SYSTEM IDENTIFICATION AROUND PERIODIC ORBITS WITH APPLICATION TO STEADY STATE HUMAN WALKING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

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2013

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ABSTRACT

To walk without falling down, human walking motions, by definition, need to be stable and humans need the ability to recover from perturbations. This thesis is about methods for analyzing human walking dynamics, as well as the application of these methods to human walking data. During steady human walking, each step is similar to every other step, but the steps are not all identical. If the deviations from periodic motion are due to random perturbations, the near-periodic steady state behavior has information about the dynamics of the underlying dynamical system near the nominal periodic orbit.

In Chapter 2, we review and develop statistical techniques to construct models of the dynamics (that is, perform ‘system identification’) near the periodic motion from noise-driven near-periodic data. Our principal construction is a sequence of Poincare sections, transverse to the periodic orbit, in the neighborhood of the periodic orbit and linearized dynamics of the state from one Poincare section to the next, resulting in a piecewise linear dynamical system around the periodic orbit. We apply these methods to synthetic data derived from the simulation of known discrete and continuous dynamical systems, both univariate and multivariate, with periodic orbits perturbed by various kinds of noise, to demonstrate effectiveness of these methods. We obtain asymptotic relations on how the accuracy of the inference procedure in simple situations depends on how many Poincare sections we use. This method of
describing and inferring dynamics around periodic orbits is applied and verified by using data from a noise-driven van der Pol oscillator.

In Chapter 3, the collection of human walking data is described. In the experiments, five subjects were asked to walk on a treadmill at specific speeds, both in steady state and as well as performing some time-varying movements. Marker-based motion capture data (movement of body segments) was collected for all the walking trials.

In Chapter 4, we analyze the collected human walking data by investigating the “top view dynamics.” We use a step-to-step model, which considers the foot placement and the upper body motion to represent walking, and compute this model from data using least squares method. We compute the effects of the upper body movements on the foot placements. In particular, we show how a higher-than-normal sideways speed or position of the upper body is corrected by stepping to the side more (using a bigger step width) and using a smaller step length. A higher-than-normal forward speed of the upper body is corrected by taking a slightly longer step. While these features of human walking have previously been conjectured and used in models of walking stability, we do not know of a rigorous data-based derivation of such features. We use the step-to-step model to examine the stability (return map eigenvalues) of human walking. We extend the model to predict motions of blindfolded people over long-durations – people walking with no visual feedback provided – showing how they may eventually deviate from a straight line.

In Chapter 5, we analyze the walking data by looking at dynamics in the sagittal plane (“side-view dynamics”). Here, we use a 2D kinematic model of the body to first infer from data and then simulate the dynamics steady walking and perturbed walking. The comparison between the simulation results and experiment data is given; model-based prediction performs better at predicting transients than the null
hypothesis of natural variability. With the confidence of this agreement, we predict
human walking under different perturbations. We again show how humans step longer
or shorter when, for example, a push forward or backward is applied on the swing or
stance leg during steady walking.

In future work, building on the inference of 2D dynamics of human walking here,
we hope to improve the statistical methods, extend the dynamical models to 3D, and
also infer controllers that map body state to muscle action.
ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my research advisor, Professor Manoj Srinivasan. It is due to his patience, enthusiasm, philosophy, vision, and direction, that this thesis sees the light of the day. It is my good fortune that I have been able to work with him. Many thanks to my committees Professor Carlos Castro, Professor Ahmet Kahraman and Professor Krishnaswamy Srinivasan, for agreeing to serve on my thesis committee despite their demanding schedules. Also, thanks to Dr.Alsion Sheets for her valuable discussions. Their comments and suggestions improved the quality of the thesis.

Thanks to the volunteers, for their patience in participating in the experiments.

I offer my deepest gratitude to my parents and my brother for their lifelong encouragement, support and love. Finally, I would like to express my love and appreciation to my wife, Tingting Mao, and my daughter, Alice Wang, for their support, patience, and endurance during the whole educational process.
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PUBLICATIONS


**FIELDS OF STUDY**

Major Field: Mechanical Engineering

Specialization: Biomechanics, Nonlinear dynamics, Vibration, System Identification
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<td>Steady human running is only approximately periodic. The angle change and angular rate of one body segment (shank) is shown and we see that each cycle is similar to but not exactly the same as every other cycle, resulting a band of trajectories. It is thought that at least some of this variability is due to process noise.</td>
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<td>1.2</td>
<td><strong>Periodic motions and Poincare sections.</strong> a) A periodic orbit (thick red line) is shown with one Poincare section $S$, intersecting $S$ at the point $x^<em>$ shown. The points $x^{(i)}$ and $x^{(i+1)}$ are successive intersections of a non-periodic trajectory with the section $S$. The Poincare section is shown as being planar, but could be a curved manifold as long as it is transversal (not tangential) to the periodic orbit. b) A periodic orbit shown with a sequence of $M$ Poincare sections $S_j$, intersecting them at $x_j^</em>$ respectively. The $i^{th}$ intersection of a non-periodic transient trajectory with the $j^{th}$ section $S_j$ is denoted $x_j^{(i)}$. Also in [97, 99, 98].</td>
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<td>2.1</td>
<td>Idealized schematic of discrete-time dynamical systems with one or two or multiple stages (“sections”). Each stage may be thought of as the transition from one section to the next around a periodic orbit. When there is only one stage, it is analogous to having only one Poincare section.</td>
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2.2 Identifying a noise-driven linear discrete dynamical system.
a) Time series obtained by simulating $x^{(i+1)} = Jx^{(i)} + \epsilon^{(i)}$, with $J = 0.5$, initial state $x_1 = 1$, and two different noise standard deviations: $\sigma = 1$ (thick black line) and $\sigma = 1/5$ (thin red line). b) The time series from panel (a) corresponding to $\sigma = 1$ is re-plotted versus the previous iterate; that is, $x^{(i+1)}$ versus $x^{(i)}$. The best fit line in the sense of least squared residual is shown: $x^{(i+1)} = 0.52x^{(i)} - 0.096$. Thus the least squares estimate of $J$ from this synthetic data is the line’s slope 0.52, compared to the true value 0.5. c) The (error) variance in the estimates of $J$ decrease with increasing length of time series data $N$. At each data length $N$, fifty different time series were used to obtain fifty different estimates of $J$ (plotted as gray circles). Also plotted is the curve (blue dashed line) corresponding to one standard deviation of the theoretically estimated variability in the estimate of $J$, roughly $1/N$ assuming large $N$ as given in Equation 2.2.2.

2.3 Discrete linear system with multiple sections. a) We generated 100 realizations of synthetic time series data of length $N = 100$ using a discrete linear dynamical system with $2^6 = 64$ sections, noise $\sigma = 1$, and $J_{1\rightarrow2,\ldots,64\rightarrow1}$ in the range 0.7 to 1.25 (Equation B.3.1). The net eigenvalue $J$ was estimated using the data at $M = 1$ to 128 sections for each of the 100 time series realizations (light gray circles). We see that the variance in estimates of $J$ reduces with number of sections used $M$ until $M = 64$. For $M = 2^7 = 128$, the data were obtained using interpolation between the 64 sections and therefore the $J$ estimates had a correlation-induced bias, as discussed in section 2.3.4. b) Improvement in error variance with increasing number of sections $M$ (Equations 2.2.4 and B.3.3). Notice that the improvement in error is not that great after about 10 sections. c) The asymptotic improvement in error variance as a function of $J$ (Equation 2.2.4). It appears that this asymptotic improvement approximately scales with $J$.

2.4 Noise-driven van der Pol oscillator. a) Shows 25 cycles of a simulated noise-driven van der Pol oscillator, Equation 2.3.1 with $k = 1$, $\mu = 0.1$, and $\sigma_f = 0.3$. Intersection of the noisy trajectory with six different Poincare sections $S_j$ are shown. b) The distribution of the states $x^{(i)}_1$ on the horizontal Poincare section $S_1$ from panel (a) obtained from $N = 10000$ cycles of the noisy trajectory. It seems bell-shaped and can be reasonably approximated by a Gaussian, although there may be a slight skew. c) The standard deviation of the distribution of states on each Poincare section $S_j$ for 200 Poincare sections around the nominal periodic orbit.
2.5 **Using a time-based stroboscope instead of state-based Poincare sections.** Even though the system is close to periodic, considering the system at constant time intervals apart (here, equal to the period of the unperturbed van der Pol) results in states that are not localized in the state-space (bold red points). Instead, the points drift around the average periodic orbit. This is because the system is perturbed by noise constantly and is neutrally stable to perturbations along the periodic orbit.

2.6 **Analyzing the van der Pol oscillator using data at a single section.**

a) The eigenvalue of the Poincare map on section $S_1$ is obtained by finding the best linear map between consecutive iterates $x_1^{(i)}$ and $x_1^{(i+1)}$. b) Even using data at only one section, we have the choice of which Poincare section $S_j$ to use. Here, we consider $M = 100$ Poincare sections $S_j$ that are all straight lines through the origin as in Figure 2.4a, intersecting the trajectories at different points along the nominal cycle. The eigenvalue estimate is plotted as a function of the index of the Poincare section used. We used only $N = 100$ cycles, hence the large deviation from the correct value. c) A number of Poincare sections passing through *same* point on the nominal periodic orbit, making different angles with the trajectory, are considered. When the Poincare sections make small angles with the average trajectory, the noisy transient trajectory may sometimes entirely miss intersecting them, resulting in fewer data points to analyze. d) Estimating the eigenvalue of the Poincare map using the different sections from panel (c) give roughly the same value, except at the extreme of Poincare section angles, when the number of data points is less by about 11%.
2.7 Estimating eigenvalues using multiple sections. a) (Mean) eigenvalue estimates as a function of number of sections used for a single data set, from three different methods: ordinary least squares, many section mapping with bias subtraction and iteratively weighted least square estimate. The correct value is about 0.53. b) The ordinary least squares method produces unbiased estimates for low number of sections, but there is a systematic positive bias for larger number of sections. This bias is due to correlations between data from neighboring sections. The variance of the estimates decreases with increasing number of sections. c) The first order bias in the ordinary least squares method is estimated from the data approximately and subtracted giving an approximately unbiased estimate despite correlations between neighboring sections. d) An iteratively re-weighted least squares method is an approximation to the Maximum Likelihood Estimator and gives an unbiased estimate. Both the bias subtraction and the iterative least squares method have slightly higher variance asymptotically than the ordinary least squares method. This figure used \( N = 100 \) cycles, hence the high variance in the estimates.

2.8 Predicting dynamical responses using a patchwork of piecewise models. A simulation of the noise-free dynamical system from (4,0) is shown in black dotted line and a prediction based on a linear factorized Poincare map is shown in green line. Another prediction from a quadratic factorized Poincare map model is shown in red line. We used \( M = 34 \) sections for the factorized Poincare map, and \( N = 100 \) cycles to construct the data-based model.

2.9 Isochrons in the plane. a) The time difference between successive intersections of a Poincare section is roughly linearly related to the state difference from that on the nominal periodic orbit (Equation 2.3.10). This linear relation depends on the orientation of the Poincare sections. The local isochrons are those Poincare sections that states starting from one reach the next simultaneously, so that the linear relation between time difference and state difference is identically zero. b) The correct local isochron, obtained from the original ODE, and the isochron estimated from the noise-driven data are shown for one data set with \( N = 100 \) cycles. c) The distribution of data-estimated isochron angles are shown, as obtained from 100 different synthetic data sets of the van der Pol oscillator.
2.10 **Spline estimates of slowly varying parameters in a univariate discrete system.** We consider Eq. 2.4.1 with continuous functions $J^{(i)} = 0.5 + 0.3 \sin(9b \, i)$ and $x^*(i) = 0.1 \cos(4b \, i)$, where $b = 10^{-4}$. Both these functions are shown (green lines) along with estimates of these functions by (1) assuming them to be piecewise linear splines with about 10 pieces (black line) (2) assuming them to be a single constant (red). We used about $N = 10^4$ synthetic data points for the inference.

2.11 **Van der Pol oscillator with slowly drifting spring constant.**

a) Non-periodic trajectories of oscillator with $k = 0.8 - 0.6(1 - i/100)$, showing a slow deformation of an almost periodic orbit. b) ‘Assuming periodicity,’ estimate of the Poincare map Jacobian using data at each Poincare section individually. For different sections, the estimates are dramatically different. c) Estimate of the Poincare map Jacobian using many Poincare sections. A spurious abrupt drop in the estimated $J$ occurs with the number of Poincare sections $M$ is increased.

2.12 **Van der Pol oscillator with slowly drifting spring constant.**

We used splines to estimate the parameter changes as a function of time. In particular, the linear map from a section back to itself is assumed to change slowly as in Eq. 2.4.1. Four different curves are shown in each panel. The ‘actual parameter values’ (green line) for $J$ and periodic point $x^*$ as functions of the cycle number $i$, are obtained from the exact periodic orbits for the appropriate $k$ during that cycle. The estimates assuming $J$ and periodic point $x^*$ were constant gives very incorrect results. The moving average estimates, assuming $J$ and $x^*$ were constant over a moving window of 40 cycles gives reasonable estimates for $x^*$ but not for $J$. The piecewise linear spline estimate which takes the drift into account seems to catch the trends reasonably.

2.13 **An algorithm for obtaining a sequence of about $M$ Poincare sections, which are nearly normal to the trajectories.**

2.14 **Transient predictions of five coupled van der Pol oscillator.**

It shows a comparison between predicted response from using factorized Poincare maps and true response obtained from an accurate ODE simulation, for two different initial conditions that are at different distances from the periodic orbit. Notice that there are substantial transients and that the factorized Poincare map representation captures a lot of these transients. See also [99].
3.1 **Experiment of human walking.** a) There are 8 high speed cameras for the marker based motion capture system focused on a treadmill in the middle of room; b) The reflective markers are placed on different segment of the subject. Cameras capture the time-history of 3D marker positions.

3.2 **Reference frame of static posture.** FA means fore-aft direction, SS means side-to-side direction, and V means vertical direction. Green dots are the reflective markers, blue dots represent the knee and ankle position by averaging the lateral and medial markers. Red dots represent hip joint center from function joint-axis experiment.

3.3 **Finding the hip joint centers.** FA means fore-aft direction of human body. SS means side-to-side direction of human body. V is the vertical direction. a) The leg is doing “star” diagram motion, with 0, 45, 90, 135, 180 and circular motion. b) Calculate the hip joint center in the trunk coordinate.

3.4 **Raw marker positions captured by motion capture system.** A total of 25 markers were placed on lower body of the subject. Each segment has three non-collinear markers. There are seven segments (e.g., left foot, left shank), which use 21 markers. The other four markers are placed on the joint positions (i.e., knees and ankles). Red markers are placed on right leg, blue markers are on left leg and dark green makers are placed on the trunk.

4.1 **Picture indicating the mapping variables.** The step-to-step model depicts human walking by considering foot positions and upper body movements. The variables are: left foot placements, \(x_{lf}^{(i)}, y_{lf}^{(i)}\), right foot placements, \(x_{rf}^{(i)}, y_{rf}^{(i)}\), body yaw angle (in the horizontal plane) \(\theta\), body positions \(x_b, y_b\), body velocities \(\dot{x}_b, \dot{y}_b\). The top view of the step-to-step model is given in a, and the side view of the model is depicted in b.

4.2 **Walking with foot positions** are shown along with the subject orientation, when the subject is walking towards the right. The coordinate \(x, y\) with black is the ground-fixed coordinate. The coordinate \(u, v\) is the local coordinate of the trunk orientation.

4.3 **Experiment data of relative foot positions** a) the relative right foot positions to the left foot position, b) the relative left foot positions to the right foot position.
4.4 **Coefficient of correlations.** Correlation coefficients computed with input variables being the body yaw angle, displacement, velocities and output variables being the foot position. For panels (a,c,e,g), the input variables are defined during the left foot stance. For panels (b,d,f,h), the input variables are defined during the right foot stance. The horizontal axis is ‘trial number’ – with a total 15 trials, 3 trials per subject. The two correlation coefficients shown for each trial correspond, respectively, to the ‘SS’ is the side-to-side directional displacement \(x\) of the foot position (black dots) and ‘FA’ is the fore-aft directional displacement \(y\) of the foot position (green dots).

4.5 **Distribution of the estimated relation between the body yaw angles and the next step’s foot placements.** The same notations are used: FA - fore-aft; SS - side-to-side. Body yaw angle represents the upper body angle in the top view dynamics. The unit for body yaw angle is degree, and the unit for foot placement is millimeter.

4.6 **Distribution of the estimated relation between the body side-to-side position and the next step’s foot placements.** The units for both body position and foot placement are millimeter. It shows that the side-to-side body position and the side-to-side foot placement are strongly positively related.

4.7 **Distribution of the estimated relation between the body side-to-side velocity and the next step’s foot placements.** The unit for body side-to-side velocity is millimeter per second. It shows that the side-to-side body velocity and the side-to-side foot placement are weakly positively related, and the side-to-side body velocity and the fore-aft foot placement are weakly negatively related.

4.8 **Distribution of the estimated relation between the body fore-aft velocity and the next step’s foot placements.** The unit for body fore-aft velocity is millimeter per second. The unit for the foot placements is millimeter. The figure indicates that the fore-aft body velocity and the fore-aft foot placement are weakly positively related.

4.9 **Distribution of the estimated relation between the body movements and the next step’s foot placements for 5 subjects.** For each subject at each speed, there are 1000 bootstraps samples. These plots include the data for 5 subjects. Each subject has three trials for three speeds. The ratios between mean and standard deviation are calculated based on the distribution and given in Table 4.1.
4.10 Prediction of foot placement  
a) Thirty independent trials of the 100 step blind-folded walking are given. The prediction shows that people can not keep straight walking without any references. 
b) Two predictions of 10000 steps blind-folded walking are given.

5.1 A planar kinematic model is shown to represent mechanical structure of human walking.  
a) Tibia and thigh lengths are fixed according to static measurements of the subject. Four degrees of freedom, $\theta_1, \theta_2, \theta_3, \theta_4$, are used to describe the walking; 
b) The phase portrait projections of tibia angles $\theta_1, \theta_4$ are given to show the nearly periodic motion;  
c) The phase portrait projections of thigh angles $\theta_2, \theta_3$. The units for all angles are radians. The units for all angular velocity are rad/s.

5.2 Poincare sections and the nominal periodic motion. 
The first Poincare section comes from the left foot striking on the ground during every stride. All Poincare sections (point clusters in yellow) are roughly perpendicular to the trajectory of walking data. Nominal periodic motion is the dashed line inside the nearly periodic band. It is found by averaging the intersection data on all Poincare sections. The unit for all angles is radians. The unit for all angular velocity is rad/s.

5.3 Projections of the phase portrait to show the comparison between the prediction and perturbed walking data.  
I.C. means the initial condition. The periodic bands represent the steady walking data. Cyan lines show the experimental (self-imposed) large step walking transient, and then back to steady walking. In this transient, the phase portrait projections for thigh angle of the large step walking are significantly away from periodic band (steady walking) in panels c and d. We can see that after the large step, the subject mostly recover in one step. The yellow point is picked as the initial condition for prediction. The predictions, from the developed factorized Poincare map are given in black lines. The unit for all angles is radians. The unit for all angular velocity is rad/s.

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CHAPTER 1
INTRODUCTION

1.1 Introduction to stability of human walking

When people want to go from one place to another, and are not in a hurry, they walk. Even though people are able to walk with very different movements (say, with exaggerated leg movements, or even walking sideways or backwards), usually, they walk with the natural movement we see every day. This walking motion is approximately the same for everybody. In this natural walking motion, as one leg (called the stance leg) is on the ground, the other leg (called the swing leg) swings forward. Then, the swing leg strikes the ground, becomes the stance leg, and the stance leg becomes the swing leg. This motion looks simple, but it is actually a complicated process involving musculo-skeletal dynamics and nervous system controls [40].

Usually, people walk stably. Here, we define “stable” approximately as walking in the same pattern repeatedly, without falling down. People can quickly resume normal walking even under certain perturbations, for example, a small push on the body or a trip of the foot. Most healthy adults have stable walking, but are we all equally stable? We know that elder people are more likely to fall down than young adults, while they are walking [65, 77]. So we may not all be walking equally stably. How stable is our walking then? Can we quantify our stability?
Such stability information can be important to understand our steady walking and walking under perturbations. For example, can we quantify how large a perturbation we can absorb without falling down or the probability that a given perturbation can make us falling down? How fast do we recover from perturbations during walking? In this thesis, we will address some of the above questions by using both theoretical and experimental methods. We first develop some numerical methods to analyze system with nearly periodic motions. Then, we verify that these methods work well by synthetic data from oscillators with known properties. We then collect human walking data with motion capture system. We apply the developed numerical method to human data, to extract information about the system’s dynamics to show stability and other dynamics of human walking.

1.2 Literature review of stability of human locomotion

Many researchers have investigated the stability of human locomotion and from various different perspectives. The research of human locomotion can be classified into two categories: theory and experiment. Theory and experiment in the study of human locomotion have developed closely through mutual cross-fertilization.

**Theoretical work** are mainly focused on mathematical models or simple mechanical models. These models can be passive models or models under active control. Passive walking means to “walk” stably without any controls. For instance, a passive biped can ‘walk’ down a shallow slope, and this walking motion is quite similar to human walking. Different simple passive models can be found in [63]. A three-dimensional passive dynamic walking robot has been built as in [18].

The passive dynamic walking models provide insights to the stability and other dynamics of human locomotion. For example, a two-link biped model has been discussed in [24]. The rigid body biped model could walk down a shallow slope, powered
only by gravity. The stability of this passive walking model depended on the slope, which indicated the effect of gait speed on human walking. A two-link model with upper body also achieved a stable walking in [102]. By adding the upper body, the model had a better energy efficiency and was slightly more robust against perturbations. A 3-D passive walker was used by [53] to show why people may be more likely to fall down laterally than the fore-aft direction, and then used to demonstrate the stabilization of lateral motion.

However, passive dynamic models and robots are only marginally stable. Robustly stable walking requires feedback control. In humans, the feedback is based on sensory information from three modalities (visual, vestibular, and kinesthetic inputs), knowledge, and prior experience. They all play critical roles in the control of dynamic stability [71]. In locomotion models, it has been shown that stable locomotion can be easily achieved by applying the feedback control, such as sensory reflex control [46], linear PD control [6] and sliding mode control [73].

When applied active controls, many other models were proposed to represent and simulate human locomotion. Some simple models can be found in [2]. Gubina et. al. [30] used feedback control on biped model and obtained stable walking cycle. The biped model could stand large disturbances with the feedback control. A more complex model included muscle and neural control, for example, the model of the neuro-musculo-skeletal system. Hemami and Dinneen [41] developed a control strategy similar to marionettes to move stably. The actuators are analogous to muscle forces in natural systems. The model of neuro-musculo-skeletal system in [89] applied active controls on 20 muscles and a neural rhythm generator. This paper showed that the stable limit cycle of locomotion could be generated by the global entrainment between the musculo-skeletal system, the neural system and environment. Moreover, Ogihara and Yamazuki [69] applied feedback control neuro-musculo-skeletal model
and optimized the neural network to maximize the walking distance with minimum energy consumption.

The research of active controls for human locomotion was extended to obtain energy-optimum controls. Researchers used these models to find the optimized control strategies for different locomotion, with respect to energy cost [17, 57, 5, 83]. Inverted pendulum model, in which the center of mass vaults over the rigid stance limb, has been considered in [54], which showed that it is beneficial for the stance leg to behave like a pendulum with respect to energy cost. Srinivasan in [85] also showed walking as inverted pendulum has the minimal energy cost among all other possible ways of walking. The realization of inverted pendulum walking involves active control from muscles. The optimized control strategies typically provide only the feed forward forces but do not focus on the walking stability. Nevertheless, they are very helpfully in robot design and controls. When applying active controls, the walking process could be more stable than real human walking, depending on the control strategy.

Experiments of human locomotion showed the natural human behaviors and real reactions under different circumstances. Different parameters have been investigated to show their effects on the stability of human locomotion. Lateral stability (that is, not falling sideways) during forward stepping has been experimentally investigated in [76]. This paper showed that the older individuals could fall down due to lateral instability and they adjusted foot placement to avoid the lateral fall down. Moreover, Townsend in [91] showed that the adjustment of the discrete foot placement can stabilize human locomotion based on the feedback information. The gait speed also affects the gait stability. England and Granata in [22] claimed that slower walking velocities lead to increases in stability.

Steady state experiments. The steady locomotion experiment is to record the human locomotion under constant constraints (e.g., speed). When people walk or
run, every step is different to every other step. Tri-axial accelerometers and elec-
trogoniometers were used to record human walking and the step to step differences
can be observed in [47, 19]. Stride-to-stride fluctuations of human walking could also
be seen in steady locomotion as in [38]. Experiments of cockroach locomotion can
be found in [75], which had similar nearly-periodic motions to human locomotion.
This paper used methods of first return map to analyze the nearly periodic motion.
However, the results had variations. More discussions of such research can be found
in section 1.3.

**Perturbation experiments.** In perturbation experiments, researchers applied
perturbations during steady locomotion and observed the transient human movement.
Lateral perturbations, such as a push and pull on the body during walking on the
treadmill were studied in [43]. It showed that the balance can be recovered quickly in
one or two steps under lateral push with magnitude of 9 kg · ms$^{-1}$. Perturbations in
the sagittal plane (plane of walking) by platform motion can be found in [68, 103, 90].
Perturbations from the platform motion can be different kinds of perturbations. It
could be the perturbations on the stance leg, for example, longitudinally translating
the foot of the stance leg (28 cm), translating the foot of stance leg upward or down-
ward (25 cm) in [68]. It could be perturbations on the swing leg, for example, the
swing leg struck on a plate with a constant velocity of 40 cm/s to simulate a forward
slip could be found in [90]. Experiments of stumbling over obstacles could be found in
[78, 77]. Kinematics of the foot during slips and gait adaptations during load carry-
ing on level and inclined surfaces were experimentally investigated respectively in [16]
and [15]. These above perturbation experiments showed the human adaption under
different circumstances. All the above papers show that people can quickly adjust
themselves after perturbations in one step or two. The experiments revealed the de-
tails of body motion when perturbations applied, such as the center of mass of trunk
in [43], muscle reactions in [78]. However, while these perturbation experiments provide valuable information, these papers provided general quantitative descriptions of what happens in response to perturbations, but no mathematical models of stability were developed from the experiments.

### 1.3 Review of data based methods for stability of human locomotion

In this section, we review research that is closest to the subject of this thesis, namely stability analysis based on steady walking motions.

While there is no such thing as truly steady state in a real system, many engineering and biological systems have roughly periodic behavior in their nominal steady state. Examples include human walking at roughly constant walking speed, periodically spiking neurons, the motion of the pistons in an internal combustion engine at roughly constant vehicle speed, planetary motion, circadian rhythms, and other coupled oscillators. If we had differential equations describing the motion of these systems, we could understand the dynamics and stability of such periodic motions by analyzing the differential equations in the neighborhood of the nominal periodic orbit. Without knowledge of governing differential equations, we may use measured data from the dynamical system to construct models representing the dynamics around the periodic orbit. The method to treat such roughly periodic behaviors in this thesis is about such system identification [58].

One way to obtain information about dynamics around the periodic orbit is to perform the experimental equivalent of a finite difference approximation — that is, perform “perturbation experiments”, in which one observes the consequences of many discrete perturbations from the periodic orbit.
An alternative to discrete perturbation experiments presents itself in systems which have inherent noise that incessantly perturbs the system, so that the steady state behavior is not exactly periodic, but is only approximately periodic. (Such noise is often called “process noise”, as distinct from “measurement noise.”) In this case, the small natural perturbations explore a small region around the periodic orbit, and therefore, the data from such noise-driven systems have some information about the dynamics in the neighborhood of the periodic orbit.

Steady human locomotion exhibits almost-but-not-quite periodic behavior. In human walking, while every stride is similar to every other stride, the strides are not quite identical. See Figure 1.1 for a low-dimensional projection of trajectories from human running data, showing nearly periodic motion, resulting in a ‘band’ of trajectories. While it is not yet entirely clear what the dominant sources of this stride-to-stride variability in human walking and running are, but it likely is a mixture of noisy muscle forces which drive the limbs [81], sensory noise on which the neural feedback is based [67], and perhaps imperceptibly small external perturbations. There is also an alternative hypothesis that such aperiodicity could be deterministic [40, 39, 19]. In any case, in this thesis, we only focus on systems in which the approximately periodic motion is a consequence of noise perturbing an otherwise smooth nonlinear dynamical system with a periodic orbit. Further, we study only systems in which the noise is relatively small compared to the overall motion, so that the pattern of near-periodic response can still be observed.

The literature on system identification near periodic orbits appears to be limited, discussed briefly below, and some of this literature uses the notion of a Poincare map, as a partial representation of the dynamics near a periodic orbit [32]. (See also section 2.6 for other related work and discussion.) Recall that a Poincare section is any surface transverse to the periodic orbit, as depicted in Figure 1.2a, and that a
Figure 1.1: Steady human running is only approximately periodic. The angle change and angular rate of one body segment (shank) is shown and we see that each cycle is similar to but not exactly the same as every other cycle, resulting a band of trajectories. It is thought that at least some of this variability is due to process noise.

Figure 1.2: **Periodic motions and Poincare sections.** a) A periodic orbit (thick red line) is shown with one Poincare section $S$, intersecting $S$ at the point $x^*$ shown. The points $x^{(i)}$ and $x^{(i+1)}$ are successive intersections of a non-periodic trajectory with the section $S$. The Poincare section is shown as being planar, but could be a curved manifold as long as it is transversal (not tangential) to the periodic orbit. b) A periodic orbit shown with a sequence of $M$ Poincare sections $S_j$, intersecting them at $x_j^*$ respectively. The $i^{th}$ intersection of a non-periodic transient trajectory with the $j^{th}$ section $S_j$ is denoted $x_j^{(i)}$. Also in [97, 99, 98].
Poincare map is the function that maps an initial point $x$ on a Poincare section to the next intersection with the Poincare section of the trajectory starting from $x$ (Figure 1.2a).

Hurmuzlu and co-workers [47, 48] seem to have been the first to use this idea, in the context of human walking. They assumed that human walking variability was noise-driven and obtained a linear approximation to the Poincare map from walking data — by fitting a linear model between successive intersections of the human walking data with specific Poincare sections. They used natural events during walking like ‘heel strike’ and ‘toe off’ to define their Poincare sections (also, as in [e.g., 63, 84]) and linear least squares for inferring the coefficients in the linear approximation. From such inferred linearized Poincare maps, they could obtain the corresponding eigenvalues, which tell us about the local stability of the periodic orbit. Dingwell and co-authors have used variants of this technique in a series of articles since then [e.g., 20].

Revzen and Guckenheimer [75] recently revisited this idea, in which, instead of considering explicit transverse surfaces for Poincare section, they (equivalently) used data-derived phase coordinates that they proposed in an earlier article [74]; they used this method to examine the stability of cockroach locomotion. Along the way, Revzen and Guckenheimer compute the so-called Floquet coordinates corresponding to the dynamics near the periodic motion, providing a simple description of the continuous dynamics around the periodic orbit; they also provide estimates of the error in their inferred Jacobians and their eigenvalues by “bootstrapped” [21] versions of the data.

Incidentally, in both these works, such as [47, 75], the set of eigenvalues of the Jacobians derived from different Poincare sections or corresponding to different phase variable values are different. But it is easily shown that the Jacobians for different Poincare sections are related by similarity transformations and therefore must have
the same set of eigenvalues. The difference in the data-derived eigenvalues is, among
other factors, likely due to the effect of noise affecting the different sections differently.
This suggests a natural way to improve the estimates of the Jacobian, which is to use
data from multiple sections simultaneously to obtain eigenvalues that are naturally consistent.

In all of the afore-mentioned work using Poincare maps, the data at any given
Poincare section is used independently of the data at any other section. In the
thesis, we offer a complement to this prior work by devising system identification
techniques by using a large number of Poincare sections, as shown in Figure 1.2b and
simultaneously considering the data at all of these Poincare sections. In particular,
we represent the dynamics around the periodic orbit as a factorized Poincare map,
consisting of mapping from one transverse section to the next. Using this patchwork
of discrete mappings, we can simulate transient responses to perturbations away from
the periodic orbit. Further, we show how one can use the factorized Poincare map
representation to obtain other many quantities associated with periodic motions,
such as a linear time-periodic ODE approximation, isochrons, and phase response
curves. After the mathematical and numerical validation of such method, we apply
this method of factorized Poincare map to analyze human walking with experiment
data.

1.4 Thesis Objective

Our goal in this thesis is to develop a data-driven method to investigate the dynamics
of human walking. Two specific objectives are listed as bellow:

1. Our first objective is to develop methods to estimate dynamical models and sta-
bility information near noise-driven periodic motions. Even though there have
been some previous work [47, 20, 75], the methods have not been rigorously examined. We will introduce the idea of representing the continuous dynamics as mappings between multiple Poincare sections and then inferring these mappings. We examine in detail using synthetic data from both discrete and continuous systems, how well these methods work under various assumptions, providing some proofs for certain results.

2. Our second objective is to use the methods described above to construct a data-derived model not only to represent steady human locomotion, but also predict other various transient responses. We compare the simulations from this data-derived model to perturbation experiments. A successful model is one that can predict human locomotion in various novel circumstances, including responses to external perturbations. If we do succeed in deriving such a model, it would be a substantial contribution as there have been no complete and reliable data-based models of human locomotion dynamical system. This model, along with all the dynamic information around the periodic orbit will benefit walking and running robot designers. Such information may also be very important for designing better prosthetic devices.

1.5 Thesis Layout

In Chapter 2, we discuss the statistical inference techniques and show the validation by using both discrete and continuous examples. The formalism of factorized Poincare maps presented herein allows us to use classical statistical inference techniques developed in the statistics literature, in particular ARMA [34, 45, 50] and its specialization to periodic dynamics PARMA [95]. In section 2.2, we discuss the dynamics of discrete linear dynamical systems driven by additive noise, and describe how data from such
systems may be used to infer the parameters of the linear system. This section is partly a review of classical system identification and statistics literature and partly an adaptation of this literature to discrete iterated systems with multiple ‘stages’ (analogous to multiple Poincare sections). We also discuss how the error in the estimates depend on the noise in the dynamical system, the size of data, and the number of Poincare sections used. In section 2.3, we use synthetic data from a noise-driven van der Pol oscillator and large number of Poincare sections, to infer a piecewise linear representation of the dynamics around the periodic orbit. We demonstrate issues with the inference procedure, such as systematic bias, and show how these issues can be handled. Finally, we show how the synthetic-data-inferred piecewise linear system may be used to predict the transient response to large perturbations, that compare well with the simulation of the original noise-free differential equation. In section 2.4, we discuss how one would handle data from a dynamical system with a slow continuous parameter drift, so that there is no exact limit cycle but a slowly changing nearly periodic motion. We discuss what happens to the inferred quantities if the drift is not recognized in the inference. The van der Pol oscillator has a two dimensional state space and essentially one dimensional Poincare maps. In section 2.5, we outline how the methods we used for the low dimensional van der Pol oscillator can be extended to analyze multivariate data, and demonstrate the performance of such a procedure on synthetic multivariate data from noise-driven (multivariate) coupled van der Pol oscillators. Throughout this thesis, our statistical tool is least squares regression in its various forms, including its generalizations, weighted least squares and maximum likelihood estimation.

Appendix A provides a self-contained discussion of factorized Poincare maps in the context of deterministic dynamical systems, including a related literature review. This appendix is closely related to Chapter 2, but could be read independently of
Chapter 2. Appendix B supports Chapter 2, by documenting some standard results in time series analysis and statistical inference, and also adapting them to our current context.

In chapter 3, we discuss the collection of human walking data by using a marker-based motion capture system. We describe how we go from 3D position of the various markers on the body to position and orientation of the body. We discuss our protocols for both steady state walking and walking with self-imposed perturbations. We briefly discuss the measurement errors and errors in the estimated joint angles, and note that these are smaller than the kinematic variability between strides.

In Chapter 4, we analyze the collected human walking data by investigating the top view dynamics. The top view dynamics of human walking considers only foot placements and upper body/trunk motions. In this chapter, we first introduce a step-to-step model to demonstrate the top view dynamics. Poincare sections are defined according to the foot stances. Then, by analyzing the data on these Poincare sections, we show relationship between upper body/trunk motions and foot placements. In particular, we show how a higher-than-normal sideways speed or position of the upper body is corrected by stepping to the side more (using a bigger step width) and using a smaller step length. A higher-than-normal forward speed of the upper body is corrected by taking a slightly longer step. While these features of human walking have previously been conjectured and used in models of walking stability, we do not know of a rigorous data-based derivation of such features. We use the step-to-step model to examine the stability (return map eigenvalues) of human walking. We extend the model to predict motions of blind-folded people over long-durations; that is, for people walk with no visual feedback provided. The model predictions show that without visual feedback, they may eventually deviate away from a straight line.
In Chapter 5, we construct a 2D kinematic model to show human walking in the sagittal plane (forward walking plane). The 2-D kinematic model ignores the lateral (side-to-side) motion. We first apply the developed method of factorized Poincare sections to discretize the continuous walking data. A piecewise model is constructed based on the factorized Poincare sections from the steady walking data. We use self-imposed perturbed walking data to show a comparison between the results of the piecewise model and real perturbed walking. The simulated results matches the experimental data qualitatively, and in some respects, quantitatively. Finally, we predict human walking under different perturbations. Two kinds of perturbed walking are given as examples: i) push/pull on the swing legs, and ii) push/pull on the stance legs. We show how humans step longer or shorter when, for example, a push forward or backward is applied on the swing or stance leg during steady walking.

In Chapter 6, the conclusions of this thesis are presented. The future work is proposed as extensions of this research.

1.6 Significance of this research

This research fills a gap in developing a data-driven method to analyze human locomotion by using experimental data. The previous applications of such method by different researchers, such as [47, 20, 74] were largely using a single Poincare section. We improved the method from single Poincare section method to multiple Poincare sections. We examined the data-driven method mathematically and numerically. First, mathematically, we proved how this method works in the context of simple model systems. We did the parameter study to determine the optimal parameters to better perform this method, such as number of sections, number of periodic data, orientation of Poincare sections etc. Second, numerically, we used noise-driven van der Pol oscillator as an example to testify the method. The results
has a good agreement between mathematical results and numerical demonstrations. The mathematical proof and the numerical simulations not only show the validity of the method in deriving dynamical information from nearly periodic motion, but also indicates us that we could use the optimal parameters to find the best results despite of the inevitable noisy variation.

This research studied human walking, in terms of stability, periodic walking and walking under perturbations. By applying the data-driven method, we considered natural walking events, such as foot striking on the ground, as Poincare sections and made use of all of them together to generate a sequence of Poincare sections human walking. We obtain human stability information and construct a piecewise linear model from this sequence of comprehensive Poincare sections. We numerically simulated human steady walking by using this model. The simulated transients compared well with the measured self-imposed transients of the experiments. Finally, we predicted various disturbed human walking under different perturbations.
CHAPTER 2
SYSTEM IDENTIFICATION OF NOISE-DRIVEN SYSTEM

Human walking, like other nearly periodic motion, is not perfect periodic. In the appended chapter, we show how to use patch-work of factorized Poincare section to discretize continuous periodic dynamics. In this chapter, we will consider noise-driven dynamical systems to generate the similar nearly periodic response. We shows how to derive stability information from the nearly periodic response and construct a legitimate piecewise model by using the factorized Poincare section.

2.1 Noise-driven systems and Factorized Poincare maps

2.1.1 Noise-driven dynamical systems.

Our formal goal in this chapter is to analyze data from noise-driven smooth dynamical systems of the form:

\[
\frac{dX}{dt} = F(X, \mu, \epsilon)
\]  (2.1.1)

where \( t \) is time, \( X \in \mathbb{R}^q \) represents the state of the system, \( F \in \mathbb{C}^1 \) is a continuous differentiable function, \( \mu \) contains all system parameters (constants), and \( \epsilon \) contains all noise terms. For simplicity, we treat the presence of noise formally, without the measure theoretic and functional analytic machinery of stochastic differential equations.
We further assume the following properties about the noise $\epsilon$. By noise, we mean a function which has a stochastic aspect and a fluctuation time-scale substantially smaller than the period of the periodic orbit. In the absence of noise (i.e. $\epsilon = 0$), the system is autonomous (time-invariant, no explicit time-dependence). This autonomous system has an asymptotically stable (isolated) periodic orbit with time period $T$. When $\epsilon \neq 0$, the noise is small enough that the overall motion in the presence of noise is close to the original periodic motion. Large noise is not considered because it may cause the near-periodicity to disappear. Without loss of generality, we assume zero mean for the noise, as any systematic bias in the noise may be treated as part of the original noise-free dynamical system. Further, to ensure time-invariance, the noise mean and covariance may be dependent on the state $X$ or the “phase” around the periodic orbit [74], but not explicitly on time.

2.1.2 Multiple sections and patchwork models

For completeness, we discuss the notion of factorized Poincare maps, our proposed representation of the dynamics around the periodic orbit, along with the notation we will use for the rest of this thesis. For a more detailed elaboration of factorized Poincare maps, see appended chapter, from which the following is an abridged excerpt.

As shown in Figure 1.2b, we consider a series of $M$ Poincare sections $S_j$ ($j = 1 \ldots M$) around the periodic orbit, intersecting the periodic orbit transversally at $x_j^*$ respectively. In the neighborhood of the periodic orbit, the sections are defined by equations $S_j(x) = 0$.

The mapping from section $S_j$ to section $S_k$ is denoted $P_{j\rightarrow k}$. The point $x_j^{(i)}$ is the
intersection of the trajectory with the section $S_j$ during the $i$th cycle, and $t_j^{(i)}$ is the corresponding time of the intersection, so that $x_j^{(i)} = x(t_j^{(i)})$. Thus, for $j < k$,

$$P_{j\rightarrow k} : x_j^{(i)} \rightarrow x_k^{(i)}$$

and for $j \geq k$,

$$P_{j\rightarrow k} : x_j^{(i)} \rightarrow x_k^{(i+1)}.$$

We use the subscript $j$ to represent the Poincare section being considered and the parenthetical superscript $(i)$ to index the successive cycles.

Thus, the full Poincare map $P_{j\rightarrow j}$ (that is, the first return map), the mapping from $S_j$ back to itself, may be factorized as a composition of the maps between consecutive sections:

$$P_{j\rightarrow j} = P_{j-1\rightarrow j} \circ P_{j-2\rightarrow j-1} \circ \cdots \circ P_{j\rightarrow j+1}. \quad (2.1.2)$$

We use the term factorized Poincare map to refer to the above representation, and we call the component maps as factor Poincare maps or simply, factor maps. Factorized Poincare maps have been used in stability analyses for special dynamical systems [4, 1, 60, 35], but have not been used in system identification or in a model-independent fashion as here. (Also related is the use of a continuum of Poincare sections by Manchester and others [61, 62].)

A linear approximation of the map $P_{j\rightarrow j+1}$ satisfies:

$$x_{j+1} - x_j = J_{j\rightarrow j+1} \left(x_j - x_j^*\right),$$

where $J_{j\rightarrow j+1}$ is the Jacobian of $P_{j\rightarrow j+1}$.

The time taken by a particular trajectory from one Poincare section $S_j$ to the next $S_{j+1}$ is dependent on the initial condition $x_j^{(i)}$ on $S_j$: that is, $t_{j+1}^{(i)} - t_j^{(i)} = Q_{j\rightarrow j+1}(x_j^{(i)})$, which has the following linearization:

$$t_{j+1}^{(i)} - t_j^{(i)} = t_{j+1}^* - t_j^* + \left. \frac{\partial Q}{\partial x} \right|_{x_j^*} (x_j^{(i)} - x_j^*) \quad (2.1.3)$$
where $t^*_j$ is the time on the periodic orbit, starting from some reference point on the periodic orbit. We call this the **time transition map**.

### 2.2 Identification of Discrete dynamical system with multiple stages

Because we eventually reduce the continuous-time dynamical system (hereafter called a continuous dynamical system) into a discrete-time dynamical system (hereafter called a discrete dynamical system) by using Poincare sections, we will now consider the inference of noise-driven discrete dynamical systems given data from them.

For simplicity, in this section, we consider **univariate** discrete dynamical systems. Further, we consider discrete dynamical systems with one or two or many “stages”, as outlined in Figure 2.1, analogous to having one or two or many Poincare sections. We study how the accuracy of system identification changes with the number of stages.

#### 2.2.1 One section

Consider the discrete dynamical system in one state variable

$$x^{(i+1)} = J \cdot x^{(i)} + \epsilon^{(i)}, \quad (2.2.1)$$

in which $x$ and $J$ are scalars ($\in \mathbb{R}$), $\epsilon^{(i)}$ is a normally distributed noise term with mean zero and variance $\sigma^2$ i.e., $\mathcal{N}(0, \sigma^2)$, and the index $i$ denotes time. This dynamical system is represented schematically in Figure 2.1a, and can be considered to be an idealization of a first return map $P_{j\rightarrow j}$. Given an initial condition far away from zero, the state $x^{(i)}$ decays to a stationary random process centered around zero.

Equation 2.2.1 is identical to the so-called auto-regressive model in the statistics literature (in particular, AR(1)). The classical statistics literature (e.g., [45, 34, 10, 50, 58]) has much to say about the dynamics and the identification of the parameters.
in such models and their various generalizations. Appendix B.1 reviews some key results and notions in greater detail, which we briefly recapitulate below.

Given time-series data from such a system, as in Figure 2.2a, we can obtain an estimate \( \hat{J} \) of \( J \) from the data, by minimizing the sum of squared residuals of the equations \( x^{(i+1)} = \hat{J} \cdot x^{(i)} \), a linear least squares problem. Figure 2.2b shows that this estimate is the slope of the best fit line relating \( x^{(i+1)} \) and \( x^{(i)} \); see the appendix for analytical formulas.

The accuracy of the estimate \( \hat{J} \) is given by the error variance \( \psi \) of the estimates from different data sets:

\[
\psi = E \left( (\hat{J} - J)^2 \right) = \frac{1 - J^2}{N} \quad (2.2.2)
\]

Figure 2.2c shows estimates \( \hat{J} \) for various synthetic data sets of different lengths \( N \), showing that the error variance decreases as \( (1/N) \). Note the independence of the estimate variance \( \psi \) to the data variance \( \sigma^2 \), due to linearity of the dynamics, resulting in the estimation being invariant to scaling the data.
2.2.2 Multiple sections

What if there are ‘multiple stages’ to the dynamics? For instance, the two-stage process depicted in Figure 2.2b, an idealization of having two Poincare sections around a periodic orbit, is defined by equations of the form:

\[
x_2^{(i)} = J_{1 \rightarrow 2} x_1^{(i)} + \epsilon_1^{(i)}; \quad x_1^{(i+1)} = J_{2 \rightarrow 1} x_2^{(i)} + \epsilon_2^{(i)}.
\] (2.2.3)

Figure 2.2: Identifying a noise-driven linear discrete dynamical system. a) Time series obtained by simulating \( x^{(i+1)} = Jx^{(i)} + \epsilon^{(i)} \), with \( J = 0.5 \), initial state \( x_1 = 1 \), and two different noise standard deviations: \( \sigma = 1 \) (thick black line) and \( \sigma = 1/5 \) (thin red line). b) The time series from panel (a) corresponding to \( \sigma = 1 \) is re-plotted versus the previous iterate; that is, \( x^{(i+1)} \) versus \( x^{(i)} \). The best fit line in the sense of least squared residual is shown: \( x^{(i+1)} = 0.52x^{(i)} - 0.096 \). Thus the least squares estimate of \( J \) from this synthetic data is the line’s slope 0.52, compared to the true value 0.5. c) The (error) variance in the estimates of \( J \) decrease with increasing length of time series data \( N \). At each data length \( N \), fifty different time series were used to obtain fifty different estimates of \( J \) (plotted as gray circles). Also plotted is the curve (blue dashed line) corresponding to one standard deviation of the theoretically estimated variability in the estimate of \( J \), roughly \( 1/N \) assuming large \( N \) as given in Equation 2.2.2.

As in the previous section, the individual stage Jacobians \( J_{1 \rightarrow 2} \) and \( J_{2 \rightarrow 1} \) may be estimated from the data \( x_1^{(i)} \) and \( x_2^{(i)} \), using linear least squares on the two equations in Eq. 2.2.3 respectively.
If we wanted to estimate $J_{1\rightarrow1}$, we can estimate it two ways: we can either estimate it directly from just $x_{1}^{(i)}$ (that is, just the data at $S_1$), or by individually estimating $J_{1\rightarrow2}$ and $J_{2\rightarrow1}$ from all the data, and using their product: $\hat{J}_{1\rightarrow1} = \hat{J}_{2\rightarrow2} = \hat{J}_{1\rightarrow2}\hat{J}_{2\rightarrow1}$.

It is shown in the appendix that the latter procedure, which uses all the data available at both the sections, has a lower error variance for the estimates $\hat{J}_{k\rightarrow k}$. Further, using the product of the factor Jacobians implies the equality of the estimates $\hat{J}_{k\rightarrow k}$ independent of $k$.

Thus, at least for this idealized model, going from one section to two sections increases the accuracy of the first return map.

More generally, say we have $M$ stages as in Figure 2.1c. We show in the Appendix B.3 that the error variance decreases with the number of stages $M$ and the length of the data (number of cycles $N$). The dependence on the data length is still $(1/N)$ asymptotically. The dependence of the accuracy on the number of stages $M$ is more complicated: it does decrease but for a given $N$, approaches a constant asymptotically. For large $M$ and $N$, and assuming equality of the individual $J_i$ for simplicity, we show that the asymptotic error variance decreases in the following manner:

$$\psi = \frac{J^2}{N} \cdot \frac{M(1 - J^{2/M})}{J^{2/M}} \quad \text{and} \quad \psi_{\infty} = \frac{-2J^2\log(J)}{N}.$$  \hspace{1cm} (2.2.4)

Figure 2.3b shows how the improvement in error variance compared to using only one section $\psi_M/\psi_1$ changes with the number of sections $M$ and the multiplier $J$. Figure 2.3c shows that when $J$ is smaller, more improvement may be obtained — the improvement factor seems, to a good approximation, linear in $J$.

Having established that increasing the number of sections has the potential to increase the accuracy of the first return map in an idealized setting, we next turn to adapting it to synthetic data from a continuous dynamical system.
Figure 2.3: **Discrete linear system with multiple sections.** a) We generated 100 realizations of synthetic time series data of length $N = 100$ using a discrete linear dynamical system with $2^6 = 64$ sections, noise $\sigma = 1$, and $J_{1 \rightarrow 2, \ldots, 64 \rightarrow 1}$ in the range 0.7 to 1.25 (Equation B.3.1). The net eigenvalue $J$ was estimated using the data at $M = 1$ to 128 sections for each of the 100 time series realizations (light gray circles). We see that the variance in estimates of $J$ reduces with number of sections used $M$ until $M = 64$. For $M = 2^7 = 128$, the data were obtained using interpolation between the 64 sections and therefore the $J$ estimates had a correlation-induced bias, as discussed in section 2.3.4. b) Improvement in error variance with increasing number of sections $M$ (Equations 2.2.4 and B.3.3). Notice that the improvement in error is not that great after about 10 sections. c) The asymptotic improvement in error variance as a function of $J$ (Equation 2.2.4). It appears that this asymptotic improvement approximately scales with $J$. 
2.3 System identification of a noise-driven van der Pol oscillator

Figure 2.4: **Noise-driven van der Pol oscillator.** a) Shows 25 cycles of a simulated noise-driven van der Pol oscillator, Equation 2.3.1 with $k = 1$, $\mu = 0.1$, and $\sigma_f = 0.3$. Intersection of the noisy trajectory with six different Poincare sections $S_j$ are shown. b) The distribution of the states $x_1^{(i)}$ on the horizontal Poincare section $S_1$ from panel (a) obtained from $N = 10000$ cycles of the noisy trajectory. It seems bell-shaped and can be reasonably approximated by a Gaussian, although there may be a slight skew. c) The standard deviation of the distribution of states on each Poincare section $S_j$ for 200 Poincare sections around the nominal periodic orbit.

We now apply the parameter inference methods presented in the previous section (section 2.2) and Appendix B.3 to a noise-driven continuous dynamical system, using one or more Poincare sections. We use synthetic data from a noise-driven van der Pol oscillator. Our intention in this section is to illustrate the basic issues in a simpler setting. We present more general methods in section 2.5, applicable to data from higher dimensional dynamical systems.
2.3.1 Generating synthetic data

The van der Pol oscillator is a simple nonlinear ODE with a stable limit cycle. The noise-driven van der Pol oscillator that we consider is

\[
\ddot{y} - \mu(1 - y^2)\dot{y} + ky = f(t),
\] (2.3.1)

where \( f(t) \) is a continuous piecewise-linear function of time representing the noisy external forcing; the function \( f(t) \) is linear between equally spaced time-grid points with \( \Delta t = T/100 \), where \( T \) is the limit cycle period, and the function value \( f(p\Delta t) \) at the grid points indexed by \( p \) are drawn from the normal distribution \( \mathcal{N}(0, \sigma_f) \), where \( \sigma_f \) is independent of time. We have chosen to not explicitly use idealized Wiener processes to model the noise terms. This is because it is likely that real noise has some time- or phase- correlations that may be captured by picking an appropriate sized \( \Delta t \) in the definition of \( f(t) \). We will see below that such correlations make the inference more challenging.

Figure 2.4a shows non-periodic but nearly periodic noise-driven trajectories from this dynamical system. Instead of a single periodic orbit, the nearly periodic noise-driven motion forms a ‘periodic band.’

2.3.2 Poincare sections

We first define Poincare sections around the periodic orbit. We consider two different possibilities for the Poincare sections: (1) state-based and (2) time-based.

State-based Poincare sections.

General algorithms for automatically generating a large number of state-based Poincare sections around a given periodic orbit are found in the appended Chapter and also in section 2.5. State-based sections have no explicit time-dependence.
For the example here, for simplicity, the Poincare sections are defined as lines through the origin with slope $a_j$:

$$\dot{y} + a_j y = 0; \quad \text{for } j = 1, 2, \ldots, M \text{ as in Figure 2.4a} \tag{2.3.2}$$

The state of the system on the $j^{th}$ Poincare section $S_j$ during the $i^{th}$ cycle is denoted by $x_j^{(i)}$, computed here as the distance along the Poincare section from the origin:

$$x = \sqrt{y^2 + \dot{y}^2}.$$

Figure 2.4a shows six linear Poincare sections, intersecting the band of trajectories. Figure 2.4b gives the distribution of $x_1^{(i)}$ for $10^4$ consecutive intersections of the non-periodic trajectory with section $S_1$ over a long simulation. This histogram plot shows that the intersection data looks like a normal distribution, although it is likely that the distribution is not exactly normally distributed asymptotically because of nonlinearities in the dynamical system.

It is unclear if the distribution will be normal for even a linear differential equation. It is known that for a linear stochastic differential equation with additive white noise, starting at a single point for the initial condition or an initial normal distribution of states, the ensemble of possible states at any future time $t$ has a normal distribution [e.g., 55]. However, even for such linear ODEs, we do not know of a corresponding theorem of normality for the distribution of states obtained by intersecting with a linear or a nonlinear Poincare section.

Finally, we see that the variances of the intersections $x^{(i)}$ at 200 different Poincare sections around the periodic orbit (similar to Eq. 2.3.2) are shown in Figure 2.4c. The variance at a section will be dependent on the dynamics around the orbit, the phase-dependence of the noise, and the orientation of the Poincare section with respect to the periodic orbit.
**Time-based Poincare sections are not useful.**

For noise-free systems whose period is known, one can use time-based sections, rather like snapshots with a stroboscope, to obtain a discrete dynamical system. This technique is especially used for stability analysis of systems with explicit time-periodic forcing e.g., [100].

Figure 2.5a-c shows how this method can be ineffective when applied to an autonomous system with noise. In this figure, knowing the exact period $T$ of the noise-free van der Pol oscillator (to about 10 decimal places), we have plotted the state at time intervals of $T$. It can be seen that the snapshots of the data thus obtained drift all around the nominal periodic motion (Figure 2.5c), without being restricted to regions with similar phase. The drift in phase is likely due to the fact that the dynamics along the periodic motion is neutrally stable, so perturbations along the periodic motion do not grow or decay, resulting in a brownian motion-like behavior for the phase of the snapshots. Given an ensemble of closely spaced initial conditions with similar phase, the variance of the phase increases with time, until it asymptotes to some non-compact-support distribution – arguably (approximately) uniform – over the whole circle.

Thus, the state obtained at these time-based snapshots are dominated by the brownian motion, rather than the actual dynamics about the periodic motion. Therefore, for the rest of this thesis, we only use state-based Poincare sections. Further, for the same reason, we argue against approximation of the local dynamics near periodic orbits using an explicitly time-dependent model, for instance, a time-periodic linear ODE. Hence our focus on factorized Poincare maps.
Figure 2.5: **Using a time-based stroboscope instead of state-based Poincare sections.** Even though the system is close to periodic, considering the system at constant time intervals apart (here, equal to the period of the unperturbed van der Pol) results in states that are not localized in the state-space (bold red points). Instead, the points drift around the average periodic orbit. This is because the system is perturbed by noise constantly and is neutrally stable to perturbations along the periodic orbit.

### 2.3.3 System identification of the Poincare map using data at one Poincare section

We *model* the stochastic transition from one section to another, say section $S_j$ back to itself, by stochastic maps of the form:

$$x_j^{(i+1)} - x_j^* = J_{j\to j} \cdot (x_j^{(i)} - x_j^*) + \epsilon_j^{(i)},$$  \hspace{1cm} (2.3.3)

where $x_j^*$ is the state on the periodic cycle when noise-free and $J_{j\to j} (=J)$ is the appropriate Jacobian. With these assumptions, $\epsilon_j^{(i)}$ may not be exactly zero mean because of nonlinearities, but we implicitly model it as such.

Thus, we have two quantities to be estimated in Equation 2.3.3, namely $x_j^*$ and $J$. As in section 2.2.1, given the data $x_j^{(i)}$ at the section, the unknowns $x_j^*$ and $J$ may be obtained as minimizing the summed squared residuals of the noise-free version of Eq. 2.3.3. It can be shown that the least squares estimate of $x_j^*$ is simply the mean value of $x_j^i$, i.e., $x_j^* = E(x_j^i)$.  

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Figure 2.6: **Analyzing the van der Pol oscillator using data at a single section.**

a) The eigenvalue of the Poincare map on section $S_1$ is obtained by finding the best linear map between consecutive iterates $x^{(i)}_1$ and $x^{(i+1)}_1$. Even using data at only one section, we have the choice of which Poincare section $S_j$ to use. Here, we consider $M = 100$ Poincare sections $S_j$ that are all straight lines through the origin as in Figure 2.4a, intersecting the trajectories at different points along the nominal cycle. The eigenvalue estimate is plotted as a function of the index of the Poincare section used.

We used only $N = 100$ cycles, hence the large deviation from the correct value. c) A number of Poincare sections passing through same point on the nominal periodic orbit, making different angles with the trajectory, are considered. When the Poincare sections make small angles with the average trajectory, the noisy transient trajectory may sometimes entirely miss intersecting them, resulting in fewer data points to analyze. d) Estimating the eigenvalue of the Poincare map using the different sections from panel (c) give roughly the same value, except at the extreme of Poincare section angles, when the number of data points is less by about 11%. 

![Graphical representation of the analysis of the van der Pol oscillator using data at a single section.](image_url)
Figure 2.6a shows the least square estimate $\hat{J}$ of $J$, which equals to the slope of the best fit line of the noise data. With the linear sections as defined in Equation 2.3.2, Figure 2.6b shows the estimates of $\hat{J}$ by using 100 different radial Poincare sections of the form Eq. 2.3.2 around the periodic orbit, similar to those in Figure 2.4a. We note that the estimates are all slightly different. However, it can be shown that the eigenvalue of Poincare map $J$ is independent of the Poincare section used. (More generally, for multivariate Poincare maps, the Jacobians at different transverse sections are related by similarity transformations, and therefore, have the same eigenvalues).

The discrepancy in the estimates $\hat{J}$ from different sections around the periodic orbit is likely on account of how the interaction of the noise and nonlinear dynamics affects the data at different sections. We note that prior work on such Poincare map inference [47, 75] also reported different eigenvalues for Poincare maps at different sections. Which among these different estimates is the best estimate to the true eigenvalue? The discrepancy arises because the data at the different sections are not all used together to estimate $J$, but independently of each other. Later in this chapter, as in section 2.2.2, we will employ methods that provide a single estimate of $J$ by combining data at many sections.

Note that the deviation from the true value in Figure 2.6b is due to use of a relatively small data set with $N = 100$ cycles.

Even at a single point on the periodic orbit, the choice of the transverse Poincare section is not unique. Figure 2.6c shows an example with many Poincare sections with different orientations. Notice that a Poincare section with too large or too small a slope is not able to intersect the whole band of trajectories, so that some cycles are missed. As a result, the sample size on such a Poincare section is smaller than a Poincare section with moderate slope value. We can see the different estimates $\hat{J}$ from Poincare section with different slopes in Figure 2.6d. The difference in estimates
at two ends is due to the smaller sample size. When the slopes are moderate, the estimates are obtained with equal data size, and the estimates changes only slightly. It appears that the Poincare section with slope zero gives relatively better estimates (in this case). This may be because the Poincare section with zero slope has a slightly smaller variance than the data on other Poincare sections. Even though we have shown that the estimate $\hat{J}$ is independent of the variance of the noise in section 2.2 (equation 2.2.2) for a linear discrete system, here we are using the method on a linearized nonlinear system – the smaller variance of the data is closer to the assumption of linearity.

### 2.3.4 System identification using many sections

We now discuss obtaining approximations to the Poincare map and the factorized Poincare map, simultaneously using the many Poincare sections around the periodic orbit. We show that a naive application of least squares inference, gives biased estimates of the net Jacobian, and show how this bias may be corrected or avoided.

**Using ordinary least squares produces biased estimates**

When we have many sections $S_1$ to $S_M$, we infer the $M$ corresponding Jacobians, namely $J_{j\rightarrow j+1}$, from one section $S_j$ to $S_{j+1}$, similar to section 2.2.2 and Appendix B.3.

Using intersections of the trajectories with each Poincare section gives us multi-stage equations analogous to equation B.3.1, except, now, each equation has an offset $x_j^*$ corresponding to the intersection of the nominal periodic motion with the Poincare section (as in equation 2.3.3). That is,
\[ x_2^{(i)} - x_2^* = J_{1 \to 2} \left( x_1^{(i)} - x_1^* \right) + \epsilon_1^{(i)} \]

\[ \ldots \ldots \ldots \]

\[ x_M^{(i)} - x_M^* = J_{M-1 \to M} \left( x_{M-1}^{(i)} - x_{M-1}^* \right) + \epsilon_{M-1}^{(i)} \]

\[ x_{1}^{(i+1)} - x_1^* = J_{M-1} \left( x_M^{(i)} - x_M^* \right) + \epsilon_M^{(i)} \]

Given the data \( x_j^{(i)} \), minimizing the squared residuals of the equations gives the ordinary least squares estimates of \( J_{j \to j+1} \) and \( x_j^* \). Multiplying the individual \( J_{j \to j+1} \) gives the total \( J \):

\[ J = J_{k \to k} = J_{k-1 \to k} J_{k-2 \to k-1} \ldots J_{k-k+1} \]

Figure 2.7b shows this ordinary least square estimate from 100 different noisy synthetic data sets, each with \( N = 100 \) cycles of data. We see how the estimate of the total Jacobian \( J \) changes with the number \( M \) of sections used. The solid line is the mean value of the estimated eigenvalues at different number of sections. The variance of the estimated eigenvalues is decreasing with \( M \) but not as dramatically as we might expect from our earlier results in section B.3 and Figure 2.3.

Most significantly, when the number of sections is large – e.g., 30 or more – a systematic bias appears in the ordinary least squares estimates of the total \( J \). Further, the value of bias increases almost linearly with the number of sections \( M \). Thus our initial expectation, based on section 2.2.2, that the error would decrease with the number of sections seems misguided.

**Bias due to correlated noise**

The reason for the biased estimates in Figure 2.7b is that the noise between two neighboring intervals is correlated. For the two-stage process in Equation 2.2.3, if the
noise terms $\epsilon_1^{(i)}$ and $\epsilon_2^{(i)}$ are dependent, i.e. $\text{cov}(\epsilon_1^{(i)}, \epsilon_2^{(i)}) \neq 0$, the estimates $\hat{J}_{1\rightarrow2}$ and $\hat{J}_{2\rightarrow1}$ of $J_{1\rightarrow2}$ and $J_{2\rightarrow1}$ can be shown to be, respectively, approximately:

$$
\hat{J}_{1\rightarrow2} = J_{1\rightarrow2} + \frac{\text{cov}(\epsilon_1^{(i)}, \epsilon_2^{(i)})}{\text{var}(x_2^{(i)})} \quad \text{and} \\
\hat{J}_{2\rightarrow1} = J_{2\rightarrow1} + \frac{\text{cov}(\epsilon_1^{(i)}, \epsilon_2^{(i)})}{\text{var}(x_1^{(i)})}.
$$

(2.3.5)  
(2.3.6)

In other words, asymptotically, $\hat{J}_{1\rightarrow2} \neq J_{1\rightarrow2}$ with the bias term depending on the noise correlations.

Where do the noise correlations come from? Note that we used a piecewise linear noise force $f(t)$ on the van der Pol oscillator in equation 2.3.1, and $f(t)$ does not change arbitrarily fast because of the non-infinitesimal $\Delta t$. This means that when two neighboring sections are close together, the noise term on the two intervals are not independent but are substantially correlated. The effect of the correlations increases when the number of Poincare sections increases, as is seen in Figure 2.7b and also the red line in Figure 2.7a. The gray shadowed area in Figure 2.7a is the regime for $M$ in which the effect of correlations and the bias are both small.

**Removing the bias using estimated bias subtraction.**

One simple way to eliminate the bias is to estimate the bias and subtract the estimated bias from the estimates. If we assume that only consecutive Poincare sections have correlated noise, and there is no correlation between non-consecutive Poincare sections, the biased estimate of $\hat{J}_{j\rightarrow j+1}$ from ordinary least square method is approximately:

$$
\hat{J}_{j\rightarrow j+1} = J_{j\rightarrow j+1} + \frac{\text{cov}(\epsilon_j, \epsilon_{j-1})}{\text{var}(x_j)}
$$

(2.3.7)

where $\epsilon_j$ is the noise term. If one can independently estimate the bias term in the above equation, we can simply subtract it from the ordinary least square estimate
to reduce the bias. The noise term above can be estimated from the residuals of the ordinary least square estimate as:

$$\hat{\epsilon}_j = x_{j+1} - \hat{J}_{j\to j+1} \cdot x_j. \quad (2.3.8)$$

where $\hat{J}_{j\to j+1}$ is the ordinary least squares estimate.

We estimate the bias term using equations 2.3.7 and 2.3.8, and subtract the estimated bias from the ordinary least squares estimate $\hat{J}_{j\to j+1}$ to obtain the corresponding bias subtracted estimates. Then, multiplying the individual $\hat{J}_{j\to j+1}$, we may obtain $\hat{J}_{k\to k}$.

The bias subtracted estimates are given in Figure 2.7a,c as the black line. In Figure 2.7c, the distribution of the bias subtracted estimates for 100 different data sets is shown. It can be seen that the heuristic bias subtraction essentially kills the bias even for large number of Poincare sections, and can approximate the true eigenvalue $J$ well. The variance of the estimates, however, does not decrease as much as in previous ordinary least square estimate does, as presumably, the estimated bias term contributes some variance to the error in the new estimate. The only remedy to reducing this variance is to increase the data length $N$.

**Using Maximum Likelihood Estimation or iteratively re-weighted least square estimates.**

So far, we have used the ordinary least squares method for inference of the relevant variables from data. In the statistics literature, a gold-standard inference technique is the Maximum Likelihood Estimation (MLE). In MLE, the unknown variables are estimated to be that which will maximize the probability of observing the given data. Ordinary least squares is actually identical to Maximum Likelihood Estimation under certain assumptions, for instance, of exact linearity of the correct model, the noise being identically normally distributed, and the initial condition being known [34].
Figure 2.7: **Estimating eigenvalues using multiple sections.** a) (Mean) eigenvalue estimates as a function of number of sections used for a single data set, from three different methods: ordinary least squares, many section mapping with bias subtraction and iteratively weighted least square estimate. The correct value is about 0.53. b) The ordinary least squares method produces unbiased estimates for low number of sections, but there is a systematic positive bias for larger number of sections. This bias is due to correlations between data from neighboring sections. The variance of the estimates decreases with increasing number of sections. c) The first order bias in the ordinary least squares method is estimated from the data approximately and subtracted giving an approximately unbiased estimate despite correlations between neighboring sections. d) An iteratively re-weighted least squares method is an approximation to the Maximum Likelihood Estimator and gives an unbiased estimate. Both the bias subtraction and the iterative least squares method have slightly higher variance asymptotically than the ordinary least squares method. This figure used $N = 100$ cycles, hence the high variance in the estimates.
Further, MLE reduces to an appropriately weighted ordinary least squares if the noise terms have different known covariances.

Because the bias arises from not taking into account the noise covariance structure, a Maximum Likelihood Estimate that takes into account this covariance structure would give an estimate with better properties [34, 10]. However, in a typical problem, we do not know the noise covariances before the system identification. Thus, the noise covariances would be part of the unknowns. In this case, one can obtain a maximum likelihood estimate by solving for both the model parameters (the $J$ and $x^*$’s) and the noise covariances that together maximize the likelihood of observing the data, assuming normally distributed noise. If we can compute the likelihood function, we can solve the estimation problem as a nonlinear optimization problem.

Here, we use a simpler procedure, called the iteratively re-weighted least squares technique, to approximate the MLE. Please see Appendix B.4 for a brief introduction to MLE and the iteratively reweighted least squares, and how these methods are adapted to the problem at hand.

The result of iteratively reweighted least squares is plotted in Figure 2.7a with blue line and the estimates from 100 different data sets are plotted in Figure 2.7d. It can be observed that iteratively reweighted least squares produces unbiased estimates even when the number of Poincare sections is quite large. While the variance in the estimates in Figure 2.7d does decrease with $M$, it is not clear to us why the variance decreases much slower than in Figure 2.7b for ordinary least squares.

In summary, we have outlined techniques that provide unbiased estimates for the linearization of the Poincare map, and thus, reasonable linear approximations to the maps from one section to the next. These $M$ linear maps constitute our approximation of the factorized Poincare map.
Figure 2.8: **Predicting dynamical responses using a patchwork of piecewise models.** A simulation of the noise-free dynamical system from \((4, 0)\) is shown in black dotted line and a prediction based on a linear factorized Poincare map is shown in green line. Another prediction from a quadratic factorized Poincare map model is shown in red line. We used \(M = 34\) sections for the factorized Poincare map, and \(N = 100\) cycles to construct the data-based model.

### 2.3.5 Simulation and prediction of transients using the inferred factorized Poincare map

Having inferred a factorized Poincare map representation, namely the mapping from one section to the next for a large number of sections, we can use this representation to perform ‘simulations’ of the dynamical system from initial conditions in the neighborhood of the periodic orbit.

Given a factorized Poincare map representation, it is straightforward to perform simulations starting from one of the Poincare sections considered. We just apply the maps in sequence. Then, using the sequence of predicted states on the sections, we may interpolate to obtain a continuous time representation of the solution.

In appended chapter, we show how to extend this idea to simulate from initial
conditions close to the periodic orbit, not just on the considered Poincare sections, but between them.

We illustrate the effectiveness of such predictions in Figure 2.8. And the blue dotted gray lines are the prediction from the piecewise linear model. We see a fair match between prediction and true transient response, even though the initial condition (4, 0) is nearly 100% off the equilibrium point on the periodic orbit (2.001, 0). For larger perturbations, the transient predictions of piecewise linear model will have less accuracy.

2.3.6 Piecewise quadratic models

To improve on the predictive accuracy of the linear models, a piecewise quadratic model may be inferred from the noisy data. In the univariate mappings for the van der Pol oscillator, we used:

\[ y^{(i+1)}_j - y^*_j = J_{j\rightarrow j+1} \cdot (y^{(i)}_j - y^*_j) + H_{j\rightarrow j+1} (y^{(i)}_j - y^*_j)^2, \]  

(2.3.9)

We see in Figure 2.8 that the transient trajectory simulated from quadratic model matches the true transient trajectories much better than the linear models. In the multivariate situation, we would simply generalize the above equations with appropriate Hessians; see [97].

2.3.7 Other derived dynamical quantities from the factorized Poincare map

The factorized Poincare map, along with the time-transition map from section to section, is effectively, an approximation of the dynamics near the periodic orbit. This means that from this representation, we may derive other quantities that describe the dynamics near the periodic orbit. In [97], we describe how to compute a number of such quantities. In particular, we may compute the coefficients of the linear
Figure 2.9: **Isochrons in the plane.** a) The time difference between successive intersections of a Poincare section is roughly linearly related to the state difference from that on the nominal periodic orbit (Equation 2.3.10). This linear relation depends on the orientation of the Poincare sections. The local isochrons are those Poincare sections that states starting from one reach the next simultaneously, so that the linear relation between time difference and state difference is identically zero. b) The correct local isochron, obtained from the original ODE, and the isochron estimated from the noise-driven data are shown for one data set with $N = 100$ cycles. c) The distribution of data-estimated isochron angles are shown, as obtained from 100 different synthetic data sets of the van der Pol oscillator.
time-periodic ODE of the form $\dot{y} = A(t)y$ that describes the local dynamics of the deviations $y$ from the periodic orbit.

Two other closely related quantities of interest are the **phase response curves** and **isochrons**. Phase response curves define how much the phase of the system changes when a small perturbation is given to one of the state variables. They are helpful in constructing local ‘phase dynamics’ models of coupled oscillators. See [11, 23, 97].

Finally, an isochron is a special set of Poincare sections transversal to the periodic orbit, defined as follows. Take a set of initial conditions on a Poincare section $S_0$ and propagate the system forward in time for exactly one period of the periodic motion. The set of new states will form another Poincare section $S_1$ transversal to the periodic orbit. Generically, $S_1$ will be close to but different from $S_0$. A Poincare section is called an isochron if $S_1 = S_0$; in other words, an isochron gets mapped into itself after exactly one period. Isochron are the level sets of the phase near an oscillator [31, 101], so that the above description defines what means ‘phase’ as well, in the neighborhood of a periodic orbit.

While these quantities can be computed from the factorized Poincare map as in [97], they may also be more directly obtained from the noise-driven data. For instance, we describe below, how we would compute the isochrons around the periodic orbit, directly from the noisy data, directly using the definition of isochrons above. We only estimate the local (linear) isochron around the nominal periodic orbit, namely the tangent to the actual isochron on the nominal periodic motion. Recall from Eq. 2.1.3 that for any Poincare section, the time difference between two successive intersections of a nonperiodic trajectory differs from the true time period $T^*$ by the following equation to first order:

$$t_j^{(i+1)} - t_j^{(i)} = T^* + \alpha (x_j^{(i)} - x_j^*)$$  \hspace{1cm} (2.3.10)
where \( x_j^{(i)} \) is state on the Poincare section, and \( t_j^{(i)} \) is time for the state \( x_j^{(i)} \). Note that, \( t_j^{(i+1)} - t_j^{(i)} \) is the time difference between states \( x_j^{(i+1)} \) and \( x_j^{(i)} \). If the Poincare section is the local isochron, the time difference between states equals the true period time \( T^* \) to first order. i.e., \( \alpha = 0 \)

Given a Poincare section, the coefficient \( \alpha \) in the above equation can be estimated by least squares regression from the noise-driven data, as usual. Then, finding the isochron is equivalent to finding the Poincare section for which \( \alpha = 0 \). Figure 2.9 shows the procedure and results of the estimation of isochrons on the noise-driven van der Pol oscillator. At each point on the periodic orbit, we compute \( \alpha \) for a number of Poincare sections, and then interpolate to find the Poincare section with \( \alpha = 0 \).

Notice that because of the noise-driven data, we compute a range of estimates for the isochrons at every point, for different data sets from the same dynamical system (Figure 2.9c).

### 2.4 Systems with slow parameter drift

Before we discuss higher dimensional dynamical systems with multivariate Poincare maps, we discuss what happens if we apply the aforementioned methods to systems which do not, in the absence of noise, have a limit cycle.

One mechanism through which one may get deterministic aperiodicity with near-periodicity is through a slow change of parameters in a dynamical system with a periodic orbit. For instance, the van der Pol oscillator (Eq. 2.3.1) with a slowly changing coefficient \( k \). We will consider this example later in this section. In human locomotion, one might get such slow parameter drift as human muscles get fatigued. In oscillations involving metals, one might get slow drift of material properties due to accumulation of dislocations and other fatigue mechanisms.
First, we consider the univariate discrete system with continuous time-varying parameters:

\[ x^{(i+1)} - x^{*(i+1)} = J^{(i)} \cdot (x^{(i)} - x^{*(i)}) + \epsilon^{(i)}, \]  

(2.4.1)

where \( x^{*(i)} \) and \( J^{(i)} \) are now time-varying functions (that is, functions of \( i \)), whereas they have been constants thus far in the thesis. Thus, we wish to estimate the functions of \( i \), \( x^{*(i)} \) and \( J^{(i)} \), which we do as minimizing the following residuals:

\[ E = \sum_{i=1}^{N} \left( x^{(i+1)} - x^{*(i+1)} - J^{(i)} \cdot (x^{(i)} - x^{*(i)}) \right)^2 \]  

(2.4.2)

Of course, if \( x^{*(i)} \) and \( J^{(i)} \) were arbitrary functions of \( i \), we would track the noisy data exactly and would be over-fitting. Thus, if we knew that \( x^{*(i)} \) and \( J^{(i)} \) were slowly-varying functions, we may parameterize these functions by, say, splines with a small number of knots. Note that given a spline representation of \( x^{*(i)} \) and \( J^{(i)} \), their values at \( i \) are linear functions of the spine’s knot values. Thus, the above minimization problem would be a positive quadratic function of the spine’s knot values, giving a unique minimum. In our implementation, we simply used a generic nonlinear optimizer (fminunc in MATLAB) for the minimization. An example of univariate discrete system with slowly varying coefficients, inferred as piecewise linear splines, is given in Figure 2.10. We see that while the inferred spline is a good approximation of the actual functions. Note that the number of spline knots need to be picked \textit{a priori}, based perhaps on prior knowledge of the functions. There may also be data-based methods in which we pick the number of spline knots by comparing the residuals from the best fit to known noise properties for a range of knot numbers.

We now apply these ideas to the van der Pol oscillator, with noisy forcing, with a slowly varying parameter, the spring constant \( k = k(t) \). If \( k \) were constant, the periodic orbit would be different for different \( k \). Now, as \( k \) changes smoothly, as
Figure 2.10: **Spline estimates of slowly varying parameters in a univariate discrete system.** We consider Eq. 2.4.1 with continuous functions $J^{(i)} = 0.5 + 0.3\sin(9b\,i)$ and $x^*(i) = 0.1\cos(4b\,i)$, where $b = 10^{-4}$. Both these functions are shown (green lines) along with estimates of these functions by (1) assuming them to be piecewise linear splines with about 10 pieces (black line) (2) assuming them to be a single constant (red). We used about $N = 10^4$ synthetic data points for the inference.

shown in Figure 2.11a, the orbit is close to periodic in any one cycle but gradually changes shape in a manner that reduces the velocity amplitude.

Say we did not know that there was slow parameter drift and applied the ordinary least square method directly to derive the Jacobian of the Poincare map at one Poincare section. Then, if we used the mean at the section as the estimate of the periodic point, we obtain estimates of the Jacobian at different Poincare sections to be dramatically different from each other, as shown in Figure 2.11b. In particular, we observe a continuous variation of the Jacobian estimate. Further, if we used a large number $M$ of Poincare sections, constructed the individual maps from section to section, without recognizing the parameter drift, the total Jacobian so inferred exhibits an dramatic drop almost to zero for a range of section counts (Figure 2.11c).

Of course, we may fix the inference artifacts of Figure 2.11b-c by recognizing that the underlying dynamical system does have some slow parameter drift. Thus, we
model the map from a section back to itself using slowly changing coefficients as in Equation 2.4.1. By using piecewise linear splines with a small number of knots for both the periodic point and the Jacobians, we obtain reasonable models for the map (Figure 2.12).

2.5 Multivariate Poincare mappings

Having discussed in some detail the various issues relating to inferring factorized Poincare maps for the low dimensional van der Pol oscillator, we now turn to higher dimensional systems which have multivariate Poincare maps.
Figure 2.12: **Van der Pol oscillator with slowly drifting spring constant.** We used splines to estimate the parameter changes as a function of time. In particular, the linear map from a section back to itself is assumed to change slowly as in Eq. 2.4.1. Four different curves are shown in each panel. The ‘actual parameter values’ (green line) for $J$ and periodic point $x^*$ as functions of the cycle number $i$, are obtained from the exact periodic orbits for the appropriate $k$ during that cycle. The estimates assuming $J$ and periodic point $x^*$ were constant gives very incorrect results. The moving average estimates, assuming $J$ and $x^*$ were constant over a moving window of 40 cycles gives reasonable estimates for $x^*$ but not for $J$. The piecewise linear spline estimate which takes the drift into account seems to catch the trends reasonably.
2.5.1 Defining Poincare sections in high dimensions

Analysis of higher dimensional data poses challenges not present in the analysis of two-dimensional van der Pol oscillator data. In particular, how the Poincare sections should be defined automatically from a long time-series noisy data becomes less obvious.

Say we have nearly periodic multivariate data $Z(t)$, which is drawn from a single continuous simulation or data collection. Using this $Z(t)$, we propose the following procedure for the definition of $M$ well-scaled Poincare sections roughly equally spaced around the periodic orbit, depicted also in Figure 2.13(i)-(ix):

i. If appropriate, scale all the state variables in $Z$ into a reasonable range, for instance, $[-1, 1]$.

ii. Consider one arbitrarily defined Poincare section as an initial Poincare section. For example, because of the scaling above, $Z_1 = 0$ is guaranteed to intersect the periodic orbit at least twice. Find all the intersections of the trajectory $Z(t)$ with the chosen section.

iii. Given that the section intersects the trajectory at least twice every cycle, perhaps more times, we cluster the intersections into distinct groups, so that points within a cluster have with similar positions on the periodic orbit. This clustering may be performed automatically with, say, k-means clustering [37] on the set of intersections. Pick any one cluster. Rank the points by their intersection times, and verify that the time difference between consecutive points is about equal (this verifies that the clustering has been successful). The mean time difference serves as an initial estimate $\hat{T}^*$ of the total time period.

iv. The initial section may not be well-conditioned. Therefore, in steps (iv)-(vi), we make the orientation of this section close to normal to the periodic orbit.
First, from every point on this cluster of points, consider the states of the system a little forward in time, say by $h$, where $h$ is large enough that we do not differentiate noise, but sufficiently smaller than the proposed time interval $\hat{T}^*/M$ between Poincare sections.

v. The average directions of these short trajectory pieces gives an estimate of the tangent to the periodic trajectory.

vi. We define a new Poincare section as the plane perpendicular to the estimated tangent to the trajectory. We replace the initial Poincare section by this normal Poincare section, which will serve as section $S_1$. Obtain intersections of the trajectory with the Poincare section $S_1$.

vii. Propagate the data forward in time by $T^*/M$. We obtain a cluster of points, which will be the rough location of the next Poincare section $S_2$.

viii. With this cluster of points, repeat steps (iv) to (vi), to obtain a new Poincare section $S_2$, nearly normal to the trajectories.

ix. Now, repeat steps (vii)-(viii) to obtain a sequence of Poincare sections $S_j$ all around the periodic orbit, approximately normal to the orbit in the chosen coordinates. We stop when we either have $M$ sections or our latest constructed section is too close to $S_1$.

Once the sections are defined, one may, in principle slide them along the periodic orbit in case better separation between them is required. For instance, we suggest the following section refinement strategy. As in [97], we construct a periodic cubic spline through the mean intersections of the trajectory at each section. If the tangent to the spline is substantially different from the mean tangent to the trajectories at a section, we might consider adding more sections.
Figure 2.13: An algorithm for obtaining a sequence of about \( M \) Poincare sections, which are nearly normal to the trajectories.
2.5.2 Application to synthetic data

We have found that the above algorithm obtains reasonable Poincare sections, robustly, for both real data (from human locomotion) and for synthetic data as below. We briefly discuss its application to synthetic data drawn from the following multivariate system consisting of coupled van der Pol oscillators, which in the considered regime, had a periodic orbit:

\[
\ddot{Y} - \mu (I - Y^2) \dot{Y} + kY = F(t)
\]  \hspace{1cm} (2.5.1)

where \( \mu \) is the matrix of damping coefficients, and \( k \) is the matrix of coupled spring coefficients. We used:

\[
\mu = \begin{bmatrix}
0.1 & 0 & 0 & 0 & 0 \\
0 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0.3 & 0 & 0 \\
0 & 0 & 0 & 0.4 & 0 \\
0 & 0 & 0 & 0 & 0.5
\end{bmatrix}
\]  \hspace{1cm} (2.5.2)

\[
k = \begin{bmatrix}
1 & -0.1 & -0.1 & -0.1 & -0.1 \\
-0.1 & 1 & -0.1 & -0.1 & -0.1 \\
-0.1 & -0.1 & 1 & -0.1 & -0.1 \\
-0.1 & -0.1 & -0.1 & 1 & -0.1 \\
-0.1 & -0.1 & -0.1 & -0.1 & 1
\end{bmatrix}
\]  \hspace{1cm} (2.5.3)

To make the data a bit less well-conditioned, we introduced a linear nonsingular transformation from data \( Y \) to new data \( Z \) as \( Z = RY \).

Using the above algorithm for constructing Poincare sections, we used ordinary linear least squares for constructing the linear maps between sections. Having obtained the factorized Poincare map representation, we compare simulations of the
noise-less dynamical system with that obtained from the factorized Poincare map, an example of which is shown in Figure 2.14. For some initial conditions, larger phase errors are obtained when the time-transition map is not used for correcting the time at the intersections.

The adaptation of the bias correcting methods, discussed earlier for the univariate Poincare maps earlier, is beyond the scope of this thesis, but see the discussion in the following section for outlines.

Figure 2.14: **Transient predictions of five coupled van der Pol oscillator.** It shows a comparison between predicted response from using factorized Poincare maps and true response obtained from an accurate ODE simulation, for two different initial conditions that are at different distances from the periodic orbit. Notice that there are substantial transients and that the factorized Poincare map representation captures a lot of these transients. See also [99].
2.6 Discussion

In this chapter, we developed a data-driven method to obtain local dynamical models near periodic orbits. In particular, we obtain representations of local dynamics in the form of factorized Poincare maps and time transition maps, using a large number of Poincare sections around the periodic orbit. We discussed the various computational and statistical issues in the context of univariate inference, and then briefly discussed the multivariate context. We noted that from the factorized Poincare map representation, other local quantities of interest, such as linear time-periodic ODE approximations, Floquet multipliers, phase response curves, and (planar) isochrons may be derived.

The basic assumption was that we had access to a time-series from a single long simulation or data acquisition from a dynamical system with a periodic orbit, constantly being perturbed by some small noisy force. Numerical experiments show that the methods discussed herein work well when the noise does not appear additively in the differential equation, but in a nonlinear or state-dependent manner (e.g., \(k = \text{constant} + \text{noise in a van der Pol oscillator}\)).

While we used and modified well-tested methods for inferring discrete dynamical systems from the statistical literature, there is an opportunity for extending these methods to the differential equation setting directly. One analog of the inference problems here would be an extended Kalman filter [87], but modified to the time-periodic setting with appropriate periodicity constraints. However, because of the drift phenomena, documented here in Figure 2.5, associated with time-based representations of noise-driven nonlinear oscillators, we would suggest a phase-based modification of the formalism, if possible. Also, we note that there is a small literature on using “harmonic transfer functions” to represent the dynamics near a periodic motion, or equivalently, as a linear time-periodic system (Tim Kiemel and Noah Cowan, personal
communication, and [66, 80]). It may be possible to adapt the methods here, using
factorized Poincare maps, to discrete-time analogs of harmonic transfer functions.

The problem of inferring the sequence of discrete maps from one section to the
next, with periodicity in both the mappings and the noise properties, has been the
subject of a few investigations in the statistics literature. Generalizing the more com-
mon ARMA models [34], they are referred to as PARMA models, short for Periodic
Auto-Regressive Moving Average [94, 59, 44]. It may be possible to directly adapt
some of the methods from that literature, with perhaps small modifications, to the
problem at hand.

In section 2.3.4, we discussed methods for eliminating bias due to unknown corre-
lated between sections. We draw attention to an alternate piecewise model and infer-
ence procedure that also takes into account the correlations in noise (Peter Craigmile,
personal communication, and [12]). In this alternate model, we assume that the noise
between two sections depends linearly on the noise between the previous two sections
in some linear manner, with some uncorrelated noise $\delta$ added. e.g., $\epsilon_{j+1} = C_j \epsilon_j + \delta$.
Thus, the noise process is modeled to have some dynamics, which automatically re-
sults in autocorrelation, which may model correlations between neighboring sections
for appropriate $C_j$. The number of unknowns to be estimated is now increased, with
the parameters $C_j$ defining the dynamics of the noise process also to be determined.

In this chapter, we have not explicitly discussed the presence of measurement error
in the time-series data, but the inference methods used here are known to work well
even in their presence. The measurement error could also be more naturally taken
into account in an MLE framework. Of course, the (error) variance in the relevant
estimates would increase with increase in measurement error, as we have found in our
numerical experiments.
In the previous chapter, we discussed a data-driven method to derive dynamic information from nearly periodic responses, and have demonstrated the usefulness of this method by applying it to synthetic data. We wish to now apply this method on human walking data, which is also observed to be a nearly periodic motion, as every step is similar but slightly different to every other steps in time history. In this chapter, the collection of human walking data is described in detail. We carried out the collection of human walking data in the Movement Lab in the Mechanical and Aerospace Engineering in the Ohio State University (http://movement.osu.edu). The experimental measurements will be analyzed in the next two chapters.

The protocols were approved by the Ohio State University Institutional Review Board. Human subjects participated with informed consent. The subjects (N = 5, 5 males), whose mean age was 23, with average 78.9 kilograms weight and 1.8 meters height (standard deviation: 12.7 kilograms and 7 centimeters).

3.1 Experiment setup for recording human walking movements

A marker-based motion capture system (VICON Inc.) was used to capture human walking as shown in Figure 3.1a. The system has eight cameras focused on one
treadmill in the middle of the room. We adjusted the views and focuses of the cameras, so that the combined view of all cameras covers the necessary volume above the treadmill. We attached reflective markers on different segments of the subject, like foot, tibia, thigh, trunk as shown in Figure 3.1b. All markers are typically visible to at least three cameras. The time series data for the 3D location of all markers were captured by the high speed cameras (capable of 500Hz; we used 100Hz).

Figure 3.1: **Experiment of human walking.** a) There are 8 high speed cameras for the marker based motion capture system focused on a treadmill in the middle of room; b) The reflective markers are placed on different segment of the subject. Cameras capture the time-history of 3D marker positions.
3.2 Experiment procedure

Human walking is a very complicated process, with all body parts moving. For simplicity, here we treat trunk, thigh, tibia and foot as rigid bodies, and that they are connected by six ideal joints, two ankles, two knees and two hips. The experiment thus should be able to capture the rigid body motions and the locations of all the six joint centers.

We carried out three types of trials: static posture, functional joint-axis experiment and dynamic walking. Trials of static posture provide the reference frame for all following dynamic experiments. The functional joint axis experiment helps to locate hip joint positions, which will be also added to the reference frame. The reference frame shows the distances between markers. Trials of dynamic walking include two experiments: steady walking at specific speeds and self-imposed perturbed walking.

3.2.1 Static posture

Before every experiment, we calibrated the Vicon marker-based motion capture system by using a standard five-marker calibration wand. After calibration, 29 markers in total are placed on the subject with adhesive tapes: 3 on each foot, 3 on each tibia, 3 on each thigh, 3 on trunk, 2 on each ankle and 2 on each knee. The markers of static posture are shown as green points in Figure 3.2.

The three markers on each segment (i.e. foot, tibia, thigh and trunk) are non-collinear, so that the markers can fully depict the segment translations and rotations. For each ankle, two markers are placed in medial and lateral (i.e. inside and outside). The average of these two markers is used as the ankle joint center. For each knee, two markers are placed in the medial and lateral side of the knee. They are located according to bone contact surface between femur and tibia [9]. The average of these
two markers is used to represent knee joint center. The ankle and knee positions are shown as blue dots in the Figure 3.2.

There are two things need to be noted: 1) the real ankle and knee joints consist of complex (partly rolling) contact of one bone surface with another. The single joint center we considered here is a kinematic simplification. 2) the positions of ankle and knee joint, located in static posture, would be used as references to calculate joint centers for dynamic walking data.

Figure 3.2: **Reference frame of static posture.** FA means fore-aft direction, SS means side-to-side direction, and V means vertical direction. Green dots are the reflective markers, blue dots represent the knee and ankle position by averaging the lateral and medial markers. Red dots represent hip joint center from function joint-axis experiment.
3.2.2 Functional joint-axis experiment for hip joint

Beside the ankle and knee joint centers, the hip joint center is another important joint center that needs to be calculated. Using hip bone to physically locate the joint position is difficult and easily misleading for the hip location [9]. There are two ways to find the hip joint centers from the walking data:

i. If the thigh was a rigid body, the distances between thigh markers and hip joint center would be constants. If we know the distances, we can use the distances to calculate hip joint center from thigh markers. Of course, a real thigh is not rigid, so we can minimize the relevant residuals (described below) to find the hip positions. This calculation is independent for each frame of the walking motion.

ii. By treating the distances $l_1, l_2, l_3$ between the three known thigh markers and hip joint center as unknown variables, we could apply optimization to find the hip joint center $(x^{(i)}_{\text{hip}}, y^{(i)}_{\text{hip}}, z^{(i)}_{\text{hip}})$ and the distances $l_1, l_2, l_3$ simultaneously, in order to minimize the error $E$, as stated in following equation:

$$
E = \sum_{k=1}^{3} \sum_{i=1}^{N} ((x^{(i)}_{\text{hip}} - x^{(i)}_k)^2 + (y^{(i)}_{\text{hip}} - y^{(i)}_k)^2 + (z^{(i)}_{\text{hip}} - z^{(i)}_k)^2 - l^2_k)^2 \quad (3.2.1)
$$

where $i$ is the time-frame index, $N$ is the total number of time-frames, $k$ is the index for three markers. This calculation can be performed by using ‘fmincon’ function in MATLAB.

The second method of optimization would generate dynamic hip position using all frames in one calculation. This method does not require any other reference frames, but only the dynamic walking data. However, such method is very time consuming because of the large number of unknown variables (the position of hip at every time
instant). Therefore, in this research, we will apply the first method to find hip joint center.

The first method requires the distances from thigh markers to hip joint center. We apply this so-called “functional joint axis method” to calculate the hip joint centers [92]. This method is described below.

**Finding hip joint centers.** During the functional experiment, the subjects were asked to stand by one foot, and swing the other leg. To help subjects to stabilize, they can hold on other static objects (e.g., chairs) with their hands. Subjects were asked to swing their legs in a “star” pattern as in [92]. For example, the trajectory of one foot marker on the swing leg is given in Figure 3.3a. The leg swung along planes with angles 0, 45, 90, 135, 180 degrees to the sagittal plane and finally went a half circle.

Ideally, if the subject stands perfectly still, by treating the swing thigh as an rigid body, the center of the marker arc will be the hip joint center. But the hip joint is moving anyway during the experiment. Therefore, in order to eliminate the motion, we need to set up a time-varying local coordinate for trunk, and transform all the recorded thigh markers data into the local trunk coordinates. The centers of the localized data give the hip joint centers.

As we noted before, the assumption of a single hip joint center is a simplification. There is no solution if we try to solve hip joint center \((x_{\text{hip}}, y_{\text{hip}}, z_{\text{hip}})\) by the following equations:

\[
(x_{\text{hip}} - x_k^{(i)})^2 + (y_{\text{hip}} - y_k^{(i)})^2 + (z_{\text{hip}} - z_k^{(i)})^2 = l_k^2
\]

(3.2.2)

where \(k = 1, 2, 3\) is the index for thigh marker. And \(i = 1, \cdots, 6\) for any six linearly independent frames. Here we apply optimization algorithms to find the nominal hip joint center \((x_{\text{hip}}, y_{\text{hip}}, z_{\text{hip}})\) by minimizing the same objective function as in Equation 3.2.1. We include all the time-frames in the functional joint axis experiment.
Note that all the data has been transformed into a local coordinate system of the trunk, so that the hip joint center is considered as a fixed point. Figure 3.3a shows the hip joint center as a “center” for the markers data.

By adding the hip joint center, the static “reference” frame is given in Figure 3.2. This frame includes all the marker positions and six joint centers positions. The distances between joint center and markers are constant on each segment. These distances would help to find the dynamic hip joint center for the following walking experiments. The objective equation for the optimization is the same as the Equation 3.2.1, except the distances $l_1, l_2, l_3$ are known variables for this case.

![Diagram showing marker motion and finding hip joint center](image)

Figure 3.3: **Finding the hip joint centers.** FA means fore-aft direction of human body. SS means side-to-side direction of human body. V is the vertical direction. a) The leg is doing “star” diagram motion, with 0, 45, 90, 135, 180 and circular motion. b) Calculate the hip joint center in the trunk coordinate.
3.2.3 Collection of dynamic walking data

During the walking trials, we removed medial markers on both ankles and both knees, so only one marker each was left for each ankle and knee. Because during a dynamic experiment, the medial markers are easily blocked from cameras, the data would be partially missed anyway.

We performed two kinds of walking experiments: steady walking and self-imposed perturbed experiments. **Steady walking** experiment required subject to walk steadily on the treadmill with three chosen speeds: 1.0, 1.2 and 1.4 m/s. These three speeds are generally considered as relative slow, medium and relative fast speeds of human walking. At each speed, the subject was asked to walk without any perturbations. For each trial speed, we recorded 5 minutes data by Vicon motion capture system. During this 5 minutes, the environment in the room has no change, and the subject has no self-imposed perturbation on purpose. So the 5 minutes walking at given speed is called “steady walking”. An example of the raw marker positions for walking experiment is shown in Figure 3.4a. Figure 3.4b shows the positions of the markers in the sagittal plane (the plane containing the upward and forward directions) when a subject was walking.

**Self-imposed perturbed experiment** was designed as a compliment to the steady walking experiment. We chose the medium 1.2 m/s speed out of the three speeds for this experiment. Here, we do not apply any external perturbations. Instead we ask subjects to deviate from steady state on their own (for example, taking a single large step) and then naturally come back to steady state. The timing of self-imposed perturbation is entirely controlled by the subjects. In other words, the subject will apply the perturbation whenever he/she feels comfortable to do so. Three kinds of self-imposed perturbation were examined:

i. Large step self-imposed perturbation: after the subject in his “steady” walking,
a) markers placement for walking

b) steady human walking

Figure 3.4: **Raw marker positions captured by motion capture system.** A total of 25 markers were placed on lower body of the subject. Each segment has three non-collinear markers. There are seven segments (e.g., left foot, left shank), which use 21 markers. The other four markers are placed on the joint positions (i.e., knees and ankles). Red markers are placed on right leg, blue markers are on left leg and dark green makers are placed on the trunk.

he would take a larger step whenever he wanted to. After the large step, the subject was asked to walk “steady” again.

ii. Small step self-imposed perturbation: similar to large step perturbation, we asked the subject to take a small step instead during “steady” walking.

iii. Walking back and forth on the treadmill: we asked the subject to walk back and forth on the treadmill. It is equivalent as a speed change of walking.

As noted earlier, the motion capture data we obtained is high-dimensional, 25 markers × 3 dimensions = 75 dimensional vector at 100 Hz. In the analysis in the following chapters, we will build different models to analyze the steady walking and the perturbed walking data. The “joint centers” are the critical variables to show the dynamics.
### 3.2.4 Measurement errors

There are two types of measurement errors: i) motion capture system measurement error, ii) errors from soft issue movement. The maximum errors are given in Table 3.1.

The motion capture system error is no larger than 0.1 mm for each marker, and it only depends on system calibration and resolution. This error was estimated by looking at the distance fluctuations between two markers on a right wand. Compared to other error sources below, this error can be safely ignored.

The second error comes from soft tissue movement. As we noted earlier, we treat all segments (e.g., tibia, thigh) as rigid bodies. There are no relative motions for markers, if they are on the rigid body. However, the markers move relatively to each other because of tissue movement during walking. The errors of soft tissue movement can be measured by the distance changes between two markers on one segment. The maximum soft tissue movement happens on the thigh markers, resulting in a maximum of 15 mm changes in distance between markers during walking.

In addition, the simplification from bone surface contact to idealized joint center also brings errors in data analysis. When we consider legs are moving in the fore-aft plane as rigid bodies, orientation (angle) for each segment (e.g., tibia and thigh) would be one of the most important variables to describe human walking.

<table>
<thead>
<tr>
<th>Type of errors</th>
<th>maximum value</th>
</tr>
</thead>
<tbody>
<tr>
<td>mocap measurement error, mm</td>
<td>0.1-0.5</td>
</tr>
<tr>
<td>error from soft tissue, mm</td>
<td>15</td>
</tr>
<tr>
<td>error for joint center distances, mm</td>
<td>25</td>
</tr>
<tr>
<td>angle error, degree</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Table 3.1: Measurement errors
angles are calculated from the joint center positions. As a result of measurement errors and simplification errors, we estimated the maximum error for joint center position to be about 25 mm. The resulting maximum angle error can be calculated by \( \tan^{-1}\left(\frac{\text{joint center error}}{\text{segment length}}\right) \), the maximum value for the angle errors is 3.5 degrees. Note that this is assuming the unlikely worst-case scenario of joint center error being perpendicular to the segment length (whereas they are actually close to being aligned, so the typical error may be much smaller). This error becomes a bit smaller when we use three markers with an optimization, and most of this error is likely to be systematic rather than random errors (as they are dynamical in origin). This (approximate) maximum error is much smaller than the thigh segment angle range of 50 degrees.

Notice that both the soft-tissue and the joint center errors are typically slow-varying, periodic and systematic, rather than noisy and random. For our analysis in later chapters, we need that the random noisy errors be much smaller than the stride to stride variability (namely the width of the periodic band). The random errors mainly from mocap errors might be only a degree, whereas the stride to stride range of thigh angle variation is about 6 degrees, at the phase at which this range is minimum. That this variation is likely to be much larger than the random component in our angle estimates means that the joint angle variation has some dynamical information.
CHAPTER 4
TOP VIEW DYNAMICS

In this chapter, we analyze human walking by looking at the top view dynamics. In this top view dynamics, we only consider the horizontal position of the foot placements, and the horizontal position and yaw orientation of the upper body. We first introduce the Poincare sections for the top view dynamics and generate return maps for the repeated walking strides. We examine properties of these maps, in particular, show how people adjust their next foot positions based on their upper body positions and velocities. Finally, the data-derived dynamical model is adopted to make predictions of blind-folded walking, to show how people would walk if there is no external visual reference.

4.1 Step-to-step model

The step-to-step model describes human walking by using foot placements and upper body/trunk movements. This model ignores details of the leg motions. The upper body/trunk is simplified as a planar rigid body. Thus, there are nine variables in this model: left foot placements, \( x_{lf}^{(i)}, y_{lf}^{(i)} \), right foot placements, \( x_{rf}^{(i)}, y_{rf}^{(i)} \), body yaw angle (in the horizontal plane) \( \theta \), body positions \( x_b, y_b \), body velocities \( \dot{x}_b, \dot{y}_b \), see Figure 4.1. The displacement \( x \) is in the walking direction (fore-aft direction). The displacement \( y \) is in the lateral direction (side-to-side direction). Here, \( i \) is the index
of walking strides. \( \ell f \) means the left foot, \( rf \) means the right foot, and \( b \) represents the upper body. From the walking data, we use ankle positions to represent foot positions. Body yaw angles, displacements and velocities are calculated from three trunk markers as shown in Figure 3.2.

4.2 Defining the Poincare sections

In the step-to-step model, body yaw angle \( (\theta) \), body displacement \( (x_b, y_b) \), and velocity \( (\dot{x}_b, \dot{y}_b) \) are considered as continuous changing variables. The foot placements are considered as discrete variables, from foot positions on the treadmill to foot positions
on the treadmill. We discretize the continuous steady walking by using two Poincare sections. The first Poincare section is defined as when the left foot and upper body have the equal fore-aft position, i.e., \( y_{lf}^{(i)} = y_b \), during left foot stance. The second Poincare section is defined when as the right foot and upper body has the equal fore-aft position, i.e., \( y_{rf}^{(i)} = y_b \), during right foot stance.

We can describe the steady walking in one single return map by using first Poincare section (similar to Equation B.2.2) as:

\[
P^{(i)}_{1 \rightarrow 1}: (x_{lf}^{(i)}, y_{lf}^{(i)}, \theta_1^{(i)}, x_{b1}^{(i)}, \dot{x}_{b1}^{(i)}, \dot{y}_{b1}^{(i)}) \rightarrow (x_{lf}^{(i+1)}, y_{lf}^{(i+1)}, \theta_1^{(i+1)}, x_{b1}^{(i+1)}, \dot{x}_{b1}^{(i+1)}, \dot{y}_{b1}^{(i+1)})
\]

(4.2.1)

or by using the second Poincare section (similar to Equation B.2.3) as:

\[
P^{(i)}_{2 \rightarrow 2}: (x_{rf}^{(i)}, y_{rf}^{(i)}, \theta_2^{(i)}, x_{b2}^{(i)}, \dot{x}_{b2}^{(i)}, \dot{y}_{b2}^{(i)}) \rightarrow (x_{rf}^{(i+1)}, y_{rf}^{(i+1)}, \theta_2^{(i+1)}, x_{b2}^{(i+1)}, \dot{x}_{b2}^{(i+1)}, \dot{y}_{b2}^{(i+1)})
\]

(4.2.2)

where integer \( i = 1, 2, \cdots \) is the index of walking strides. The variables in the above mapping (Equations 4.2.1 and 4.2.2) are considered in the coordinate fixed on the treadmill belt. For each of the above mappings, we can build a single mapping mathematical model to represent one walking stride.

Alternatively, we break one stride into two steps: i) from the section of the left foot placement to the section of the right foot placement and ii) then from the section of the right foot back to section of the left foot placement, as shown in Figure 4.2. Two mappings can be proposed as:

\[
P^{(i)}_{1 \rightarrow 2}: (x_{lf}^{(i)}, y_{lf}^{(i)}, \theta_1^{(i)}, x_{b1}^{(i)}, \dot{x}_{b1}^{(i)}, \dot{y}_{b1}^{(i)}) \rightarrow (x_{rf}^{(i)}, y_{rf}^{(i)}, \theta_2^{(i)}, \dot{x}_{b2}^{(i)}, \dot{x}_{b2}^{(i)}, \dot{y}_{b2}^{(i)})
\]

(4.2.3)

and

\[
P^{(i)}_{2 \rightarrow 1}: (x_{rf}^{(i)}, y_{rf}^{(i)}, \theta_2^{(i)}, x_{b2}^{(i)}, \dot{x}_{b2}^{(i)}, \dot{y}_{b2}^{(i)}) \rightarrow (x_{lf}^{(i+1)}, y_{lf}^{(i+1)}, \theta_1^{(i+1)}, x_{b1}^{(i+1)}, \dot{x}_{b1}^{(i+1)}, \dot{y}_{b1}^{(i+1)})
\]

(4.2.4)

where the mapping \( P^{(i)}_{1 \rightarrow 2} \) is from the left foot stance to the right foot stance. The mapping \( P^{(i)}_{2 \rightarrow 1} \) is from the right foot stance back to the left foot stance. The full cycle
of steady walking is a combination of the two mappings in sequence. We ignore all the intermediate data between foot strikes, but only consider the data on the Poincare sections.

In the description so far, we have implicitly considered all variables in some ‘absolute’ coordinates, say fixed to the ground, or fixed to the treadmill belt. Instead, we now consider the step in a ‘body-fixed coordinate’. For each step, the origin point of the body coordinate can be defined on the fixed foot. First, although the data of human walking was collected when subjects walked on the treadmill, we can subtract the treadmill velocity from the walking data. The treadmill belt velocity is known and constant for each steady walking. After subtraction, the foot positions during stance are fixed constants. Figure 4.3a shows the relative right foot positions to the left foot positions, if we place all the left foot position on the origin. Figure 4.3b shows the relative left foot positions to the right foot positions, if we place all the right foot positions on the origin. We describe the positions in a rotated coordinate frame fixed to the yaw orientation of the upper body.

In this upper body-fixed coordinate, the fixed foot placements are known variables. In the following discussions in this chapter, all the variables are considered in the body-fixed coordinate. The first return map of the left foot is simplified as:

\[ P_{1 \rightarrow 1}^{(i)} : (\theta_1^{(i)}, u_b^{(i)}, \dot{u}_b^{(i)}, \ddot{u}_b^{(i)}) \rightarrow (u_{l_f}^{(i+1)}, v_{l_f}^{(i+1)}, \theta_1^{(i+1)}, u_b^{(i+1)}, \dot{u}_b^{(i+1)}, \ddot{u}_b^{(i+1)}) \]  \hspace{1cm} (4.2.5)

and the first return map of the right foot is simplified as:

\[ P_{2 \rightarrow 2}^{(i)} : (\theta_2^{(i)}, u_b^{(i)}, \dot{u}_b^{(i)}, \ddot{u}_b^{(i)}) \rightarrow (u_{r_f}^{(i+1)}, v_{r_f}^{(i+1)}, \theta_2^{(i+1)}, u_b^{(i+1)}, \dot{u}_b^{(i+1)}, \ddot{u}_b^{(i+1)}) \]  \hspace{1cm} (4.2.6)

where \( u, v \) are the side-to-side and fore-aft displacements in the body coordinate respectively. And the mappings for two steps are simplified as:

\[ P_{1 \rightarrow 2}^{(i)} : (\theta_1^{(i)}, u_b^{(i)}, \dot{u}_b^{(i)}, \ddot{u}_b^{(i)}) \rightarrow (u_{r_f}^{(i)}, v_{r_f}^{(i)}, \theta_2^{(i)}, u_b^{(i)}, \dot{u}_b^{(i)}, \ddot{u}_b^{(i)}) \]  \hspace{1cm} (4.2.7)
\[ P^{(i)}_{2→1} : (\theta^{(i)}, u^{(i)}, \dot{u}^{(i)}, v^{(i)}, \dot{v}^{(i)}) \rightarrow (u^{(i+1)}_{lf}, v^{(i+1)}_{lf}, \theta^{(i+1)}, u^{(i+1)}_{b1}, \dot{u}^{(i+1)}_{b1}, \dot{v}^{(i+1)}_{b1}) \]  (4.2.8)

Similar to Equations 4.2.1 and 4.2.2 in the Chapter 2, we can use one return map to construct a linear model as:

\[
\begin{bmatrix}
  u^{(i+1)}_{rf} \\
  v^{(i+1)}_{rf} \\
  \theta^{(i+1)}_{b1} \\
  v^{(i+1)}_{b1} \\
  \dot{u}^{(i+1)}_{b1} \\
  \dot{v}^{(i+1)}_{b1}
\end{bmatrix} = J_{1→1} \begin{bmatrix}
  \theta^{(i)}_1 \\
  u^{(i)}_{b1} \\
  \dot{u}^{(i)}_{b1} \\
  \theta^{(i)}_{b1} \\
  v^{(i)}_{b1} \\
  \dot{v}^{(i)}_{b1}
\end{bmatrix}; \text{ or } \begin{bmatrix}
  u^{(i+1)}_{rf} \\
  v^{(i+1)}_{rf} \\
  \theta^{(i+1)}_2 \\
  u^{(i+1)}_{b2} \\
  \dot{u}^{(i+1)}_{b2} \\
  \dot{v}^{(i+1)}_{b2}
\end{bmatrix} = J_{2→2} \begin{bmatrix}
  \theta^{(i)}_2 \\
  u^{(i)}_{b2} \\
  \dot{u}^{(i)}_{b2} \\
  \theta^{(i)}_{b2} \\
  v^{(i)}_{b2} \\
  \dot{v}^{(i)}_{b2}
\end{bmatrix} \text{ for } i = 1, 2 \cdots (4.2.9)
\]

Similar to univariate mapping Equation 2.2.1 in the Chapter 2, we can use a piecewise model (two linear pieces) to construct the step-to-step model as:

\[
\begin{bmatrix}
  u^{(i)}_{rf} \\
  v^{(i)}_{rf} \\
  \theta^{(i)}_2 \\
  u^{(i)}_{b1} \\
  \dot{u}^{(i)}_{b1} \\
  \dot{v}^{(i)}_{b1}
\end{bmatrix} = J_{1→2} \begin{bmatrix}
  \theta^{(i)}_1 \\
  \dot{v}^{(i)}_{b1} \\
  \dot{u}^{(i)}_{b1} \\
  \theta^{(i+1)}_2 \\
  \dot{u}^{(i+1)}_{b1} \\
  \dot{v}^{(i+1)}_{b1}
\end{bmatrix}; \text{ and } \begin{bmatrix}
  u^{(i+1)}_{rf} \\
  v^{(i+1)}_{rf} \\
  \theta^{(i+1)}_1 \\
  u^{(i+1)}_{b2} \\
  \dot{u}^{(i+1)}_{b2} \\
  \dot{v}^{(i+1)}_{b2}
\end{bmatrix} = J_{2→1} \begin{bmatrix}
  \theta^{(i)}_2 \\
  \dot{u}^{(i)}_{b2} \\
  \dot{v}^{(i)}_{b2} \\
  \theta^{(i+1)}_1 \\
  \dot{u}^{(i+1)}_{b1} \\
  \dot{v}^{(i+1)}_{b1}
\end{bmatrix} \text{ for } i = 1, 2 \cdots (4.2.10)
\]

where \( J_{1→1} \) and \( J_{2→2} \) are 6×4 Jacobian matrices of the return maps for walking by using the left foot position and the right foot position respectively. Jacobian \( J_{1→2} \) is a 6×4 matrix for the first step, from the left foot stance to the right foot stance. Jacobian \( J_{2→1} \) is a 6×4 matrix for the second step, from the right foot stance to the left foot stance.
Figure 4.2: **Walking with foot positions** are shown along with the subject orientation, when the subject is walking towards the right. The coordinate $x,y$ with black is the ground-fixed coordinate. The coordinate $u_i,v_i$ is the local coordinate of the trunk orientation.

Figure 4.3: **Experiment data of relative foot positions** a) the relative right foot positions to the left foot position, b) the relative left foot positions to the right foot position
4.3 Correlation coefficients between the body movements and the foot placements

Before we discuss the Jacobian matrices described in the previous section, we examine the simple “correlation coefficients” between the body movements and the subsequent foot placements. There are four variables considered for the body movements: body yaw angle ($\theta$), body side-to-side position ($x_b$), body side-to-side velocity ($\dot{x}_b$) and body fore-aft velocity ($\dot{y}_b$). Note that the body fore-aft position is not considered, because the Poincare section defined in section 4.2 requires that the body fore-aft position equals to the stance foot position.

Recall that there are five subjects, walking with three different speeds. Thus, there are 15 trials to be analyzed in total. In the Figure 4.4, trial indices 1-3 correspond to the three different speeds (i.e., 1.0 m/s, 1.2 m/s and 1/4 m/s) of subject-1, trial indices 4-6 correspond to subject-2, trial indices 7-9 correspond to subject-3, and so on.

The coefficient of correlation, also known as the correlation coefficient, is used to show the strength of linear dependence between two variables. The range of this coefficient is from -1 to +1. A correlation coefficient of +1 means one variable equals to the other variable scaled by a positive number, and a correlation coefficient of -1 means that one variable equals to the other variable scaled by a negative number.

The coefficients of the correlations between the body movements and the foot placements are separately calculated for two steps: left foot stance and right foot stance. The calculations are performed on the two Poincare sections, as defined in section 4.2.

The coefficients of the correlation between the body yaw angle ($\theta_1$) and the right foot placements ($u_{rf}, v_{rf}$) are given in Figure 4.4a, and the coefficients of correlation between the upper body yaw angle ($\theta_2$) and the right foot placement ($u_{rf}, v_{rf}$) are
Figure 4.4: **Coefficient of correlations.** Correlation coefficients computed with input variables being the body yaw angle, displacement, velocities and output variables being the foot position. For panels (a,c,e,g), the input variables are defined during the left foot stance. For panels (b,d,f,h), the input variables are defined during the right foot stance. The horizontal axis is ‘trial number’ – with a total 15 trials, 3 trials per subject. The two correlation coefficients shown for each trial correspond, respectively, to the ‘SS’ is the side-to-side directional displacement $x$ of the foot position (black dots) and ‘FA’ is the fore-aft directional displacement $y$ of the foot position (green dots).
given in Figure 4.4b. It can be observed that there is no strong correlation indicated between the upper body yaw angle and the foot placement.

For the step of left foot stance, the coefficients of the correlation between the body side-to-side position ($u_{b1}$) and the right foot placement ($u_{rf}, v_{rf}$) are given in Figure 4.4c. The coefficients of the correlation between the body side-to-side position ($u_{b2}$) and the right foot placement ($u_{lf}, v_{lf}$) are given in Figure 4.4d. The values of the dark points are all positive and close to 1. It indicates that the body side-to-side position is strongly positively correlated with the foot side-to-side position. In other words, if the side-to-side distance between the center of mass of body and the standing foot position is larger, the next step of steady walking will have larger side-to-side distance. The body side-to-side position has no strong effect on the fore-aft step length, as we can see from the Figures 4.4c and 4.4d.

The coefficients of correlation between the body side-to-side velocity ($\dot{u}_b$) and the foot placements ($u_{rf}, v_{rf}$) are given in Figures 4.4e and 4.4f. There is no strong evidence to show the correlation.

The coefficients of correlation between the body fore-aft velocity ($\dot{v}_b$) and the foot placements ($u_{rf}, v_{rf}$) are given in Figures 4.4g and 4.4h. There is a weak positive correlation between the body fore-aft velocity ($\dot{v}_b$) and the fore-aft step length ($u_{rf}$), since the coefficients of the correlations between these two are all above zero. It indicates that the larger body fore-aft velocity may cause larger fore-aft step length.

4.4 Relation (Jacobians) between the body movements and the foot placements

The estimates for the relations. We can estimate the values of the relation between the body parameters and the foot placements. We use the mapping in the
Equation 4.2.10 and apply the least square method to estimate the matrices $J_{1-2}$ and $J_{2-1}$ from the human walking data. The dimensions of matrices $J_{1-2}$ and $J_{2-1}$ are $6 \times 4$. The Jacobian matrix $J_{1-2}$ is given as:

$$
J_{1-2} = 
\begin{bmatrix}
\frac{\partial u_f^{(i)}}{\partial \theta_f^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_f^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_f^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_f^{(i)}} \\
\frac{\partial u_f^{(i)}}{\partial \theta_1^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_1^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_1^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_1^{(i)}} \\
\frac{\partial u_f^{(i)}}{\partial \theta_2^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_2^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_2^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_2^{(i)}} \\
\frac{\partial u_f^{(i)}}{\partial \theta_3^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_3^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_3^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_3^{(i)}} \\
\frac{\partial u_f^{(i)}}{\partial \theta_4^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_4^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_4^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_4^{(i)}} \\
\frac{\partial u_f^{(i)}}{\partial \theta_5^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_5^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_5^{(i)}} & \frac{\partial u_f^{(i)}}{\partial \theta_5^{(i)}} \\
\end{bmatrix}
$$

(4.4.1)

The Jacobian matrix $J_{2-1}$ is given as:

$$
J_{2-1} = 
\begin{bmatrix}
\frac{\partial u_f^{(i+1)}}{\partial \theta_f^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_f^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_f^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_f^{(i+1)}} \\
\frac{\partial u_f^{(i+1)}}{\partial \theta_1^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_1^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_1^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_1^{(i+1)}} \\
\frac{\partial u_f^{(i+1)}}{\partial \theta_2^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_2^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_2^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_2^{(i+1)}} \\
\frac{\partial u_f^{(i+1)}}{\partial \theta_3^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_3^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_3^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_3^{(i+1)}} \\
\frac{\partial u_f^{(i+1)}}{\partial \theta_4^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_4^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_4^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_4^{(i+1)}} \\
\frac{\partial u_f^{(i+1)}}{\partial \theta_5^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_5^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_5^{(i+1)}} & \frac{\partial u_f^{(i+1)}}{\partial \theta_5^{(i+1)}} \\
\end{bmatrix}
$$

(4.4.2)

The parameters in the first two rows show the relations between body parameters and foot placements. The whole matrices $J_{1-2}$ and $J_{2-1}$ will be estimated, but only the coefficients in the first two rows are plotted in the following figures (Figures 4.5, 4.6, 4.7, 4.8) and discussions. We plot the distribution of estimates for each coefficient, based on bootstrap resampling as described below.
Figure 4.5: **Distribution of the estimated relation between the body yaw angles and the next step’s foot placements.** The same notations are used: FA - fore-aft; SS - side-to-side. Body yaw angle represents the upper body angle in the top view dynamics. The unit for body yaw angle is degree, and the unit for foot placement is millimeter.
Figure 4.6: **Distribution of the estimated relation between the body side-to-side position and the next step’s foot placements.** The units for both body position and foot placement are millimeter. It shows that the side-to-side body position and the side-to-side foot placement are strongly positively related.
Figure 4.7: Distribution of the estimated relation between the body side-to-side velocity and the next step’s foot placements. The unit for body side-to-side velocity is millimeter per second. It shows that the side-to-side body velocity and the side-to-side foot placement are weakly positively related, and the side-to-side body velocity and the fore-aft foot placement are weakly negatively related.
Figure 4.8: **Distribution of the estimated relation between the body fore-aft velocity and the next step’s foot placements.** The unit for body fore-aft velocity is millimeter per second. The unit for the foot placements is millimeter. The figure indicates that the fore-aft body velocity and the fore-aft foot placement are weakly positively related.
**Bootstrapping.** The human walking data has a lot of noise. The Jacobian matrices $J_{1\rightarrow2}$ and $J_{2\rightarrow1}$ are estimated from this noisy data. Hence we have to estimate the errors in the Jacobian matrices. One way of doing this is called “bootstrapping”, where we “re-sample” from the data with replacement to generate new data sets that have the same lengths as the data sets. We recompute the Jacobians $J_{1\rightarrow2}$ and $J_{2\rightarrow1}$ for each of the bootstrap samples. The distribution of recomputed Jacobians gives an estimate of the variances/errors in the estimates of $J_{1\rightarrow2}$ and $J_{2\rightarrow1}$. In the following results, we apply bootstrapping method to randomly re-sample 1000 times from the existing experiment data.

Here, we take one trial from one subject, walked with the speed of 1.2 m/s, as an example to demonstrate the results of bootstrapping. The distribution of the estimate of the relation between the body yaw angle and the foot placements is given in Figure 4.5. The distribution covers zero. It means that there is no evidence to show the effect of body yaw angle on foot placements.

The distribution of the estimate of the relation between the body side-to-side position and the foot placement is given in Figure 4.6. Clearly, the value of the relation between the body side-to-side position and the foot side-to-side placement is positive and the distribution is far from zero. It indicates a strong positive relation between the body side-to-side position and the foot side-to-side placement. It indicates that a large body side-to-side position very likely leads to a large side-to-side step length. The conclusion is the same as the observation in the calculation of the coefficients of correlations in the Figure 4.4.

Similarly, the estimates of the effects of the body velocities on the foot placements are given in Figures 4.7 and 4.8. We can observe the positive relations i) between the body side-to-side velocity and the foot side-to-side position, and ii) between the body fore-aft velocity and the foot fore-aft position. The relations are obvious but not as
strong as the relation between the body side-to-side position and the foot side-to-side placement. We can also see a negative relation between the body side-to-side velocity and the foot fore-aft position. It means that if we have a larger side-to-side velocity on body, it may lead to a larger side-to-side step length, but a smaller fore-aft step length.

Recall that there are 5 subjects. We collect walking data at three different speeds for each subject. For each subject at each speed, we apply bootstrapping method to randomly re-sample 1000 times. Merging all the bootstrapping samples, the overall distribution is given in Figure 4.9.

In general, if the distribution of the estimate is positively/negatively away from zero, it indicates a positive/negative relation. If the distribution is centered on zero, there is no relation indicated. To quantify how far the distribution from zero, the ratio between mean and standard deviation of the distribution is introduced. After assembling all the 15 trials walking data together, the ratios are summarized in Table 4.1. There are several observations need to be noted:

i. The mapping from the body side-to-side position to the next step’s side-to-side position has the largest mean-to-standard-deviation ratio (8.6). It is a strong evidence to show that the large body side-to-side position leads to the large side-to-side length.

ii. The mapping from the body side-to-side velocity to the next step’s side-to-side position has a positive mean-to-standard-deviation ratio (2.8). The mapping from the body fore-aft position to the next step’s fore-aft position has a positive mean-to-standard-deviation ratio (2.2). It means that the large side-to-side/fore-aft velocity likely leads to the large side-to-side/fore-aft step length.

iii. The mapping from the body side-to-side position to the next step’s fore-aft
Foot SS placement | Foot FA placement
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Body yaw angle</td>
<td>-0.8</td>
</tr>
<tr>
<td>Body SS position</td>
<td>8.6</td>
</tr>
<tr>
<td>Body SS velocity</td>
<td>2.8</td>
</tr>
<tr>
<td>Body FA velocity</td>
<td>-0.4</td>
</tr>
</tbody>
</table>

Table 4.1: The ratio between mean and standard distribution for the distributions of the estimates of the relation between the body movement and the foot placement.

position has a negative mean-to-standard-deviation ratio (-1.3). The mapping from the body side-to-side velocity to the next step’s fore-aft position also has a negative mean-to-standard-deviation ratio (-1.2). It indicates that the large side-to-side position or velocity causes the small fore-aft step length.

### 4.5 Stability and return map eigenvalues

We know that healthy human walking is stable, because people walk without falling down. Here, we wish to quantify the stability of human walking by applying the inferred Jacobians. The stability of the top view dynamics is calculated in two ways: i) by using the data of foot placements, and ii) by using the data of upper body motion. The details are described in the below sections. To be clear, because the results below are based on Jacobians with a reduced number of states, not necessarily capturing all the important state variables, the following should only be considered exploratory analysis too see if there might be meaningful patterns.
Figure 4.9: Distribution of the estimated relation between the body movements and the next step’s foot placements for 5 subjects. For each subject at each speed, there are 1000 bootstraps samples. These plots include the data for 5 subjects. Each subject has three trials for three speeds. The ratios between mean and standard deviation are calculated based on the distribution and given in Table 4.1.
4.5.1 Stability eigenvalues from the foot placements

One way to analyze human walking is to look only at the foot placements. The left foot position are denoted as \( x^{(i)}_{\ell f}, y^{(i)}_{\ell f} \). The right foot position are denoted as \( x^{(i)}_{rf}, y^{(i)}_{rf} \), where \( i \) is the index for strides. The first return map of the left foot position can be defined as:

\[
P^{(i)}_{1-1} : (x^{(i)}_{\ell f}, y^{(i)}_{\ell f}) \rightarrow (x^{(i+1)}_{\ell f}, y^{(i+1)}_{\ell f})
\]  

(4.5.1)

and the first return map of the right foot position is defined as:

\[
P^{(i)}_{2-2} : (x^{(i)}_{rf}, y^{(i)}_{rf}) \rightarrow (x^{(i+1)}_{rf}, y^{(i+1)}_{rf})
\]  

(4.5.2)

The Jacobian of the above first return maps are defined as:

\[
J^{(i)}_{1-1} = \begin{bmatrix}
\frac{\partial x^{(i+1)}_{\ell f}}{\partial x^{(i)}_{\ell f}} & \frac{\partial x^{(i+1)}_{\ell f}}{\partial y^{(i)}_{\ell f}} \\
\frac{\partial y^{(i+1)}_{\ell f}}{\partial x^{(i)}_{\ell f}} & \frac{\partial y^{(i+1)}_{\ell f}}{\partial y^{(i)}_{\ell f}}
\end{bmatrix}
\]

(4.5.3)

and

\[
J^{(i)}_{2-2} = \begin{bmatrix}
\frac{\partial x^{(i+1)}_{rf}}{\partial x^{(i)}_{rf}} & \frac{\partial x^{(i+1)}_{rf}}{\partial y^{(i)}_{rf}} \\
\frac{\partial y^{(i+1)}_{rf}}{\partial x^{(i)}_{rf}} & \frac{\partial y^{(i+1)}_{rf}}{\partial y^{(i)}_{rf}}
\end{bmatrix}
\]

(4.5.4)

If we consider the strides as two steps, we have two mappings, one is from the left foot stance to the right foot stance:

\[
P^{(i)}_{1-2} : (x^{(i)}_{\ell f}, y^{(i)}_{\ell f}) \rightarrow (x^{(i)}_{rf}, y^{(i)}_{rf})
\]

(4.5.5)

another mapping is the right foot stance to the left foot stance:

\[
P^{(i)}_{2-1} : (x^{(i)}_{rf}, y^{(i)}_{rf}) \rightarrow (x^{(i+1)}_{\ell f}, y^{(i+1)}_{\ell f})
\]

(4.5.6)
The Jacobian for two steps are defined as:

\[
J_{1\rightarrow 2} = \begin{bmatrix}
\frac{\partial x_r^{(i)}}{\partial x_f^{(i)}} & \frac{\partial x_r^{(i)}}{\partial y_f^{(i)}} \\
\frac{\partial x_c^{(i)}}{\partial y_f^{(i)}} & \frac{\partial y_r^{(i)}}{\partial y_f^{(i)}} \\
\frac{\partial y_r^{(i)}}{\partial x_f^{(i)}} & \frac{\partial y_r^{(i)}}{\partial y_f^{(i)}} \\
\frac{\partial y_c^{(i)}}{\partial y_f^{(i)}} & \frac{\partial y_c^{(i)}}{\partial y_f^{(i)}}
\end{bmatrix}
\quad \text{and} \quad
J_{2\rightarrow 1} = \begin{bmatrix}
\frac{\partial x_r^{(i+1)}}{\partial x_f^{(i)}} & \frac{\partial x_r^{(i+1)}}{\partial y_f^{(i)}} \\
\frac{\partial x_c^{(i+1)}}{\partial y_f^{(i)}} & \frac{\partial y_r^{(i+1)}}{\partial y_f^{(i)}} \\
\frac{\partial y_r^{(i+1)}}{\partial x_f^{(i)}} & \frac{\partial y_r^{(i+1)}}{\partial y_f^{(i)}} \\
\frac{\partial y_c^{(i+1)}}{\partial y_f^{(i)}} & \frac{\partial y_c^{(i+1)}}{\partial y_f^{(i)}}
\end{bmatrix}
\quad (4.5.7)
\]

These two steps in sequence together is a stride. The overall Jacobian matrix of the whole stride is defined as:

\[
J = J_{2\rightarrow 1} \cdot J_{1\rightarrow 2}
\quad (4.5.8)
\]

The eigenvalues of the overall Jacobian \( J \) indicate the stability of the walking. The eigenvalues of Jacobian \( J_{1\rightarrow 2} \) and \( J_{1\rightarrow 2} \) show the stability for the two steps respectively.

By applying the linear model as described in section B.1 in the Chapter 2, eigenvalues can be calculated by using foot placements data, obtained from the steady human walking data. The maximum eigenvalues of the return map are given in Tables 4.2 and 4.3, the values are all less than 1, which indicate the stable human walking. Most of the values are in the range of 0.3 and 0.5. The maximum eigenvalues for all other subjects are close to each other, except that the third subject has relative lower eigenvalues (0.2 to 0.4).

For the different speeds 1.0 m/s, 1.2 m/s and 1.4 m/s , there are no significant differences in maximum eigenvalues, as we can see from Tables 4.2 and 4.3. The reason may be that three walking speeds are close.

The maximum eigenvalues of Jacobian, as the product of \( J_{1\rightarrow 2} \) and \( J_{2\rightarrow 1} \), are given in Table 4.4. We can see that there is a big drop when comparing Table 4.4 to Tables 4.2 and 4.3. From the work in the Chapter 2, the two section mappings of Jacobian \( J \) may have more accurate estimates, than the estimates of one section mapping, \( J_{1\rightarrow 1} \) or \( J_{2\rightarrow 2} \).
Table 4.2: Maximum eigenvalues of Jacobian $J_{1 \rightarrow 1}$ ($\pm$ bootstrap-based standard deviation) by using the left foot positions. The first return map is constructed from one left foot strike to the next left foot strike.

<table>
<thead>
<tr>
<th>Subject</th>
<th>1.0 m/s</th>
<th>1.2 m/s</th>
<th>1.4 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.47 ±0.06</td>
<td>0.73 ±0.05</td>
<td>0.47 ±0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.57 ±0.07</td>
<td>0.67 ±0.06</td>
<td>0.48 ±0.09</td>
</tr>
<tr>
<td>3</td>
<td>0.36 ±0.15</td>
<td>0.48 ±0.09</td>
<td>0.44 ±0.06</td>
</tr>
<tr>
<td>4</td>
<td>0.48 ±0.07</td>
<td>0.52 ±0.07</td>
<td>0.51 ±0.07</td>
</tr>
<tr>
<td>5</td>
<td>0.49 ±0.08</td>
<td>0.47 ±0.09</td>
<td>0.52 ±0.07</td>
</tr>
</tbody>
</table>

Table 4.3: Maximum eigenvalues of Jacobian $J_{2 \rightarrow 2}$ ($\pm$ bootstrap-based standard deviation) by using the right foot positions. The first return map is constructed from one right foot strike to the next right foot strike.

<table>
<thead>
<tr>
<th>Subject</th>
<th>1.0 m/s</th>
<th>1.2 m/s</th>
<th>1.4 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.44 ±0.06</td>
<td>0.73 ±0.05</td>
<td>0.53 ±0.06</td>
</tr>
<tr>
<td>2</td>
<td>0.51 ±0.06</td>
<td>0.58 ±0.06</td>
<td>0.53 ±0.08</td>
</tr>
<tr>
<td>3</td>
<td>0.27 ±0.10</td>
<td>0.27 ±0.08</td>
<td>0.40 ±0.07</td>
</tr>
<tr>
<td>4</td>
<td>0.38 ±0.08</td>
<td>0.49 ±0.08</td>
<td>0.52 ±0.05</td>
</tr>
<tr>
<td>5</td>
<td>0.46 ±0.08</td>
<td>0.54 ±0.07</td>
<td>0.40 ±0.06</td>
</tr>
<tr>
<td>Subject</td>
<td>1.0 m/s</td>
<td>1.2 m/s</td>
<td>1.4 m/s</td>
</tr>
<tr>
<td>---------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>0.11 ±0.05</td>
<td>0.37 ±0.07</td>
<td>0.12 ±0.04</td>
</tr>
<tr>
<td>2</td>
<td>0.16 ±0.05</td>
<td>0.26 ±0.05</td>
<td>0.18 ±0.08</td>
</tr>
<tr>
<td>3</td>
<td>0.42 ±0.10</td>
<td>0.04 ±0.02</td>
<td>0.05 ±0.03</td>
</tr>
<tr>
<td>4</td>
<td>0.04 ±0.02</td>
<td>0.03 ±0.02</td>
<td>0.08 ±0.05</td>
</tr>
<tr>
<td>5</td>
<td>0.05 ±0.03</td>
<td>0.04 ±0.03</td>
<td>0.08 ±0.05</td>
</tr>
</tbody>
</table>

Table 4.4: Maximum eigenvalues of Jacobian $J = J_{1\rightarrow 2} \cdot J_{2\rightarrow 1}$ (± bootstrap-based standard deviation) by using the foot positions, using products of two mappings constructed: i) the left foot strike to the right foot strike $J_{1\rightarrow 2}$ and ii) the right foot strike to the left foot strike $J_{2\rightarrow 1}$.

4.5.2 Stability eigenvalues from the body motions

To compare the eigenvalues estimated by using the foot position, we can examine the eigenvalues of human walking by looking at the body movements. We cut out the variables of the foot placements in Equations 4.2.9 and 4.2.10. Then Jacobians $J_{1\rightarrow 1}, J_{2\rightarrow 2}, J_{1\rightarrow 2}$ and $J_{2\rightarrow 1}$ become $4 \times 4$ matrices. The $J_{1\rightarrow 1}$ are the Jacobians of the first return map for body movement at left foot stance. Jacobian $J_{2\rightarrow 2}$ represent the first return map of right foot stance. The maximum eigenvalues of $J_{1\rightarrow 1}$ and $J_{2\rightarrow 2}$ are given in Tables 4.5 and 4.6 respectively. The maximum eigenvalues of products $J_{2\rightarrow 1} \cdot J_{1\rightarrow 2}$ are given in Table 4.7.

By comparing to the stability results calculated by using foot placements, as given in the Tables 4.2, 4.3 and 4.4, we can observe that the maximum eigenvalues calculated by using the foot placements are similar to that calculated by using the body movements. We see that the maximum eigenvalues of first return map are roughly from 0.3 to 0.5, which are close to the results of stability in the paper [47]. However, the maximum eigenvalues of using two Poincare sections (at the left foot
<table>
<thead>
<tr>
<th>Subject</th>
<th>1.0 m/s</th>
<th>1.2 m/s</th>
<th>1.4 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.40 ±0.07</td>
<td>0.31 ±0.06</td>
<td>0.32 ±0.06</td>
</tr>
<tr>
<td>2</td>
<td>0.29 ±0.06</td>
<td>0.38 ±0.07</td>
<td>0.32 ±0.06</td>
</tr>
<tr>
<td>3</td>
<td>0.66 ±0.09</td>
<td>0.30 ±0.07</td>
<td>0.38 ±0.07</td>
</tr>
<tr>
<td>4</td>
<td>0.30 ±0.07</td>
<td>0.40 ±0.08</td>
<td>0.54 ±0.06</td>
</tr>
<tr>
<td>5</td>
<td>0.33 ±0.08</td>
<td>0.33 ±0.06</td>
<td>0.29 ±0.06</td>
</tr>
</tbody>
</table>

Table 4.5: Maximum eigenvalues of Jacobian $J_{1-1}$ (± bootstrap-based standard deviation) by using the body movements. Here the foot positions are ignored. The first return map is constructed from body movement back to body movement. However, the Poincare section includes the left foot position. The Poincare section is defined when the left foot and upper body have the same fore-aft displacement.

stance and at the right foot stance), are about 0.1. They are much smaller than the eigenvalues of using single Poincare sections.

4.6 Predictions of blind-folded walking

Of course, in our experiments, subjects walked with their eyes open. But we can ask how people might walk when they are blind-folded – having no external references. There is no route designed for people to follow. The experiment of blind-folded walking is found in papers [7, 82].

Here, we want to simulate this blind-folded walking by using the steady walking data. The step-to-step model in section 4.1 is applied.

It should be noted that two steps during one steady walking stride are asymmetric. The differences in stability calculations are already given in Tables 4.2 and 4.3. We can see it more directly from the statistics of the experiment data, see Table 4.8.
<table>
<thead>
<tr>
<th>Subject</th>
<th>1.0 m/s</th>
<th>1.2 m/s</th>
<th>1.4 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.33 ±0.06</td>
<td>0.50 ±0.07</td>
<td>0.36 ±0.06</td>
</tr>
<tr>
<td>2</td>
<td>0.36 ±0.07</td>
<td>0.31 ±0.07</td>
<td>0.34 ±0.06</td>
</tr>
<tr>
<td>3</td>
<td>0.75 ±0.06</td>
<td>0.44 ±0.08</td>
<td>0.46 ±0.07</td>
</tr>
<tr>
<td>4</td>
<td>0.33 ±0.06</td>
<td>0.33 ±0.07</td>
<td>0.45 ±0.06</td>
</tr>
<tr>
<td>5</td>
<td>0.36 ±0.06</td>
<td>0.30 ±0.06</td>
<td>0.27 ±0.06</td>
</tr>
</tbody>
</table>

Table 4.6: Maximum eigenvalues of Jacobian $J_{2\rightarrow2}$ ($\pm$ bootstrap-based standard deviation) by using the body movements. The first return map is constructed from body movement back to body movement, when right foot is on the ground. The Poincare section is defined when the right foot and upper body have the same fore-aft displacement.

<table>
<thead>
<tr>
<th>Subject</th>
<th>1.0 m/s</th>
<th>1.2 m/s</th>
<th>1.4 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.14 ±0.04</td>
<td>0.14 ±0.04</td>
<td>0.16 ±0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.19 ±0.06</td>
<td>0.18 ±0.05</td>
<td>0.18 ±0.05</td>
</tr>
<tr>
<td>3</td>
<td>0.50 ±0.09</td>
<td>0.17 ±0.05</td>
<td>0.20 ±0.04</td>
</tr>
<tr>
<td>4</td>
<td>0.20 ±0.05</td>
<td>0.16 ±0.05</td>
<td>0.31 ±0.06</td>
</tr>
<tr>
<td>5</td>
<td>0.15 ±0.04</td>
<td>0.20 ±0.04</td>
<td>0.14 ±0.04</td>
</tr>
</tbody>
</table>

Table 4.7: Maximum eigenvalues of Jacobian $J_{1\rightarrow2} \cdot J_{2\rightarrow1}$ by using the body movements. The calculation here is similar to the calculation in the Table 4.4, except that we use the upper body data here. There are two Poincare sections defined as: i) left foot stance and ii) right foot stance.
The two steps have different step lengths and variances. The side-to-side step length has a similar mean, because the subject was forced to walking in straight line on the treadmill belt.

A piecewise model can be constructed according to Equation 4.2.10. As we mentioned in section 4.2, there are two pieces in the model: i) from the left foot stance to the right foot stance, and ii) from the right foot stance to the left foot stance. The piecewise model includes the step asymmetry.

Predictions can be made by applying the piecewise model to simulate the top view dynamics of human walking. The predicted walking all started at the original point. The model used the body-fixed coordinates, but the results are given as in the ground-fixed coordinate according to Equation 4.6.1. The transformation from the positions in the body coordinate \( u^{(i)}, v^{(i)} \) to the positions in treadmill coordinate \( x^{(i)}, y^{(i)} \) is:

\[
\begin{bmatrix}
  x^{(i)} \\
  y^{(i)}
\end{bmatrix}
= \begin{bmatrix}
  \cos(\alpha^{(i)}) & -\sin(\alpha^{(i)}) \\
  \sin(\alpha^{(i)}) & \cos(\alpha^{(i)})
\end{bmatrix}
\begin{bmatrix}
  u^{(i)} \\
  v^{(i)}
\end{bmatrix}; \text{ for } i = 1, 2, \cdots \tag{4.6.1}
\]

where \( \alpha^{(i)} \) is the instant ‘absolute’ angle between the body-fixed coordinate and the ground-fixed coordinate. Since there is no external reference, the angle \( \alpha^{(i)} \) between the body-fixed coordinate and the ground-fixed coordinate is an accumulation of upper body yaw angles in the top view dynamics. The person is always stabilizing the yaw relative to the previous step, and not absolute yaw, so this absolute angle \( \alpha \) may be expected to drift around.

The predictions of 100 strides blind walking are given in Figure 4.10a, showing that the model does not walk straight over 100 steps. The side-to-side offset would become more severe over more steps. Figure 4.10b shows two trials of 10000 strides walking. The prediction results are qualitatively similar to the experimental results of blind-folded walking. The long-distance blind-folded walking results, see Figure 4.10b,
SS mean of step 1   SS std of step 1   SS mean of step 2   SS std of step 2
205.31          18.66            -205.48         17.36

FA mean of step 1  FA std of step 1  FA mean of step 2  FA std of step 2
659.95           19.78            621.55          19.40

Table 4.8: An example of the asymmetric steps. The unit is millimeter. “SS” means the side to side direction. “FA” means the fore and after direction. “std” is the standard deviation. Step 1 is for the step when left foot stance, and right foot strikes on the treadmill, and step 2 is the next step, when right foot stances and left foot strikes.

are qualitatively similar to the experiments in paper [82]. The short-distance walking results, see Figure 4.10a, are comparable to the experiments in paper [7], however the real experiment shows twice as much deviation from straight line walking, perhaps due to other sources of noise not modeled here – after all, we should note that our model is built based on straight walking data (on the treadmill), which may not allow sufficient flexibility to express all possible noise sources.
Figure 4.10: **Prediction of foot placement** a) Thirty independent trials of the 100 step blind-folded walking are given. The prediction shows that people can not keep straight walking without any references. b) Two predictions of 10000 steps blind-folded walking are given.
CHAPTER 5
SAGITTAL PLANE DYNAMICS

During walking on the treadmill, the lateral motions are much smaller than the fore-aft motions. Thus, in this chapter, we use a 2-D kinematic model to describe the walking in the sagittal plane (i.e. the vertical plane in the walking direction).

As described in the Chapter 3, using a motion capture system, we have collected data in two types of human walking: steady state walking and walking with self-imposed transients. We extract dynamic information from the steady walking data and construct a piecewise model to simulate human walking using methods from Chapter 2. This model successfully simulates steady walking and compares well with the experiment of perturbed walking, at least qualitatively. To compare with future experiments, we make predictions, based on the model, showing how people would recover from the various perturbations during the steady walking.

5.1 Planar 2D model

From the experiment data, we project the human walking motion onto the sagittal plane, see Figure 3.4b. The planar kinematic model describes human locomotion based only on the joint centers. This model ignores the side-to-side motion for simplicity.

Figure 5.1a shows the planar kinematic model in its sagittal plane. When one foot
Figure 5.1: A planar kinematic model is shown to represent mechanical structure of human walking. a) Tibia and thigh lengths are fixed according to static measurements of the subject. Four degrees of freedom, $\theta_1, \theta_2, \theta_3, \theta_4$, are used to describe the walking; b) The phase portrait projections of tibia angles $\theta_1, \theta_4$ are given to show the nearly periodic motion; c) The phase portrait projections of thigh angles $\theta_2, \theta_3$. The units for all angles are radians. The units for all angular velocity are rad/s.

is in contact with the ground (modeled as a pin joint), the model simplifies human walking into a 8 dimensional phase space, or a 4 degrees of freedom system. The 4 degrees of freedom are the angles for the different segments, i.e., left tibia $\theta_1$, left thigh $\theta_2$, right thigh $\theta_3$ and right tibia $\theta_4$. In this model, the foot motion and upper body motion are ignored.

The nearly periodic motion can be observed in the phase portrait projections of tibia angles $(\theta_1, \theta_3)$ in Figure 5.1b or thigh angles $(\theta_2, \theta_4)$ in Figure 5.1c. Every step is similar to every other step, but different. Note that, a perfect periodic motion will generate a closed periodic orbit in the phase portrait projection, rather than the 'periodic band' shown in Figures 5.1b and 5.1c. The developed data-driven method using factorized Poincare maps, described in the Chapters 2, can be applied to analyze this nearly periodic motion.
5.2 Defining the Poincare sections

The factorized Poincare map will discretize continuous walking into a discrete periodic dynamical system. The Poincare sections are defined according to the description in the section 2.5. The basic steps are described below:

i. We first normalize all steady walking data into the scale of $[-1, 1]$. In the 2-D kinematic model, we have 8 variables: left tibia angle $\theta_1$, left thigh angle $\theta_2$, right thigh angle $\theta_3$, right tibia angle $\theta_4$, left tibia angular velocity $\dot{\theta}_1$, left thigh angular velocity $\dot{\theta}_2$, right thigh angular velocity $\dot{\theta}_3$ and right tibia angular velocity $\dot{\theta}_4$. All of the above 8 variables are normalized. We construct models in this normalized variables space, but re-scale back to original variables when we wish to make biological predictions or comparisons.

ii. From steady walking, we detect when the left foot strikes on the treadmill. The corresponding intersections are considered as the initial data to find the first Poincare section.

iii. The walking period time $\hat{T}^\ast$ can be calculated as the step length divided by the known treadmill speed.

iv. We make the orientation of the first Poincare section close to normal to the periodic orbit. For every point on this initial data (when left foot strikes on the treadmill), we propagate the data a little forward in time, say by $h$, where $h$ is large enough that we do not differentiate noise, but sufficiently smaller than the proposed time interval ($\hat{T}^\ast/M$) between Poincare sections. We defined the first section $S_1$ using the estimated tangent to the trajectory of walking data, as given in Figure 5.2.

v. Propagate the data forward in time by $T^\ast/M$. Note that, the number of sections
$M$ can be an arbitrary integer. We obtain a cluster of points, which will be the rough location of the next Poincare section, $S_2$.

vi. Repeat steps iv-v, we can find the all Poincare sections $S_3, S_4, \cdots$. See Figure 5.2.

In the Figure 5.2, the cluster of points (yellow) corresponding to the first five sections of an example with $M = 10$ is shown using projections of the state space. The cluster of intersections on each Poincare section are all respectively on the plane perpendicular to the trajectories of walking data. Note that the cluster does not look perpendicular to the trajectories only because these are low dimensional projections.

If we take the average of the data on each Poincare section, the discrete average points on all sections represent the nominal periodic motion. The nominal periodic motion is given as the dashed dark line in the Figure 5.2. The nominal periodic motion is perfectly periodic, in which every stride repeats every other stride. Note that, although the number of Poincare sections can be defined arbitrarily, the nominal periodic motion, computed as the average of data on each Poincare section, does not really change – with higher $M$, the nominal motion is simply defined with more grid points.

The mapping from one Poincare section $S_j$ to next Poincare section $S_{j+1}$ is:

$$P_{j\rightarrow j+1} : \Theta^{(i)}_j \rightarrow \Theta^{(i)}_{j+1}$$

where $\Theta^{(i)}_j = [\theta^{(i)}_{1j}, \theta^{(i)}_{2j}, \theta^{(i)}_{3j}, \theta^{(i)}_{4j}, \dot{\theta}^{(i)}_{1j}, \dot{\theta}^{(i)}_{2j}, \dot{\theta}^{(i)}_{3j}, \dot{\theta}^{(i)}_{4j}]$. Integer $i = 1, 2, \cdots$ is the index for the strides of human walking, and $j = 1, 2, \cdots, M$ is the index for the Poincare section. The corresponding linear model can be constructed as:

$$\Theta^{(i)}_{j+1} - \Theta^{\ast}_{j+1} = J_{j\rightarrow j+1}(\Theta^{(i)}_j - \Theta^{\ast}_j)$$

where $\Theta^{\ast}_{j+1}$ and $\Theta^{\ast}_j$ are the averages of data on Poincare sections $j$ and $j+1$ respectively. $J_{j\rightarrow j+1}$ is the Jacobian matrix. Using linear least squares as detailed in chapter 2, we can estimate the various Jacobian matrices.
After computing the intersections of the long time-series orbit with the Poincare sections, we construct the piecewise mathematical model to represent the steady walking, according to the sequence of $P_{1\rightarrow 2}, P_{2\rightarrow 3}, \cdots, P_{M\rightarrow 1}$ in Equation 5.2.1.

5.3 Comparison between simulation and experimental perturbation

Once we have derived the mappings between the Poincare sections, we can make predictions regarding the transients from perturbed initial conditions. Among the walking data under self-imposed perturbations, the ‘recovery from large step’ has a clear walking adjustment under perturbations compared to steady walking. Thus, we choose the self-imposed large step walking as an example, to show the agreement between simulation of the developed method and the experimental data.

Recovery from large step. The transient corresponding to the recovery from a large step walking is shown on each joint angle’s phase portrait projection, given as the cyan lines in the Figure 5.3. It can be observed that, as a result of the large step, the left and right thighs are significantly away from the steady walking data (periodic band).

The initial condition for the transient walking simulation is depicted as yellow points in the Figure 5.3. The initial point is picked from roughly the middle of the actual large step. By using an initial condition from the measured transient, we can compare directly with the model-based predictions. The predictions of the piecewise model are given in black lines in Figure 5.3.

The comparisons between of time history model-based predictions and true experiment data are presented in Figure 5.4. First, it can be observed that during recovery
Figure 5.2: **Poincare sections and the nominal periodic motion.** The first Poincare section comes from the left foot striking on the ground during every stride. All Poincare sections (point clusters in yellow) are roughly perpendicular to the trajectory of walking data. Nominal periodic motion is the dashed line inside the nearly periodic band. It is found by averaging the intersection data on all Poincare sections. The unit for all angles is radians. The unit for all angular velocity is rad/s.
Figure 5.3: **Projections of the phase portrait to show the comparison between the prediction and perturbed walking data.** I.C. means the initial condition. The periodic bands represent the steady walking data. Cyan lines show the experimental (self-imposed) large step walking transient, and then back to steady walking. In this transient, the phase portrait projections for thigh angle of the large step walking are significantly away from periodic band (steady walking) in panels c and d. We can see that after the large step, the subject mostly recover in one step. The yellow point is picked as the initial condition for prediction. The predictions, from the developed factorized Poincare map are given in black lines. The unit for all angles is radians. The unit for all angular velocity is rad/s.
Figure 5.4: **Comparisons of all variables in time domain.** The red lines are for transient recovery from a self-imposed large step. The green lines are the predicted responses from our factorized Poincare map model. The initial condition for the predictions was taken to be equal to the initial state of the experimental data.
step, the transient responses of the left and right thigh angles have the largest difference to the thigh angles ($\theta_2, \theta_4$) in the steady walking, while the transient response of tibia angles ($\theta_1, \theta_3$) has a smaller difference to steady walking. In general, the phase and amplitude between simulation and experiment all match with each other qualitatively, but we do see some quantitative differences.

We wish to explore the prediction and the data from a statistical perspective. We choose the left and thigh angles to show the relation between steady walking, perturbed walking data, and the predicted transient. The comparison is given in the Figure 5.5. The histogram plot shows the distribution of the left and right thigh angle data in steady state for each Poincare section. They look roughly like Gaussian distributions. Cyan bars are for the experimental large step transient. Red bars represent for the transient predictions of the piecewise model. Two things can be observed: i) the predictions follow the experimental data qualitatively, and ii) both prediction and experimental data are systematically bounded away from the distribution of steady walking data. In other words, the predicted transient explains the measured transient better than ‘natural steady state variability’ (even if not perfectly).

5.4 Eigenvalues of the Poincare map

We estimate the eigenvalues of the Poincare map by using i) one Poincare section and ii) multiple sections. The one section Poincare mapping uses only the data on one Poincare section. We estimate Jacobian from the data on the section according to Equation B.1.5. Such one Poincare section can be defined anywhere on the nearly periodic orbit. So, we defined 36 Poincare sections evenly distributed on the nearly periodic orbit and estimate the Jacobian on each Poincare section separately. Figure 5.6a shows the maximum amplitude of the eigenvalues of Jacobian matrix on the 36 different Poincare sections. We see a large variation of the maximum amplitudes.
Figure 5.5: **Histogram of predictions of the left and right thigh angles.** The histogram plot shows the distribution of the left and right thigh angle data in steady state for each Poincare section. Cyan bars are for the experimental large step transient. Red bars represent for the transient predictions of the piecewise model.
This variation is due to the noise. The average maximum amplitude of eigenvalues is around 0.6.

Alternatively, we can use multiple sections to estimate the Poincare map. We estimate local Jacobians according to Equation 5.2.2 and calculate the global Jacobian. The maximum amplitude of eigenvalues by using multiple Poincare sections is given in Figure 5.6b. We observe an abrupt drop of the maximum amplitude of eigenvalues when we increase the number of sections from 1 to 5. Then the maximum amplitude is approximate 0.1 for the large number of Poincare sections. The maximum amplitude (0.1) of the multiple Poincare sections is significantly smaller than the maximum amplitude (0.6) of one section Poincare mapping.

The results are consistent with maximum eigenvalues in previous chapter, by using the foot placements and the body movements in the step-to-step model. According to work done in the Chapter 2, the multiple Poincare sections provides a more reliable estimate of the Jacobian – on the other hand, see appendix for situations in which using multiple Poincare sections could be less accurate. An eigenvalue of 0.1 means that a perturbation reduces to one tenth of its size in one walking cycle. While this low eigenvalue may be close to reality if most of the perturbation decays in a couple of steps, it is not clear what the source of the large discrepancy between single section and multiple section based eigenvalues.

5.5 Transient responses to different types of predictions

To compare with future experiments, we make predictions, based on the 2D kinematic model, showing how people would recover from the various perturbations during the steady walking. Transient predictions of human walking under various perturbations can be achieved using the piecewise model computed in section 5.2. The dynamics evolves to steady state according to Equation 5.2.2.
Below, we examine briefly what transients are predicted if perturbations are applied to the stance leg or the swing leg.

**Perturbation on swing leg.** A perfect periodic walking is given in Figure 5.7a. The periodic walk is simulated according to the nominal periodic motion in the Figure 5.2. Figure 5.7b predicts transient responses of human walking after a forward push on the swing leg. A larger step is observed after the forward push. Figure 5.7c predicts transient response of human walking after a backward push on the swing leg, which results in a smaller step.

**Perturbation on stance leg.** The predictions of applying perturbation on the stance leg are given in Figure 5.8. The results are similar to the prediction results of the perturbations on the swing leg. Figure 5.8a is the repeat of Figure 5.7a, in order to show the difference between the walking under perturbation on the stance legs and
the normal walking. Figure 5.8b is the predicted walking under a forward push on
the stance leg and Figure 5.8c is the predicted walking under a backward push on
the stance leg. The forward push leads to a larger step and the backward push leads
to a smaller step.

In these predictions, we observe that walking people (at least according to our
inferred model), under either forward push or backward push, can adjust themselves
very quickly in nearly one step by changing step length. Those observations of quick
step length adjustment are consistent with our anecdotal experiences.

Note that these predictions of responses to perturbations were obtained without
performing any perturbation experiments on human subjects. Rather, the dynam-
ics near the periodic orbit was inferred by using noisy nearly periodic steady state
experimental data in the absence of any explicit external perturbations.
Figure 5.7: Prediction of transient human walking perturbed on swing leg. RS: right leg stance; LS : left leg stance. The yellow lines represent the right legs, and purple lines stand for the left legs. The dark green lines mean where we apply the perturbation (forward/backward push on the swing leg). a) shows a perfect normal walk, which obtained with the nominal periodic motion. b) shows a transient response with a forward push on swing leg. We observed a larger step after the push as we expected; c) shows a transient response with a backward push on the swing leg, which results a smaller step.
Figure 5.8: Prediction of transient human walking perturbed on stance leg. RS: right leg stance; LS: left leg stance. The yellow lines represent right legs, and purple lines stand for left legs. The dark green lines mean where we apply the perturbation (forward/backward push on the swing leg). a) shows a perfect normal walk for reference, same as Figure 5.7. b) shows a transient response with a forward push on stance leg. We observed a larger step after the push as we expected; c) shows a transient response with a backward push on the stance leg, which results a smaller step.
CHAPTER 6
CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

In this thesis, we have made several contributions in analyzing human walking, broadly divided into two categories: (1) general mathematical/computational methods (2) analysis of human data and biological inferences.

Contributions to mathematical and computational methods.

i. To analyze the dynamics near a periodic motion, we discussed the procedure to develop a patchwork of finitely many discrete dynamical systems that we called **factorized Poincare maps**. The Poincare map is a standard technique in the analysis of periodic motions, which reduces the continuous-time dynamics to a discrete dynamical system [32, 88, 100]. By using a large finite number of Poincare sections, one can arbitrarily closely approximate the continuous nonlinear dynamics around the periodic orbit with a patchwork discrete model.

ii. We developed a data-driven method to analyze nearly periodic motion driven by noisy perturbations. Our data-driven method uses the factorized Poincare maps, in which we can discretize continuous periodic or nearly periodic motion using a sequence of Poincare sections – building on previous articles in this area [47, 20, 75] which (mostly) only considered one Poincare section at a time,
independent of others. Under certain conditions, we proved that the multiple Poincare sections can generate the estimations with less variances than the estimations from one Poincare section. We discussed the various computational and statistical issues in the context of univariate inference, showing how to improve accuracy and consistency of the statistical estimates, and then also discussed the multivariate context. We verified the method by applying to synthetic data of noise-driven van der Pol oscillator, both one DOF and multi-DOF coupled oscillators. The estimate of the eigenvalues and their variances were found to be consistent with that suggested by theory. Once a model is created, we can simulate transient and steady responses using the model, and these simulations from the developed method agreed to the true response of van der Pol oscillator.

**Contributions to human movement.** We collected human walking data by using the marker-based motion capture system. We reviewed and showed how we obtained position and orientation of the body from 3D position of the various markers on the body. The worst case measurement errors and errors in the estimated joint angles were calculated, and shown to be smaller than natural joint variability during walking.

i. We analyzed the top-view dynamics of collected human walking data, and computed a step-to-step model. A mapping from body movement to the next step’s foot position was computed. We can see a very strong positive relation between the body side-to-side position and the foot side-to-side position. That is, when the side-to-side body position is higher than usual, humans took a larger sideways step to correct the deviation. Positive relations also exist i) between the body side-to-side velocity and the foot side-to-side position, and ii) between the
body fore-after velocity and the foot fore-after position. Moreover, the body side-to-side position is negatively related to the foot fore-aft position.

ii. We predicted the paths of people walking when blind-folded by using the step to step model, which seems to show less deviation from straight line walking than that that in observed in previous experiments, suggesting that other factors need to be taken into account.

iii. We considered the dynamics of human walking in the sagittal plane (forward walking direction). From the steady walking, we extracted the dynamic walking information, including nominal periodic walking and stability information. A piecewise model was constructed based on the factorized Poincare sections approach. We simulated transients to steady state using this model, and these transients compared well with the measured self-imposed transients.

iv. Further, we examined the perturbed walking under two different perturbations: i) a push/pull on the swing leg and ii) a push/pull on the stance leg. The predicted walking agree to our daily experience, that during normal walking, we quickly recover from external perturbations in one step.

6.2 Future work

We have developed a data-driven method and applied it to human walking data. Potential future work that builds on the work in this thesis is discussed below:

i. We can develop a factorized Poincare map description of lateral dynamics (as we have for sagittal plane dynamics). We can compare predicted responses under lateral perturbations to actual lateral perturbation experiments, applying lateral push/pull on body.
ii. A three dimensional model can be built to represent human walking. The 3D model could be interesting when the lateral motion becomes more significant. The data-driven method with factorized Poincare sections can be directly applied by using the same procedure described in this thesis. A piecewise model can be constructed to correlate the lateral perturbation experiments and then be used to predict how people would recover under lateral perturbations.

iii. The data-driven method can be extended to analyze human running as well. The step-to-step model and sagittal 2D model can be employed to analyze running data. Similarly, the analysis can be executed in the different aspects: stability estimate, dynamical relations, comparison to experiment data and predictions under various circumstances.

iv. We have only considered “dynamics” — that is, the evolution of body states. But we can simply extend our methods to infer the “controller” that the human uses for controlling human walking. To derive the controller, we would have to use a mechanics-based multi-body model of the human body, with muscles, and then try to find the mapping from body states to muscle forces that gives rise to the same dynamics as those inferred in this thesis. Such an inferred controller could then be used to design prosthetic devices and walking robots with truly human-like dynamics.

v. A more rigorous understanding of data-driven system identification method when applied to real data is desired. Although, we have offered a couple of explanations, it is still unclear why the return map eigenvalues drop dramatically when one goes from using multiple sections, and therefore, some of the eigenvalue estimates offered here could be considered exploratory in nature. Errors in our statistical inference could probably be reduced by increasing the ‘signal’
by applying external perturbations instead of relying on small internal perturbations. Even when we have small error estimates obtained from bootstrap resampling, such error estimates are reliable only so far as we have assumed the correct model structure and model order to fit to the data: we may be missing delay terms, significant state variables like foot angle, or even, dynamics that cannot be linearized, like dead-bands. Future work may add some of these elements to see if they parsimoniously explain some of the unexplained variability.
Appendix A

PATCH-WORK OF FACTORIZED POINCARE MAPS

A.1 Introduction

Deterministic models of stable periodic behavior in science and engineering often consist of nonlinear differential equations with asymptotically stable periodic motions. Examples include coupled neural oscillators [23, 27, 11, 62], human, animal and robot locomotion [63, 84], circadian rhythms [26], frictional [29] and aeroelastic oscillations [70], etc.

In this chapter, we elaborate on a discrete approximation of the continuous dynamics near the periodic motion that is complementary to a nonlinear differential equation representation — namely, a patchwork of finitely many discrete dynamical systems that we term factorized Poincare maps. Briefly, the Poincare map is a standard technique in the analysis of periodic motions that reduces the continuous-time dynamics to a discrete dynamical system [32, 88, 100], obtained by paying attention to the system only when a trajectory intersects a surface in state-space called the Poincare section. By sufficiently highly factorizing this Poincare map, say, by using a large finite number of Poincare sections, one can arbitrarily closely approximate the continuous nonlinear dynamics around the periodic orbit with a patchwork discrete model.

Here, we show how such factorized Poincare maps can be computed, and how,
once computed, can be used in lieu of the underlying continuous differential equations to derive various quantities of interest associated with periodic motions and periodic dynamics: for instance, approximations to the Poincare map (including the Floquet multipliers), linear time-periodic ODE approximation to the underlying nonlinear ODE, phase-response curves, and isochrons. Further, we show how to use the discrete approximation to simulate the system in continuous time from arbitrary initial conditions close to the periodic orbit.

The central idea of factorized Poincare maps and using multiple Poincare sections is not novel, but appears to have not been elaborated upon in the practical form presented in this chapter. For instance, a few articles have used factorized Poincare maps with a small number of Poincare sections (usually less than five) to examine various dynamical systems in chemistry, impact mechanics, astronomy, and legged locomotion e.g., [56, 35, 1, 3, 4, 52, 60]. In such work, the Poincare maps and its factors are selected by special consideration of the system under analysis rather than through an automated algorithm as presented here. At the other extreme, Manchester and others recently used a similar idea [61, 62] and obtained representations of continuous dynamics transverse to the periodic orbit [79] using, essentially, infinitely many (that is, a one-parameter family of) Poincare sections. They were also able to obtain linear differential equations representing “transverse linearizations.” Their formalism could be considered a continuum version of our finite construction described below.

A.2 Limit cycles, transverse sections and Poincare maps.

We will primarily consider autonomous (time-invariant) nonlinear dynamical systems:

\[ \frac{dx}{dt} = f(x), \ x \in \mathbb{R}^n \]  

(A.2.1)
which has an isolated periodic orbit, a limit cycle, as shown in Figure 1.2. Say the limit cycle parameterized by time is $x_p(t)$, such that $x_p(t) = x_p(t+T)$, where $T$ is the period of the limit cycle. Extensions to dynamical systems with explicit time-dependence are straight-forward. The formalism here is most useful for asymptotically stable periodic orbits.

A **Poincare section** is a surface in state space that intersects the periodic orbit transversally (that is, not touching tangentially, Figure 1.2a). Sometimes we will simply call such a surface a **transverse section**. The periodic orbit intersects the Poincare section at $x^*$. The Poincare map $P(\cdot)$, also called the first return map, is the function taking some point $x^{(i)}$ in the neighborhood of the periodic orbit on the Poincare section to the next intersection $x^{(i+1)}$ of the trajectory of the dynamical system with the Poincare section:

$$x^{(i+1)} = P(x^{(i)}) \tag{A.2.2}$$

The point $x^*$ where the periodic orbit intersects the Poincare section is a fixed point of this discrete dynamical system ($x^* = P(x^*)$) and the stability of the periodic orbit of the continuous dynamical system is identical to that of the stability of the fixed point $x^*$ of the Poincare map.

The linearization of the Poincare map around $x^*$ is given by:

$$x^{(i+1)} - x^* = J \cdot (x^{(i)} - x^*) \tag{A.2.3}$$

where $J$ is the Jacobian of the function $P(\cdot)$. This linearization, specifically the eigenvalues of the Jacobian $J$ (the Floquet multipliers), is generically sufficient to establish local stability of the periodic motion.
A.3 Multiple sections and patchwork models

While the Poincare map does capture the dynamics around the periodic orbit, often in many applications, we need predictions of the state all around the periodic orbit, not just at one transverse section. This necessity naturally motivates the patchwork model below involving a large number of transverse sections.

As in Figure 1.2b, we consider a series of $M$ Poincare sections $S_j \ (j = 1 \ldots M)$ all around the periodic orbit, intersecting the periodic orbit transversally at $x^*_j$ respectively. In the neighborhood of the periodic orbit, the sections are defined by equations $S_j(x) = 0$.

The Poincare map $P_{j \to j}$ from section $S_j$ all the way back to $S_j$ is defined as:

$$P_{j \to j} : x_j^{(i)} \to x_j^{(i+1)}, \quad \text{for } i = 1, 2, \ldots$$

(A.3.1)

where $i$ denotes the $i^{th}$ cycle. The point $x_j^{(i)}$ is the intersection of the trajectory with the section $S_j$ during the $i^{th}$ cycle, and $t_j^{(i)}$ is the corresponding time of the intersection, so that $x_j^{(i)} = x(t_j^{(i)})$. Throughout this thesis, we use the subscript $j$ to represent the Poincare section being considered and the parenthetical superscript $(i)$ to index the successive cycles.

A transient trajectory sufficiently close to the periodic orbit intersects the transverse sections $S_j$ in the same sequence as does the periodic orbit: that is,

$$..., 1, 2, \ldots, M-1, M, 1, 2, \ldots, \text{etc.}$$

The mapping from a point on section $S_j$ to the next intersection on section $S_k$ is denoted $P_{j \to k}$. Thus the full Poincare map $P_{j \to j}$ may be factorized as a composition of component maps between consecutive sections:

$$P_{j \to j} = P_{j-1 \to j} \circ P_{j-2 \to j-1} \circ \cdots \circ P_{j \to j+1}.$$  

(A.3.2)
We use the term **factorized Poincare map** to refer to the above representation, and we call the component maps as **factor Poincare maps** or simply, **factor maps**. Of course, the factorization is far from unique. Choosing different Poincare sections gives rise to correspondingly different factorizations of the Poincare map $P_{j\rightarrow j}$.

**Tracking time from section to section.** The time taken by a particular trajectory from one Poincare section $S_j$ to the next $S_{j+1}$ is dependent, generically on the initial condition $x_j^{(i)}$ on $S_j$, represented by the function $Q$ here. That is,

$$t_{j+1}^{(i)} - t_j^{(i)} = Q_{j\rightarrow j+1}(x_j^{(i)}).$$

(A.3.3)

We will call $Q$, the **time transition map**. Note that this function $Q$ degenerates to a constant function when $S_j$ and $S_{j+1}$ are special Poincare sections called **isochrons**, discussed later in section A.7.2.

Note that the time transition map is decoupled from the state transition map here because of no explicit time-dependence on $f$. For non-autonomous systems, we would have to write coupled equations $^1$.

### A.4 Local approximations

While Poincare sections need not be planar, we will restrict our discussion below to planar Poincare sections in the original coordinates $x$. Consider that the unit normal to section $S_j$ is $n_j$ so that the planar Poincare section is: $n_j \cdot (x - x_j^*) = 0$.

---

$^1$In fact, this coupling is the only change needed to adapt the formalism above to explicitly time-dependent systems.
A.4.1 Linear maps expressed in original coordinates

A local linear approximation of the mapping from one transverse section to the next can be written as:

\[ x_{j+1}^{(i)} - x_{j+1}^* \approx J_{j \rightarrow j+1} \cdot (x_j^{(i)} - x_j^*) , \]  

(A.4.1)

where \( J_{j \rightarrow j+1} \) is the local Jacobian of the map \( P_{j \rightarrow j+1} \).

Note that as expressed in equation A.4.3, the Jacobian \( J_{j \rightarrow j+1} \) acts on the state deviation \( (x^{(i)} - x^*) \) represented in the original coordinates. Therefore the action of the Jacobian matrix is uniquely determined only when the state is on the input transverse section \( S_j \). Given this indeterminacy of the full Jacobian matrix, it is convenient and always possible to obtain a \( J_{j \rightarrow j+1} \) such that its null space is the normal to the input transverse section \( S_j \) and its range space is exactly the output Poincare section, so that its operation on the state on the input Poincare section \( S_j \) is correct to first order. Here we are implicitly making the identification between the plane \( S_j \) passing through \( x_j^* \) and the linear subspace obtained by translating this space so that the origin is at \( x^* \).

A.4.2 Linear maps expressed in local coordinates

It is often more convenient to define a local coordinate system on each transverse section, with origin at the fixed point \( x_j^* \) so that the deviations from it are represented by \( y_j \in \mathbb{R}^{n-1} \).

\[ y_j = Z_j \cdot (x_j - x_j^*) \]  

(A.4.2)

where the rows of \( Z_j \in \mathbb{R}^{(n-1)\times n} \) constitute an orthonormal basis for the subspace \( S_j \). We explain how the basis \( Z_j \) is obtained in section A.4.5, but see Figure A.1 for an example of a continuously and periodically varying series of frames around the periodic orbit.
The reduced Jacobian $\bar{J}$ can be written in this local coordinate system, so that

$$y^{(i)}_{j+1} \approx \bar{J}_{j\rightarrow j+1} \cdot y^{(i)}_j,$$  \hspace{1cm} (A.4.3)

ignoring higher order terms.

If we define $t^*_j$ as the time on the periodic orbit from some reference time, modulo the time period $T$, then we may track time approximately from section $S_j$ to $S_{j+1}$ using the following linearization:

$$t^{(i)}_{j+1} - t^{(i)}_j = t^*_{j+1} - t^*_j + K_{j\rightarrow j+1} \left( x^{(i)}_j - x^*_j \right).$$ \hspace{1cm} (A.4.4)

where $K_{j\rightarrow j+1} = \partial Q_{j\rightarrow j+1}/\partial x|_{x^*_j}$, the Jacobian of the time transition map. This equation can also be reduced to the local coordinate system:

$$t^{(i)}_{j+1} - t^{(i)}_j = t^*_{j+1} - t^*_j + \bar{K}_{j\rightarrow j+1} \cdot y^{(i)}_j.$$ \hspace{1cm} (A.4.5)

### A.4.3 Linearized Poincare map

The linearization of the Poincare map, namely the total Jacobian matrix $J_{j\rightarrow j}$ of the map from section $S_j$ back to $S_j$ is the product of all factor Jacobian matrices as:

$$J_{j\rightarrow j} = J_{j-1\rightarrow j} \cdots J_{j+1\rightarrow j+2} \cdot J_{j\rightarrow j+1}.$$ \hspace{1cm} (A.4.6)

As the $J$’s above are in $\mathbb{R}^{n\times n}$, the resulting $J_{j\rightarrow j}$ will have a zero eigenvalue corresponding to an eigenvector $n_j$ if the individual Jacobians have nullspaces as described above.

In local coordinates, the expression would be

$$\bar{J}_{j\rightarrow j} = \bar{J}_{j-1\rightarrow j} \cdots \bar{J}_{j+1\rightarrow j+2} \cdot \bar{J}_{j\rightarrow j+1}.$$ \hspace{1cm} (A.4.7)

Note that the Jacobian matrix $J_{j\rightarrow j}$ for section $S_j$ is related to another Jacobian matrix $J_{k\rightarrow k}$ for section $S_k$ for $(k \neq j)$ by a similarity transformation:

$$J_{j\rightarrow j} = (J_{k\rightarrow j}) \cdot J_{k\rightarrow k} \cdot (J_{k\rightarrow j})^{-1}.$$
Therefore, as is well known, while the Poincare maps at different sections \( S_j \) and \( S_k \) are different, their Jacobians \( \bar{J}_{j\rightarrow j} \) and \( \bar{J}_{k\rightarrow k} \) have the same eigenvalues and are related by a similarity transformation. These eigenvalues are, of course, all but one of the Floquet multipliers corresponding to the local linear approximation. The unit Floquet multiplier corresponding to the progression along the periodic orbit is not obtained as these these Jacobians are reduced to the transverse sections.

### A.4.4 Quadratic maps

A **quadratic approximation** would involve a correction to equation A.4.3 of the form

\[
\begin{bmatrix}
\langle y_j^{(i)} , \bar{H}_{j\rightarrow j+1}[1] \cdot y_j^{(i)} \rangle \\
\vdots \\
\langle y_j^{(i)} , \bar{H}_{j\rightarrow j+1}[n-1] \cdot y_j^{(i)} \rangle \\
\end{bmatrix}
\tag{A.4.8}
\]

involving as many different Hessians as there are output dimensions, namely \( n - 1 \), and yet another for the time transition map.

### A.4.5 Defining continuous and periodic local frames

The discussions in the previous sub-sections required the definition of local coordinate frames defined on the Poincare sections to define the maps between successive sections.

Ideally, the local coordinate frame should be continuously varying and periodic around the periodic orbit. That is, the axes defining the local frames should ideally change only a little between closely spaced consecutive sections.

If the local frame is continuously varying, the factor Poincare maps and the corresponding Jacobians \( \bar{J} \) between arbitrarily close sections will be near-identity in this local coordinate representation, and the corresponding Hessians will approach zero.
If the local coordinates do not vary continuously, nor will the Poincare maps in the local representations. If the $Z_j$ for neighboring sections vary dramatically, the corresponding Jacobians $\bar{J}$ will have to incorporate large rotations to compensate for the dramatic change in local coordinates.

Figure A.1a shows a sequence of frames around a periodic orbit that has two discontinuities. This sequence of frames was obtained by performing Gram-Schmidt at each section with an initial non-orthogonal basis consisting of the section normal $n_j$ as the first vector and a given constant set $A$ of $n - 1$ vectors. Except for the first vector being the section normal, the same set $A$ is used for all the sections. While this procedure gives continuously varying frames at most points on the orbit, discontinuities arise when the section normal is not independent of the set $A$. It seems non-exceptional for this simple algorithm to produce frames with discontinuities.

To obtain a continuously varying basis, we start by obtaining an initial frame by applying the following algorithm.

i. For the first section $S_1$, the basis $Z_1$ is picked by applying Gram-Schmidt process with an initial basis consisting of $n_1$ and $n - 1$ other randomly chosen vectors.

ii. For all the subsequent sections $S_j$, we perform Gram-Schmidt on a basis consisting of $n_j$ as the first vector and the rows of $Z_{j-1}$ as the other $n - 1$ vectors. Because $n_j$ is close to $n_{j+1}$, $Z_j$ will be close to $Z_{j+1}$ by continuity of the Gram-Schmidt process.

The result of this algorithm on a curve in 3D is shown in Figure A.1b.

The above algorithm is a close cousin of algorithms to obtain frames that are variously called “rotation-minimizing” [96], obtained by “parallel transport” [36], and are “relatively parallel” [8]. An example of frames obtained by this process is shown in Figure A.1b. In this thesis, we will call such frames as parallel transport.
Figure A.1: Local coordinate frames are shown with red and black coordinate axes. a) An orthogonal basis is obtained for each section by performing Gram-Schmidt with the section normal as the first vector and a constant set \( A \) of linearly independent vectors. b) An orthogonal basis is obtained by performing Gram-Schmidt with the section normal and the orthogonal basis from the previous section. c) The frames obtained from the parallel transport procedure are rotated in the plane of the section, so that the aperiodicity is eliminated. d) The discontinuities and aperiodicities of the first two methods are shown by measuring the distance between successive frames. e) The distance between successive frames for the parallel transport method with periodicity correction.
frames. These parallel transport frames have been found to be better behaved than
the generalized Frenet-Serret frames due to Jordan [51], which often vary more dra-
matically over the periodic orbit [36, 96]. However, the parallel transport algorithm
while producing a continuous set of frames while moving forward does not guarantee
periodicity; that is, the frame obtained after one circuit around the periodic orbit
will not generically be the same as the initial frame (in 3D, a sufficient condition for
periodicity is zero torsion [36]).

We now describe a way to make the basis periodic. Consider the 3D case in
Figure A.1b with 2D sections. Say the angle in the plane between the initial frame
at $S_1$ and the frame obtained after one period of parallel transport from $S_1$ is $\alpha$.
The aperiodicity, namely the discontinuity $\alpha$, can be eliminated by distributing the
discontinuity $\alpha$ uniformly around the periodic orbit by applying a rotation about the
normal to every frame; the $j^{th}$ frame is rotated by an angle $(j - 1)\alpha/M$.

In higher dimensions, we first find the orthogonal matrix $R$ that characterizes the
transformation to the initial frame $Z_1$ at $S_1$ from the frame $Z_{1T}$ obtained at $S_1$ after
one parallel transport circuit around the orbit. This discontinuity $R$ is eliminated
by distributing it around the periodic orbit by rotating the $j^{th}$ frame by the matrix
$(R^{1/M})^{(j-1)}$. See Appendix A.4.6 for details of how the relevant real matrix roots and
matrix powers are computed reliably and efficiently.

Note that the procedure ensures continuity (in the discrete sense) by construction,
but we do not know if it would ensure differentiability at the first section. The Frenet-
Serret frames would ensure periodic differentiability of the frames, but these frames
are defined uniquely for only generic curves, which have the property that, at every
point, its tangent to the curve and its successive vector derivatives form a linearly
independent set.
a) Solve BVP to find periodic orbit

b) Represent periodic orbit by periodic spline (say) using finitely many knots

c) Define Poincare sections & continuous local frames

d) Propagate a number of neighboring trajectories & compute intersections

e) Compute linear or quadratic maps using regression or finite differences

Figure A.2: A procedure for obtaining linear and quadratic maps between sections. We might improve upon its accuracy, by using, say, automatic differentiation, or use other approximations of the periodic orbit. See section A.5.

A.4.6 Computing roots of an orthogonal matrix

Higham [42] notes that the questions of existence, uniqueness and computation of real square roots of general real matrices are non-trivial. However, it is clear from their geometric meaning that an orthogonal matrix has real square roots and more generally, real $M^{th}$ roots for integer $M > 2$.

For simplicity, the computations in this chapter requiring the definition of periodic local frames, we used the number of sections to be a power of 2: $M = 2^m$. Then, $R^{1/M}$ may be computed by a sequence of matrix square roots. Note that for a generic proper orthogonal matrix (determinant = 1), there are generically $2^{|m/2|}$ proper orthogonal square roots, where $[·]$ is the floor function. These square roots may be obtained by considering first the diagonalization of the orthogonal matrix $R = VDV^{-1}$ and
then computing the various square roots $D^{(1/2)}$ of the diagonal matrix such that each complex conjugate pair of eigenvalues of $R$ leads to 2 conjugate eigenvalue pairs $c \pm id$ and $-c \pm id$. One then obtains $R^{(1/2)} = VD^{(1/2)}V^{-1}$. Note that some $D^{(1/2)}$ may have determinant -1, and we explicitly check for such to avoid them. As an aside, for orthogonal $R$ we note that $V^{-1} = V^\dagger$, the conjugate transpose. Given this plethora of square roots, we simply pick one at each stage to obtain one $R^{1/M}$ for $M = 2^m$.

In the algorithm in Section A.4.5, we had to determine $R^{p/M}$. Of course, this may be obtained by $(2^{1/M})^p$ using $p$ matrix multiplications. However, we instead obtain $R^{p/M}$ with no more than $\log_2 p$ multiplications by representing $p$ in binary or equivalently as sum of powers of two. This improves accuracy of these products for large $M$.

A.5 Finding the linear and quadratic maps

We now describe how to obtain the piecewise linear or quadratic model from differential equations with a limit cycle. The periodic orbit is first obtained to high accuracy by solving a boundary value problem, say, using a single shooting method or multiple shooting for more complex or higher dimensional problems. For instance, a single shooting approach is to solve for initial conditions $x(0)$ and time period $T$ such that the following fixed point equation is satisfied: $x(T) - x(0) = 0$ (see Figure A.2a). Alternatively, one can use an arbitrary Poincare section, define a Poincare map through numerical integration of the ODE until the Poincare section is intersected again, and then solve a fixed point problem for the Poincare map. For the examples in this thesis, both the integration of the ODEs and stopping the integration at a Poincare section to high accuracy is accomplished using MATLAB’s ODE suite (in particular ode45) and its event detection capabilities, and the fixed point problems as above were solved.
using \texttt{fsolve}. These methods are relatively standard and could be improved with the use of, say, automatic differentiation [33].

Once an accurate initial condition on the periodic orbit is obtained, we obtain $M$ points on the periodic orbit. Here we use points equally spaced in time, but they need not be — they could be chosen adaptively to ensure a given level of model accuracy. With these $M$ points on the periodic orbit, we construct a periodic cubic spline that has continuous and periodic first and second derivatives (Figure A.2b). This cubic spline approximation ensures high order representation of the periodic orbit between the $M$ points [72].

At each of these $M$ points on the periodic orbit, we construct a transverse Poincare section. As noted earlier, we use the tangents to the periodic orbit as the normals to the sections, but this is not essential for any of the following. Defining the sequence of planar Poincare sections around the periodic orbit requires only the definition of a periodically varying normal vector $\mathbf{n}(t)$ to the plane [61, 62], satisfying certain properties, the key among which is transversality: $\mathbf{n}(t) \cdot \mathbf{e}_t(t) \neq 0$, where $\mathbf{e}_t(t)$ is the local tangent. For sufficiently smooth systems, sections that are exactly orthogonal to the periodic orbit may be sufficient: $\mathbf{n}(t) = \mathbf{e}_t(t)$. See Figure A.2c. Because of our cubic spline representation, we obtain the tangents to the cubic spline approximation analytically, and these tangents would be continuous, differentiable and periodic by construction.

Next, we construct maps from one planar section to the next. For simplicity, we use a simulation-based approach, analogous to finite difference approximations. Most simply, to obtain a linear approximation, to estimate the Jacobian, it is sufficient to propagate $n – 1$ initial conditions from one section, sufficiently close to the periodic point $\mathbf{x}^*$ and obtain the intersections with the next section. The Jacobian from $\mathbf{y}_j^{(i)}$ to $\mathbf{y}_{j+1}^{(i)}$ is obtained by linear least squares regression. Similarly, to obtain the Jacobian
and the Hessian simultaneously, one uses $n + n(n - 1)/2$ initial conditions and fits a quadratic model between the input and the output (Figure A.2d,e). Alternatively but similarly, one can obtain finite difference approximations to the Jacobian and the Hessian to the various partial derivatives by similarly propagating sufficiently many trajectories from one section to the next. For the examples in this thesis (i.e., for Figures A.5, A.6), we used central difference approximations for the various partial derivatives.

While these simulation-based derivative estimates of the maps would be orders of magnitude less accurate than the ODE solution itself, such accuracy may be sufficient for most purposes (if the ODE accuracy is $\epsilon$, the central difference derivative accuracy would be roughly $\epsilon^{2/3}$). If greater accuracy is required, one can always propagate appropriate adjoint equations to the original differential equations to obtain these derivatives, possibly employing automatic differentiation [33].

Further, the linear and quadratic approximations need not necessarily be identified with a Taylor series about the periodic point as above. Instead, one can obtain the best linear or quadratic approximation over a given non-infinitesimal neighborhood around the periodic orbit. Approximations to such best approximations may be obtained, for example, by propagating a large number of initial conditions (many more than the minimum necessary) from one section to the next, and fitting a linear or a quadratic function between the inputs and the outputs in the sense of a $p$-norm (say).
A.6 Continuous time simulations using factorized Poincare maps

In this section, we outline some methods for generating predictions of the continuous-time dynamical system from the piecewise model. That is, we ‘simulate’ the dynamical system, using the discrete model.

A.6.1 Discrete and continuous time predictions starting from a Poincare section

Given an initial condition on a Poincare section in our piecewise model, we can simply apply the obtained piecewise model to the initial condition, and obtain a sequence of iterates on successive Poincare section. Thus, if all we need are predictions on the Poincare sections, the task is trivial (Figure A.3(i)-(ii)).

Once a sequence of state and time predictions are obtained on successive Poincare sections, we may use some interpolation technique to obtain a continuous time approximation of the state evolution. Below, we outline an interpolation technique that has the property that if the initial condition is on the periodic orbit, the prediction is exactly the periodic orbit, as approximated by the periodic spline described above.

First, obtain the state and time deviations of the transient from the periodic orbit, on the Poincare sections, using the factorized Poincare map, for sufficiently many sections into the future (Figure A.3(ii)). We used about $M$ sections. Then, treat both the state and time deviations of the transient as functions of the corresponding time on the periodic orbit at those sections (Figure A.3(iii)). Now, by defining a dense grid for the time on the periodic orbit, we can interpolate the state deviations and time deviations for the transient (we used non-periodic natural cubic splines, but any interpolation would do).
Figure A.3: Given an initial condition $x_0$ already on a considered section, we can obtain a continuous time simulation of the forward trajectory. (i) First find the state deviation from the periodic orbit $y_0$. (ii) Then, using the linear or quadratic maps, obtain the state deviations $y$ and the time $t$ on the successive sections, sufficiently many sections. (iii) We consider both $y$ and time $t$ on the transient as functions of the corresponding time on the periodic orbit, and interpolate to obtain smooth transients. (iv) We add the state deviations to the periodic orbit at the corresponding time on the periodic orbit to obtain the transient prediction in the ambient space.
Now, we add these state and time deviations to the corresponding states and times on the periodic orbit, to obtain an interpolated prediction of the transient (Figure A.3(iv)). It is clear that this procedure, if started with the initial condition on the periodic orbit, will return the periodic orbit, as the state and time deviations predicted from the factorized Poincare maps will all be zero.

Examples of such successfully interpolated ‘continuous time’ predictions are described in the following subsection and Figure A.5, which considers the more general situation of initial conditions not on a Poincare section.

A.6.2 Simulating transients with initial conditions not on a Poincare section

What do we do when an initial condition provided does not lie on a Poincare section in our model (e.g., Figure A.4a)? We first determine an “intermediate section” $S_{IC}$ that is normal to the periodic orbit and passes through the initial condition of interest. Here, we found this intermediate section $S_{IC}$ by finding the point on the periodic orbit that is the (globally) closest to the initial condition (Figure A.4b). This optimization problem was solved as follows: take a reasonably fine grid on the periodic orbit and find the point on this grid closest to the initial condition. Then, using this point as an initial seed, we used a standard unconstrained local optimizer (based on Newton’s method) that finds the point on the periodic orbit that locally minimizes the distance to the initial condition; this is univariate optimization along the cubic spline. Associated with the closest point on the periodic orbit is the corresponding time $t^*$ on the periodic orbit.

Note that we are making the assumption that the initial condition is sufficiently close to the periodic orbit, so that the section associated with the closest point on the
orbit is representative of the further dynamics for the initial condition (we remark later about what happens when the initial condition is too far away).

Now, say we the initial condition has been associated with a corresponding point on the periodic orbit and an “intermediate” normal section $S_{IC}$, and that this intermediate section $S_{IC}$ is between two regular sections $S_p$ and $S_{p+1}$. Then, our strategy is to find an initial condition on $S_p$ such that the forward simulation using the method from section A.6.1 from that initial condition passes through the initial condition of interest. See Figure A.4c. To obtain the appropriate initial condition on $S_p$, we solve a simple boundary value problem using the simulation from sub-section A.6.1.

A.6.3 A three-dimensional example

To illustrate the results from the methods outlined in the previous sections visually, we use a simple three dimensional dynamical system, an extension of the van der Pol oscillator, governed by:

$$\ddot{x} = \mu(1-x^2)\dot{x} - x, \quad \dot{z} = -\lambda z.$$  \hspace{1cm} (A.6.1)

We used $\mu = 0.1$ and $\lambda = 0.3$. This dynamical system preserves the planar limit cycle of the van der Pol oscillator with an exponentially decaying $z$. The limit cycle is globally asymptotically attracting, by construction.

For the illustrations in Figure A.5, we obtained both a piecewise linear and quadratic model of the local dynamics around the limit cycle, with $M = 32$ sections. Figure A.5a-c shows comparisons of the continuous time predictions using the linear and quadratic piecewise models, with a high accuracy numerical solution of the differential equations. Note that the initial conditions $(4, 0, 0)$ and $(3, 0, 0)$ are ‘far away’ from the limit cycle, which is well-known to pass through, roughly, $(2, 0, 0)$. In Figure A.5a,c, the difference between a linear and quadratic model is more apparent,
Figure A.4: If an initial condition $x_0$ (blue circle) not on one of the $M$ sections is given, we first find an intermediate section (green disk) through the initial condition, as passing through the point on the periodic orbit closest to $x_0$. Then, we solve for the initial condition on the previous section, such that the continuous time transient from this initial condition passes through $x_0$. 
whereas in Figure A.5b, which is from an initial condition closer to the limit cycle, the difference between linear and quadratic is negligible.

As would be expected, when the dynamical system is more nonlinear, say with higher $\mu$, the simulation from the factorized Poincare map has greater error for a given distance from the limit cycle. Further, because the limit cycle has fast and slow regions, and also, regions of greater and lesser curvature, the distance from the limit cycle for which the piecewise models are accurate depends on the phase of the initial condition.

Figure A.5: **Continuous time simulation with discrete model.** All panels correspond to Equation A.6.1. a) Three simulations, one directly using the differential equation (gray, solid line), and the other two using a linear (light blue, short dashes) and quadratic (red, long dashed) factorized Poincare map, with $M = 32$. Simulations from initial condition $(4, 0, 0)$. b) Two simulations, one using the differential equation and another with a piecewise linear factorized Poincare map with $M = 32$. c) Same as panel-a, depicted in state-space.
A.7 Linearized ODE, phase response curves and isochrons

We have seen that the factorized Poincare map representation can be used as an approximate representation of the dynamics around the periodic orbit – in that, this representation is sufficient to obtain simulations of the original dynamical system from arbitrary initial conditions, sufficiently close to the periodic orbit. Therefore, this representation should be sufficient to derive other quantities and representations related to the dynamics close to the periodic orbit, such as a local linear ODE, phase response curves, isochrons, etc. In this section, we show simple methods of computing these from the factorized Poincare maps.

A.7.1 Linearized ODE

First, the dynamics near the limit cycle $x_p(t)$ of a smooth nonlinear dynamical system in Eq. A.2.1 can be approximated by a linear time-periodic differential equation of the form:

$$\frac{du}{dt} = A(t)u,$$

where $u = x(t) - x_p(t)$, the deviation from the limit cycle and $A(t)$ is the time-periodic local Jacobian $\partial f/\partial x$ at the periodic orbit $x_p(t)$.

Figure A.6a shows the comparison between one element of $A(t)$ for the van der Pol oscillator, with that derived using the linear factorized Poincare map (gray circles). The Jacobian $A(t)$ was obtained using the factorized Poincare map using the following simplistic algorithm. First, from a small group of initial conditions (at least $N$ initial conditions) from a small neighborhood of a point $x_p(t)$ on the periodic orbit is propagated forward in time using the factorized Poincare map using methods from section A.6. Then, compute the time-derivatives of the simulated state deviation $\dot{u}$ by differentiating the splines. Then, $A$ is obtained in the neighborhood of the point
\( x_p(t) \) by using linear regression on equation A.7.1, as both \( u \) and \( \dot{u} \) are known for at least \( N \) generic points. Repeating this procedure for every point on the orbit \( x_p(t) \), we obtain \( A(t) \) as in Figure A.6a.

### A.7.2 Phase response curves and isochrons

Phase response curves [23, 11] measure how the ‘phase’ \( \phi \) of an oscillation changes when a state variable is perturbed. Phase response curves are used especially in neuroscience to characterize dynamical properties of neurons experimentally and also obtain reduced mathematical (phase oscillator) models of neural networks.

For completeness, we provide a brief definition of the notion of phase. Given a point on the periodic orbit associated with \( t = 0 \), the phase of any other point on the periodic orbit is just the time \( t \) at that point (modulo the period \( T \)). It is sometimes conventional to use normalized phase \( \phi = t/T \) or \( 2\pi t/T \); we do no such. Before we define phase for a point not on the periodic orbit, we define “isochrons.” An isochron is a special Poincare section (generally non-planar), associated with every point on the periodic orbit, with the property that initial conditions started together on an isochron will return back to the isochron simultaneously, exactly after one period \( T \) [31] \(^2\). In other words, points on an isochron ‘move together’ with the corresponding point on the periodic orbit, and all these points are said to have the same phase as the corresponding point on the periodic orbit. This is how the notion of a phase is extended to points not on the period.

The isochrons as defined above are level-sets of the phase function \( \phi(x) \). Perturbations from the periodic orbit change the phase of a system by moving the state to a different isochron. The infinitesimal phase response curves (iPRC’s) are the partial

\(^2\)Equivalently, the isochron is the fixed point of the mapping of a Poincare section by the flow through exactly a time duration \( T \).
derivatives of the phase function $\phi(x)$ with respect to the individual state variables: $\partial \phi / \partial x_i$.

The high accuracy iPRC’s using the ordinary differential equation are computed at each point on the periodic orbit using appropriate adjoint equations as outlined in [11, 23]. For the factorized Poincare map, we compute them using finite differences, again as outlined [11]. Consider some Poincare section $S$ along the periodic orbit, say at the point of interest. To compute the iPRC at a point on the periodic orbit, we make a small perturbation in one of the state variables and compute the time $T'$ it takes for the system to intersect the section $S$. In the absence of the perturbation, the nominal time to the section would be the period $T$. The difference in arrival time $T - T'$ is the phase change due to the perturbation in $x_i$, and we compute the relevant partial derivative using finite differences. We see in Figure A.6b that the iPRC’s obtained thus, using simulations from the linear factorized Poincare map, agree well with the iPRC computed using the adjoint equation to the original differential equation.

Noting that the isochrons are simply the level sets of the phase function, the normal to the isochron is given by the gradient of the phase $\nabla \phi$, which is simply a vector containing the three iPRCs. Thus, we obtain the isochrons as a consequence of computing the iPRCs.

A.8 Discussion

In the following subsections, we remark about certain issues that might arise with the practical use of factorized Poincare maps.
Figure A.6: **Other quantities related to local dynamics.** Both panels correspond to the van der Pol oscillator (Eq. A.6.1). The solid (red) line is directly from nonlinear differential equation and the (gray) circles are from the linear factorized Poincare map with \( M = 32 \) sections. a) An element of the Jacobian \( A(t) \) from the linear time-periodic ODE (Eq. A.7.1). b) The three phase response curves, the partial derivatives of the phase with respect to perturbations in the three state variables.

A.8.1 Asymptotic bias errors from infinite compositions

Making predictions and performing simulations with factorized Poincare maps requires the multiplication of a large number of matrices (when considering linear approximations), or more generally, the composition of many factor maps. If each individual factor map is only estimated approximately, what is the composition of a large number of such maps?

We consider a simple example. Imagine that the univariate function \( r(\cdot) \) is a composition of \( M \) identical functions \( s(\cdot) \), so that

\[
r(x) = s(s\ldots s(x)\ldots) = s^M(x)
\]
and $r(0) = 0$ and $s(0) = 0$. Then the first derivatives $r'(0) = J$ (say) and $s'(0)$ are related by:

$$J = r'(0) = s'(0)^M \text{ or } s'(0) = J^{1/M}.$$

and the second derivatives are related by:

$$r''(0) = s''(0) \cdot J^{(M-1)/M} \cdot \frac{1 - J}{1 - J^{(1/M)}}.$$

Now, let us say that we wish to estimate $r'(0) = J$ by first estimating $s'(0)$ and raising it to the $M$th power. We wish to understand how an error in $s'(0)$ translates to an error in $r'(0)$ as $M \to \infty$. As $M$ changes, the function $r(x)$ is kept constant, and $s(x)$ changes appropriately. Note that $s(x)$ becomes near-identity as $M \to \infty$; in particular,

$$s'(0) = J^{1/M} \to 1, \text{ (assuming } J > 0).$$

In the following, we denote the estimates with an over-bar, so the estimate of $s'(0)$ is $\bar{s}'(0)$, etc. We now consider various models for error in $\bar{s}'(0)$.

**Case 1. Constant error.** Say the error $w$ in the estimate $\bar{s}'(0)$ is independent of $M$. That is, $\bar{s}'(0) = s'(0) + w$. Then the estimate $\bar{r}'(0)$, ignoring higher order terms in $w$, is given by:

$$\bar{r}'(0) = s'(0)^M \sim s'(0)^M + wMs'(0)^{M-1} = J + wMJ^{(M-1)/M}.$$

Thus, asymptotically, the error $(\bar{r}'(0) - J)$ grows linearly in $M$.

**Case 2. Error from forward difference approximations.** Say, the estimate $\bar{s}'(0)$ is obtained using forward differences, using function evaluations which have a small error $\bar{s} - s \sim \epsilon$. This error $\epsilon$ could be round-off, or due to the integration
accuracy if the evaluation of \( s \) requires a numerical integration. Then the error in \( \bar{s}'(0) \) is given by, approximately,

\[
\bar{s}'(0) - s'(0) = \frac{\bar{s}(h) - \bar{s}(0)}{h} - s'(0) \sim s''(0) h + \frac{\epsilon}{h}.
\]

There are two options for choosing the forward difference step size \( h \). One can either choose a (small) constant \( h \) or choose an \( h \) that attempts to minimize the above error \( h \sim \sqrt{\epsilon/s''} \).

For now, say \( h \) is constant. Then, the error from equation A.8.1 is given by

\[
MJ^{(M-1)/M} \left( s''(0) h + \frac{\epsilon}{h} \right).
\]

For very large \( M \), this error is dominated by the second term, and the error grows linearly with \( M \). However, let us consider \( \epsilon = 0 \). Then the error is

\[
\bar{r}'(0) - J = MJ^{(M-1)/M} h \cdot s''(0)
\]

\[
= \frac{hr''(0)}{1 - J} \cdot M \left( 1 - J^{(1/M)} \right)
\]

As \( M \to \infty \), the above error estimate asymptotes to a constant:

\[
\frac{hr''(0)}{1 - J} \cdot \log J
\]

Note that the error in \( \bar{r}'(0) \) asymptotes to the above constant even though the error in \( \bar{s}'(0) \) goes to zero, because \( s''(0) \to 0 \), but not fast enough to beat \( M \to \infty \). This asymptotic bias error can be large when \( J \) is close to zero or one.

Thus, while using factorized Poincare maps, care must me taken to trade-off ability to approximate the dynamics with greater refinement along the periodic orbit versus the errors committed by composing too many approximate mappings. To keep the error from compositions low, the accuracy of the factor maps must increase appropriately with \( M \).
Figure A.7: a) The van der Pol limit cycle ($\mu = 0.4$) is shown with 50 Poincare sections, equally spaced in time, normal to the curve, and of equal lengths. Sections longer than the radii of curvature would intersect with neighboring sections. b) For two neighboring sections $S_1$ and $S_2$, transitions from $S_1$ to $S_2$ predict the wrong flow direction when applied to states ‘beyond’ the intersection of the two sections.

A.8.2 Bounded Poincare sections and trust regions

Any approximation to the dynamics around the periodic orbit is likely to have a limited domain of good approximation properties. The following are purely geometric restrictions on the region of validity; we do not discuss possible dynamical restrictions on the region of validity (such as perhaps the basin of attraction of the limit cycle).

Some limitations to the validity of a factorized Poincare map are posed by the curvature of the periodic orbit itself. If one considered orthogonal and planar Poincare sections, neighboring sections arbitrarily close to each other will intersect at the local centers of curvature. Figure A.7a shows sections of a given length close to intersecting each other near regions where the van der Pol limit cycle has high curvature. Figure A.7b shows two consecutive sections $S_1$ and $S_2$ intersecting. Given initial conditions on section $S_1$, a transition to $S_2$ will predict flow in the incorrect direction (clockwise)
beyond the intersection of the two sections. Thus, using the factorized Poincare map based on these sections beyond the intersection point would give qualitatively incorrect results. Therefore, in the construction of models and in the subsequent application of the derived models, we used initial conditions that are not further than the local radius of curvature from the periodic orbit.

To take this idea further, we can define a tube around the periodic orbit, within which the factorized Poincare map approximations are deemed acceptable. For simplicity, say this tube has constant radius. So that each point inside the tube is associated with a unique closest point on the periodic orbit, we might require that the tube cannot have self-intersections or have non-smoothness (kinks). Non-smoothness is eliminated by having the radius of the orthogonal sections at most equal to the local radius of curvature. Defining the local section radius so as to avoid any non-local self-intersections is more complex. Limiting the radius to the local radius of curvature does not eliminate non-local self-intersections (for example, consider a periodic orbit with a nearly flat region). Gonzalez et al [28] have defined sufficient conditions for global non-intersection based on a new concept called global curvature (its global minimum) for three dimensional space curves. We mention a related literature on “ideal knots” (e.g., [49, 86]) which concerns the finding of knot shapes of a given type that admit the maximum radius of the tube around the knot without self-intersections or non-smoothness.

A.8.3 Other factors contributing to approximation quality

As is conventional in approximation of functions, it may be appropriate to judiciously choose the location of the $M$ sections, so as to accomplish the best approximation of the dynamics with a given number of sections. We may need more sections when
either the limit cycle changes direction quickly (high derivatives of its tangent) and when the dynamics around the limit cycle is fast.

We have mostly considered planar Poincare sections that are orthogonal to the limit cycle. But other planar or even curved sections may provide improvements in approximation quality and indeed, enlarge the region over which the factorized Poincare map give reasonable predictions. Ideal sections would be those that are sufficiently far away from losing transversality, while at the same time, perhaps ensuring that the relevant matrices (Jacobians, etc) are well-conditioned.

As noted earlier, isochrons, a special set of Poincare sections, have the property that flow from one isochron to another takes constant time irrespective of the initial condition on the first isochron. Thus, using isochrons, or even the tangent plane to the isochron, may simplify both representation of the factorized Poincare maps and also the interpolations described in section A.6 by not having to track time. However, in some systems, isochrons could be close to tangential.

We have implicitly discussed smooth sufficiently differentiable systems. Many systems of interest with limit cycles are non-smooth (hybrid systems) — for instance, models of legged locomotion with contact and collisions,[63, 84] and stick-slip oscillators with Coulomb friction. Such systems have boundaries in state space across which the equations of motions change (the vector field is not smooth) or there is a discontinuity in state. Factorized Poincare maps are possible for such systems as well. But if the maps between sections are to be smooth, care must be taken to introduce key Poincare sections that are coincident with the surfaces of non-smoothness in state space.

Finally, the goodness of the mapping from one Poincare section to the next can be improved by considering piecewise models instead of the monolithic linear and quadratic models considered here. For instance, Varosi et al [93] considered simplicial
decompositions of a Poincare section and suggested a piecewise linear approximation of the (first return) Poincare map that was able to better capture nonlinear dynamics far away from the limit cycle. Of course, more efficient high-dimensional nonlinear representations such as radial basis functions [13] may also be pursued.

A.9 Conclusion

We have elaborated on ‘factorized Poincare maps’ as a representation of the dynamics in the neighborhood of a periodic orbit. With this representation, we were able to perform many calculations that we might wish to perform of the nonlinear differential equation, at least approximately.

We believe that the factorized Poincare map representation may be especially useful in the context of ‘system identification.’ There has been recent interest in system identification around periodic motions, including from long time-series data from noise-driven dynamics [74, 75, 14, 98], with applications to human and animal locomotion. The formalism of factorized Poincare maps presented herein allows us to use classical statistical inference techniques developed in the statistics literature, in particular ARMA [34, 45, 50] and its specialization to periodic dynamics PARMA [95]. We explore such applications in a sequel in the Chapter 2. Further, we remark that it is possible to generalize the factorized Poincare map formalism here to controlled dynamical systems (e.g., $x = f(x, u)$), with external control inputs $u$. Such controlled factorized Poincare maps may be used as possibly simpler substrates for designing controllers for periodic motions, for instance, designing controllers for legged robots. The factorized Poincare map will be applied to investigate noisy system in the Chapter 2.
Appendix B

ADDITIONAL NOTES ON SYSTEM IDENTIFICATION

B.1 Univariate mapping with one section

Consider the discrete dynamical system in one state variable

\[ x^{(i+1)} = J \cdot x^{(i)} + \epsilon^{(i)}, \]  

(B.1.1)

in which \( x \) and \( J \) are scalars (\( \in \mathbb{R} \)), \( \epsilon^{(i)} \) is a normally distributed noise term with mean zero and variance \( \sigma^2 \) i.e., \( \mathcal{N}(0, \sigma^2) \), and the index \( i \) denotes time. This dynamical system is represented schematically in Figure 2.1a. Equation B.1.1 is identical to the classical AR(1) model in the statistics literature, where AR stands for auto-regressive.

For completeness, in this subsection, we briefly review key dynamical properties of this system and how to identify the model parameters given data from the system. More detailed and more general treatments can be found in the classical statistics, time-series analysis, econometrics, and/or the engineering system identification literature [e.g., 45, 34, 10, 50, 58]. In the following subsections (sections B.2 and B.3), we extend the standard analysis to discrete systems to multiple stages.

**Dynamics with no noise.** When there is no noise (\( \sigma = 0 \)), \( x^* = 0 \) is a fixed point of Equation 2.2.1. When \( |J| < 1 \), the fixed point is globally asymptotically stable and \( x^{(i)} \) converges to zero from any initial condition \( x^{(1)} \). For \( |J| > 1 \), the system is unstable and \( x^{(i)} \) diverges to infinity.
Dynamics with noise. Figure 2.2a shows two simulations of Equation 2.2.1 for two noise levels $\sigma$. In the presence of noise, the state does not ever converge to $x^*$ but it may have steady states for mean and variance. When $|J| < 1$, the mean state converges to $x^* = 0$ from any initial condition:

$$E(x^{(i+1)}) = J \cdot E(x^{(i)}) \quad \text{and} \quad \lim_{i \to \infty} E(x^{(i)}) = 0 \quad \text{(B.1.2)}$$

where the expectation $E(\cdot)$ is over multiple realizations of the noise process $\epsilon$. Squaring Equation B.1.1 and taking expectations on both sides, we can show that the steady state variance $\gamma^2$ of $x^{(i)}$ is given by

$$E\left((x^{(i+1)})^2\right) = J \cdot E\left((x^{(i)})^2\right) + \sigma^2 \quad \text{and} \quad \lim_{i \to \infty} E\left((x^{(i)})^2\right) = \gamma^2 = \frac{\sigma^2}{1 - J^2}. \quad \text{(B.1.3)}$$

The asymptotic variance $\gamma^2$ of $x^{(i)}$ is bounded if and only if $|J| < 1$. When $J = 1$, $x^{(i)}$ is a Brownian motion (random walk) for which the variance increases linearly with time: $\gamma^2(i) \sim i \cdot \sigma^2$. In the following, we assume $|J| < 1$.

System identification using ordinary least square method. Given time series data $x^{(i)}$ for $i = 1 \ldots N$ such as in Figure 2.2a, thought to have been generated by Equation B.1.1, the system identification problem is to find the parameters $J$ and $\sigma$ that best fits the data. The multiplier $J$ may be determined by solving an ordinary least squares problem, determining $\hat{J}$ and $\hat{b}$ that minimizes

$$g(J, b) = \sum_{i=1}^{N-1} \left((x^{(i+1)})^2 - J \cdot x^{(i)} - b\right)^2. \quad \text{(B.1.4)}$$

Here, the constant term $b$ is assuming that the Equation 2.2.1 is generalized so that the fixed point $x^*$ is not necessarily zero, but is instead $-b/(1 + J)$. The least squares estimates citeWeisberg85 of $J$ and $b$, namely $\hat{J}$ and $\hat{b}$, are the slope and the $y$-intercept.
of the best fit line to the scatter plot of \(x^{(i+1)}\) versus \(x^{(i)}\) as shown in Figure 2.2b. The corresponding analytical expressions for the estimates are as follows. If

\[
\mathbf{X}_{in} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(N-1)} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & \mathbf{X}_{in} \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_{out} = \begin{bmatrix} x^{(2)} \\ \vdots \\ x^{(N)} \end{bmatrix},
\]

then

\[
\begin{bmatrix} \hat{b} \\ \hat{J} \end{bmatrix} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{X}_{out} = \mathbf{A} \backslash \mathbf{X}_{out},
\]

where \(\mathbf{1}\) is a column vector of ones, of the same size as \(\mathbf{X}_{in}\). Of course, \((\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T\) is the left pseudo-inverse of \(\mathbf{A}\), and in MATLAB notation, the \((\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{X}_{out}\) can be computed with the so-called backslash operator: \(\mathbf{A} \backslash \mathbf{X}_{out}\).

Completely equivalent to simultaneously solving for \(\hat{b}\) and \(\hat{J}\) as above, we may subtract from \(\mathbf{X}_{in}\) and \(\mathbf{X}_{out}\) their respective means \(\mathbf{X}_{in}\) and \(\mathbf{X}_{out}\), and then fit a line through the origin relating the two variables. That is, \(\hat{J} = (\mathbf{X}_{in} - \mathbf{X}_{in}) \backslash (\mathbf{X}_{out} - \mathbf{X}_{out})\).

**How good is the estimate?** The estimate \(\hat{J}\) depends on the data and is typically different from the true value \(J\). In particular, noting that \(\mathbf{X}_{in} = 0\) asymptotically, the least square estimate \(\hat{J}\) can be written as:

\[
\hat{J} = \mathbf{X}_{in} \backslash \mathbf{X}_{out} = \mathbf{X}_{in} \backslash (J \cdot \mathbf{X}_{in} + \Sigma) = J + \mathbf{X}_{in} \backslash \Sigma,
\]

where \(\Sigma\) is the noise vector \([\epsilon^{(1)} \ldots \epsilon^{(N-1)}]^T\). Thus, the error in the estimate is \(\mathbf{X}_{in} \backslash \Sigma\).

An estimator of a variable is called **unbiased** when its estimate approaches its true value asymptotically as the amount of data \(N\) increases. The estimate \(\hat{J}\) is unbiased because \(\mathbf{X}_{in}\) and \(\Sigma\) are independent. Figure 2.2c shows a distribution of \(\hat{J}\) estimates obtained with different synthetic time series data sets and we see that this distribution is unbiased asymptotically (at least visually). Further, the error variance
of the estimates decreases with the length $N$ of the time series data. It can be shown that the error variance $\psi$ of $\hat{J}$ is:

$$\psi = E\left( (\hat{J} - J)^2 \right) = \frac{1 - J^2}{N}$$

(see Figure 2.2c). (B.1.7)

The above expression requires large $N$ if the initial condition $x^{(1)}$ is specified and is not drawn from the asymptotic stationary distribution. When $|J|$ is closer to one, a larger $N$ is required for a given accuracy of the expression in Equation B.1.7. See [34] for a chapter on how to handle multivariate systems with unit eigenvalues.

**B.2 Univariate mapping with two sections**

Before we consider discrete dynamical systems with arbitrary number of stages or sections, we consider a two stage discrete dynamical system, shown schematically in Figure 2.1b. Here, the state $x^{(i)}_1$ at section $S_1$ is transformed to the state $x^{(i)}_2$ at section $S_2$, then back to a state $x^{(i+1)}_1$ on section $S_1$, and so on; the super-script $i = 1, 2, \ldots$ denotes the index of the cycle, each cycle consisting of two transitions. This two section dynamic system is defined by the following two equations:

$$x^{(i)}_2 = J_{1\to2} x^{(i)}_1 + \epsilon^{(i)}_1, \quad x^{(i+1)}_1 = J_{2\to1} x^{(i)}_2 + \epsilon^{(i)}_2.$$  

(B.2.1)

The noise terms $\epsilon^{(i)}_1, \epsilon^{(i)}_2 \in \mathbb{R}$ are independent normally distributed. The two equations in Equation B.2.1 can be synthesized into a single equation directly from $x^i_1$ to $x^{i+1}_1$ as:

$$x^{(i+1)}_1 = J_{1\to2} J_{2\to1} x^{(i)}_1 + J_{2\to1} \epsilon^{(i)}_1 + \epsilon^{(i)}_2;$$

(B.2.2)

or directly from $x^i_2$ to $x^{i+1}_2$ as:

$$x^{(i+1)}_2 = J_{2\to1} J_{1\to2} x^{(i)}_2 + J_{1\to2} \epsilon^{(i)}_2 + \epsilon^{(i+1)}_1;$$

(B.2.3)

both obtained by rearranging Equation 2.2.1.
The net multiplier \( J = J_{1\rightarrow 2} \cdot J_{2\rightarrow 1} \) can now be estimated one of three different ways. We can:

(a) just use the data \( x^{(i)}_1 \) at section \( S_1 \),

(b) just use the data \( x^{(i)}_2 \) at section \( S_2 \), or

(c) use the data at both sections \( S_1 \) and \( S_2 \) to individually estimate \( J_{1\rightarrow 2} \) and \( J_{2\rightarrow 1} \) and then find \( J = J_{1\rightarrow 2} \cdot J_{2\rightarrow 1} \).

The three methods will thus find \( J \) so as to minimize the squared residuals of the Equations B.2.2, B.2.3, or B.2.1 respectively.

All three methods produce asymptotically unbiased estimates under the assumption of independent noise, but the three estimates have different error variance. Using Equation 2.2.2, the error variance of methods (a) or (b) — using the data at only one of the two sections — is seen to be:

\[
\psi_1 = \frac{1 - (J_{1\rightarrow 2} \cdot J_{2\rightarrow 1})^2}{N} = \frac{1 - J^2}{N} \tag{B.2.4}
\]

The error variance for method (c) — using data from both sections and minimizing the squared residuals of Equation B.2.1 — can be shown to be:

\[
\psi_2 = \frac{1 - J^2}{N} \left( \frac{J_{1\rightarrow 2}^2}{1 + J_{1\rightarrow 2}^2} + \frac{J_{2\rightarrow 1}^2}{1 + J_{2\rightarrow 1}^2} \right) \tag{B.2.5}
\]

When \( |J| = |J_{1\rightarrow 2}J_{2\rightarrow 1}| < 1 \), we can show that \( J_{1\rightarrow 2}^2/(1 + J_{1\rightarrow 2}^2) + J_{2\rightarrow 1}^2/(1 + J_{2\rightarrow 1}^2) < 1 \), demonstrating that \( \psi_2 < \psi_1 \). That is, one obtains estimates with lower error variance when using data at both sections rather than at just one section. In addition, it is easy to prove that under the constraint \( J_{1\rightarrow 2}J_{2\rightarrow 1} = \text{constant} \), the minimum variance happens when \( |J_{1\rightarrow 2}| = |J_{2\rightarrow 1}| = \sqrt{|J|} \). Further, again for a given \( J \), the worst \( \psi_2 \) is obtained if one of \( J_{1\rightarrow 2} \) or \( J_{2\rightarrow 1} \to \infty \) and this upper bound is identical to \( \psi_1 \). That is, under the assumptions above, using two sections in this manner is never worse – never produces a higher error variance – than using just one section.
B.3 Univariate mapping with many sections

Now we consider the univariate mapping with $M$ sections, depicted schematically in Figure 2.1c. The system goes from one section to the next, and then looping back from section $S_M$ to section $S_1$. The transition between sections are given by the following $M$ equations:

\[
x_2^{(i)} = J_{1 \rightarrow 2} x_1^{(i)} + \epsilon_1^{(i)};
\]

\[
\ldots
\]

\[
x_M^{(i)} = J_{M-1 \rightarrow M} x_{M-1}^{(i)} + \epsilon_{M-1}^{(i)};
\]

\[
x_1^{(i+1)} = J_{M-1} x_M^{(i)} + \epsilon_M^{(i)};
\]

where $i = 1, 2, \ldots, N$ is the cycle number, $J_{1 \rightarrow 2}, J_{2 \rightarrow 3}, \ldots, J_{M-1}$ are scalars, and as usual, $\epsilon_j^{(i)}$ are from identically independent normal distributions. As in Equations B.2.2 & B.2.3, we can combine the equations in Equation B.3.1 into a single equation that expresses the transition from the state at section $S_j$ back to section $S_j$ as follows:

\[
x_1^{(i+1)} = J x_1^{(i)} + \eta_1^{(i)};
\]

\[
\ldots
\]

\[
x_j^{(i+1)} = J x_j^{(i)} + \eta_j^{(i)};
\]

\[
\ldots
\]

\[
x_M^{(i+1)} = J x_M^{(i)} + \eta_M^{(i)},
\]

in which $\eta_j$ is a known linear combination of the noises ($\epsilon$’s) in Equation B.3.1; for instance, $\eta_1^{(i)} = \epsilon_1^{(i)} \prod_{k \neq 1}^{M} J_{k \rightarrow k+1} + \ldots + \epsilon_M^{(i)}$ and so on.

Similar to the two section case, we now have a few different ways of estimating $J$. For instance, we may choose to just use the time series data $x_j^{(i)}$ corresponding to one of the sections, say $S_j$, and find $\hat{J}$ that minimizes the corresponding squared residual in Equation B.3.2. We know from Equation 2.2.2 that the variance of each
such estimate will be \((1 - J^2)/N\) (note that this variance does not rely on the size of the noise term).

Alternatively, we may use the data at all \(M\) sections, find the individual \(\hat{J}_j\) by minimizing the squared residuals in Equation B.3.1, and then computing \(\hat{J} = \hat{J}_{1\to2}\hat{J}_{2\to3} \ldots \hat{J}_{M\to1}\). The expression for the error \(\psi_M\) is complicated. Here, we just show the expressions for \(\psi_3\).

\[
\psi_3 = \frac{1 - J^2}{N} \left( \frac{J^2_{2\to3}J^2_{3\to1}}{J^2_{3\to1}J^2_{1\to2} + J^2_{1\to2} + 1} + \frac{J^2_{1\to2}J^2_{2\to3}}{J^2_{1\to2}J^2_{2\to3} + J^2_{2\to3} + 1} \right).
\]

We believe that \(\psi_M\) is minimized when the \(\hat{J}_{j\to j+1}\) are all equal as can be easily verified for \(\psi_3\) above. When the \(\hat{J}_{j\to j+1}\) are equal to \(a = J^{1/M}\), the expression for \(\psi_M\) reduces to:

\[
\psi_M = \frac{1 - J^2}{N} \cdot \frac{M(a^2)^{M-1}}{1 + a^2 + (a^2)^2 + \ldots + (a^2)^{M-1}} = \frac{J^2}{N} \cdot \frac{M(1 - J^{2/M})}{J^{2/M}}. \tag{B.3.3}
\]

Figure 2.3b shows how the improvement in error variance compared to using only one section \(\psi_M/\psi_1\) changes with the number of sections \(M\) and the multiplier \(J\). When \(J\) is smaller, more improvement may be obtained. Recall, the \(J = 0\) limit is when the dynamic system Equation B.3.1 degenerates to pure random noise and the \(J = 1\) limit is when the dynamic system degenerates to a Brownian motion.

When we have a large number of sections, that is, as \(M\) tends to infinity, we obtain the following limit from Equation B.3.3:

\[
\psi_\infty = -\frac{2J^2\log(J)}{N} \quad \text{and} \quad \psi_\infty = -\frac{2J^2\log(J)}{1 - J^2}. \tag{B.3.4}
\]

The above expression for \(\psi_\infty\) shows that we cannot drive the error \(\psi_M\) to zero by increasing \(M\). The only way to get the error to zero is to increase the number of
cycles as well: \( N \to \infty \). But for a given number of cycles \( N \), the ratio \( \psi_\infty / \psi_1 \) is the improvement we obtain by considering arbitrarily many sections compared to just one section. The expression in Equation B.3.4, plotted in Figure 2.3c shows how this asymptotic improvement depends on \( J \); more improvement for smaller \( J \). The improvement seems almost linear in \( J \). Thus, when \( J = 0.1 \), we may reduce the error variance by a factor of 10. But when \( J = 0.9 \), we would reduce the error by only about 10%.

B.4 Iteratively reweighted least squares and maximum likelihood estimation adapted to multiple section inference

In this appendix, we outline the relationship between ordinary least squares and maximum likelihood estimation, and then, discuss the iteratively re-weighted least squares method as an approximation to MLE, so as to obtain asymptotically unbiased estimates. The following also applies, as a special case, to the inference of the multivariate maps in section 2.5.2.

Linear regression with correlated noise.

Following Example 1 of [64] closely, consider first the following linear regression problem with correlated noise:

\[
Y^{(i)} = X^{(i)} \beta + \epsilon \tag{B.4.1}
\]

where \( Y^{(i)} \in \mathbb{R}^p \) is the \( i^{th} \) observation from \( N \) observations, \( p \) is the dimension of each data point, \( X^{(i)} \in \mathbb{R}^{p \times q} \) the values of the independent variables for the \( i^{th} \) observation, \( \beta \in \mathbb{R}^q \) contains the regression coefficients to be determined, and \( \epsilon \in \mathbb{R}^p \) is a vector containing the noise terms. Say the noise vector \( \epsilon \) is a general multivariate normal distribution, with correlations between elements so that the covariance matrix
\( E(\epsilon\epsilon^T) = V \) is not necessarily diagonal. The noise terms in different observations are uncorrelated. (The rest of this sub-section is applicable to the inference of a single mapping, either univariate or multivariate.)

If the covariance matrix \( V \) is known, the “correct” weighted least squares problem to solve, to find \( \beta \) is to minimize:

\[
g(\beta) = \sum_{i=1}^{N} (Y^{(i)} - X^{(i)}\beta)^T V^{-1} (Y^{(i)} - X^{(i)}\beta). \tag{B.4.2}
\]

One rationale for the above weighted least squares problem comes from Maximum Likelihood Estimation. Given normality of noise, the logarithm of the likelihood \( L \) of observing the data is given by:

\[
\ell = \log(L) = -\frac{Np}{2} \log 2\pi - \frac{N}{2} \log |V| \ldots
- \frac{1}{2} \sum_{i=1}^{N} (Y^{(i)} - X^{(i)}\beta)^T V^{-1} (Y^{(i)} - X^{(i)}\beta). \tag{B.4.3}
\]

Given the covariance matrix \( V \), maximizing the likelihood \( L \) or the log-likelihood \( \ell \) are equivalent to minimizing the weighted sum of squared residuals \( g(\beta) \) in Eq. B.4.2.

If the covariance matrix \( V \) is not known \textit{a priori}, the maximum likelihood estimate is obtained by maximizing the log-likelihood \( \ell \) with respect to both \( \beta \) and \( V \). No general closed form solutions exist, but one can solve this optimization problem numerically, using some nonlinear programming technique cite[e.g.,]Noc99,Gil02. Short of performing such a simultaneous optimization, one can perform a sequence of weighted least squares problems, each of which has a constant weighting matrix \( \hat{V} \), which gets updated iteratively; this procedure is called \textbf{iteratively reweighted least squares} citeRubin83,Meng93. The typical procedure is starting with step 1 below and repeating steps 2 and 3 until convergence.

1. Assume the initial weighting matrix \( \hat{V} \) to be identity \( I \).
2. Obtain the least squares estimate $\hat{\beta}$ based on the current weighting matrix $V$.

$$
\hat{\beta} = \left\{ \sum_{i=1}^{N} X^{(i)T} \hat{V}^{-1} X^{(i)} \right\}^{-1} \left\{ \sum_{i=1}^{N} X^{(i)T} \hat{V}^{-1} Y^{(i)} \right\}
$$

(B.4.4)

3. Update the weighting matrix to the estimated error covariance.

$$
\hat{V} = \frac{1}{N} \sum_{i=1}^{N} \left( Y^{(i)} - X^{(i)} \hat{\beta} \right) \left( Y^{(i)} - X^{(i)} \hat{\beta} \right)^T.
$$

(B.4.5)

It can be shown that the updates for $\hat{\beta}$ and $\hat{V}$ in equations B.4.4 and B.4.5 maximize the log-likelihood $\ell$ from equation B.4.3, holding $\hat{V}$ and $\hat{\beta}$ constant respectively. Thus, the above iteration is effectively applying the standard optimization technique called “coordinate descent” [25] to maximizing $\ell$, alternating between $\beta$ and $V$. When the iterations converge to an extremum of $\ell$, one will have obtained the maximum likelihood estimates of $\beta$ and $V$ citeMeng93. In numerical experiments with synthetic data using Eq. B.4.1, we have found that the above iterations converged to the same values for $\hat{\beta}$ and $\hat{V}$, up to about 4 significant digits, when compared to full numerical optimization using MATLAB’s general nonlinear programming solvers (we used the function fminunc).

Finally, we note that in the above regressions, we have assumed no constant term; for instance, instead of Eq. B.4.1, we could have assumed:

$$
Y^{(i)} = X^{(i)} \beta + \gamma + \epsilon
$$

(B.4.6)

where $\gamma$ is some constant offset, to be determined. It can be shown that the both the least squares estimate and the maximum likelihood estimate of $\gamma$ is simply the mean of $(Y^{(i)} - X^{(i)} \hat{\beta})$ over the observations.

$$
\hat{\gamma} = \bar{Y} - \bar{X} \hat{\beta}
$$

(B.4.7)

where the overbar denotes averages over the observations. If $\gamma$ is not known to be zero, as is the case in all our inference problems (i.e., we do not know where the
periodic point of the Poincare map is), we can obtain the estimate $\hat{\beta}$ by replacing $Y^{(i)}$ by $(Y^{(i)} - \bar{Y})$ and $X^{(i)}$ by $(X^{(i)} - \bar{X})$ in equations B.4.3, B.4.4 and B.4.5.

**Adapting to multiple sections inference: the univariate case.**

We briefly describe how we adapted the above iterations to approximate the maximum likelihood estimate of the $J$’s when we have multiple sections and periodicity. The following was used in section 2.3.4 and only describes the univariate case.

Recall again the equations mapping the state from one Poincare section to the next when there are $M$ sections.

\[
x_2^{(i)} - x_2^* = J_{1-2} (x_1^{(i)} - x_1^*) + \epsilon_1^{(i)}
\]

\[
\ldots \ldots \ldots 
\]

\[
x_M^{(i)} - x_M^* = J_{M-1-M} (x_{M-1}^{(i)} - x_{M-1}^*) + \epsilon_{M-1}^{(i)}
\]

\[
x_1^{(i+1)} - x_1^* = J_{M-1} (x_M^{(i)} - x_M^*) + \epsilon_M^{(i)}.
\]

For the periodic orbit $x_j^*$, we simply use the mean of the intersections $x_j^{(i)}$. Then, we only have to estimate the $J$’s. The $J$’s are estimated by identifying equation B.4.8 with equation B.4.1, with

\[
Y^{(i)} = \begin{bmatrix} x_2^{(i)} - x_2^* \\ \vdots \\ x_M^{(i)} - x_M^* \\ x_1^{(i+1)} - x_1^* \end{bmatrix}, \quad X^{(i)} = \text{diag} \begin{bmatrix} x_1^{(i)} - x_1^* \\ \vdots \\ x_{M-1}^{(i)} - x_{M-1}^* \\ x_M^{(i)} - x_M^* \end{bmatrix},
\]

\[
\beta = \begin{bmatrix} J_{1-2} \\ \vdots \\ J_{M-1-M} \\ J_{M-1} \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1^{(i)} \\ \vdots \\ \epsilon_{M-1}^{(i)} \\ \epsilon_M^{(i)} \end{bmatrix}
\]

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where ‘diag’ of an array is a square diagonal matrix with the elements in the array on the diagonal. By making the identifications in the above equation, we are implicitly making the approximation that the noise terms for any two section-to-section transition within a given cycle are correlated, but the noise terms for section-to-section transitions that belong to different cycles, $\epsilon^{(i)}$ and $\epsilon^{(i+1)}$ are uncorrelated. This is not quite true, as at least the noise terms $\epsilon^{(i)}_M$ and $\epsilon^{(i+1)}_1$ are correlated. But ignoring this correlation appears not take away from the good properties of the estimator, as transitions between sections well-separated in time have low noise correlations.
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