Photon Counting as a Probe of Superfluidity in a Two-Band Bose Hubbard System Coupled to a Cavity Field

A Thesis

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By

Sara Rajaram, B.A.

Graduate Program in Physics

The Ohio State University

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Master’s Examination Committee:

Dr. Nandini Trivedi, Advisor

Dr. Louis DiMauro
Abstract

We show that photon number measurement can be used to detect superfluidity for a two-band Bose-Hubbard model coupled to a cavity field. The atom-photon coupling induces transitions between the two internal atomic levels and results in entangled polaritonic states. In the presence of a cavity field, we find different photon numbers in the Mott-insulating versus superfluid phases, providing a method of distinguishing the atomic phases by photon counting. Furthermore, we examine the dynamics of the photon field after a rapid quench to zero atomic hopping by increasing the well depth. We find a robust correlation between the field’s quench dynamics and the initial superfluid order parameter, thereby providing a novel and accurate method of determining the order parameter.
Dedicated to the students at The Ohio State University
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Vita

June 2006 .......................... High School Diploma,
Lake Highland Preparatory School

May 2010 .......................... B.A. Physics, B.A. Mathematics,
Washington University in St. Louis

Sept. 2010-March 2011 ............... Graduate Teaching Assistant,
The Ohio State University

March 2011-present ..................... Graduate Research Fellow,
The Ohio State University

April 2012 .......................... National Science Foundation Graduate
Research Fellowship Awarded

Publications

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6.2 Exact Diagonalization Quench Analysis for Nonzero Inter-species Interaction
Chapter 1: Introduction

The understanding of atom-photon interaction progressed tremendously in recent years due to advances in engineering cavity Quantum Electrodynamic (cQED) devices. The cavity walls and geometry enable existence of standing wave modes in the cavity, thereby allowing the light to interact with intracavity matter for a long duration. As a result, experimentalists were able to observe and manipulate the quantum mechanical nature of photon-atom interaction, opening up a new frontier of what is experimentally possible and theoretically exciting. Experimental proposals, once conceivable only on paper, were brought to fruition in the laboratory.

We offer the theory for an experiment that will utilize this progress to explore open questions in condensed matter physics. We conceive of putting a lattice of bosons in a cavity with near-resonant light. Through numerical simulation, we find that the system exhibits an interesting quantum phase transition between Mott-insulating (MI) and superfluid (SF) phases. Past lattice experiments on bosonic phase transitions probed the phases by destruction of the lattice. However, the resulting “interference pattern” is a coarse and imprecise manner of deducing the atomic state. Here, we provide a novel experimental method to accurately probe the phase without turning off the lattice. Our proposed method only requires photon number measurement, thereby enabling deduction of a critical quantity by an easy procedure.
Our work is at the growing interface between condensed matter physics and atomic/molecular/optical (AMO) physics. As explained above, we advance condensed matter physics by allowing experimentalists to accurately observe a quantum phase transition without destroying the lattice. Furthermore, we propose harnessing the technology provided by AMO to understand a many body atomic system strongly interacting with light. We expect our theoretical investigations will guide future experiments in this burgeoning arena of interacting matter-light systems.

The following section reviews the theory of a single intracavity atom interacting with a field and the related experimental verification. We then summarize the experimental observation of a cloud of many atoms strongly interacting with a field. We wrap up the introductory material by discussing our theoretical model and outlining the remainder of the thesis.

1.1 The Single Cavity Jaynes-Cummings Model

To begin, let us start with a basic model describing a single two-level system interacting with a field inside a cavity, the Jaynes-Cummings Model [1, 2, 3]. Throughout this thesis, we work in units where \( \hbar = c = 1 \). In the presence of any field, the atomic momentum operator is given by \( \mathbf{p} \to \mathbf{p} - e\mathbf{A} \), where \( \mathbf{A} \) is the field’s vector potential. Then, the Hamiltonian is given by,

\[
H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + V(r) + \sum_k \omega_k \psi_k^{\dagger} \psi_k
\]

where \( V(r) \) encapsulates the trapping potential. The last term is the energy contribution from all the field modes in the cavity, where the \( \{\omega_k\} \) are the mode energies. The operators, \( \psi_k \) and \( \psi_k^{\dagger} \), denote the respective photon annihilation and creation
operators in mode $k$. We can re-write the Hamiltonian as,

$$H = H_A + H_I + \sum_k \omega_k \psi_k^\dagger \psi_k$$  \hspace{1cm} (1.2)

$H_A$ denotes the bare atomic Hamiltonian in the absence of light coupling:

$$H_A = \frac{\mathbf{p}^2}{2m} + V(r)$$  \hspace{1cm} (1.3)

$H_I$ refers to the atom-light interaction terms:

$$H_I = -\frac{e}{2m}(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p})$$  \hspace{1cm} (1.4)

We neglected a term proportional to $e^2 A^2$ because it is small due to the $e^2$ factor. From here, we take additional steps to simplify the interaction term. The vector potential, in quantized notation, is given by,

$$\mathbf{A}(r) = \sum_k \gamma_k (\psi_k e^{ik \cdot r} + \psi_k^\dagger e^{-ik \cdot r})$$

where the $\{\gamma_k\}$ coefficients depend upon the light properties. However, $r$ is on the order of the bohr radius, $10^{-11} m$, and for visible light, $k$ is on the order of $10^6 m^{-1}$. Thus, $\mathbf{k} \cdot \mathbf{r}$ is on the order of just $10^{-4}$, so $e^{\pm ik \cdot r} \approx 1$. Then the vector potential reduces to,

$$\mathbf{A} = \sum_k \gamma_k (\psi_k + \psi_k^\dagger)$$  \hspace{1cm} (1.5)

Now that we simplified the vector potential, next let us simplify the atomic momentum operator. Let the set of vectors $\{|n\rangle\}$ denote the eigenvectors of the bare atomic Hamiltonian $H_A$, with eigenenergies $\{\epsilon_n\}$. In other words, $H_A|n\rangle = \epsilon_n |n\rangle$. We make use of this complete basis by defining these projection operators: $\sigma_{nm} = |n\rangle \langle m|$. In doing so, we can use the following representation for the momentum:

$$\mathbf{p} = \sum_{n,m} \langle n|\mathbf{p}|m\rangle \sigma_{nm}.$$  \hspace{1cm} At this point we make the very reasonable and common assumption that the atoms are two level systems. Meaning, there is no possible coupling to any higher levels because the required transition frequencies are far from any of
the cavity field modes. Denoting the ground state with the letter “a” and the excited
with “b”, the respective energies are $\epsilon_b$ and $\epsilon_a$ (see Figure 1.1). As a result of this
simplification, the $\sigma_{nm}$ operators become the familiar Pauli spin operators. The two
level assumption simplifies the momentum to,

$$p = \langle b|p|a\rangle \sigma_{ba} + \langle a|p|b\rangle \sigma_{ab}$$  \hspace{1cm} (1.6)

This is further reducible by exploiting the following commutation relation, $\frac{p}{m} = [r, H_A]$. Taking the inner product of both sides between the $b$ and $a$ states, we get,

$$\langle b|r|a\rangle = im(\epsilon_b - \epsilon_a)\langle b|r|a\rangle = -m\omega_0D$$  \hspace{1cm} (1.7)

The operator $D$ is the atomic dipole operator, defined $D = -i\langle b|r|a\rangle$. Plug equation
1.7 into equation 1.6 to arrive at the following momentum operator representation:

$$p = -m\omega_0D(S_+ + S_-)$$  \hspace{1cm} (1.8)

where $S_+ = \sigma_{ba}$, the atomic raising operator causing transitions from the ground to
excited state. Likewise, $S_- = \sigma_{ab}$ is the atomic lowering operator. Plugging equations
1.8 and 1.5 into the interaction terms, equation 1.4 becomes \( H_I = \sum_k g_k (\psi_k^{\dagger} + \psi_k) (S_+ + S_-) \), where each \( g_k \) is proportional to the atomic dipole operator dotted with the polarization of mode \( k \). That is, \( g_k \propto \omega_0 \mathbf{D} \cdot \mathbf{e}_k \).

However, we can actually simplify the interaction terms further. We first restrict ourselves to just a single cavity mode by assuming that the atomic resonance, \( \omega_0 \), is close to only one mode. Now, expand the interaction term: \( H_I = g (\psi^{\dagger} S_- + \psi S_+ + \psi^{\dagger} S_+ + \psi S_-) \). The first term describes emission, creation of a photon accompanied by atomic transition from the excited state to the ground state. The second term is the reverse process of absorption. We assume that the photon mode is close in energy to the atomic resonance, so the resulting energy change \( \Delta E = |\omega - \omega_0| \) from either of these two terms is small. Heisenberg’s time-energy uncertainty relation tells us that \( \Delta E \Delta t = \text{constant} \), so if \( \Delta E \) is small then \( \Delta t \) must be large. Therefore, the photons are long lived in the system when they are coupled to the atoms by either of the first two terms.

However, the third term describes creation of a photon and transition to the excited state, and the fourth term is annihilation of a photon accompanied by transition to the ground state. Both of these processes produce a large energy change, \( \Delta E = \omega + \omega_0 \), which results in short lifetime photons. We can ignore these latter two terms that produce short lifetime photons because they have very little effect on the system (the Rotating Wave Approximation). Then the interaction term neatly becomes, \( H_I = g (\psi^{\dagger} S_- + \psi S_+) \). The form of this is very intuitive; an atom interacts with the field by emitting or absorbing photons.

As previously noted, the bare atom is just a two-level system, so the atomic Hamiltonian in operator form is simply given by: \( H_A = \omega_0 S_z \), where \( S_z \) is the usual
diagonal Pauli spin operator in the $z$ direction. Plugging in the simplified forms for $H_A$ and $H_I$, we obtain the Jaynes-Cummings Hamiltonian:

$$H_{JC} = \omega_0 S_z + g(\psi^\dagger S_- + \psi S_+) + \omega\psi^\dagger \psi$$  \hspace{1cm} (1.9)

Before proceeding to describe the outcomes of this model, we state our notation for number states. For much of this work, we will denote number states in the following manner: $|n_{ph}, n_b, n_a\rangle$, where the first entry is photon number, the second entry indicates number of atoms in the “b” (excited) state, and the third entry gives the number of atoms in the “a” (ground) state. For this model of just one atom, $n_a + n_b$ is always 1, meaning the atom is either in the ground state ($n_a = 1, n_b = 0$) or excited state ($n_a = 0, n_b = 1$).

Now we ask, what are the eigenstates of this interacting atom-field system? The number of excitations, $n$, is a conserved quantity of $H_{JC}$, so the basis states are just $\{|n,0,1\rangle, |n−1,1,0\rangle\}$. The resulting eigenstates of this 2 by 2 matrix are the nondegenerate polariton states: one lower branch with energy $E_−^{(n)}$ and one upper branch with energy $E_+^{(n)}$. A polariton is an entangled atom-photon quasiparticle and is defined as a linear combination of the aforementioned basis states (see Figure 1.2 for a schematic). From here on, the upper branch polariton will be denoted $|n, +\rangle$, and the lower branch will be denoted $|n, −\rangle$.

These polariton eigenstates arise because the Jaynes-Cummings Model treats the cavity field in a fully quantized manner when deriving the interaction term. Due to field quantization, the atoms and photons are on the same footing, paving the way for the atom-field entangled polariton states. Experimental verification of the polariton exploits this entanglement to observe unique phenomena. Rabi oscillations
Figure 1.2: The eigenstates, known as polaritons, of a system consisting of exactly one atom interacting with a field in a cavity with exactly $n$ excitations. As shown, the polaritons are entangled atom-photon quasiparticles. The lower branch has energy $E^{(n)}_-$ and the upper has energy $E^{(n)}_+$. are one such phenomenon, so we wrap up this section by describing a Rabi oscillation experiment.

Consider placing an initially excited atom into a cavity with some resonant field. The number probability distribution of the field is given by $P_{ph}(n)$. For example, $P_{ph}(n)$ can be a Poissonian distribution corresponding to a coherent state or a Kronecker $\delta$-function picking out a single number state. It is a simple exercise to time evolve the polariton states by solving the Schrödinger equation. From here, one can calculate the probability, $P_{ba}(t)$, to find the atom in the ground state at time $t$ [4]:

$$P_{eg}(t) = \sum_n P_{ph}(n) \sin^2(\chi\sqrt{n + 1}t)$$

where $\chi$ is the “Rabi frequency,” a quantity dependent on $g$ and the field-atom detuning.
Figure 1.3: From Brune, et al. [4]. Figure shows Rabi oscillations from placing an excited Rydberg atom in an empty cavity (A) and in a cavity with a coherent field (B-D).

Figure 1.3 exhibits the first experimental evidence of these oscillations in cQED devices [4]. Figure 1.3A shows the vacuum Rabi oscillations that arise from placing the atom into an empty cavity (with small thermal fluctuations). For figures 1.3B-D, the cavity contained coherent fields with varying amplitudes. The oscillations and resulting discrete Fourier transforms are consistent with theoretical predictions. If the eigenstates of the cavities were not polaritons, then there would not be a sinusoidal probability to exist in the ground state. Rather, the atom would just decay to the ground state after some time and then remain in the ground state. This important experiment evidences the quantum mechanical nature of the cavity field and the resulting entangled polariton states. In subsequent chapters, we will show that our
model predicts polaritonic entanglement as well, so this early experimental progress forms a strong foundation for our proposal.

From here, we add the next layer of complexity by considering a cloud of many atoms inside the same cavity.

1.2 The Dicke Model

Now expand the system to consider a gas of $N$ identical two-level atoms interacting with the cavity field [5]. The Dicke Hamiltonian is given by:

$$H_D = \omega\psi^\dagger\psi + \omega_0 \sum_i S^i_z + \frac{g}{\sqrt{N}} \sum_i (\psi S^i_+ + \psi^\dagger S^-)$$ (1.11)

where the summations are over all atoms in the cloud. A naive expectation is that the atoms always absorb and emit photons independently, so the field amplitude would be proportional to $N$ and phase incoherent. However, above a critical coupling, interaction with the field actually induces collective atomic excitations and emissions, resulting in a phase coherent cavity field [6, 7, 8, 9]. If we initially prepare all $N$ atoms in their excited states, the emitted field has a peak amplitude proportional to $N^2$, further indicating the collective atomic behavior. This well known “super-radiant” phase occurs above critical coupling, $g_c = \sqrt{\omega_0\omega}$. Super-radiance is accompanied by the atoms self-organizing into a checkerboard density pattern.

Dissipation and difficulty reaching criticality stalled experimental verification of super-radiance. Very recently, experimentalists overcame these challenges and observed this phase transition in an open, dissipative system [10, 11]. See Figure 1.4 for the groundbreaking observation of this phase transition in a Rb Bose-Einstein Condensate interacting with light. As the figure shows, a sudden buildup of the intracavity
Figure 1.4: From Baumann, *et al.* [10, 11]. The dashed line in the lower panel shows that atom-field coupling strength, $g$, increases with time due to increasing pump power. The lower panel also indicates the photon number. The upper two panels show the atomic expansion pattern for varying coupling strengths. Above some critical coupling, super-radiance occurs, with its onset evidenced by: 1) the presence of a large intracavity field and 2) formation of atomic momentum components indicating spatial self-organization.

The photon number signifies the onset of super-radiance. Additionally, formation of momentum components in the atomic expansion pattern indicates the self-organization.

The important implication is that one can strongly couple matter and light in an open, dissipative cavity. Such progress indicates the feasibility of studying large atomic systems entangled with light; thus we are encouraged to add the next layer of complexity. Now we ask, what if we put the $N$ atoms in a lattice? Will theory predict a quantum phase transition? How would the cavity field reflect the quantum phase transition? Before delving into these questions further, we briefly review atomic phase transitions of bare lattice bosons.
1.3 The Bose-Hubbard Model

The Bose-Hubbard Hamiltonian models bosonic atoms confined to a lattice, with nearest neighbor tunneling and repulsive local interactions. Define the local bosonic annihilation and creation operators as $a_i$ and $a_i^\dagger$ respectively. Then the Hamiltonian is given by:

$$H_{BH} = -J \sum_{\langle i,j \rangle} (a_i^\dagger a_j + h.c.) + U \sum_i n_i(n_i - 1)$$  \hspace{1cm} (1.12)

The nearest neighbor tunneling amplitude is $J$, and the local repulsion strength is $U$. At sufficiently high $J/U$, the bosons are superfluid, delocalized across the lattice and phase coherent. At low $J/U$, the system is Mott-insulating, characterized by a fixed integer number of atoms per site and large phase fluctuations. Therefore, the competition between the kinetic term ($J$) and the local interaction term ($U$) leads to the MI-SF quantum phase transition [12]. From a pure science standpoint, this model is interesting because it predicts a striking reconstruction of the many body ground state through the development of large fluctuations. The model is also practically important because the bosons can represent Cooper pairs in superconducting systems or Helium atoms on a substrate.

Experimentalists typically study this model by placing the bosons in an optical lattice, a lattice formed by shining counter-propagating laser beams on the atomic gas. The laser beams create a periodic potential in each spatial direction, and atoms exist at the potential minima. These minima thereby serve as artifical lattice sites. Beneficially, experimentalists can actually tune the ratio $J/U$, by increasing (or decreasing) the intensity of the lasers forming the lattice, effectively increasing (or decreasing)
Figure 1.5: From Greiner, et al. [14]. The results indicate a quantum phase transition from the superfluid to Mott-insulating phase in Rb atoms confined to a three-dimensional optical lattice. In panel a, the lattice is absent and the interference pattern is that of a Bose-Einstein Condensate. The panels b-h show atomic interference patterns for increasing lattice depths. Initially the system is superfluid, signified by peaks in the interference pattern. With increasing well depth, we see a buildup in higher order peaks due to increased localization of the atomic wavefunctions. Gradually, the peaks wash out and we are left with an incoherent background, indicating the Mott-insulating state.

the lattice depth. Increasing the lattice depth amounts to localizing an atom more on a particular lattice site, thereby increasing $J/U$ and driving the system more Mott-insulating. With this fantastic capability, the optical lattice provides a clean testing ground to both realize quantum phases and to tune across the phase boundary just by adjusting the laser strength [13, 14].

Figure 1.5 shows a famous experimental realization of the MI-SF phase transition in an optical lattice. The experimentalists prepared a system with some initial lattice depth. They then abruptly turned off the lattice and allowed the atoms to expand in free space for a long time (several milliseconds). They then imaged where the atoms ended up on a screen some distance away. By this expansion process, the atomic wavefunctions interfere, leading to an interference pattern dependent on the initial quantum phase. Precisely, the interference pattern is equivalent to a Fourier transform of the many body atomic wavefunction. If the wavefunction was superfluid,
the atoms were initially delocalized across the lattice. In Fourier space, superfluidity corresponds to localized peaks at all reciprocal lattice vectors (Figure 1.5b). On the other hand, a Mott-insulating state is characterized by local atomic wavefunctions, resulting in a delocalized Fourier pattern with many components in $k$ space (Figure 1.5h). Panel a shows the interference pattern of a Bose-Einstein Condensate without a lattice. As we proceed from panel b to panel h, we see a smooth transition from the SF phase to the MI phase as the atoms become increasingly localized.

With these experimental results, we come to the heart of the problem we aim to address. The ensuing section explains the disadvantages of relying on interference patterns and introduces our proposal to resolve the issues for a unique quantum phase transition.

### 1.4 The Dicke-Hubbard Model

While tuning across the Bose-Hubbard phase boundary is easy, the actual measurements of the phases are imprecise. In particular, theoretical work suggests that interference peaks are not actually sufficient evidence of superfluidity because such peaks may exist above the critical temperature [15]. Furthermore, there is still an open question of how to measure the order parameter, the quantity characterizing the quantum phase. These interference pattern measurements often lack sufficient sensitivity due to noisy backgrounds in the data, so it is difficult to obtain a robust measurement of the order parameter.

Fortunately, experimentalists can acquire time resolved photon statistics with high precision, so our work aims to circumvent these experimental roadblocks by modeling the atoms coupled to photons. We then seek to understand how the intracavity light
reflects the phase and the order parameter magnitude, enabling insight into the phase by photon statistics alone. Our work is unique and advantageous because photon measurements are quantitatively accurate and do not require turning off the lattice. Furthermore, the quantum phases we propose to probe involve two species of bosons, the ground and excited bosons, rather than the unispecies transition encapsulated by the Bose-Hubbard Model. Therefore, we contribute to an understanding of the novel phases arising from a matter-light system.

In addition to the aforementioned experiments, numerous others succeeded in studying light-matter interactions at the quantum level [16, 17, 18, 19]. Notably, the group of Thompson and collaborators succeeded in placing an optical lattice inside a cavity, albeit a thermal system [19]. Thus, we expect realization of our proposal in the imminent future, and our work will guide and strengthen these future experiments.

To meet our ends, we propose to marry the Dicke and Bose-Hubbard Hamiltonians and focus on the “Dicke-Hubbard Hamiltonian”. This model describes atoms in a lattice coupled to a single mode field inside a cavity. Like the Bose-Hubbard model, the atoms tunnel to nearest neighbor sites and can thereby take on superfluid character. Like the Dicke model, the atoms experience a local interaction with the cavity field.

As in previous works [20, 21, 22], we find a rich phase diagram for this system, including two component superfluidity and Mott-insulating states. Some of these past investigations [20, 21] approximate the cavity field to be coherent, whereas we do not make this assumption. As a result, we find differing shapes and positions of the phase boundaries when light is present. However, the coherent state approach produces Mott-insulating eigenstates that are not entangled. We show in Chapter
7 that this description fails to predict oscillatory photon dynamics after an abrupt increase to a large optical lattice depth. This coherent state prediction is in direct disagreement with an exact computation on a small system (Chapter 5), suggesting the entanglement is an important piece of the picture. We explore these oscillations by a theoretical analysis that allows light-matter entanglement, and we show that the entanglement is key to witnessing the oscillations. As we will show, these photon dynamics are a useful probe of the order parameters. Thus, we consider our approach to solving the Dicke-Hubbard Model beneficial for our goals. This abrupt increase to a large lattice depth, known as a quench, has been investigated experimentally in the context of the Bose-Hubbard Model [23, 24], so performing a quench is experimentally feasible, at least in the absence of a cavity field.

Our main results are the following:

1. We show that in the presence of a cavity field, the intracavity photon number is lower in the superfluid phase than in the Mott insulating phase. Photon counting thereby provides a coarse method of differentiating the atomic phases.

2. After the quench, the photon number oscillates as a function of time as a result of the entanglement. The amplitudes of the oscillations increase with the initial value of atomic tunneling parameter and correspondingly the initial order parameter. Thus, one can deduce the initial order parameter by recording the photon number dynamics after the quench.

Our paper is organized as follows. In Chapter 2, we begin with the theoretical model and explain the mean field approximation used to solve the model. Chapter 3 contains the equilibrium computations, exhibiting the first main result. In Chapter
4, we proceed to examine the quench dynamics and provide the second main result. In Chapter 5, we corroborate the results of Chapters 3 and 4 by a complementary approach, exact diagonalization of a small system. In Chapters 2-5, we consider only the case with zero inter-species interactions; Chapter 6 goes beyond this assumption by considering the cases of intermediate and large inter-species interactions. Finally, Chapter 7 explains why the coherent state approach fails to predict the oscillatory photon number dynamics after the quench.
Chapter 2: The Dicke-Hubbard Hamiltonian

Using the same notation as a previous work [21], we begin with the following theoretical model. Let $H_a$ and $H_b$ denote the bare atomic Hamiltonians of the ground and excited states, respectively. Let $H_{\text{field}}$ denote the energy contributions from the photon field and the photon-atom coupling. Then, the complete system Hamiltonian, assuming negligible cavity dissipation and spontaneous emission, is given by:

$$H = H_a + H_b + H_{\text{field}} - \mu_1 N_1 - \mu_2 N_2$$

(2.1)

where,

$$H_a = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + \text{h.c.}) + \epsilon_a \sum_i \hat{n}^a_i$$

(2.2)

$$H_b = -J \sum_{\langle ij \rangle} (b_i^\dagger b_j + \text{h.c.}) + \epsilon_b \sum_i \hat{n}^b_i$$

(2.3)

$$H_{\text{field}} = \omega \psi^\dagger \psi + \frac{g}{\sqrt{N_{\text{sites}}}} \sum_i (b_i^\dagger a_i \psi + b_i a_i^\dagger \psi^\dagger)$$

(2.4)

Here, $a_i$ and $b_i$ are the respective ground state and excited state bosonic annihilation operators at site $i$, and $\psi$ is the annihilation operator of the cavity field. As with the aforementioned models, the on-site field-atom coupling, $\frac{g}{\sqrt{N_{\text{sites}}}} (b_i^\dagger a_i \psi + b_i a_i^\dagger \psi^\dagger)$, allows for atomic transition to its excited state by absorption of a photon and the reverse process of emission. The parameter $J$ is the atomic nearest neighbor tunneling
amplitude, assumed to be the same for both species. The field energy is $\omega$, the excited state energy is $\epsilon_{b}$, and the ground state energy is $\epsilon_{a}$.

Note that, unlike the Bose-Hubbard model, there are no finite local repulsive terms. We assume that intra-species interaction is hardcore, meaning at most one of each atomic species can populate a single lattice site. For now, we further assume the inter-species interaction is zero. We will show in Chapter 6 that the results do not qualitatively depend on the strength of reasonably finite inter-species interactions. We make these choices because we seek to understand how competition between the tunneling and the local atom-photon interaction, the ratio $J/g$, drives the quantum phase transition.

Finally, we work in the grand canonical ensemble with two chemical potentials corresponding to two conserved quantities: the total number of atoms and the total number of excitations. The total number of atoms is $N_1 = \sum_i n_i^a + n_i^b$, with chemical
potential, $\mu_1$. In defining the total number of excitations, $N_2$, we follow reference [21] in order to allow comparison with this closely related work. Therefore, let $N_2 = \psi^{\dagger}\psi + \sum_i \frac{1}{2}(n^b_i - n^a_i + 1)$, with chemical potential $\mu_2$. Physically, we correlate the value of $\mu_2$ with the strength of an external pump laser that is responsible for maintaining the cavity field. That is, a more strongly driven system is expected to have a larger value of $\mu_2$ and thereby more excitations.

We are primarily interested in the dependence of the system on $J/g$, as we want to focus on the relation between the atomic phases and the light-matter coupling. Thus, we choose $g$ to be our unit. To keep consistency and allow comparisons with previous works [20, 21], we also fix the following parameters: $\mu_1/g = -0.6$ and $\delta/g = 1$, where $\delta = \epsilon_b - \epsilon_a - \omega$. Note also that, at least when the lattice is absent, an experimental scheme [25] exists to reach the regime $g \approx \delta$. According to this theory, one can tune these parameters by inducing Raman transitions between stable atomic states and controlling the lasers’ energies and Rabi frequencies. Feasibly, this regime is within reach when the lattice is present as well. We briefly consider departures from $g = \delta$ in the following section but otherwise stick to this choice throughout the thesis.

To make the atomic tunneling terms more tractable, we employ a mean field approximation. Following known methods [26, 27], we first make the following exact transformations:

\begin{align}
a_i = \langle a \rangle + (a_i - \langle a \rangle) = \langle a \rangle + \delta a \\
b_i = \langle b \rangle + (b_i - \langle b \rangle) = \langle b \rangle + \delta b
\end{align}

We then make this approximation: the fluctuations, $\delta a$ and $\delta b$, from the mean fields, $\langle a \rangle$ and $\langle b \rangle$, are very small. Then we can throw away terms of order $\delta a^2$ and $\delta b^2$ to arrive at the mean field Hamiltonian $H_{MF}$.
\[ H_{MF} = \sum_i H^a_{MF} + H^b_{MF} + \omega \psi_i^\dagger \psi_i + \]
\[ + g(b_i^\dagger a_i \psi_i + b_i a_i^\dagger \psi_i^\dagger) - \mu_1 n_{i1} - \mu_2 n_{i2} \]

(2.7)

with

\[ H^a_{MF} = \epsilon_a n_i a_i - zJ \phi_a (a_i^\dagger + a_i) + zJ|\phi_a|^2 \]

(2.8)

\[ H^b_{MF} = \epsilon_b n_i b_i - zJ \phi_b (b_i^\dagger + b_i) + zJ|\phi_b|^2 \]

(2.9)

where we define \( \phi_a = \langle a \rangle \) and \( \phi_b = \langle b \rangle \), the respective superfluid order parameters for the lower and upper level atoms. The factor \( z \) is the “coordination number” of the lattice, twice the dimension for simple lattices. The above atomic Hamiltonians have an intuitive interpretation; each site behaves independently, and atomic number fluctuations from a particular site occur by coupling to mean atomic fields, \( \phi_a \) and \( \phi_b \). When \( \phi_a = \phi_b = 0 \), there is no transfer of atoms between a particular lattice site and the mean field. Thus, number fluctuations are zero and each atom is localized to a particular lattice site; the system is Mott-insulating. On the other hand, larger mean fields correspond to stronger atomic hopping terms, resulting in larger on-site number fluctuations characteristic of superfluidity. Therefore, the order parameter magnitudes directly quantify the superfluidity of the system. For this reason, measuring the order parameters in experiment is crucial to studying the transition.

Before proceeding, note that we also consider \( \psi_i \) to be a local operator, rather than global. To justify this, note that \( \psi = \frac{1}{\sqrt{N}} \sum_{i} e^{i\mathbf{k} \cdot \mathbf{r}_i} \psi_i \). However, \( |\mathbf{k} \cdot \mathbf{r}_i| \) is negligibly small, allowing the replacement \( \psi \rightarrow \sum \psi_i \). The following sections exhibit the results of our mean field assumptions.
Chapter 3: Equilibrium Phases and Expectation Values

We numerically minimize $H_{MF}$ with respect to $\phi_a$ and $\phi_b$ to determine the equilibrium phases in the $\frac{\mu_2}{g} - \frac{J}{g}$ plane. See Figure 3.1 for the phase diagram. The “normal” regime, in which light is absent, is in strong qualitative agreement with prior works [20, 21]. For the remainder of this work we will focus on the upper half of the phase diagram, in which $\langle \psi_i^\dagger \psi_i \rangle$ is nonzero, because we want to establish the relationship between the phases and photon number. As noted, our phase diagram differs from references [20, 21] in the shape and position of the phase boundary in this light-entangled regime due to a different set of assumptions to treat the light. Furthermore, $H_{MF}$ predicts a first order phase transition in this region (see Figure 3.2a), whereas [20, 21] predict a second order phase transition.

Within mean field, the light-entangled MI ground state is a lower branch polariton state on each site, $|S_{MI}\rangle = \Pi_{i=1}^{N_{\text{sites}}} |n_{ph},-\rangle_i$, with $n_{ph}$ dictated by the Hamiltonian parameters. This is clear from setting $\phi_a = \phi_b = 0$ in $H_{MF}$ and then noting that $H_{MF} \rightarrow \sum_i H_i^{JC}$, where $H_i^{JC}$ is simply the Jaynes-Cummings Hamiltonian on site $i$. In contrast to the MI state of the one-species Bose-Hubbard Model, this state is characterized by $n^i_a + n^i_b = 1$. Looking at the upper right quadrant of Figure 3.1, the light-entangled superfluid phase is a delocalized state, thereby necessarily a linear combination of multiple polariton states.
Figure 3.1: The phase diagram shows four distinct phases in the $\mu_2/g - J/g$ plane. Here $\mu_2$ controls the number of excitations, $J$ is the inter-site tunneling rate, and $g$ is the atom-photon coupling. The lower half of the phase diagram has no excitations, leading to a typical MI-SF transition of uni-species hardcore bosons. For larger $\mu_2$, we predict a field-matter entangled system. As one tunes $J/g$ in this region, there is a MI-SF transition of both atomic species.

Looking at Figure 3.2b, we also see that the MI to SF phase transition is associated with a drop in the photon number density, and the number decreases continuously as we raise $J/g$ in the SF regime. This behavior results from nonlocal hopping taking precedence over the local light-atom interaction with increasing $J/g$. Thus, the photon number tracks the atomic phase, so photon number measurement is an easy experimental method to accurately detect superfluidity.

Turning to Figure 3.3a, we see that this trend is actually reversed at very low $g$; at sufficiently low hopping, there are no intracavity photons and the number of photons increases with hopping after some critical hopping value. The results are sensible because large $J$ favors the existence of delocalized upper level atoms and thereby intracavity photons to mediate transitions to the upper level. At very large $g$, the atom-field interaction completely dominates over the hopping interaction, so the
Figure 3.2: a) Plot shows how the mean field order parameters of the “a atoms” and “b atoms” vary with $zJ/g$, indicating a first order transition when the system is light entangled. b) Photon number density, $\langle n_{ph} \rangle$, as a function of hopping strength, $zJ/g$. c) Measure of photon absorption, $|\langle \psi_i b_i^a a_i \rangle|/\sqrt{n_{ph} n_b n_a}$, as a function of hopping strength, $zJ/g$. Plots a-c indicate that the Mott insulator to superfluid phase transition is accompanied by a change in both the intracavity photon number and the photon absorption strength.

Figure 3.3: These plots provide insight into the range of $g/\delta$ over which the system retains the trends of Figures 3.2a and 3.2b. In both plots, we see that these trends remain true for a large intermediate window of $g/\delta$ values. We do not normalize the photon absorption strength, Figure b, as we had done in Figure 3.2b because the photon number is zero or very low at low $g$.

system is a Mott-insulator of polaritons regardless of the hopping strength. Thus, both the low $J$ curve and the large $J$ curve converge in the large $g$ regime. Each
plateau corresponds to a different polariton state, $|n_{ph}, -\rangle$, where $n_{ph}$ depends on the value of $g$. Aside from these low $g$ and high $g$ regimes, there is fortunately a large intermediate range of couplings which maintain the trends of Figure 3.2. Thus, we proceed using the parameters of Figure 3.2 because the trends are not sensitively dependent on the value of $g/\delta$ as long as we stick to an intermediate value.

From here, we provide a method of elucidating the order parameters.
Chapter 4: Quench Dynamics as a Probe to Determine the Initial Order Parameters

The above features as well as past works [28, 29] provide methods to distinguish the two atomic phases. However, these works do not tell us how light observables reflect the order parameter magnitudes in a transition driven by changing $J/g$. That is, as one increases the hopping, the order parameters increase, and we would like a method to probe this trend via light observables. To do so, we need a quantity that varies sensitively with $J$. Looking at Figure 3.2c, we see that the normalized value of $|\langle \psi_i b_i^\dagger a_i \rangle|$, a measure of the strength of photon absorption, does in fact decrease sensitively with $J$. As with the average photon number, Figure 3.3b confirms there is a large intermediate range over which this trend is valid. Unfortunately, $|\langle \psi_i b_i^\dagger a_i \rangle|$ is not a directly measurable quantity. We therefore consider the equation of motion of the intracavity photon number. The number dynamics directly depend on $\psi_i b_i^\dagger a_i$ according to,

$$\frac{d(\psi_i^\dagger \psi_i)}{dt} = ig(\psi_i b_i^\dagger a_i - \psi_i^\dagger b_i a_i)$$  \hspace{1cm} (4.1)

Furthermore, the time evolution of $\psi_i b_i^\dagger a_i$ depends on the order parameters, $\phi_a$ and $\phi_b$:

$$\frac{d(\psi_i b_i^\dagger a_i)}{dt} = i[H_a, \psi_i b_i^\dagger a_i] + i[H_b, \psi_i b_i^\dagger a_i] + ...$$  \hspace{1cm} (4.2)
\[
\frac{d(\psi_i b_i^\dagger a_i)}{dt} = iz \phi_{\alpha} \psi_i b_i^\dagger + ... + -iz \phi_{\beta} \psi_i a_i + ...
\] (4.3)

Therefore the absorption strength is the bridge between an observable, the photon number dynamics, and the desired quantity, the superfluid order parameters. Here we propose a way of accessing the information contained in the photon field’s time evolution.

Our method exploits the quench dynamics to capture how the field-matter entanglement affects the photon number dynamics. Consider a system in the light-entangled superfluid regime. In this regime, we can assume that the time scale of the atomic hopping is much greater than the rate of photon leakage out of the cavity. That is, for a dissipation rate, \( \kappa \), we can assume \( \kappa << J \), so the system is approximately in the ground state of the mean field Hamiltonian, \( H_{MF} \). Now consider abruptly raising the lattice depth such that \( J \approx 0 \). The system retains memory of its initial state, and this state time evolves with the zero-hopping Hamiltonian, \( H_{final} \). At this point we can expect the dissipation through the cavity walls to become an important factor, both in the observation of the system and its effect on the system dynamics. Therefore, we phenomenologically include the dissipation through the cavity walls by tacking on a \( -i \kappa \psi_i^\dagger \psi \) term to \( H_{final} \). Thus, the resulting effective Hamiltonian, \( H_{final} \), guiding the time evolution is given by:

\[
H_f = \sum_i (\epsilon_a n_{ia} + \epsilon_b n_{ib} + \omega \psi_i^\dagger \psi_i + \mu_1 n_{i1} - \mu_2 n_{i2} - i \kappa \psi_i^\dagger \psi_i)
\] (4.4)

Figure 4.1a shows the average intracavity photon number density as a function of time after the quench. Both in the absence of dissipation (red) and in the presence of
Figure 4.1: a) Quench dynamics of the intracavity photon number density, found from time evolving an initial superfluid state. The red curve assumes zero dissipation. The blue curve accounts for photon leakage by averaging over 10000 wave function Monte Carlo realizations ($\frac{\kappa}{g} = .05$). The dissipation modifies the photon dynamics by causing an overall decay, but the oscillation is preserved for sufficiently small $\kappa$. b) Fourier transform of the zero-dissipation-dynamics indicates two dominant frequencies. The corresponding amplitudes of photon oscillation are denoted $A_1$ and $A_2$. c) Figure shows the amplitudes, $A_1$ and $A_2$, in the presence of dissipation for varying initial hopping. We see that the amplitudes closely correspond to the superfluid order parameters. This trend offers a novel experimental method to deduce the order parameters from photon number dynamics alone. Note: for the plots obtained from Monte Carlo, error bars are smaller than the plot line width.

dissipation (blue) we see oscillatory photon number. We accounted for dissipation by time evolving with wave function Monte Carlo [30, 31]. The probability of detecting a photon outside the cavity is directly proportional to the mean intracavity photon number; thus these intracavity oscillations are expected to result in oscillatory photon counting outside the cavity.

To understand the oscillatory behavior, we write the time evolved SF state of the system as a linear combination of the eigenstates of $H_{final}$:

$$|S\rangle(t) = c_0 e^{-iE_0 t} |0, 0, 0\rangle + c_{001} e^{-iE_{001} t} |0, 0, 1\rangle + c_{011} e^{-iE_{011} t} |0, 1, 1\rangle + \ldots \quad (4.5)$$

$$\ldots + \sum_{k=1}^{k_{max}} \left[ c_{k1} e^{-iE^{(k)}_t} |\eta_{ph} = k, -\rangle + c_{k2} e^{-iE^{(k)}_+ t} |\eta_{ph} = k, +\rangle \right]$$
where \( |n_{ph} = k, -\rangle \) and \( |n_{ph} = k, +\rangle \) are the lower and upper branch polariton states with \( k \) excitations. \( E_{-}^{(k)} \) and \( E_{+}^{(k)} \) are their respective energies, and \( k_{\text{max}} \) is the maximum photon number. The number states \( (|0, 0, 1\rangle \) and \( |0, 1, 1\rangle \) denote \( |n_{ph}, n_{b}, n_{a}\rangle \). The ellipsis indicates that there are many other number states than the ones shown above, but we ignore these states because they do not contribute to the photon number oscillation. From here, \( \langle n_{ph}\rangle(t) \) is simply:

\[
\langle n_{ph}\rangle(t) = A_0 - \sum_{k=1}^{k=k_{\text{max}}} A_k \cos(\Omega_k t)
\]

with \( \Omega_k = E_{+}^{(k)} - E_{-}^{(k)} \). Each frequency is the energy difference between the upper and lower polariton states with \( k \) excitations. Further, we can see that oscillation occurs only when the initial state has nonzero weight in both of these polariton states; more precisely, both \( c_{k1} \) and \( c_{k2} \) must be nonzero in order for the number to oscillate at frequency \( \Omega_k \). For the particular parameters of Figure 4.1, the initial (mean field) state of the system has weight in the following states: \( |n_{ph} = 1, -\rangle \), \( |n_{ph} = 1, +\rangle \), \( |n_{ph} = 2, -\rangle \), and \( |n_{ph} = 2, +\rangle \). Therefore, two frequencies are present in the quench dynamics (Figure 4.1b). The dominant frequency corresponds to the \( k = 1 \) polariton, and the subtler frequency corresponds to the \( k = 2 \) polariton.

Next we explore how the dynamics change as the initial order parameters change. Consider a system with low initial \( J \) such that the initial system is Mott insulating. As noted in the previous section, the initial state here is simply a polariton on each site, \( |S_0\rangle = \prod_{i=1}^{N} |n_{ph}, -\rangle_i \). If there is zero dissipation, this state is just an eigenstate of \( H_{\text{final}} \), so all amplitudes are zero. In the presence of dissipation, there is a small offset from zero.

Now consider starting in the superfluid regime. At smaller initial \( J \), the initial state has more weight concentrated in one particular polariton state over the others.
That is, from the set of \{c_{k1}, c_{k2}\} one particular mode dominates over the others. For larger initial $J$, the initial state has its weight diffused over multiple polariton states, so all the modes from the set \{c_{k1}, c_{k2}\} contribute. The validity of these statements is reflected in Figure 3.2c, where $|\langle \psi_i | b_i^\dagger a_i \rangle|$ decreases with $\frac{\hbar J}{\hbar g}$ as the nonlocal kinetic term increasingly dominates over the local atom-photon interaction. Thus, if we start in the superfluid regime, the amplitudes, $A_k \propto c_{k1}c_{k2}$, increase with the hopping.

Figure 4.1c exactly exhibits this correspondence between the order parameter and the amplitudes of oscillation. The amplitudes, $A_1$ and $A_2$, are extracted by fitting the Monte Carlo curves to the following:

$$
\langle n_{ph} \rangle(t) = A_0 - (A_1 \cos(\Omega_1 t) + A_2 \cos(\Omega_2 t))e^{-ct} \tag{4.7}
$$

We emphasize that Figure 4.1c relates to the dynamics of the intracavity photon number density. The total number within the cavity scales with the number of lattice sites. As a result, the amplitude trends in Figure 4.1c would be far more pronounced in actual experiment. Thus, we come to the central conclusion of this section; the correspondence between the initial order parameters and the photon number quench dynamics allows elucidation of the initial quantum phase from photon statistics alone.

Next, we show that the equilibrium and quench trends are in qualitative agreement with exact diagonalization results.
Chapter 5: Exact Diagonalization Results

This section exhibits the results of a complementary approach to solving the system Hamiltonian, equation 2.1. Rather than approximate the Hamiltonian, we treat the Hamiltonian and all its inter-site fluctuations exactly but only for a small system. The number of basis states goes as $4^{N_{\text{sites}}}(N_{\text{ph}}^{\text{max}})$, where $N_{\text{ph}}^{\text{max}}$ is some maximum number of photons at which we choose to truncate our Hilbert space. We choose $N_{\text{ph}}^{\text{max}}$ to be sufficiently greater than the average number of photons. Due to the exponential scaling of the Hilbert space, the systems we diagonalize are just a few sites in size, so the results will not capture a quantum phase transition. However, the exact diagonalization results should show trends consistent with the quantum phase transition captured by mean field.

Refer to Figure 5.1 to see the $N_{\text{sites}} = 9$ exact diagonalization results, with periodic boundary conditions, for photon number density and photon absorption strength (normalized). We see that both of these quantities decrease with increasing $J/g$, in agreement with the mean field results of Chapter 3. As expected, exact diagonalization does not agree perfectly with mean field and exhibits sharp cusps and imperfections due to finite size effects. Nonetheless, Figure 5.1 tells us that these quantities track the atomic correlations as predicted by mean field.
Figure 5.1: The above shows equilibrium expectation values, calculated from $N_{\text{sites}} = 9$ exact diagonalization. a) Figure exhibits $\sum_i \langle \psi^\dagger b_i a_i \rangle$, normalized by the number of photons, “b atoms”, and “a atoms”. b) Figure shows the average photon number density. Both plots display patterns in qualitative agreement with the mean field results of Chapter 3; that is, these quantities both decrease with the hopping strength. The parameters were chosen consistently with the plots of Chapter 3, with $g$ normalized by the number of sites.

Now we compare the $g/\delta$ dependence from exact diagonalization, Figure 5.2, with the mean field calculation, Figure 3.3. As with the mean field result, we see that the aforementioned trends are reversed at very low $g$; photon number and photon absorption strength are both zero for very low $J$ and above some critical $J$, they increase with increasing $J$. Furthermore, there is a large regime of intermediate $g$ over which the trends of Figure 5.1 are maintained, again in agreement with mean field. Unlike the mean field plots of Figure 3.3, we do not see any plateaus corresponding to polariton eigenstates because exact diagonalization expectedly complicates the picture by accounting for all fluctuations that accompany even very low hopping strengths. Also for this reason, the low $J$ and high $J$ curves do not seem converge in any reasonably high coupling regime, as they do in the mean field case. Putting aside these discrepancies, Figures 3.3 and 5.2 indicate that the two methods agree in
Figure 5.2: Average photon number density (a) and photon absorption strength (b) as a function of $g/\delta$, calculated from $N_{\text{sites}} = 9$ exact diagonalization. The plots indicate there is a large window of $g/\delta$ values over which the system retains the trends of Figure 5.1. Thus, the photon number density and absorption strength track the atomic state’s $J$ dependence as long as $g/\delta$ is in this large window.

the existence of a large intermediate $g$ regime over which photon number and photon absorption strength decrease with $J$. Thus, we can conclude that these quantities track the phase transition, without sensitive dependence on $g/\delta$ as long as we operate within the large window.

Now we discuss the exact diagonalization quench results. Due to computational constraints, we are restricted to 7 sites and we ignore dissipation. Figure 5.3 displays the photon number density as a function of time after the quench for increasing values of the initial hopping. The quench number dynamics clearly exhibit an increase in the number of frequencies and in the overall oscillation amplitude as initial hopping increases. Beyond $J_{\text{initial}} = 1.4$, very little change is visible because the superfluid density tapers off above sufficiently large $J_{\text{initial}}$. 

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Figure 5.3: The figures show the quench dynamics of the intracavity photon number density for varying initial atomic hopping. The plots were obtained by time evolving the exact ground states of a $N_{\text{sites}}=7$ system. As we scan from low to large $J_{\text{initial}}$, we see an overall increase in oscillation amplitude and an increase in the number of frequencies. Such patterns are qualitatively consistent with mean field expectations, suggesting means to infer the initial atomic hopping and thereby the superfluid density from the quench dynamics.

How can we analyze the dynamics visible in Figure 5.3? Exact diagonalization has an inherently much larger Hilbert space than the mean field case. Thus, many more frequencies must contribute to the dynamics. Analysis of the many amplitudes is a far more complicated task because the large number of modes interfere, complicating the Fourier data. To provide a robust analysis, we ask ourselves what quantity would be easy and meaningful for an experimentalist to measure from conducting such an experiment. The experimentalist can easily quantify the trends that are clearly visible in Figure 5.3 by recording how much the photon number is displaced from its original value during its time evolution. Therefore, Figure 5.4 plots this quantity, the
maximum photon density – minimum photon density, for increasing values of initial hopping.

In order to show how this quantity tracks superfluidity, we show the superfluid fractions of the lower (red) and upper (orange) atomic states on the same plot. The superfluid fractions are calculated by imposing a phase gradient at the boundaries of the lattice and calculating the resulting change in energy ([32, 33]). The gradient creates a “superflow” across the boundary, and those atoms that travel with this superflow are in the superfluid. The kinetic energy of the superflow increases the energy of the system relative to the system without the phase gradient. The size of the energy increase directly corresponds to the size of the superfluid fraction. A necessary condition is that $\theta \ll \pi$, so that the change in energy is entirely due to superflow and not other excitations.

A boundary phase gradient corresponds to transforming the Hamiltonian’s boundary hopping elements in this fashion: $(-J a_1^\dagger a_N + h.c.) \rightarrow (-J e^{i\theta} a_1^\dagger a_N + h.c.)$ and likewise for the upper level hopping terms. Instead, it is common to employ a unitary transformation to equivalently transform all hopping terms by the the phase factor, $e^{i\theta/N_{\text{sites}}}$:

$$\sum_{\langle ij \rangle} (-J a_i^\dagger a_j + h.c.) \rightarrow \sum_{\langle ij \rangle} (-J e^{i\theta/N_{\text{sites}}} a_i^\dagger a_j + h.c.)$$ (5.1)

and similarly for the upper level hopping terms. Then, the superfluid fraction is given by:

$$\rho_x = \frac{N_{\text{sites}}(E_{\theta_x} - E)}{J_\theta^2}$$ (5.2)

where $x = \{a, b\}$, indicating which hopping operators (upper or lower level) experience the phase gradient, $\theta_x$. Note that we do not normalize the fractions by the number
of those atoms in the system, so “fraction” here means a fraction of the lattice size, $N_{\text{sites}}$. Figure 5.4 shows the expected rise of the superfluid fractions with hopping. As with mean field, we see that a property of the photons, the maximum density displacement from the $t = 0$ value, closely tracks the superfluid fraction. We see rapid rise of this quantity with the rapid rise of the superfluid fraction, and then both quantities level out together. These statements are just a more quantitative description of the visible trends in Figure 5.1.

Therefore, we have confirmation that exact treatment of a small system qualitatively corroborates the mean field results. Thus far, we have assumed zero inter-species interaction. In the following section we discuss how this interaction may affect the quench results.
Chapter 6: Quench Dynamics for Nonzero Inter-species Interactions

Consider adding an on-site repulsive interaction between the upper level atoms and the lower level atoms. That is, we add on the term $\frac{U_{ab}}{2}n_{ia}n_{ib}$ to the Hamiltonian, equation 2.1. How do we expect finite values of $U_{ab}$ to alter the previous results? The inter-species interaction is a localizing interaction that forces the system to prefer the atom either in its ground or excited state. As a result, the interaction adds an energy cost if both atomic species are delocalized across the lattice because then the upper and lower level are sharing each lattice site. Thus, at a mean field level, reasonable finite $U_{ab}$ values should further support the localized polaritons of the MI state. Thus, we expect that for a decent range of finite $U_{ab}$, the repulsive interaction should shift the critical point to a larger value, but the trends should otherwise be qualitatively similar.

Reference [20] comes to a similar conclusion by employing a BCS approach to approximate $n_{ia}n_{ib}$ with $(\Delta_i^2 + (\Delta_i b_i^\dagger a_i + h.c.))$. The pairing order parameter, $\Delta_i = \langle a_i^\dagger b_i \rangle$, has a self-consistency constraint. The authors then note that $\Delta_i$ is simply a “local photon”. As we treat photons locally in our mean field model anyway, this interaction term is absorbable into the local atom-photon coupling terms. The
Figure 6.1: Plots a and b are mean field results for finite inter-species interactions. a) Shows correspondence between the amplitudes of photon oscillation and the superfluid order parameters for $U_{ab}=2$. The trends are very similar to the $U_{ab}=0$ case, indicating that the proposed measurement scheme is valid for many finite values of $U_{ab}$. b) The analogous plot for $U_{ab}=35$. Here we see the amplitudes and $\phi_b$ flatten, rather than increase, in the superfluid regime. The trend here disagrees with exact diagonalization, indicating a breakdown of the mean field solution for very large values of $U_{ab}$. In both plots, the amplitudes were scaled by a factor of 5 and error bars are smaller than the size of the markers.

Figure 6.1a validates this expectation. The amplitudes of oscillation still track the superfluid order parameters, but the onset of superfluidity occurs at a higher value of $J/g$ than in the $U_{ab}=0$ case of Figure 4.1c. The corresponding exact diagonalization result, Figure 6.2a, is in qualitative agreement.

Now we ask, does this reasoning break down at some sufficiently large $U_{ab}$? Indeed it must because the BCS approach of reference [20] is only valid for weak interactions. Furthermore, within our mean field model, a very large $U_{ab}$ should prefer a system with fewer upper level atoms to reduce the on-site interaction between the two atomic
Figure 6.2: Plots a and b are $N_{\text{sites}} = 7$ exact diagonalization results for finite inter-species interactions. a) As with the mean field case, a small finite inter-species interaction does not cause the trends to deviate significantly from the $U_{ab} = 0$ plots. b) A large $U_{ab}$ does indeed cause a large change to the trends. The plot shown here is in complete disagreement with the analogous mean field plot of Figure 6.1b. Thus, our work is inconclusive in ascertaining the effect of very large $U_{ab}$ but small finite $U_{ab}$ does not appear to alter our main results.

species. Looking at the $\frac{U_{ab}}{g} = 35$ result of Figure 6.1b, we see agreement with this reasoning because $\phi_b$ flattens out to a roughly constant value in the superfluid regime rather than increasing. The amplitude $A_2$ is correspondingly flat, whereas $A_1$ does not appear to track the order parameters at all. However, the exact diagonalization result, Figure 6.2b, disagrees completely with the mean field trend. The origin of the disagreement may be that mean field cannot capture nontrivial atomic density patterns. For example, perhaps large $U_{ab}$ causes the upper level and lower level atoms to occupy spatially alternating sites. Seeking conclusive answers to the effects of large $U_{ab}$ may be worthwhile topics of exploration in future works.
Chapter 7: The Resulting Dynamics in a Coherent Field Ansatz

As noted, references [20, 21] approach the system Hamiltonian by employing a variational state in which the field is a coherent state and the atomic state is the same at every site. Now we calculate the photon number quench dynamics predicted by this variational ansatz. As we will show, this approach does not result in oscillatory behavior, in direct disagreement with exact diagonalization results.

Within this framework, reference [21] gives us that the eigenstates of the Mott insulator are the set \( \{ |\gamma_-\rangle|\rangle, |\gamma_+\rangle|\rangle, |\gamma_1\rangle|n_b = 1, n_a = 1\rangle, |\gamma_0\rangle|n_b = 0, n_a = 0\rangle \}. The states, \( \{ |\gamma_\pm\rangle, |\gamma_1\rangle, |\gamma_0\rangle \} \), are coherent fields. Also,

\[
|\rangle = \alpha|n_b = 0, n_a = 1\rangle + \beta|n_b = 1, n_a = 0\rangle \quad (7.1)
\]
\[
|\rangle = \beta|n_b = 0, n_a = 1\rangle - \alpha|n_b = 1, n_a = 0\rangle \quad (7.2)
\]

where \( \alpha \) and \( \beta \) depend upon the Hamiltonian parameters.

Immediately after the quench, a superfluid state has the following time dependence:

\[
|S_{SF}\rangle(t) = c_1 e^{-iE_- t}|\gamma_-\rangle|\rangle + c_2 e^{-iE_+ t}|\gamma_+\rangle|\rangle + c_3 e^{-iE_1 t}|\gamma_1\rangle|1, 1\rangle + c_4 e^{-iE_0 t}|\gamma_0\rangle|0, 0\rangle
\]

(7.3)
From here, the expectation value of the photon number density is given by, \( \langle n_{ph} \rangle(t) = |c_1|^2 \gamma_-^2 + |c_2|^2 \gamma_+^2 + |c_3|^2 \gamma_1^2 + |c_4|^2 \gamma_0^2 \), a time independent quantity. Thus, quenching even a deep superfluid state would not produce interesting dynamics. Our mean field model produces the oscillatory behavior due to the Mott-insulator’s light-matter entanglement, making the photon number operator non-diagonal in the basis of the MI eigenstates. As a result, oscillation occurs as long as the initial state is a linear combination of two MI eigenstates with the same excitation number. Unlike the MI eigenstates of our mean field approach, none of the above states are light-matter entangled, thereby failing to produce the expected dynamics. Given the contrast of these results with exact time evolution of the 7 site system, we conclude that our mean field model is an advantageous description of the system.
Appendix A: Mean Field Phase Boundaries

Looking at Figure 3.1, the Mott insulating state with no excitations is the state $|0, 0, 1\rangle$, where the notation indicates $|n_{ph}, n_b, n_a\rangle$. The Mott insulating state with excitations is the polariton state, $|n_{ph}, -\rangle = \alpha_-|n_{ph}, 0, 1\rangle + \beta_-|n_{ph} - 1, 1, 0\rangle$. Here, $\alpha_-$ and $\beta_-$ depend on $g, n_{ph}, \omega, \tilde{\epsilon}_b$, and $\tilde{\epsilon}_a$, and are obtained by diagonalizing the Jaynes-Cummings Hamiltonian. Close to but above the boundary, $n_{ph} = 1$. The phase boundary between the two Mott-insulating states is then straightforwardly obtained by setting $\langle 0, 0, 1|H_{MF}|0, 0, 1\rangle = \langle n_{ph} = 1, -|H_{MF}|n_{ph} = 1, -\rangle$.

Now we calculate the phase boundary between the Mott-insulating state with no excitations and the superfluid state (of “a” atoms) with no excitations. Near the phase boundary, we consider the hopping term to be a small perturbation. Therefore, the state near the boundary is given by:

$$|\psi\rangle = |0, 0, 1\rangle - \frac{J\phi_a}{\epsilon_a - \mu_1 - \frac{\mu_2}{2}}|0, 0, 0\rangle$$ (A.1)

Now we solve the self-consistency equation, $\langle \psi|a|\psi\rangle = \phi_a$ to get,

$$\frac{\mu_2}{2} = -(\epsilon_a - \mu_1) - J$$ (A.2)

The remaining phase boundaries were determined numerically.
Bibliography


