WAVE MOTION IN ELASTIC-PLASTIC SOLIDS BY SPACE-TIME CONSERVATION ELEMENT AND SOLUTION ELEMENT METHOD

Thesis

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ABSTRACT

The objective of the present work is the development of a theoretical and numerical framework to simulate elastic-plastic waves in solids. The governing equations used include the momentum equation, continuity equation and the constitutive relation for plasticity in the Eulerian frame. The constitutive equation relates deviatoric stresses with the plastic strain. The linear constitutive relation is made non-linear by the use of Jaumann rate. The final set of equations is a system of first order, fully coupled partial differential equations with source terms.

To solve these equations, the space-time Conservation Element and Solution Element (CESE) method is used. Numerical results of two dimensional elastic-plastic waves would be reported in the thesis. The results would be validated by comparisons with available experimental results and published works. The model can be further developed to aid in modeling properties and real world performance more accurately. This framework can be used to model a variety of applications like explosive welding and ultrasonic welding processes.
To my parents Venkatesan Rajagopalan, Thangam Venkatesan, brother Gokulkrishnan Venkatesan and my girlfriend Yamini.
I would like to express my gratitude to my advisor Professor Sheng-Tao John Yu. I am very thankful to him for giving me an opportunity to work on this thesis. His willingness to help, patience, ideas and valuable inputs have constantly provided me an impetus and the means to face any technical challenge. I would also take this opportunity to thank Professor Daniel Albert Mendelsohn for agreeing to be my committee member. I want to thank my family members and my girlfriend for continuously supporting me along the way. I am grateful to my parents for granting me the privilege of pursuing my masters degree. I am very much indebted to my lab mates, Dr. Yung-Yu Chen, Dr. Lixiang Yang and Mr. Po-Hsien Lin. Their knowledge and technical support played an important role in improving my understanding of the subject. Lastly, I want to thank my friend Deepak Ravindran for his encouragement and support.
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CHAPTER 1
INTRODUCTION

1.1 Motivation

The thesis deals with the numerical simulation of stress waves in elastic-plastic solids. The subject of wave propagation has been investigated by the likes of Poisson, Navier, Stokes, Kevin and Rayleigh. With the exception of seismological studies, the topic of wave propagation in solids underwent research at a very slow pace until the 1930s. The spike in interest in this topic resulted from a number of reasons. Kolsky[9] lists some of the reasons as the (i) use of high frequency waves in the study of processes governing deformation in solids at a microscopic level, (ii) availability of experimental apparatus for production and detection of stress waves, (iii) need for data on material behavior at high rates of loading and (iv) increase in military research pertaining to the response of structures to large forces for short period of time.

The concept of stress waves is useful in a number of applications such as impact of metal structures, explosive welding, dynamic fracture and high rate machining. Other important applications include the measurement of dynamic moduli, Non Destructive Testing (NDT) and medical imaging. In this thesis, I have taken Explosive Welding as an example to illustrate the use of the mathematical model developed in simulating stress waves in solids.
1.2 Explosive Welding

Explosive welding is one of the few processes with the capability to join a variety of both similar and dissimilar combination of metals which cannot be joined by other processes due to differing melting points Akbari Mousavi and Al-Hassani[10].

During the 2nd world war, metal shrapnel often got welded to metal structures such as tank bodies. It was soon discovered, the oblique high energy impact resulted in a weld. This concept was soon commercialized and it became popularly known as explosive welding.

The welding process consists of three stages Akbari Mousavi and Al-Hassani[11]:

- Detonation of explosive charge
- Deformation and acceleration of the flyer plate
- Collision between the plates.

Cladded plates are in good demand from chemical and nuclear industries due to their good corrosion resistance and mechanical properties. This technique forms a solid state weld between the two metals by plasticizing them. There is no melting involved and the weld is clean and quick.

The typical setup for explosive welding is shown in figure (1.1). Any welding process requires a huge amount of energy. Explosive welding uses the high density energy contained in the explosive and distribute over a wide surface area to form the weld. Conversely, huge amount of energy can be concentrated to produce welds over small areas. The welding occurs when there is an oblique high velocity impact or collision of two metal plates.

Many of the studies have used a finite element to simulate the explosive welding process. In this thesis, we will be using a unique numerical framework called the Conservation Element Solution Element (CESE) method to model the same.
1.3 Objective and Approach

The objective of this thesis is to develop a theoretical model comprising of a hyperbolic system of equations to simulate the propagation of stress waves in an elastic-plastic material. For this purpose, we have selected an application namely Explosive welding to validate and illustrate the use of this model in one dimension. The theoretical model will be solved using a numerical framework namely CESE method. The equations will be formulated and adapted for the CESE based solver which is implemented through a FORTRAN program.

The theoretical model consists of the governing equations - continuity, momentum and constitutive relations. We have assumed that there is no effect of temperature on the system so the model is isothermal. Since, the explosive welding takes place in a short time due to the high speed nature of the process, there is little scope for heat
transfer which allows to neglect the effect of heat. The deformations encountered in the problem are of small amplitude which is consistent with this assumption.

Hooke’s law will be used for linear elasticity. For the plastic part, a relation will be derived using von-mises yield function and hence the associated flow rule. The system of partial differential equations is hyperbolic in nature and is closed using an equation of state relating density and pressure.

1.4 Organization

The remainder of the thesis is organized as follows. Chapter 2 deals with CESE method. It provides an introduction to the method and elaborates on how the 1D and 2D versions can be used to solve the respective model equations to get the result.

Chapter 3 deals with the formulation of the mathematical models. The formulation pertains to one-dimensional and two-dimensional hyperbolic models for elastic-plastic waves in solids. This model has been implemented using the CESE method to simulate stress waves in Explosive Welding.

Chapter 4 discusses the numerical solutions of the explosive welding process obtained in one dimension. Chapter 5 presents the conclusions and future works of the thesis.
CHAPTER 2
THE SPACE-TIME CONSERVATION ELEMENT AND SOLUTION ELEMENT METHOD

In this thesis, the governing equations are solved by using the space time Conservation and Solution Element (CESE) method. One of the main challenges is solving the hyperbolic system of equations numerically. Usually, the flux is conserved over a fixed domain in traditional finite volume methods. The rate of change of a quantity in a fixed spatial domain i.e the control volume CV, is equal to the flux across the boundary S(V). The differential form of a conservative law is given as:

\[
\frac{\partial u}{\partial t} + \nabla \cdot f = 0 \tag{2.1}
\]

where \( u \) is the density of the conserved flow variable and \( f \) is the spatial flux vector. Using Reynold’s transport theorem, we can obtain the integral form as:

\[
\frac{\partial}{\partial t} \int_V u \, dV + \oint_{S(V)} f \cdot dS = 0 \tag{2.2}
\]

where \( dV \) is spatial volume element in \( V \), \( ds = d\sigma \mathbf{n} \) with \( d\sigma \) and \( \mathbf{n} \) as the area and the unit outward normal vector of the surface, respectively. Integrating Eq. (2.2), we
\[
\left[ \int_{V} u \, dV \right]_{t=t_f} - \left[ \int_{V} u \, dV \right]_{t=t_s} + \int_{t_s}^{t_f} \left( \oint_{S} (V \cdot f) \, dS \right) \, dt = 0
\]  

(2.3)

The CESE method doesn’t use the above formulation based on the Reynold’s transport theorem. A unified treatment of space and time will allow for consistent integration in calculation and guarantee local and global flux balance. We will be focussing on the one dimensional CESE method in the next section.

### 2.1 One-Dimensional CESE Method

Since this is a space time numerical method, let time and space be the two orthogonal axes in the euclidean space \(E_2\) i.e \(x_1 = x, x_2 = t\). Let \(h \equiv (f, u)\). Using Gauss divergence theorem, we get,

\[
\int_{S(V)} h \, ds = 0
\]  

(2.4)

\(h\) denotes the total flux leaving the control volume through its surface which becomes zero. To perform the above integration, we use the CESE method. Figure 2.1 depicts the schematic of the CESE method. The CESE method is set of 1D, 2D and 3D codes for solving conservation laws. The codes are parallelized and large scale simulations could be performed. In the present thesis, the simulations have been carried out on a single standalone machine. The CESE method as the name suggests introduces the concept of two different elements: Conservation element (CE) and Solution element (SE). Over each CE, the flux is conserved and the flux in integral form is applied. For every SE, the unknown variable solutions are assumed to be continuous and a predefined function is used to represent the profile. Figure 2.2(a) shows the space time...
mesh. Solutions of the unknowns are stored at the mesh nodes. Since time is treated together with space, the mesh is staggered in time. So, in time-marching calculations, the solution variables skip neighbouring SEs in time. The solution propagates only in one direction. Figure 2.2(b) depicts the space-time flux conservation in a CE. If the solutions at time step n-1/2 are known, the flux conservation will help find the solution at (j,n).

In my thesis, there is a use of source terms. So, lets consider equations with source terms:

$$\frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} = s_m, m = 1, 2 \quad (2.5)$$

where source terms are functions of the unknown variables $u_m$ and their spatial derivatives. For any $(x, t) \in \text{SE}(j, n)$, $u_m(x, t)$, $f_m(x, t)$ and $h_m(x, t)$, are approximated by
Fig. 2.2: CESE method in one dimension. (a) Zigzagging Solution Element(SE). (b) Integration over Conservation Element(CE) to solve $u_i$ and $(u_x)_i$. 

\( u^*(x, t; j, n), f^*(x, t; j, n), \) and \( h^*(x, t; j, n) \). We have,

\[
u^*_m(x, t; j, n) = (u_m)^n_j + (u_{mx})^n_j(x - x_j) + (u_{mt})^n_j(t - t^n),
\]
\[
f^*_m(x, t; j, n) = (f_m)^n_j + (f_{mx})^n_j(x - x_j) + (f_{mt})^n_j(t - t^n),
\]
\[
h^*_m(x, t; j, n) = (f^*_m(x, t; j, n), u^*_m(x, t; j, n)),
\]

where

\[
(u_{mx})^n_j = \left(\frac{\partial u_m}{\partial x}\right)^n_j,
\]
\[
(f_{mx})^n_j = \left(\frac{\partial f_m}{\partial x}\right)^n_j = (f_{m, l})^n_j(u_{lx})^n_j,
\]
\[
(u_{mt})^n_j = \left(\frac{\partial u_m}{\partial t}\right)^n_j = -(f_{mx})^n_j = -(f_{m, l})^n_j(u_{lx})^n_j,
\]
\[
(f_{mt})^n_j = \left(\frac{\partial f_m}{\partial t}\right)^n_j = (f_{m, l})^n_j(u_{lx})^n_j = -(f_{m, l})^n_j(f_{lp})^n_j(u_{px})^n_j,
\]

and \((f_{m, l})^n_j = (\partial f_m / \partial u_l)^n_j\) is the Jacobian matrix. Assume that, for any \((x, t) \in SE(j, n)\), \(u_m = u^*_m(x, t; j, n)\) and \(f_m = f^*_m(x, t; j, n)\) satisfy Eq. (2.5), i.e.,

\[
\frac{\partial u^*_m(x, t; j, n)}{\partial t} + \frac{\partial f^*_m(x, t; j, n)}{\partial x} = s^*_m(x, t; j, n),
\]

where we assume that \(s^*_m\) is constant within \(SE(j, n)\), i.e., \(s^*_m(x, t; j, n) = (s_m)^n_j\).

Eq.(2.6) becomes

\[
(u_{mt})^n_j = -(f_{mx})^n_j + (s_m)^n_j.
\]

As \((f_{mx})^n_j\) is a function of \((u_m)^n_j\) and \((u_{mx})^n_j\); and \((s_m)^n_j\) is also a function of \((u_m)^n_j\),
Eq. (2.5) and Eq. (2.7) imply that \((u_{mt})^n_j\) is also a function of \((u_m)^n_j\) and \((u_{mx})^n_j\). Aided by the above equations, we determine that the only unknowns are \((u_m)^n_j\) and \((u_{mx})^n_j\) at each mesh point \((j, n)\). Next, we impose space-time flux conservation over CE\((j, n)\) to determine the unknowns \((u_m)^n_j\). Refer to Fig. 2.3. Assume that \(u_m\) and \(u_{mx}\) at mesh points \((j - 1/2, n - 1/2)\) and \((j + 1/2, n - 1/2)\) are known and their values are used to calculate \((u_m)^n_j\) and \((u_{mx})^n_j\) at the new time level \(n\). By imposing the flux balance over CE\((j, n)\), i.e.,

\[
\oint_{S(CE(j, n))} h^*_m \cdot ds = \int_{CE(j, n)} s^*_m d\Omega,
\]

one obtains

\[
(u_m)^n_j - \frac{\Delta t}{4}(s_m)^n_j = \frac{1}{2}\left[(u_m)^{n-1/2}_{j-1/2} + (u_m)^{n-1/2}_{j+1/2}ight] + \frac{\Delta t}{4}(s_m)^{n-1/2}_{j-1/2} + \frac{\Delta t}{4}(s_m)^{n-1/2}_{j+1/2} + (p_m)^{n-1/2}_{j-1/2} - (p_m)^{n-1/2}_{j+1/2}\]

\[(2.8)\]

where

\[
(p_m)^n_j = \frac{\Delta x}{4}(u_{mx})^n_j + \frac{\Delta t}{\Delta x}(f_m)^n_j + \frac{\Delta t^2}{4\Delta x}(f_{mt})^n_j.
\]

Using the marching variables at the mesh nodes \((j - 1/2, n - 1/2)\) and \((j + 1/2, n - 1/2)\), the right-hand side of Eq. (2.8) can be explicitly calculated. Since \((s_m)^n_j\) on the left hand side of Eq. (2.8) is a function of \((u_m)^n_j\), Newton’s method can be used to solve
Fig. 2.3: Modified CESE method schematic in one dimension. (a) Staggered mesh.
(b) SE(j,n) (in yellow) and CE(j,n)s
for \((u_m)^n_j\). The initial guess of the Newton iterations is

\[
(\bar{u}_m)^n_j = \frac{1}{2} \left[ (u_m)^{n-1/2}_j + (u_m)^{n-1/2}_{j+1/2} \right] + \frac{\Delta t}{4} (s_m)^{n-1/2}_{j-1/2} + \frac{\Delta t}{4} (s_m)^{n-1/2}_{j+1/2} + (p_m)^{n-1/2}_{j-1/2} - (p_m)^{n-1/2}_{j+1/2},
\]

i.e., the explicit part of the solution of \((u_m)^n_j\).

The solution procedure for \((u_{mx})^n_j\) at node \((j, n)\) follows the standard \(a-\varepsilon\) scheme Chang[4] with \(\varepsilon = 0.5\). To proceed, we let

\[
(u_{mx})^n_j = \frac{(u_{mx})^n_j + (u_{mx})^n_j}{2}, \quad (2.9)
\]

where

\[
(u_{mx})^n_j = \pm \frac{(u_m)^{n-1/2}_{j\pm 1/2} - (u_m)^n_j}{\Delta x/2}, \\
(u_m)^n_{j\pm 1/2} = (u_m)^{n-1/2}_{j\pm 1/2} + \frac{\Delta t}{2} (u_{mt})^{n-1/2}_{j\pm 1/2}.
\]

For solutions with discontinuities, Eq. (2.9) is replaced by a re-weighting procedure to add artificial damping at the jump

\[
(u_{mx})^n_j = W \left( (u_{mx})^n_j, (u_{mx})^n_j, \alpha \right),
\]

where the re-weighting function \(W\) is defined as:

\[
W(x_-, x_+, \alpha) = \frac{|x_+|^\alpha x_- + |x_-|^\alpha x_+}{|x_+|^\alpha + |x_-|^\alpha}.
\]
and $\alpha$ is an adjustable constant. The complete discussion of the one-dimensional CESE method can be found in Chang[4].

### 2.2 Two-Dimensional CESE Method

The standard conservative form in two dimensions can be expressed as:

$$\frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} + \frac{\partial g_m}{\partial y} = \mu_m, \ m = 1, 2, ..., 6$$

(2.10)

where $f_m$ and $g_m$, are functions of the independent conservative variables $u_m$. Let $x_1 = x, x_2 = y$ and $x_3 = t$ be the coordinates of the euclidean space $E_3$. Using the Gauss divergence theorem, we get

$$\oint_{S(V)} h_m \cdot ds = \int_V \mu_m dV, m = 1, 2, ..., 6$$

(2.11)

In the above equation, $S(V)$ and $ds$ were defined by Eq. (2.4) and figure 2.1, respectively. Also, $h_m \equiv (f_m, g_m, u_m)$.

#### 2.2.1 Conservation element and Solution element

In two dimensions, the domain on the $x - y$ plane is segregated as convex quadrilaterals and two neighboring quadrilaterals anywhere have a common side. Refer to figure 2.4(a). The centroids and vertices of the quads are represented by circles and dots, respectively. Q is the centroid of a quadrilateral $B1B2B3B4$. The underscore makes the points unique in the computational domain on the $xy$ plane and differentiates from those in the space-time domain. The centroids of the neighboring quadrilaterals of the quadrilateral $B1B2B3B4$ are represented by points $A1, A2, A3$ and $A4$. The polygon $A1B1A2B2A3B3A4B4$ has a centroid which is marked by point...
Q* which is marked by a cross in figure 2.4(a). Point Q*, which does not coincide with point Q, is the solution point of the centroid Q. Likewise, points \( A_1^*, A_2^*, A_3^* \) and \( A_4^* \) are the solution points of the centroids \( A_1, A_2, A_3 \) and \( A_4 \) respectively.

Let us consider the figure 2.4(b) which shows the space-time mesh. Here at the nth time level, \( t = n \Delta t \), where \( n = 0, 1/2, 1, 3/2, \ldots \). By definition, \( Q^*, A_1^*, A_2^*, A_3^*, A_4^* \), belong on the nth time level and are associated to be the space-time solution mesh points of points \( Q, A_1, A_2, A_3, A_4 \) respectively. \( Q'^*, A_1'^*, A_2'^*, A_3'^*, A_4'^* \), belong on the time level \( n - 1/2 \) and are associated to be the space-time solution mesh points of points \( Q', A_1', A_2', A_3', A_4' \) respectively.

Using the above definitions, we can proceed with the geometry of the CE and the SE associated with point \( Q^* \). The known flow solution at points in the time level \( n-1/2 \), represented by superscript prime can be used to calculate the numerical solution of the flow variable \( u_m \) at the nth time level. In order to integrate Eq.(2.11), four Basic Conservation Elements (BCEs) of point \( Q^* \) are created and represented by \( BCE_l(Q) \), with \( l = 1, 2, 3 \) and 4. The aforementioned BCEs are the space-time cylinders \( A_1 B_1 Q B_4 A_1' B_1' Q' B_4', A_2 B_2 Q B_1 A_2' B_2' Q' B_1', A_3 B_3 Q B_2 A_3' B_3' Q' B_2 \) and \( A_4 B_4 Q B_3 A_4' B_4' Q' B_3' \), respectively. The conservation element pertaining to point \( Q \), represented by \( CE(Q) \), is the space-time cylinder \( A_1 B_1 A_2 B_2 A_3 B_3 A_4 B_4 A_1' B_1' A_2' B_2' A_3' B_3' A_4' B_4' \), which is the union of the four BCEs. Also, the SE of point \( Q^* \), represented by \( SE(Q^*) \), is the union of \( CE(Q) \) and plane segments \( QQ'' B''_1 B_1, QQ'' B''_2 B_2, QQ'' B''_3 B_3, \) and \( QQ'' B''_4 B_4 \) as well as their immediate surroundings.

2.2.2 Approximations with Solution Element

Let us represent the space-time mesh points which are defined as the centroids of quadrilaterals and the space-time mesh points which are defined as the solution points, shown in figure 2.4(a), by O and \( O^* \), respectively. For any \( Q^* \in O^* \) and any \((x, y, t) \in O\).
Fig. 2.4: Mesh in two dimensions. (a) x-y plane with grid points. (b) SE and CE
SE($Q^*$), the flow variables and flux vectors, which is $u_m(x, y, t)$, $f_m(x, y, t)$, $g_m(x, y, t)$, and $h_m(x, y, t)$, are approximated to their respective numerical counterparts, i.e., $u^*_m(x, y, t; Q^*)$, $f^*_m(x, y, t; Q^*)$, $g^*_m(x, y, t; Q^*)$, and $h^*_m(x, y, t; Q^*)$, on the basis of the first-order Taylor series expansion with respect to $Q^*(x_{Q^*}, y_{Q^*}, t^n)$. For any $m = 1, 2, 3, 4, 5, 6$, let

$$u^*_m(x, y, t; Q^*) \equiv (u_m)_{Q^*} + (u_{mx})_{Q^*}(x - x_{Q^*}) + (u_{my})_{Q^*}(y - y_{Q^*}) + (u_{mt})_{Q^*}(t - t^n) \quad (2.12)$$

Where the space-time solution mesh point $Q^*$ is represented by coordinates $(x_{Q^*}, y_{Q^*}, t^n)$. $(u_m)_{Q^*}$, $(u_{mx})_{Q^*}$, $(u_{my})_{Q^*}$ and $(u_{mt})_{Q^*}$ are the numerical analogues of the values of $u_m$, $\partial u_m/\partial x$, $\partial u_m/\partial y$ and $\partial u_m/\partial t$ at point $Q^*$, respectively which are constant in SE($Q^*$). Using chain rule, $(f_{mx})_{Q^*}$, $(g_{mx})_{Q^*}$, $(f_{my})_{Q^*}$, $(g_{my})_{Q^*}$, $(f_{mt})_{Q^*}$ and $(g_{mt})_{Q^*}$ are defined similar to the 1D case as $(f_{mx})_j^n$ and $(f_{mt})_j^n$. We define

$$f^*_m(x, y, t; Q^*) \equiv (f_m)_{Q^*} + (f_{mx})_{Q^*}(x - x_{Q^*}) + (f_{my})_{Q^*}(y - y_{Q^*}) + (f_{mt})_{Q^*}(t - t^n) \quad (2.13)$$

$$g^*_m(x, y, t; Q^*) \equiv (g_m)_{Q^*} + (g_{mx})_{Q^*}(x - x_{Q^*}) + (g_{my})_{Q^*}(y - y_{Q^*}) + (g_{mt})_{Q^*}(t - t^n) \quad (2.14)$$

and

$$h^*_m(x, y, t; Q^*) \equiv (f^*_m(x, y, t; Q^*), g^*_m(x, y, t; Q^*), u^*_m(x, y, t; Q^*)) \quad (2.15)$$
for any \( m = 1, 2, 3, 4, 5, 6 \). By definition, \((f_m)Q^*, (g_m)Q^*, (f_{mx})Q^*, (g_{mx})Q^*, (f_m)Q^*, (g_{mx})Q^*\) and \((g_{mt})Q^*\) are functions of \((u_m)Q^*, (u_{mx})Q^*, (u_{my})Q^*\) and \((u_{mt})Q^*\) only, for any \( m = 1, 2, 3, 4, 5, 6 \).

Let us assume that for any \((x, y, t) \in \text{SE}(Q^*)\), and any \( m = 1, 2, 3, 4, 5, 6 \)

\[
\frac{\partial u_m^*(x, y, t; Q^*)}{\partial t} + \frac{\partial f_m^*(x, y, t; Q^*)}{\partial x} + \frac{\partial g_m^*(x, y, t; Q^*)}{\partial y} = \mu_m(Q^*) \tag{2.16}
\]

which is the numerical analogue of Eq.(2.10). Here the source term \( \mu_m^* \) is assumed to be a constant within \( \text{SE}(Q^*) \) and the value of \( \mu_m^* \) is found by \((u_m)Q^*\). Using Eqs. (2.12)-(2.14), Eq.(2.16) implies that for any \( m = 1, 2, 3, 4, 5, 6 \)

\[
(u_{mt})Q^* = -(f_{mx})Q^* - (g_{mx})Q^* + (\mu_m)Q^* \tag{2.17}
\]

Note that, as in the 1D case, in order to solve Euler equations which has stiff source term, for any \( l = 1, 2, 3, 4 \), \((u_{mt})A_l^*\) are evaluated without the source term effect, which is

\[
(u_{mt})A_l^* = -(f_{mx})A_l^* - (g_{mx})A_l^* \tag{2.18}
\]

With the aid of the above equations, we can show that \( Q^* \) are \((u_m)Q^*, (u_{mx})Q^*\) and \((u_{my})Q^*\), \( m = 1, 2, 3, 4, 5, 6 \) analogous to the 1D case, are the only independent discrete solution variables pertaining to the space-time solution point.

### 2.2.3 Evaluation of \( u_m \)

Now let us evaluate space-time flux in \( E_3 \). Let \( G \) be a plane segment lying within \( \text{SE}(Q^*) \). Let \( A \) be the area of \( G \), \((x_c, y_c, t_c)\) be the coordinates of the centroid of \( G \), and \( n \) be a unit vector normal to \( G \). As \( u_m^*(x, y, t; Q^*), f_m^*(x, y, t; Q^*), g_m^*(x, y, t; Q^*) \)
are linear in x, y, and t, Eq. (2.10) means that

\[ \int_{\Gamma} \mathbf{h}_m^* \, d\mathbf{s} = \mathbf{h}_m^*(x_c, y_c, t_c; Q^*) \mathbf{A} \mathbf{n} \]  

(2.19)

where \( d\mathbf{s} = d\mathbf{s}_n \), \( d\mathbf{s}_n \) is the area of a surface element on G.

The union of SE\((Q^*)\) and SE\((A_i^*)\), \(1 = 1, 2, 3, 4\) is bounded by CE\((Q)\). (a) the octagon \(A_1B_1A_2B_2A_3B_3A_4B_4\) belongs to SE\((Q^*)\); (b) the quadrilaterals \(A'_1B'_1Q'B'_4\), \(A'_1B'_4A_1\), and \(A'_1B'_1B_1A_1\) belong to SE\((A_1^*)\); (c) the quadrilaterals \(A'_2B'_2Q'B'_1\), \(A'_2B'_1B_1A_2\), and \(A'_2B'_2B_2A_2\) belong to SE\((A_2^*)\); (d) the quadrilaterals \(A'_3B'_3Q'B'_2\), \(A'_3B'_2B_2A_3\), and \(A'_3B'_3B_3A_3\) belong to SE\((A_3^*)\); and (e) the quadrilaterals \(A'_4B'_4Q'B'_3\), \(A'_4B'_3B_3A_4\), and \(A'_4B'_4B_4A_4\) belong to SE\((A_4^*)\).

Let us evaluate the surface vector of each bounded face of CE\((Q)\). Let S be the area of the octagon \(A_1B_1A_2B_2A_3B_3A_4B_4\), then surface vector of the top face of CE\((Q)\) is \((0, 0, S)\) as unit outward normal vector is \((0, 0, 1)\).

Now let us consider the bottom face of CE\((Q)\), which is comprised of four quadrilaterals, namely, \(A'_1B'_1Q'B'_4\), \(A'_2B'_2Q'B'_1\), \(A'_3B'_3Q'B'_2\), and \(A'_4B'_4Q'B'_3\). Let \(x^l, y^l\) and \(S^l, 1 = 1; 2; 3; 4\), represent the coordinates of the centroids and the areas of the four quadrilaterals, respectively. So for any \(l = 1, 2, 3, 4\), \((x^l, y^l, t^{n-1/2})\) are the coordinates of the aforementioned four centroids, and \((0, 0, -S^l)\) are the surface vectors of the four quadrilaterals, respectively. Moreover, as the area of the bottom face of CE\((Q)\) is the same as that of the top face, we can conclude that \(S = \sum_{l=1}^{4} S^l\).

Lastly, take into account the side faces of CE\((Q)\), which is, \(A'_1B'_1B_4A_1\), \(A'_1B'_1B_1A_1\), \(A'_2B'_2B_2A_2\), \(A'_3B'_3B_3A_3\), \(A'_4B'_4B_4A_4\), and \(A'_4B'_4B_4A_4\), which belong to SE\((A_i^*)\), \(l = 1, 2, 3, 4\), respectively. Suppose that the eight side faces be assigned indices \((k, l)\), respectively. Hence, the \((1, 1)\) and \((2, l)\) side faces will be part of SE\((A_i^*)\). The spatial projection of each side face is a line segment on the x-y plane.
Suppose that $\lambda_k^l$, $(n_{kx}^l, n_{ky}^l)$ and $(x_k^l, y_k^l)$, respectively, represent the length, the unit outward normal, and the coordinates of the midpoint of the spatial projection (on the x-y plane) of the $(k, l)$ side face. So, since each side face is sandwiched between the $(n - 1/2)$th and the nth time levels, we can conclude that the surface vector and the coordinates of the centroid of the $(k, l)$ side face, are given by $(\Delta t/2)\lambda_k^l(n_{kx}^l, n_{ky}^l, 0)$ and $(x_k^l, n_{ky}^l, t^n - \Delta t/4)$, respectively.

Using Eq.(2.19), the flux of $h_m^*$ leaving each face of CE(Q) can be calculated in terms of the independent marching variables at points $Q^*$ and $A_l^* \ast$, $l = 1, 2, 3, 4$. For eg., since $(x_{Q^*}, y_{Q^*}, t^n)$ are the coordinates of the centroid $Q^*$ of the top face of CE(Q); $u_m^*(x_{Q^*}, y_{Q^*}, t^n; Q^*) = (u_m)_{Q^*}$ (Eq.(2.12)); and the surface vector of the top face is $(0, 0, S)$, the flux of $h_m^*$ leaving CE(Q) through its top face is $(u_m)_{Q^*} \ast S$. To proceed, we employ space-time flux balance over CE(Q):

$$\oint_{S(CE(Q))} h_m^* ds = \int_{CE(Q)} \mu_m dV, m = 1, 2, 3, 4, 5, 6. \quad (2.20)$$

With the aid of the above discussion, it can be shown that

$$(u_m)_{Q^*} \frac{\Delta t}{2} (\mu_m)_{Q^*} = \frac{1}{S} (\sum_{l=1}^{4} \Delta t), m = 1, 2, 3, 4, 5, 6. \quad (2.21)$$

where, for any $l = 1, 2, 3, 4$,

$$R_m^l = S^l u_m^*(x_l^l, y_l^l, t^{n-1/2}; A_l^\ast) - \sum_{k=1}^{2} \frac{\Delta t}{2} \left[ n_{kx}^l f_m^\ast \left( x_k^l, y_k^l, t^n - \frac{\Delta t}{4}; A_l^\ast \right) + n_{ky}^l g_m^\ast \left( x_k^l, y_k^l, t^n - \frac{\Delta t}{4}; A_l^\ast \right) \right] \quad (2.22)$$

The functions $u_m^*(x, y, t; A_l^\ast)$, $f_m^*(x, y, t; A_l^\ast)$, and $g_m^*(x, y, t; A_l^\ast)$ are defined using Eqs.(2.12)-(2.14), respectively, with the symbols $Q^*$ and $t^n$ in these equations being substituted by $A_l^\ast$ and $t^{n-1/2}$, respectively. Using the values of the marching variables
at $t = t^{n-1/2}$, the right hand side of Eq. (2.21) can be readily solved. $(\mu_m)_{Q^*}$ is a function of $(u_m)_{Q^*}$. So, $(u_m)_{Q^*}$ can be solved with Newtons method.

### 2.2.4 Evaluation of $u_{mx}$ and $u_{my}$

Similar to the 1D case, $(u_m x)_{Q^*}$ and $(u_m y)_{Q^*}$ are evaluated with a finite-difference approach. A spatial translation of the quadrilateral $A_1^o A_2^o A_3^o A_4^o$ is performed so that which results in the coincidence of the centroid of the new quadrilateral Ao Ao Ao Ao with $Q^*$. Refer to figure 2.5. Let us denote the centroid of the quadrilateral $A_1^o A_2^o A_3^o A_4^o$ by $A^*$ and its spatial coordinates by $(x_{A^*}, y_{A^*})$. Hence for any $l = 1, 2, 3, 4, (x_{A_l^o}, y_{A_l^o})$, the spatial coordinates of $A_l^o$ are given by

$$x_{A_l^o} = x_{A_l^*} + x_{Q^*} - x_{A^*} \text{ and } y_{A_l^o} = y_{A_l^*} + y_{Q^*} - y_{A^*} \quad (2.23)$$

Let

$$(u_m)_{A_l^o} \equiv u_m^*(x_{A_l^o}, y_{A_l^o}, t^n; A_l^o), m = 1, 2, 3, 4, 5, l = 1, 2, 3, 4. \quad (2.24)$$

Consider the three points in the xyu space with the coordinates $(x_{Q^*}, y_{Q^*}, (u_m)_{Q^*}), (x_{A_l^o}^*, y_{A_l^o}^*, (u_m)_{A_l^o})$, for any $m = 1, 2, 3, 4, 5, 6$ respectively. The values of $\partial u / \partial x$ and $\partial u / \partial x$ on the plane that intercepts the three points are given by

$$(u_{mx})_{Q^*} \equiv \Delta_x / \Delta \text{ and } (u_{my})_{Q^*} \equiv \Delta_y / \Delta (\Delta \neq 0), \quad (2.25)$$

where

$$\Delta \equiv \begin{vmatrix} x_{A_1^*} - x_{Q^*} & y_{A_1^*} - y_{Q^*} \\ x_{A_2^*} - x_{Q^*} & y_{A_2^*} - y_{Q^*} \end{vmatrix}, \quad (2.26)$$

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Fig. 2.5: Spatial translation of quadrilateral $A_1^* A_2^* A_3^* A_4^*$

$$\Delta_x \equiv \begin{vmatrix} (u_m)_{A_1^*} - (u_m)_{Q^*} & y_{A_1^*} - y_{Q^*} \\ (u_m)_{A_2^*} - (u_m)_{Q^*} & y_{A_2^*} - y_{Q^*} \end{vmatrix}, \quad (2.27)$$

and

$$\Delta_x \equiv \begin{vmatrix} (u_m)_{A_1^*} - (u_m)_{Q^*} & x_{A_1^*} - x_{Q^*} \\ (u_m)_{A_2^*} - (u_m)_{Q^*} & x_{A_2^*} - x_{Q^*} \end{vmatrix}, \quad (2.28)$$

Here, $(u_m^{(k)})_{Q^*}$ and $(u_m^{(k)})_{Q^*}$, $k = 2, 3, 4$, are defined by replacing the points $A_1^*$ and $A_1^*$ in the above operations with $A_2^*$ and $A_3^*$, $A_3^*$ and $A_4^*$ and $A_4^*$ and $A_1^*$, respectively.

For each $m = 1, 2, 3, 4, 5, 6$, $(u_{mx})_{Q^*}$ and $(u_{my})_{Q^*}$ can then be calculated by

$$(u_{mx})_{Q^*} = \frac{1}{4} \sum_{k=1}^{4} (u_{mx})^{(k)}_{Q^*}, \text{ and } (u_{my})_{Q^*} = \frac{1}{4} \sum_{k=1}^{4} (u_{my})^{(k)}_{Q^*}, \quad (2.29)$$
For a flow with discontinuities, the above equation may be substituted by a reweighting procedure, i.e.,

\[
(u_{mx})_Q^* = \frac{\Sigma_{k=1}^4 \left[ (W_m^{(k)})^\alpha (u_{mx}^{(k)})^\alpha \right]}{\Sigma_{k=1}^4 (W_m^{(k)})^\alpha} \quad \text{and} \quad (u_{my})_Q^* = \frac{\Sigma_{k=1}^4 \left[ (W_m^{(k)})^\alpha (u_{my}^{(k)})^\alpha \right]}{\Sigma_{k=1}^4 (W_m^{(k)})^\alpha}
\]

(2.30)

where \( \alpha \geq 0 \) is an adjustable constant,

\[
\theta_{mk} \equiv \sqrt{\left( (u_{mx}^{(k)})_Q^* \right)^2 + \left( (u_{my}^{(k)})_Q^* \right)^2}, \quad m = 1, 2, 3, 4, 5, 6, k = 1, 2, 3, 4
\]

(2.31)

and

\[
W_m^{(1)} \equiv \theta_{m2}\theta_{m3}\theta_{m4}, \quad W_m^{(2)} \equiv \theta_{m3}\theta_{m4}\theta_{m1}, \quad W_m^{(3)} \equiv \theta_{m4}\theta_{m1}\theta_{m2}, \quad W_m^{(4)} \equiv \theta_{m1}\theta_{m2}\theta_{m3}
\]

(2.32)

Specifically, for any \( k = 1, 2, 3, 4 \), \( \theta_{mk} = 0 \), then \( (u_{mx})_Q^* \) and \( (u_{my})_Q^* \) are set to be 0. Generally \( a = 1 \) or \( a = 2 \), and Eqs. (2.30) becomes Eqs. (2.29) if \( a = 0 \). Usually a small positive number like \( 10^{-50} \) is added to the denominator of Eqs. (2.30) in order to avoid division by zero.
CHAPTER 3
MODEL EQUATIONS FOR STRESS WAVES IN ELASTIC-PLASTIC SOLIDS

The model equations for the wave motion consist of the continuity, momentum and the constitutive equations. The objective rate is used to ensure that the constitutive equation is invariant under change of frame of reference. To close the system of equations obtained, we use the equation of state to relate pressure and density. Since the model is isothermal, we have not included the energy equation.

The model can be validated by using it to find the numerical solution of the 1-D impact problem. The CESE method would be used to find the solution. One of the things this model hopes to achieve is to highlight the effectiveness of the CESE method in modelling elastic-plastic wave motion in solids.

Since we are using the model for simulating wave motion in an application such as explosive welding, this will prove it is safe to use a simplified isothermal model for such an application. Cao[3] has delved into theoretical model for many 1D and 2D problems similar to the one under consideration.
3.1 Constitutive Relations

To begin simulating the stress wave propagation in elastic-plastic media, we must develop a suitable constitutive equation to model the material response. For simplicity, we can focus on infinitesimal deformations initially and then generalize for finite deformations.

The material in question can be assumed to be isotropic, metallic, homogeneous and non-porous. For simplicity, we assume that material is strain rate independent. There are some important points we need to consider in this relation:

- Decompose the infinitesimal stress tensor
- Use the yield criteria to determine how the solid responds. Here, we make use of the von Mises yield criterion.
- Strain hardening rules govern the way how resistance to plastic flow changes with plastic strain. In this case, we assume the material undergoes isotropic hardening.
- Finally, the plastic flow rule determines the relationship between stress and plastic strain.

Let us begin by decomposing the infinitesimal stress tensor:

\[ \epsilon_{ij} = \epsilon_{ij}^{e} + \epsilon_{ij}^{p} \]  

(3.1)

where \( \epsilon_{ij} \) is the strain, \( \epsilon_{ij}^{e} \) and \( \epsilon_{ij}^{p} \) are the elastic and plastic strain respectively. Eq. (3.1) can be written in incremental form:

\[ d\epsilon_{ij} = d\epsilon_{ij}^{e} + d\epsilon_{ij}^{p} \]  

(3.2)
where \( \text{d} \epsilon_{ij} \), \( \text{d} \epsilon^e_{ij} \) and \( \text{d} \epsilon^p_{ij} \) are the total, elastic and plastic strain increments respectively. Using the above equations, we can write the constitutive equation for the linear elastic part for a homogeneous isotropic material:

\[
\epsilon^e_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \tag{3.3}
\]

Again, which can be expressed in incremental form as

\[
\text{d} \epsilon^e_{ij} = \frac{1 + \nu}{E} \text{d} \sigma_{ij} - \frac{\nu}{E} \text{d} \sigma_{kk} \delta_{ij} \tag{3.4}
\]

In the above equation, \( \sigma_{ij} \) is the stress tensor, \( \text{d} \sigma_{ij} \) is the stress increment, \( E \) is the Young’s modulus and \( \nu \) is the poisson ratio. \( E \) and \( \nu \) are material dependent constants.

To determine how the material responds, we chose the von mises yield criteria earlier. This states that plastic yielding will occur only when the second invariant \( J'_2 \) of the deviatoric stress tensor \( S_{ij} \) reaches a critical value \( K^2 \):

\[
J'_2 - K^2 = 0 \quad \text{Plastic deformation} \tag{3.5}
\]

\[
J'_2 < K^2 \quad \text{Elastic deformation} \tag{3.6}
\]

Here,

\[
J'_2 = \frac{1}{2} S_{ij} S_{ij} = \frac{1}{6} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 \right] + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2
\]

\[
= \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \tag{3.7}
\]
where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the principal stresses. Therefore,

\[
\frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] = K^2
\] (3.8)

To determine \( K \), for a simple tension test, \( \sigma_2 = \sigma_3 = 0 \). So, \( \sigma_1 = \sigma_Y \).

\[
\frac{1}{3} \sigma_Y^2 = K^2
\] (3.9)

\[
K = \frac{\sigma_Y}{\sqrt{3}}
\] (3.10)

The von Mises stress criterion can now be written as

\[
\frac{1}{2} S_{ij} S_{ij} = \frac{1}{3} \sigma_Y^2
\] (3.11)

When a stress state at a point satisfies the above yield criterion, then the point undergoes plastic deformation or else elastic deformation. All the stress states corresponding to yielding can be represented by a yield surface in the stress space. The yield function for this surface can be written as:

\[
F(\sigma_{ij}, \epsilon_{ij}^p) = 0
\] (3.12)

where \( \epsilon_{ij}^p \) is the plastic strain. The yield surface depends on the stress \( \sigma_{ij} \). The subsequent yield surfaces would depend on the plastic strain as the yield stress is a function of \( \epsilon_{ij}^p \) since the material undergoes isotropic strain hardening. The flow rule
for the plastic flow is given by:

\[ \dot{\epsilon}_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (3.13) \]

where \( f(\sigma_{ij}, \epsilon_{ij}^p) \) is the plastic flow potential. The deformation of metals can be characterized better by an associated flow rule. In the case of an associated flow rule, the plastic flow potential and yield function are associated i.e \( f=\bar{F} \). Therefore,

\[ \dot{\epsilon}_{ij}^p = \lambda \frac{\partial \bar{F}}{\partial \sigma_{ij}} \quad (3.14) \]

which can be written as

\[ d\epsilon_{ij}^p = d\lambda \frac{\partial \bar{F}}{\partial \sigma_{ij}} \quad (3.15) \]

where \( d\epsilon_{ij}^p \) is the plastic strain increment, \( \lambda \) is a proportional parameter which will be determined later. The above equation is in conjunction with the fact that \( d\epsilon_{ij}^p \) is perpendicular to the yield surface. For time dependent plasticity, it is meaningless to use

\[ F(\sigma_{ij}, \epsilon_{ij}^p) > 0 \quad (3.16) \]
Therefore, for work hardening materials, we have

\[
F < 0 \quad \text{elastic deformation}
\]

\[
F = 0, \quad \frac{dF}{d\sigma_{ij}} d\sigma_{ij} > 0 \quad \text{plastic loading}
\]

\[
F = 0, \quad \frac{dF}{d\sigma_{ij}} d\sigma_{ij} = 0 \quad \text{neutral loading}
\]

\[
F = 0, \quad \frac{dF}{d\sigma_{ij}} d\sigma_{ij} < 0 \quad \text{elastic unloading}
\]

Plastic strain occurs only when there is plastic loading. To deduce the value of \(d\lambda\), we consider the plastic loading case throughout this section. During plastic deformation, the stress point in the stress space must remain on all subsequent yield surfaces as a result of isotropic hardening. This implies the following consistency condition:

\[
dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \bar{\varepsilon}_{ij}^p} d\bar{\varepsilon}_{ij}^p = 0 \tag{3.17}
\]

Using equations (3.15) and (3.17), we get:

\[
d\lambda = -\frac{\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij}}{\frac{\partial F}{\partial \bar{\varepsilon}_{kl}^p} \frac{\partial F}{\partial \sigma_{kl}} } \tag{3.18}
\]

Using the von Mises yield criterion which gives the yield function as:

\[
F(S_{ij}, \bar{\varepsilon}^p) = \frac{1}{2} S_{ij} S_{ij} - \frac{1}{3} (\sigma^y(\bar{\varepsilon}^p))^2 = 0, \tag{3.19}
\]

where \(J_2 = S_{ij} S_{ij}/2\) is the second invariant of the deviatoric stress tensor \(S_{ij} = \sigma_{ij} - \sigma_{kk} \delta_{ij}/3\), and \(\sigma^y(\bar{\varepsilon}^p)\) is the yield stress in uniaxial tension. Using this, we can
obtain the following results:

\[
\frac{dF}{d\sigma_{ij}} = S_{ij}, \quad S_{ij}d\sigma_{ij} = \frac{2}{3}\sigma^y d\sigma^y, \quad \frac{dF}{d\bar{\varepsilon}^p} = -\frac{2}{3}\sigma^y \frac{d\sigma^y}{d\bar{\varepsilon}^p} \quad (3.20)
\]

Using the above equations in (3.17), we get,

\[
d\lambda = \frac{d\sigma^y}{\frac{d\sigma^y}{\bar{\varepsilon}_{kl}} S_{kl}} \quad (3.21)
\]

The loading history of the material affects the yield function. The von mises criterion emphasizes that the effective plastic strain is relevant to the yield function. The length of the path is more important with respect to the path of loading itself. The von mises criterion assumes that the effective stresses greater than a value which causes failure in the uniaxial case will result in plastic deformation. The effective stress \( \bar{\sigma} \) and the effective plastic strain are defined by:

\[
\bar{\sigma} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \quad (3.22)
\]

and

\[
\bar{\sigma} = \sqrt{\frac{2}{3} \epsilon_{ij}^p \epsilon_{ij}^p} \quad (3.23)
\]

Equations (3.22) and (3.23) imply that the effective stress can be used as an equivalent to the yield stress in uniaxial tension:

\[
\bar{\sigma}(\epsilon_{ij}^p) = \sigma_y(\epsilon_{ij}^p) \quad (3.24)
\]
which gives us:

\[ F(S_{ij}, \epsilon_{ij}^P) = \frac{1}{2} S_{ij} S_{ij} - \frac{1}{3} \bar{\sigma} (\epsilon_{ij}^P)^2 = \sigma_y(\epsilon_{ij}^P) = 0 \]  

(3.25)

and also

\[ d\lambda = \frac{d\bar{\sigma}}{d\epsilon_{ij}^P} S_{kl} \]  

(3.26)

We can expand the denominator of the previous equation using chain rule as follows:

\[ \frac{d\bar{\sigma}}{d\epsilon_{kl}^P} = \frac{d\bar{\sigma}}{d\epsilon_{kl}^P} \frac{d\epsilon_{kl}^P}{dW^p} \frac{dW^p}{d\sigma} \frac{d\sigma}{d\epsilon_{ij}^P} S_{kl} \]  

(3.27)

where \( dW^p \) is the plastic work increment given by

\[ dW^p = S_{ij} d\epsilon_{ij}^P dW^p = \bar{\sigma} d\epsilon^P \]  

(3.28)

Substituting Eq. (3.27) into Eq. (3.26), we get

\[ d\lambda = \frac{3}{2} \frac{d\bar{\sigma}}{d\epsilon_{ij}^P} \frac{d\bar{\sigma}}{d\epsilon_{ij}^P} S_{ij} \]  

(3.29)

Using the above result along with the flow rule (3.15) gives us the plastic strain increment

\[ d\epsilon_{ij}^P = \frac{3}{2} \frac{d\bar{\sigma}}{d\epsilon_{ij}^P} S_{ij} \]  

(3.30)

Since the material undergoes linear strain hardening, the yield stress can be given as
the following linear equation:

$$\sigma^y(\varepsilon^p) = \sigma^y_o + B_{SH} \varepsilon^p \quad (3.31)$$

where the strength coefficient $B_{SH}$ and the intial yield stress $\sigma^y_o$ are material dependent constants. From Eqs. (3.24) and (3.31) we get

$$\frac{d\bar{\sigma}}{d\bar{\varepsilon}^p} = B_{SH} \quad (3.32)$$

Therefore,

$$d\varepsilon^p_{ij} = \frac{3}{2} \frac{d\bar{\sigma}}{B_{SH} \bar{\sigma}} S_{ij} \quad (3.33)$$

The infinitesimal plasticity concept allows for additive decomposition of elastic and plastic strain increments. Linearly combining Eqs. (3.4) and (3.33), we get:

$$d\varepsilon_{ij} = \frac{1 + \nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} + \frac{3}{2} \frac{d\bar{\sigma}}{B_{SH} \bar{\sigma}} S_{ij} \quad (3.34)$$

We can rewrite the previous equation as,

$$\frac{1 + \nu}{E} d\sigma_{ij} = d\varepsilon_{ij} + \frac{\nu}{E} d\sigma_{kk} \delta_{ij} - \frac{3}{2} \frac{d\bar{\sigma}}{B_{SH} \bar{\sigma}} S_{ij} \quad (3.35)$$

After taking the inner product of the deviatoric stress $S_{ij}$ and Eq. (3.35), using eqs. (3.20) and (3.25), gives us the effective stress increment

$$d\bar{\sigma} = \frac{S_{ij} d\varepsilon_{ij}}{\left(\frac{3}{2} \frac{1 + \nu}{E} + \frac{1}{B_{SH}}\right) \bar{\sigma}} \quad (3.36)$$

Taking the trace of Eq. (3.4) gives us the relation between mean stress increment
and volumetric strain increment:

\[ d\sigma_{ii} = \frac{E}{1 - 2\nu} d\epsilon_{ii} \]  

(3.37)

Substituting Eqs. (3.36) and (3.37) into Eq. (3.34), we get

\[ dS_{ij} = 2\mu d\epsilon_{ij} - \frac{2}{3}\mu d\epsilon_{kk}\delta_{ij} - 3\mu \left( \frac{B_{2n}}{2\mu} + \frac{3}{2} \right) S_{mn} S_{mn} S_{ij} \]  

(3.38)

where the shear modulus is \( \mu = \frac{E}{2(1 + \nu)} \), E is the Young’s modulus and \( \nu \) is the poisson’s ratio. Also we have used \( dS_{ij} = d\sigma_{ij} - \frac{1}{3} d\sigma_{kk}\delta_{ij} \). The previous equation can be expressed in rate form as

\[ \dot{S}_{ij} = 2\mu \dot{\epsilon}_{ij} - \frac{2}{3}\mu \dot{\epsilon}_{kk}\delta_{ij} - 3\mu \left( \frac{B_{2n}}{2\mu} + \frac{3}{2} \right) S_{mn} S_{mn} S_{ij} \]  

(3.39)

To formulate a plastic constitutive equation for finite deformation, we need to generalize the infinitesimal elastic-plastic constitutive equation to finite deformations. We can do this by doing the following: (i) replace the infinitesimal strain rate \( \dot{\epsilon}_{ij} \) with its finite Eulerian analog \( D_{ij} \), rate of deformation. (ii) replace the stress tensor \( \sigma_{ij} \) with its finite Eulerian analog \( T_{ij} \), Cauchy stress. (iii) Use an objective rate \( D/Dt \) for the stress. We can use the Jaumann rate as the objective rate to make sure the constitutive equation is invariant under any arbitrary superposed rigid body motion. The Jaumann rate is given by:

\[ \frac{DS_{ij}}{Dt} = \frac{\partial S_{ij}}{\partial t} + v_k \frac{\partial S_{ij}}{\partial x_k} - W_{ik} S_{kj} + S_{ik} W_{kj} \]  

(3.40)
where

\[ D_{ij} = \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]

is the symmetric part of the velocity gradient and

\[ W_{ij} = \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \]

is the skew part of the velocity gradient. The application of the Jaumann rate to eq. (3.39) gives us the final constitutive equation:

\[ \frac{D}{Dt} S_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu D_{kk}\delta_{ij} - 3\mu \left( \frac{\beta + \mu}{2\mu} + \frac{3}{2} \right) S_{mn}S_{mn} S_{ij} \]

3.2 Governing Equations

Based on the constitutive relations derived in the previous section, the three dimensional governing equations for elastic-plastic wave motion formulated in the Eulerian frame are: (i) Conservation of mass:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \]

(ii) Conservation of linear momentum:

\[ \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p - S_{ij}) = 0 \]

(iii) Elastic-plastic constitutive relation:

\[ \frac{D}{Dt}(S_{ij}) = 2\mu D_{ij} - \frac{2}{3}(\mu D_{kk}\delta_{ij}) - \beta(s) S_{kl}D_{kl}S_{ij} \]
where \( s = S_{mn}S_{mn} \) and
\[
\beta(s) = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } \frac{dF}{d\sigma_{ij}} < 0 \\
0 & \text{if } F = 0 \text{ and } \frac{dF}{d\sigma_{ij}} = 0 \\
\frac{6\mu^2}{(3\mu + B_{SH}) s} & \text{if } F = 0 \text{ and } \frac{dF}{d\sigma_{ij}} > 0 
\end{cases}
(3.45)
\]

The above four rows signify elastic deformation, elastic unloading, neutral loading and plastic loading respectively. The Cauchy stress components and pressure are \( T_{ij} = -p + S_{ij}, p = -\sum_{i=1}^{3} T_{ii}/3 \). In the above equations (3.42)-(3.45), \( \rho \) is the density, \( v_i \) is the velocity, \( \bar{p} = -T_{kk}/3 \) is the mean pressure, and \( D_{ij} = (\partial v_i / \partial x_j + \partial v_j / \partial x_i) / 2 \) is the symmetric part of the velocity gradient, all of which are functions of time \( t \) and the position \( \mathbf{x} = \{x_1, x_2, x_3\} \) of a typical material particle in the current configuration referred to a fixed Cartesian basis. The shear modulus \( \mu \) is a prescribed material constant. Using the \( J_2 \) flow plastic theory, we get,
\[
\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} = d\left(\frac{1}{2} S_{ij} S_{ij} - K^2\right),
(3.46)
\]
where \( K \) is a constant for perfect plastic materials and in general it is
\[
K^2 = \frac{1}{3}(\bar{\sigma}(\varepsilon^p_{ij}))^2.
\]

Then Eqs. (3.44) and (3.45) can be written as
\[
\frac{D}{Dt} S_{ij} = 2\mu D_{ij} - \frac{2}{3} \mu D_{kk} \delta_{ij} - \beta(s) S_{kl} D_{kl} S_{ij},
(3.47)
\]
where \( s = S_{mn}S_{mn} \) and
\[
\beta(s) = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } d(\frac{1}{2} S_{ij} S_{ij} - k^2) \leq 0 \\
\frac{6\mu^2}{(3\mu + B_{SH}) s} & \text{if } F = 0 \text{ and } d(\frac{1}{2} S_{ij} S_{ij} - k^2) > 0 
\end{cases}
(3.48)
\]
Lets formulate the equations for the 1-dimensional and 2-dimensional cases in the following sections.

### 3.2.1 1-D Formulation

The equations to be considered include the mass, momentum conservation equations and the constitutive equation. For a dimensional strain problem, the Cauchy stress tensor can be divided into the deviatoric stress and pressure:

\[
T = \begin{bmatrix}
T_{11} & 0 & 0 \\
0 & T_{22} & 0 \\
0 & 0 & T_{33}
\end{bmatrix} = \begin{bmatrix}
-p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{bmatrix} + \begin{bmatrix}
S_{11} & 0 & 0 \\
0 & S_{22} & 0 \\
0 & 0 & S_{33}
\end{bmatrix} \tag{3.49}
\]

where the deviatoric stress matrix can be expressed as

\[
S = \begin{bmatrix}
S_{11} & 0 & 0 \\
0 & -\frac{1}{2}S_{11} & 0 \\
0 & 0 & \frac{1}{2}S_{11}
\end{bmatrix} \tag{3.50}
\]

Since we consider the wave motion as one dimensional, \(S_{11}\) is directly related to \(S_{22}\) and \(S_{33}\) as shown in the above equation. The velocity components for this case are:

\[
v = (u, v, w) = (u, 0, 0) \tag{3.51}
\]

Using eqs. (3.49), (3.50) and (3.51), we can expand eqs. (3.42) and (3.43) into the following scalar forms:

\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial x} = 0, \tag{3.52}
\]

\[
\frac{\partial \rho v_1}{\partial t} + \frac{\partial}{\partial x} (\rho v_1 v_1 + p - S_{11}) = 0. \tag{3.53}
\]
where \( p \) is the mean stress and \( S_{11} \) is a component of the deviatoric stress. Using the definition for Jaumann rate, we can calculate the objective time derivative of the deviatoric stress tensor i.e, \( DS/Dt \):

\[
\frac{DS_{ij}}{Dt} = \frac{\partial S_{ij}}{\partial t} + v_k \frac{\partial S_{ij}}{\partial x_k} - W_{ik} S_{kj} + S_{ik} W_{kj}
\]  

(3.54)

where \( W_{ij} = (\partial v_i / \partial x_j - \partial v_j / \partial x_i) \) is the skew part of the velocity gradient. With the given velocities in Eq. (3.51), we have

\[
S = \begin{bmatrix}
\frac{\partial u}{\partial x} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  

(3.55)

Since \( W = O \), \( SW = WS = O \). Therefore, using the above results, we can expand Eq. (3.54) into the following scalar form:

\[
\frac{\partial p S_{11}}{\partial t} + \frac{\partial p v_1 S_{11}}{\partial x} = \frac{4}{3} \mu \rho \left( 1 - \frac{\beta}{(1 + \frac{B_{SH}}{3\mu})} \right) \frac{\partial u}{\partial x}
\]  

(3.56)

where

\[
\beta = \begin{cases}
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } d(\frac{1}{2} S_{ij} S_{ij} - k^2) \leq 0 \\
1 & \text{if } F = 0 \text{ and } d(\frac{1}{2} S_{ij} S_{ij} - k^2) > 0
\end{cases}
\]  

(3.57)

where \( J_2 \) yielding criterion has been used for this one dimensional stress case and \( B_{SH} \) is the modulus for the elastic-linear hardening plastic material. To close Eqs. (3.42), (3.43) and (3.57), we use the following equation of state to relate pressure to density:

\[
p = k \ln \frac{\rho}{\rho_o}
\]  

(3.58)
where \( p \) is the pressure, \( k \) is the bulk modulus, \( \rho \) is the density and \( \rho_o \) is the initial density. The governing equations (3.42), (3.43) and (3.57) in the conservative form can be written in the following vector form:

\[
\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = S \tag{3.59}
\]

where

\[
Q = (\rho, \rho u, \rho S_{11})^T,
\]

\[
F = (\rho u, \rho uu + p - S_{11}, \rho S_{11} u)^T,
\]

\[
S = (0, 0, \frac{4}{3} \mu \rho (1 - \beta/(1 + B_{SH}/3\mu)))^T
\]

In order to use the model for numerical simulations, we can recast them in the following form:

\[
\frac{\partial Q}{\partial t} + A \frac{\partial Q}{\partial x} = S \tag{3.60}
\]

where the Jacobian matrix \( A \) and source term are given by

\[
A = \frac{\partial F}{\partial Q} = \begin{pmatrix} 0 & \frac{S_{11}}{\rho} & 1 & 0 \\ -u^2 + \frac{k}{\rho} + \frac{S_{11}}{\rho} & 2u & -1 & 0 \\ -uS_{11} & S_{11} & u \end{pmatrix} \tag{3.61}
\]

and

\[
S = (0, 0, \frac{4}{3} \mu \rho (1 - \frac{\beta}{1 + B_{SH}/3\mu}))^T \tag{3.62}
\]
In order to study the mathematical structure of system of equations, we need to obtain the eigenvalues of the Jacobian matrix $A$. To do this, we will rewrite the above hyperbolic system in non-conservative form. The non-conservative variables vectors is:

$$\tilde{Q} = (\rho, u, S_{11})^T.$$  \hfill (3.63)

We premultiply both sides of Eq. (3.60) by a matrix $M$:

$$M \frac{\partial Q}{\partial t} + MAM^{-1}M \frac{\partial Q}{\partial x} = MS$$  \hfill (3.64)

where

$$M = \frac{\partial \tilde{Q}}{\partial Q} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & 1 & 0 \\ -\frac{S_{11}}{\rho} & 0 & 1/\rho \end{pmatrix}$$  \hfill (3.65)

and

$$M^{-1} = \frac{\partial Q}{\partial \tilde{Q}} = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ S_{11} & 0 & \rho \end{pmatrix}$$  \hfill (3.66)

Using Eqs. (3.65) and (3.66), we rewrite Eq. (3.64) as

$$\frac{\partial \tilde{Q}}{\partial t} + \tilde{A} \frac{\partial \tilde{Q}}{\partial x} = \tilde{S}$$  \hfill (3.67)
where
\[
\tilde{A} = MAM^{-1} = \begin{pmatrix}
u & \rho & 0 \\
\frac{k}{\rho_2} & u & 1 \\
0 & 0 & u \\
\end{pmatrix}
\] (3.68)

and \( S = (0, 0, \frac{4}{3} \mu (1 - \beta/(1 + B_{SH}/3\mu))^T \) We can modify Eq. (3.67) by moving the source term to the left side and we get
\[
\frac{\partial \tilde{Q}}{\partial t} + \tilde{A} \frac{\partial \tilde{Q}}{\partial x} = 0 \tag{3.69}
\]

where
\[
\tilde{A} = \begin{pmatrix}
u & \rho & 0 \\
\frac{k}{\rho_2} & u & 1 \\
0 & -\frac{4}{3} \mu (1 - \beta/(1 + B_{SH}/3\mu) & u \\
\end{pmatrix}
\] (3.70)

The eigen values can then be easily derived and they are:
\[
\lambda_1 = u, \lambda_{2,3} = u \pm c = u \pm \sqrt{K + \frac{4}{3} \mu (1 - \frac{\beta}{1 + B_{SH}/3\mu})}, \tag{3.71}
\]

where \( \mu \) is the Shear modulus, \( K \) is the Bulk modulus \( B_{SH} \) is the strength coefficient. It is evident that the wave speed is \( \sqrt{(k + \frac{4}{3} \mu)/\rho} \) by putting \( \beta = 0 \).

3.2.2 2-D Formulation

To formulate the equations for the two dimensional case, we will expand the eqs. (3.42), (3.43) and (3.44):
\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \tag{3.72}
\]
\[
\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p - S_{ij}) = 0 \quad (3.73)
\]

\[
\frac{D}{Dt}(S_{ij}) = 2\mu D_{ij} - \frac{2}{3}(\mu D_{kk}\delta_{ij}) - \beta(s)S_{kl}D_{kl}S_{ij}, \quad (3.74)
\]

where \(\rho\) is the density, \(\mathbf{v} = (u, v, w)\) is the velocity vector, \(\mathbf{D}\) is the symmetric part of the velocity gradient matrix, \(\mathbf{S}\) is the deviatoric stress tensor. The deviatoric stress tensor is given by

\[
\mathbf{S} = \begin{pmatrix}
S_{11} & S_{12} & 0 \\
S_{21} & S_{22} & 0 \\
0 & 0 & -(S_{11} + S_{22})
\end{pmatrix} \quad (3.75)
\]

The velocity components are

\[
v = (u, v, w) = (u, v, 0) \quad (3.76)
\]

Using Eqs. (3.75) and (3.76), we expand Eqs. (3.72)-(3.74) into the following scalar equations:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \quad (3.77)
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu + p - S_{11})}{\partial x} + \frac{\partial (\rho uv - S_{12})}{\partial y} = 0 \quad (3.78)
\]

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho vw - S_{12})}{\partial x} + \frac{\partial (\rho vv + p - S_{22})}{\partial y} = 0 \quad (3.79)
\]
Using the definition for Jaumann rate, we can calculate the objective time derivative of the deviatoric stress tensor i.e, DS/Dt:

$$\frac{DS_{ij}}{Dt} = \frac{\partial S_{ij}}{\partial t} + v_k \frac{\partial S_{ij}}{\partial x_k} - W_{ik} S_{kj} + S_{ik} W_{kj}$$ \hspace{1cm} (3.80)

where

$$D_{ij} = \left( \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right)$$

$$W_{ij} = \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

Having known the velocities in Eq. (3.76), we have

$$D = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & 0 \\ \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & 0 \\ \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
Using the \( D \), \( W \) and \( S \) formulations, we have

\[
\text{SW} = \begin{pmatrix}
\frac{1}{2} S_{12} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) & \frac{1}{2} S_{11} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & 0 \\
\frac{1}{2} S_{22} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) & \frac{1}{2} S_{21} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\text{WS} = \begin{pmatrix}
\frac{1}{2} S_{21} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & \frac{1}{2} S_{22} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & 0 \\
\frac{1}{2} S_{11} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) & \frac{1}{2} S_{12} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\text{SD} = \begin{pmatrix}
S_{11} \frac{\partial u}{\partial x} + \frac{1}{2} S_{12} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & S_{12} \frac{\partial v}{\partial y} + \frac{1}{2} S_{11} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) & 0 \\
S_{21} \frac{\partial u}{\partial x} + \frac{1}{2} S_{22} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & S_{22} \frac{\partial v}{\partial y} + \frac{1}{2} S_{21} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\text{DS} = \begin{pmatrix}
S_{11} \frac{\partial u}{\partial x} + \frac{1}{2} S_{21} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & S_{12} \frac{\partial v}{\partial y} + \frac{1}{2} S_{11} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) & 0 \\
S_{21} \frac{\partial u}{\partial y} + \frac{1}{2} S_{11} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) & S_{22} \frac{\partial v}{\partial y} + \frac{1}{2} S_{12} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Using the above equations we can expand Eq. (3.80) into the following scalar forms:

\[
\frac{DS_{11}}{Dt} = \frac{\partial S_{11}}{\partial t} + u \frac{\partial S_{11}}{\partial x} + v \frac{\partial S_{11}}{\partial y} - S_{12} ( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} )
\] (3.81)

\[
\frac{DS_{22}}{Dt} = \frac{\partial S_{22}}{\partial t} + u \frac{\partial S_{22}}{\partial x} + v \frac{\partial S_{22}}{\partial y} + S_{12} ( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} )
\] (3.82)
\[
\frac{DS_{12}}{Dt} = \frac{DS_{21}}{Dt} = \frac{\partial S_{12}}{\partial t} + u \frac{\partial S_{12}}{\partial x} + v \frac{\partial S_{12}}{\partial y} + \frac{1}{2} (S_{11} - S_{22}) \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \tag{3.83}
\]

Substituting Eqs. (3.81)-(3.83) into the constitutive equation (3.44), we get:

\[
\frac{\partial S_{11}}{\partial t} + u \frac{\partial S_{11}}{\partial x} + v \frac{\partial S_{11}}{\partial y} - S_{12} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) - \frac{4}{3} \mu \frac{\partial u}{\partial x} + \frac{2}{3} \mu \frac{\partial v}{\partial y} + \beta(s)(\mathbf{S.D})S_{11} = 0 \tag{3.84}
\]

\[
\frac{\partial S_{22}}{\partial t} + u \frac{\partial S_{22}}{\partial x} + v \frac{\partial S_{22}}{\partial y} + S_{12} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \frac{2}{3} \mu \frac{\partial u}{\partial x} - \frac{4}{3} \mu \frac{\partial v}{\partial y} + \beta(s)(\mathbf{S.D})S_{22} = 0 \tag{3.85}
\]

\[
\frac{\partial S_{12}}{\partial t} + u \frac{\partial S_{12}}{\partial x} + v \frac{\partial S_{12}}{\partial y} + \frac{1}{2} (S_{11} - S_{22}) \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) - \mu \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \beta(s)(\mathbf{S.D})S_{12} = 0 \tag{3.86}
\]

where \( \beta(s) \) is defined by Eq. (3.48) and \( \mathbf{S.D} \) is given by

\[
\mathbf{S.D} = S_{11} \frac{\partial u}{\partial x} + S_{12} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + S_{22} \frac{\partial v}{\partial y} \tag{3.87}
\]

Eqs. (3.84)-(3.86) are in non-conservative form. Since the other model equations are in conservative form, using conservative variables \((\rho S_{11}, \rho S_{22}, \rho S_{12})^T\), we get

\[
\frac{\partial (\rho S_{11})}{\partial t} + \frac{\partial (\rho u S_{11})}{\partial x} + v \frac{\partial (\rho v S_{11})}{\partial y} = \left( \frac{\partial (\rho u S_{12})}{\partial y} - u \frac{\partial (\rho S_{12})}{\partial y} \right) - \left( \frac{\partial (\rho v S_{12})}{\partial x} - v \frac{\partial (\rho S_{12})}{\partial x} \right)
\]

\[
+ \frac{4}{3} \mu \left( \frac{\partial (\rho u)}{\partial x} - u \frac{\partial \rho}{\partial x} \right) - \frac{2}{3} \mu \left( \frac{\partial (\rho v)}{\partial y} - v \frac{\partial \rho}{\partial y} \right) + \beta(s)(\mathbf{S.D})\rho S_{11} = 0 \tag{3.88}
\]
\[
\frac{\partial (\rho S_{22})}{\partial t} + \frac{\partial (\rho u S_{22})}{\partial x} + v \frac{\partial (\rho v S_{22})}{\partial y} = \left( \frac{\partial (\rho v S_{12})}{\partial x} - v \frac{\partial (\rho S_{12})}{\partial x} \right) - \left( \frac{\partial (\rho u S_{12})}{\partial y} - u \frac{\partial (\rho S_{12})}{\partial y} \right)
+ \frac{4}{3} \mu \left( \frac{\partial (\rho u)}{\partial y} - v \frac{\partial \rho}{\partial y} \right) - \frac{2}{3} \mu \left( \frac{\partial (\rho u)}{\partial x} - u \frac{\partial \rho}{\partial x} \right) + \beta(s)(S.D)\rho S_{22} = 0 \quad (3.89)
\]

\[
\frac{\partial (\rho S_{12})}{\partial t} + \frac{\partial (\rho u S_{12})}{\partial x} + v \frac{\partial (\rho v S_{12})}{\partial y} = \frac{1}{2} \left\{ \left( \frac{\partial (\rho v S_{11})}{\partial x} - v \frac{\partial (\rho S_{11})}{\partial x} \right) - \left( \frac{\partial (\rho u S_{11})}{\partial y} - u \frac{\partial (\rho S_{11})}{\partial y} \right) \right\}
- \frac{1}{2} \left\{ \left( \frac{\partial (\rho v S_{22})}{\partial x} - v \frac{\partial (\rho S_{22})}{\partial x} \right) - \left( \frac{\partial (\rho u S_{22})}{\partial y} - u \frac{\partial (\rho S_{22})}{\partial y} \right) \right\}
+ \mu \left( \frac{\partial (\rho u)}{\partial y} - u \frac{\partial \rho}{\partial y} \right) + \mu \left( \frac{\partial (\rho v)}{\partial x} - v \frac{\partial \rho}{\partial x} \right) + \beta(s)(S.D)\rho S_{12} = 0 \quad (3.90)
\]

To close the above system of equations, we use the equation of state to relate pressure to density as done for 1 dimensional case:

\[
p = k \ln \frac{\rho}{\rho_o} \quad (3.91)
\]

where \( p \) is the pressure, \( k \) is the bulk modulus, \( \rho \) is the density and \( \rho_o \) is the initial density. For the formulation, the governing equations (3.77)-(3.79) and the constitutive relations (3.88)-(3.90) in conservative form should be cast into the following vector form:

\[
\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S} \quad (3.92)
\]
where the conservative variable vector is defined as
\[
Q = (\rho, \rho u, \rho v, \rho S_{11}, \rho S_{22}, \rho S_{12})^T,
\]
\[
F = (\rho u, \rho uu + p - S_{11}, \rho uv - S_{12}, \rho uu S_{11} + \rho u S_{22}, \rho u S_{12})^T,
\]
\[
G = (\rho v, \rho uv - S_{12}, \rho vv + p - S_{22}, \rho v S_{11} + \rho v S_{22}, \rho v S_{12})^T
\]

the source term on the right hand side of the equation is given by
\[
S_s = \begin{pmatrix}
0 \\
0 \\
\frac{4}{3} \mu \left[ \frac{\partial p u}{\partial y} - u \frac{\partial p S_{12}}{\partial y} \right] - \frac{2}{3} \mu \left[ \frac{\partial p v}{\partial y} - v \frac{\partial p S_{12}}{\partial y} \right] \\
\frac{4}{3} \mu \left[ \frac{\partial p v}{\partial x} - v \frac{\partial p S_{12}}{\partial x} \right] - \frac{2}{3} \mu \left[ \frac{\partial p u}{\partial x} - u \frac{\partial p S_{12}}{\partial x} \right] + \beta(s)(S.D)\rho S_{11}
\end{pmatrix}
\]
\[
S_s = \begin{pmatrix}
0 \\
0 \\
\frac{4}{3} \mu \left[ \frac{\partial p v}{\partial x} - v \frac{\partial p S_{22}}{\partial x} \right] - \frac{2}{3} \mu \left[ \frac{\partial p u}{\partial x} - u \frac{\partial p S_{22}}{\partial x} \right] + \beta(s)(S.D)\rho S_{22}
\end{pmatrix}
\]
\[
S_s = \begin{pmatrix}
\frac{1}{2} \left[ \frac{\partial p v S_{11}}{\partial x} - v \frac{\partial p S_{11}}{\partial x} \right] - \frac{1}{2} \left[ \frac{\partial p v S_{22}}{\partial x} - v \frac{\partial p S_{22}}{\partial x} \right] \\
+ \mu \left[ \frac{\partial p v}{\partial y} - v \frac{\partial p S_{12}}{\partial y} \right] + \mu \left[ \frac{\partial p u}{\partial y} - u \frac{\partial p S_{12}}{\partial y} \right] + \beta(s)(S.D)\rho S_{12}
\end{pmatrix}
\]

We can rewrite the above system as
\[
\frac{\partial Q}{\partial t} + A \frac{\partial Q}{\partial x} + B \frac{\partial Q}{\partial y} = S_s
\]
where jacobian matrices A and B are defined by:

$$A = \frac{\partial F}{\partial Q} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
A_{21} & \frac{2 + q_2}{q_1} + \frac{\partial p}{\partial q_3} & \frac{\partial p}{\partial q_4} & \frac{1}{q_1} & \frac{\partial p}{\partial q_5} & \frac{\partial p}{\partial q_6} \\
-\frac{q_2q_3}{q_1^2} + \frac{q_6}{q_1^2} & \frac{q_3}{q_1} & \frac{q_2}{q_1} & 0 & 0 & -\frac{1}{q_1} \\
-\frac{q_2q_4}{q_1^2} & \frac{q_4}{q_1} & 0 & \frac{q_2}{q_1} & 0 & 0 \\
-\frac{q_2q_5}{q_1^2} & \frac{q_5}{q_1} & 0 & 0 & \frac{q_2}{q_1} & 0 \\
-\frac{q_2q_6}{q_1^2} & \frac{q_6}{q_1} & 0 & 0 & 0 & \frac{q_2}{q_1}
\end{bmatrix}$$

(3.95)

where $$A_{21} = -\frac{q_2^2}{q_1^2} + \frac{\partial p}{\partial q_1} + \frac{q_4}{q_1^2}$$ The above jacobian matrix can be rewritten as

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
A_{21} & 2u + \frac{\partial p}{\partial q_3} & \frac{\partial p}{\partial q_4} & \frac{1}{\rho} & \frac{\partial p}{\partial q_5} & \frac{\partial p}{\partial q_6} \\
-vu + \frac{S_{12}}{\rho} & v & u & 0 & 0 & -\frac{1}{\rho} \\
-uS_{11} & S_{11} & 0 & u & 0 & 0 \\
-uS_{22} & S_{11} & 0 & 0 & u & 0 \\
-uS_{12} & S_{12} & 0 & 0 & 0 & u
\end{bmatrix}$$

(3.96)
where $A_{21} = -u^2 + \frac{\partial p}{\partial q_1} + \frac{S_{11}}{\rho}$ and jacobian matrix $B$ is given as

\[
B = \frac{\partial G}{\partial Q} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{q_2 q_3}{q_1^2} + \frac{q_6}{q_1^2} & q_3 & q_2 & 0 & 0 & -\frac{1}{\rho} \\
\frac{B_{31}}{\rho} & 2q_3 q_1 & \frac{\partial p}{\partial q_3} & \frac{\partial p}{\partial q_4} & \frac{\partial p}{\partial q_5} & -\frac{1}{\rho} \\
q_3 q_4 & 0 & q_4 & q_3 & 0 & 0 \\
q_3 q_5 & 0 & q_5 & q_3 & 0 & 0 \\
q_3 q_6 & 0 & q_6 & 0 & q_3 & 0 \\
\frac{q_3 q_6}{q_1^2} & 0 & q_1 & 0 & 0 & q_3 \\
\end{bmatrix}
\]

(3.97)

where $B_{31} = -\frac{q_2^3}{q_1^2} + \frac{\partial p}{\partial q_1} + \frac{q_5}{q_1^2}$ the above jacobian matrix can be rewritten as

\[
B = \frac{\partial G}{\partial Q} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
-uv + \frac{S_{12}}{\rho} & v & u & 0 & 0 & -\frac{1}{\rho} \\
B_{31} & 2v + \frac{\partial p}{\partial q_3} & \frac{\partial p}{\partial q_4} & \frac{\partial p}{\partial q_5} & -\frac{1}{\rho} & \frac{\partial p}{\partial q_6} \\
-v S_{11} & 0 & S_{11} & v & 0 & 0 \\
-v S_{22} & 0 & S_{22} & 0 & v & 0 \\
-v S_{12} & 0 & S_{12} & 0 & 0 & v \\
\end{bmatrix}
\]

(3.98)

where $B_{31} = -v^2 + \frac{\partial p}{\partial q_1} + \frac{S_{22}}{\rho}$ In the Jacobian matrices $A$ and $B$, the derivative of pressure can be calculated using equation of state:

\[
p = k \ln \frac{\rho}{\rho_o}
\]

(3.99)
The same equation can be written as $p = k \ln \frac{q_1}{\rho_o}$. We can use the above equation to substitute the derivatives of pressure in the jacobian matrices. The derivatives are:

$$\frac{\partial p}{\partial q_1} = k \frac{q_1}{\rho}$$

and

$$\frac{\partial p}{\partial q_2} = \frac{\partial p}{\partial q_3} = \frac{\partial p}{\partial q_4} = \frac{\partial p}{\partial q_5} = \frac{\partial p}{\partial q_6} = 0$$

Substituting the above results in the jacobian matrices, we get

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-u^2 + k \frac{S_{11}}{\rho} & 2u & 0 & -\frac{1}{\rho} & 0 & 0 \\
-u v + \frac{S_{12}}{\rho} & v & u & 0 & 0 & -\frac{1}{\rho} \\
-u S_{11} & S_{11} & 0 & u & 0 \\
-u S_{22} & S_{11} & 0 & 0 & u & 0 \\
-u S_{12} & S_{12} & 0 & 0 & 0 & u \\
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
-u v + \frac{S_{12}}{\rho} & v & u & 0 & 0 & -\frac{1}{\rho} \\
-v^2 + k \frac{S_{22}}{\rho} & 0 & 2v & 0 & -\frac{1}{\rho} & 0 \\
-v S_{11} & 0 & S_{11} & v & 0 & 0 \\
-v S_{22} & 0 & S_{22} & 0 & v & 0 \\
-v S_{12} & 0 & S_{12} & 0 & 0 & v \\
\end{bmatrix}$$
We can do a similar mathematical analysis of the system by calculating the eigen values of the Jacobian matrices A and B, but it is complex and beyond the scope of this thesis. For simulation purposes, we can use the eigen values obtained in the 1-D model for calculating the CFL.
CHAPTER 4
EXPLOSION WELDING

4.1 Introduction

One of the applications of the constitutive model developed in chapter 3 is explosive welding. One method of joining a wide variety of both similar and dissimilar metals is explosive welding. It is a highly capable welding process used in fusing metals which cannot be joined by other processes.

This technique is quite popular in improving the corrosion resistance of carbon steel by cladding it with a thin layer of corrosion resistant material such as stainless steel, titanium, zirconium etc. The geometries possible through this process are limited to plates and tubes.

This technique forms a solid state weld between the two metals by plasticizing them. There is no melting involved and the weld is clean and quick. Jetting during the process cleans both surfaces to produce the clean weld. Jetting is process of explosion of material from both surfaces due to the high velocity impact.

The typical setup for explosive welding is shown in figure (4.1). Any welding process requires a huge amount of energy. Explosive welding uses the high density energy contained in the explosive and distribute over a wide surface area to form the weld. Conversely, huge amount of energy can be concentrated to produce welds over small areas. The welding occurs when there is an oblique high velocity impact or
collision of two metal plates. This collision is governed by the collision angle $\beta$ and collision velocity $V_p$. When the two metal plates collide, a high velocity forward jet is formed which takes away contaminants from both plates and cleans the two surfaces. This phenomena is highly useful in removing oxide films which inhibit the process of bonding between the surfaces.

The pressure at the collision point needs to be high for a considerable amount of time to allow inter-atomic bonds. It is important to select the right explosive and hence get the correct detonation velocity to obtain good welds. The detonation of the explosive produces a shockwave with velocities ranging between 3000 to 6000 m/s. This causes the flyer plate to impact the base plate at high velocities resulting in high pressure which in turn causes plastic deformation of the metals. This allows the metals to flow and form a bond. The detonation of the explosive does generate a considerable amount of heat but time is insufficient to allow for heat transfer \cite{7}. There is no significant change in temperature hence we will be using the isothermal model developed in chapter 3. The bonding process is carefully controlled based

Fig. 4.1: Typical Explosive Welding Setup \cite{10}
on a number of parameters which will be discussed in the subsequent section. In this chapter, the explosive welding process will be numerically simulated using the aforementioned constitutive model.

### 4.2 Explosive welding parameters

In addition to the explosive selection and the detonation velocity, there are a number of parameters governing the process. The collision velocity $V_c$ and the plate velocity $V_p$ should be less than the velocity of sound in either welding component [6][5]. The detonation velocity in a common explosive is typically higher than the velocity of sound in engineering materials. Usually, an inert substance such as sand or perlite is used to get the detonation velocity in the correct range [6][5]. The explosive should allow for uniform detonation so that the collision velocity is uniform throughout the weld.

The discontinuities in shock fronts Findik[7] can produce tensile stresses of sufficient magnitude so as to fracture the bond. This will be case where the velocity of the shock front exceeds the velocity sound in the material. The sonic velocity is given by:

$$U_s = \sqrt{\frac{E'}{\rho}} \quad (4.1)$$

where $U_s$ is the sonic velocity, $E$ is the elastic modulus and $\rho$ is the material density. Since most metals have sonic velocity in the range of 2000 to 6000 m/s, it is desirable to use explosives with detonation velocities in this range. The schematic (4.2) gives an idea about the different parameters affecting the explosive welding process. Flyer plate velocities can be calculated from Gurneys model[44]. The model assumes that
the flyer plate will accelerate perpendicular to the direction of detonation and reach a terminal velocity before impacting the base plate. We will be making use of Gurney equation for our numerical simulations.

This thesis will not be focussing on parameters governing the quality of the weld. The simulations will help us understand the variation of pressure, density, velocity and stress on impact.

4.3 Governing Equations

The governing equations consist of the continuity, momentum and the constitutive equations. Since the model is isothermal, we have not included the energy equation. To close the hyperbolic system of equations, we use a simple equation of state used by Cao[3] relating pressure and density. Akbari Mousavi and Al-Hassani[2] used the Jones-Wilkins-Lee (JWL) and Williamsburg equation of state for the detonation products for the PETN and ANFO explosives respectively. Since, we are not modeling
the detonation process but the impact phase of the welding process, we can use a simpler equation of state.

Using governing equations for the 1D model from chapter 3, in scalar form:

(i) Conservation of mass:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_1}{\partial x} = 0, \quad (4.2)
\]

(ii) Conservation of linear momentum:

\[
\frac{\partial \rho v_1}{\partial t} + \frac{\partial}{\partial x} (\rho v_1 v_1 + p - S_{11}) = 0,
\]

where \( p \) is the mean stress and \( S_{11} \) is a component of the deviatoric stress. (iii) Elastic-plastic constitutive relation:

\[
\frac{\partial \rho S_{11}}{\partial t} + \frac{\partial \rho v_1 S_{11}}{\partial x} = \frac{4}{3} \mu \rho (1 - \beta/(1 + \frac{B_{SH}}{3\mu})) \frac{\partial u}{\partial x}, \quad (4.4)
\]

where

\[
\beta = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } d(\frac{1}{2}S_{ij}S_{ij} - k^2) \leq 0 \\
1 & \text{if } F = 0 \text{ and } d(\frac{1}{2}S_{ij}S_{ij} - k^2) > 0
\end{cases}
\]

(4.5)

where \( J_2 \) yielding criterion has been used for this one dimensional stress case and \( B_{SH} \) is the modulus for the elastic-linear hardening plastic material. To close Eqs. (4.2), (4.3) and (4.4), we use the following equation of state to relate pressure to density:

\[
p = k \ln \frac{\rho}{\rho_o}, 
\]

(4.6)
where $s = S_{mn}S_{mn}$ and

$$
\beta(s) = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } d\left(\frac{1}{2}S_{ij}S_{ij} - k^2\right) \leq 0 \\
\frac{6\mu^2}{(3\mu + BS_H)s} & \text{if } F = 0 \text{ and } d\left(\frac{1}{2}S_{ij}S_{ij} - k^2\right) > 0 
\end{cases}
$$  \hspace{1cm} (4.7)

where $p$ is the pressure, $k$ is the bulk modulus, $\rho$ is the density and $\rho_o$ is the initial density. The governing equations (4.2), (4.3) and (4.4) in the conservative form can be written in the following vector form:

$$
\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = S
$$  \hspace{1cm} (4.8)

where

$$
Q = (\rho, \rho u, \rho S_{11})^T,
F = (\rho u, \rho uu + p - S_{11}, \rho S_{11}u)^T,
S = (0, 0, \frac{4}{3}\mu \rho (1 - \beta/(1 + BS_H/3\mu)))^T
$$

We need to calculate the wave speed of the stress waves inside the material to adjust the simulation parameters for appropriate spatial and temporal resolution. The wave speed can be found by analysing the mathematical structure of the above system of equations. The eigen values for the above system is by eq.(4.9):

$$
\lambda_1 = u, \lambda_{2,3} = u \pm c = u \pm \sqrt{\frac{K + \frac{4}{3}\mu (1 - \frac{\beta}{1 + BS_H/3\mu})}{\rho}},
$$  \hspace{1cm} (4.9)

The above system of hyperbolic equations have been implemented in the CESE solver to conduct numerical simulations for explosive welding which will be covered in the next section.
4.4 Numerical Simulation

The governing equations in the previous section are adapted to a FORTRAN program based on the Conservation Element Solution Element (CESE) method. This CESE based solver can be used to solve a system of hyperbolic system of equations. The computer program in question is a one dimensional solver. In such a case, the mesh is a line divided into numerous segments. Since the CESE method is a set of program codes for solving conservation laws, it is very much suited to Computational Fluid Dynamics. The CESE method treats space and time in a unified manner. In configuring the program, we need to adjust the mesh size $\Delta x$ and the time step $\Delta t$ in accordance with the wavelength and time period of the wave so as to effectively capture them.

As discussed earlier, explosive welding depends on a number of parameters, namely: (1) Base plate and Flyer plate thickness, (2) Impact velocity of the flyer plate, (3) Material properties of both plates, (4) Standoff distance and (5) Dynamic angle $\beta$. Akbari-Mousavi and Al-Hassani[10] have conducted experiments on explosive welding using different metal combinations and parameters. Also, they have conducted 2D numerical simulation using a 2D commercial general purpose finite difference software called AUTODYN. The results obtained were in good agreement with the experimental results. They made use of the Williamsburg equation of state[10] and the Johnson-Cook constitutive equation [8] to model the explosive welding process.

The objective of this simulation is to compare it to results obtained by Akbari-Mousavi and Al-Hassani [[10]]. This simulation will give us an idea about the stress wave propagation through the two materials during the explosive welding process.
4.4.1 Results

The simulation will model the welding between two stainless steel plates. Both the base plate and the flyer plate are 6mm thick. The impact velocity of the flyer plate is $650 \text{ m/s}$ and two plates are parallel to each other. The initial pressures $p$ and deviatoric stress component $S_{11}$ are zero for both the plates. We will be using the elastic-plastic constitutive model developed earlier. The material is elastic-plastic with a strain hardening constant of 275 MPa. The remaining material properties are summarized in table (4.1). The parameters of the simulation are summarized in table (4.2).

<table>
<thead>
<tr>
<th>Table 4.1: Material properties of 1006 Stainless steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>Yield Stress(MPa)</td>
</tr>
<tr>
<td>Bulk modulus(GPa)</td>
</tr>
<tr>
<td>Shear modulus(GPa)</td>
</tr>
<tr>
<td>Density($Kg/m^3$)</td>
</tr>
<tr>
<td>Strain Hardening constant(MPa)</td>
</tr>
</tbody>
</table>

(4.2) The boundary conditions on the left end of the flyer plate and on the right end of the base plate are set as non reflecting boundaries. This allows all the waves to
pass through the plates without any reflection. The absence of reflections will make the interaction between the two plates much more clearer.

The domain is discretized uniformly into 5000 cells and the time step for the simulation is set to 2.5 nanoseconds. Based on the wave speed, the spatial domain and the time step, the CFL number is regulated to be 0.608. The total duration of the simulation is 7.5 $\mu$s. Figure (4.3) shows the schematic for explosive welding for the simulation.
Fig. 4.4: Pressure distribution of two parallel stainless steel plates - 1D explosive welding simulation
Fig. 4.5: Simulation of two 6-mm-thick stainless steel plates impacting at a velocity of 650 m/s and an inclination of 15 degrees pressure distribution Akbari Mousavi and Al-Hassani[10]
Fig. 4.6: Velocity distribution of two parallel stainless steel plates - 1D explosive welding simulation
Fig. 4.7: Simulation of two 6-mm-thick stainless steel plates impacting at a velocity of 650 m/s and an inclination of 15 degrees. Velocity distribution. Akbari Mousavi and Al-Hassani[10]
Fig. 4.8: Normal Stress distribution of two parallel stainless steel plates - 1D explosive welding simulation
Fig. 4.9: Simulation of two 6-mm-thick stainless steel plates impacting at a velocity of 650 m/s and an inclination of 15 degrees. Normal stress distribution. Akbari Mousavi and Al-Hassani [10]
The distributions of contact pressures, stresses and contact velocities are plotted for the given simulation parameters. Also, plotted are the distributions by Akbari Mousavi and Al-Hassani[10] for 2D simulations which are in good agreement with experiments conducted by them. Figures (4.4)-(4.9) refer to the aforementioned plots. Figure (4.10) depicts the density distribution of the two stainless steel plates after impact.
4.5 Discussion

4.5.1 Pressure distribution

Figures (4.4) and (4.5) indicate that the pressure is maximum at the collision point or point of impact. The distribution obtained using the 1D simulation is in good agreement with the 2D simulation which also takes into consideration effects of heat. The pressure drops steeply as the stress wave diffuses into the base plate.

The distribution is very much symmetrical in the two plates which implies the absence of jetting. Jetting is evident in the 2D simulation as the point of high pressure appears closer to the base plate than the flyer plate. Jetting causes material to be expelled from in between the plate due to the high velocity impact. This reduces the pressure on the flyer plate in this case.

This simulation confirms the fact that the maximum pressure is present at the point of impact.

4.5.2 Velocity distribution

The velocity like pressure will be maximum at the collision point. Figure (4.7) confirms this fact and is in line with experiments Akbari Mousavi and Al-Hassani[10]. In the 1D simulation result, the velocity initially is high around $650\text{m/s}$ which is the impact velocity and then reduces to approximately to $325\text{m/s}$.

The velocity at the collision point for 2D simulation is extremely high in comparison to the 1D simulation. This can be due to the fact that the 1D model is inadequate in depicting the effects of the detonation shock. This translates to a higher velocity.
4.5.3 Stress distribution

The normal stress distribution is negative due to downward vertical velocity of both plates producing compressive stresses. The high stresses at the impact region causes liquid like flow of material due to the it yielding. This points to the mechanism by which the solid state weld could be formed.

4.5.4 Assumptions and other parameters

We have made a comparison between simulations with one key difference apart from the fact that one of them is 2D. The 2D simulation was carried out with a dynamic angle of $\beta = 15\text{degrees}$. We have assumed, for a 1D simulation, since plate surface will experience multiple infinitesimal points of impact which can be approximated as having head on impact, we can neglect the effect of dynamic angle.

We have also neglected the effect of stand-off distance as the flyer plate can accelerate to a terminal velocity greater than the impact velocity which we have used from the experimental investigation done in Akbari Mousavi and Al-Hassani[10].

4.6 Conclusions

The numerical simulations successfully depicted the conditions at the point of impact between the two plates. The current chapter summarizes the findings using the one dimensional developed in chapter 3. The success of this model in simulating explosive welding encourages us to use the 2D formulation in chapter 3 for more accurate results. Appropriate mesh and time steps were used to conduct the simulations so as to obtain proper results. The results depict the formation and propagation of waves in solids.

The constitutive and governing equations were developed in previous chapters and were implemented using CESE method based solver called Solvcon. This setup
was then used to tackle the one dimensional explosive welding problem. This affirms the fact that this model could be used for other applications with stress waves in elastic-plastic solids.

Use of an isothermal model such as this for a high speed impact problem could be useful in some cases but not all. Since the process is really fast, the scope for heat transfer is low thus, allowing us to neglect the use of energy equations in our formulation.
CHAPTER 5
CONCLUSIONS AND FUTURE WORKS

The present research is an extension of previous works on non linear stress waves in solids. Conservation laws and elastic-plastic constitutive relations have been developed and coupled together to form a theoretical model. Initially, the one-dimensional model equations were formulated and the two dimensional model equations were built upon them. The equations pertain to the propagation of stress waves in elastic-plastic solids. Explosive welding was taken as a case study to apply the theoretical model. The one dimensional equations have been expressed in conservative and non-conservative form. Also, the eigen value analysis has been carried out for the one-dimensional set of equations.

The system of equations consist of the continuity, momentum and constitutive equations. To solve the given hyperbolic system of equations, we use the Conservation Element Solution Element (CESE) method. The equations were adapted and plugged into the CESE solver to solve the explosive welding problem in one dimension.

Chapters 3 discusses the model equations which is the main feature of this research. In addition to these features, the major points to note include:

- Wave propagation in metals during the explosive welding process was simulated and analyzed. The 1D consitutive equations developed in chapter 3 were utilized in successfully simulating the process.
• A basic eigen value analysis of the one dimensional hyperbolic system of equations was provided. The wave speed can be discerned from the eigen values.

• Constitutive equations were derived for both one- and two-dimensional cases. Moreover, Jaumann rate was used to make sure the constitutive equations were invariant under any arbitrary superposed rigid body motion.

• We have adapted the CESE method to solve a solid mechanics problem namely nonlinear stress waves in solids.

There have been many unsuccessful attempts to use Finite Element methods to simulate the propagation of stress waves in solids. The use of conservative laws coupled with constitutive equations to form a hyperbolic system of equations represents a new method of analyzing the problem. The identification of the variables involved in simulating the explosive welding problem paves the way for manufacturers to get a better understanding of the process and hence make process improvements in the future. A wider use of this numerical method in analyzing other manufacturing processes will open new avenues for research and development in manufacturing.

There are a number of improvements and suggestions which can be incorporated into the current research to make it more effective and useful to the industry. Future works include:

• The explosive welding process could be more accurately simulated using the 2D formulation coupled with the CESE solver. This would allow to see the intermixing of the material. The 2D simulation would enable us to simulate the effects of the detonation shock which would add to the accuracy of the result.

• More simulations could be conducted with dissimilar metals and with different thickness for the flyer and base plate.
• We can add the effect of friction and also simulate welding with different dynamic angles which has been neglected in the present case.


