INTERNAL SET THEORY AND EULER’S INTURODUCTIO IN ANALYSIN INFINITORUM

THESIS

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Abstract

In Leonhard Euler’s seminal work *Introductio in Analysin Infinitorum* (1748), he readily used infinite numbers and infinitesimals in many of his proofs. We aim to reformulate a group of proofs from the *Introductio* using concepts and techniques from Abraham Robinson’s celebrated Nonstandard Analysis (NSA); in particular, we will use Internal Set Theory, Edward Nelson’s distinctive version of NSA. We will specifically examine Euler’s proofs of the Euler formula, the Euler product, the Wallis product and the divergence of the harmonic series. All of these results have been proved in subsequent centuries using epsilontic arguments. In some cases, the epsilontic arguments differ significantly from Euler’s original proofs. We will compare and contrast the epsilontic proofs with those we have developed by following Euler more closely through NSA. We claim that NSA possesses the tools to provide appropriate proxies of some—but certainly not all—of the inferential moves found in the *Introductio*. 
Dedication

For Glenna R. Joyce
Acknowledgments

I want to first thank Stewart Shapiro and Neil Tennant for encouraging me to pursue graduate studies in mathematics. During my course work, Neil especially persistently urged me to continue even when my own motivation was wearing thin. I also appreciate their reading various incarnations of this document, even when it was incomplete and still littered with my own editorial comments. I also want to thank Salvatore Florio for his companionship as we marched numerous of the mathematics courses together.

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Last, I want to offer my greatest thanks to my advisor, Warren Sinnott. First, I had the good fortune of having him as my graduate introductory algebra teacher in
the academic year of 2009-2010, which was essential in reading certain mathematical papers necessary for my dissertation in Philosophy. He very patiently dedicated time to slogging through the basics of Nonstandard Analysis with me in the Summer of 2010. Since then, he has read numerous drafts and caught some very serious mathematical errors. I sincerely appreciate his patience with this reckless philosopher.

I dedicate my thesis to Glenna R. Joyce. I had the pleasure of being one of several Joyce Scholars starting in the Fall of 2000 at Ohio State. If I had not had this opportunity and was forced to work through school, it is very difficult for me to imagine finishing a mathematics major, let alone pursue a double major in Philosophy. In that way, my ability to continue onto graduate work in each of these fields is very much a result—direct or indirect—of her financial support.

Curious to learn more about her, I had a very difficult time finding information about such a generous and otherwise mysterious woman. I want to thank Anne Krabacher from the Honors and Scholars Center for retrieving this information about Ms. Joyce for me. In an article from OSU News dated September 12, 1995, written by Mabel Freeman, one can learn the following about this sympathetic woman:

Glenna Stengel was a seamstress before her marriage to William Joyce. Her husband came to Columbus in 1910 from Shawnee, Ohio, and started the Wyandotte Pop Company and the Milbrook Distillery at West Second Avenue and Perry Street. Joyce later started the Joyce Products Co. and Beverage Management Inc. One of the company’s products was a carbonated lemon soda that later became the popular soft drink 7-Up.

After Mr. Joyce’s death in 1933, Glenna Joyce moved from the Ohio State area to Upper Arlington, where she was a member of St. Agatha Roman Catholic Church. Glenna Joyce was vice president and a major shareholder of Joyce Products Co. when she died in 1960. (Freeman, [13])
Vita

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Chapter 1

Introduction

And therefore to get rid of Quantities by the received Principles of Fluxions or of Differences is neither good Geometry nor good Logic.

—Bishop Berkeley, *The Analyst*

[T]here are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.

—Kurt Gödel, March 1973

Infinitesimal quantities have had an extraordinary history. Infinitesimals made their debut some time in the early Renaissance period but played an indispensable role in analysis long into the nineteenth century. Even in spite of Bishop Berkeley’s assiduous—though, at times, acerbic—denunciation of the use of infinitesimal magnitudes, mathematicians continued to use them, especially on the continent. Following the pioneering work of Cauchy, Weierstrass, Dedekind and Cantor, mathematical analysis was placed on a foundation that eschewed infinitesimal magnitudes. Some have suggested that the rigorization of the calculus in the nineteenth century was

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1 See [2], §XXX.
2 Quoted by Robinson in *Non-standard Analysis*, [31], p. x.
due to a foundational crisis bound up with the usage of infinitesimals.\textsuperscript{3} Others have suggested that the introduction of the limit definition was a necessary advancement required simply by mathematical interests alone.\textsuperscript{4} Either way, it did not take long for them to make a dramatic reintroduction. Bertrand Russell writes:

> It is a singular fact that, in proportion as the infinitesimal has been extruded from mathematics, the infinite has been allowed a freer development. (Russell, [35], 302)

Such freedom allotted to the infinite is exactly what made way for a rigorous return of the infinitesimal. A variety of work surfaced in the mid-twentieth century placing the infinitesimal on footing as sure as Cantor’s infinite. This has occurred most notably Abraham Robinson’s nonstandard analysis.\textsuperscript{5}

The present study is devoted to discovering (in a small way) the relationship between Robinson’s work and several fascinating proofs found in Euler’s significant contribution to analysis, \textit{Introductio in Analysin Infinitorum} of 1748. This will proceed primarily by examining the extent to which Euler’s inferences concerning the infinite can be reconstructed within nonstandard analysis. We will use Edward Nelson’s development of nonstandard analysis, known as Internal Set Theory (IST). In effect, Nelson applies Robinson’s techniques to all of Zermelo-Fraenkel Set theory with choice (ZFC), rather than simply to a first-order theory of the real numbers. Below, we will briefly examine some of the history of the calculus and its relationship to Robinson’s work. Following this, in chapter 2, we will present nonstandard analysis (NSA) in line with Internal Set Theory. Lastly, in chapter 3, we will apply

\textsuperscript{3}This portrayal is often viewed as the party line on the early calculus. Such a cartoon picture is found in a variety of sources, including Russell’s \textit{Principles of Mathematics}, [35] and Boyer’s \textit{The History of the Calculus and its Conceptual Development}, [3]. Both of these studies are valuable for various reasons, even if the melodramatic narrative provided is slightly misleading.

\textsuperscript{4}See the extended discussion by Imre Lakatos in “Cauchy and the Continuum: The Significance of Non-standard analysis for the History and Philosophy of Mathematics,” [24], and also Philip Kitcher in chapter 10 of \textit{The Nature of Mathematical Knowledge}, [21].

\textsuperscript{5}See also Detlef Laugwitz’s “Curt Schmieden’s approach to infinitesimals. An eye-opener to the historiography of analysis,” [25].
nonstandard analysis to a group of proofs in the Introductio that make special use of infinite and infinitesimal numbers. In particular, we will be evaluating the extent to which (IST) captures the “essence” of Euler’s proof. We will also compare these proofs to more “orthodox” proofs, viz. those that do not depend upon infinitely large or small values. We discover below that it’s a mixed bag: some of his proofs fit naturally within (IST) and others do not.

Chapter 1.1

Historical Remarks

As indicated above, infinitesimals have faced a hard road. Probably the most well-known and also earliest attack on infinitesimals comes from Bishop Berkeley’s The Analyst, with its rather forward subtitle, A Discourse Addressed to an Infidel Mathematician. The primary thesis of this text is that faith in Christian doctrine is no worse than the claims and techniques of the Calculus. Although ultimately unpersuasive, the idea is that, if this infidel mathematician doesn’t reject the Calculus when there are conceptual problems, why is he so hard on Christianity?

As for Berkeley’s particular complaints about the Calculus, he primarily complains in a variety of ways that infinitesimals are treated equivocally—zero in one place and nonzero in another. He famously mocked this equivocation as follows:

And what are these Fluxions? The Velocities of evanescent Increments?
And what are these same evanescent Increments? They are neither finite Quantities, nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities? (Berkeley, [2], §XXXV.)

This equivocation undermines the truth-preserving spirit of mathematical demonstration. As such, the scientific fruit of the calculus is not enough: the ends might be good but do not justify the means. He writes:
That although, in the present case, by inconsistent Suppositions Truth may be obtained, yet that such Truth is not demonstrated: That such Method is not conformable to the Rules of Logic and right Reason: That, however useful it may be, it must be considered only as a Presumption, as a Knack, an Art or rather an Artifice, but not a scientific Demonstration. (Berkeley, [2], §XXV.)

In spite of these and other earlier criticisms, infinitesimals remained steadfast. Eighteenth century Continental mathematicians used infinite and infinitesimal quantities with abandon, including our subject, Euler. Continental Mathematicians followed Leibniz’s lead in allowing mathematical activity to operate somewhat autonomously, allowing their work to proceed with less concern for the physical or geometrical applications when compared to their English counterparts. Along these lines, Euler has been argued to be revolutionary in his protoformalistic approach to mathematics; that is, he allowed symbolic representations to take on a mathematical life of their own. In spite of his liberality in manipulation of symbols, he had an uncanny ability to avoid error. Not all of his work has survived scrutiny, but between his extraordinary fecundity and preternatural sensitivity to the infinitely large and small, many of Euler’s results are with us today as mathematical benchmarks.

As the eighteenth turned into the nineteenth century, an urgency about how to understand infinite series arose; an especial distrust of divergent series and their “so-

\footnote{For example, Bernard Nieuwentijt critically engaged the work of both Leibniz and Newton. Leibniz’s response can be found translated into English in [15].}

\footnote{Much of the following comes from insights in Kitcher, [21]. Craig Fraser also provides the following striking observation along these lines:}

The calculus of Euler and Lagrange differs from later analysis in its assumptions about mathematical existence. The relation of this calculus to geometry or arithmetic is one of correspondence rather than representation.

In other words, the calculus was an independently valuable mathematical enterprise, untethered by geometry or arithmetic.

Regarding the contrast between British and Continental mathematics, we ironically find in the years preceding Newton and Leibniz, John Wallis—an English mathematician—was scrutinized for being too cavalier by Fermat—a Continental mathematician.

\footnote{Most notoriously, Euler was among those who attempted to find solutions to divergent series. Some discussion of this is found in Kitcher’s [21]. Extensive discussion is found in Giovanni Ferraro’s [11].}
lutions” emerged. With the arrival of men like Bolzano and Cauchy, serious reform was initiated. It is noteworthy that Cauchy’s work was meant to extricate mathematics from the thorn bush of divergent series; nevertheless, even he ran afoul of another brier patch of infinite series. It took nearly the rest of the century for the reform to be complete between Weierstrass’ epsilontic \((\epsilon,\delta)\) treatments of limits and continuity and the Cantor-Dedekind arithmetization of the continuum.

By the time the twentieth century arrived, many viewed the work of these men as the culmination of all the work from both Newton and Leibniz and onward. Such anachronism is tempting. Looking backwards in time, people often miss the valleys that exist between each historical peak. Major contributors to the history of ideas like Newton and Leibniz, Berkeley and the triumvirate of Cantor, Dedekind and Weierstrass loom large. Influential figures in the philosophy and history of mathematics—notoriously, Bertrand Russell—drew straight lines from start to finish in the history of analysis, ignoring long stretches of time, including the one in which Euler lived.

Chapter 1.2

Optimism and Historiography

“The history of a subject usually is written in the light of later developments.” (Robinson, [31], 260) This sentence opens Robinson’s historical chapter in the monograph, *Non-standard Analysis*. Such a beginning only suggests that he plans to take aim at the historiographical tradition discussed above. He describes this tradition as follows:

As the theory of limits became firmly established, the use of infinitely small and infinitely large quantities in Analysis became discredited and

\[ \sum \frac{1}{n} \sin nx \]

provides a counterexample to one of his proofs that the infinite sum of continuous functions is continuous. For lengthy discussion of this, see Lakatos’ “Cauchy and the Continuum,” [24].
survived only as *a manner of speaking*; e.g., in the statement that a variable *tends to infinity*. The significance of subsequent developments in the theory of non-archimedean fields was confined entirely to Algebra. (Robinson, [31], 261, emphasis added).

As much as Robinson’s critique of his immediate predecessors is accurate, we must not commit the same errors and read of all history in terms of NSA. In some places Robinson expresses a measure of caution in his enthusiasm, in others, he writes things like, “It is shown in this book that Leibniz’ ideas can be *fully vindicated* . . .” (Robinson, [31], 2, emphasis added)

We will discover here that NSA does not vindicate everything of the infinitesimal calculus.\(^\text{10}\) In those portions of Euler’s work examined here, there are some cases where his proofs can be followed rather closely using the infinitesimal and infinite numbers of NSA; there are other cases where the contemporary limit-based treatment does just as good of a job. Admittedly, there are no examples among the cases we examine where NSA *fails* to prove something. After all, NSA is a conservative extension of standard analysis. However, the kind of full vindication of seventeenth and eighteenth century analysis that Robinson envisions is a bit overstated.

We may forgive Robinson for his sanguine proclamations: NSA is a genuine mathematical achievement. At the time *Non-standard Analysis* was written, it provided a much needed exoneration of the hitherto exiled infinitesimal. And, as far as this work is concerned, we view it as a significant credit to Robinson’s work that even *some* of Euler’s corpus can be rationally reconstructed within NSA.

\(^{10}\)To be fair, Robinson does not directly mention Euler, but Euler is a mathematical descendant of Leibniz. Also, the frontispiece to the *Introductio*, entitled “Analyse des Infiniment Petits” is Robinson’s own frontispiece to *Non-standard Analysis*. Such a choice by Robinson is a clear nod to Euler and his notably frequent use of the infinitely large and small.
Chapter 2

Internal Set Theory

In “Internal Set Theory: A New Approach to Nonstandard Analysis,” Nelson developed IST as an alternative to the rather involved ultraproduct presentation of NSA. Essentially, it is an extension of Zermelo-Fraenkel set theory with Choice (ZFC). The axiomatic and logical details follow.

Chapter 2.1

The Axioms of ZFC

The following are the well-known first-order axioms of ZFC. We will give a casual gloss followed by a completely formal characterization (allowing for definitions). Note that the two schemata may have parameters over which one must universally quantify in the usual way. We will ignore such details here.

Emptyset There is a set, written $\emptyset$, that has no members.

Formally: $(\exists x)(\forall y) \ y \notin x$.

Extensionality If $x$ and $y$ have the same members, then $x$ and $y$ are identical.

Formally: $(\forall y)(\forall z)((\forall x)(x \in y \leftrightarrow x \in z) \rightarrow y = z)$.

Pairset For any $x$ and $y$, there is a set with exactly $x$ and $y$ as members, written $\{x, y\}$. Formally: $(\forall x)(\forall y)(\exists z)(x \in z \land y \in z \land (\forall w)(w \in z \rightarrow (w = y \lor w = x)))$. 7
Unions For any set $x$, there is a set containing the members of the members of $x$, written $\cup x$. Formally: $(\forall x)(\exists y)(\forall z)(\forall w)((w \in z \land z \in x) \rightarrow w \in y)$.

Infinity There is a set $x$ such that $\emptyset \in x$ and for any $y \in x$, $\{y\} \cup y \in x$.
Formally: $(\exists x)(\emptyset \in x \land (\forall y)(y \in x \rightarrow \cup\{y\} \in x))$.

Power-set For any set $x$, there is a set, written $\mathcal{P}(x)$, such that any subset of $x$ is a member of $\mathcal{P}(x)$. Formally: $(\forall x)(\exists y)((\forall z)((w \in z \land z \in x) \rightarrow w \in y))$.

Foundation For any nonempty set $x$, there is some member of $x$ that is disjoint from $x$.
Formally: $(\forall x)(x \neq \emptyset \rightarrow (\exists y)(\forall z)(y \in x \land \neg(\exists z)(z \in y \land z \in x)))$.

Choice For any set $x$ of nonempty pairwise disjoint sets, there is a set with exactly one member of each member of $x$.
Formally: $(\forall x)((\forall w)(w \in x \rightarrow w \neq \emptyset) \land (\forall y)(\forall z)(\forall w)(w \in z \land z \in x \land y \neq z) \rightarrow (\exists y)(\forall z)(z \in x \rightarrow (\exists y)(w \in z \land w \in y)))$.

Separation (schema) For any set $x$, and any first-order formula $\varphi(y)$, with $y$ free, there is a set with all and only members of $x$ satisfying $\varphi$, written $\{y \in x : \varphi(y)\}$.
Formally: Supposing $\varphi(y)$ has $y$ free, $(\forall x)(\exists y)(\forall y)((y \in x \land \varphi(y)) \leftrightarrow y \in z)$.

Replacement (schema) For any $x$ and any function $f$, there is a set that is the image of $x$ under $f$, written $\{f(y) : y \in x\}$.
Formally: Supposing $\varphi(x,y)$ is a first-order formula with $x$ and $y$ free, $(\forall x)(\exists! y)\varphi(x,y) \rightarrow (\forall z)(\exists w)(\forall x)(\forall y)((x \in z \land \varphi(x,y)) \leftrightarrow y \in w)$.

Chapter 2.2

Internal Set Theory

First, in order for the following definitions to make sense, we must introduce a few notions. Below, the concept of finiteness is extremely significant. Stereotypically, finiteness is defined in one of two ways. The first way, which will not be our way, requires understanding Dedekind infinity: a set $X$ is infinite $\mathcal{I}$ just in case $X$ can be
injected into a proper subset of $X$. So, $X$ is $\text{finite}_D$ just in case $X$ is not $\text{infinite}_D$.

The other way, our way, will be to define finiteness in terms of the natural numbers. A set $X$ is $\text{finite}$ just in case there is a bijection between $X$ and an initial segment of $\mathbb{N}$.\footnote{The natural numbers must be defined first. $\mathbb{N}$ satisfies the following first-order statement where $X$ is free: $\emptyset \in X$ and for any $y \in X, y \cup \{y\} \in X$ and for any $Z$ such that $\emptyset \in Z$ and for any $y \in Z, y \cup \{y\} \in Z$, then $X \subseteq Z$. Furthermore, we have that $\mathbb{N}$ is well-ordered under $\in$. In order to capture that some $X \subseteq \mathbb{N}$ is an initial segment, we simply express that there is a largest element $\lambda \in X$. That is, there is some $\lambda \in X$ and for any $z \in \lambda$, then $z \in X$.}

Another notion we must introduce is that of being $\text{Standard}$: a completely new predicate into the language of set theory, defined implicitly by the following axioms:

**Idealization (schema)** Let $\psi(x, y)$ be a formula with $x$ and $y$ free and no occurrences of the $\text{Standard}$ predicate. Then,

For all standard finite $z$, there is an $x$ such that for all $y \in z$, $\psi(x, y)$

\[\iff\]

There is an $x$ such that for any standard $y$, $\psi(x, y)$.

**Standardization (schema)** Let $\varphi(z)$ be any formula without $y$ free. Then, for any standard $x$, there is a standard $y$ such that for all standard $z$,

\[z \in y \iff z \in x \text{ and } \varphi(z).\]

**Transfer (schema)** Let $t_1, \ldots, t_k$ be standard and $\psi$ have no occurrences the $\text{Standard}$ predicate. Then,

For every standard $x$, $\psi(x, t_1, \ldots, t_k)$

\[\iff\]

For every $x$, $\psi(x, t_1, \ldots, t_k)$.

A helpful mnemonic for remembering the axioms of Internal Set Theory (IST) is that the axioms have the same initials as the theory's acronym: $\text{I}(\text{dealization})$, \text{idealization},
S(standardization) and T(ransfer).

First, let us consider the Transfer axiom. It essentially says that if something is true of all standard $x$, then it is true of everything simpliciter.$^2$ We will provide a concrete use of Transfer momentenarily. Before that, we must introduce Standardization. This is meant to mimic the Separation axiom from above. This is not a replacement, because Separation still holds for sentences $\psi$ with no use of the predicate Standard (or, of course, anything that requires Standard for its definition). Full Separation is not available, viz. where the full language of IST is the range of the formulas in the schemata. Its presence would interfere with Transfer. By way of concrete example, full Separation applied to all formulas of the expanded language, would allow us to define the set of all infinitesimals, $I$ (numbers $|i| < r$, for all standard $r > 0$). On the other hand, Transfer enables us to say that every bounded set of reals has a least upper bound, since every standard bounded set of reals has a least upper bound. To see the conflict, suppose the set of infinitesimals has a least upper bound, $b$. If $b \in I$, then $3b/2 \in I$ and $3b/2 > b$. So, $b$ is not an upper bound—contradiction. If $b \notin I$, then $b/2 \in I$. But, for any $c \in I$, $2c \in I$; i.e. $b \in I$—contradiction. For this reason, we must carefully restrict set-formation when the predicate Standard is involved.

Lastly, we turn to Idealization. Put simply, it indicates “where” in the universe there are nonstandard items. Most notably, Idealization is essential for the existence of illimited or “infinite” numbers (values $k$ such that $|k| > r$ for all standard $r > 0$). To see this, let $\psi$ be $\{x, y \in \mathbb{N} \text{ and } x > y\}$. Beginning with the top-side of the Idealization biconditional, any standard finite set satisfying $\psi$ is a subset of $\mathbb{N}$. Since there is a bound for each such set (e.g., the next integer greater than its largest member), then we may conclude that there is some $x \in \mathbb{N}$ greater than any standard

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$^2$It is worth noting that in ordinary NSA, a transfer principle is actually proved to hold for the constructed model underneath the theory of NSA. The transfer result comes from Los' [26]. Nelson builds the transfer principle right into the axioms.
A few logical remarks are in order. A very natural worry is whether IST is consistent. It turns out that it is and in fact, IST is a conservative extension of ZFC. In other words, if \( \varphi \) is \textit{in the language of ZFC only} and \( IST \vdash \varphi \), then \( ZFC \vdash \varphi \) (the other direction follows since IST \textit{extends} ZFC).

As already indicated in the above discussion of \textbf{Standardization}, all schema in ZFC must remain restricted to the language of ZFC. The axioms I, S and T tell us how to use the predicate \textit{Standard} alongside ZFC without contradictions. So, we may not go back and allow the statements \( \varphi \) of the \textbf{Separation} and \textbf{Replacement} schemata include the predicate \textit{Standard}.

\section*{Chapter 2.3}

\textbf{Nonstandard Analysis in (IST)}

From here on out, there is not a significant amount of difference between (IST) and traditional nonstandard analysis. Probably the most significant difference is (more-or-less) notational and could cause serious confusion for one familiar with more conventional presentations of NSA. Here, well-known infinite sets all contain nonstandard members. For example, what is usually written as \( ^*\mathbb{N} \) in ordinary NSA is simply written \( \mathbb{N} \) here. IST cannot tell the difference since there is no set \( \{ x \in \mathbb{N} : x \text{ is standard} \} \) (cf. discussion above about \textbf{Standardization}). Past this, things are almost identical.

The following concepts and theorems will be cited below in the specific work on Euler.

- For any \( c \in \mathbb{C} \), \( c \) is \textit{limited} \( \iff \) \( \exists r \in \mathbb{R} \) such that \( |c| < r \).
- For any \( c \in \mathbb{C} \), \( c \) is \textit{illimited} \( \iff \) \( c \) is not limited.
• For any $c \in \mathbb{C}$, $c$ is infinitesimal $\iff$ for all standard $r \in \mathbb{R}$, $r > 0, |c| < r$.

• For any $x, y \in \mathbb{C}$, $x$ and $y$ are infinitely close, (written, $x \simeq y$) $\iff |x - y| = \epsilon$ for some infinitesimal $\epsilon \geq 0$.

• For any $x, y \in \mathbb{C}$, $x$ is infinitely smaller than $y$ (written, $x \ll y$) $\iff df m|x| \leq |y|$ for some positive illimited $m \in \mathbb{R}$.

• Sequence Theorem: Suppose that $(a_n)$ is a standard sequence such that $a_n \to a$ for standard $a$. It follows that for any illimited $n$, $a_n \simeq a$.

\[\text{\footnotesize\textsuperscript{3}One might also use this definition to express that $y$ is infinitely larger than $x$.}\]
Chapter 3

Nonstandard Analysis and the *Introductio*

Chapter 3.1

Euler’s Formula

We begin with one of the most surprising mathematical discoveries that appropriately bears Euler’s name: the Euler formula, *viz.* that for any $x \in \mathbb{R}$, $e^{ix} = \cos x + i \sin x$. Below, the inferences follow as closely as possible to Euler’s original proofs.

3.1.1 Basic Outline

Euler’s work actually begins by defining $e$ as follows:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad (3.1.1)$$

This definition will become useful in future settings. Since Euler develops this definition as a special case of much more general mathematical facts, we will begin from this general setting as well.$^{1}$

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$^{1}$This section depends largely on Dunham’s text [5]. It is supplemented by our own reading of Blanton’s translation of the *Introductio*, [7]. The same is true of §§3.2.1, 3.3.1, and 3.4.1.
Suppose $\epsilon$ is infinitely small. So,

$$\alpha^\epsilon = 1 + \psi_\alpha, \text{ for any } \alpha > 0$$

(3.1.2)

and $\psi_\alpha$ also infinitely small. Let $j_\alpha$ satisfy $\psi_\alpha = j_\alpha \epsilon$. It follows that for any $n$ that,

$$\alpha^{\epsilon n} = (1 + j_\alpha \epsilon)^n = \sum_{k=0}^{n} \binom{n}{k} (j_\alpha \epsilon)^k$$

Let $z$ be some finite value (possibly complex), whereby $n = \frac{z}{\epsilon}$ and so $n$ must be infinite. Substituting $\epsilon = \frac{z}{n}$ above, we get,

$$\alpha^z = \left(1 + \frac{j_\alpha z}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{j_\alpha z}{n}\right)^k.$$  

(3.1.3)

Note that each term of the sum in (3.1.3) may be rewritten so that

$$\frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!} \left(\frac{j_\alpha z}{n}\right)^k =$$

$$\frac{n(n-1)(n-2)\cdots(n-(k-1)) (j_\alpha)^k z^k}{n^k}.$$  

We note however that since $n$ is infinite,

$$\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n^k} = 1$$

(3.1.4)

It follows that substituting $z = 1$ into (3.1.3) that

$$\alpha = \sum_{k=0}^{n} \frac{(j_\alpha)^k}{k!}$$
Euler then substitutes $j_\alpha = 1$ to define $e$ as we now know it, as in (3.1.1):

$e = \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}$  \hspace{1cm} (3.1.5)

Now that a definition of $e$ is in place, returning to (3.1.3) with $j_\alpha = 1$ and therefore $\alpha = e$, it will be useful for us to have that,

$e^z = \left(1 + \frac{z}{n}\right)^n$.  \hspace{1cm} (3.1.6)

Lemma 3.1.1 For any $x \in \mathbb{R}$,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Proof. Euler starts with DeMoivre’s Theorem; that is,

$$\forall n \geq 0, (\cos \theta \pm i \sin \theta)^n = \cos n\theta \pm i \sin n\theta.$$  \hspace{1cm} (3.1.9)

Fixing $n$, if we sum and divide by 2, we have,

$$\cos n\theta = \frac{1}{2} \left( (\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n \right).$$  \hspace{1cm} (3.1.7)

Similarly, if we subtract and divide by $2i$, we get,

$$\sin n\theta = \frac{1}{2i} \left( (\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n \right).$$  \hspace{1cm} (3.1.8)

Now, let $\theta$ be infinitely small and pick $n$ so that $x = n\theta$ is finite. To proceed, we must note that,

$$\sin \theta = \theta \quad \& \quad \cos \theta = 1.$$  \hspace{1cm} (3.1.9)
Substituting into (3.1.7) and (3.1.8) yields:

\[ \cos n\theta = \frac{1}{2} \left( (1 + i\theta)^n + (1 - i\theta)^n \right), \]
\[ \sin n\theta = \frac{1}{2i} \left( (1 + i\theta)^n - (1 - i\theta)^n \right). \]

Substituting \( \theta = x/n \) delivers,

\[ \cos x = \frac{1}{2} \left[ \left(1 + \frac{ix}{n}\right)^n + \left(1 + \frac{(-ix)}{n}\right)^n \right], \]
\[ \sin x = \frac{1}{2i} \left[ \left(1 + \frac{ix}{n}\right)^n - \left(1 + \frac{(-ix)}{n}\right)^n \right]. \]

Note then that putting \( z = \pm ix \) into (3.1.6) delivers our result:

\[ \cos x = \frac{1}{2} (e^{ix} + e^{-ix}) \]
\[ \sin x = \frac{1}{2i} (e^{ix} - e^{-ix}). \]

\[ \square \]

The following theorem falls out rather naturally from here.

**Theorem 3.1.2** For any \( x \in \mathbb{R}, e^{ix} = \cos x + i \sin x \)

**Proof.** Note that from Lemma 3.1.1, we have,

\[ \frac{e^{ix} - e^{-ix}}{2} = i \sin x; \]

whereas,

\[ \frac{e^{ix} + e^{-ix}}{2} = \cos x. \]

Summing these two delivers the desired result, because the left-hand side is exactly
\(e^{ix}\). □

### 3.1.2 (IST) Proof

Below, we will unravel specific transitions made by Euler that require special care, some of which cannot be vindicated straightforwardly. First, we start with (3.1.2):

\[\alpha^\epsilon = 1 + \psi_\alpha\] when \(\epsilon\) is infinitesimal. Since \(\epsilon \simeq 0\), then \(\alpha^\epsilon \simeq 1\). It follows that \(\alpha^\epsilon = 1 + \psi_\alpha\) where \(\psi_\alpha\) is also infinitesimal and depends upon \(\alpha\).

Now, we turn to the most straightforward of them all: (3.1.5). Euler himself defines \(e\) to be \(\sum_{k=0}^{\infty} \frac{1}{k!}\). This is of no serious consequence, since, the Sequence Theorem guarantees that \(\sum_{k=0}^{n} \frac{1}{k!} \simeq \sum_{k=0}^{\infty} \frac{1}{k!}\).

Before dealing with the specifics of the above lemmas, we will discuss (3.1.4). Below, fix illimited positive integer \(n\) and let \(k \neq n\). We need to divide into three cases: (i) \(k\) is limited, (ii) \(k\) is illimited and \((n - k)\) is limited and (iii) \(k\) and \((n - k)\) are both illimited. We will see that Case (iii) will be a problem.

Case (i). First, we see that

\[
\frac{n(n-1)\cdots(n-(k-1))}{n^k} = 1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)
\]

But, since for each integer \(\ell \in [1, k - 1]\), \(\ell/n \simeq 0\) and that \(k\) is infinitely smaller than \(n\), it follows that the right-hand side has a limited number of multiplicands. Therefore,

\[
1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \simeq 1.
\]

Case (ii). If \(n - k\) is limited, then

\[
\frac{n - (k - 1)}{n} \simeq 0.
\]

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We note that in this case that for each $\ell \leq n$, $0 \leq \frac{n-\ell}{n} \leq 1$. In other words, a product of such values will never exceed 1. From these facts, it follows that,

$$\frac{n(n-1)\cdots(n-(k-1))}{n^k} \simeq 0$$

However, this is only in the special case when $k$ is very large.

Case (iii). This is the particular case where Euler’s reasoning stymies. Providing an estimate for values in this range would prove difficult. However, since $k$ is illimited, we may not as readily conclude as before that,

$$1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \simeq 1,$$

nor may we conclude that the far-right multiplicands are nearly zero in which case,

$$\frac{n(n-1)\cdots(n-(k-1))}{n^k} \simeq 0.$$

As such, the proper route to a definition of $e$ as in (3.1.5) would most likely require further investigation or that we simply assume it as an unmotivated constant.

We now consider (3.1.9): $\cos \theta = 1$ and $\sin \theta = \theta$, for infinitesimal $\theta$. Unfortunately, the analogues of these results in IST are unhelpful; viz. that $\cos \theta \simeq 1$ and $\sin \theta \simeq \theta$, for infinitesimal $\theta$. Substitution (as needed to follow Euler perfectly) cannot be carried out when mere infinite closeness holds. Most notoriously, even though $1 + 1/n \simeq 1$ for illimited $n$, it does not follow that $(1 + 1/n)^n \simeq 1^n = 1$; rather, $(1 + 1/n)^n \simeq e$. So, let us take care with these expressions.

Recall that the relevant infinitesimal is $x/n$ for limited $x$ and illimited $n$. Note that $\cos(x/n) = 1 + \delta_1$ and $\sin(x/n) = x/n + \delta_2$ for infinitesimals $\delta_1, \delta_2$. In order to handle things appropriately, we need to get a sense for just how small $\delta_1, \delta_2$ are. The
The easiest way would be to examine their Taylor expansions. However, Euler’s technique for providing Taylor series for sine and cosine depends on the exact same substitution of \( \cos \theta = 1 \) and \( \sin \theta = \theta \) for infinitesimal \( \theta \). In order to stay within the confines of Euler’s resources avoid an implicit circularity, we establish the following related lemma instead:

**Lemma 3.1.3** For all \( y \in [-2, 2] \),

\[
1 - y^2 \leq \cos y \leq 1 - \frac{y^2}{4},
\]

and for all \( y \in [-4, 4] \),

\[
|y - y^3/16| \leq |\sin y| \leq |y - y^3/16|
\]

**Proof.** Beginning with \( \cos y \), we note that \( 1 - y^2 \) and \( 1 - \frac{y^2}{4} \) have roots at \( y = \pm 1 \) and \( y = \pm 2 \), whereas \( \cos y \) has roots at \( y = \pm \pi/2 \). Also, \( \frac{d}{dy} \cos y = -\sin y \), and \( \sin y = 0 \) when \( y = 0 \), so the local maximum is at \( y = 0 \), \( \cos 0 = 1 \). Similarly, \( \frac{d}{dy}(1 - y^2) = -2y \) and \( \frac{d}{dy}(1 - \frac{y^2}{4}) = -y/2 \). Hence, again, the local maximum is at \( y = 0 \), indeed also with value 1. Now, \( 1 - (2)^2 = -3 \); hence,

\[
1 - (2)^2 \leq 0 = \cos(\pi/2) \leq \cos(2) \leq \cos(3\pi/4) = \frac{-\sqrt{2}}{2} \leq 0.
\]

Since \( \cos(-x) = \cos y \), this completes the demonstration that for all \( y \in [-2, 2] \),

\[
1 - y^2 \leq \cos y \leq 1 - \frac{y^2}{4}.
\]

We now turn to \( \sin y \). We will focus on \( y \in [0, 4] \) and drop the absolute value, since \( \sin(-y) = -\sin y \). All three functions have value 0 at \( y = 0 \). Furthermore, they each have another zero at \( y = 1, \pi, 4 \) respectively. The local maximum of \( \sin y \) is the root of \( \frac{d}{dy} \sin y = \cos y \), that is, \( y = \pi/2 \). Furthermore, each of \( y - y^3 \) and \( y - y^3/16 \)
has a local maximum at $y = 1/\sqrt{3}$ and $y = 4/\sqrt{3}$ respectively (again, by examining the roots of their derivative functions.) The following inequality will give us a sense of where these values fall, with the maxima in *italics*:

$$0 < \frac{1}{\sqrt{3}} < 1 < \frac{\pi}{2} \leq 2 = \frac{4}{2} < \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3} < \frac{9}{3} = 3 < \pi$$

Let us note a few things: First, $1 = \sin(3\pi/2) < \sin 4 < 0 = \sin(\pi)$ and $4 - 4^3 = -60$. So, clearly at $y = 4$, the inequality holds. Furthermore, using the first part of the lemma, we have for $y \in [0, 2]$,

$$\left(\frac{d}{dy}\right) y - y^3 = 1 - 3y^2 \leq 1 - y^2 \leq \cos y \leq 1 - \frac{y^2}{4} \leq 1 - \frac{3y^2}{16} = \left(\frac{d}{dy}\right) \left(y - \frac{y^3}{16}\right)$$

Since all three functions being have values at $y = 0$ and their derivative functions exhibit the expected inequality, we have the desired inequality for $y \in [0, 2]$.

We should also examine the desired inequality on the various maxima near $y = 2$. Provided it holds in those places, by the continuity of each function, our inequality will hold everywhere on the relevant interval.

<table>
<thead>
<tr>
<th>$y = \pi/2$</th>
<th>$y - y^3$</th>
<th>$\frac{\pi}{2}(1 - \frac{\pi^2}{4}) &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \pi/2$</td>
<td>$\sin y$</td>
<td>1</td>
</tr>
<tr>
<td>$y - y^3/16$</td>
<td>$\frac{\pi}{2}(1 - \frac{\pi^2}{64}) \geq \frac{\pi^2}{2} \geq \frac{162}{128} &gt; 1$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y = 4/\sqrt{3}$</th>
<th>$y - y^3$</th>
<th>$\frac{4}{\sqrt{3}}(1 - \frac{16}{3}) &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 4/\sqrt{3}$</td>
<td>$\sin y$</td>
<td>$0 = \sin \pi \leq \sin(4/\sqrt{3}) \leq \sin(\pi/2) = 1$</td>
</tr>
<tr>
<td>$y - y^3/16$</td>
<td>$\frac{4}{\sqrt{3}}(1 - \frac{1}{3}) = \frac{4}{\sqrt{3}} \cdot \frac{2}{3} = \frac{4}{3} \cdot \frac{2}{\sqrt{3}} &gt; \frac{4}{3} &gt; 1$</td>
<td></td>
</tr>
</tbody>
</table>
From the previous lemma, we have for infinitesimal \( x/n \),

\[
1 - \left( \frac{x}{n} \right)^2 \leq 1 + \delta_1 \leq 1 - \left( \frac{x}{2n} \right)^2,
\]

and

\[
\left( \frac{x}{n} \right)^3 - \left( \frac{x}{n} \right)^3 \leq \left( \frac{x}{n} \right)^3 + \delta_2 \leq \left( \frac{x}{n} \right)^3 - \frac{1}{16} \left( \frac{x}{n} \right)^3.
\]

From these we have

\[
\frac{r}{n^2} \leq \delta_1 \leq \frac{s}{n^2} \quad \text{and} \quad \frac{t}{n^3} \leq \delta_2 \leq \frac{u}{n^3}
\]

(3.1.10)

for limited \( r, s, t, u \in \mathbb{R} \). These approximations on \( \delta_1 \) and \( \delta_2 \) will be significant below.

Returning to (3.1.7) and (3.1.8), we need to establish the following.

**Lemma 3.1.4** For limited \( x \) and illimited \( n \),

\[
\left( \cos\left( \frac{x}{n} \right) \pm i \sin\left( \frac{x}{n} \right) \right)^n \simeq \left( 1 \pm i \left( \frac{x}{n} \right) \right)^n.
\]

**Proof.** We begin with the following substitutions. \( \left( \cos\left( \frac{x}{n} \right) \pm i \sin\left( \frac{x}{n} \right) \right)^n = \left( 1 + \delta_1 \pm i(x/n + \delta_2) \right)^n = \left( 1 \pm i(x/n) + (\delta_1 \pm i\delta_2) \right)^n = \left( 1 \pm i(x/n) \right)^n + \]

\[
\binom{n}{1}(\delta_1 \pm i\delta_2)(1 \pm i(x/n))^{n-1} + \binom{n}{2}(\delta_1 \pm i\delta_2)^2(1 \pm i(x/n))^{n-2} + \cdots + (\delta_1 \pm i\delta_2)^n.
\]

We must treat this final expanded binomial expression with care. There are \( n + 1 \) (illimitedly many) summands, so we must ensure that together they are infinitesimal. First, we will show that for \( k = 1 \) the value is infinitesimal. But, for \( k > 1 \), we will show that the summands are especially small.

First, we note that \( (1 \pm i(x/n))^n \simeq e^{\pm ix} \), i.e. \( |(1 \pm i(x/n))^n| \) is limited. It follows
that for \( k \leq n \),

\[
0 \leq \left| (1 \pm i(x/n))^{n-k} \right| \leq (1 + |x|/n)^{n-k} \leq (1 + |x|/n)^n \approx e^{|x|}.
\]

Again, its complex magnitude must be limited. Therefore, starting with \( k = 1 \), we have,

\[
\left| \binom{n}{1} (\delta_1 \pm i\delta_2) (1 \pm i(x/n))^{n-1} \right| = |dn(\delta_1 \pm i\delta_2)| \leq \frac{2sdn}{n^2} = \frac{2sd}{n}
\]

The right-hand side is clearly infinitesimal.

Now, we show that for all \( k > 1 \),

\[
\binom{n}{k} (\delta_1 \pm i\delta_2)^k (1 \pm i(x/n))^{n-k} \ll \frac{1}{n} \tag{3.1.11}
\]

Examining the general case, we have,

\[
\binom{n}{k} |(\delta_1 \pm i\delta_2)^k| \cdot |(1 \pm ix/n)^{n-k}| \leq e^{|x|} \tag{3.1.12}
\]

Due to the fact that the right hand multiplicand stays limited, all interesting behavior involves the interaction between the \( \delta \)'s and \( \binom{n}{k} \). As with any binomial expression, the place where \( \binom{n}{k} \) is largest is in the middle, \( \text{viz.} \) when \( k = \lceil n/2 \rceil \). If the \( \delta \)'s dominate \( \binom{n}{k} \) there, they will undoubtedly do so for all other relevant values of \( k \). It suffices then to show (3.1.11) for \( k = \lceil n/2 \rceil \). To simplify, fill in (3.1.12), and note that,

\[
\binom{n}{\lceil n/2 \rceil} |(\delta_1 \pm i\delta_2)^{\lceil n/2 \rceil}| \cdot |(1 \pm ix/n)^{n-\lceil n/2 \rceil}| \leq \binom{n}{\lceil n/2 \rceil} 2e^{|x|}|\delta_1|^{\lceil n/2 \rceil} \leq 2 \cdot n \cdots (n - \lceil n/2 \rceil) e^{|x|}|\delta_1|^{\lceil n/2 \rceil}
\]
In this case, the first term of the product is the largest, it suffices to recognize the following:

\[
2^n \left| e^{\frac{x}{n^2}} \right| \delta_1 \leq 2^n \left| e^{\frac{s}{n^2}} \right| \frac{s}{n^2} = \frac{2e^{\left| s \right|/n^2}}{n^2} \ll \frac{1}{n}
\]

To review the path we have taken, we have that

\[
\left( \cos(x/n) \pm i \sin(x/n) \right)^n = \left( 1 \pm i(x/n) \right)^n + \frac{n}{1} \left( \delta_1 \pm i\delta_2 \right) \left( 1 \pm i(x/n) \right)^{n-1} \]

\[
\cong 0 \\
+ \frac{n}{2} \left( \delta_1 \pm i\delta_2 \right)^2 \left( 1 \pm i(x/n) \right)^{n-2} + \cdots + \left( \delta_1 \pm i\delta_2 \right)^n \simeq \left( 1 \pm i(x/n) \right)^n.
\]

\[\square\]

### 3.1.3 Orthodox Proof

A most elegant proof comes from Strang’s *Calculus*, [40]. Since \( e^{ix} \) is complex, there is a polar coordinate representation of it, for some \( r, \theta \) pair:

\[
e^{ix} = r(\cos \theta + i \sin \theta).
\]

(3.1.13)

Differentiating both sides yields:

\[
ie^{ix} = \frac{dr}{dx} \left( \cos \theta + i \sin \theta \right) + r \left( -\sin \theta \frac{d\theta}{dx} + i \cos \theta \frac{d\theta}{dx} \right).
\]

(3.1.14)

Now, we substitute (3.1.13) into the left-hand side:

\[
ir(\cos \theta + i \sin \theta) = \frac{dr}{dx} \left( \cos \theta + i \sin \theta \right) + r \left( -\sin \theta \frac{d\theta}{dx} + i \cos \theta \frac{d\theta}{dx} \right).
\]

(3.1.15)
Now, let us compare the complex and real components of (3.1.15). For the complex components, we have:

\[ r \cos \theta = \left( \frac{dr}{dx} \sin \theta + r \cos \theta \frac{d\theta}{dx} \right); \quad (3.1.16) \]

whereas, the real component is

\[ -r \sin \theta = \left( \frac{dr}{dx} \cos \theta - r \sin \theta \frac{d\theta}{dx} \right). \quad (3.1.17) \]

Using simple linear combinations, multiply (3.1.16) by \( \cos \theta \) and (3.1.17) by \( -\sin \theta \) and combine to yield:

\[ -r \left( \cos^2 \theta + \sin^2 \theta \right) = \frac{dr}{dx} \left( \sin \theta \cos \theta - \cos \theta \sin \theta \right) - r \frac{d\theta}{dx} \left( \cos^2 \theta + \sin^2 \theta \right) \]

It follows that \( d\theta/dx = 1 \). Substituting this back into either (3.1.16) or (3.1.17) yields that \( dr/dx = 0 \). It follows that \( r = c \) and \( \theta = x + d \) for constants \( c, d \in \mathbb{R} \). Now, since \( e^{i\theta} = 1 \), we substitute all values into (3.1.13) to yield: \( 1 = c \cos d \) and \( 0 = c \sin d \). It follows that \( c = 1 \) and \( d = 2k\pi \) for some integer \( k \). But \( \theta \in [0, 2\pi) \), so put \( d = 0 \).

Hence, \( r = 1 \) and \( \theta = x \), yielding the desired result. \( \square \)

Chapter 3.2

A Proof of Harmonic Divergence

Euler proves that the Harmonic Series diverges in a stereotypically quick manner. First, we begin with,

\[ \ln \left( \frac{1}{1 - x} \right) = \sum_{k=1}^{\infty} \frac{x^k}{k} \]
Put $x = 1$, and we have that $\sum \frac{1}{k} = \ln(\frac{1}{1-1}) = \ln \infty = \infty$. Let us slow things down some and take a closer look.

### 3.2.1 Basic Outline

We start by establishing the relevant Taylor expansion used by Euler in his proof.

**Lemma 3.2.1** For $x < 1$,

$$\ln \left( \frac{1}{1 - x} \right) = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad (3.2.1)$$

**Proof.** Recall (3.1.6), which states that $e^z = (1 + \frac{z}{n})^n$. Let $\epsilon = z/n$ so that $e^{\epsilon n} = (1 + \epsilon)^n$. Now, define $y$ so that it satisfies,

$$y = (1 + \epsilon)^n - 1, \quad (3.2.2)$$

with relevant definitions from above remaining intact. It follows that $\epsilon + 1 = (1 + y)^{\frac{1}{n}}$. Now, $\epsilon + 1 = (1 + y)^{\frac{1}{n}}$ can be expanded using Newton’s generalized binomial theorem:

$$(1 + y)^{\frac{1}{n}} = \sum_{k=0}^{\infty} \binom{1/n}{k} y^k = 1 + \sum_{k=1}^{\infty} \frac{(1/n)(1/n - 1) \cdots (1/n - (k - 1))}{k!} y^k.$$  

Switching the order of each term in the numerator yields,

$$1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(1/n)(1 - 1/n) \cdots ((k - 1) - 1/n)}{k!} y^k.$$  

To tighten up, subtract 1 and reduce to get,

$$\epsilon = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n - 1) \cdots ((k - 1)n - 1)}{n^k k!} y^k. \quad (3.2.3)$$
Using (3.2.2), $1 + y = (1 + \epsilon)^n$. Using the fact that $(1 + \epsilon)^n = e^{\epsilon n}$, it follows that

$$\ln(1 + y) = \epsilon n.$$ 

Multiply everything by $n$ in (3.2.3) delivers,

$$\ln(1 + y) = \epsilon n = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n-1) \cdots ((k-1)n-1)}{n^{k-1}k!} y^k. \quad (3.2.4)$$

Note that for $k > 1$, each term may be written in the following form,

$$\frac{(n-1) \cdots ((k-1)n-1)}{2n \cdots kn}.$$

Since $n$ is infinite, it follows that

$$\frac{(n-1) \cdots ((k-1)n-1)}{2n \cdots kn} = \frac{1}{2} \frac{2}{3} \cdots \frac{k-2}{k} \frac{1}{k} = \frac{1}{k}. \quad (3.2.5)$$

Substituting the above reduction delivers the familiar result:

$$\ln(1 + y) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{y^k}{k}. \quad (3.2.6)$$

Finally, put $y = -x$ in (3.2.6),

$$\ln(1 - x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(-x)^k}{k} = \sum_{k=1}^{\infty} (-1)^{2k+1} \frac{x^k}{k} = - \sum_{k=1}^{\infty} \frac{x^k}{k}$$

Using laws of logarithms, (3.2.1) follows naturally.
3.2.2 (IST) Proof

Most of the above follows from ordinary algebra. There is one exception: (3.2.5):

\[
\frac{(n-1)2n-1}{2n} \cdot \frac{(k-1)n-1}{3n} \cdots \frac{k-2k-1}{kn} = \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{k-1}{k} = \frac{1}{k}
\]

Within (IST) we represent this in terms of the following: Let \( n \in \mathbb{N} \) be illimited. We want for all \( k < n \),

\[
\prod_{j=2}^{k} \frac{(j-1)n-1}{jn} \approx \frac{1}{k}.
\]

We will see that in spite of our desires, we are not yet in a position to defend this claim. To see where things go astray, we note that,

\[
\prod_{j=2}^{k} \frac{(j-1)n-1}{jn} = \prod_{j=2}^{k} \left( \frac{a_j}{j} + \frac{-1}{jn} \right),
\]

where we define sequences \( a_j \) and \( b_j \). So,

\[
\prod_{j=2}^{k} \frac{(j-1)n-1}{jn} = \prod_{j=2}^{k} \left( a_2a_3 \ldots a_k + b_2a_3 \ldots a_k + \cdots + b_2b_3 \ldots b_k \right). \quad (3.2.7)
\]

Now, for \( \prod a_i \), the product is just \( \frac{1}{k} \) by cancellation. We need to ensure that the remainder is infinitesimal. We divide into cases again, (i) \( k \) is limited and (ii) \( k \) is illimited. In cases (i), the sum will be less than or equal to \( (2^{k-1} - 1)/n \approx 0 \), which is what we want. We now examine case (ii) where \( k \) is illimited. For any other summand, we note that its absolute value falls in the following interval:

\[
\left[ \frac{1}{k!n^k}, \frac{1}{n} \right].
\]
Unfortunately, it is not clear how one might proceed because the behavior across the range of cases is not sufficiently lucid. If all summands behaved like the lefthand side, it would be infinitesimal; if all summands were like the righthand side, \((2^k - 1)/n\) could even be illimited given the size of \(2^k\).

Finally, we justify the theorem directly, assuming Lemma (3.2.1). We need to show that a statement such as \(\ln\left(\frac{1}{1-\beta}\right) = \infty\) can be vindicated within (IST). In other words, we want to know the value of \(\ln\left(\frac{1}{1-\beta}\right)\) for all \(\beta < 1\) and \(\beta \approx 1\) and how it relates to the harmonic series.

**Theorem 3.2.2** The harmonic series diverges.

**Proof.** Note that \(1 - \beta \approx 0\) when \(\beta \approx 1\), and so \(\frac{1}{1-\beta} = w\), where \(w \in \mathbb{R}\) is positive illimited. Suppose for contradiction that \(\ln w\) is limited. It follows that there is some standard \(r \in \mathbb{R}\) so that \(\ln w < r\). It follows that \(w < e^r\) and \(e^r\) is a standard value. It follows that \(\ln w\) and hence \(\sum_{k=1}^{\beta^k} \frac{k}{k}\) are illimited. Since \(\beta < 1\), it follows that \(\beta^k < 1\).

Since for each \(k \in \mathbb{N}\), we have that \(\frac{\beta^k}{k} < \frac{1}{k}\), we see that

\[
\ln w = \sum_{k=1}^{\infty} \frac{\beta^k}{k} < \sum_{k=1}^{\infty} \frac{1}{k}
\]

Clearly, if \(\ln w\) is illimited, so is the harmonic series; i.e. it diverges. □

### 3.2.3 Orthodox Proof

A proof using limits that follows a similar pathway is not obviously available. The natural analogue using limits requires first establishing the requisite uniform convergence result. The more commonly known way of proving the harmonic series is as follows.

**Theorem 3.2.3** The harmonic series diverges.
**Proof.** We begin by first observing the following. For any positive integer $k$,

$$\frac{1}{2^{b_k}} \leq \frac{1}{k}, \text{ where } b_k = \lceil \log_2 k \rceil,$$  

(3.2.8)

where $\lceil x \rceil = \inf \{n \in \mathbb{Z} : n \geq x\}$, the ceiling function. To see this, note that $b_n = \lceil \log_2 n \rceil \geq \log_2 n$; whence it follows that $2^{b_n} \geq n$. The observation (3.2.8) follows from this. Now, the harmonic series diverges if the series $\sum 2^{-b_k}$ does. It suffices then to establish this latter fact.

**Lemma 3.2.4** The series $\sum_{k=1}^{\infty} 2^{-b_k}$ is unbounded.

**Proof.** We show this by demonstrating the unboundedness of a special subsequence of partial sums: for all $n \geq 1$,

$$\sum_{k=1}^{2^n} 2^{-b_k} = 1 + \frac{n}{2},$$  

(3.2.9)

from which our result follows since the $n/2$ is obviously unbounded. We prove this by induction. **Basis:** For $n = 1$, $b_1 = 0$, $b_2 = 1$, so the lefthand side is $2^{-b_1} + 2^{-b_2} = 1 + 2^{-1} = 3/2$. The righthand side is $1 + 1/2 = 3/2$. **Inductive Step:** Suppose that for all $m \leq n$, (3.2.9) holds. Now, clearly,

$$\sum_{k=1}^{2^{n+1}} 2^{-b_k} = \sum_{k=1}^{2^n} 2^{-b_k} + \sum_{k=2^n+1}^{2^{n+1}} 2^{-b_k}$$

But by inductive hypothesis, this is equal to,

$$1 + \frac{n}{2} + \sum_{k=2^n+1}^{2^{n+1}} 2^{-b_k}.$$  

(3.2.10)

Note that the far-right side summation has $2^n$ summands, since $2^n + 2^n = 2^{n+1}$; i.e. $2^{n+1} - 2^n = 2^n$. Furthermore, for any integer $k \in (2^n, 2^{n+1}]$, $b_k = n + 1$. Therefore,
(3.2.10) is
\[
1 + \frac{n}{2} + 2^n 2^{-(n+1)} = 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n + 1}{2}.
\]

\[\square\]

Chapter 3.3

The Wallis Product

The Wallis product is a relatively early result, produced by John Wallis in 1655 linking \(\pi\) with a special infinite quotient of odd and even integers. Euler uses the insights gained regarding the infinite expansion of sine to deliver the desired result. Compared to Euler, Wallis’ own proof is even more alien in technique to modern readers. It seems that at this stage in mathematics, revealing the process of discovery was treated by some—though not all—as an adequate proof of a given truth of mathematics. From our understanding, Euler’s proof was well-received from the get-go despite any potential misgivings from contemporary mathematicians.\(^2\)

3.3.1 Basic Outline

Proposition 3.3.1 The Wallis Product:

\[
\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1}
\]

\(^2\)Boyer and Merzbach discuss this in their History of Mathematics, [27]. Fermat was openly critical of Wallis’ lack of rigor. Apparently, Wallis’ proof using the technique of interpolation left even his contemporaries cold. Nevertheless, his proofs were not so bad as to be thrown out by everyone. Wallis’ results like this one have been celebrated ever since.
Proof. Fix the infinite polynomial (or infinite series in modern language) \( P(x) \), as follows:

\[
P(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}
\]

Now, \( P(x) \) may be written as follows

\[
P(x) = \frac{x}{x} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} = \frac{1}{x} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.
\]

Since the Taylor expansion of sine \(^3\) is the far-right multiplicand, it follows that,

\[
P(x) = \frac{\sin x}{x}
\]

At this point, one notes that \( P(x) \)'s zeros can be revealed by the zeros of \( \sin x \), namely all and only those \( x = k\pi \) for \( k \in \mathbb{Z} \setminus \{0\} \). Using the fundamental theorem of algebra, it follows that, \(^4\)

\[
P(x) = \prod_{k=1}^{\infty} \left( 1 - \frac{x}{k\pi} \right) \left( 1 + \frac{x}{k\pi} \right) = \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{k^2\pi^2} \right).
\]

(3.3.1)

Now, put \( x = \frac{\pi}{2} \), so \( P(\frac{\pi}{2}) = \)

\[
\frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{2}{\pi} = \prod_{k=1}^{\infty} \left( 1 - \frac{(\frac{\pi}{2})^2}{k^2\pi^2} \right) = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{(2k)^2} \right) = \prod_{k=1}^{\infty} \left( \frac{(2k-1)(2k+1)}{(2k)^2} \right);
\]

that is,

\[
\frac{2}{\pi} = \prod_{k=1}^{\infty} \left( \frac{(2k-1)(2k+1)}{(2k)^2} \right).
\]

\(^3\)Euler himself actually produces the Taylor expansion of sine in his characteristically wild way without any recourse to derivatives.

\(^4\)Euler produces this result in §158 of the Introductio.

\(^5\)The clever insight of substituting here is found in §§184-5 of the Introductio.
3.3.2 (IST) Proof

Euler’s proof of the Wallis product is so elegant that it seems almost too good to be true. To some extent, it is too good to be true. After a certain point, NSA gives out.

Instead of speaking of an infinite polynomial, in (IST), let $Q(x)$ be the following polynomial of degree $2n$, for a fixed illimited valued $n$:

$$Q(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k + 1)!}$$

By the fundamental theorem of algebra, $Q(x)$ may be written as follows:

$$Q(x) = c \prod_{k=1}^{n} \left( 1 - \frac{x}{a_k} \right) \left( 1 + \frac{x}{a_k} \right), \text{ for } c, a_k \in \mathbb{C}.$$ 

This is a noteworthy advantage that cannot be recaptured in orthodox analysis, since this is actually a polynomial with roots, unlike $P(x)$ above. Carrying forward, since the zeroth term of $Q(x)$ is 1, $c = 1$.

As before, the taylor expansion of sine guarantees that,

$$Q(x) = \frac{1}{x} \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k + 1)!} \simeq \frac{\sin x}{x}.$$ 

Ideally, we would simply use the roots from $\sin x/x$ to apply to the illimited polynomial $Q$. Unfortunately, this tells us only when $Q(x) \approx 0$, and is therefore not especially helpful in determining $Q$’s roots.\(^6\) So, $Q$ has its own roots, but to tie these to the needed trigonometric values would require significantly more investigation. At minimum, to expand Euler’s proof would require proceeding along a circuitous path not tread by Euler himself—not even in spirit.

\(^6\)In “The correctness of Euler’s method for the factorization of the sine function into an infinite product,” V.G. Kanovei provides two proofs of Euler’s factorization of sine in NSA. These proofs pass through a slightly different inferential path than the one we are examining here.
3.3.3 Orthodox Proof

It is possible to provide an orthodox justification of (3.3.1). Here we give a more direct proof of Wallis’ product that proceeds by integrating \( \sin^n(x) \) as \( n \) goes to infinity.\(^7\)

We must establish first the following lemma to be used below.

Lemma 3.3.2 For all positive integers \( n > 2 \),

\[
\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \quad (3.3.2)
\]

Proof. Using integration by parts, one may easily establish that

\[
\int \sin^n(x) \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx
\]

It follows that \( \int_0^{\pi/2} \sin^n x \, dx = \)

\[
-\frac{1}{n} \sin^{n-1}(\pi/2) \cos(\pi/2) + \frac{1}{n} \sin^{n-1}(0) \cos(0) + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx
\]

\( \square \)

Lemma 3.3.3 For all positive integers, \( n \),

\[
\int_0^{\pi/2} \sin^{2n+1} x \, dx = \prod_{k=1}^n \frac{2k}{2k+1}. \quad (3.3.3)
\]

Proof. We proceed by induction on \( n \). Basis: use (3.3.2), \( \int_0^{\pi/2} \sin^3 x \, dx = 2/3 \int_0^{\pi/2} \sin x \, dx = 2/3 \).

Inductive Step: Suppose for some \( n > 0 \),

\[
\int_0^{\pi/2} \sin^{2n+1} x \, dx = \prod_{k=1}^n \frac{2k}{2k+1}.
\]

\(^7\)This proof is laid out in parts as several consecutive exercises in Spivak’s [39].
By (3.3.2) and the inductive hypothesis,

\[
\int_0^{\pi/2} \sin^{2(n+1)} x \, dx = \frac{2n + 2}{2n + 3} \int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2n + 2}{2n + 3} \prod_{k=1}^{n} \frac{2k}{2k + 1} = \prod_{k=1}^{n+1} \frac{2k}{2k + 1}.
\]

\Box

**Lemma 3.3.4** For all positive integers, \( n \),

\[
\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k - 1}{2k}.
\] (3.3.4)

**Proof.** We will proceed by induction on \( n \). **Basis:** We use (3.3.2), \( \int_0^{\pi/2} \sin^2 x \, dx = 1/2 \int_0^{\pi/2} dx = 1/2 \cdot \pi/2 \). **Inductive Step:** Suppose for some \( n > 0 \),

\[
\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k - 1}{2k}.
\]

By (3.3.2) and the inductive hypothesis,

\[
\int_0^{\pi/2} \sin^{2(n+1)} x \, dx = \frac{2n + 1}{2n + 2} \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{2n + 1}{2n + 2} \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k - 1}{2k} = \frac{\pi}{2} \prod_{k=1}^{n+1} \frac{2k - 1}{2k}.
\]

\Box

**Theorem 3.3.5** *The Wallis Product:*

\[
\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k}{2k - 1} \cdot \frac{2k}{2k + 1}
\]

**Proof.** Using Lemmas (3.3.3) and (3.3.4), it follows that for any positive integer \( n \),

\[
\frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} \prod_{k=1}^{n} \frac{(2k)(2k)}{(2k - 1)(2k + 1)} = \frac{\pi}{2}
\]
Now, we claim that the left-hand quotient of integrals closes in on 1 as \( n \) increases. In fact, we claim that for all positive integers \( n \), that,

\[
1 \leq \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} \leq 1 + \frac{1}{2n}, \tag{3.3.5}
\]

whence the result follows.

To establish left-hand inequality of (3.3.5), first we show that for all \( n \int_0^{\pi/2} \sin^{n+1} x \, dx \leq \int_0^{\pi/2} \sin^n x \, dx \). This follows easily from the fact that for every \( x \in [0, \pi/2] \), \( 0 \leq \sin x \leq 1 \) and so \( \sin^{n+1} x \leq \sin^n x \) on the same interval.

To establish the right-hand inequality of (3.3.5). By applying Lemma (3.3.2) to the denominator of the following, we get,

\[
\frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} = \frac{2n + 1}{2n} \left( \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n-1} x \, dx} \right) \leq 1 + \frac{1}{2n} = 1 + \frac{1}{2n}
\]

\( \square \)

**Chapter 3.4**

**Euler’s Product**

This particular example suggests that not all of Euler’s flights of fancy can be recovered in an interesting way within (IST). In order to simplify readability in the below proofs, let \( \mathfrak{P} \) be the set of prime numbers.

**Theorem 3.4.1** For any positive integer, \( s > 1 \)

\[
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{p \in \mathfrak{P}} \frac{1}{1 - p^{-s}}.
\]
I will here quote directly from Euler as another illustration of the rapid pace with which his proofs proceed.

270. Let us now consider the expression

\[
\frac{1}{(1 - \alpha z)(1 - \beta z)(1 - \gamma z)(1 - \delta z)(1 - \epsilon z)\cdots}
\]

When the division is carried out, we obtain the series

\[
1 + A z + B z^2 + C z^3 + D z^4 + E z^5 + F z^6 + \cdots.
\]

It is clear that the coefficients \(A, B, C, D, E, F\), etc. depend on the numbers \(\alpha, \beta, \gamma, \delta, \epsilon, \) etc. in the following way: \(A\) is the sum of the numbers taken singly; \(B\) is the sum of the products taken two at a time; \(C\) is the sum of the products taken three at a time; \(D\) is the sum of the products taken four at a time; etc., where we do not exclude products of the same factor.

271. When we let \(z = 1\) the expression

\[
\frac{1}{(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)(1 - \epsilon)\cdots}
\]

is equal to the sum of 1 and the series of all the numbers which arise from \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta\), etc. taken either singly or the product of two, or more at a time, where we do not exclude the possibility that some of these terms may be equal . . .

272. For this reason, the series always has an infinite number of terms, whether the product has an infinite or finite number of factors. For example,

\[
\frac{1}{(1 - \frac{1}{2})} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

where all the numbers occur which are powers of \(\frac{1}{2}\). Then we have

\[
\frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \cdots,
\]

where numbers do not occur unless they are products of powers of 2 and 3, that is, unless they are divisible by 2 or 3.
273. If for $\alpha, \beta, \gamma, \delta$, etc. we substitute the reciprocals of all of the primes, and let

$$P = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})\cdots}$$

then $P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \cdots$, where all numbers, whether prime or products of primes, occur. Since every natural number is either a prime or a product of primes, it is clear that all integers will appear in denominators.

274. The same happens if any power of the primes is used. We let

$$P = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\cdots}$$

then $P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{9^n} + \cdots$, where all natural numbers occur with no exception.

3.4.1 Basic Outline

The reader will have noticed that in spite of Euler’s fast-paced proof that the substance of the argument revolves around the fundamental theorem of arithmetic and expression for the geometric series, whereby,

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a}, \quad 0 \leq a < 1. \quad (3.4.1)$$

In effect, if $N$ is an infinite number, $p \in \mathbb{P}$ and $s > 1$,

$$\sum_{k=0}^{N} p^{-sk} = \frac{1}{1 - p^{-s}} \quad (3.4.2)$$

---

8Euler, [7], pp. 230-232.
Now, multiply out all $p \in \mathbb{P}$ on both sides.

$$\prod_{p \in \mathbb{P}} \sum_{k=0}^{N} p^{-sk} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}. \tag{3.4.3}$$

But the left hand-side when expanded is simply $\sum 1/n^s$.

### 3.4.2 (IST) Proof

Unfortunately, very little interesting material emerges from the use of (IST). Indeed, a particular inference is at best not obvious and at worst outright spurious. It is worth examining it either way. The reader will discover that in the next section, (IST) proof closely resembles the orthodox proof. In fact, our ability to manipulate genuine identities in the orthodox proof gives it the edge over (IST). From the Basic Outline, we aim to must provide justification for both (3.4.2) and (3.4.3).

Since $0 \leq p^{-s} < 1$ for any $p \in \mathbb{P}, s > 1$,

$$\sum_{k=0}^{\infty} p^{-sk} = \frac{1}{1 - p^{-s}}$$

Fix some limited $N \in \mathbb{N}$. It follows that

$$\sum_{k=0}^{\infty} p^{-sk} \simeq \sum_{k=0}^{N} p^{-sk}.$$

Therefore,

$$\sum_{k=0}^{N} p^{-sk} \simeq \frac{1}{1 - p^{-s}}. \tag{3.4.4}$$

Now, multiply both sides of (3.4.4) by all primes $p \leq N$. It is at this point that
we discover a break down. Ideally, we would like to infer that,

$$\prod_{p \in \mathbb{P}, p \leq N} \sum_{k=0}^{N} p^{-sk} \simeq \prod_{p \in \mathbb{P}, p \leq N} \frac{1}{1 - p^{-s}}. \quad (3.4.5)$$

The problem here is that infinite products of infinitesimally close values are not infinitesimally close. For example, for all $k$, set $x_k = 1$ and $y_k = (1 + 1/N)$, using the same $N$ from above. Unfortunately,

$$\prod_{k=1}^{N} x_k = 1 \not\simeq e \simeq \left(1 + \frac{1}{N}\right)^N = \prod_{k=1}^{N} y_k.$$

If this inference can be made faithfully, we would continue as follows. First, let us examine the left-hand side of (3.4.5). Expand it to deliver,

$$\prod_{p \in \mathbb{P}, p \leq N} \sum_{k=0}^{N} p^{-sk} = (1 + p^{-s} + \cdots + p^{-Ns}) \cdots (1 + p^{-s} + \cdots + p^{-\pi(N)s}) \quad (3.4.6)$$

where $\pi(n)$ is the cardinality of the set $\{p \in \mathbb{P} : p \leq n\}$. Define

$$\mathfrak{T}^m_n = \left\{ \langle \ell_1, \ldots, \ell_{\pi(n)} \rangle \in \{1, \ldots, m\}^{\pi(n)} : n < \prod_{i=1}^{\pi(n)} p_i^{\ell_i} \right\}.$$

By the uniqueness of decomposition, multiplying through the right-hand side of (3.4.6) delivers,

$$\sum_{k=1}^{N} \frac{1}{k^s} + \sum_{\ell \in \mathfrak{T}^m_N} \prod_{i=1}^{\pi(N)} p_i^{-s\ell_i}.$$

The left summand is infinitesimally close to $\zeta(s)$. Now, since

$$\sum_{k=1}^{N} \frac{1}{k^s} \simeq \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s},$$
it follows that
\[ \sum_{k=N+1}^{\infty} \frac{1}{k^s} \simeq 0, \]

But clearly,
\[ 0 \leq \sum_{\ell \in \mathcal{T}_N} \prod_{i=1}^{\pi(N)} p_i^{-s \ell_i} \leq \sum_{k=N+1}^{\infty} \frac{1}{k^s} \simeq 0 \]

Hence, returning to (3.4.5),
\[ \zeta(s) \simeq \prod_{\substack{p \in \mathcal{P} \leq N}} \frac{1}{1 - p^{-s}}. \]

Now, we may generalize on any illimited \( N \), so that,
\[ \zeta(s) \simeq \prod_{\substack{p \in \mathcal{P} \leq N}} \frac{1}{1 - p^{-s}} \text{ for all illimited } N. \]

In other words,
\[ \zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}. \]

\( \square \)

### 3.4.3 Orthodox Proof

Here we develop in slightly greater detail a proof found in Ireland and Rosen’s [17].

The reader will notice that the major insights from Euler are just as easily captured here as in (IST). We begin with the insight regarding geometric series noted above in (3.4.1): for \( p \in \mathcal{P}, s > 1, \)
\[ \sum_{k=0}^{\infty} p^{-sk} = \frac{1}{1 - p^{-s}}, \]

since \( p^{-s} < 1 \). Now, fix some \( N \in \mathbb{N} \) and it follows that
\[ \prod_{\substack{p \in \mathcal{P} \leq N}} \sum_{k=0}^{\infty} p^{-sk} = \prod_{\substack{p \in \mathcal{P} \leq N}} \frac{1}{1 - p^{-s}}. \]
Focusing on the left-hand side, the fundamental theorem of arithmetic yields,

\[
\prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} p^{-sk} = \sum_{k=0}^{N} \frac{1}{k^s} + \sum_{\ell \in \mathcal{S}_N} \prod_{i=1}^{\pi(N)} p_{\ell_i}^{-s\ell_i},
\]  

(3.4.9)

where we define

\[
\mathcal{S}_n = \{ (\ell_1, \ldots, \ell_{\pi(n)}) \in \mathbb{N}^{\pi(n)} : n < \prod_{i=1}^{\pi(n)} p_i^{\ell_i} \}.
\]

But \( R_N(s) \to 0 \), since for \( s > 1 \), \( \sum \frac{1}{n^s} \) converges and

\[
R_N(s) \leq \sum_{N+1}^{\infty} n^{-s}.
\]

Combining (3.4.8) and (3.4.9), and letting \( N \to \infty \), it follows as desired that,

\[
\prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}} = \sum_{k=0}^{\infty} \frac{1}{n^s}.
\]

\[\square\]
In this study, we hope to have shown that a fair amount of Euler’s creative and revolutionary proofs may be reconstructed within nonstandard analysis. Although there are many, many more proofs of Euler’s besides these here, it is significant that NSA provides at least some sense that Euler’s inferences were, if not erroneous, overly compact. We do not mean to claim that Euler somehow anticipated NSA, but we do think that NSA leaves less question as to the inferences that have been previously regarded as “questionable.”

\[\text{Dunham makes remarks like these throughout his text. He writes, “As we shall see, some of Euler’s arguments were questionable, and others were simply wrong.” (Dunham, [5], xvii)}\]
Bibliography


