15: The Definition of $d$

Figure 6.16: Steps to Prove $cd$ is Trivial: Part 1
We elaborate on Figure 6.16. For the first equality in the figure, we have used an isotopy to stretch the maximum upwards to the top of the diagram. In the second equality, we have changed the interior parenthesis. As we have previously discussed, in the tangle category, changing the interior parenthesis produces an equivalent tangle. Our goal in this second step is to produce appropriate parenthesis to apply the Hexagon axiom in the tangle category as depicted in Figure 4.10. Next consider Figure 6.17.

We elaborate on Figure 6.17. We moved from Figure 6.16 to Figure 6.17 by applying the Hexagon Axiom, as stated. The first equality is an isotopy. The second equality is a change of parenthesis. In the third equality, we use Move I on the first strand.

The argument to show that $dc$ is also trivial is essentially the same. In this second case, we first use the Hexagon axiom and then an isotopy which moves the maximum downwards. We change the parenthesis as necessary, and uncross the diagram. After uncrossing the diagram, an isotopy and additional change of parenthesis moves the diagram into one straight strand and one strand with a wiggle. We straighten the
wiggle with Move I, hence we again have that the product of these two tangles is trivial in the tangle category.

\[ R^{-1} = \sum X_k X_i \beta S(Y_i) S(s_j) S(Y''_k) S(Y_m) \alpha Z_m Z_k \otimes X_m Y'_m t_j Z_i \] (6.3.13)

\[ = \sum X_k X_i \beta S(Y_m Y''_k s_j Y_i) \alpha Z_m Z_k \otimes X_m Y'_m t_j Z_i \] (6.3.14)

Remark 6.3.6. As we did with the element \( u \), we can write down many equivalent formulations for \( R^{-1} \) using equivalent tangle diagrams.

Lemma 6.3.7. The functor \( \tilde{\nu}_H \) factors through Move IV, depicted in Figure 6.18.

Proof. As usual, we will prove that \( \Psi : Tgl^{(1)} \to DTgl \) factors through Move IV. We first consider the top equality of tangles in Figure 6.18. We will show the equivalence of these tangles in the category \( Tgl^{(1)} \) using our earlier moves and Lemma 6.3.4, which is equivalent to Move IV. We will use this equivalence of tangles to produce an algebraic proof that the beaded diagrams are the same.

Consider the equality of tangles given in Figure 6.19. The first equality is given by inserting a “wiggle” into the lower part of the diagram using Move I. The second equality is given by using an isotopy to move the maximum upwards. The third equality is a rearrangement of parenthesis in the diagram in order to isolate the diagram for \( c^{-1} \) on the right-hand side of Figure 6.15. The final equality replaces the right-hand of Figure 6.15 with the left-hand side, or uses the equality of tangles.
we just proved for $c^{-1}$. This gives the desired diagram, and hence the two tangles are indeed equivalent in $T gl^0$ using only the moves we have previously verified and Lemma 6.3.4.

To prove the equality of tangles for the bottom equality in Figure 6.18, we turn each diagram in our previous proof upside-down. This proof is still valid, because if we turn our diagrams for $c^{-1}$ upside-down, we produce the same equivalence.

We translate the proof only on tangles to an algebraic proof. Consider the beaded diagram obtained by using the first part of the functor $\tilde{\nu}_H$ and the top equality in Figure 6.18. See Figure 6.20. Our goal is to show that

$$\sum X_bX_{a\beta}S(Y_a)S(s_i)S(Z_b) \otimes Y_b\iota_iZ_a = \sum s_jY_d\beta S(Z_d) \otimes \iota_jX_d.$$

(6.3.15)

Figure 6.18: Move IV on $T gl^0$
Figure 6.19: Equivalence of Move IV Tangles
First, we use the identity

$$
\sum S(\bar{X}_c)\alpha \bar{Y}_c \beta S(\bar{Z}_c)
$$

(6.3.16)

and insert this equation into the left-hand side of (6.3.15). The left-hand side becomes

$$
\sum \bar{X}_b X_a \beta S(Y_a) S(s_i) S(\bar{Z}_b) \otimes \bar{Y}_b t_i Z_a
$$

$$
= \sum \bar{X}_b X_a \beta S(Y_a) S(s_i) \bar{S}(\bar{X}_c) \alpha \bar{Y}_c \beta S(\bar{Z}_c) S(\bar{Z}_b) \otimes \bar{Y}_b t_i Z_a.
$$

(6.3.17)

Next, we change the parenthesis according to the same change performed in Figure 6.19. That is, the simplest way to see that the two sides of the next equation are equal is to use $\Psi$ on images three and four in Figure 6.19, which is a change of parenthesis. Then (6.3.17) becomes

$$
\sum X_b X_a \beta S(Y_a) S(s_i) S(Y_b') S(Y_c) \alpha Z_c Z_b \bar{Y}_d \beta S(\bar{Z}_d) \otimes X_c Y_b' t_i Z_a \bar{X}_d
$$

(6.3.18)

Finally, we use the equation for $\mathcal{R}^{-1}$ found in (6.3.13) and substitute in (6.3.18) to see

$$
\sum \bar{s}_j \bar{Y}_d \beta S(\bar{Z}_d) \otimes \bar{t}_j \bar{X}_d,
$$

(6.3.19)
which is precisely our goal in (6.3.15). The proof for the other half of Move IV is similar - we flip over the tangle diagrams and again use them to produce an algebraic proof. Thus, $\tilde{\nu}_H$ factors through Move IV.

\[ \]

**Remark 6.3.8.** Since we have that the tangles of Move IV are equivalent using the new tangle for $c^{-1}$ of Figure 6.15 and moves we have already proven, this proof is unnecessary. Move IV is equivalent to the new definition of $c^{-1}$ in Figure 6.15.

We next move towards proving that our functor factors through the Fenn-Rourke move, equivalent to the Second Kirby Move. First, however, some preliminaries.

**Lemma 6.3.9.** The element $\mathcal{M}$ satisfies the given relations.

\[
\mathcal{M} = \Delta(v^{-1})(v \otimes v) \quad (6.3.20)
\]

\[
\mathcal{M}^{-1} = \Delta(v)(v^{-1} \otimes v^{-1}) \quad (6.3.21)
\]

**Proof.** The second equation is the inverse of the first, hence we need only prove the first equation. We verify this relation using only isotopy on the tangles, then applying the functor $\Psi$. As our functor $\tilde{\nu}_H$ respects isotopy, the algebraic relation will be proved by the relation on the tangles. The isotopy between the two tangles is depicted in Figure 6.21.

Lemma 6.2.11 gives us that the first tangle in the figure maps to $\Delta(v^{-1})$ under $\Psi$, as we have applied the tangle functor $\Delta_S$ to the single-strand twist with the component $S$ the twist itself. In the middle tangle, we use the Hexagon Axiom and change parenthesis. Note that we have indicated only the minimal bracketing in the middle diagram. More changes of parenthesis are needed than are marked; we omit intermediate steps in changing the parenthesis. Finally, the image of the third tangle would be decorated with the element $\mathcal{M}(v^{-1} \otimes v^{-1})$, the desired result. \[ \]
Lemma 6.3.10. The relation depicted in Figure 6.22 holds under the functor $\tilde{\nu}_H$, where the labeling \("-1\) on the circle represents a negative twist, or a framing number of $-1$.

Note that with a framing number, when we insert the twist it must be properly parenthesized.
Proof. For the first equality in Figure 6.22 we use Move IV, and therefore the functor \( \tilde{\nu}_H \) satisfies this first relation. We must investigate the second equality. The tangle on the right-hand side of Figure 6.22 maps under \( \Psi \) to a single strand with one bead labeled \( v \), and one closed circle with one bead on the right-hand side labeled \( v^{-1} \), along with the usual beads at the minimum and maximum. That is, applying \( \hat{\Psi} \) we have \( \lambda(\alpha v^{-1} S^{-1}(\beta))v \). We wish to show that the middle diagram of Figure 6.22 maps to this same element under \( \hat{\Psi} \circ \Psi \). Consider Figure 6.23. Recall that \( v^{-1} \) is central, with \( S(v^{-1}) = v^{-1} \), hence this bead may be moved freely through any decorated tangle. Also recall that we suppress summation indices for \( p_L \) and \( q_L \) and write \( p_L = \sum \tilde{p}_1^1 \otimes \tilde{p}_2^2 \).

We may collect these beads as follows. On the closed circle, we have one bead labeled \( \tilde{q}^1 t_j v^{-1} s_i \tilde{p}^1 \) and on the straight strand we have one bead labeled \( \tilde{q}^2 s_j t_i \tilde{p}^2 \). Applying the map \( \hat{\Psi} : DTgl \to H\text{-mod} \), we have

\[
\sum \lambda(\tilde{q}^1 t_j v^{-1} s_i \tilde{p}^1)\tilde{q}^2 s_j t_i \tilde{p}^2. \tag{6.3.22}
\]
We know that \( v \) is in the center of \( H \), so we may move it anywhere in the product. Also, we substitute the element \( \mathcal{M} = \mathcal{R}^{\text{op}} \mathcal{R} = \sum t_j s_i \otimes s_j t_i \) and use the notation \( \mathcal{M} = \sum M^1 \otimes M^2 \). We rewrite (6.3.22) as
\[
\sum \lambda (v^{-1} \tilde{q}^1 M^1 \tilde{p}^1) \tilde{q}^2 M^2 \tilde{p}^2.
\] (6.3.23)

We simplify (6.3.23) into the desired result.
\[
\sum \lambda (v^{-1} \tilde{q}^1 M^1 \tilde{p}^1) \tilde{q}^2 M^2 \tilde{p}^2
= \sum \lambda (v^{-1} \tilde{q}^1 M^1 \tilde{p}^1) v v^{-1} \tilde{q}^2 M^2 \tilde{p}^2
= v(\lambda \otimes \text{id})[(v^{-1} \otimes v^{-1}) q_L M_p L]
\] (6.3.26)

Next, we have previously seen that \( \mathcal{M} = \Delta (v^{-1})(v \otimes v) \), so we may rewrite (6.3.26) as follows.
\[
= v(\lambda \otimes \text{id})[(v^{-1} \otimes v^{-1}) q_L \Delta (v^{-1})(v \otimes v) p_L]
\] (6.3.27)
\[
= v(\lambda \otimes \text{id})(q_L \Delta (v^{-1}) p_L)
\] (6.3.28)

Finally, the property of \( \lambda \) in Lemma 3.3.13 allows us to rewrite (6.3.28).
\[
= v \cdot \lambda (\alpha v^{-1} S^{-1}(\beta))
\] (6.3.29)

This is the desired result.

\[ \square \]

We would like to see that \( \tilde{\nu}_H \) factors through the Second Kirby Move, defined for links as follows.

**Definition 6.3.11.** [9], M.2. Let \( L \) be a framed link. Let \( C \) be a component of \( L \) with framing \( n \) and let \( C' \) be a parallel curve to \( C \). Let \( C'' \) be another component of \( L \). Replace \( C'' \) by the connected sum \( C'' \#_b C' \), where \( b \) is some band connecting \( C' \) to \( C'' \) and otherwise disjoint from the link. Replace \( n'' \), the framing of \( C'' \), by
\[
n'' + n \pm 2l(C, C'')
\] (6.3.30)
for \( l \) the linking number and taking the plus sign if and only if the band can be oriented to agree with some compatible orientations of \( C \) and \( C'' \). This procedure is called the Second Kirby Move.

**Remark 6.3.12.** In Definition 6.3.11, we use the following terminology: we slide component \( C'' \) over component \( C \).

We will replace the Second Kirby Move by the Fenn-Rourke Move.

**Definition 6.3.13.** [9], M.1. Let \( L \) be a link, and \( C \) an unknotted component of \( L \) labeled \( \pm 1 \). Let \( N \) be a tubular neighborhood of \( C \) and \( D \) a spanning disc with boundary a longitude of \( \partial N \). Cut \( S^3 - N \) along \( D \). If \( C \) is labeled \( +1 \), twist one side of \( D \) in a left-handed anticlockwise manner through \( 2\pi \) and reglue. If \( C \) is labeled \( -1 \), take the opposite twist and reglue. The component \( C \) is now unlinked from the rest of the diagram. This procedure is called the Fenn-Rourke Move.

If \( C'' \) is another component of \( L \) with framing \( n'' \), after performing the Fenn-Rourke Move, the framing on \( C'' \) will be given by

\[
    n'' \pm l(C, C'')^2, \tag{6.3.31}
\]

where \( l \) denotes the linking number and the plus or minus depends on whether \( C \) has framing plus or minus \( 1 \).

**Remark 6.3.14.** The effect of the Fenn-Rourke move is the following. If \( n \) vertical strands pass through the component \( C \), these \( n \) strands receive a full \( 2\pi \)-twist, and \( C \) is unlinked from these strands.

Theorems 2, 3, 6, and 8 in [9] give proofs in various situations that the Fenn-Rourke Move is equivalent to the Second Kirby Move. Since our manifold is not closed and compact, we need to also modify the First Kirby Move - instead of addition or removal of isolated closed components, we must take the Modified First Kirby Move - addition or removal of an isolated Hopf link in which one component has zero framing.
Remark 6.3.15. All of the statements and results thus far have been made only for links. We would like to extend these definitions to tangles in the following manner. Tangles have both open components, isomorphic to intervals, and closed components. For both the Second Kirby Move and the Fenn-Rourke Move, we restrict the component $C$ so that it must be a closed component of our tangle. The other component $C''$ may be either open or closed.

The proofs in [9] of the equivalence of these two moves are done locally, and in fact the graphics may be taken as tangles instead of links. Hence, the equivalence of the two moves still holds in our tangle category.

Therefore, we would like to extend Lemma 6.3.10 to the case of an arbitrary number of strands. This will prove the Fenn-Rourke Move. To do this, we essentially take the $n$-fold coproduct of our earlier result and use induction. Recall the discussion of a coproduct on tangles from Section 4.4. We again let $\Delta_R^{(n)}$ denote an $n$-fold coproduct which is concentrated on the right-hand side, this time using $\Delta$ in $H$. We have the following inductive definition.

\[
\Delta_R^{(2)}(x) = \Delta(x) = \sum x' \otimes x'' \quad (6.3.32)
\]
\[
\Delta_R^{(3)}(x) = (1 \otimes \Delta)(\Delta(x)) = \sum x' \otimes (x'')' \otimes (x'')'' \quad (6.3.33)
\]
\[
\Delta_R^{(n)}(x) = (1 \otimes ... \otimes 1 \otimes \Delta)\Delta_R^{(n-1)}(x) \quad (6.3.34)
\]

We may apply this $n$-fold coproduct to the quasi-Yang Baxter equation, generalizing the Hexagon Axiom to $n + 1$ elements. We discussed a corresponding result in $Tgl^{(l)}$ in Section 4.4, allowing us to separate $n$ over- or under-crossing strands and evaluate them individually.

Lemma 6.3.16. Let $a \in H$ be central with the property that $S(a) = a$. Let $\Omega = \Omega^1 \otimes \Omega^2$ be defined as follows.

\[
\Omega = q_L M_p L \quad (6.3.35)
\]
Under the functor $\hat{\Psi} \circ \Psi : Tgl^{(0)} \to Vect$, the strands of the tangle in Figure 6.24 correspond to

$$\lambda(a\Omega^1)\Delta^{(n)}_R(\Omega^2).$$

(6.3.36)

**Figure 6.24: Set Up for the Fenn-Rourke Move**

**Remark 6.3.17.** This hybrid-type diagram in Figure 6.24 indicates that the bead $a$ will be applied under $\Psi$. When we apply the lemma, the bead $a$ will come from the framing of the closed component, and hence will satisfy the conditions of Lemma 6.3.16.

**Proof.** First, recall from Section 4 that we may deal with $n$ strands crossing over or under a single strand by applying the tangle coproduct $\Delta^{(n)}_R$ to the appropriate component of our crossing map. In this case, we choose the component $S$ to be the vertical strand of the center tangle in Figure 6.22. The resulting tangle is essentially
that pictured in Figure 6.24. Under our functor $\Psi$ for the upper crossing in Figure 6.24 we apply the factors of $(\text{id} \otimes \Delta_R^{(n)})(\mathcal{R})$ to the $n + 1$ strands and for the lower crossing we apply the factors of $(\Delta_R^{(n)} \otimes \text{id})(\mathcal{R})$ to the $n + 1$ strands.

To simplify the figure, instead of our usual bead decorations, we will decorate with horizontal lines labeled with the element whose factors would be used for the beads at that level. See Figure 6.25.

![Diagram](image.png)

Figure 6.25: The Quasi-Hennings Procedure Applied to the Tangle in Figure 6.24

We collect the beads on the vertical strands and on the right-hand side of the circular component. First, note that we can simplify the calculations at the top and bottom of the diagram. We move the $\beta$ across the maximum and combine it with the beads directly below, and then combine the $\alpha$ with the beads directly above.

\[
(1 \otimes S^{-1}(\beta) \otimes 1 \otimes \cdots \otimes 1)(S^{-1} \otimes 1 \otimes \Delta_R^{(n)})(\Phi^{-1}) = 1 \otimes (1 \otimes \Delta_R^{(n)})p_L \tag{6.3.37}
\]

\[
(S \otimes 1 \otimes \Delta_R^{(n)})(\Phi)(1 \otimes \alpha \otimes 1 \otimes \cdots \otimes 1) = 1 \otimes (1 \otimes \Delta_R^{(n)})q_L \tag{6.3.38}
\]
The bead $a$ may be placed anywhere in the diagram as we have assumed it is in the center of $H$ and $S(a) = a$. The two horizontal lines directly above and below the bead $a$ in Figure 6.25 represent two applications of $\mathcal{R}$ on opposite strands, hence may be combined to give the element
\[
1 \otimes (1 \otimes \Delta_R^{(n)})(\mathcal{M}). \tag{6.3.39}
\]
Note that the left-most coordinates in all of these elements are all the identity in $H$.

We combine all of this information together to see that we have
\[
(1 \otimes \Delta_R^{(n)})q_L \cdot (1 \otimes \Delta_R^{(n)})(\mathcal{M}) \cdot (1 \otimes \Delta_R^{(n)})p_L
= (1 \otimes \Delta_R^{(n)})(q_L \mathcal{M} p_L).
\tag{6.3.40}
\]

By the definition of $\Omega$, we rewrite (6.3.41) as
\[
(1 \otimes \Delta_R^{(n)})(\Omega). \tag{6.3.42}
\]
Multiplying by $a$ and applying the cointegral $\lambda$ on the first factor gives our final result in $\text{Vect}$:
\[
\lambda(a \Omega^1)\Delta_R^{(n)}(\Omega^2), \tag{6.3.43}
\]
which is the desired result.

\[\square\]

**Lemma 6.3.18.** The functor $\tilde{\nu}_H$ factors through the Fenn-Rourke move.

**Proof.** This is the result of the cases $a = v$ and $a = v^{-1}$ in Lemma 6.3.16.

A framing of $-1$ on the circular component of Figure 6.24 corresponds to taking $a = v^{-1}$. Lemma 6.3.16 gives that the strands are decorated with the element
\[
\lambda(v^{-1} \Omega^1)\Delta_R^{(n)}(\Omega^2). \tag{6.3.44}
\]
In Lemma 6.3.10, we proved that

\[ \lambda(v^{-1}\Omega^1)\Omega^2 = v \cdot \lambda(\alpha v^{-1}S^{-1}(\beta)). \]  

(6.3.45)

Applying the \( n \)-fold coproduct to both sides of this equation gives

\[ \lambda(v^{-1}\Omega^1)\Delta^{(n)}_R(\Omega^2) = \Delta^{(n)}_R(v)\lambda(\alpha v^{-1}S^{-1}(\beta)). \]  

(6.3.46)

Under \( \hat{\Psi} \circ \Psi \), the element on the left-hand side of 6.3.46 corresponds to the tangle on the left-hand side of Figure 6.26, while the element on the right-hand side of 6.3.46 corresponds to the tangle on the right-hand side of Figure 6.26. That is, the map \( \tilde{\nu}_H \) factors through the Fenn-Rourke move in the case of \( a = v^{-1} \). To complete the proof,

we must also consider the case of a framing of +1, or \( a = v \). We essentially reverse all of the crossings. Lemma 6.3.10 becomes

\[ \lambda(a\tilde{\Omega}^1)\tilde{\Omega}^2 = v^{-1}\lambda(\alpha vS^{-1}\beta), \]  

(6.3.47)
where
\[ \tilde{\Omega} = \tilde{\Omega}^1 \otimes \tilde{\Omega}^2 = q_L \mathcal{M}^{-1} p_L. \] (6.3.48)

Applying $\Delta_R^{(n)}$ to both sides of this equation gives
\[ \lambda(v\tilde{\Omega}^1)\Delta_R^{(n)}(\tilde{\Omega}^2) = \Delta_R^{(n)}(v^{-1})\lambda(\alpha v S^{-1}(\beta)), \] (6.3.49)

which is the desired result if we reverse the crossings in Figure 6.26. Hence, our functor $\tilde{\nu}_H$ factors through the Fenn-Rourke move, as desired.

The other case of Lemma 6.3.16 with which we are concerned is the case $a = 1$, and this gives us a formula for an integral $\Lambda$.

**Lemma 6.3.19.** The following gives a formula for an integral $\Lambda \in H$.
\[ \Lambda = \lambda(\alpha S^{-1}(\beta))(\lambda \otimes 1)(q_L \mathcal{M} p_L) \] (6.3.50)

**Proof.** We have just shown in Lemma 6.3.18 that our functor factors through the Fenn-Rourke move. Using the fact that the Fenn-Rourke Move is equivalent to the Second Kirby Move, we have the equivalence of the tangles in Figure 6.27. The

![Figure 6.27: The Tangle of Figure 6.28 Gives an Integral](image)
assumption that $\mathcal{M}$ is non-degenerate is equivalent to the statement that the tangle on the left-hand side of Figure 6.28 must map to an integral under the first part of our functor $\tilde{\nu}_H$. See [4], where the result is given for ordinary Hopf algebras. The proof is the same in the quasi-Hopf case. Applying the functor $\hat{\Psi} \circ \Psi$ to the tangle

![Tangle Diagram](image)

Figure 6.28: A Tangle for the Integral

on the left-hand side of Figure 6.28 gives the algebraic formula for $\Lambda$ in (6.3.50).

Remark 6.3.20. We have included the coefficient $\lambda(\alpha S^{-1}(\beta))$ from the closed circle as part of the definition of the integral.

Remark 6.3.21. Since we have assumed that $H$ is unimodular, this integral must be both a left and right integral in $H$.

Lemma 6.3.22. The functor $\tilde{\nu}_H$ factors through the addition or removal of a Hopf link where one component has zero framing, or the Modified 1st Kirby Move.
Proof. Via a result in [4], it suffices to show that two disjoint closed circles, one with +1 framing and the other with −1 framing, cancel. We have the tangles in Figure 6.29, with the first part of our functor applied.

Collecting the beads on the right-hand side of each closed component, we see that the resulting field element is

$$\lambda(\alpha v S^{-1}(\beta))\lambda(\alpha v^{-1} S^{-1}(\beta)),$$

which is trivial by assumption.

We now come to our final move.

Lemma 6.3.23. The functor $\tilde{\nu}_H$ factors through the $\sigma$ Move, depicted in Figure 6.30.

Proof. Note that the diagram for the $\sigma$-move should be properly parenthesized, which we will do shortly. See Figure 6.31 for the beaded diagram produced by $\Psi$. We have used several isotopies. First, we move the minimum and the maximum into side-by-side positions, then stretch the minimum downward and the maximum upward. Next, we shrink the center circle so that it surrounds the inner strands. Finally, we change the parenthesis to simplify our calculations. Note that when collecting beads,
the two changes of parenthesis labeled with $*$ have $\epsilon$ applied on one factor, hence are the identity. Thus, the corresponding beads are not included in the beaded diagram.

Collecting the beads and applying $\hat{\Psi}$ to this diagram, we have the following map.

$$x \mapsto \lambda(S(x)S(\bar{X}_i)q_1\bar{Y}_i^1\Lambda'p_1)q^2\bar{Y}_i^2\Lambda''p^2S(\bar{Z}_i) \quad (6.3.52)$$

One property of the elements $p_R$ and $p_L$ is that we have

$$\Lambda'p_1 \otimes \Lambda''p^2 = \Lambda^{'p_1} \otimes \Lambda''p^2 \quad (6.3.53)$$

for any right integral $\Lambda$, see [3], equation (3.21). We substitute this into (6.3.52).

$$x \mapsto \lambda(S(x)S(\bar{X}_i)q_1\bar{Y}_i^1\Lambda'p_1)q^2\bar{Y}_i^2\Lambda''p^2S(\bar{Z}_i) \quad (6.3.54)$$

Next, we note that the reassociator here is multiplied by $\Delta(\Lambda)$ as follows.

$$\bar{X}_i \otimes \bar{Y}_i^1 \Lambda' \otimes \bar{Y}_i^2'' \Lambda'' \otimes \bar{Z}_i = (1 \otimes \Delta \otimes 1)(\Phi^{-1}) \cdot (1 \otimes \Delta(\Lambda) \otimes 1) \quad (6.3.55)$$

This can be rewritten as

$$(1 \otimes \Delta \otimes 1)(\Phi^{-1} \cdot (1 \otimes \Lambda \otimes 1)) = \bar{X}_i \otimes \Delta(\bar{Y}_i\Lambda) \otimes \bar{Z}_i. \quad (6.3.56)$$
Since $\Lambda$ is also a left integral, however, we know that $\bar{Y}_{i}\Lambda = \epsilon(\bar{Y}_{i})\Lambda$, so our expression becomes

$$\bar{X}_{i} \otimes \epsilon(\bar{Y}_{i})\Delta(\Lambda) \otimes \bar{Z}_{i} = 1 \otimes \Delta(\Lambda) \otimes 1,$$

since $(1 \otimes \epsilon \otimes 1)(\Phi^{-1}) = 1 \otimes 1 \otimes 1$. 

Figure 6.31: The Map $\bar{\nu}_{H}$ Applied to the $\sigma$-Move
Substituting this result into (6.3.54), we have the following.

\[ x \mapsto \lambda(S(x)\bar{q}^1\Lambda'\bar{p}^1)\bar{q}^2\Lambda''\bar{p}^2 \quad (6.3.58) \]

\[ = \bar{q}^2\lambda(S(x)\bar{q}^1\Lambda'\bar{p}^1)\Lambda''\bar{p}^2 \quad (6.3.59) \]

Finally, we use (3.3.11) with \( h = S(x)\bar{q}^1 \) to simplify (6.3.59). This gives the final simplification of (6.3.52).

\[ x \mapsto \bar{q}^2\epsilon(\beta)\lambda(\Lambda)S^{-1}(S(x)\bar{q}^1\beta) \quad (6.3.60) \]

\[ = \lambda(\Lambda)\bar{q}^2S^{-1}(\beta)S^{-1}(\bar{q}^1)x \quad (6.3.61) \]

\[ = \lambda(\Lambda)\bar{Z}_jS^{-1}(\beta)S^{-1}(\bar{Y}_j)S^{-1}(\alpha)\bar{X}_jx \quad (6.3.62) \]

\[ = \lambda(\Lambda)S^{-1}(S(\bar{X}_j)\alpha\bar{Y}_j\beta S(\bar{Z}_j))x \quad (6.3.63) \]

\[ = \lambda(\Lambda)S^{-1}(1)x \quad (6.3.64) \]

\[ = \lambda(\Lambda)x = x \quad (6.3.65) \]

Note that we have assumed \( \lambda(\Lambda) = 1 \). Also, in (6.3.62) we have substituted the definition of \( q_L \), given originally in (3.2.34). Hence the relation on tangles in Figure 6.30 holds algebraically under the functor \( \tilde{\nu}_H \), as desired. \( \square \)

We now have all the required steps to prove the main theorem.

**Theorem 6.3.24.** The functor \( \tilde{\nu}_H \) is a TQFT.

**Proof.** By definition, the functor \( \tilde{\nu}_H \) is a TQFT as long as it is well-defined. The lemmas we have proved in this section show that equivalent tangles in \( Tgl^l(\) are mapped to the same element of \( Vect \). Recall the equivalence relations in \( Tgl^l(\) given in (6.2.3) - (6.2.9).

First, if two tangles are isotopic, they are decorated with the same beads under the first part of the functor. The only cause for concern is moving a maximum or minimum through a change of parenthesis, but this is possible with the assumptions
on $Tgl^{(1)}$ concerning moving morphisms through change of parenthesis. See Figure 4.13 and the relevant discussion.

Lemma 6.3.1 gives us that the functor preserves the first two tangle moves. Lemma 6.3.2 gives us that the functor preserves Move III. Lemma 6.3.7 gives that the functor preserves Move IV.

The framed 1st Reidemeister Move, or Move R, is given in Figure 6.32. We see from the figure that the functor $\tilde{\nu}_H$ sends this tangle to the identity map, hence the functor factors through this move as desired.

![Figure 6.32: Framed 1st Reidemeister Move](image)

**Remark 6.3.25.** The parenthesis have been suppressed in Figure 6.32. Recall, however, that we have included any change of parenthesis necessary into the element $v$, so that we replace the entire curl, including all change of parenthesis, by the element $v$ or $v^{-1}$, as appropriate.
Rebracketing of tangles is equivalent in the tangle category to using the Pentagon Axiom and Coherence Theorem. Any quasi-Hopf algebra also has a Pentagon Axiom and hence a Coherence Theorem, and we have seen that these coincide. Hence the functor $\tilde{\nu}_H$ factors through any rebracketing.

We need next that the two Kirby moves are satisfied; instead we use the Modified First Kirby Move given in Lemma 6.3.22 and the Fenn-Rourke Move of Lemma 6.3.18. These two lemmas prove that the functor $\tilde{\nu}_H$ factors through each of these moves.

Finally, Lemma 6.3.23 gives that our functor factors through the $\sigma$-move. These are all of the moves in $Tgl^{()}$, hence this concludes the proof.

6.4 Example: Computations on $D^{\omega}[G]$

We would like to apply our construction to a concrete case. Since our ultimate goal is to prove the equivalence of the Hennings TQFT applied on $D^{\omega}[G]$ to the Dijkgraaf-Witten TQFT, we will apply all of our previous calculations to the specific case of $D^{\omega}[G]$. Refer to Section 3.4 for calculations on many of the special elements.

Remark 6.4.1. Once we prove that all of our original assumptions are satisfied, all of our calculations in this section are redundant. We present the calculations here as they help us to understand $D^{\omega}[G]$ and may help us to understand the TQFT in general.

Our definitions for the integral and cointegral as well as their relevant properties are given in Section 3.4. We verify that our assumptions about integrals and cointegrals are satisfied.
Lemma 6.4.2. The following equalities hold.

\[ \lambda(\Lambda) = 1 \quad (6.4.1) \]

\[ \lambda(\alpha v S^{-1}(\beta)) \lambda(\alpha v^{-1} S^{-1}(\beta)) = 1 \quad (6.4.2) \]

Proof. First, for (6.4.1) we have the following.

\[ \lambda(\Lambda) = \lambda(\sum_{h} \frac{e}{h}) = \sum \omega^{-1} \delta_{h,e} = 1 \quad (6.4.3) \]

For (6.4.2) we first recall that \( v = \beta u \) (see Lemma 3.4.16), \( S(\beta) = \beta^{-1} \) (see Lemma 3.4.11), and \( S^2(x) = \beta^{-1} x \beta \) (see Lemma 3.4.12). We consider the two elements on the left-hand side of (6.4.2) separately. For the first, we have the following.

\[ \lambda(\alpha v S^{-1}(\beta)) = \lambda(\beta u \beta^{-1}) \quad (6.4.4) \]

\[ = \lambda(S^{-2}(u)) \quad (6.4.5) \]

\[ = \lambda(u) \quad (6.4.6) \]

\[ = \lambda(\sum g \frac{g}{g^{-1}}) \quad (6.4.7) \]

\[ = \sum \omega^{-1} \delta_{g^{-1},e} = 1 \in k \quad (6.4.8) \]

For the second, we have the following.

\[ \lambda(\alpha v^{-1} S^{-1}(\beta)) = \lambda(v^{-1} \beta^{-1}) \quad (6.4.9) \]

\[ = \lambda(\sum g \frac{g}{g} \omega^{-1} \frac{h}{h} \frac{e}{e}) \quad (6.4.10) \]

\[ = \lambda(\sum \omega_{g} \frac{g}{g}) \quad (6.4.11) \]

\[ = \sum \omega_{g} \omega^{-1} \delta_{g,e} = 1 \in k \quad (6.4.12) \]

Together, we see \( \lambda(\alpha v S^{-1}(\beta)) \lambda(\alpha v^{-1} S^{-1}(\beta)) = 1 \cdot 1 = 1 \) as desired. \hfill \Box

All of our assumptions in (6.1.1) are now satisfied.

Our goal is to follow the verification in Section 6.3. Note that all of this work is covered by the general case, but the algebraic calculations in the case of \( D\omega[G] \) are much simpler than the general case, and may shed light on some of the formulae.
Lemma 6.4.3. The functor $\tilde{\nu}_H$ factors through Move I and Move II.

Proof. This is Lemma 6.3.1 in the general case. Recall Figure 6.11a. We have the following computation for the left-hand side of the figure.

$$\sum X_i \alpha S(Y_i) \beta Z_i = \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.13)$$

$$= \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.14)$$

$$= \sum \omega(g, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.15)$$

$$= \sum \omega(g, g^{-1}_e, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.16)$$

$$= \sum \omega(g, g^{-1}_e, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.17)$$

$$= \sum \omega(g, g^{-1}_e, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.18)$$

$$= \sum g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.19)$$

On the right side of the figure, we have the following similar computation.

$$\sum Z_i S^{-1}(\beta) S^{-1}(Y_i) S^{-1}(\alpha) X_i \sum (6.4.20)$$

$$= \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.21)$$

$$= \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.22)$$

$$= \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.23)$$

$$= \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.24)$$

$$= \sum \omega(c, b, a) \omega^{-1} g^{-1}_e c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.25)$$

$$= \sum c \alpha S^{-1}(b^{-1}_e) g^{-1}_e c \sum (6.4.26)$$

Thus both cases of Move I are verified.

For Move II, recall Figure 6.11b. Recall from (3.4.46) that

$$\mathcal{R}^{-1} = \sum s_i \otimes \bar{t}_i = \sum \theta(y^{-1}xy, y^{-1}, y)^{-1} x \otimes y^{-1} \sum (6.4.27)$$
Then the functor $\Psi$ sends the left-hand side of Figure 6.11b to the right-hand side of Figure 6.33.

![Diagram of Move II Beads]

Figure 6.33: Move II Beads

We verify that $R^{-1}R = 1 \in D^2[G]^{\otimes 2}$.

\[
\sum s_is_j \otimes \bar{t}_it_j = \sum \theta(y^{-1}xy, y^{-1}, y)^{-1} x \prod_y g \prod_h \otimes y^{-1} \prod_e h \prod_e \tag{6.4.28}
\]

\[
= \sum \theta(y^{-1}xy, y^{-1}, y)^{-1} \theta(x, y, h) \delta_{x,y} \theta(y^{-1}, e, e) \delta_{y^{-1}, h} x \prod_{y,h} \otimes y^{-1} \prod_e \tag{6.4.29}
\]

\[
= \sum \theta(y^{-1}xy, y^{-1}, y)^{-1} \theta(x, y^{-1}, y) \delta_{x,y} \prod_e x \otimes y^{-1} \prod_e \tag{6.4.30}
\]

\[
= \sum \theta(x, y, y^{-1})^{-1} \theta(x, y^{-1}, y) \prod_e x \otimes y^{-1} \prod_e \tag{6.4.31}
\]

\[
= \sum x \prod_e \otimes y^{-1} \prod_e \tag{6.4.32}
\]

\[
= 1 \otimes 1 \tag{6.4.33}
\]

Hence Move II is verified. $\square$

**Lemma 6.4.4.** The functor $\tilde{\nu}_H$ factors through Move III.
Proof. This is Lemma 6.3.2 in the general case. As depicted in Figure 6.14, we must verify that the quasi-Yang-Baxter equation holds for our definitions in $D^\omega[G]$. The left-hand side of the quasi-Yang-Baxter equation is the following.

$$
\Phi_{321} R_{23} \Phi_{13}^{-1} R_{13} \Phi_{213} R_{12} \\
= \sum \omega(c_1, b_1, a_1) \omega(c_2, b_2, a_2)^{-1} \omega(c_3, b_3, a_3) c_1 \downarrow e \quad c_2 \downarrow e \quad m \downarrow n \quad b_3 \downarrow e \quad k \downarrow l
$$

(6.4.34)

$$
\otimes b_1 \downarrow e \quad g \downarrow h \quad a_2 \downarrow e \quad a_3 \downarrow e \quad l \downarrow e \quad \otimes a_1 \downarrow e \quad h \downarrow e \quad b_2 \downarrow e \quad n \downarrow c_3 \downarrow e
$$

(6.4.35)

$$
= \sum \omega(c_1, b_1, a_1) \omega(c_2, b_2, a_2)^{-1} \omega(c_3, b_3, a_3) \delta_{c_1,c_2} \delta_{b_4,k} \delta_{b_1,g} \delta_{a_2,a_3} \delta_{a_1,h} \delta_{b_2,n}
$$

(6.4.36)

$$
c_2 \downarrow e \quad m \downarrow n \quad k \downarrow l \quad \otimes g \downarrow h \quad a_3 \downarrow e \quad l \downarrow e \quad \otimes h \downarrow e \quad n \downarrow c_3 \downarrow e
$$

(6.4.37)

$$
= \sum \omega(c_2, g, h) \omega(c_2, n, a_3)^{-1} \omega(c_3, k, a_3) \delta_{c_2,n} \delta_{a_3,t} \delta_{h,n}
$$

(6.4.38)

$$
m \downarrow h \quad \otimes g \downarrow h \quad \otimes h \downarrow e
$$

(6.4.39)

$$
= \sum \omega(hkh^{-1}, g, h) \omega(hkh^{-1}, h, h^{-1} gh)^{-1} \omega(h, k, h^{-1} gh) \theta(hkh^{-1}, h, h^{-1} gh)
$$

(6.4.40)

$$
hkh^{-1} \downarrow gh \quad \otimes g \downarrow h \quad \otimes h \downarrow e
$$

(6.4.41)
Similarly, the right-hand side of the quasi-Yang-Baxter equation is the following.

\[ \mathcal{R}_{12} \Phi_{312} \mathcal{R}_{13} \Phi_{132}^{-1} \mathcal{R}_{23} \Phi_{123} \]  \hfill (6.4.42)

\[ = \sum \omega(c_1, b_1, a_1) \omega(c_2, b_2, a_2)^{-1} \omega(c_3, b_3, a_3) g \downarrow_{h} b_1 \downarrow_{e} m \downarrow_{n} a_2 \downarrow_{e} a_3 \downarrow_{e} \]  \hfill (6.4.43)

\[ \otimes h \downarrow_{e} c_1 \downarrow_{e} c_2 \downarrow_{e} \]  \hfill (6.4.44)

\[ = \sum \omega(c_1, b_1, a_1) \omega(c_2, b_2, a_2)^{-1} \omega(c_3, b_3, a_3) \delta_{g, h b_1, h^{-1}} \delta_{b_2, l} \delta_{a_1, n} \delta_{b_2, l} \]  \hfill (6.4.45)

\[ = \sum \omega(h, h^{-1} g h, n) \omega(k, l, n^{-1} m n)^{-1} \omega(c_3, b_3, a_3) \delta_{m, n a_2 n^{-1}} \delta_{h, c_1} \delta_{c_2, k} \delta_{a_1, n} \delta_{b_2, l} \]  \hfill (6.4.46)

\[ = \sum \omega(h, h^{-1} g h, n) \omega(k, l, n^{-1} m n)^{-1} \omega(n, b_3, a_3) \delta_{g, h m h^{-1}} \delta_{h, b_2, l} \delta_{a_1, n} \delta_{b_2, l} \]  \hfill (6.4.47)

To continue, in (6.4.47) we replace \( g \) by \( h k h^{-1} \), \( h \) by \( g \), and \( n \) by \( h \) and simplify.

\[ = \sum \omega(g, g^{-1} h k h^{-1} g, h) \omega(h, h^{-1} g^{-1} h k h^{-1} g h)^{-1} \]  \hfill (6.4.48)

\[ = \sum \omega(g, g^{-1} h k h^{-1} g, h) \omega(g, h, h^{-1} g^{-1} h k h^{-1} g h)^{-1} \]  \hfill (6.4.49)

\[ = \sum \omega(g, g^{-1} h k h^{-1} g, h) \omega(g, h, h^{-1} g^{-1} h k h^{-1} g h) \]  \hfill (6.4.50)

Since (6.4.50) and (6.4.41) are identical, we have the quasi-Yang-Baxter equation, as desired.

Next, we verify our formula for the inverse of the \( \mathcal{R} \)-matrix as well as move IV.
Lemma 6.4.5. The formula (6.3.13) for the inverse of the $R$-matrix found from the result of Lemma 6.3.4 simplifies to the definition of $R^{-1}$ in (3.4.46).

Proof. The proof is by computation.

\[
R^{-1} = \sum \omega(h^{-1}g, h^{-1}g^{-1}, g)\omega(h^{-1}, g^{-1}, g)\omega(h^{-1}g, h^{-1}g^{-1}h, h^{-1})
\]

\[
\omega_{g}^{-1}\gamma(h^{-1}, h^{-1}g, h^{-1}g^{-1}h)^{-1}\theta(h^{-1}g, h^{-1}, h)^{-1}g\big|_{h^{-1}}\otimes g\big|_{e} (6.4.51)
\]

\[
= \sum \theta(h^{-1}g, h^{-1}, h)^{-1}g\big|_{h^{-1}}\otimes g\big|_{e} (6.4.52)
\]

This is the formula for $R^{-1}$ that we specified in (3.4.46). □

Lemma 6.4.6. The functor $\tilde{\nu}_{H}$ factors through Move IV.

Proof. This is our verification of Lemma 6.3.7. Consider the beaded diagram given in Figure 6.20. This beaded diagram corresponds under $\Psi$ to the algebraic equality in (6.3.15), and in our specific case this equation is as follows.

\[
\sum \omega(a_{3}, a_{2}, a_{1})\omega(b_{3}, b_{2}, b_{1})^{-1}\omega_{k}^{-1}b_{1}\big|_{e} a_{1}\big|_{e} k\big|_{e} a_{2}^{-1}\big|_{e} S(g\big|_{h}) b_{1}^{-1}\big|_{e} \otimes b_{2}\big|_{e} h\big|_{e} a_{3}\big|_{e}
\]

\[
= \sum \omega(d_{3}, d_{2}, d_{1})^{-1}\theta(y^{-1}xy, y^{-1}, y)^{-1}\omega_{j}^{-1}x\big|_{y} d_{2}\big|_{e} j\big|_{e} d_{3}^{-1}\big|_{e}
\]

\[
\otimes y^{-1}\big|_{e} d_{3}\big|_{e} (6.4.53)
\]

The left-hand side of (6.4.53) simplifies to the following.

\[
\sum \theta(g^{-1}, h, h^{-1})^{-1}\gamma(h, g^{-1}, g)^{-1}\omega(h, h^{-1}g, h^{-1}g^{-1}h)
\]

\[
\omega(g, h, h^{-1}g^{-1}h)^{-1}\omega_{h}^{-1}(h^{-1}g^{-1}h)\big|_{h^{-1}}\otimes h\big|_{e} (6.4.54)
\]

\[
= \sum \theta(y^{-1}xy, y^{-1}, y)^{-1}\gamma(y^{-1}, y^{-1}xy, y^{-1}x^{-1}y)^{-1}\omega(y^{-1}, x^{-1}, x)
\]

\[
\omega(y^{-1}x^{-1}y, y^{-1}, x)^{-1}\omega_{y}^{-1}x\big|_{y} \otimes y^{-1}\big|_{e} (6.4.55)
\]

\[
= \sum \omega(y^{-1}x^{-1}y, y^{-1}xy, y^{-1})^{-1}\theta(y^{-1}xy, y^{-1}, y)^{-1}\omega_{y^{-1}}x\big|_{y} \otimes y^{-1}\big|_{e} (6.4.56)
\]
The right-hand side of (6.4.53) simplifies to

\[
\sum \omega(y^{-1}x^{-1}y, y^{-1}xy, y^{-1})^{-1}\theta(y^{-1}xy, y^{-1}, y)^{-1}\omega_{y^{-1}xy} x \nabla y^{-1} \otimes y^{-1} \nabla_{e},
\]

which matches the left-hand side. This verifies Move IV in one of the two cases; the other is very similar. \(\Box\)

Lemma 6.4.7. The following relations hold.

\[
\mathcal{M} = \Delta(v^{-1})(v \otimes v) \tag{6.4.58}
\]
\[
\mathcal{M}^{-1} = \Delta(v^{-1} \otimes v^{-1}) \tag{6.4.59}
\]

Proof. This will verify Lemma 6.3.9 for \(D^\omega[G]\). We instead verify the equivalent relation \(\mathcal{M}(v^{-1} \otimes v^{-1}) = \Delta(v^{-1})\). The calculation is as follows.

\[
\mathcal{M}(v^{-1} \otimes v^{-1}) = \sum x \nabla y \nabla g \otimes xyx^{-1} \nabla x \nabla h \nabla h
\]
\[
= \sum \theta(x, y, g)\theta(xy, x, h)\delta_{x,gy}x^{-1}\delta_{y,h}x \nabla yx^{-1} \nabla xy \nabla xh
\]
\[
= \sum \theta(x, y, y^{-1}xy)\theta(xy, x, y) x \nabla yx^{-1} \nabla xy \nabla xy \nabla x
\]
\[
= \sum \theta(s, s^{-1}r, r^{-1}sr)\theta(rs, s, s^{-1}r) s \nabla rs^{-1} \nabla sr \nabla r
\]
\[
= \sum \gamma(r, rs^{-1}, s) s \nabla rs^{-1} \nabla r
\]
\[
= \Delta(v^{-1})
\]

Recall that (6.4.59) is the inverse of (6.4.58). \(\Box\)

We defined the element \(\Omega\) in (6.3.35), in \(D^\omega[G]\) this is the element

\[
\Omega = \sum \omega(ghg^{-1}, g, g^{-1})^{-1}\omega_{h^{-1}gh}\omega(h^{-1}, h^{-1}gh, h^{-1}g^{-1}h)
\]
\[
g \nabla h \otimes ghg^{-1} \nabla g
\]

(6.4.66)
We are concerned with evaluating Lemma 6.3.16 for \( n = 1 \) on three values, given below.

\[
a = 1 = \sum_{e} x_{e} \quad (6.4.67)
\]

\[
a = v = \sum_{x^{-1}} \omega_{x}^{-1} x_{x^{-1}} \quad (6.4.68)
\]

\[
a = v^{-1} = \sum_{x} x_{x} \quad (6.4.69)
\]

**Lemma 6.4.8.** For the three values of \( a \) just stated, we have the following.

\[
\lambda(\Omega_{1} \cdot 1) \Delta(\Omega_{2}) = \Lambda \quad (6.4.70)
\]

\[
\lambda(\Omega_{1} \cdot v) \Delta(\Omega_{2}) = v^{-1} \quad (6.4.71)
\]

\[
\lambda(\Omega_{1} \cdot v^{-1}) \Delta(\Omega_{2}) = v \quad (6.4.72)
\]

**Proof.** For convenience, denote \( \psi(a) := \lambda(a\Omega_{1}) \Delta(\Omega_{2}) \). We simplify this map \( \psi \) for an arbitrary element \( \sum c_{x,y} x_{y} \in D^{e}[G] \) where \( x \) and \( y \) commute.

\[
\psi(\sum c_{x,y} x_{y}) = \sum \lambda(\Omega_{1} c_{x,y} x_{y}) \Omega_{2} \quad (6.4.73)
\]

\[
= \sum \omega(ghg^{-1}, g, g^{-1})^{-1} \omega_{h}^{-1} gh \omega(h^{-1}, h^{-1} gh, h^{-1} g^{-1} h)
\]

\[
c_{x,y} \lambda(g_{x} x_{y} \omega_{h}^{-1} g_{h}) \quad (6.4.74)
\]

\[
= \sum \omega(y^{-1} x y^{-1} x^{-1} y, y^{-1} x y, y^{-1} x^{-1} y)^{-1} \omega_{z} \omega(y, x, x^{-1})
\]

\[
c_{x,y} \theta(z, y, y^{-1}) \omega_{z}^{-1} x_{y}^{-1} y^{-1} x^{-1} y \quad (6.4.75)
\]

\[
= \sum \omega(y^{-1}, x, x^{-1})^{-1} \omega \omega(y^{-1}, x, x^{-1}) \theta(x, y, y^{-1}) \omega_{x}^{-1}
\]

\[
c_{x,y} y^{-1} \quad (6.4.76)
\]

\[
= \sum \theta(x, y, y^{-1}) c_{x,y} y^{-1} \quad (6.4.77)
\]

Substituting \( a = 1 \), we have \( c_{x,y} = 1 \) and \( x = x, y = e \). Then

\[
\psi(1) = \sum c_{x} \quad (6.4.78)
\]

142
Substituting \( a = v \), we have \( c_{x,y} = \omega_x^{-1} \) and \( x = x, y = x^{-1} \). Then

\[
\psi(v) = \sum \omega_x \omega_y^{-1} x \mid_x \omega_{x^{-1}} x \mid_x = \sum x \mid_x = v^{-1}. \tag{6.4.79}
\]

Finally, substituting \( a = v^{-1} \), we have \( c_{x,y} = 1 \), and \( x = x, y = x \). Then

\[
\psi(v^{-1}) = \sum \omega_x x^{-1} \mid_x = \sum \omega_x x^{-1} \mid_x = v. \tag{6.4.80}
\]

These are the desired results. The result for \( a = 1 \) verifies Lemma 6.3.19.

\[\square\]

**Lemma 6.4.9.** The functor \( \tilde{\nu}_H \) factors through the Fenn-Rourke Move.

**Proof.** This is our verification of Lemma 6.3.18. The outcomes for the elements \( v \) and \( v^{-1} \) in Lemma 6.4.8 give us our verification of the Fenn-Rourke move on one strand, or Lemma 6.3.10. The Fenn-Rourke move on \( n \) strands proceeds by induction as in the general case. \(\square\)

**Remark 6.4.10.** In the general case, Lemma 6.3.22 concerning the addition or removal of an isolated Hopf link was reduced to an assumption. As we have verified all of our assumptions, this lemma holds.

**Lemma 6.4.11.** The functor \( \tilde{\nu}_H \) factors through the \( \sigma \) Move.

**Proof.** The general proof of this fact is Lemma 6.3.23. See Figure 6.30 and Figure 6.31. Since we have \( \Lambda = \sum e \mid_g \), we have

\[
\Delta(\Lambda) = \sum \gamma(g, u, u^{-1}) u^{-1} \mid_g \otimes u \mid_g. \tag{6.4.81}
\]
The map corresponding to the $\sigma$-move is given in (6.3.58). We use an arbitrary basis element $x \downarrow y \in H$ and see the following.

\[
x \downarrow y \mapsto \sum \omega(b, a, a^{-1})^{-1}\omega(c, c^{-1})\gamma(g, u, u^{-1})
\]

\[
\lambda(S(x \downarrow y) u^{-1} \downarrow g \downarrow e) b \downarrow u \downarrow g \downarrow e \quad (6.4.82)
\]

\[
= \sum \omega_u^{-1}\omega(c, c^{-1})\gamma(g, u, u^{-1})
\]

\[
\lambda(S(x \downarrow y) u^{-1} \downarrow g \downarrow e) u \downarrow g \downarrow e \quad (6.4.83)
\]

\[
= \sum \omega_u^{-1}\omega(g^{-1}, u^{-1}, g)\gamma(g, u, u^{-1})\lambda(S(x \downarrow y) u^{-1} \downarrow g) \quad (6.4.84)
\]

\[
= \sum \omega_u^{-1}\theta(x^{-1}, y, y^{-1})^{-1}\gamma(y, x^{-1}, x)^{-1}\gamma(g, u, u^{-1})
\]

\[
\lambda(y^{-1} x^{-1} y^{-1} \downarrow u^{-1} \downarrow g) u \downarrow g \quad (6.4.85)
\]

\[
= \sum \omega_u^{-1}\theta(x^{-1}, y, y^{-1})^{-1}\gamma(y, x^{-1}, x)^{-1}\gamma(g, x, x^{-1})\theta(y^{-1} x^{-1} y, y^{-1}, g)
\]

\[
\lambda(y^{-1} x^{-1} y \downarrow g) \quad (6.4.86)
\]

\[
= \sum \omega_u^{-1}\theta(x^{-1}, y, y^{-1})^{-1}\gamma(y, x^{-1}, x)^{-1}\gamma(g, x, x^{-1})
\]

\[
\omega(y^{-1} x^{-1} y \downarrow g) \theta(y^{-1} x^{-1} y, y^{-1}, g) \delta_{y,g} \downarrow g \quad (6.4.87)
\]

\[
= \omega(x^{-1}, y, y^{-1})^{-1}\gamma(y, x^{-1}, x)^{-1}\gamma(g, x, x^{-1})\omega(y^{-1} x)
\]

\[
\theta(y^{-1} x^{-1} y, y^{-1}, y) \downarrow y \quad (6.4.88)
\]

\[
= x \downarrow y \quad (6.4.89)
\]

Note that equation (6.4.89) may be verified by the definitions of $\theta$, $\gamma$, and $\omega$. Thus, the $\sigma$-move is verified.

\[\square\]

This concludes our verification of the main theorem in Section 6.3.
CHAPTER 7
EQUIVALENCE OF TQFTS

We would like to prove that the Hennings TQFT applied to the quasi-Hopf algebra \( D^\omega[G] \) is equivalent to the Dijkgraaf-Witten TQFT. Towards this end, we begin by proving that the equivalence holds in the case where \( \omega \) is trivial, or \( \omega = 1 \). We defined the Hennings TQFT for \( D[G] \) in Section 5.2 and the Dijkgraaf-Witten TQFT in Section 5.3. Our main theorem will be the following.

**Theorem 7.1.3.** Let \( \nu_H \) refer to the Hennings functor and \( \nu_T \) refer to the Dijkgraaf-Witten functor. We have a natural isomorphism between these functors. That is, take \( f \) to be the map of Definition 7.1.1 and \( M : \Sigma_m \to \Sigma_n \) to be a cobordism. Then the following diagram commutes.

\[
\begin{array}{ccc}
D[G]^{\otimes m} & \xrightarrow{\nu_H(M)} & D[G]^{\otimes n} \\
\downarrow f & & \downarrow f \\
C(\Hom(\pi_1(\Sigma_m), G)) & \xrightarrow{\nu_T(M)} & C(\Hom(\pi_1(\Sigma_n), G))
\end{array}
\] (7.0.1)

### 7.1 Case: \( \omega = 1 \) or \( D[G] \)

Recall that \( k \) is a field of characteristic zero. Let \( G \) be a finite group and \( \Sigma_n^\bullet \) the once-punctured surface of genus \( n \). We would like to see that the Hennings TQFT applied to \( H = D[G] \) over \( k \) is equivalent to the Dijkgraaf-Witten TQFT for the trivial cocycle. As these are both functors, we produce a natural transformation
between them. Thus the desired equivalence will be given by Theorem 7.1.3. First, however, some preliminaries.

**Definition 7.1.1.** Define a map \( f : C(\text{Hom}(\pi_1(\Sigma_n^\bullet), G)) \rightarrow D[G]^\otimes n \) by

\[
f : \phi \mapsto \sum_{\gamma \in \text{Hom}(\pi_1(\Sigma_n^\bullet), G)} \phi(\gamma)(a_1) \otimes \cdots \otimes \phi(a_n) \otimes (b_1) \otimes \cdots \otimes \phi(b_n)
\]

where \((a_1, b_1, \ldots, a_n, b_n)\) are the generators of \(\pi_1(\Sigma_n^\bullet)\) and \(\phi(\gamma) \in k\).

**Remark 7.1.2.** Recall that for \(G\) a finite group, \(\text{Hom}(\Sigma_n^\bullet, G) \simeq (G^\times)^{2n}\).

For convenience, we denote the images of the generators as

\[
\{\gamma(a_1) = g_1, \gamma(b_1) = h_1, \ldots, \gamma(a_n) = g_n, \gamma(b_n) = h_n\}.
\]

(7.1.2)

Since we have a one-to-one correspondence between a homomorphism \(\gamma : \pi_1(\Sigma_n^\bullet) \rightarrow G\) and the images \(\{g_1, h_1, \ldots, g_n, h_n\}\) of the generators of the fundamental group, we may write \(\phi(\gamma) = \phi(g_1, h_1, \ldots, g_n, h_n)\). We then rewrite our map \(f\).

\[
f : \phi \mapsto \sum_{\gamma \in \text{Hom}(\pi_1(\Sigma_n^\bullet), G)} \phi(g_1, h_1, \ldots, g_n, h_n) \otimes g_1 \otimes \cdots \otimes g_n \otimes h_1 \otimes \cdots \otimes h_n
\]

(7.1.3)

Our graphics will have the puncture “spread out” as the base of the torus, and we will consider this puncture to be the basepoint for the fundamental group. For example, see Figure 7.1.

![Figure 7.1: Genus One Punctured Torus, \(\Sigma_1^\bullet\)](image-url)
Recall (or see [24, 15]) that the mapping class group $\Gamma(\Sigma_n)$ is generated by Dehn Twists. We label these Dehn Twists as $\{A_i : 1 \leq i \leq n\}$ for the meridian twists, $\{B_j : 1 \leq j \leq n\}$ for the longitude twists, and $\{C_k : 1 \leq k \leq n - 1\}$ for the twists between two holes, where $C_k$ stretches between the $k$th and the $k + 1$st hole. See Figure 5.2. The standard convention when working with elements of the mapping class group is to find the image of a generator under a Dehn Twist by either always making a “right-hand turn” or always making a “left-hand turn”. Our convention will be to always make a left-hand turn when crossing a twist.

Suppose we write a word $w_T$ which describes a twist $T$ in terms of the generators of the fundamental group. Each time a curve $\gamma$ crosses the twist $T$, we can then simply multiply the word corresponding to $\gamma$ with either $w_T$ or $w_T^{-1}$ to find the word for the resulting element of the fundamental group. If the crossing is positive in terms of $\gamma$, we use $w_T$, and if the crossing is negative in terms of $\gamma$, we multiply by $w_T^{-1}$. This choice corresponds precisely to our choice of the left-hand turn.

Recall that the corresponding tangles for Dehn Twists $A_i$, $B_j$ and $C_k$ are given by the tangles $A_i$, $B_j$ and $C_k$ in Figure 5.3. See [19, 18] for more information.

Finally, we use the standard generators for the fundamental group of $\Sigma_n^*$ as depicted in Figure 7.2. The fundamental group $\pi_1(\Sigma_n)$ is given by

$$\pi_1(\Sigma_n) = \langle a_1, b_1, ..., a_n, b_n | [a_1, b_1] ... [a_n, b_n] \rangle,$$

(7.1.4)

where $[a, b]$ denotes the commutator. Since we fix a specific basepoint for $\Sigma_n^*$ on the boundary or puncture, we have

$$\pi_1(\Sigma_n^*) = \langle a_1, b_1, ..., a_n, b_n \rangle.$$

(7.1.5)

As usual, we choose the multiplicative notation for the operation in the group.

We are now ready to prove our main theorem for this section.
Theorem 7.1.3. Let $\nu_H$ refer to the Hennings functor and $\nu_T$ refer to the Dijkgraaf-Witten functor. We have a natural isomorphism between these functors. That is, take $f$ to be the map of Definition 7.1.1 and $M: \Sigma^\bullet_m \to \Sigma^\bullet_n$ to be a cobordism. Then the following diagram commutes.

\[
\begin{array}{ccc}
D[G]^\otimes_m & \xrightarrow{\nu_H(M)} & D[G]^\otimes_n \\
f \downarrow & & \downarrow f \\
C(\text{Hom}(\pi_1(\Sigma^\bullet_m), G)) & \xrightarrow{\nu_T(M)} & C(\text{Hom}(\pi_1(\Sigma^\bullet_n), G))
\end{array}
\]

(7.1.6)

Proof. As we have reduced calculations on TQFTs to calculations on elements of the mapping class group plus special cobordisms, we need to show only that the diagram commutes for the generators of the mapping class group as well as the special cobordisms $H^n_+$ and $H^n_-$. For convenience, we split the proof into steps. The first is a simplification given in Lemma 7.1.4. We look at the Dehn Twists in Lemma 148.
Lemma 7.1.4. For a Dehn Twist $D$, $f \circ \nu_T(D)$ is the following.

\[ \phi \mapsto \sum \phi(\gamma(D_*(a_1)), \gamma(D_*(b_1)), \ldots, \gamma(D_*(a_n)), \gamma(D_*(b_n))) \]

\[ g_1 \mathbin{\bigotimes} \ldots \mathbin{\bigotimes} g_n \]

(7.1.7)

Proof. We use the definitions of the various maps. See Section 5.3.

\[ f \circ \nu_T(D) : \phi \mapsto \phi_D \mapsto \sum \phi_D(\gamma) \mathbin{\bigotimes} \mathbin{\bigotimes} \mathbin{\bigotimes} g_1 \mathbin{\bigotimes} \ldots \mathbin{\bigotimes} g_n \]

(7.1.8)

\[ = \sum \phi(\gamma \circ D_*) \mathbin{\bigotimes} \mathbin{\bigotimes} \mathbin{\bigotimes} g_1 \mathbin{\bigotimes} \ldots \mathbin{\bigotimes} g_n \]

(7.1.9)

\[ = \sum \phi(\gamma(D_*(a_1)), \gamma(D_*(b_1)), \ldots, \gamma(D_*(a_n)), \gamma(D_*(b_n))) \]

\[ g_1 \mathbin{\bigotimes} \ldots \mathbin{\bigotimes} g_n \]

(7.1.10)

This is the desired result. \qed

Remark 7.1.5. Since the special cobordisms $H^+_n$ and $H^-_n$ may also be formulated in terms of the images of the generators of the fundamental group, we may also apply Lemma 7.1.4 to these cobordisms. In this case, we replace $D_*(a_i)$ by the image of $a_i$ under the given cobordism, and $D_*(b_j)$ by the image of $b_j$ under the given cobordism.

Lemma 7.1.6. Theorem 7.1.3 holds for the generator $A_i$, $1 \leq i \leq n$.

Proof. Let $i$ be arbitrary. The Hennings map for the tangle $A_i$ is easy to compute given our earlier work. The strand labeled $i^-$ in Figure 5.3 receives the element $v$, while the map is the identity on all other strands. That is, we have

\[ \nu_H(A_i) : x_1 \otimes \ldots \otimes x_n \mapsto x_1 \otimes \ldots \otimes vx_1 \otimes \ldots \otimes x_n. \]

(7.1.11)
The top half of our diagram is then given by the following computation.

\[
\nu_H(A_i) \circ f : \phi \mapsto \sum \phi(g_1, h_1, \ldots, g_n, h_n) \frac{g_1 \otimes \cdots \otimes g_n}{h_1} \quad (7.1.12)
\]

\[
\mapsto \sum \phi(g_1, h_1, \ldots, g_n, h_n) \frac{g_1 \otimes \cdots \otimes v \cdot g_i \otimes \cdots \otimes g_n}{h_i} \quad (7.1.13)
\]

Recall that the element \( v \in D[G] \) is central and that \( v = \sum_{m} m^{-1} \frac{1}{m} \). We substitute in (7.1.13). Note that now we sum over all \( m \in G \) as well as all \( \gamma \in \text{Hom}(\pi_1(\Sigma_n^\bullet), G) \).

\[
= \sum \phi(g_1, h_1, \ldots, g_n, h_n) \frac{g_1 \otimes \cdots \otimes g_i \otimes \cdots \otimes g_n}{h_i} \quad (7.1.14)
\]

\[
= \sum \phi(g_1, h_1, \ldots, g_n, h_n) \frac{g_1 \otimes \cdots \otimes g_i \otimes \cdots \otimes g_n}{h_i \otimes \cdots \otimes h_n} \quad (7.1.15)
\]

Since we are summing over all values of \( g_i, h_i \in G \), we may make a substitution at the \( i \)th element: let \( k = g_i^{-1}h_i \), so that \( h_i = g_i k \). To make the notation more clear, we can then substitute again \( k = h_i \) to recover our original variables. We then find

\[
\sum \phi(g_1, h_1, \ldots, g_i, g_i h_i, \ldots, g_n, h_n) \frac{g_1 \otimes \cdots \otimes g_i \otimes \cdots \otimes g_n}{h_1 \otimes \cdots \otimes h_n} \quad (7.1.16)
\]

We now compute the bottom half of our diagram, and compare with the result in equation (7.1.16). The twist \( A_i \) has a non-trivial intersection with the generator \( b_i \); all other intersections are the identity. We consider the intersection of \( A_i \) and \( b_i \), depicted in Figure 7.3.

Following the left-hand turn convention we have established, we see that this means our twist \( A_i \) sends \( b_i \) to the element \( a_i b_i \) of the fundamental group. That is,

\[
a_j \mapsto a_j, \quad 1 \leq j \leq n
\]

\[
A_i : \quad b_j \mapsto b_j, \quad j \neq i
\]

\[
b_i \mapsto a_i b_i
\]

(7.1.17)

completely describes the behavior of the generators of the fundamental group under the action of the Dehn Twist \( A_i \). In other words, the word \( w_{A_i} \), corresponding to \( A_i \) is \( a_i^{-1} \) and we apply \( w_{A_i}^{-1} \) to \( b_i \) since the crossing is negative.
Using the fact that $\gamma(a_k) = g_k$ and $\gamma(b_k) = h_k$ for all $k$ along with our earlier general computation in Lemma 7.1.4, the lower half of the diagram is then

$$f \circ \nu_T(A_i) : \phi \mapsto \sum \phi(g_1, h_1, ..., g_i, h_i, ..., g_n, h_n) \downarrow_{h_1} \otimes ... \otimes \downarrow_{h_n}$$  \hspace{1cm} (7.1.18)

and thus the diagram commutes.

\[ \square \]

**Lemma 7.1.7.** Theorem 7.1.3 holds for the generator $B_j$, $1 \leq j \leq n$.

**Proof.** The Hennings procedure applied to the tangle given in Figure 5.3 for $B_j$ results in the beaded diagram given in Figure 7.4.
Let $\mathcal{R}^{-1} = \sum \tilde{s}_i \otimes \tilde{t}_i$. The beads in the figure are $b_1 = \sum S(\tilde{s}_i)\tilde{t}_i v^{-1}$ and $b_2 = \sum v^{-1} \tilde{s}_j S(\tilde{t}_i)$. In the case of $D[G]$, we know that $\mathcal{R}^{-1} = \sum g_h \otimes h^{-1} e$, so these elements simplify to
\[
b_1 \otimes b_2 = \sum_{k,m} k \downarrow_{mk} \otimes m \downarrow_{mk}.
\] (7.1.19)

According to the Hennings algorithm, the map corresponding to the decorated tangle is therefore
\[
x_1 \otimes \ldots \otimes x_n \mapsto \sum x_1 \otimes \ldots \otimes m \downarrow_{mk} \lambda(S(x_j)^k \downarrow_{mk}) \otimes \ldots \otimes x_n.
\] (7.1.20)

To simplify our calculations, we will omit the $n - 1$ factors on which the map is the identity. The top half of our diagram is then as follows, where we sum over all $k, m \in G$ as well as all $\gamma \in \text{Hom}(\pi_1(\Sigma_n^*)^), G$.
\[
\nu_H(B_j) \circ f : \phi \mapsto \sum \phi(\ldots, g_j, h_j, \ldots) \otimes g_j \downarrow_{h_j} \otimes \ldots\]
\] (7.1.21)
\[
\mapsto \sum \phi(\ldots, g_j, h_j, \ldots) \otimes m \downarrow_{mk} \lambda(S(g_j \downarrow_{h_j})^k \downarrow_{mk}) \otimes \ldots\]
\] (7.1.22)
\[
= \sum \phi(\ldots, g_j, h_j, \ldots) \otimes m \downarrow_{mk} \lambda((h_j^{-1}g_j^{-1}h_j^{-1}) \downarrow_{(h_j^{-1})}^k \downarrow_{(h_j^{-1})}) \otimes \ldots\]
\] (7.1.23)
\[
= \sum \phi(\ldots, g_j, h_j, \ldots) \otimes m \downarrow_{mg_j^{-1}} \lambda((h_j^{-1}g_j^{-1}h_j^{-1}) \downarrow_{(h_j^{-1})}^k \downarrow_{(h_j^{-1}mg_j^{-1})}) \otimes \ldots\]
\] (7.1.24)
\[
= \sum \phi(\ldots, g_j, h_j, \ldots) \otimes h_jg_j \downarrow_{h_j} \otimes \ldots\]
\] (7.1.25)

After substitutions similar to the case of $A_i$, we find the final result for the top of the diagram.
\[
\nu_H(B_j) \circ f : \phi \mapsto \sum \phi(\ldots, h_j^{-1}g_j, h_j, \ldots) \otimes g_j \downarrow_{h_j} \otimes \ldots\]
\] (7.1.26)

For the other half of the diagram, we first examine the intersections of the twist $B_j$ with the generators. The only generator that intersects $B_j$ is $a_j$, and this intersection is depicted in Figure 7.5.
Our left-hand turn convention gives us the following description of the action of $B_j$ on the generators of the fundamental group.

\[ a_i \mapsto a_i, \quad i \neq j \]
\[ B_j : \quad b_i \mapsto b_i, \quad 1 \leq i \leq n \]
\[ a_j \mapsto b_j^{-1}a_j \quad (7.1.27) \]

Combining this information with Lemma 7.1.4, we see that the bottom half of the diagram gives us the following.

\[
f \circ \nu_T(B_j) : \phi \mapsto \sum \phi(..., h_j^{-1}g_j, h_j, ...) \otimes g_j \mu_{h_j} \otimes ...
\quad (7.1.28)\]

As this matches (7.1.26), we have the desired result.

\[ \square \]

**Lemma 7.1.8.** Theorem 7.1.3 holds for the generator $C_k$, $1 \leq k \leq n - 1$.

**Proof.** First, we calculate the Hennings map associated to $C_k$, and then look at the word associated with $C_k$ in the fundamental group. The beads associated with $C_k$ in this case are as given in Figure 7.6.
This gives us that the map corresponding to the tangle $C_k$ is as follows.

$$
\nu_H(C_k) : x_1 \otimes \ldots \otimes x_n \mapsto \sum x_1 \otimes \ldots \otimes x_k S ((v^{-1})' \otimes (v^{-1})'' x_{k+1} \otimes \ldots \otimes x_n
$$

(7.1.29)

Again, we will suppress the coordinates on which $\nu_H$ acts as the identity. Using

$$
\Delta(v^{-1}) = \sum_{x,y} \frac{x}{y} \otimes \frac{y}{x}
$$

(7.1.30)

in $D[G]$, we simplify the map.

$$
\nu_H(C_k) : \ldots \otimes x_k \otimes x_{k+1} \otimes \ldots \mapsto \sum \ldots \otimes x_k S \left( \frac{x}{y} \right) \otimes \frac{y}{x} x_{k+1} \otimes \ldots
$$

(7.1.31)

$$
= \sum \ldots \otimes x_k \left( (x^{-1} y^{-1} x^{-1} y_{x^{-1} y^{-1}}) \otimes \frac{y}{x} x_{k+1} \otimes \ldots \right)
$$

(7.1.32)

This means that the top half of our diagram is given by the following. Here, we sum
over all \( x, y \in G \) as well as all \( \gamma \in \text{Hom}(\pi_1(\Sigma_0^n), G) \).

\[
\nu_H(C_k) \circ f : \phi \mapsto \sum \phi(\ldots, g_k h_k, g_{k+1} h_{k+1}, \ldots) \otimes g_k \downarrow g_{k+1} \downarrow \otimes \ldots
\]

\[
\mapsto \sum \phi(\ldots, g_k h_k, g_{k+1} h_{k+1}, \ldots)
\]

\[
\ldots \otimes g_k \downarrow (x^{-1} y^{-1} x^{-1} y x) \downarrow x^{-1} y^{-1} \otimes y \downarrow g_{k+1} \downarrow h_{k+1} \downarrow \otimes \ldots
\]

(7.1.33)

\[
= \sum \phi(\ldots, g_k h_k, g_{k+1} h_{k+1}, \ldots)
\]

\[
\ldots \otimes g_k \downarrow \otimes (w g_{k+1} w^{-1}) \downarrow w h_{k+1} \downarrow \otimes \ldots
\]

(7.1.34)

where \( w = g_{k+1} h_k^{-1} g_k^{-1} h_k \). Here, we make several substitutions to clean up the formula and return to the original variables, as we have done for \( A_i \) and \( B_j \). Our final formula is the following.

\[
\nu_H(C_k) \circ f : \phi \mapsto \sum \phi(\ldots, g_k h_k w, w^{-1} g_{k+1} w, w^{-1} h_{k+1}, \ldots)
\]

\[
\ldots \otimes g_k \downarrow \otimes g_{k+1} \downarrow h_{k+1} \downarrow \otimes \ldots
\]

(7.1.35)

Next, we consider the intersections of the curve \( C_k \) with the generators of the fundamental group. We see that the only generators which will intersect \( C_k \) are \( b_k, a_{k+1} \), and \( b_{k+1} \). Their intersections can be seen in Figure 7.7.

Let \( w_C \) denote the word corresponding to \( C_k \) in terms of the generators of the fundamental group. Using our left-hand turn convention, we can now see that the full behavior of the twist \( C_k \) as a map on the fundamental group is given by the following.

\[
a_i \mapsto a_i, \quad i \neq k + 1
\]

\[
b_i \mapsto b_i, \quad i \neq k, k + 1
\]

\( C_k : \)

\[
b_k \mapsto b_k w_C
\]

\[
b_{k+1} \mapsto w_C^{-1} b_{k+1}
\]

\[
a_{k+1} \mapsto w_C^{-1} a_{k+1} w_C
\]

(7.1.36)
Figure 7.7: Intersections of $C_k$ with Generators
Figure 7.8: Deforming the Twist $C_k$
To find the word \( w_C \) we deform the element of the fundamental group corresponding to the twist \( C_k \) as seen in Figure 7.8, and we find that the word \( w_C = a_{k+1}b_k^{-1}a_k^{-1}b_k \).

Finally, we compute the lower half of the diagram using this information and Lemma 7.1.4.

\[
f \circ \nu_T(C_k) : \phi \mapsto \sum \phi(..., g_k, h_k w^{-1}, w g_{k+1} w^{-1}, w h_{k+1}, ...)
\]

\[
... \otimes g_k \quad \otimes g_{k+1} \quad \otimes h_k \quad \otimes h_{k+1} \quad \otimes ...
\]  

(7.1.37)

As this matches (7.1.35), we have proved the result.

\[\square\]

**Lemma 7.1.9.** Theorem 7.1.3 holds for the special cobordisms \( H_n^+ \) and \( H_n^- \).

**Proof.** First, we compute the Hennings maps on \( H_n^+ \) and \( H_n^- \). The tangles for these cobordisms, as discussed in [18] and depicted in Figure 9 of [19] are given in Figure 7.9. We will first consider the case of \( H_n^- \).

![Figure 7.9: Tangles for the Special Cobordisms](image)

The map associated with \( H_n^- \) is as follows.

\[
H_n^- : x_1 \otimes ... \otimes x_{n+1} \mapsto x_1 \otimes ... \otimes x_n \lambda(v^{-1})\lambda(S(x_{n+1}))
\]  

(7.1.38)
We simplify (7.1.38). First, we have

$$\lambda(v) = \lambda \left( \sum g \left\downarrow g^{-1} \right\downarrow \right) = \sum \delta_{g^{-1}, e} = 1$$  \hspace{1cm} (7.1.39)

$$\lambda(v^{-1}) = \lambda \left( \sum g \left\downarrow g \right\downarrow \right) = \sum \delta_{g, e} = 1$$  \hspace{1cm} (7.1.40)

Next, Lemma 2.2.7 gives us $\lambda(S(x)) = \lambda(G^2 x)$. In $D[G]$ we have $G^2 = 1$ for the special grouplike element $G$, so we have $\lambda(S(x)) = \lambda(x)$. That is, equation (7.1.38) simplifies to the following.

$$H_n^- : x_1 \otimes \ldots \otimes x_{n+1} \mapsto x_1 \otimes \ldots \otimes x_n \lambda(x_{n+1})$$  \hspace{1cm} (7.1.41)

We may now compute the top half of our diagram as follows.

$$\nu_H(H_n^-) \circ f : \phi \mapsto \sum \phi(g_1, h_1, \ldots, g_{n+1}, h_{n+1}) g_1 \left\downarrow h_1 \right\downarrow \otimes \ldots \otimes g_{n+1} \left\downarrow h_{n+1} \right\downarrow$$

$$\mapsto \sum \phi(g_1, h_1, \ldots, g_{n+1}, h_{n+1}) g_1 \left\downarrow h_1 \right\downarrow \otimes \ldots \otimes g_{n} \left\downarrow h_{n} \right\downarrow \lambda\left( g_{n+1} \left\downarrow h_{n+1} \right\downarrow \right)$$

$$= \sum \phi(g_1, h_1, \ldots, g_{n+1}, e) g_1 \left\downarrow h_1 \right\downarrow \otimes \ldots \otimes g_{n} \left\downarrow h_{n} \right\downarrow$$  \hspace{1cm} (7.1.42)

For the bottom half of our diagram, recall the discussion in Section 5.3. On the generators of the fundamental group, the cobordism $H_n^-$ is given by

$$\nu_T(H_n^-) : \tilde{\phi} \mapsto \phi,$$  \hspace{1cm} (7.1.43)

where

$$\phi(g_1, h_1, \ldots, g_n, h_n) = \sum_{g_{n+1}} \tilde{\phi}(g_1, h_1, \ldots, g_n, h_n, g_{n+1}, e).$$  \hspace{1cm} (7.1.44)

Using Lemma 7.1.4, the bottom half of the diagram is as follows.

$$f \circ \nu_T(H_n^-) : \tilde{\phi} \mapsto \phi \mapsto f(\phi)$$

$$= \sum \phi(g_1, h_1, \ldots, g_n, h_n) g_1 \left\downarrow h_1 \right\downarrow \otimes \ldots \otimes g_{n} \left\downarrow h_{n} \right\downarrow$$

$$= \sum_{g_1, \ldots, h_n} \sum_{g_{n+1}} \tilde{\phi}(g_1, h_1, \ldots, g_n, h_n, g_{n+1}, e) g_1 \left\downarrow h_1 \right\downarrow \otimes \ldots \otimes g_{n} \left\downarrow h_{n} \right\downarrow$$  \hspace{1cm} (7.1.45)
Since (7.1.42) is the same as (7.1.45), the result is true in the case of $H^-_n$.

Now, we consider the case of $H^+_n$. The tangle for $H^+_n$ is given in Figure 7.9. The beaded diagram corresponding to this tangle is given in Figure 7.10.

![Figure 7.10: Hennings Procedure applied to $H^+_n$](image)

The map corresponding to this beaded diagram is as follows.

$$H^+_n : x_1 \otimes ... \otimes x_n \mapsto x_1 \otimes ... \otimes x_n \otimes \lambda(v) \Lambda$$ (7.1.46)

We know that $\lambda(v) = 1$ by (7.1.39), and $\Lambda = \sum e_k$. Thus, we can simplify equation (7.1.46) to the following.

$$H^+_n : x_1 \otimes ... \otimes x_n \mapsto \sum_k x_1 \otimes ... \otimes x_n \otimes e_k$$ (7.1.47)

We compute the top half of our diagram in this case.

$$\nu_H(H^+_n) \circ f : \phi \mapsto \sum \phi(g_1, h_1, ..., g_n, h_n) g_1 \otimes_h \otimes ... \otimes g_n \otimes_h$$

$$\mapsto \sum \phi(g_1, h_1, ..., g_n, h_n) g_1 \otimes_h \otimes ... \otimes g_n \otimes_h \otimes e_k$$ (7.1.48)
We now turn our attention to the bottom half of the diagram by considering the action of $H_n^+$ on the generators of the fundamental group, again discussed in Section 5.3. We can write this map algebraically as

$$\nu_T(H_n^+) : \phi \mapsto \tilde{\phi}, \quad (7.1.49)$$

where

$$\tilde{\phi}(g_1, h_1, ..., g_{n+1}, h_{n+1}) = \delta_{g_{n+1},e} \phi(g_1, h_1, ..., g_n, h_n). \quad (7.1.50)$$

We then use Lemma 7.1.4 to compute the bottom half of the diagram.

$$f \circ \nu_T(H_n^+) : \phi \mapsto \tilde{\phi}$$

\[\begin{align*}
&\mapsto \sum \tilde{\phi}(g_1, h_1, ..., g_n, h_n, g_{n+1}, h_{n+1}) \underbrace{g_1 \otimes \cdots \otimes g_{n+1}}_{h_1} \\
&= \sum \delta_{g_{n+1},e} \phi(g_1, h_1, ..., g_n, h_n) \underbrace{g_1 \otimes \cdots \otimes g_n}_{h_1} \underbrace{\otimes g_{n+1}}_{h_{n+1}} \\
&= \sum_{h_{n+1}} \phi(g_1, h_1, ..., g_n, h_n) \underbrace{g_1 \otimes \cdots \otimes g_n \otimes e}_{h_{n+1}} \underbrace{\otimes}_{h_{n+1}} \quad (7.1.51)
\end{align*}\]

Since in equation (7.1.48) we sum over all values of $k$ and in equation (7.1.51) we sum over all values of $h_{n+1}$, these two equations represent the same element, hence the square commutes in this case. This concludes the proof.

$\square$

### 7.2 Further Work

We are now in a position to formulate our main conjecture.

**Conjecture 7.2.1.** Let $G$ be a finite group and $\omega : G \times G \times G \to U(1)$ a 3-cocycle. The Hennings TQFT applied with the quasi-Hopf algebra $D^\omega[G]$ is equivalent to the Dijkgraaf-Witten TQFT applied with the same group $G$ and 3-cocycle $\omega$.

This conjecture is the natural extension of the work done in Section 7.1. A conjecture that these two objects are equivalent is not unexpected, but could not be
formulated without a precise definition of the Hennings TQFT for quasi-Hopf algebras. We have presented such a definition, hence may now formulate this conjecture, which will be the subject of a future paper.
BIBLIOGRAPHY


