A Study of the Rigidity of Regular Polytopes

A Thesis

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Abstract

This paper will look at the basic notions behind rigidity. We will consider bar frameworks and plate hinge structures of regular convex polytopes in many dimensions and determine their rigidity.

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Chapter 1: Introduction

Imagine building a bookcase. There are solid shelves, sides, a top, and a bottom, but you forget to add a back to the bookcase. That would not be a very good bookcase. Without a back, the whole structure would be flexible, able to lean to one side or the other, and once it was under the weight of the books it was built to support, it would collapse.

Once you see that your creation is defective, however, you add a back and find that now the bookcase, no longer flexible, but rigid, is able to support all of your books.

The rigidity, or inability to change shape without breaking, of a structure depends on multiple factors. Rigidity of a structure depends on what material it it made out of, what forces are being placed on the structure, and finally, the geometry of the structure.

Example 1. *Imagine the square on the left built with rigid rods and flexible vertices. If put under enough pressure, the square will deform into the second structure. If another diagonal rod is added like in the figure on the right, however, like the bookcase, the structure is rigid.*

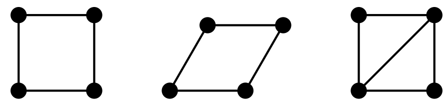

Figure 1.1

The study of the geometry of rigid structures dates back to 1766 when Euler made his conjecture that "a closed spatial figure allows no changes, as long as it is not ripped apart." This closed spatial figure that he is talking about is a closed polyhedral surface built out of polygonal plates which are all attached by hinges. Though there were countless examples that seemed to confirm this conjecture, it was never proved.

Forty-seven years later, however, Cauchy did prove a part of the conjecture. He confirmed that a *convex* closed spatial figure allows no changes, as long as it is not ripped apart.

In 1975, Gluck showed that Euler's conjecture is almost always true. He showed that a closed, simply connected, generic spatial figure allows no changes, as long as it is not ripped apart.

Just two years later in 1977, Robert Connelly finally gave a counterexample to Euler's conjecture.

This paper looks at the rigidity theorems that have thus been proved, and it applies them to specific types of structures, regular, convex polytopes.

Chapter 2: An Introduction to Graphs, Polytopes, and Rigidity

In this section we are going to talk about basic notions and definitions concerning rigidity.

Definition. A **graph** G is a set V of vertices and a set E of edges where each edge in E is a pair of vertices in V.

Example 2. *Consider* $V = \{1, 2, 3, 4, 5, 6, 7\}$ *and*

$E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (6, 7), (1, 2)\}$,

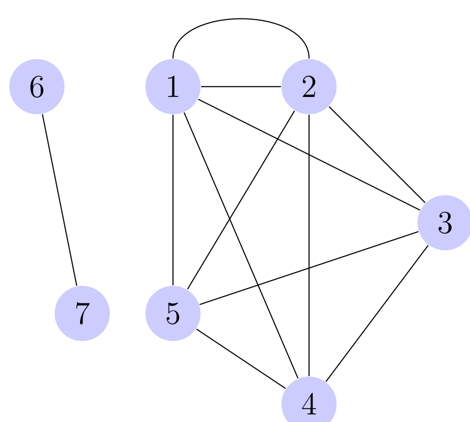

Figure 2.1: An example of a graph.

Definition. A graph is called **simple** if there are no loops and no pair of vertices is connected by more than one edge.

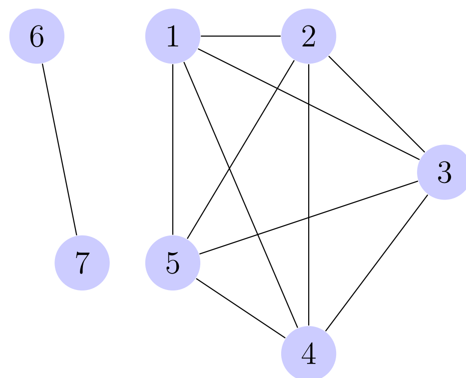

Figure 2.2: An example of a simple graph.

Definition. A graph is called **connected** if there exists a sequence of edges and vertices between any given pair of vertices.

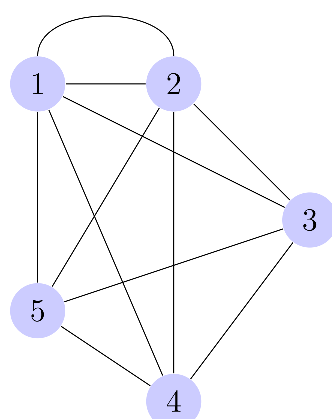

Figure 2.3: An example of a connected graph.

Definition. A graph is called **planar** if it can be drawn in the plane such that no edges cross except at the vertices.

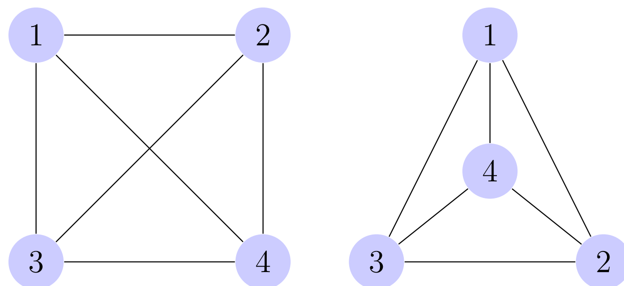

Figure 2.4: An example of a planar graph. Note that these two graphs are identical and planar. While the one on the left has edges that cross, with some manipulation it can be represented as the one on the right with no edges crossing.

Assume from here on that all graphs are planar.

At this point we'll need Euler's famous formula about planar graphs, so that we can later use it for proofs of the rigidity theorems:

Theorem 2.0.1 (Euler's Formula)**.** *Let G be a connected plane graph with v vertices, e edges, and f faces. Then the the formula $v - e + f = 2$ holds.*

Proof. Proceed by induction on e. If $e = 0$, then in this somewhat degenerate case we have one vertex and one face (the whole plane) showing us that $v - e + f = 2$.

For the inductive step, assume that G has $n > 0$ edges, and that Euler's formula holds for all graphs with $n - 1$ edges.

Case 1: Suppose G has no bounded regions. Then $f = 1$ and $e = v - 1$ because there is only one edge attached to each vertex. Thus, $v - e + f = v - (v - 1) + 1 = 2$.

Case 2: Suppose G has bounded regions. Then remove an edge from a bounded region, giving a new planar graph G', which has $n-1$ edges, v vertices, and $f-1$ faces. Since G' has $n-1$ edges, by the inductive assumption Euler's formula holds for G'. Thus,

$$\begin{aligned} 2 &= v - (n-1) + (f-1) \\ &= v - n + f. \end{aligned}$$

Therefore, Euler's formula also holds for G. □

Definition. A **polygon** is a closed 2-dimensional figure that is bounded by a finite number of line segments.

Definition. A **polyhedron** is a 3-dimensional solid that is bounded by a finite number of polygons.

Every convex polyhedron can be represented as a connected planar graph.

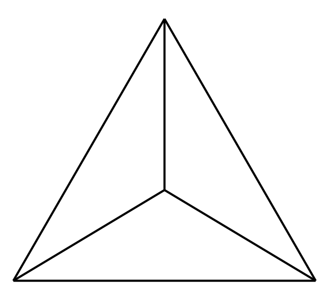

Figure 2.5: A pyramid represented by a planar graph.

Thus, we can see that Euler's formula can be applied to convex polyhedra. For every convex polyhedron with v vertices, e edges, and f faces. Then the the formula $v - e + f = 2$ holds.

Definition. An ***n*-polytope** is an n-dimensional region bounded by a finite number of $(n-1)$-dimensional polytopes.

Definition. An n-polytope is **convex** if, for each pair of points inside the polytope, the segment connecting the two points is also inside the polytope.

Let us now talk about specific types of polytopes, namely regular polytopes.

Definition. A **regular 2-polytope** is a polygon that has internal angles of the same measure and sides of all the same length.

Ludwig Schlafli developed an unconventional way to represent certain types of n-polytopes. To start, a regular 2-polytope can be represented as $\{p\}$, where p is the number of edges of the polygon. For a 3-polytope, $\{p, q\}$ has q p-sided regular polygons around each vertex. Similarly, $\{p, q, r\}$ represents a 4-polytope with r $\{p, q\}$ polyhedra around each edge. An n-polytope can then be represented recursively as $\{p_1, p_2, ..., p_{n-1}\}$, where p_1 is the number of edges of each face, p_2 is the number of faces surrounding each vertex, p_3 is the number of polyhedra around each edge, and so on. One should note that a symbol does not guarantee the existence of that specific polytope. For instance $\{4, 5\}$ would represent a 3-polytope that has 5 squares around each vertex. Since the internal angle of a square is $90°$, it is clearly impossible to fit 5 squares around a single vertex.

Definition. A **facet** of an n-polytope is an $(n-1)$-face. A facet of $\{p_1, p_2, ..., p_{n-1}\}$ can be represented with a Schlafli symbol as $\{p_1, p_2, ..., p_{n-2}\}$

A facet of a 3-polytope is is a 2-polytope, and a facet of a 4-polytope is a 3-polytope.

Definition. The **vertex figure** of an n-polytope, $\{p_1, p_2, ..., p_{n-1}\}$, is the $(n-1)$-polytope represented by the Schlafli symbol $\{p_2, p_2, ..., p_{n-1}\}$

The vertex figure of a polygon, is simply a line segment, and the vertex figure of a polyhedron is a polygon. For example, the vertex figure of a cube, represented as $\{4,3\}$, is an equilateral triangle, represented as $\{3\}$.

Definition. A convex n-polytope is **regular** if the $(n-1)$-facets, $\{p_1, p_2, ..., p_{n-2}\}$, of the polytope are regular and if the vertex figure, $\{p_2, p_3, ..., p_{n-1}\}$, of the polytope is regular.

Euler's formula was also generalized for higher dimensions by Ludwig Schlafli. Note that for line segments, the formula $v = 2$ always holds; for convex polygons, $v - e = 0$ is always true. Euler's generalized formula for any convex n-polytope is

$$\sum_{i=0}^{n-1}(-1)^i N_i = 1 - (-1)^n \tag{2.1}$$

where N_i is the number of i-dimensional facets in the n-polytope (i.e. N_0 =number of vertices, N_1 =number of edges,...).

2.0.1 Cauchy's Rigidity Theorem

While Euler's conjecture that all closed polyhedral structures built with plates and hinges are rigid has never been proved, Cauchy was able to prove that all *convex* plate and hinge polyhedra are rigid. Now we give a corollary to Euler's formula.

Corollary 2.0.2. *Consider a polyhedron G with its edges labeled $+$ or $-$. Then there exists a vertex and a cycle containing this vertex on the polyhedron with at most two sign-changes.*

Proof. Define a corner of the polyhedron as where two edges meet on a face at a vertex. Let c be the number of corners of all the faces of G where sign alterations occur.

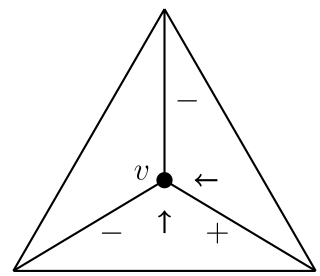

Figure 2.6: The arrows point to corners with sign changes.

Seeking a contradiction, assume that every vertex has more than two sign changes in the cyclic order of the edges around the vertex. Then $c \geqslant 4v$. Note that this is the case because there must be an even number of sign alternations at every vertex and there are at least three corners at each vertex.

Now consider that if a face of G has $2k$ or $2k+1$ edges, there can be at most $2k$ sign changes. Letting f_k be the number of faces of G with k edges, first note that the number of edges can be given as

$$e = \frac{1}{2}(3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + 8f_8 + \cdots)$$

and the number of faces as

$$f = f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + \cdots$$

Based on the assumption that $c \geqslant 4v$, it must be that

$$
\begin{aligned}
4v \leqslant c &\leqslant 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + \cdots \\
&\leqslant 2f_3 + 4f_4 + 6f_5 + 8f_6 + 10f_7 + \cdots \\
&= 2(3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + \cdots) \\
&- 4(f_3 + f_4 + f_5 + f_6 + f_7 + \cdots) \\
&= 4e - 4f
\end{aligned}
$$

This yeilds the result that $e \geqslant v + f$, which contradicts Euler's formula. Hence, it must be that there exists a vertex with at most two sign-changes in the path going around the edges of the vertex. □

Definition. An n-polytope is **rigid** if all embeddings in $\mathbb{R}^n$ that preserve the edges and connections of the polytope are isometries.

Lemma 2.0.3 (Cauchy's Arm Lemma)**.** *Let P and P' be two planar or spherical polygons, both with n-sides. If all but one of the sides are of the same length, and if all the remaining angles of P' are smaller than or equal to the corresponding angle in P, then the length of the omitted side of P' is less than or equal to the one in P.*

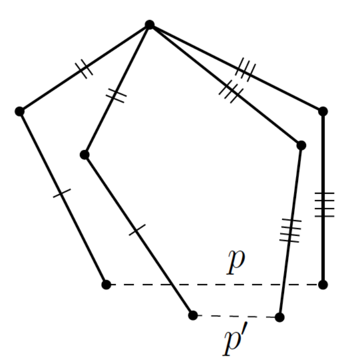

Figure 2.7: The remaining side of P' is smaller than that of P.

Equality holds if and only if all the angles are equal.

Proof. Proceed by induction on n. If $n = 3$, then an application of the law of cosines shows that if two sides remain the same and one angle increases or decreases, then the remaining side will increase or decrease, respectively. Now let $n \geqslant 4$. Let a_i and a'_i denote an angle of P and P', respectively, and let p_i and p'_i denote the vertices. If $a_i = a'_i$ for $2 \leqslant i \leqslant n-1$, then p_i and p'_i can be cut off by creating new sides $p_{i-1}p_{i+1} = p'_{i-1}p'_{i+1}$. see Figure 2.8

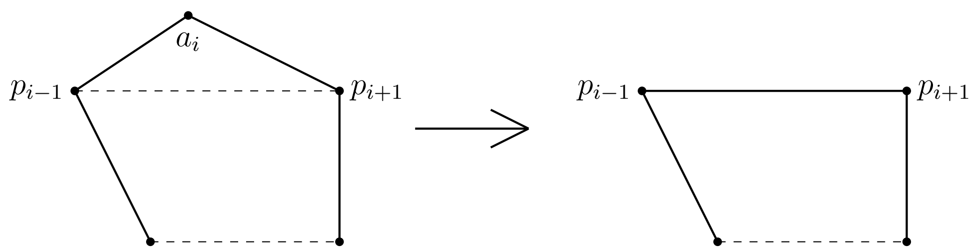

Figure 2.8: cutting off p_i.

Repeatedly doing this will reduce P and P' to the $n = 3$ case. Thus, assume $a_i < a'_i$ for $2 \leqslant i \leqslant n-1$

We now create a new polygon, P^* from P by replacing the angle a_{n-1} with the largest possible angle a^*_{n-1} such that $a^*_{n-1} \leq a'_{n-1}$ and P^* is still convex. We do this by replacing the vertex p_n with p^*_n.

If $a^*_{n-1} = a'_{n-1}$, then we can conclude that the length of the edge between p_1 and p_n is less than that between p'_1 and p'_n.

Otherwise, when $\overline{p_1p^*_n} > \overline{p_1p_n}$, we end up with

$$\overline{p_2p_1} + \overline{p_1p^*_n} = \overline{p_2p^*_n}$$

In other words, p_1, p_2, and p_n^* are collinear. Also note that, by induction on n, and by ignorning the vertices p_1 and p_1', we have that

$$\overline{p_2 p_n^*} \leq \overline{p_2' p_n'}$$

This finally yeilds

$$\overline{p_1' p_n'} \geq \overline{p_2' p_n'} - \overline{p_1' p_2'} \geq \overline{p_2' p_n^*} - \overline{p_1 p_2} = \overline{p_1 p_n^*} > \overline{p_1 p_n}$$

□

Theorem 2.0.4 (Cauchy's Rigidity Theorem)**.** *If a 3-dimensional convex polyhedron has rigid faces (like a shape with cardboard faces) then the polyhedron itself is rigid, even if the edges are connected with flexible "hinges."*

Proof. Seeking a contradiction, suppose there was a convex polyhedron that flexed. Call the polyhedron P before the flex and P' after. Since the polyhedron flexes, not all the corresponding angles between the faces of P and P' can be equal, i.e. some interior angles must be opening or closing. If, when flexed, an interior angle of P opens, label its corresponding edge "+".

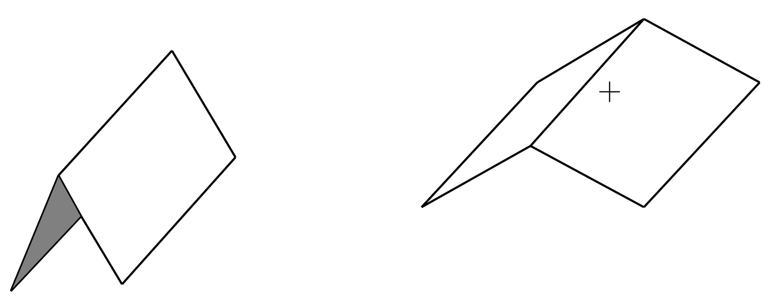

Figure 2.9: Interior angle opening.

If, on the other hand, an interior angle closes, label the corresponding edge "$-$".

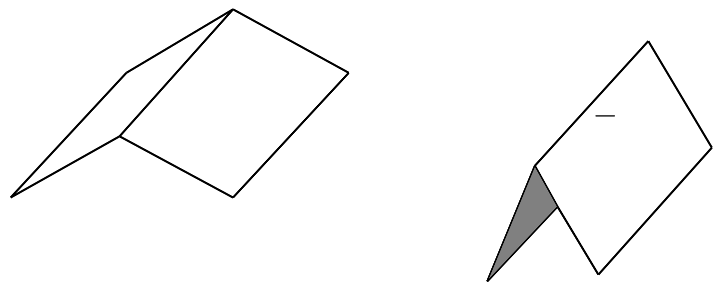

Figure 2.10: Interior angle closing.

To ensure that all edges are marked, also mark the angles that are unchanged with a $+$. From Corollary 2.0.2, we know that we can find a vertex p such that it is connected to at least one $+$ or $-$ edge, and such that, in cyclic order, there are at most two changes between $+$ and $-$ edges. Let $p \in P$ and $p' \in P'$ be corresponding vertices. Now intersect P with a small sphere S_ε with the vertex p at the center and with a radius of ε.

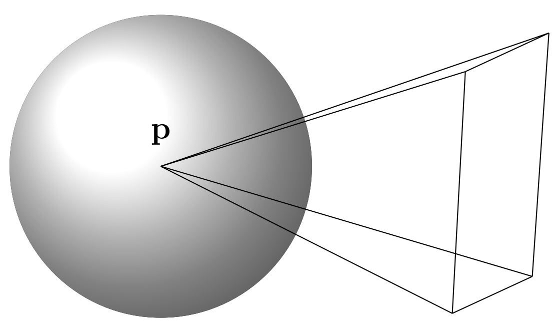

Figure 2.11: S_ε intersected with convex polyhedron.

Also intersect P' with the sphere S'_ε, which has the same radius, and is centered at the corresponding vertex p'. Inside S_ε and S'_ε there are convex spherical polygons Q and Q' such that the edge lengths of the polygons are equal to the face angles in the polyhedra, and the angles of the polygons are equal to the dihedral angles of the polyhedra. Since S_ε and S'_ε have the same radius, and since the faces of P and P' are the same, the corresponding edge lengths of Q and Q' are equal. However, since the dihedral angles are changed through flexing, at least one of the angles of the polygons must differ.

Now if an angle in Q is smaller than the corresponding angle in Q', mark it with a $+$, and if an angle in Q is larger than the corresponding angle in Q', mark it with a $-$. Mark the angle with a 0 if the corresponding angles are equal. It is important to note that the angles of Q cannot be marked all $+$, all $-$, or all 0. This would either lead to a contradiction by Cauchy's Arm Lemma if the angles were all $+$ or all $-$, or, if all the angles were 0, this would contradict the original assumption that P flexes into P'. Let us call a $+/-$ transition one sign change. Because of the choice in p, it is known that there are at most two sign changes. If there are two sign changes, one can draw a chord across Q separating the $+$ and $-$ changes. Then using Cauchy's Arm Lemma, one can again reach a contradiction. □

2.0.2 Infinitesimal Rigidity and the Rigidity Matrix

Cauchy's rigidity theorem showed that all convex polyhedra with rigid faces are rigid. What about polyhedra that do not have rigid faces? Consider polyhedra without solid faces that are constructed only with rigid bars to be used as edges and that have flexible joints as vertices. Let us call these constructions *bar frameworks*. Consider a finite collection of v vertices $\mathbf{p} = (p_1, p_2, ..., p_v)$. Each vertex p_i is in $\mathbb{R}^n$.

Definition. Let G be a graph on the vertices $p_1, p_2, ..., p_v$, where each edge is a rigid bar.The pair G and $\mathbf{p}$, denoted $G(\mathbf{p})$, is a **bar framework** in $\mathbb{R}^n$.

When is a bar framework of a polyhedron rigid? What about of an n-polytope? Consider that the length of each edge of a bar framework is given by $|p_i - p_j|$, the Eulcidean distance between two vertices.

Let $G(\mathbf{p})$ and $G(\mathbf{q})$ be two bar frameworks.

$G(\mathbf{p})$ and $G(\mathbf{q})$ are *equivalent* if they have the same edge lengths.

$G(\mathbf{p})$ and $G(\mathbf{q})$ are *congruent* if $|p_i - p_j| = |q_i - q_j|$ for all $1 \leq i \leq v$ and $1 \leq j \leq v$.

$G(\mathbf{p})$ is *globally rigid* if for every $G(\mathbf{q})$ equivalent to $G(\mathbf{p})$, $G(\mathbf{q})$ is also congruent to $G(\mathbf{p})$. It turns out that global rigidity is a very strong property and determining whether a bar framework has that property is a bit tricky. Fortunately, there are other properties that are easier to demonstrate.

Let $\mathbf{q} = q_1, q_2, ..., q_v$ be a configuration that represents a flex of $\mathbf{p}$. Then $G(\mathbf{q})$ represents the bar framework of $G(\mathbf{p})$ *after* a flex. A flex is said to be infinitesimal if for all $p_i, p_j \in \mathbf{p}$ and $q_i, q_j \in \mathbf{q}$

$$(p_i - p_j) \cdot (q_i - q_j) = 0.$$

A bar framework is then said to be *infitesimally rigid* if all its flexes are infinitesimal.

R. Connelly and Asimow and Roth explain how one can determine if a bar framework is infinitesimally rigid. Consider the collection of vertices $\mathbf{p} = (p_1, p_2, ..., p_v)$ as a single point in $\mathbb{R}^{nv}$. Also suppose that the graph G has e edges.

Determining whether $G(\mathbf{p})$ is infinitesimally rigid involves solving e linear equations with nv unknowns. We can do this by constructing a **rigidity matrix**.

Each row of the rigidity matrix corresponds to an edge of $G(\mathbf{p})$, and the columns correspond to the vertices. A bar framework in $\mathbb{R}^n$ with v vertices and e edges will have a rigidity matrix with e rows and nv columns. So an entry in row e and column p_i would be

$$p_i - p_j \text{ if the edge } \{p_i p_j\} \text{ is incident to } p_i$$

$$0 \text{ if the edge } \{p_i p_j\} \text{ is not incident to } p_i$$

Here is a simple example.

Example 3. *Consider a triangle bar framework. There are* 3 *vertices in* $\mathbb{R}^2$ *and* 3 *edges. Note that since the vertices are in* $\mathbb{R}^2$*, each* $p_i = \{x_i, y_i\}$

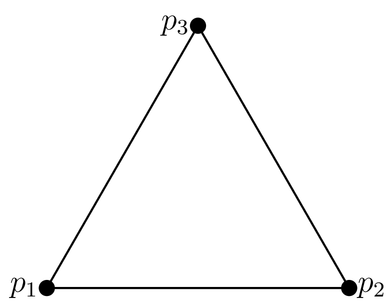

Figure 2.12: A bar framework of a triangle in $\mathbb{R}^2$

This framework has the following 3×6 *rigidity matrix. Each vertex* p_i *is at a coordinate* (x_i, y_i)*. Choosing different* (x_i, y_i)*, we can represent any triangle's rigidity matrix as*

$$\begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 \end{bmatrix}$$

Example 4. *A bar framework representing a tetrahedron has* 4 *vertices in* $\mathbb{R}^3$ *and* 6 *edges.*

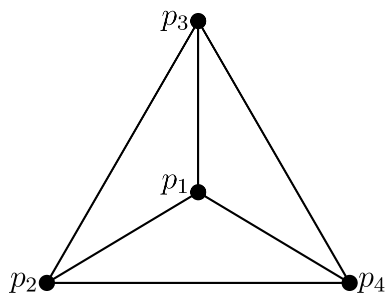

Figure 2.13: A bar framework of a tetrahedron in $\mathbb{R}^3$

Remember that each $p_i \in \mathbb{R}^3$*, so each* $p_i = (x_i, y_i, z_i)$*. The tetrahedron has the following* 6×12 *rigidity matrix.*

$$\begin{bmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 & x_2-x_1 & y_2-y_1 & z_2-z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 & 0 & 0 & 0 & x_3-x_1 & y_3-y_1 & z_3-z_1 & 0 & 0 & 0 \\ x_1-x_4 & y_1-y_4 & z_1-z_4 & 0 & 0 & 0 & 0 & 0 & 0 & x_4-x_1 & y_4-y_1 & z_4-z_1 \\ 0 & 0 & 0 & x_2-x_3 & y_2-y_3 & z_2-z_3 & x_3-x_2 & y_3-y_2 & z_3-z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2-x_4 & y_2-y_4 & z_2-z_4 & 0 & 0 & 0 & x_4-x_2 & y_4-y_2 & z_4-z_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_3-x_4 & y_3-y_4 & z_3-z_4 & x_4-x_3 & y_4-y_3 & z_4-z_3 \end{bmatrix}$$

One can get important information about the rigidity of an n-polytope from its rigidity matrix. Let $\mathbf{q} = q_1, q_2, ..., q_v$ be another configuration that represents the flex of $G(\mathbf{p})$; it can be represented as a column vector in $\mathbb{R}^{nv}$. Now consider $R(\mathbf{p})\mathbf{q}$ The row representing the $\{i, k\}$ edge will be $(p_i - p_j) \cdot (q_i - q_j)$. Thus, recalling that a flex is infinitesimal if, $(p_i - p_j) \cdot (q_i - q_j) = 0$, we can see that a bar framework is infinitesimally rigid if $R(\mathbf{p})\mathbf{q} = 0$.

$$R(\mathbf{p}) = \begin{pmatrix} \overset{i}{\cdots} & \cdots & \overset{j}{\cdots} \\ (p_i - p_j) & \cdots & (p_j - p_i) \\ \cdots & \cdots & \cdots \end{pmatrix} \mathbf{q} = \begin{bmatrix} q_i \\ \vdots \\ q_j \end{bmatrix}$$

The following theorem tells us exactly when that is the case.

Theorem 2.0.5. *For a bar framework $G(\boldsymbol{p})$ in $\mathbb{R}^n$ with v vertices, $G(\boldsymbol{p})$ is infinitesimally rigid in $\mathbb{R}^n$ if and only if the rank of the rigidity matrix $R(\boldsymbol{p})$ is $vn - \frac{1}{2}n(n+1)$. Otherwise, the framework is flexible*

Finally, there is another theorem that connects infinitesimal rigidity to rigidity [4].

Theorem 2.0.6. *A bar framework in $\mathbb{R}^n$ is rigid if and only if it is infinitesimally rigid.*

One can now determine the rigidity of any triangle bar framework. Let M be the rigidity matrix for any triangle with vertices at the coordinates (x_i, y_i).

$$M = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 \end{bmatrix}$$

To determine the rank of the given matrix, one can look at all of its 3rd order minors. There are 20 such minors. If we can prove that at least one of those minors has determinant not equal to zero, then M has maximal rank of 3.

With some calculations, one can show that 12 of the 20 minors have determinants of the form

$$(x_i - x_j)(x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2)$$

or

$$(y_i - y_j)(x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2)$$

for $1 \leq i \leq 3$, $1 \leq j \leq 3$, and $i \neq j$.

Now since we are representing any triangle, we must assume that the coordinates of each vertex, (x_1, y_1), (x_2, y_2), and (x_3, y_3) are not collinear. This means that not all $(x_i - x_j) = 0$ and not all $(y_i - y_j) = 0$. It also means that

$$\frac{x_2 - x_1}{y_2 - y_1} \neq \frac{x_3 - x_1}{y_3 - y_1}.$$

With some manipulation, the above equation also says that

$$x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2 \neq 0$$

Thus, there must be at least one 3rd order minor with a determinant of the form

$$(x_i - x_j)(x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2)$$

or

$$(y_i - y_j)(x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2)$$

not equal to zero. Therefore, we have shown rank of the rigidity matrix M is $3 = (3)(2) - \frac{1}{2}(2)(2+1)$, and thus by Theorem 2.0.5, the framework of any triangle is rigid.

Note that if we had not required all the coordinates be collinear, then the determinants of all the 3rd order minors could have been zero, and then the rank of the matrix would have been less than 3.

Example 5. *Similarly, the rank of the rigidity matrix in Example 4 is* $6 = (4)(3) - \frac{1}{2}(3)(3+1)$*, and thus by Theorem 2.0.5, the framework is rigid.*

Chapter 3: Rigidity of Regular Polytopes

3.1 Rigidity of Regular Convex 2-polytopes

First, let us consider regular convex 2-polytopes or polygons. Recall that a regular convex polygon is one where all the edges are of equal length and the interior angles are equal and measure less than 180 degrees. By constructing regular polygons, beginning with the triangle and then increasing the number of edges, on can see that there are infinitely many regular convex 2-polytopes. We can denote polygons with a Schlafli symbol $\{p\}$, where p is the number of edges.

So which of the infinitely many 2-polytopes are rigid? Consider the polygons with solid faces and edges, i.e. with rigid 1-dimensional and 2-dimensional component; all of these polygons are rigid. But now consider the regular convex polygons built as bar frameworks, with rigid edges like bars, but which do not have a rigid face. Of the infinitely many regular convex polygons, the only such one that is rigid is the equilateral triangle.

3.2 Regular Convex 3-polytopes

Definition. A **platonic solid** is a regular, convex polyhedron with faces that are congruent, regular polygons, and the same number of faces meet at each vertex.

3.2.1 A Construction of the Platonic Solids

Corollary 3.2.1. *There are exactly five such platonic solids: tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron.*

Proof. Let p be the number of edges of each face and let q be the number of faces that meet at each vertex. Note that $p \geq 3$ and $q \geq 3$. Polyhedrons can then be denoted as $\{p, q\}$.

For example, a hexahedron (i.e. a cube) has sides with 4 edges, and three faces meet at each vertices. Thus a hexahedron can be denoted as $\{4, 3\}$.

Now recall Euler's Formula $v - e + f = 2$. Note that, for a platonic solid, since each face is congruent and the same number of faces meet at each vertex,

$$pf = qv = 2e$$

Thus we have that $f = \frac{2e}{p}$ and $v = \frac{2e}{q}$. So,

$$\begin{aligned} 2 &= v - e + f \\ &= \frac{2e}{q} - e + \frac{2e}{q} \end{aligned}$$

and by then dividing everything by $2e$,

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} = \frac{1}{e}$$

Since $e > 0$, $\frac{1}{e} > 0$. Thus

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} > 0$$

and

$$\frac{1}{q} + \frac{1}{p} > \frac{1}{2}$$

At this point, we can test different pairings of p and q to see which polyhedra satisfy the above equation. Recalling that $p \geq 3$ and $q \geq 3$, the only possible pairings are $\{3,3\}, \{3,4\}, \{4,3\}, \{3,5\}$, and $\{5,3\}$, giving us exactly five platonic solids. $\square$

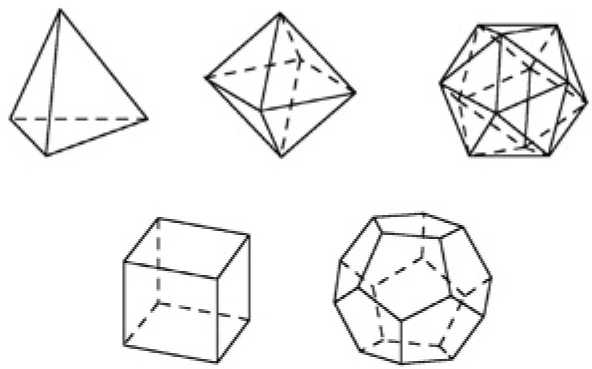

Figure 3.1: The five platonic solids, adapted from [7].

There are many proofs showing that there exist five platonic solids. Here is another proof that we can later extend to higher dimensions.

Proof. We know that a regular convex polyhedron must be made up of regular convex polygons. We also know that we can only fit a certain number of polygons around a single vertex, but how many exactly? That depends on the internal angle of the polygons being used to construct the polyhedron.The internal angle of a polygon is the angle between two edges, and the formula to find this internal angle is $180\frac{(p-2)}{p}$ degrees, where p is the number of sides of the polygon. The sum of all the internal angles around a single vertex must be less than $360°$, and there must be at least three polygons meeting at each vertex. If the sum of the internal angles were exactly $360°$, then the polygons would tile in two dimensional space like these squares tiling below.

Figure 3.2: Squares tiling in 2-dimensions.

First consider what regular polyhedra can be constructed with triangles? A triangle has an internal angle of $60°$, so there can be 3, 4, or 5 triangles meeting at each vertex. This leads to the construction of $\{3,3\}$, $\{3,4\}$, and $\{3,5\}$. A square has a $90°$ internal angle, so only 3 squares can fit around a vertex, showing the existence of $\{4,3\}$. A pentagon has a $108°$ internal angle, so the only regular polyhedron constructed with pentagons is $\{5,3\}$ All other regular polyhedra with more than five sides have internal angles greater than or equal to $120°$. No regular polyhedra can be constructed with these polygons. Thus concludes an alternate proof that there are exactly five platonic solids. $\square$

3.2.2 Rigidity of the Platonic Solids

Recall that we used Euler's formula to show that there are only five platonic solids, i.e. there are only five regular convex 3-polytopes. Let us examine the rigidity or flexibility of the five solids. First, let us consider the regular convex 3-polytopes as solids. That is, the 3 dimensional, 2 dimensional, and 1 dimensional components of the polytope are rigid. Clearly all of the platonic solids, if actual solids, are rigid.

Now consider the same regular 3-polytopes, but with the 3 dimensional component missing. Consider regular convex 3-polytopes with rigid 2-dimensional and 1-dimensional components. This would be like building the polyhedra with solid plates for faces and hinges at each edge. By Cauchy's rigidity theorem, we know that any convex polyhedron with rigid faces, is rigid. Thus, since every platonic solid is convex by definition, we can see that regular convex 3-polytopes with rigid edges and faces, are rigid.

But what about if the faces are not rigid? What if we consider the platonic solids with the 3 dimensional and 2 dimensional components missing? Since we could build these type of structures with solid bars, we will refer to this type of structure as a bar framework. Then the tetrahedron, the octahedron, and the icosahedron are ridid; however, the hexahedron and the dodecahedron are flexible. Note that the 3-polytopes with triangulated faces are rigid, while the rest are flexible. This can be determined by using the rigidity matrix or by simply building and checking each structure, as I have done.

It is interesting to note that the only rigid regular convex 3-polytopes with no rigid faces are the tetrahedron and octahedron, which both have faces that are all equilateral triangles, which, as we recall, is the only regular 2-polytope that is rigid as a bar framework. In other words, rigid bar frameworks of 3-polytopes are constructed from rigid bar frameworks or 2-polytopes.

In 1916 M. Dehn proved the following theorem that confirms our results.

Theorem 3.2.2. *A bar framework of any convex 3-polytope with all faces triangles is infinitesimally rigid and thus rigid in* $\mathbb{R}^3$.

3.3 Regular Convex 4-polytopes

Now that we have looked at the rigidity of polytopes in three dimensions, let us consider 4-polytopes.

Definition. An **4-polytope** is a 4-dimensional region bounded by a finite number of 3-dimensional regions. It is made up of vertices, edges, polygons, and polyhedra.

Definition. A **regular, convex** 4**-polytope**, the four dimensional analog of the platonic solid, is a 4-polytope that is both convex and regular. It is bounded by a finite number of platonic solids of the same size and type.

The Schlafli symbol for a regular polychoron is $\{p, q, r\}$ where p is the number of edges of each face, q is the number of faces that meet at each vertex, and r is the number of regular polyhedral cells around each edge.

We can use the generalized Euler's formulas to find that Euler's formula in the fourth dimension is $N_0 - N_1 + N_2 - N_3 = 0$ where N_0 is the number of vertices, N_1 is the number of edges, N_2 is the number of faces, and N_3 is the number of cells or polyhedra that make up the 4-polytope.

3.3.1 A Construction of Regular Convex 4-polytopes

Corollary 3.3.1. *There are exactly six regular, convex* 4*-polytopes: regular simplex, hypercube,* 16*-cell,* 24*-cell,* 120*-cell, and* 600*-cell.*

Proof. Consider a regular, convex polychoron $\{p, q, r\}$, which has cells that are platonic solids $\{p, q\}$ with r cells surrounding each edge.

Clearly, we can only fit a certain number of platonic solids around an edge. The dihedral angle of a polyhedron is the internal angle between two faces. The sum of the dihedral angles surrounding an edge, therefore, must be less than 2π. For a platonic solid, the dihedral angle θ can be expressed as

$$\cos(\theta) = \frac{\cos(\phi) - \cos(\phi)\cos(\phi)}{\sin(\phi)\sin(\phi)}$$

and in terms of p and q,

$$\theta = 2 \arcsin \left(\frac{\cos \frac{\pi}{q}}{\sin \frac{\pi}{p}} \right)$$

Since r of these θ meet at each edge of a regular 4-polytope, $r\theta < 2\pi$, and we can get

$$\arcsin(\cos \frac{\pi}{q}) / \sin \frac{\pi}{p} < \frac{\pi}{r} \tag{3.1}$$

or

$$\cos \frac{\pi}{q} < \sin \frac{\pi}{p} \sin \frac{\pi}{r} \tag{3.2}$$

This inequality is analogous to the one we used to find the five platonic solids. We could go through and find the six solutions to this inequality, or we could construct the six regular 4-polytopes individually by looking at the dihedral angles of each type of platonic solid.

Consider the possible polychoron that can be composed with tetrahedron cells, $\{3, 3\}$ How many possible cells can there be around each edge? The tetrahedron has a dihedral angle of a little less than 71°. Thus, we must have at least three, but no more than five tetrahdra around each edge. So, we can construct the first three regular, convex 4-polytopes: $\{3, 3, 3\}$, $\{3, 3, 4\}$, and $\{3, 3, 5\}$.

What regular, convex 4-polytopes can be constructed with the cube, $\{4, 3\}$? The cube has a dihedral angle of 90°, so we can build a 4-polytope with three cubes surrounding each edge, giving us $\{4, 3, 3\}$.

The octahedron, $\{3, 4\}$ has a dihedral angle slightly less than 110°; thus we may fit only three cells around an edge. This produces $\{3, 4, 4\}$.

Similarly, the dodecahedron has a dihedral angle just less than 117°, and so only threee cells may fit around an edge. This gives the sixth and final regular, convex 4-polytope $\{5, 3, 3\}$.

Note that since the icosahedron has a dihedral angle of about 138°, there are not regular, convex polychoron with icosahedron cells.

What happens if $r\theta = 2\pi$? Consider the 4-polytope cells that are cubes, which have a dihedral angle of 90°. If for cells surround each edge, then what results is cubes tiling in 3 dimensions, and this is called a cubic honeycomb. There can be an infinite number of cells.

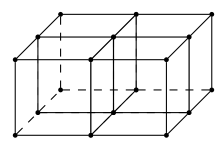

Figure 3.3: Cubic Honeycomb

Thus, we have shown that there are exactly six regular, convex 4-polytopes. □

3.3.2 Rigidity of regular convex 4-polytopes

Consider bar frameworks of the regular convex 4-polytopes. Which of them are rigid? We can use the rigidity matrices of the 4 polytopes to determine their rigidity.

The 4-polytope represented by the Schlafli symbol $\{3, 3, 3\}$, also referred to as the 5-cell, has 5 vertices in $\mathbb{R}^4$ and cells that are tetrahedra. The rank of the rigidity matrix of the 5-cell is 10, which is equal to $(5)(4) - \frac{1}{2}(4)(4+1)$. Thus, by Theorem 2.0.5, the 5-cell is rigid.

The 4-polytope represented by the Schlafli symbol $\{4, 3, 3\}$, also referred to as the 8-cell, has 16 vertices in $\mathbb{R}^4$ and cells that are cubes. The rank of the rigidity matrix of the 8-cell is 32, which is less than $(16)(4) - \frac{1}{2}(4)(4+1)$. Thus, by Theorem 2.0.5, the 8-cell is flexible.

The 4-polytope represented by the Schlafli symbol $\{3, 3, 4\}$, also referred to as the 16-cell, has 8 vertices in $\mathbb{R}^4$ and cells that are tetrahedra. The rank of the rigidity matrix of the 5-cell is 22, which is equal to $(8)(4) - \frac{1}{2}(4)(4+1)$. Thus, by Theorem 2.0.5, the 16-cell is rigid.

The 4-polytopes represented by the Schlafli symbols $\{3, 4, 3\}$, $\{5, 3, 3\}$, and $\{3, 3, 5\}$, referred to as the 24-cell, 120-cell, and 600-cell, respectively were too large for me to construct rigidity matrices.

Of the ones that I confirmed, note that the regular 4-polytopes that are rigid are the ones that have cells that are tetrahedra, which are rigid frameworks in $\mathbb{R}^3$. On the other hand, $\{4, 3, 3\}$, which is flexible, has cells that are cubes, which are flexible in $\mathbb{R}^3$.

Based off of these observations, I would conjecture that bar frameworks of the 24-cell, made up of octahedra, and of the 600-cell, made up of tetrahedra, are rigid in $\mathbb{R}^4$, while the bar framework of the 120-cell, made up of dodecahedra, is flexible.

In 1984, Whitely extended Dehn's theorem to higher dimensions, proving the following theorem.

Theorem 3.3.2. *A bar framework of any convex n-polytope with all 2-dimensional faces triangles is infinitesimally rigid and thus rigid in $\mathbb{R}^n$.*

3.4 Rigidity of Regular Convex n-polytopes

There are also a finite number of regular, convex n-polytopes. To show this, let us first take a look at the regular 5-polytopes, approaching their construction in the same way that we did for 4-polytopes. While before we looked at dihedral angles, or the angle that two face make at a edge, now we will consider "4-angles" or the angle made by the 4-polytopes meeting at the 3-polytopes. Just as the sum of the dihedral angles at an edge in the 4-polytope had to be less than 360°, the sum of the 4-angles that meet at a polyhedron must also be less than 360°.

Consider $\{3,3,3\}$, which is made up of three tetrahedra at each edge; it has an interior angle of 75.5°. We can then construct two regular 5-polytopes that are made up of the $\{3,3,3\}$ polytope:$\{3,3,3,3\}$ and $\{3,3,3,4\}$. $\{4,3,3\}$, since its cells are cubes, has an interior angle of 90°. There is thus just one regular 5-polytope, $\{4,3,3,3\}$ made up of hypercubes. Note that $\{4,3,3,4\}$ would tile in 4 dimensions.

The other four regular 4-polytopes all have dihedral angles that are greater than or equal to 120°, and so we cannot construct any regular 5-polytopes with those polychora.

Recall that if $\{x_1, x_2..., x_n\}$ is a regular polytope, then $\{x_1, x_2..., x_{n-1}\}$ and $\{x_2, x_3..., x_n\}$ must also be regular polytopes. Observing that the only regular polytopes in 5 dimensions are :$\{3,3,3,3\}$, $\{3,3,3,4\}$, $\{4,3,3,3\}$, we can see that the only regular polytopes in dimension $n > 4$ are of the form $\{3,3,...,3,3\}$, $\{3,3,...,3,4\}$, $\{4,3,...,3,3\}$. These are referred to as the n-simplex, n-dimensional cross-polytope, and n-cube, respectively. There can be no more than three regular n-polytopes for $n > 4$.

We can then see by Theorem 3.3.2 that the only bar frameworks of regular n-polytopes that are rigid are of the form $\{3,3,...,3,3\}$ and $\{3,3,...,3,4\}$.

Chapter 4: Other Rigidity Theorems

4.0.1 A Counterexample to Euler's Conjecture

Euler's conjecture from 1766 that all closed polyhedral structures constructed with polygonal plates and hinges was never been proved. Cauchy was able to prove that *convex* polyhedral structures constructed with polygonal plates and hinges are rigid. Then, in 1975, H. Gluck showed that almost all closed, simply connected, generic plate and hinge polyhedral structures are rigid. Up until 1977, though not yet proven, it seemed that Euler's conjecture was true.

In 1977, R. Connelly finally developed a counterexample to Euler's conjecture. He found a non-convex polydron with triangular faces that is flexible. I built my own model of this polyhedron with the flat net in Figure 4.1 to confirm that it is indeed flexible.

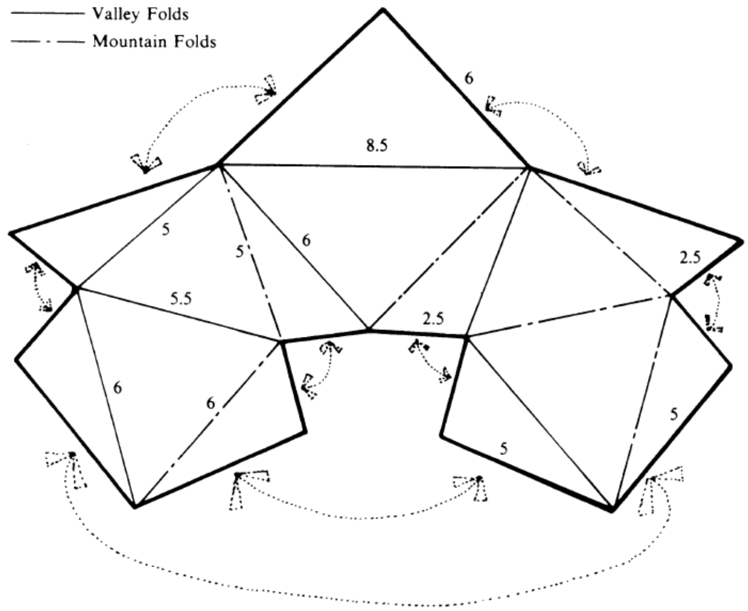

Figure 4.1: The flat net of Connelly's counterexample, constructed by cutting and folding, adapted from [3].

4.0.2 Alexandrov's Theorem and Non-Convex Polyhedra

Though Euler's conjecture was finally shown to be false, much about the rigidity and flexibility of frameworks and plate-hinge structures was discovered and proved.

After finding a counterexample to Euler's conjecture Connelly D. Sullivan conjectured that a polyhedron that does flex maintains a constant volume. It was not until 1997 that Connelly and A. Waltz were able to prove this.

While Dehn proved that A bar framework of any convex 3-polytope with all faces triangles is infinitesimally rigid and thus rigid in $\mathbb{R}^3$, A.D. Alexandrov proved in the 1940's the following theorem.

Theorem 4.0.1. *All triangulated convex polyhedral surfaces with all the vertices lying only in the natural edges are rigid.*

This is simply a sharpening of Dehn's theorem, saying that to triangulate a polyhedron, vertices can be added to its natural edges, and this will produce a rigid structure.

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